2007

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PHYSICAL REVIEW A **76**, 033401 (2007)

Sufficient condition for the coherent control of *n*-qubit systems

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(Received 3 January 2007; revised manuscript received 1 March 2007; published 4 September 2007)

We study quantum systems with even numbers N of levels that are completely state controlled by unitary transformations generated by Lie algebras isomorphic to sp(N) of dimension $N(N+1)/2$ as discussed by Albertini and D'Allesandro [IEEE Trans. Autom. Control **48**, 1399 (2003)]. These Lie algebras are smaller than the corresponding su(N) with dimension N^2-1. We show that this reduction constrains the field-free Hamiltonian to have symmetric energy levels. An example of such a system is an n-qubit system with state-independent interaction terms. Using Clifford's geometric algebra to represent the quantum wave function of a finite system, we present an explicit example of a two-qubit system that can be controlled by the elements of the Lie algebra sp(4) [isomorphic to spin(5) and so(5)] with dimension 10 rather than su(4) with dimension 15, but only if its field-free energy levels are symmetrically distributed about an average. These results enable one to envision more efficient algorithms for the design of fields for quantum-state engineering in certain quantum-computing applications, and provide more insight into the fundamental structure of quantum control.

DOI: 10.1103/PhysRevA.76.033401 PACS number(s): 32.80.Qk, 03.67.−a, 03.65.Fd

I. INTRODUCTION

The coherent control of an N-level quantum system is of interest in fields such as chemical dynamics [1], quantum information processing [2], and quantum communication [3]. It is well known [4,5] that for an N-level system to be completely controllable, it is sufficient that the free-evolution Hamiltonian, along with the interaction Hamiltonian (which could involve a sequence of steps) and all possible commutators among them, form a Lie algebra of dimension N^2, which in general is taken to be $u(N)$. Recently, it has been shown [6] that state-to-state controllability can be achieved with a Lie algebra isomorphic to sp(N) with dimension $N(N+1)/2$. [We use the notation sp(N) for the algebra of the group Sp(N) of $N\times N$ unitary symplectic matrices, as for example in the text by Jones [7]. Other authors denote the same group by Usp(N) [8] or Sp($N/2$) [6].]

In this paper, we show by calculating the Cartan subalgebra that this reduction places a restriction on the types of systems that can be state-to-state controlled. Specifically, not only do the systems have to have an even number of energy levels [6], their field-free energy levels must be symmetrically distributed about an average. An example of such a system is a multiqubit system with state-independent interactions that has $N=2^n$ energy levels. This result in quantum control is important both for developing optimal control schemes in quantum computing [9,10] and for finding algorithms to calculate applied fields for quantum-state engineering [11].

The control equations can be derived from the time-dependent Schrödinger equation

$$\dot{x}(t) = \left(A + \sum_{i=1}^{m} u_i(t)B_i\right)x(t), \tag{1}$$

where the state vectors $x(t)\in\mathbb{C}^n$ give the amplitudes in a basis of free-evolution eigenstates, A and B_i are constant matrices, and the real scalar functions $u_i(t)$ are the control fields. The evolution of an N-level system can be studied by integrating the corresponding matrix equation in which $x(t)$ is replaced by a matrix $X(t)$, each column of which represents an independent state; one follows the evolution of $X(t)$ from the identity matrix $X(0)=I$. If A and B_i are anti-Hermitian, the solutions of $x(t)$ have constant norms $|x(t)|$ and can thus be viewed as lying on a sphere, and the groups that define the complete controllability of Eq. (1) for general systems are those summarized in [4].

In this paper, we study and independently demonstrate a sufficient condition suggested by Refs. [4,6,12] for establishing controllability of a common class of systems that uses sp(N) Lie algebras, which are smaller, namely, of dimension $N(N+1)/2$, compared with N^2-1 for su(N) or N^2 for u(N). We show that the Cartan subalgebra of sp(N) restricts its application to systems where the free-evolution Hamiltonian has a symmetric distribution of energy levels about an average. These systems are a subset of the general ones discussed in Refs. [4,5]. As an example, we illustrate explicitly that a system with four levels (a two-qubit system) is controllable with sp(4), [which is isomorphic to the spin(5) and so(5) algebras, and which has 10 dimensions and is thus smaller than su(4) with its 15 dimensions] only if the field-free energies are of the form E_1, E_2, $-E_2$, and $-E_1$. Similarly, a system with eight levels (a three-qubitsystem) is controllable with a Lie algebra of dimension 36, significantly smaller than su(8) with its 63 dimensions, only if the energies are symmetrically distributed about an average. This result implies that the state-controllability results of Ref. [6] are applicable only to some qubit implementations such as in trapped ions [13,14], and not to other implementations such as NMR [15].

