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ANALYSIS OF A BUILD-OPERATE-TRANSFER SCHEME FOR ROAD FRANCHISING

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ABSTRACT

Private provision of public roads through build-operate-transfer (BOT) contracts is increasing around the world. Under a BOT contract, a private firm would build a road, charge tolls to road users for a period, and then transfer the road to the government. By viewing a BOT contract as a combination of three variables of concession period, road capacity and toll charge, we study optimal BOT contracts which maximize social welfare and allow the private sector an acceptable profit. We also study how to reach optimal BOT contracts, either through bilateral negotiations between public and private sectors, or through competitive auctions.

1 INTRODUCTION

How should society go about expanding its road systems? Who would decide where to provide more road capacity, and how much more? Where would funds for expansion come from? The recent world-wide tendency toward the introduction of commercially and privately provided public roads proves to be an efficient answer to these questions (Roth, 1996). Private provision of roads is driven by a number of factors. A primary motivation is a widespread belief that the private sector is inherently more efficient than the public sector, and therefore builds and operates facilities at less cost than the public sector. Also, the public sector, facing taxpayer resistance, may simply be unable to finance facilities that the private sector would be willing and able to undertake for a profit. In addition, if new road space is provided as an "add-on" to an existing network system, and if road users find it worthwhile to patronize this new road and pay charges, and if the charges cover all costs (including congestion and environmental costs), all may gain benefit, and there would be no obvious losers. Even those who do not use these new roads would benefit from reduced congestion on the old ones (Mills, 1995). For these reasons, the commercial and private provision of transport infrastructure has attracted fast-growing interest in recent years and is being employed to finance modern road systems.

Private provision of roads is typically made through a so-called build-operate-transfer (BOT) contract. Under a BOT contract, the private sector would build and operate the road at its own expense and in turn should receive the revenue from road toll charge for a period, and then the road will be transferred to the government. BOT projects use the market criterion of

profitability for road development and rely on the voluntary participation of private investors, who hope to benefit financially from their participation. A BOT contract generally involves three fundamental decision variables: the concession period, the road capacity and the toll charge. These three variables are crucial for both the private firm and the government to reach their respective objectives: the private firm wishes to undertake the road project for a maximum profit throughout the concession period; while the government aims to maximize social welfare throughout the whole life of the road by awarding the road concession contract. The concession period, representing the number of years for operating the road by the private firm, directly governs the total toll revenue of the private firm as well as the total social welfare gain during the life of the road (the concession period and the post-concession period). The selected road capacity makes impacts on both the private firm's profit and the total social welfare in either direct or indirect manners. First, the road capacity determines the road construction cost, the major investment cost of the private firm for the road project; second, the road capacity affects the congestion degree and thereby the travel time on the road, which in turn affects the travel demand and, as a result, the revenue and the social welfare. The toll charge under the operation of the private sector to a large extent determines the total revenue that the private sector receives and the social welfare gain as well during the concession period. In summary, each of the three fundamental variables of concession period, road capacity and toll charge, plays an important role in forming a feasible BOT contract. Their values determine how, and under what circumstances, a road BOT project is feasible and profitable, and how the project will benefit the private investor, the road users and the whole society.

Most previous analyses of BOT road projects have focused on capacity choice and toll setting and the resulting profitability and social welfare gain; the concession period is usually assumed to be given and fixed (unit cost of capacity per unit period is thus a given constant). An important result in the early literature is the self-financing theorem for congestion pricing and capacity choice of a single road in a first-best environment in which the toll is set equal to the difference between the marginal social cost and the marginal private cost of a trip. Under certain technical conditions, the revenues from optimal congestion pricing just suffice to finance the fixed costs associated with the optimal capacity supply (Mohring and Harwitz, 1962; Mohring, 1976; Keeler and Small, 1977). In a general traffic equilibrium context, Yang and Meng (2000) looked into the profitability and social welfare gain of a single BOT road in a network; Yang and Meng (2002) further showed that the self-financing theorem still holds for each road individually in a full network and consequently to the network in aggregate, provided each link is optimally priced and all capacities are optimized. Recently, Ubbels and Verhoef (2004) and Verhoef (2005) analyzed capacity choice and toll setting by private investors in a competitive bidding framework organized by the government. They considered concessionaire selection based on the various criteria of maximization of capacity or patronage, minimization of tolls or minimization of toll revenues, and compared the resulting welfare gains (or losses) from each criterion.

The recent worldwide experiences of BOT contracts showed that the fixed-term contract suffers certain pitfalls especially when there is uncertainty about future traffic demand: (a) the frequent use of government guarantees, thereby reducing incentives to control construction costs, and (b) government bailouts for almost every franchise that faces financial trouble (Engel et al., 1997). For this reason, flexible-term contract for road franchising has been proposed, and it can be implemented with a fairly straightforward mechanism. For example in a least-present-value-of-revenue auction, the bidding variable is the present value of toll revenues and the lowest bid wins and the franchise ends when the amount has been

collected (Engel et al., 2001 and Nombela and de Rus, 2004). In this case the linkage between traffic uncertainty and revenue uncertainty is effectively broken; the contract term is endogenously determined by the realized level of future demand, so it is shortened in condition of high demand, and extended if traffic levels are low. These flexible-term contract analyses mainly focus on the concession period and demand uncertainty. A critical simplifying assumption is made in that traffic congestion is ignored or the travel time and thus traffic demand is independent of the road capacity or initial investment cost for the new road (equivalent to assuming that the road capacity was predetermined and large enough).

In contrast to previous researches, in this paper we explicitly regard a BOT contract as a combination of all the three fundamental variables: the concession period, the road capacity, and the toll charge. By assuming that the government and the private sector both have perfect information on the project cost and future traffic demand, we investigate the problem of how to set an optimal BOT contract to maximize social welfare, while allowing for an acceptable level of profit to the private sector. We shall classify and analyze the full information "first-best" and "second-best" BOT contracts: the former refers to the case where the social welfare-maximizing point is located in the profitable domain of the aforementioned three fundamental variables, and thus a socially optimum BOT contract can be formed between the government and the private firm; the latter refers to the case where the welfare-maximizing point is located in the unprofitable domain of the three fundamental variables, and thus the government's choice for maximizing welfare is subjected to an active profitability constraint. For the second-best BOT problem, we show that the government should choose the whole life of the road as the concession period in order to maximize social welfare under the constraint of a minimum acceptable profit. Moreover, we introduce a two-player sequential game model between the government and the private firm in negotiating and reaching a feasible BOT contract. A strategy is proposed for the government to obtain a socially optimal BOT contract in which the government just needs to set a minimum level of travel demand, while leaving the private sector to freely choose a preferable combination of road capacity, toll charge and concession period in realizing a travel demand not less than the minimum level. We also examine the conditions under which competitive auctions can lead to optimal BOT contracts.

The structure of this paper is as follows. Section 2 introduces the first-best problem in the absence of profitability constraint and the BOT problem with explicit consideration of concession period. Section 3 systematically examines the properties of an optimal BOT contract for both the first-best and second-best cases. Section 4 proposes a strategy for the government to obtain an optimal BOT contract in a bilateral negotiation game, and also demonstrates that competitive auctions can lead to optimal BOT contracts. Conclusions are given in Section 5.

2 THE FIRST-BEST PROBLEM AND THE BOT PROBLEM

Assume that a new public road is to be built. Let $y \geq 0$ be the capacity of the new road, $q \geq 0$ be the travel demand and $B(q)$ be the inverse demand function (or the marginal benefit function), and $t(q, y)$ be the travel time function. Note that q and y are measured in number of vehicles per unit period. The following demand-supply equilibrium condition always holds:

$$B(q) = p + \beta t(q, y) \tag{1}$$

where p is the toll charged on each user of the road and β is the value of time (VOT) to convert time into equivalent monetary cost (we consider homogeneous users only). Condition (1) simply means that travel demand for the new road is determined by the generalized travel cost. From eqn. (1), we have the following price function

$$p(q, y) = B(q) - \beta t(q, y) \tag{2}$$

where price (toll) p is viewed as a function of traffic volume q and road capacity y. Let $I(y)$ be the road construction cost function. The following basic assumptions are made on the properties of $B(q)$, $I(y)$ and $t(q, y)$.

Assumption 1
(a) The inverse demand function $B(q)$ is a continuously decreasing and differentiable function of q for $q \geq 0$.
(b) The road construction cost function $I(y)$ is a continuously increasing and differentiable function of y for $y \geq 0$.
(c) The travel time function $t(q, y)$ is a continuously differentiable function of (q, y) for $q \geq 0$ and $y \geq 0$; for any $q > 0$, $t(q, y)$ decreases with y; for any $y > 0$, $t(q, y)$ is a convex and increasing function of q.

Consider that the road has a life of $\bar{T} > 0$. Then the first-best problem for this new road project is to choose the capacity of the road and set the toll level (and thereby control the traffic volume) to maximize the total social welfare throughout the lifetime of the road.

