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An inferential community: Poincaré's mathematicians

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ABSTRACT: Inferential communities are communities using specific substantial argumentative schemes. The religious or scientific communities are examples. I discuss the status of the mathematical community as it appears through the position held by the French mathematician Henri Poincaré during his famous arguments with Russell, Hilbert, Peano and Cantor. The paper focuses on the status of complete induction and how logic and psychology shape the community of mathematicians and the teaching of mathematics.

KEYWORDS: inferential community, Poincaré, logicism, complete induction, intuition, creativity, education.

1. INTRODUCTION

This paper is a contribution to a more general reflexion about the didactical use of argument, a topic which receives less interest than dialectical argumentation. An argument can be both didactical and dialectical, especially when "dialectical" is taken loosely as a synonym of "interactive", "dialogical" or "pragmatic", words which are themselves often used loosely. But a didactical context usually presumes a strong and acknowledged epistemic asymmetry missing in paradigmatic dialectical situations, typically seen as symmetrical even when a partial epistemic asymmetry is assumed. Although the distinction between both kinds of arguments can also be seen as gradual, a normative symmetry, especially logical, is assumed in typical dialectical situations such as democratic political debates, juridical decision or debates between experts of the same field. But in a didactical context it might be unfortunate to make too sharp a distinction between logical and epistemic resources since epistemic commitments can be expressed by material implications. And contrary to many dialectical arguments, didactical arguments are not directed "against" an opponent or a sceptic: they are not essentially controversial and rather help to make a community.

I am interested here in a specific inferential community, namely a human community united by the sharing and the use of substantial and relevant argumentative schemes. This community is the community of mathematicians as seen by the French mathematician and physicist Henri Poincaré during his famous controversies with Russell, Hilbert, Peano and Cantor—among others—about the foundations of mathematics (Schmid 2001).

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2. A SPECIFIC PILLAR OF MATHEMATICAL WISDOM?

We certainly have a vague idea of a community of mathematicians which does not explicitly call to the notion of specific schemes of reasoning. Remembering Plato's Meno we may think that anybody is at least a potential member. This view is also held by contemporary ethnomathematicians who stress the universality of mathematical practices.[1]

But can we seriously speak here of a universal community? For the fact that the study of mathematics is difficult and selective supports the opposite view that not everybody is a mathematician: only a (high) degree in mathematics would open the door of this aristocratic society based on demonstrative achievements and inferential know-how which matters as much, and maybe more, as propositional knowledge.

A diplomatic way to solve the tension between the (hyper)democratic and the aristocratic views is to say that both are right: the community is widely open but highly hierarchical. So, let us grant that great and little mathematicians are mathematicians but the great ones are more specific. However, this compromise raises the challenge to conciliate inferential commonwealth and class society. So, let us turn now to Poincaré's solutions.

2.1 Poincaré on mathematical proofs

Most of Poincaré's papers on mathematical reasoning were written in the context of the debates about the foundations of mathematics and logicism, namely the view that mathematics is a deductive science that can be reduced to logic.

Poincaré was a stark opponent to logicism. To understand his point of view it is important to keep in mind that he takes the words "analytic", "logical" and "deductive" to be synonyms and that "logic" amounts to "syllogism". The core of his argument is that mathematics uses specific reasonings that are essential for its construction but cannot be reduced to logic that he claimed to be "essentially barren" (Poincaré 1894, 1902: 31). Here is his argument based on the creativity of mathematics:

Logic cannot teach us anything. Mathematics is both rigorous and fertile. If it were reducible to logic it would not be fertile. Therefore, it is not reducible to logic.

Poincaré makes one step further by claiming that mathematical reasoning is not really deductive because "in some way it participates to the nature of inductive reasoning and this is why it is fertile". It has "a kind of creative virtue" which makes it different from the syllogism (Poincaré 1902: 25). Otherwise, it would amount to "a vast tautology" (Poincaré 1902: 32).

This last comment reminds us that the debate on logicism explicitly involved the opposition between Leibniz and Kant. Poincaré claimed to be a kantian although he was not an orthodox one. In his important set of papers published in 1905 and 1906 and called "Mathematics and logic" (Poincaré 1905b, 1906a, 1906b), he reports that his logicist and leibnizian colleague Louis Couturat claimed that Russell and Peano's works had "finally settled the debate between Leibniz and Kant" by showing that there are no synthetic a priori judgements2. In turn, Poincaré attacks the leibnizian side by claiming that logicism

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[1] D'Ambrosio [2001] gives an overview of this program. According to him, Western Mathematics is only a branch of the more general field of Ethnomathematics.