II. SUFFICIENT CONDITION FOR STATE CONTROLLABILITY

The wave function Ψ is constructed as a unitary transformation of a reference or pass state [16], represented in Clifford's geometric algebra by the primitive projector P. The unitary transformation is an exponential operator of anti-Hermitian elements of the Lie algebra for the system

$$\Psi = e^{\mathbf{a}}P, \quad \mathbf{a} \in \text{ Lie algebra}, \tag{2}$$

and P can be represented by the singular matrix

$$P = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{3}$$

with 0's everywhere except at the upper left diagonal position. One can verify the normalization $\mathrm{tr}(\Psi^{\dagger}\Psi) = \mathrm{tr}(Pe^{-a}e^{a}P) = \mathrm{tr}(P) = 1$. In this form, the wave function, as an element of the Clifford algebra, represents an arbitrary single state of the system as a square matrix, corresponding to X mentioned in the previous paragraph but with a single nonvanishing column on the leftmost side. The form (2) is equivalent to a column-matrix representation of the spinor Ψ, but as seen below its algebraic form is useful in manipulations of the unitary operations acting on the pass state.

Our sufficient condition for a Lie algebra that governs the pure-state control of a quantum system is based on the following: the parametrization of the wave function using unitary exponential operators $e^{\mathbf{a}}$ of the Lie algebra defines a complete control scheme if we are able to reach an arbitrary ray in the complete state space. We illustrate the procedure first in general terms and then give explicit examples.

We require that, for any pair of basis states ψ_j, ψ_k of the state space, there exists an anticommuting pair of anti-Hermitian elements $\mathbf{a}_{kj}, \mathbf{b}_{kj}$ of the algebra that relates them:

$$\psi_k = \mathbf{a}_{kj}\psi_j = -i\mathbf{b}_{kj}\psi_j,$$

$$\mathbf{a}_{kj} = -\mathbf{a}_{kj}^{\dagger}, \quad \mathbf{b}_{kj} = -\mathbf{b}_{kj}^{\dagger}, \quad \mathbf{a}_{kj}\mathbf{b}_{kj} + \mathbf{b}_{kj}\mathbf{a}_{kj} = 0. \tag{4}$$

The basis states have the projective form (2) of a minimal left ideal of the Clifford algebra. Assuming unit normalization $(\mathbf{a}_{kj})^2 = -1 = (\mathbf{b}_{kj})^2$, it follows that we can write $\psi_k = \exp(\mathbf{a}_{kj}\pi/2)\psi_j = -i\exp(\mathbf{b}_{kj}\pi/2)\psi_j$, and the more general superposition

$$\begin{aligned}
&\exp\left((\mathbf{a}_{kj}\cos\phi + \mathbf{b}_{kj}\sin\phi)\frac{\theta}{2}\right)\psi_j \\
&\quad = \psi_j\cos\frac{\theta}{2} + \psi_k e^{i\phi}\sin\frac{\theta}{2} \\
&\quad = \exp\left(\mathbf{c}_{kj}\frac{\phi}{2}\right)\exp\left(\mathbf{a}_{kj}\frac{\theta}{2}\right)\exp\left(-\mathbf{c}_{kj}\frac{\phi}{2}\right)\psi_j
\end{aligned} \tag{5}$$

is expressed as a continuous "rotation" with real angle parameters θ, ϕ, in state space, where

$$\mathbf{c}_{kj} = \frac{1}{2}(\mathbf{a}_{kj}\mathbf{b}_{kj} - \mathbf{b}_{kj}\mathbf{a}_{kj})$$

is another element of the Lie algebra, and we noted that $\mathbf{c}_{kj}\mathbf{a}_{kj} = \mathbf{b}_{kj}$ and $(\mathbf{c}_{kj})^2 = -1$. There are thus simple unitary operators as in Eq. (5) to transform any basis state of the system into an arbitrary linear superposition of that state and any other basis state. More generally, it can be shown [17] that products of such unitary operators allow transitions from one basis state to any linear combination of the states. One additional element $\mathbf{b}_{jj}$ is needed to simply change the complex phase of ψ_j:

$$i\psi_j = \mathbf{b}_{jj}\psi_j. \tag{6}$$

The elements $\mathbf{a}_{kj}, \mathbf{b}_{kj}, \mathbf{c}_{kj}$ are generators of the control group and represent the effect of coupling fields. Given any initial basis state ψ_j, a general state of the system is a real linear combination

$$\Psi = \sum_k (\alpha_{kj}\mathbf{a}_{kj} + \beta_{kj}\mathbf{b}_{kj})\psi_j, \quad \alpha_{kj}, \beta_{kj} \in \mathbb{R}, \tag{7}$$

of the $\mathbf{a}_{kj}$ and $\mathbf{b}_{kj}$ generators operating on ψ_j, where for notational convenience we write $\mathbf{a}_{jj} = 1$. In practice, the elements $\mathbf{a}_{kj}, \mathbf{b}_{kj}, \mathbf{c}_{kj}$ are members of the same small set. As we demonstrate below, a set of N distinct elements is sufficient to create, through its commutators, a Lie algebra of $N(N+1)/2$ dimensions.

Calculating the Lie algebra of a higher-dimensional system can require intensive computations, but there is an elegant and efficient approach using techniques of Clifford's geometric algebra. The N-level quantum system can be described using multivectors in a geometric algebra. The bivectors are well known as generators of the spin groups, and it has been shown [18] that in fact every classical Lie group can be represented as a spin group or subgroup thereof. Here we introduce the possibility of using the full set of anti-Hermitian multivectors (including, for example, trivectors and six-vectors) to generate the control group. We illustrate our method with examples of one- and two-qubit systems, and then generalize to show how the control of an n-qubit system can be achieved by a Lie algebra generally smaller than su(N) as long as the field-free energy levels are symmetrically distributed about an average.

Example: Single-qubit control

In the simplest example, Clifford's geometric algebra Cl_3 of three-dimensional Euclidean space enables us to describe a single qubit ($N=2$) [19]. In this case, Pauli spin matrices can represent the three orthonormal vectors (basis elements of grade 1): $\mathbf{e}_j = \boldsymbol{\sigma}_j$, $j=1,2,3$. The products of grade 2,

$$\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2, \quad \mathbf{e}_{23} = \mathbf{e}_2\mathbf{e}_3, \quad \mathbf{e}_{31} = \mathbf{e}_3\mathbf{e}_1, \tag{8}$$

form a basis for the bivector space and generate rotations. There is a single independent element of grade 3, namely, the trivector

$$\mathbf{e}_{123} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3, \tag{9}$$

whose matrix representation is i times the unit matrix. These elements along with the identity span the full linear space of the closed algebra Cl_3.

We can take the basis states of the qubit system to be $\psi_0 = P$ and $\psi_1 = \mathbf{e}_{13}P$. Then we note by the "pacwoman" property of projectors [19,20], namely, $\mathbf{e}_3P = P$, that $i\psi_1 = i\mathbf{e}_1P = \mathbf{e}_{23}P$ and $i\psi_0 = \mathbf{e}_{12}P$. The $N=2$ generators $\mathbf{a}_{10} = \mathbf{e}_{13}, \mathbf{c}_{10} = -\mathbf{e}_{12}$ generate the control Lie algebra spin(3), which is isomorphic to su(2), so(3), and sp(2). An arbitrary state can be expressed by

TABLE I. A matrix representation of orthonormal vectors for some dimensions. The 4×4 matrix representation for five dimensions (5D) is not faithful for the universal Clifford algebra Cl_5 (it is a homomorphism rather than an isomorphism) but does represent all bivectors uniquely and is therefore adequate for state control.

4D and 5D	7D
$\mathbf{e}_1=\boldsymbol{\sigma}_3\otimes\boldsymbol{\sigma}_1$	$\mathbf{e}_1=1\otimes\boldsymbol{\sigma}_3\otimes\boldsymbol{\sigma}_1$
$\mathbf{e}_2=\boldsymbol{\sigma}_3\otimes\boldsymbol{\sigma}_2$	$\mathbf{e}_2=1\otimes\boldsymbol{\sigma}_3\otimes\boldsymbol{\sigma}_2$
$\mathbf{e}_3=\boldsymbol{\sigma}_3\otimes\boldsymbol{\sigma}_3$	$\mathbf{e}_3=1\otimes\boldsymbol{\sigma}_3\otimes\boldsymbol{\sigma}_3$
$\mathbf{e}_4=-\boldsymbol{\sigma}_2\otimes 1$	$\mathbf{e}_4=1\otimes\boldsymbol{\sigma}_2\otimes 1$
$\mathbf{e}_5=-\boldsymbol{\sigma}_1\otimes 1$	$\mathbf{e}_5=\boldsymbol{\sigma}_3\otimes\boldsymbol{\sigma}_1\otimes 1$
	$\mathbf{e}_6=\boldsymbol{\sigma}_1\otimes\boldsymbol{\sigma}_1\otimes 1$
	$\mathbf{e}_7=\boldsymbol{\sigma}_2\otimes\boldsymbol{\sigma}_1\otimes 1$