The First-best Problem:

$$\max_{q \geq 0, y \geq 0} \bar{T} S(q, y) - I(y) \tag{3}$$

where

$$S(q, y) = \int_0^q B(w) \, dw - \beta q t(q, y). \tag{4}$$

$S(q, y)$ defined by (4) is the social welfare (in monetary unit) obtained per unit period when the travel demand and road capacity are q and y, respectively. Thus the first term in objective function (3) is the social welfare obtained during the whole life of the road, and the second term is the road construction cost, which is the only cost considered here. For simplicity and by convention, we do not consider the maintenance cost related to traffic volume. Also, for simplicity of exposition, hereafter, we do not adopt an interest rate to discount future revenues to its equivalent present values. It should be mentioned here that, *the use of a discounting rate does not alter our analysis results* as shown in Appendix A1.

Now suppose that the road is to be built through a BOT contract. Let T, such that $0 \leq T \leq \bar{T}$, be the concession period of the BOT contract. That is, the private sector will build the road and operate it for a time period T, and after that the government will operate the road for its remaining life $(\bar{T} - T)$. Then the profit that the private sector earns during the concession period is

$$P(T, q, y) = Tqp(q, y) - I(y) \tag{5}$$

where the price function $p(q, y)$ is given by (2). In profit function (5), the first term is the total revenue collected during the concession period, and the second term is the road construction cost, which is assumed to be fully born by the private investor.

Assumption 2. For any given $y > 0$, the revenue function $qp(q, y) = q(B(q) - \beta t(q, y))$ is a strictly concave function of q for $q \geq 0$.

With part (c) of Assumption 1, $t(q, y)$ is convex and thus $qt(q, y)$ is strictly convex in q for any given $y > 0$, which means that the second term of $qp(q, y)$, i.e. $-\beta qt(q, y)$, is strictly concave in q for any given $y > 0$. Thus $qp(q, y)$ is strictly concave if the first term $qB(q)$ is concave in q. In other words, with Assumption 1, Assumption 2 holds if $qB(q)$ is concave. Indeed, in the literature it is common to assume that $qB(q)$ is concave. Thus Assumption 2 is not restrictive.

For a BOT road project, the total social welfare during the lifetime of the road is
$$W = TS(q, y) + (\bar{T} - T)S(\tilde{q}, y) - I(y) \tag{6}$$
where q and $\tilde{q}$ are the travel demands under the operations of the private sector and the government, respectively. The first and the second terms of (6) are the social welfares obtained respectively during the concession period of operation by the private sector and the remaining period of operation by the government; the third term is the road construction cost.

In eqn. (6), there are four variables, i.e. T, y, q and $\tilde{q}$. Note that q and $\tilde{q}$ can be regarded as the indirect control variables by the private firm and the government, respectively, through their direct control of road toll charge. Namely, for given road capacity, q and $\tilde{q}$ can be targeted and achieved by the private firm and the government by choosing their respective toll charges.

Since the government aims to maximize social welfare, we can rewrite (6) as
$$W(T, q, y) = TS(q, y) + (\bar{T} - T)\tilde{S}(y) - I(y) \tag{7}$$
where
$$\tilde{S}(y) = \max_{\tilde{q}} S(\tilde{q}, y) = \max_{\tilde{q}} \int_0^{\tilde{q}} B(w)\mathrm{d}w - \beta\tilde{q}t(\tilde{q}, y). \tag{8}$$

Equation (8) states that $\tilde{S}(y)$ is the maximal social welfare obtained per unit period under the government operation, which is a function of the road capacity y. An intuitive understanding of $\tilde{S}(y)$ is that, after the road is transferred to the government, the government will choose an optimal toll (congestion toll) to achieve the socially optimal level of traffic demand, and the realized optimal social welfare is solely dependent upon the road capacity.

For a BOT road project, (5) and (7) state that the private profit and the social welfare are both determined by (T, q, y), i.e. the concession period, the travel demand within the concession period and the road capacity. Hence, a BOT contract between the government and the private sector is essentially to negotiate and set the values of the three variables, and we thus use the

three variable values to characterize a BOT contract. Namely, a BOT contract (T, q, y) represents the values of the concession period, the travel demand within the concession period and the road capacity to be set jointly by the government and the private sector. Note that, once (T, q, y) is fixed, the toll p is uniquely given by the price function (2), i.e. the toll level p should be set according to the price function (2) to realize the travel demand q. Therefore, regarding (T, q, y) as a BOT contract is essentially equivalent to regarding (T, p, y) as a BOT contract, and we shall use (T, q, y) for convenience of exposition.

For a BOT road project, the government is concerned about how to negotiate an optimal BOT contract with the private sector to maximize the social welfare subject to a profitability constraint. This is termed as the BOT problem and formulated below.

The BOT problem:
$$\max \ W(T, q, y) = TS(q, y) + (\overline{T} - T)\tilde{S}(y) - I(y) \tag{9}$$
subject to
$$0 \leq T \leq \overline{T}, \ q \geq 0, \ y \geq 0, \tag{10}$$
$$P(T, q, y) = Tqp(q, y) - I(y) \geq \tilde{P} \tag{11}$$
where $\tilde{P} \geq 0$ is the minimum profit margin that is acceptable to the private sector. Intuitively, constraint (11) means that, to attract private investors, the BOT contract should be profitable to some extent. We assume that the feasible region given by constraint (10)-(11) is not empty, i.e. there exists a feasible solution to the BOT problem. This guarantees the existence of an optimal solution to the BOT problem (an optimal BOT contract). We should mention here that the uniqueness of an optimal solution to the BOT or first-best problem is not guaranteed because the objective functions (3) and (9) may not be (strongly) concave, especially when the road construction function $I(y)$ is concave.

3 OPTIMAL BOT CONTRACTS

After introducing the first-best problem and the BOT problem, we now move on to examine an optimal BOT contract. We first introduce the first-best and the second-best BOT contracts, and then review the classic self-financing problem. After establishing the conditions for the existence of a first-best BOT contract, we then look into the more complicated second-best one and show that the concession period of a second-best BOT contract should be the whole life of the road. A numerical example is provided to illustrate the observations.

3.1 First-best and second-best BOT contracts

We first introduce the definition of first-best and second-best BOT contracts.

Definition 1. (First-best and Second-best BOT Contracts) Let (T, q, y) be an optimal solution to the BOT problem (9)-(11), (T, q, y) is said to be a first-best BOT contract if (q, y) solves the first-best problem (3)-(4). Otherwise (T, q, y) is said to be a second-best BOT contract.

By Definition 1 and comparing the first-best problem (3)-(4) and the BOT problem (9)-(11), we readily have the following observation.

Observation 1. A first-best BOT contract exists if and only if there exists a solution $\left(q^{*},y^{*}\right)$ that solves the first-best problem and satisfies

$$\bar{T}q^{*}p\left(q^{*},y^{*}\right)-I\left(y^{*}\right)\geq\tilde{P}.\qquad(12)$$

Furthermore, if $\left(q^{*},y^{*}\right)$ solves the first-best problem and meets (12), then $\left(T,q^{*},y^{*}\right)$ is a first-best BOT contract for any $\tilde{T}\leq T\leq\bar{T}$ where

$$\tilde{T}=\frac{I\left(y^{*}\right)+\tilde{P}}{q^{*}p\left(q^{*},y^{*}\right)}.\qquad(13)$$

Observation 1 is intuitive: if the total congestion toll revenue generated by the first-best optimum solution can cover the construction cost and generates an acceptable threshold of profit to the private sector, then the first-best optimum solution can be realized through a BOT project, and the corresponding BOT contract is a first-best one. Furthermore, if the first-best optimum toll revenue (congestion toll revenue) generated per unit period is so large that the minimum time needed to generate the predetermined threshold of profit is less than the lifetime of the road, then the concession period can be any value between the minimum required time period and the lifetime of the road, namely $\tilde{T}\leq T\leq\bar{T}$ in Observation 1.

From Definition 1 and Observation 1, it is clear that an optimal BOT contract is either a first-best one or a second-best one, but not both. A first-best BOT contract exists if and only if the first-best optimum solution satisfies condition (12). Otherwise we can only obtain a second-best BOT contract. Therefore, to solve the BOT problem (9)-(11), it is useful to first solve the first-best problem (3)-(4) and check whether condition (12) holds for a first-best optimum solution $\left(q^{*},y^{*}\right)$. If condition (12) holds, it is unnecessary to consider the BOT problem because $\left(T,q^{*},y^{*}\right)$ for any T such that $\tilde{T}\leq T\leq\bar{T}$ solves the BOT problem as described in Observation 1. In a word, the existence of a first-best BOT contract depends on whether the first-best optimum solution is profitable. Thus the question is similar to the classic self-financing problem, which, as to be seen in next subsection, is about whether the revenue from socially optimal pricing on a road can cover the capital cost of the socially optimally selected road capacity.

3.2 The classic self-financing problem

We briefly review the classic self-financing problem here, which demonstrates the existence of first-best or second-best BOT contracts.

If $q>0$ and $y>0$ at first-best optimum, then the first-order optimality conditions for the first-best problem (3)-(4) are given by

$$B(q)-\beta\left(t(q,y)+q\frac{\partial t(q,y)}{\partial q}\right)=0,\qquad(14)$$

$$\bar{T}\beta q\frac{\partial t(q,y)}{\partial y}+I'(y)=0. \tag{15}$$

Combining (14) with the price function (2) yields

$$p(q,y)=\beta q\frac{\partial t(q,y)}{\partial q}. \tag{16}$$

Equation (16) gives the first-best optimum toll, which is equal to the congestion externality.