[2] Poincaré (1905b: 815). This claim was already made by Russell (1903: section 4).

makes of mathematics "just a roundabout way to say that A is A" (Poincaré 1902: 31). Against the threat of such a "vast tautology" with a leibnizian flavour, he purports that since mathematical knowledge is neither reducible to the principle of contradiction nor empirical, it is synthetic a priori. But, as he puts it, this is not the solution but only the baptism of the problem.

According to Poincaré there are several synthetic a priori reasonings in mathematics. But he chose to focus on "complete induction", i.e. reasoning by recurrence, because it is the pillar of arithmetical wisdom, the field where mathematical thinking "has kept pure".

What is complete induction? In Poincaré's own words: "First you establish a theorem for $n = 1$, then you show that if it is true for n-1 then it is true for n, and you conclude that it is true for all integers" (Poincaré 1902 : 38).

Poincaré holds that this reasoning is present "at each step" in arithmetic, and this is why it is the mathematical reasoning "par excellence" (Poincaré 1902: 38). Many other analytical reasonings are based on it and it is the key of the definition and of the proof of the properties of the addition and the multiplication. This key being synthetic a priori it makes the community of mathematicians wide open.

Is this scheme of reasoning foreign to syllogism? No, it is based on it. But it goes further and this is the crucial point. The "essential feature" of complete induction is, "so to speak, to condense …in one single formula, an infinity of [hypothetical] syllogisms" (Poincaré 1902: 39). Without this "tool which allows the step from finite to infinite" it would be impossible to get "the least theorem … essentially new".

What difference does Poincaré make between deduction and induction? We already know that he holds "deduction" as synonymous with "syllogism" and "analytic". Hence it is repetitive and, therefore, epistemologically barren. He also takes deduction as a reasoning which, according to the common place inspired by Aristotle[3], "goes from the general to the particular". For instance, about his colleague Hermite, Poincaré reports that "strictly speaking he is not a logician" for he has "a repulsion for processes purely deductive starting from the general to the particular" (Poincaré 1905: 40). On the contrary, induction goes from the particular to the general. And this is typical of mathematics. "If we open any book of mathematics—Poincaré says—at each page the author states his intention to generalize a proposition already known". Accordingly, induction is crucial in the practice of arithmetic. And since another popular slogan attributed to Aristotle[4] and reported by Poincaré claims that "science is only about generality" (Poincaré 1902: 34), induction is essential to any science.

Before turning to some objections made to Poincaré, let us note that if we associate one inferential community to each intuitive principles of mathematics, only a community of arithmeticians should be associated to complete induction. But the situation is more complicated for when Poincaré talks about complete induction he sometimes refers to the formalized mathematical scheme and sometimes to a more informal notion involving first counting and then generalizing. And so, if complete induction is used in contexts that are not explicitly mathematical (Acerbi 2000), in what sense can we speak of a mathematical inferential community? From Poincaré's point of view it seems reasonable not

[3] The origin of this formula may be Aristotle's *Posterior analytics*, I, 18, 81 a-81 b.

[4] See Aristotle's *Posterior analytics*, I, 31, 87 b.

to separate pure arithmetic from the use of counting in other fields, including logic and day to day practical calculations. This question requires a thorough discussion that I postpone.

2.2 First objections

In his review of Poincaré's book, the inductive status of complete induction brought about one of Russell's major objections. According to him, there is neither induction nor generalization in a complete induction since the premise stating that if one number has property P its follower has it, is already a universal proposition. Then, the final inference is a deduction from a universal proposition to another (Russell 1905). Poincaré's answer came next year: the word "all" is clear when it applies to a finite number of objects but it is not when its scope is infinite, unless you admit an actual infinite. But this is precisely one of the points at stake in the debate since the endorsement of the actual infinite opens the path to the antinomies of the cantorian and the logicist programs (Poincaré 1906b: 316).