$$\Psi=\exp\left(-\mathbf{e}_{12}\frac{\phi}{2}\right)\exp\left(\mathbf{e}_{13}\frac{\theta}{2}\right)\exp\left(-\mathbf{e}_{12}\frac{\chi}{2}\right)P,$$

which, in fact, is just the Euler-angle expression for the Bloch-sphere representation of the state [19]. Note that, since the exponents form a closed Lie algebra, no generators outside of the algebra arise from an expansion of the unitary operator [11].

The energy levels of the system are generally the eigenvalues of the free-evolution Hamiltonian H_0. Without restriction, we can assume a basis for the system in which H_0 is diagonal. Since commutators (Lie products) of H_0 with the control transformations must remain within the controlling Lie algebra, we need to construct H_0 from the unit matrix plus elements of the Lie algebra. To ensure that H_0 is diagonal, its contributions from the Lie algebra are restricted to the Cartan subalgebra, defined as the largest set of commuting generators of the Lie algebra. For the two-state system, the Cartan subalgebra of su(2) comprises a single element, namely, the generator $\mathbf{e}_{12}=\sigma_1\sigma_2=i\sigma_3$. We thus construct a general free-evolution Hamiltonian for a two-level system (apart from an offset energy proportional to the unit matrix) as

$$H_0=-i\mathbf{e}_{12}\omega=\omega\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \qquad (10)$$

III. GEOMETRIC REPRESENTATION OF MULTI-QUBIT CONTROL

For systems of multiple qubits, the orthogonal unit vectors of the appropriate Clifford algebra can be represented as tensor products (Kronecker products) of the Pauli matrices as shown in Table I. Bivectors, trivectors, etc., can be obtained by the product of the unit orthogonal vectors among themselves.

Any homogeneous multivector (comprising elements of a single grade g) in the real Clifford algebra Cl_n for an n-dimensional Euclidean space can be classified as Hermitian or anti-Hermitian according to its grade. Elements of grade 0,1,4,5,8,9 or generally whenever the grade is 0 or 1 mod 4, are Hermitian, whereas those of other grades are anti-Hermitian. This is important because the bivectors on one hand, as well as the complete set of anti-Hermitian multivectors on the other hand, form Lie algebras of compact groups. In some algebras for Euclidean spaces of odd dimension, as for example in Cl_3 or Cl_7, the highest-grade multivector (the volume element) is anti-Hermitian but commutes with every element of the algebra, and it therefore must be excluded from the set of anti-Hermitian elements that, together with their commutators, form the Lie algebra.

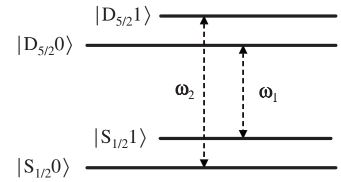

FIG. 1. Symmetric energy levels of the two-qubit system in a trapped ion [14]. The electronic states $|S_{1/2}\rangle$ and $|D_{5/2}\rangle$ are coupled with the first two levels of a harmonic oscillator.

Example: Two-qubit control

The four-level system, understood as comprising two qubits, is controlled using the bivectors plus trivectors of the Clifford algebra Cl_4 of four-dimensional Euclidean space or, equivalently, by the bivectors of a Clifford algebra for a five-dimensional Euclidean space, namely, the nonuniversal Clifford algebra $Cl_5(1+\mathbf{e}_{12345})/2$, a left ideal of Cl_5, which is isomorphic to Cl_4. These bivectors are the elements of a spin(5) algebra, which is isomorphic to so(5) and to sp(4). The primitive projector for two qubits can be represented in terms of bivectors $\mathbf{e}_{jk}$ (see Table I) by

$$P=\frac{1}{4}(1-i\mathbf{e}_{12})(1+i\mathbf{e}_{45}). \qquad (11)$$

The dimension of the control algebra is 10. Because the elements form a closed algebra, in this case spin(5), we know that no other generators are needed for state control. The Cartan subalgebra in this case is two dimensional, and its elements can be represented by diagonal matrices for two of the spin(5) elements, from which we can construct the free-evolution Hamiltonian (apart from a constant offset and with $\hbar=1$)

$$H_0=\frac{i}{2}(\omega_2+\omega_1)\mathbf{e}_{45}-\frac{i}{2}(\omega_2-\omega_1)\mathbf{e}_{12}. \qquad (12)$$