One important assumption needed for the self-financing theorem is that the travel time function $t(q,y)$ is homogeneous of degree zero in (q,y), or $t(q,y)$ is a function of the volume-capacity ratio q/y, such as the widely used Bureau of Public Road function. Function $t(q,y)$ being homogeneous of degree zero in (q,y) means that $t(\lambda q,\lambda y)=t(q,y)$ for any $\lambda>0$, then taking partial derivative with respect to λ on both sides readily gives the following equation, which holds for any (q,y).

$$y\frac{\partial t(q,y)}{\partial y}=-q\frac{\partial t(q,y)}{\partial q}. \tag{17}$$

Applying (17) to a first-best optimum solution, the first-best condition (15) is equivalent to

$$\bar{T}q\left(\beta q\frac{\partial t(q,y)}{\partial q}\right)=I(y)E_I^y \tag{18}$$

where

$$E_I^y=\frac{\mathrm{d}I(y)/I(y)}{\mathrm{d}y/y}=I'(y)\frac{y}{I(y)}.$$

E_I^y is the elasticity of cost $I(y)$ with respect to road capacity y. Furthermore, substituting the first-best optimum toll (16) into (18) gives

$$\bar{T}qp(q,y)=I(y)E_I^y. \tag{19}$$

Observe that the left-hand side of (19) is the total revenue collected from road user charges during the life of the road for the first-best optimum toll and capacity, and the right-hand side is closely related to the road construction cost. Thus equation (19) throws light on the relationship between the total toll revenue and the road construction cost. If there is a constant return to scale in road construction, namely $E_I^y=1$ or $I(y)=ky$, then equation (19) becomes $\bar{T}qp=I(y)$, which means that the revenue generated from optimum road user charge just covers the road construction cost. Note that road construction can have decreasing or increasing returns to scale, depending on the geological conditions and construction technologies. If road construction has a decreasing return to scale, then it holds that $E_I^y>1$, and we have $\bar{T}qp>I(y)$ from (19), namely that the revenue exceeds the capital cost. Similarly, with an increasing return to scale, we have $E_I^y<1$ and $\bar{T}qp<I(y)$, namely that the revenue fails to cover the construction cost.

The above self-financing result is in fact an indication of profitability in a BOT project. It is clear that, depending on specific situations (the elasticity of $I(y)$ if travel time function is homogeneous of degree zero), a first-best optimum solution may or may not exist to meet the profitability condition (12). That is to say, an optimal BOT contract for a new road may be either a first-best one or a second-best one. As a first-best BOT contract is described in

Observation 1 and characterized by the first-best optimum conditions (14)-(16), in the following subsection we examine the properties of a second-best BOT contract.

3.3 The second-best BOT contract

In this subsection we examine the properties of a second-best BOT contract. In particular, we show that a second-best solution to the BOT problem requires the concession period to be the lifetime of the road. In other words, the private sector should be allowed to operate the road for its whole life. To keep the main text here concise and concentrated, detailed mathematics in this subsection is moved to Appendix A2.

Let $\left(T^{**}, q^{**}, y^{**}\right)$ be a second-best solution to the BOT problem (9)-(11). We shall prove that $T^{**} = \bar{T}$. For convenience, we define the following revenue function, $R(q)$, and the unit-period social welfare function, $S(q)$, as functions of q for given road capacity y^{**}:

$$R(q) = qp\left(q, y^{**}\right), \tag{20}$$

$$S(q) = S\left(q, y^{**}\right). \tag{21}$$

Let q_1 be the unique q that maximizes $R(q)$, and q_0 be the unique q that maximizes $S(q)$. We have that $R(q)$ decreases with q for $q \geq q_1$, $S(q)$ increases with q for $q \leq q_0$, and q_0 corresponds to the congestion toll for given road capacity y^{**}. It also holds that $0 < q_1 < q_0$, i.e. the travel demand for profit maximization is smaller than that for social welfare maximization. Figure 1 provides a simple graphical illustration for $R(q)$, $S(q)$, q_1 and q_0 (detailed mathematical discussions are given in Appendix A2).

Lemma 1. It holds for the second-best solution $\left(T^{**}, q^{**}, y^{**}\right)$ that

(a) $q_1 \leq q^{**} < q_0$.

(b) $T^{**} R\left(q^{**}\right) = I\left(y^{**}\right) + \tilde{P}$.

(c) $\bar{T} R\left(q_0\right) < I\left(y^{**}\right) + \tilde{P}$

Proof: See Appendix A2.

Lemma 1 is mainly based on the first-order optimality conditions of the BOT problem, and thus is very intuitive. Part (a) of Lemma 1 means that the travel demand of a second-best solution should be strictly less than the social welfare maximizing level, and not less than the profit maximizing level. That is, the toll of a second-best solution is strictly higher than the optimal congestion (marginal-cost pricing) toll, and not higher than the toll level for profit maximization. Part (b) is equivalent to

$$P\left(T^{**}, q^{**}, y^{**}\right) = T^{**} R\left(q^{**}\right) - I\left(y^{**}\right) = \tilde{P}$$

which simply states that, as expected, the profitability constraint (11) is binding for a second best solution. Similarly, Part (c) means that the (unit-period) congestion toll revenue $R\left(q_0\right)$ can not generate an acceptable total profit to the private sector even if the concession period is the whole life of the road.

To prove that $T^{**} = \bar{T}$, we shall first examine how $W(T, q, y^{**})$ changes with (T, q) for $0 < T \leq \bar{T}$, $q_1 \leq q < q_0$, while (T, q) satisfying the profitability constraint $TR(q) = I(y^{**}) + \tilde{P}$. This is not a straightforward task. To see this, we first note that

$$W(T, q, y^{**}) = TS(q) + (\bar{T} - T)\tilde{S}(y^{**}) - I(y^{**}). \tag{22}$$

Then we can readily observe that $W(T, q, y^{**})$ decreases with T due to $\tilde{S}(y^{**}) > S(q)$ and increases with q for $0 < T \leq \bar{T}$, $q_1 \leq q < q_0$. On the other hand, as required by the equation $TR(q) = I(y^{**}) + \tilde{P}$, q will decrease so that $R(q)$ increases when T decreases. The intuition is that, in achieving the profitability constraint through a shorter concession period, the unit-period revenue has to be higher, which requires a higher toll charge and thus lower traffic demand. Thus, we can see that, for given road capacity $y = y^{**}$, for any (T, q) such that $0 < T \leq \bar{T}$, $q_1 \leq q < q_0$ and $TR(q) = I(y^{**}) + \tilde{P}$, a shorter T will have a direct positive impact on social welfare as $W(T, q, y^{**})$ decreases with T, but also induce a lower traffic demand and thus have an indirect negative impact on social welfare. Therefore, under the profitability constraint, the net effect of changing the concession period T on the social welfare is obscure and requires rigorous further analysis.

To proceed, we rewrite (22) as

$$W(T, q, y^{**}) = -TD(q) + \bar{T}\tilde{S}(y^{**}) - I(y^{**}) \tag{23}$$

where

$$D(q) = \tilde{S}(y^{**}) - S(q). \tag{24}$$

Intuitively, $D(q)$ is the deadweight loss function for given y^{**}, because $S(q)$ defined by (21) is the unit-period social welfare function, and $\tilde{S}(y^{**})$ by definition (8) is the maximum social welfare obtained per unit period. Since q_0 maximizes the unit-period social welfare for given y^{**}, we have $\tilde{S}(y^{**}) = S(q_0)$ and thereby $D(q_0) = 0$, i.e. q_0 causes no deadweight loss for given y^{**}. Further define

$$h(q) = \frac{D(q)}{R(q)} \tag{25}$$

which is the ratio of the deadweight loss to the revenue per unit period. Without difficulty, it can be seen that both $D(q)$ and $R(q)$ decrease with q for $q_1 \leq q < q_0$. In the following lemma we shall show that $h(q)$ decreases with q as well for $q_1 \leq q < q_0$, which is useful to prove $T^{**} = \bar{T}$ later.

Lemma 2. The function $h(q)$ defined by (25) decreases with q for $q_1 \leq q < q_0$.

Proof: See Appendix A2.

Lemma 2 is essential to the following proposition.

Proposition 1. The concession period of a second-best BOT contract is equal to the lifetime of the road. Specifically, let $\left(T^{**}, q^{**}, y^{**}\right)$ be a second-best solution, then $T^{**} = \bar{T}$.

Proof: It suffices to prove that, if $T^{**} < \bar{T}$, there exists a feasible BOT solution that gives rise to a social welfare larger than $W\left(T^{**}, q^{**}, y^{**}\right)$.

For any $(T, q) > (0, 0)$ satisfying $TR(q) = I\left(y^{**}\right) + \tilde{P}$, we have

$$T = \frac{I\left(y^{**}\right) + \tilde{P}}{R(q)}. \tag{26}$$

Since $R(q)$ decreases with q for $q \geq q_1$, (26) implies that T increases with q for $q \geq q_1$. Suppose $T^{**} < \bar{T}$, then, in view of $T^{**} R\left(q^{**}\right) = I\left(y^{**}\right) + \tilde{P}$ and $\bar{T} R\left(q_0\right) < I\left(y^{**}\right) + \tilde{P}$ from Lemma 1, there exists a unique $\bar{q}$ such that $q^{**} < \bar{q} < q_0$ and $\bar{T} R(\bar{q}) = I\left(y^{**}\right) + \tilde{P}$. We only need to prove that $W\left(\bar{T}, \bar{q}, y^{**}\right) > W\left(T^{**}, q^{**}, y^{**}\right)$.