If you try to demonstrate the principle of complete induction, Poincaré says, you will either fail or use it somewhere in your demonstration. This is his objection to Peano, Russell and Hilbert as well: all of them make an unfortunate petitio leading to an illusion. Any attempt of demonstration is doomed for it will rely on an axiom "which finally, will merely be the proposition to be shown but translated into another language" (Poincaré 1902: 41). Russell and Hilbert may have had the impression to ruin the mathematical synthetic a priori, but they failed and Kant is still alive (Poincaré 1906a: 34). And things got worse with the apparition of the antinomies. Hence, Poincaré's famous sentence: "Logistic is no longer barren, it generates antinomies" (Poincaré 1906b: 316), which was not the last word of the project to demonstrate the principle of complete induction (Boniface (2001)).

Another of Poincaré's attacks on the logicist program was that mathematics and logic make at most a federation: their trading zone cannot cancel the border between them. But this border can be moved. Poincaré acknowledges the importance and the novelty of Russell's work but wonders about its status. Is it still "logic"? He comments: "let us not be puzzled if some truths, previously declared irreducible to logic as understood previously, are now reducible to logic when taken in its new sense" (Poincaré 1905a: 828). The importation of some parts of arithmetic into logic will make its reduction easier.

Poincaré does not deny the importance of logical reasoning in mathematics but claims that the role of intuition must also be acknowledged. For him, intuition has several forms which apply to many other things than "pure logic" and can be developed by education (Poincaré 1908: 72), a property which shows it does not coincide exactly with the kantian a priori. But intuition has a drawback: it can't give us certainty and rigor (Poincaré 1905: 30-32), it just "commands itself to us with an irresistible obviousness" (Poincaré 1902: 41).

Isn't Poincaré mixing logic and psychology? Russell objects that "'The mind' must be somebody's mind … minds differ from time to time and from person to person; and psychology is not usually considered more certain than arithmetic" (Russell 1905: 65). But we could reply, on behalf of Poincaré, that to be more or less certain is not the point. Russell should not mix normative and singular psychology. According to Poincaré, the obviousness of the principle of complete induction relies on a general property of the mind "which knows itself able to conceive the indefinite repetition of the same action inasmuch it has been actually possible once" (Poincaré 1902: 41). Indefinite iteration is the basic arithmetical action and the mind "knows" it. The logicists who try to make of this inferential principle the definition of the finite number miss the point for complete induction is not a mere property of a specific object of the arithmetical domain. It is the non-logical but inferential tool which allows the building of this domain. Poincaré's view comes close to the aristocratic idea that to be a member of the community of mathematicians to know the principle is not sufficient: you must also be able to use it. But being a priori, this ability is normally open to anybody.

Some authors put forward Poincaré's reply to Russell that "there is no logic or epistemology independent of psychology" to stress Poincaré's psychologism (Poincaré 1909: 482, Goldfarb 1988) while others use it to emphasize its epistemological focus (Detlefsen 1992, 1993, Folina, 1996). Although Poincaré uses the word "psychology" quite a lot and reports many personal anecdotes, it seems to me that the somewhat equivocal term "cognitive" would make a compromise more faithful to his claim that the principles and the practice of mathematics, including learning, teaching and doing research are closely connected.

3. THE COMMUNITY OF MATHEMATICIANS

For the logicists, logic and arithmetic should be human deserts. When you read Poincaré you have the opposite impression: some aspects of the "psychology" of mathematicians are very important for the fertility of their science.

3.1 Turns of mind

Poincaré does not emphasize single psychologies but two general turns of mind which are both relevant and necessary for mathematics because its progress requires their cooperation.

These two turns of mind match rather well with the division of labour made by ancient rhetoric when it distinguished two steps in the production of a *logos*. First came the time of invention (*inventio*), discovering topics and arguments, just followed by the work of disposition (*dispositio*) putting them in a convenient order. The couple made by these two notions has also often been associated with another famous one: analysis and synthesis. And more contemporary discourses see two similar steps in the process of scientific work, discovery and justification, the former being bound to intuition understood as a more or less wild activity of the mind, while the latter is rather obedient to logic, the application of public inference rules.

Poincaré subscribes to a psychological explanation of this division of labour which was already a common place in his time: according to him some minds are rather keen on synthesis when others prefer analysis for these two tendencies are usually not well balanced in a single mind, one dominating the other. And this is also why the dichotomy appears in the long lasting social division of the mathematical labour: according to Poincaré, geometers are intuitive and visualisers while arithmeticians are more gifted to write and count.