This Hamiltonian has symmetric eigenenergies as represented in Fig. 1 for the case of two electronic levels (trapped-ion qubit) coupled with a harmonic oscillator in its two lowest levels [14]

TABLE II. Generators for transition operators in two-qubit systems (see text).

a	**c**	Transitions
0	$\mathbf{e}_{12}$	$0\leftrightarrow 0, 1\leftrightarrow 1, 2\leftrightarrow 2, 3\leftrightarrow 3,$
$\mathbf{e}_{13}$	$\mathbf{e}_{12}$	$0\leftrightarrow 1, 2\leftrightarrow 3$
$\mathbf{e}_{24}$	$\mathbf{e}_{12}$	$1\leftrightarrow 2, 0\leftrightarrow 3$
$\mathbf{e}_{35}$	$\mathbf{e}_{45}$	$0\leftrightarrow 2, 1\leftrightarrow 3$

$$H_0 = \begin{pmatrix} \omega_2 & 0 & 0 & 0 \\ 0 & \omega_1 & 0 & 0 \\ 0 & 0 & -\omega_1 & 0 \\ 0 & 0 & 0 & -\omega_2 \end{pmatrix}. \tag{13}$$

The sufficient condition for state controllability discussed in Sec. II thus leads to a class of systems *with energy levels symmetrically distributed about a center*, such as those that can be found in trapped-ion qubits [13,14].

The unitary transition operators among the eigenstates can be expressed [see Eq. (5)] in the form $\exp(\mathbf{c}\phi/2)\exp(\mathbf{a}\theta/2)\exp(-\mathbf{c}\phi/2)$, where θ determines the magnitudes of the state amplitudes and ϕ gives the relative phase. The transition between states is complete when $\theta=\pi$, as in a π pulse. Table II shows the generators $\mathbf{a},\mathbf{c}$ for each transition in the two-qubit system. Note that, with $\theta=\pm\pi/2$, the partial transitions $1\leftrightarrow 2, 0\leftrightarrow 3$ induced by the coupled-qubit bivector $\mathbf{e}_{24}$, create four entangled Bell states. We also point out that our algebraic factorization of the unitary transition operator is quite distinct from the factors of 2×2 submatrices proposed, for example, by Ramakrishna *et al.* [21].

Thus all the transitions, together with control of the relative phase, require no more than the five nonzero elements in Table II. These elements and their commutators give all ten independent elements of spin(5). However, only four of the five are required in a minimal set, since, for example, $\mathbf{e}_{45}$ can be obtained from the other four:

$$\frac{1}{2}[\mathbf{e}_{12},\mathbf{e}_{24}] = \mathbf{e}_{14},$$

$$\frac{1}{2}[\mathbf{e}_{13},\mathbf{e}_{35}] = \mathbf{e}_{15},$$

TABLE III. Lie algebras and their controllable n-qubit systems. $N=2^n$ is the number of levels and also the minimum number of elements needed to produce the entire algebra as a result of a recursive application of the Lie product.

Clifford algebra	Qubits	N	Lie algebra	Dim
Cl_3 bivectors only	1	2	su(2)	3
Cl_4 anti-Hermitian	2	4	sp(4)	10
Cl_5 bivectors only	2	4	spin(5) ≅ sp(4)	10
Cl_6 anti-Hermitian	3	8	sp(8)	36

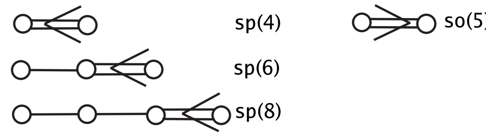

FIG. 2. Dynkin diagrams corresponding to some of the symplectic Lie algebras in low dimensions and the case for so(5), which is isomorphic to spin(5).

$$\frac{1}{2}[\mathbf{e}_{15},\mathbf{e}_{14}] = \mathbf{e}_{45} = \exp(\mathbf{e}_{15}\pi/4)\mathbf{e}_{14}\exp(-\mathbf{e}_{15}\pi/4).$$

Fewer than four is easily seen to be insufficient to generate all the elements of spin(5), so that four is the number of elements that is necessary and sufficient for state control of an arbitrary two-qubit system. The anti-Hermitian multivectors used to define controllable schemes are summarized in Table III for small systems.