Substituting (26) into (23) gives

$$W\left(T, q, y^{**}\right) = -h(q)\left(I\left(y^{**}\right) + \tilde{P}\right) + \bar{T}\tilde{S}\left(y^{**}\right) - I\left(y^{**}\right) \tag{27}$$

where $h(q) = D(q)/R(q)$ as defined by (25). In the right-hand side of (27), only q is a variable, and from Lemma 2, $h(q)$ decreases with q for $q_1 \leq q < q_0$. Thus (27) means that $W\left(T, q, y^{**}\right)$ increases with q for $q_1 \leq q < q_0$ if (T, q) satisfies $TR(q) = I\left(y^{**}\right) + \tilde{P}$. Then it follows readily from $q_1 \leq q^{**} < \bar{q} < q_0$ that $W\left(\bar{T}, \bar{q}, y^{**}\right) > W\left(T^{**}, q^{**}, y^{**}\right)$. This completes the proof.

From the above proof, the importance and essence of Lemma 2 is clear: while the social welfare decreases with T and increases with q, Lemma 2 tells us that the joint effect of increasing both T and q while satisfying the profitability constraint is positive for the social welfare. Proposition 1 states that, if the first-best optimum solution is not profitable and thus can not be obtained through a BOT project, the government should let the private sector operate the road for its whole life and earn a minimum acceptable profit to maximize the social welfare. The intuition behind this is that, given a longer concession period, the private sector can afford to build a larger road capacity with an acceptable profit margin, which in turn results in a lower travel time and a higher travel demand. Although the operation period for welfare-maximization by the government after concession period becomes shorter (i.e. zero), all factors within and after the concession period together give a larger social welfare with a longer concession period. Moreover, under some proper government regulation (to be discussed in next section), the private firm is willing to invest in an optimal amount of capacity to reduce congestion if it is allowed to operate the whole life of the road, because the benefits from reduced congestion accrue over the full lifetime of the road.

Given the requirement that $T = \bar{T}$ for a second-best BOT solution, we have the following first-order optimality conditions for the second-best BOT problem:

$$B(q) - \beta \left(t(q, y) + q \frac{\partial t(q, y)}{\partial q} \right) > 0, \tag{28}$$

$$B(q) + qB'(q) - \beta \left(t(q, y) + q \frac{\partial t(q, y)}{\partial q} \right) \leq 0, \tag{29}$$

$$\bar{T}qp(q, y) - I(y) = \tilde{P}, \tag{30}$$

$$\bar{T}\beta q \frac{\partial t(q, y)}{\partial y} + I'(y) = 0 \tag{31}$$

where conditions (28) and (29) correspond to part (a) of Lemma 1, (30) corresponds to part (b) of Lemma 1, and (31) is due to $T = \bar{T}$. Furthermore, because of $T = \bar{T}$ for a second-best BOT solution, condition (31) is exactly the same as the first-best optimum condition (15), both stating that the road capacity should be expanded to the point where the marginal cost of an extra unit of capacity is equal to the marginal value of user cost savings brought about by that investment.

3.4 A simple numerical example

Consider $B(q) = 5 - 2q$, $t(q, y) = 1 + \frac{q}{y}$, $\beta = 1$ and $\bar{T} = 1$. Then we have

$p(q, y) = 4 - 2q - \frac{q}{y}$, $S(q, y) = 4q - q^2 - \frac{q^2}{y}$, and $\tilde{S}(y) = \frac{4y}{y+1}$. The first-best problem is

$$\max_{q \geq 0, y \geq 0} \ 4q - q^2 - \frac{q^2}{y} - I(y)$$

For simplicity, let us first consider $I(y) = y$. Then, making use of the first-best optimum conditions (14)-(16), we obtain the first-best optimum solution of $(q^*, y^*) = (1,1)$ together with a first-best toll $p^* = 1$, which gives an overall maximum social welfare $W^* = 1$. It can be easily checked that $P(\bar{T}, q^*, y^*) = 0$, which means that the first-best toll revenue just covers the investment for the first-best road capacity. This is consistent with the classic self-financing theory as $t(q, y)$ is homogeneous of degree zero and $I(y) = y$ has a constant return of scale.

Suppose that the minimum acceptable profit level to a private firm is $\tilde{P} = 0.18$. Then it is clear that the first-best optimum solution is unacceptable to a private firm even if it can operate the whole life of the road. Thus a first-best BOT contract does not exist, and we have to consider the (second-best) BOT problem.

$$\max_{0 \leq T \leq 1, q \geq 0, y \geq 0} W(T, q, y) = T \left(4q - q^2 - \frac{q^2}{y} \right) + (1 - T) \frac{4y}{y+1} - y$$

subject to

$$P(T, q, y) = Tq \left(4 - 2q - \frac{q}{y} \right) - y \geq 0.18.$$

Making use of the second-best conditions (28)-(31), we can obtain the second-best BOT contract $(\bar{T}, q^{**}, y^{**}) = (1, 0.9, 0.9)$, which gives a second-best social welfare $W^{**} = 0.99$.

Besides the first-best problem and the BOT problem, here we numerically examine how the maximum social welfare changes with the concession period. To do this, we define

$$W_{max}(T) = \max_{(q,y) \in \Omega_T} W(T,q,y)$$

where

$$\Omega_T = \left\{ (q,y) : q \geq 0, y \geq 0, P(T,q,y) \geq \tilde{P} \right\}$$

Clearly, Ω_T is the feasible set of (q,y) for given T, and $W_{max}(T)$ is the maximum social welfare for given T subject to the profitability constraint. Note that Ω_T can be empty if T is too small, i.e. the profitability constraint can not be met with a too short concession period. For the numerical example here, Ω_T is empty if $T < 0.64$, i.e. there does not exist a combination of toll level and road capacity that can generate an acceptable profit to the private sector if the concession period is less than 64 percent of the lifetime of the road. Figure 2 plots how $W_{max}(T)$ changes with T for $0.64 \leq T \leq 1$. It can be seen clearly that the maximum social welfare (subject to a profitability constraint) increases with T until $T = 1$.

So far we have constructed a second-best BOT problem by assuming a positive minimum acceptable profit $\tilde{P} = 0.18$. In connection with the self-financing problem discussed in Subsection 3.2, we now look at the first-best and second-best BOT solutions by assuming $\tilde{P} = 0$ and considering three kinds of $I(y)$, with increasing, constant and decreasing return to scale, respectively. To do this, we consider $I(y) = 0.5y^2$ (decreasing return to scale), $I(y) = y$ (constant return to scale) and $I(y) = 1.25y^{0.8}$ (increasing return to scale). Similar to Figure 2, Figure 3 plots how $W_{max}(T)$ changes with T for the three cases, where the value of $W_{max}(T)$ in each case is represented as the percentage of the corresponding first-best welfare value (in each case, the y-axis value "100" represents the attainment of the first-best optimum)[1]. As seen from Figure 3, in all three cases, the maximum social welfare (subject to a profitability constraint) increases with the concession period T. For the case of decreasing return to scale in road construction, $I(y) = 0.5y^2$, the first-best optimum is obtained at a concession period of less than half lifetime of the road; for the "constant return to scale" case, $I(y) = y$, the first-best optimum is obtained at a concession period of exactly equal to the road lifetime ($T = 1$); for the "increasing return to scale" case, $I(y) = 1.25y^{0.8}$, the first-best optimum can not be obtained even if $T = 1$. These observations are exactly consistent with the classic self-financing results discussed in Subsection 3.2. In particular, for the "increasing return to scale" case, because the first-best toll revenue can not cover the capital cost of the first-best road capacity, i.e. the first-best optimum is not profitable, the maximum welfare (subject to a profitability constraint) obtained at $T = 1$ is a second-best value.

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[1] Because the three cases have different first-best welfare values, the graphical comparison among them would be unclear if Figure 3 uses the absolute value of $W_{max}(T)$ as Figure 2 does.

4 NEGOTIATING AN OPTIMAL BOT CONTRACT

So far we have examined the basic properties of an optimal BOT contract for a new road. If the first-best optimum solution is not profitable, i.e. there does not exist a first-best optimum solution satisfying the profitability condition (12), then a second-best BOT contract has to be sought, where the concession period should be the lifetime of the road, and the optimal travel demand and road capacity are determined by conditions (28)-(31). If the first-best optimum solution is profitable, a first-best BOT contract can be reached, where the socially optimal travel demand, road capacity and toll charge are determined by the first-best optimum conditions (14)-(16), and the concession period can be any T such that $\tilde{T} \leq T \leq \bar{T}$ as described in Observation 1. Keeping these observations in mind, we look into how to reach an optimal BOT contract, either through bilateral negotiations between the government and a private sector, or through competitive auctions among many private firms.