In his paper *Intuition and logic in mathematics* Poincaré even makes lively descriptions of great mathematicians, including picturesque details about their physiognomies and tempers. For instance, Bertrand "is always in action", either "fighting against an interior enemy" or "drawing with his hand the figures he studies". On the contrary, Hermite's eyes "seem to fly away from the world" for he looks for truth in himself. Very rightly, Russell pointed out that Poincaré's style is lucid, trenchant, witty, and often appears less profound than it is. Here, the deep point is that "Both kinds of mind are equally necessary to the progress of science: both logicians and intuitive minds made great achievements that the other could not have done." (Poincaré 1905: 29)

You could say that all these internal details about the community of mathematicians relate only to the context of discovery but that the important part is what comes finally out, namely rigorous demonstrations. Unfortunately, this view takes the norm for the reality. The equation between "mathematical" and "analytical" does not hold as well as Perelman wanted us to believe. The great German mathematician Felix Klein, Poincaré says, is able to publish "a mere overview" based on the kind of tinker that Pierre Duhem, the same year, found so typical of the English spirit. Klein is a man who can even take an overview for a "rigorous demonstration" or, at least, "a moral certainty". But this is not too bad when you think that this kind of loose but fertile ideas cannot even sprout in a logician mind.

The will to preserve intuition as "an antidote to logic" as Poincaré says, inspires another of his famous metaphor where the analytic-logic-local conceptual cluster works against the synthetic-intuitive-global one: "Would a naturalist who has made the study of an elephant only through a microscope, believe he has enough knowledge of this animal? It is the same in mathematics. After the decomposition of a demonstration into a host of elementary correct operations, the logician still does not know the whole reality, he will definitively miss this I-don't-know-what [*Je ne sais quoi*] making the unity of the demonstration." (Poincaré 1908: 70)

It may seems easy to turn to the synthesis after an analysis. But it is not that simple. "One is born a mathematician, one does not become one and it also seems that one is born geometer or born analyst" (Poincaré 1905: 27). This has as a consequence that the analyst cannot see the elephant but also that the border between the two turns of mind spreads beyond the aristocratic part of the mathematical community, a tendency confirmed by the behaviour of students. "Some of them prefer to solve problems "by Analysis", Poincaré says, when the others prefer "by Geometry" " for they "see into space" when the others cannot and even get lost.[5]

However, when you turn to the History of mathematics you meet a strange situation: all Ancient mathematicians seem intuitively minded, seem to be geometers. How is it possible when you also know that Euclide is obviously a logician? According to Poincaré the minds haven't changed, it is only the expectations of readers that have changed. The public of mathematics became more demanding and since intuition does not provide certainty, the logicians had to set to work to get rid of vague ideas, loose definitions and rough reasonings. According to Poincaré this safe tendency is recent, fifty years at most. But it has a serious drawback: what mathematics "won in rigor, it lost it in objectivity. It became perfectly pure by going away from the reality" (Poincaré 1908: 69) while intuition "keeps in touch with the real world" (Poincaré 1908: 73). But this general move does not mean that mathematics became "logicized", but rather "arithmeticized" (Poincaré 1908: 69; 1905: 32). Like many French mathematicians, including Descartes and D'Alembert, Poincaré holds that mathematics comes before logic.

[5] All these ideas and the distinction between intuitive and logical minds of experts and students can be found again in Poincaré [1908], Book II, Chapter II, p 65-80.

3.2 Understanding mathematics

There are (at least) two classical solutions to the vexed problem of the acquisition of axioms, namely principles or basic truths of a constituted field of knowledge. And this could also be the case with the principle of complete induction.

One is the authority of the master or, in a more anonymous version, the reliability of the field. This seems Aristotle's option when he discusses didactical arguments in the *Sophistical Refutations* (II, 165a-b). What is typical of this type of argument, he says, is that the principles belong to the discipline; they are not a matter of opinion and the student has to grant them. The other solution calls to one of the many faces of intuition: the student finds the principle by herself. But in any case, the stimulation by the master is essential. Here again, the case of the Meno is paradigmatic: without Socrates' supervision and hints would the slave have considered the diagonal of the square and found out the solution?