The Lie algebras of interest are of dimension $N(N+1)/2$, which is the same as the dimension of the symplectic Lie algebras sp(N) (for even N). The Dynkin diagrams for lower dimension are shown in Fig. 2 including the case of sp(4) to show the isomorphism with so(5) and thus with spin(5).

IV. EXPLICIT CONTROL SCHEME

The method is readily extended to higher even values of N. An explicit control scheme shows that an arbitrary superposition of states in a quantum system with an even number of energy levels symmetrically distributed about an offset

TABLE IV. Matrices representing the initial entry subset of generators, which produce the complete sp(N) Lie algebra upon recursive application of the Lie products on all the possible pairs of generators.

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \quad \cdots$$

$$\begin{pmatrix} 0 & i & 0 & \cdots & 0 & 0 & 0 \\ i & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & i \\ 0 & 0 & 0 & \cdots & 0 & i & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & i & & 0 & 0 & 0 \\ 0 & i & 0 & & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & & 0 & i & 0 \\ 0 & 0 & 0 & & i & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \quad \cdots$$

TABLE V. Matrices representing a basis of the Cartan subalgebra.

$$2i\begin{pmatrix} -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$2i\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\vdots$$

can be produced from another arbitrary superposition using a set of fields, and the Lie algebra implied by these field couplings is of dimension $N(N+1)/2$. This scheme is based on the subspace controllability theorem [22] which describes the method of transferring any superposition of states to any other superposition through a pivot state (pass state). This builds on the work done by Eberly and co-workers on the control of harmonic oscillator states [23,24].

In the general case of the even N-level system with symmetric energies, this scheme is implemented by transferring population in any superposition of states to the ground state $|0\rangle$ through a sequential application of fields. (In Table II, we show the fields connecting all energy states, and in practice some of these may correspond to qubit-qubit couplings. However, this scheme will succeed with any sequentially connected quantum transfer graph [25].) To obtain an arbitrary final-state superposition, the time-reversed sequence of fields is applied starting from $|0\rangle$. Since the system is finite, we conclude that it is arbitrarily controllable. Note that uncoupled n-qubit systems are all cases of the general even-level system with symmetric energy distributions

The control algebra for this scheme contains only $N(N+1)/2$ elements, which can be always constructed defining an initial set of N generators with representation matrices of the form shown in Table IV.

These matrices generally represent linear combinations of the anti-Hermitian generators $\mathbf{a}_{jk}$, $\mathbf{b}_{jk}$, and $\mathbf{c}_{jk}$ introduced above. For two qubits, this initial set of generators can be constructed from linear superpositions of the spin(5) generators as

$$\{-\mathbf{e}_{13},-(\mathbf{e}_{15}+\mathbf{e}_{24})/2,\mathbf{e}_{23},\ (\mathbf{e}_{14}-\mathbf{e}_{25})/2\}. \tag{14}$$

The complete algebra is then found from all the new possible independent commutators calculated recursively until the linear space is exhausted [25]. The Cartan subalgebra can be directly obtained from the initial set (Table IV) by calculating the commutator of the two elements in each column of the table, giving all $N/2$ diagonal matrices of the Cartan basis, as shown in Table V.

This basis of the Cartan subalgebra is used to define free-evolution Hamiltonians, all of which are seen to have energy levels $\pm E_k$, $k=1,2,\ldots,N/2$, symmetrically distributed around an average (the offset) energy. To give another explicit example, the general free-evolution Hamiltonian of a three-qubit system is then

$$\begin{pmatrix} E_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -E_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -E_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -E_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -E_1 \end{pmatrix}. \tag{15}$$

V. SUMMARY

A Lie algebra of $N(N+1)/2$ elements—significantly fewer than N^2 —is shown to be sufficient for arbitrary control of an even-level quantum system, specifically of $[n = \log_2(N)]$-qubit systems, but only if the energy levels are symmetrically distributed about an average energy. All elements of the algebra can be produced by commutator products from a minimal set of N elements, which is the minimum number of generators for state control of the N-level system. These results have the potential to lead to more efficient optimal-control schemes for quantum state engineering and production of entangled states in specific quantum-computing implementations.

ACKNOWLEDGMENTS

Our research is supported by NSERC, CFI, and ORF, Canada.

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