4.1 Bilateral negotiation

In this subsection we look into how to reach an optimal BOT contract through a bilateral negotiation between the government and a private sector concessionaire. In particular, we are interested in what negotiation regime and what government strategy should be adopted to reach an optimal BOT contract.

We treat the bilateral negotiation of a BOT contract as a sequential game in which the government is the regulator and leader and the private sector is the follower. In this sequential game, the government first sets (or at least puts restrictions on) the values of several regulation variables, while taking account of the responses of the private sector, and then the private sector freely chooses the values of the other decision variables. Specifically, the government first sets the values of one or two of the three variables T, y and q, and then the private sector determines the remaining ones, and finally a combination (T, q, y), i.e. a BOT contract, is obtained. With this setting, there are several different games because multiple combinations of regulation and response variables exist. We prove that an optimal BOT contract will be obtained if travel demand is a government regulation variable and the government requires travel demand to be not less than the socially optimal level (either first-best or second-best). To this end, we first give the following lemma.

Lemma 3. Let $(\bar{T}, q^*, y^*)$ be an optimal BOT contract (either first-best or second-best). Then $P(\bar{T}, q, y) < P(\bar{T}, q^*, y^*)$ for any $q > q^*$, $y > 0$.

Proof: See Appendix A3.

In the sequential game negotiation, the private sector is, as usual, assumed to be a profit-maximizer. Once the government sets (or puts restrictions on) the values of the regulation variables, the private sector will choose the values of the other variables to maximize its profit. Thus, if the government does not control the concession period, i.e. T is not a regulation variable, then the response of the private sector is always to choose $T = \bar{T}$ for making more profit. Furthermore, let $(\bar{T}, q^*, y^*)$ be an optimal BOT contract, then if the government requires $q \geq q^*$, the private sector will choose $q = q^*$ because any $q > q^*$ gives a

profit less than $P(\bar{T}, q^*, y^*)$ according to Lemma 3. This means that the privately selected travel demand level will not deviate from the socially optimal level under the government regulation $q \geq q^*$. In other words, $q \geq q^*$ is equivalent to $q = q^*$ as a government regulation. In the following proposition we prove that the privately selected road capacity is also socially optimal under the regulation $q \geq q^*$, and thus an optimal BOT contract is obtained through this regulation.

Proposition 2. Let (T^*, q^*, y^*) be an optimal BOT contract (either first-best or second-best). Then, if the government let the private sector choose (T, q, y) such that $q \geq q^*$, an optimal BOT contract will be obtained.

Proof: Given that (T^*, q^*, y^*) is an optimal BOT contract, we have that $(\bar{T}, q^*, y^*)$ is an optimal BOT contract: if (T^*, q^*, y^*) is a second-best solution, Proposition 1 states that $T^* = \bar{T}$; if (T^*, q^*, y^*) is a first-best solution, then $(\bar{T}, q^*, y^*)$ is also a first-best solution from Observation 1. Since the government does not control T, the private sector will choose $T = \bar{T}$. Also, from Lemma 3, the private sector will choose $q = q^*$ under the government regulation $q \geq q^*$. Then, with $(T, q) = (\bar{T}, q^*)$, the private sector will solve the following problem to determine y.

$$\max_{y \geq 0} P(\bar{T}, q^*, y) = \bar{T}q^* p(q^*, y) - I(y) \tag{32}$$

In view of the price function $p(q^*, y) = B(q^*) - \beta t(q^*, y)$, and that q^* is predetermined by the government, problem (32) is equivalent to

$$\min_{y \geq 0} \beta \bar{T}q^* t(q^*, y) + I(y) \tag{33}$$

Let $\hat{y}$ be the choice of the private sector, then $\hat{y}$ solves problem (33), and the resulting BOT contract is $(\bar{T}, q^*, \hat{y})$. We need to prove that $(\bar{T}, q^*, \hat{y})$ is an optimal solution to the BOT problem. For given $(\bar{T}, q^*)$, we have

$$W(\bar{T}, q^*, y) = \bar{T}\left(\int_0^{q^*} B(w)\,dw - \beta q^* t(q^*, y)\right) - I(y)$$

$$= \bar{T}\int_0^{q^*} B(w)\,dw - \left(\beta \bar{T}q^* t(q^*, y) + I(y)\right) \tag{34}$$

Note that the second term of (34) is equivalent to the objective function (33), thus y^* solves problem (33), otherwise $(\bar{T}, q^*, y^*)$ can not be an optimal solution. Since $\hat{y}$ solves problem (33), it follows readily that $W(\bar{T}, q^*, y^*) = W(\bar{T}, q^*, \hat{y})$ and thus $(\bar{T}, q^*, \hat{y})$ is an optimal BOT contract. This completes the proof.

Proposition 2 shows an appealing negotiation regime in which the government only needs to set a minimum level of travel demand, while leaving the private sector to freely choose a preferable combination of road capacity, toll charge and concession period to maximize its

profit. The key point in the above proof is that "the second term of (34) is equivalent to the objective function (33)", which is not a coincidence. Let $T = \bar{T}$, we have

$$W\left(\bar{T},q,y\right) = \bar{T}\int_0^q B\left(w\right)dw - \left(\beta\bar{T}qt\left(q,y\right) + I\left(y\right)\right) \tag{35}$$

$$P\left(\bar{T},q,y\right) = \bar{T}qB\left(q\right) - \left(\beta\bar{T}qt\left(q,y\right) + I\left(y\right)\right) \tag{36}$$

(35) and (36) have the same second term, $\left(\beta\bar{T}qt\left(q,y\right) + I\left(y\right)\right)$, which can be viewed as the total system cost with $\beta\bar{T}qt\left(q,y\right)$ being the total travel cost and $I\left(y\right)$ being the construction cost. This system cost involves the road capacity y as the only variable if the travel demand q is predetermined, and the objective functions (35) and (36) states that both the social welfare maximizer (the government) and the profit maximizer (the private sector) would like to set the value of y to minimize this total cost. Simply speaking, Proposition 2 holds because the public and the private sectors share the same interest regarding the road capacity choice when $T = \bar{T}$ and the travel demand is predetermined. In contrast, the two sectors generally do not share the same interest regarding travel demand and/or toll level choice for any predetermined road capacity. Also, it can be readily verified that, if $T < \bar{T}$, the two sectors will also have different interest regarding road capacity choice. Thus the first-best solutions with $T < \bar{T}$, if they exist, generally can not be obtained through a simple sequential game bilateral negotiation.

Proposition 2 provides a negotiation regime and a strategy for the government to obtain an optimal BOT contract, which uses only one regulation variable. It can be easily verified that adding one more regulation variable by the government will not improve the outcome. For example, let $\left(\bar{T},q^*,y^*\right)$ be an optimal BOT contract, then if the government sets $\left(T,q\right) = \left(\bar{T},q^*\right)$ or sets $\left(q,y\right) = \left(q^*,y^*\right)$, the response of the private sector will result in an optimal solution as when the government sets $q \geq q^*$ alone. In both cases, q must enter as a regulation variable to guarantee an optimal BOT contract. Otherwise, the optimality cannot be guaranteed even if the government sets the other two regulation variables $\left(T,y\right) = \left(\bar{T},y^*\right)$. If the government sets $\left(T,y\right) = \left(\bar{T},y^*\right)$, then the private sector will typically target a $q < q^{*\,2}$ by setting a higher toll level to maximize the profit, and thus the outcome will generally not be socially optimal.

It should be mentioned here that, in a sequential bilateral negotiation, choosing target travel demand as a regulation variable gives rise to an outcome that is different from choosing toll level as a regulation variable. To see this, we still consider the optimal BOT contract $\left(\bar{T},q^*,y^*\right)$. The corresponding optimal toll is $p^* = B\left(q^*\right) - \beta t\left(q^*,y^*\right)$. If the government only sets $p = p^*$, the private sector will choose $T = \bar{T}$, and then solve the following problem to determine $\left(q,y\right)$.

$$\max_{q\geq 0,\,y\geq 0} \bar{T}qp^* - I\left(y\right) \tag{37}$$

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[2] The private sector will target a $q = q^*$ only when q^* "happens" to be the profit maximizing travel demand level for given road capacity $y = y^*$, which is a sort of knife-edge case.

subject to

$$p(q, y) = p^* \tag{38}$$

To see that the response of the private sector to the government strategy $p = p^*$ is different from the response to $q = q^*$, rewrite the profit objective functions (32) for given $q = q^*$ and (37) for given $p = p^*$ as

$$P_1(y) = \bar{T}q^* p(q^*, y) - I(y) \tag{39}$$
$$P_2(y) = \bar{T}p^* q(y) - I(y) \tag{40}$$

where $q(y)$ regards the travel demand q as a function of the road capacity y for given toll p^*, and is determined by (38). About $q(y)$, taking derivatives with respect to y on both sides of (38) gives

$$\frac{dq(y)}{dy} = -\frac{\partial p(q, y)/\partial y}{\partial p(q, y)/\partial q} \tag{41}$$