Poincaré jointly discusses invention and education in the chapter of *Science and Method* devoted to mathematical creativity[6]. If it is true that complete induction "commands itself to us with an irresistible obviousness" it should be grasped immediately by anybody and the mathematical community should be universal. In logic, the situation is normatively similar since it is supposed to rely "on principles common to all men". More precisely, if you introduce the normative versus factual distinction you can be more restrictive, taking that the rules of logic are only "granted by all well-made minds" (Poincaré 1908: 24). But for Poincaré the factual situation in mathematics is different from the situation in logic for the rules of logic are generally followed while mathematical demonstrations are puzzling for many people. This wider popularity of logic reminds us of his aristocratic "one is born mathematician". And if not everybody is born mathematician, Meno's slave could have been only a happy choice.

What about complete induction? Does it command itself with an irresistible obviousness? Let us go back to Poincaré's formulations. Is it true that "to conceive the indefinite repetition of the same action inasmuch it has been actually possible once" amounts to the same as taking "the step from finite to infinite"? My view is rather that it is easier to conceive an indefinite repetition than to take the step from finite to infinite. It also seems easier to make an indefinite generalization than to apply explicitly the formalized principle of induction. Opinions may differ about it and this confirms that the notion of complete induction covers several different operations or levels of interpretation, the formalized demonstration by recurrence being only one of them.

The discrepancy between the normative and the factual claims concerning the universality of mathematical knowledge is at the very heart of Poincaré's amazement about the most scandalous paradox of mathematics:

> If Mathematics calls only upon the rules of Logic, those which are accepted by all well-made minds, if their obviousness is based on principles common to all men and that nobody would deny without being mad, how come that so many people totally resist them ? (Poincaré 1908: 25)

A presupposition of this question is again the existence of a gap between logic and mathematics for Poincaré presumes that most people are logically-minded (have "well-made minds") but that few are mathematically-minded. However he provides no empirical evi-

[6] Poincaré's views about this point are discussed in Resnik (1996) and Stenlund (1996).

dence about the first point but emphasizes that “according to the masters of secondary school” a majority of people is unable to follow the thread of a mathematical demonstration.

How is it, Poincaré goes on, that a mistake is possible and that many people who can easily follow a short reasoning about daily life cannot follow a mathematical demonstration which is only the “accumulation of little reasonings” analogous to daily ones? (Poincaré 1908: 24). According to him, lay people can follow logical steps but fail to grasp a mathematical demonstration because it involves something more; it requires to grasp the overview, the architecture.

Having first suggested that this problem comes from a lack of memory or attention, Poincaré underlines many exceptions to this explanation, including his own behaviour. He confesses to be unable to do an addition without a mistake and thinks he would make a poor chess player (Poincaré 1908: 25). The solution he finally proposes does not call to the incapacity to make any specific mathematical reasoning like complete induction, but to the distinction between the two turns of mind previously discussed. He does not deny the ability to follow each step of a demonstration, but a demonstration is not a mere “juxtaposition of syllogisms”. Most people simply cannot grasp its organization: they are unable to put the (unproblematic) syllogisms into the right order because they are not able to “guess harmonies and hidden relations”, “to distinguish and to choose” between them, to produce and use analogies. To understand a demonstration or to make a discovery, you have to be an architect, not a mason. Mathematical expertise is based on the ability to see far away. This is why mathematical research will be fertile if it borrows from domains far from each other. Not everybody can do that.

A creative mind has not to choose among all the open possibilities, he looses no time at examining barren combinations because a kind of unconscious pre-selection is at work in the dark. Poincaré reports that his own “sudden illuminations” as he says (Poincaré 1908: 29) came after an unconscious gestation which itself followed a primary conscious phase of reflection. All this is not a matter of logic but of “sensibility”, expert mathematicians having a “special sensibility” at which laymen have a tendency to laugh (Poincaré 1908: 31).

Is there finally any difference between the expert mathematician’s experience and the efficient beginner’s one when both of them grasp the structure, the plan? Poincaré does not answer this question but something appears through his writing. When talking about the expert, his key words sounds rather romantic—illumination, secret of the soul, sensibility, genius—showing that the process of expert invention is internal, individualistic and even solitary. On the contrary the didactical process depends on someone else, it is essentially interactive.

3.3 How to make it clear to the others?

Understanding and doing mathematics is neither a purely analytic nor a purely synthetic activity, it is a compromise of both. And this has to be made compatible with the fact that students have a dominant tendency in the one or the other direction. Some of them, Poincaré says, want to get into the detail of each syllogism while others prefer an overview and some others care mostly about utility. Since minds cannot be changed, a plurality of didactical methods is required to fit their various sensibilities.