From Proposition 2, we know that $y = y^*$ maximizes $P_1(y)$, and we shall show that $y = y^*$ generally does not maximize $P_2(y)$. To do so, we evaluate the derivatives of $P_1(y)$ and $P_2(y)$ at $y = y^*$. From (39)-(40), also in view of $q(y^*) = q^*$ and (41), we have

$$\frac{dP_1(y^*)}{dy} = \bar{T}q^* \frac{\partial p(q^*, y^*)}{\partial y} - I'(y^*) \tag{42}$$

$$\frac{dP_2(y^*)}{dy} = \bar{T}\frac{p^*}{-p'(q^*)}\frac{\partial p(q^*, y^*)}{\partial y} - I'(y^*) \tag{43}$$

where $p'(q^*) = \partial p(q^*, y^*)/\partial q$. With Assumption 2, $qp(q, y^*)$ is a strictly concave function of q, and since $(\bar{T}, q^*, y^*)$ is an optimal solution to the BOT problem, the slope of $qp(q, y^*)$ at $q = q^*$ is non-positive, namely

$$p(q^*, y^*) + q^* p'(q^*) \leq 0 \tag{44}$$

Note that $\partial p(q, y)/\partial q < 0$ and $\partial p(q, y)/\partial y > 0$, (44) is equivalent to

$$\frac{p^*}{-p'(q^*)} \leq q^* \tag{45}$$

From (45) and comparing the first terms of (42) and (43), we have

$$\frac{dP_2(y^*)}{dy} \leq \frac{dP_1(y^*)}{dy} \tag{46}$$

where "=" holds if and only if (44) takes equality, which is not a general case. Actually, inequality (44) taking equality means that q^* "happens" to be the profit maximizing travel demand for $y = y^*$. For the general cases that (44) and thus (46) take strict inequality, i.e. $dP_2(y^*)/dy < dP_1(y^*)/dy$, since $y = y^*$ maximizes $P_1(y)$ and $dP_1(y^*)/dy = 0$, we have $dP_2(y^*)/dy < 0$, which means that reducing y will increase $P_2(y)$. Therefore, for the optimal BOT contract $(\bar{T}, q^*, y^*)$, if the government only sets the toll level to be $p = p^*$, the private sector will typically choose a road capacity $y < y^*$, and thus the outcome will

generally not be socially optimal. The intuition is that, because the optimal toll is below the profit maximizing toll, it gives the private sector inadequate incentive to expand the road capacity to the optimal level.

4.2 Competitive auction

The last subsection shows that the road capacity and the toll level are not effective regulation variables for achieving an optimal BOT contract in a sequential game bilateral negotiation. This subsection will demonstrate that the two are effective if the road is franchised through competitive auctions among a sufficiently large number of private firms.

We first consider the case that the first-best optimum solution is not profitable and thus the optimal BOT contract is a second-best solution. Then the optimal concession period is the whole life of the road, and the government is only concerned about how to reach the optimal toll and road capacity. We consider two types of auctions: the government fixes the road capacity, and the road is awarded to the private sector that bids the lowest toll; alternatively, the government fixes the toll level, and the winner is the one that bids the largest road capacity. We assume that there are a sufficiently large number of private firms bidding for the project, and thus the bidding result gives rise to the minimum acceptable profit.

Let $\left(\bar{T}, q^{**}, y^{**}\right)$ be a second-best BOT contract with the corresponding toll level $p^{**} = B\left(q^{**}\right) - \beta t\left(q^{**}, y^{**}\right)$. To reach this optimal BOT contract, one auction mechanism is that the government fixes the concession period and the road capacity to be $(T, y) = \left(\bar{T}, y^{**}\right)$ and lets the private firms bid for the lowest toll level p, which is equivalent to bidding for the highest travel demand q. Under this auction, with a sufficiently large number of bidders, the bidding result will be the lowest toll level (the highest travel demand) that satisfies the minimum profit constraint (11). Then it can be easily verified that $p = p^{**}$ ($q = q^{**}$) will be bidding result, and thus the optimal BOT contract $\left(\bar{T}, q^{**}, y^{**}\right)$ is obtained.

Another auction mechanism is that the government fixes the concession period and the toll level to be $(T, p) = \left(\bar{T}, p^{**}\right)$ and lets the private firms bid for the largest road capacity. Note that in this case, the government fixing the toll level is different from fixing the target travel demand as shown in last subsection. Under this auction, with a sufficiently large number of bidders, the bidding result will be the largest road capacity that satisfies the minimum profit constraint (11). For fixed $(T, p) = \left(\bar{T}, p^{**}\right)$, rewrite the profit function as

$$P(y) = \bar{T} p^{**} q(y) - I(y) \tag{47}$$

where $q(y)$ is indirectly given by

$$B(q) - \beta t(q, y) = p^{**} \tag{48}$$

Then a necessary condition for y to be the bidding result is that the minimum profit constraint (11) is binding, i.e. $P(y) = \tilde{P}$, and that the profit decreases with the road capacity, i.e. $dP(y)/dy \leq 0$. Note that $P(y)$ defined by (47)-(48) here is exactly the same as $P_2(y)$ defined by (40) in last section, thus we know about $P(y)$ from the analysis on $P_2(y)$ in last

section that $P\left(y^{**}\right)=\tilde{P}$ and $dP\left(y^{**}\right)/dy \le 0$. Hence $y=y^{**}$ satisfies the necessary condition for being the bidding result, and we can call $y=y^{**}$ a "local" bidding result. To guarantee that $y=y^{**}$ is a "global" bidding result, we need that $P\left(y\right)$ defined by (47)-(48) is strictly pseudoconcave in y. With Assumption 1, pseudoconcavity of $P\left(y\right)$ means that $P\left(y\right)$ first increases and then decreases with y, which is intuitively reasonable and holds easily even if $I\left(y\right)$ is concave (note that $P\left(y\right)$ typically can not be concave when $I\left(y\right)$ is concave, and pseudoconcavity is a much weaker requirement than concavity). With the pseudoconcavity assumption on $P\left(y\right)$, since $dP\left(y^{**}\right)/dy \le 0$, any $y>y^{**}$ will give $P\left(y\right)<P\left(y^{**}\right)$, and since $P\left(y^{**}\right)=\tilde{P}$, it is clear that $y=y^{**}$ is the largest road capacity that meets the minimum profit constraint (11). Therefore, $y=y^{**}$ will be the bidding result and the optimal BOT contract $\left(\bar{T},q^{**},y^{**}\right)$ is obtained.

From the above analyses, a second-best BOT contract can be obtained through competitive auctions because competitive auctions among a sufficiently large number of private firms drive the bid winner to earn only the minimum acceptable profit, which is a major distinction from the bilateral negotiation in which the private sector tries to earn as much profit as possible. Nevertheless, this causes a problem for the first-best solutions for which the profitability constraint (11) is not binding, because a distorted result will be obtained when competitive auctions drive the bid winner to earn the minimum acceptable profit. Specifically, for a first-best solution that gives the private sector more profit than the minimum acceptable one, bidding for the lowest toll level will force the bidders to set a toll lower than the congestion toll, which is obviously not optimal, and bidding for the largest road capacity will force them to choose a road capacity larger than the optimal one. This problem is essentially caused by the property that the profitability constraint (11) is not binding. However, for the particular first-best solution $\left(\tilde{T},q^{*},y^{*}\right)$, which, like a second-best solution, just gives the minimum acceptable profit to the private sector as described in Observation 1, it can be obtained through the aforementioned two competitive auctions.

5 CONCLUSION

Three variables are essential to a BOT road project, namely the concession period, the road capacity and the toll charge, while previous researches typically neglected either the concession period or the road capacity. Motivated by this, we explicitly regarded a BOT contract as a combination of the three variables, and examined how a BOT contract should be designed to maximize the social welfare, while allowing for an acceptable level of profit to the private sector. We classified and investigated the optimal BOT contract as either a first-best or a second-best one, depending on whether the first-best optimum toll and capacity are profitable. In particular, we proved that, if the first-best optimum solution is not profitable and thus a second-best BOT contract has to be considered, the whole life of the road should be selected to be the concession period. This "lifetime concession period" result seems to be realistic because several BOT contracts around the world have been awarded for 99 years, including Highway 407 in Toronto, the Chicago Skyway and the Pocahontas Parkway (Virginia Route 495) in Richmond, Virginia.

With the properties of optimal BOT contracts well established, we provided a strategy for the government to ensure the achievement of an optimal BOT contract in the bilateral negotiation with a private sector firm. Specifically, by regarding the bilateral negotiation as a sequential game where the government is the leader and the private sector is the follower, an optimal BOT contract will be obtained if the government requires the travel demand to be not less than the optimal level (either first-best or second-best) and lets the private sector freely choose the other variables such as road capacity, toll charge and concession period. A practical interpretation of this result is that, when awarding a BOT contract for a new road, the government only needs to set a minimum service level for the road (i.e. the road should serve a traffic volume not less than certain level), and then let the private sector freely determine the rest.

We also briefly examined two widely used auction mechanisms for road franchising, i.e. bidding for the largest road capacity and bidding for the lowest toll. Under the assumption that the number of bidders is large enough, it is demonstrated that both these competitive auctions can lead to optimal BOT contracts especially for the second-best case.

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APPENDIX

A1. Impact of a discounting rate on the results

Assume time is continuous and $I(y)$ is born by the private sector at $t = 0$. Let r be an interest rate of reference used for discounting all monetary units to equivalent values at $t = 0$. Then the first-best optimum objective function (3) should be written as

$$\int_0^{\bar{T}} S(q, y) e^{-rt} dt - I(y) = \frac{1 - e^{-r\bar{T}}}{r} S(q, y) - I(y). \tag{49}$$

Similarly, the profit formulation (5) should be written as

$$P = \frac{1 - e^{-rT}}{r} q p(q, y) - I(y) \tag{50}$$

and the social welfare formulation (6) be written as

$$W = \frac{1 - e^{-rT}}{r} S(q, y) + \left(\frac{1 - e^{-r\bar{T}}}{r} - \frac{1 - e^{-rT}}{r} \right) S(\tilde{q}, y) - I(y). \tag{51}$$

Denote $L = \dfrac{1 - e^{-rT}}{r}$ and $\bar{L} = \dfrac{1 - e^{-r\bar{T}}}{r}$, then (49), (50) and (51) become

$$\bar{L} S(q, y) - I(y), \tag{52}$$

$$P(L,q,y) = Lqp(q,y) - I(y), \tag{53}$$

$$W(L,q,y) = LS(q,y) + (\bar{L} - L)S(\tilde{q},y) - I(y). \tag{54}$$

Comparing (52), (53) and (54) with (3), (5) and (6), respectively, it is then clear that the adoption of a discounting rate only requires T and $\bar{T}$ be rewritten as L and $\bar{L}$ (or T and $\bar{T}$ are re-scaled), respectively. Thus it is clear that our major results remain valid.

A2. Mathematical Analysis in Subsection 3.3

For further analysis, we first calculate the partial derivatives of $W(T,q,y)$ and $P(T,q,y)$. From the social welfare function $W(T,q,y)$ given by (7), we have

$$\frac{\partial W(T,q,y)}{\partial y} = -\left(T\beta q \frac{\partial t(q,y)}{\partial y} + I'(y)\right) + (\bar{T} - T)S'(y), \tag{55}$$

$$\frac{\partial W(T,q,y)}{\partial q} = T\left(B(q) - \beta t(q,y) - \beta q \frac{\partial t(q,y)}{\partial q}\right), \tag{56}$$

$$\frac{\partial W(T,q,y)}{\partial T} = \left(\int_0^q B(w)\,dw - \beta q t(q,y)\right) - \tilde{S}(y). \tag{57}$$

From the profit function given by (5), we have

$$\frac{\partial P(T,q,y)}{\partial y} = -\left(T\beta q \frac{\partial t(q,y)}{\partial y} + I'(y)\right), \tag{58}$$

$$\frac{\partial P(T,q,y)}{\partial q} = T\left(B(q) + qB'(q) - \beta t(q,y) - \beta q \frac{\partial t(q,y)}{\partial q}\right), \tag{59}$$

$$\frac{\partial P(T,q,y)}{\partial T} = qp(q,y) > 0. \tag{60}$$

Let (T^{**}, q^{**}, y^{**}) be a second-best solution. We shall prove that $T^{**} = \bar{T}$. For simplicity of exposition, define $R(q)$ and $S(q)$ as functions of q for given y^{**}

$$R(q) = qp(q, y^{**}), \tag{61}$$

$$S(q) = S(q, y^{**}). \tag{62}$$

$R(q)$ and $S(q)$ are the revenue function and the unit-period social welfare function, respectively, for given road capacity y^{**}. $R(q)$ is strictly concave in q according to Assumption 2. With Assumption 1, it is obvious that $S(q)$ is also strictly concave in q. With $R(q)$ and $S(q)$, we can rewrite several formulations for given y^{**} as follows.

$$W(T,q,y^{**}) = TS(q) + (\bar{T} - T)\tilde{S}(y^{**}) - I(y^{**}), \tag{63}$$

$$P(T,q,y^{**}) = TR(q) - I(y^{**}), \tag{64}$$

$$\frac{\partial W(T,q,y^{**})}{\partial q} = TS'(q), \tag{65}$$

$$\frac{\partial W\left(T,q,y^{**}\right)}{\partial T} = S\left(q\right) - \tilde{S}\left(y^{**}\right), \tag{66}$$

$$\frac{\partial P\left(T,q,y^{**}\right)}{\partial q} = TR'\left(q\right). \tag{67}$$

Let q_1 maximize the revenue function $R\left(q\right)$. Then $R'\left(q_1\right) = 0$; i.e.

$$B\left(q_1\right) + q_1 B'\left(q_1\right) - \beta t\left(q_1, y^{**}\right) - \beta q_1 \frac{\partial t\left(q_1, y^{**}\right)}{\partial q} = 0. \tag{68}$$

Let q_0 maximize the unit-period social welfare function $S\left(q\right)$. Then $S'\left(q_0\right) = 0$; i.e.

$$B\left(q_0\right) - \beta t\left(q_0, y^{**}\right) - \beta q_0 \frac{\partial t\left(q_0, y^{**}\right)}{\partial q} = 0. \tag{69}$$

By definition (8), q_0 maximizing $S\left(q\right)$ for given y^{**} simply means that $\tilde{S}\left(y^{**}\right) = S\left(q_0\right)$.

With Assumption 1, there exist unique q_0 and q_1 satisfying (68) and (69), respectively. By comparing (68) and (69), we can observe that $q_1 < q_0$. It is also readily determined that q_0 corresponds to the congestion toll for given road capacity y^{**}.

Lemma 1. It holds for the second-best solution $\left(T^{**}, q^{**}, y^{**}\right)$ that

(a) $q_1 \leq q^{**} < q_0$.

(b) $T^{**} R\left(q^{**}\right) = I\left(y^{**}\right) + \tilde{P}$.

(c) $\bar{T}R\left(q_0\right) < I\left(y^{**}\right) + \tilde{P}$

Proof:
(a) Comparing (56) with (59), we have $\partial W\left(T,q,y\right)/\partial q > \partial P\left(T,q,y\right)/\partial q$ for any feasible $\left(T,q,y\right)$ in view of $B'\left(q\right) < 0$. In particular, for the second-best solution $\left(T^{**}, q^{**}, y^{**}\right)$, $\partial W\left(T^{**}, q^{**}, y^{**}\right)/\partial q$ and $\partial P\left(T^{**}, q^{**}, y^{**}\right)/\partial q$ can not be positive or negative simultaneously, otherwise increasing or decreasing q will increase both $W\left(T,q,y\right)$ and $P\left(T,q,y\right)$, which contradicts that $\left(T^{**}, q^{**}, y^{**}\right)$ is a second-best solution. Thus it holds that $W\left(T^{**}, q^{**}, y^{**}\right)/\partial q \geq 0$ and $\partial P\left(T^{**}, q^{**}, y^{**}\right)/\partial q \leq 0$, and from (65) and (67), we simply have $S'\left(q^{**}\right) \geq 0$ and $R'\left(q^{**}\right) \leq 0$. Then it follows readily that $q_1 \leq q^{**} \leq q_0$ in view of $S'\left(q_0\right) = 0$, $R'\left(q_1\right) = 0$, and that $S\left(q\right)$ and $R\left(q\right)$ are strictly concave.

We still need to prove $q^{**} \neq q_0$. Suppose $q^{**} = q_0$, then we have $\partial W\left(T,q^{**}, y^{**}\right)/\partial q = TS'\left(q^{**}\right) = TS'\left(q_0\right) = 0$ from (65). Also, with $S\left(q^{**}\right) = S\left(q_0\right) = \tilde{S}\left(y^{**}\right)$, we have $W\left(T^{**}, q^{**}, y^{**}\right) = W\left(\bar{T}, q^{**}, y^{**}\right)$ from (63). Then, since $\left(T^{**}, q^{**}, y^{**}\right)$ is a second-best solution, $\left(\bar{T}, q^{**}, y^{**}\right)$ is also a second-best solution. Comparing (55) with (58), we have

$\partial W\left(\bar{T},q,y\right)/\partial y = \partial P\left(\bar{T},q,y\right)/\partial y$ in view of $T=\bar{T}$. In particular, for the second-best solution $\left(\bar{T},q^{**},y^{**}\right)$, it holds that $\partial W\left(\bar{T},q^{**},y^{**}\right)/\partial y = \partial P\left(\bar{T},q^{**},y^{**}\right)/\partial y = 0$. Otherwise increasing or decreasing y will increase both $W\left(T,q,y\right)$ and $P\left(T,q,y\right)$, which contradicts that $\left(\bar{T},q^{**},y^{**}\right)$ is a second-best solution. In summary, if $q^{**}=q_0$ and $\left(\bar{T},q^{**},y^{**}\right)$ is a second-best solution, then $\partial W\left(\bar{T},q^{**},y^{**}\right)/\partial q = 0$ and $\partial W\left(\bar{T},q^{**},y^{**}\right)/\partial y = 0$, which means that $\left(q^{**},y^{**}\right)$ meets the first-best optimum conditions (14)-(15). Then the second-best solution $\left(\bar{T},q^{**},y^{**}\right)$ becomes a first-best one, which is a contradiction. Thus it is proved that $q^{**} \neq q_0$.