But this does not mean that the master has to be analytic with the analytic student and synthetic with the other. First, he has to argue against what his students think they understand, against their first impressions. He ought to say: "You don't understand what you believe to understand. I have to demonstrate what you think is obvious" (Poincaré 1908: 71). Poincaré advises the master not to use premises less obvious than the conclusion; otherwise the students could think that mathematics consists in useless subtleties or fallacious exercises. This concern for beginners motivates even some of the ironical criticisms that Poincaré addressed to his professional opponents. "Against all safe psychology", against "the way the human mind has built mathematics" the cantorian want to begin his teaching with "the general properties of transfinite cardinal numbers" (Poincaré 1908: 81). People "speaking the peanian" (namely Peano's formalism) define the number "one" in a way "eminently apt to give an idea of the number one to people who would have heard nothing about it" (Poincaré 1905b: 823). Poincaré respects Hilbert's mechanical formalist program but "would not advise it to a high school student" who would drop it very soon (Poincaré 1908: 68). Finally, when the logicists do not import intuitive notions into their principles they "define what is clear by what is obscure".

This last comment may seem paradoxical for when the master demonstrates an obvious truth isn't he breaking the rule of not using premises more obscure than the conclusion? Perhaps, but not for a long time. Remember that Poincaré's conception of intuition is not conservative: intuition can be educated. Ways of arguing with students sometimes differ from professional ways because they are still defective. This is why the intuition of beginners has sometimes to keep silent and wait for a more expert doubt that only rigour will silence. And since in a probable allusion to Pascal's definition of the "geometrical spirit" Poincaré claims "you cannot demonstrate everything and you cannot define everything" (Poincaré 1908: 73), a partial call to intuition is sometimes necessary in teaching. And since what matters is right reasoning, even from false premises, you sometimes have to be satisfied with provisional arguments that, later, the master will have to rectify by giving up "the rough form left to theorems not to bother beginners" (Poincaré 1902: 35). But these rough arguments are precious goods belonging to the treasure of the community or rather to the society of mathematicians. For Poincaré's explains that his realistic but optimistic conception of mathematical education is just another consequence of the principle that ontogenesis summarizes phylogenesis (Poincaré 1908: 71): like students, ancient mathematicians were intuitive while analytic rigour has now become almost mandatory among professionals.

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Commentary on *"AN INFERENTIAL COMMUNITY: POINCARÉ'S MATHEMATICS"* by Michel Dufour

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In this part of the world, the contributions of Henri Poincaré to the foundations of mathematics and logic are considerably more cited than read. That this is less so in France—after all the university in Nancy is named in his honour—is nothing but a good thing. Michel Dufour has done us a service in recalling to our attention some of the important, and still unresolved, issues that arise from Poincaré's foundational insights. In a scant 10 pages, Dufour has covered an impressive amount of ground, to which in the time available to me here I can make only brief and selective reference.

1. PETITIO

One of the objections that Poincaré presses against logicism is that it is mired in circularity. In "Les mathématiques et la logique" (1905) Poincaré avers that the logic to which logicists want to reduce arithmetic is already arithmetically pre-loaded. Concerning the logicist definition of number, he writes:

> It is impossible to give a definition without enunciating a phrase, and difficult to enunciate a phrase without putting in it a name of a number, or at least the word "several", or at least a word in the plural. And then the slope is slippery, and at each instant one risks falling into a *petitio principii.* (1905: 821)

Frege himself forewarned against this criticism, although Poincaré would not have been aware of it. Poincaré had insisted that "one cannot speak of *x* and *y* without thinking *two.*" Couturat, picking up the Fregean reservation, argued that logically speaking the number two is not implicated in the use of two variables, and Poincaré's claim to the contrary is grounded in the presumed psychological fact, which may be true, that in circumstances such as these we do indeed think of two. But to accord to this empirical regularity any foundational standing in the semantics of pure quantification is psychologism.

Poincaré is minded to resist the accusation of psychologism, and in a way it is understandable that he would do so. Perhaps the shortest way of saying why is to recount an exchange of many years ago between one of my own students and me. The point of logicism, I said patiently, was to re-express without relevant loss all the truths of arithmetic in a language that lacked an arithmetical vocabulary. When that language is logic, I said, the gains are impressive, because the truths of logic do so much better on the score of epistemological and ontological security. My student said, "Logic isn't that language." "Why not?", I replied. He then went to the blackboard and wrote, "There is at least one *x* such that …" In this little exchange, my student was Poincaré and I, by default, an open-mouthed logicist. This, of course, is the stuff of standoff.

Zenker, F. (ed.). *Argumentation: Cognition and Community. Proceedings on the 9th International Conference of the Ontario Society for the Study of Argumentation* (OSSA), May 18-21, 2011. Windsor, ON (CD ROM), pp. 1-3.

2. VICIOUS CIRCLES

The vicious circle principle bans the use of impredicative definitions in mathematics. It first arose in 1906 in the second installment of Poincaré's "Les mathématiques et la logique". It is a generalization of Richard's own solution of the paradox that bears his name, which Poincaré then marshaled against the Burali-Forti paradox, as well as the logicist definition of number. In this strengthened version, quantifiers used to specify a set must not include that set, or anything whose definition invokes that set. Poincaré would later define a predicative property as one that involves no vicious circles.

The predicativity constraint attracted powerful objections from Zermelo, of which the more or less standard one is that Poincaré has misunderstood the nature of mathematical definition. In 1908 Zermelo writes: "A definition may very well rely upon notions that are [extensionally] equivalent to the one to be defined" (1908: 191).
Nevertheless, Russell adopted the predictivity constraint for the ramified theory of types ("Mathematical logic as based on the theory of types", 1908). Part of the reason is that Russell's logic lacked what Zermelo's objection requires, namely, the principle of extensionality. Russell's penchant for intensional entities in logic—against which Quine would rail in his 1932 thesis and every year of his life thereafter—was not inadvertent. It was principled, not least because, as Russell believed, it provided a "natural" and "positive" non *ad hoc* mechanism for the suppression of impredicativity. But, of course, it is precisely on this point that Poincaré disagreed with Russell. The ban on impredicativity is not *ad hoc* and not in need of, or susceptible of, any underlying justification. It needn't be *made* natural and positive, it already *is* natural and positive. That is to say, we grasp the predicativity constraint as self-evident or *intuitive*. But isn't this psychologism?

3. PSYCHOLOGISM

Poincaré is not a psychologicist in Frege's sense. Poincaré didn't think that arithmetic is an experimental science, or that the truths of mathematics are subjective. Poincaré's psychologism is embedded in the assumption that the epistemological legitimacy of a mathematical claim requires that it not confound our best understanding of how mathematics is learned. So, for example, if transfinite numbers are to be allowed, we must have an account of what makes them learnable. If mathematical induction is to be permitted, we must have an account of what makes it a graspable principle. More carefully, we must not endow these objects and principles with an epistemic character that would, if true, make mathematics unlearnable. Thus mathematical learning theory is deeply implicated in mathematical epistemology, whence its foundational significance.

By the philosophical lights of the present day, this is a breakaway insight, tantamount to a decision to naturalize the epistemology of mathematics; and it provides a welcome clue about how to understand the role of intuitions in Poincaré's philosophy. For intuition now is a learning-theoretic concept, not a mathematical one – at least not directly. It also helps us see the importance of Michel Dufour's emphasis on *community*. Learning mathematics is centrally a matter of a positive response to the inculcations of meanings imparted by learning the language of mathematics. Since language learning is a communal enterprise, the question "Which community?" takes on a central importance, as Dufour observes. Is the mathematical community "open" and "hyperdemocratic"—the community of us all—or is it "highly hierarchical" and "aristocratic"? It is an interesting question, and an utterly necessary one. We see in this the prefiguration of Quine's thesis that a theory of how best to understand a subject-matter will always involve a theory of how its meanings are inculcated.

Poincaré's connection to Quine, by the way, is commonly located in their approaches to conventionalism, something they both shared with Duhem. This is not quite right. The conventionalism Quine shared with Duhem is a holism of which Poincaré is no supporter. But there is another sense of conventionalism which Quine and Poincaré do share. It is the view that, under the right conditions, scientific truths can be contrived by "legislative postulation" (Quine 1960). Of course, Quine extended this same stipulationist latitude to mathematics. Poincaré would have fainted.

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