(b) $q^{**} \neq q_0$ gives $S\left(q^{**}\right) < S\left(q_0\right) = \tilde{S}\left(y^{**}\right)$, then we have $\partial W\left(T,q^{**},y^{**}\right)/\partial T < 0$ from (66), which means that $W\left(T,q^{**},y^{**}\right)$ always decreases with T for given $\left(q^{**},y^{**}\right)$. Therefore, the profitability constraint (11) is binding for the second-best solution $\left(T^{**},q^{**},y^{**}\right)$, i.e. $T^{**}R\left(q^{**}\right) = I\left(y^{**}\right) + \tilde{P}$, otherwise T can be decreased to increase $W\left(T,q,y\right)$.

(c) Suppose $\bar{T}R\left(q_0\right) \geq I\left(y^{**}\right) + \tilde{P}$, then $\left(\bar{T},q_0,y^{**}\right)$ is a feasible solution and it is readily seen that $W\left(\bar{T},q_0,y^{**}\right) > W\left(T^{**},q^{**},y^{**}\right)$, which contradicts that $\left(T^{**},q^{**},y^{**}\right)$ is a second-best solution. This completes the proof.

Lemma 2. The function $h\left(q\right)$ defined by (25) decreases with q for $q_1 \leq q < q_0$.

Proof: It suffices to prove that $h'\left(q\right) < 0$ for $q_1 \leq q < q_0$. We have

$$h'\left(q\right) = \frac{D'\left(q\right)R\left(q\right) - D\left(q\right)R'\left(q\right)}{\left(R\left(q\right)\right)^2}$$

Thus it suffices to prove that $D'\left(q\right)R\left(q\right) < D\left(q\right)R'\left(q\right)$ for $q_1 \leq q < q_0$. Since $S\left(q\right)$ is strictly concave and increasing in q for $q_1 \leq q \leq q_0$, $D\left(q\right) = \tilde{S}\left(y^{**}\right) - S\left(q\right)$ is strictly convex and decreasing in q for $q_1 \leq q \leq q_0$. And in view of $D\left(q_0\right) = 0$, we have

$$D\left(q\right) = D\left(q\right) - D\left(q_0\right) < \left(-D'\left(q\right)\right)\left(q_0 - q\right), \text{ for } q_1 \leq q < q_0 \tag{70}$$

With Assumption 2, $R\left(q\right)$ is strictly concave and decreasing in q for $q_1 \leq q \leq q_0$. And in view of $R\left(q_0\right) > 0$ (congestion toll is positive), we have

$$R\left(q\right) > R\left(q\right) - R\left(q_0\right) > \left(-R'\left(q\right)\right)\left(q_0 - q\right), \text{ for } q_1 \leq q < q_0 \tag{71}$$

Given $\left(-D'\left(q\right)\right) > 0$ and $\left(-R'\left(q\right)\right) > 0$, combining (70) and (71) leads to

$$\left(-D'\left(q\right)\right)R\left(q\right) > D\left(q\right)\left(-R'\left(q\right)\right), \text{ for } q_1 \leq q < q_0$$

which is simply $D'\left(q\right)R\left(q\right) < D\left(q\right)R'\left(q\right)$ for $q_1 \leq q < q_0$. This completes the proof.

A3. Proof of Lemma 3

Lemma 3. Let $\left(\bar{T}, q^*, y^*\right)$ be an optimal BOT contract (either first-best or second-best). Then $P\left(\bar{T}, q, y\right) < P\left(\bar{T}, q^*, y^*\right)$ for any $q > q^*$, $y > 0$.

Proof: Suppose there exist $\bar{q} > q^*$, $\bar{y} > 0$ such that $P\left(\bar{T}, \bar{q}, \bar{y}\right) \geq P\left(\bar{T}, q^*, y^*\right)$, then $\left(\bar{T}, \bar{q}, \bar{y}\right)$ is a feasible BOT contract, and it suffices to prove that $W\left(\bar{T}, \bar{q}, \bar{y}\right) > W\left(\bar{T}, q^*, y^*\right)$, which contradicts that $\left(\bar{T}, q^*, y^*\right)$ is an optimal BOT contract. To this end, we prove that

$$W\left(\bar{T}, \bar{q}, \bar{y}\right) - W\left(\bar{T}, q^*, y^*\right) > P\left(\bar{T}, \bar{q}, \bar{y}\right) - P\left(\bar{T}, q^*, y^*\right) \geq 0 \tag{72}$$

where the second inequality follows from $P\left(\bar{T}, \bar{q}, \bar{y}\right) \geq P\left(\bar{T}, q^*, y^*\right)$, and the first inequality is equivalent to

$$W\left(\bar{T}, \bar{q}, \bar{y}\right) - P\left(\bar{T}, \bar{q}, \bar{y}\right) > W\left(\bar{T}, q^*, y^*\right) - P\left(\bar{T}, q^*, y^*\right) \tag{73}$$

From the welfare function $W\left(T, q, y\right)$ given by (7) and the profit function $P\left(T, q, y\right)$ given by (5), we can see that (73) is equivalent to

$$\int_0^{\bar{q}} B(w)\, dw - \bar{q} B(\bar{q}) > \int_0^{q^*} B(w)\, dw - q^* B\left(q^*\right) \tag{74}$$

which in turn is equivalent to

$$\int_{q^*}^{\bar{q}} B(w)\, dw > \bar{q} B(\bar{q}) - q^* B\left(q^*\right) \tag{75}$$

In view of $\bar{q} > q^*$ and that $B(q)$ decreases with q, (75) holds readily as below:

$$\int_{q^*}^{\bar{q}} B(w)\, dw > \left(\bar{q} - q^*\right) B(\bar{q}) > \bar{q} B(\bar{q}) - q^* B\left(q^*\right)$$

This completes the proof.

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CAPTIONS

Figure 1. Graphical illustration for $R(q)$, $S(q)$, q_1 **and** q_0

Figure 2. Maximum welfare as a function of concession period

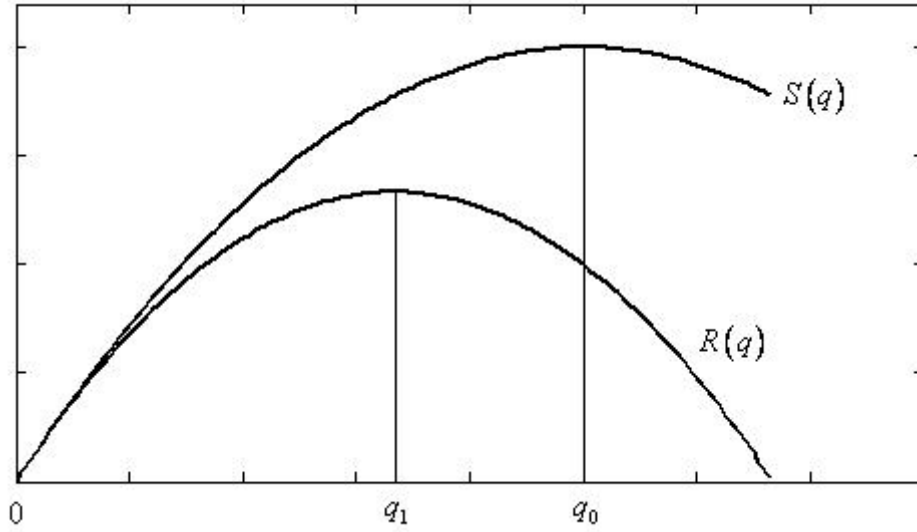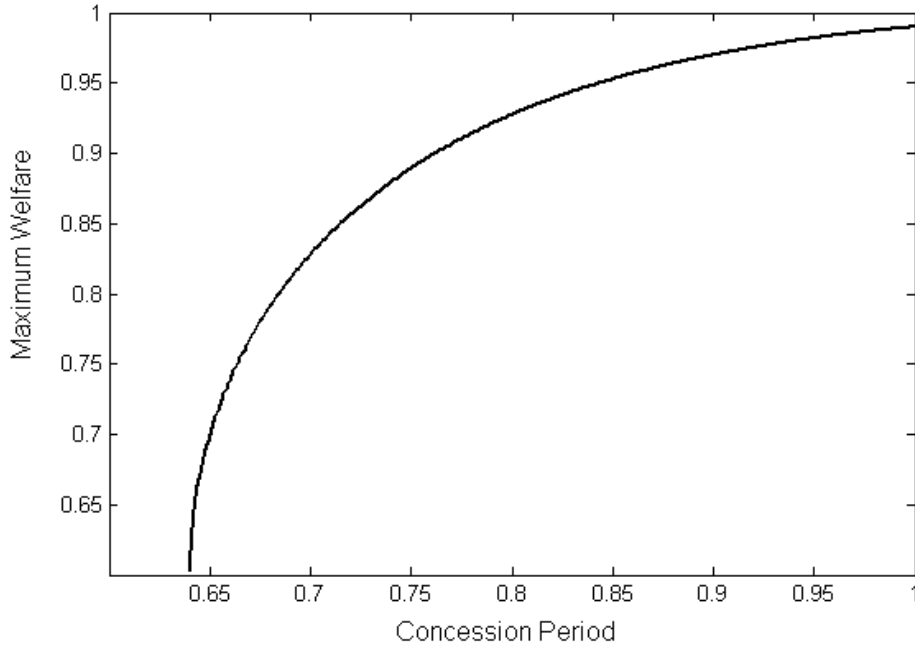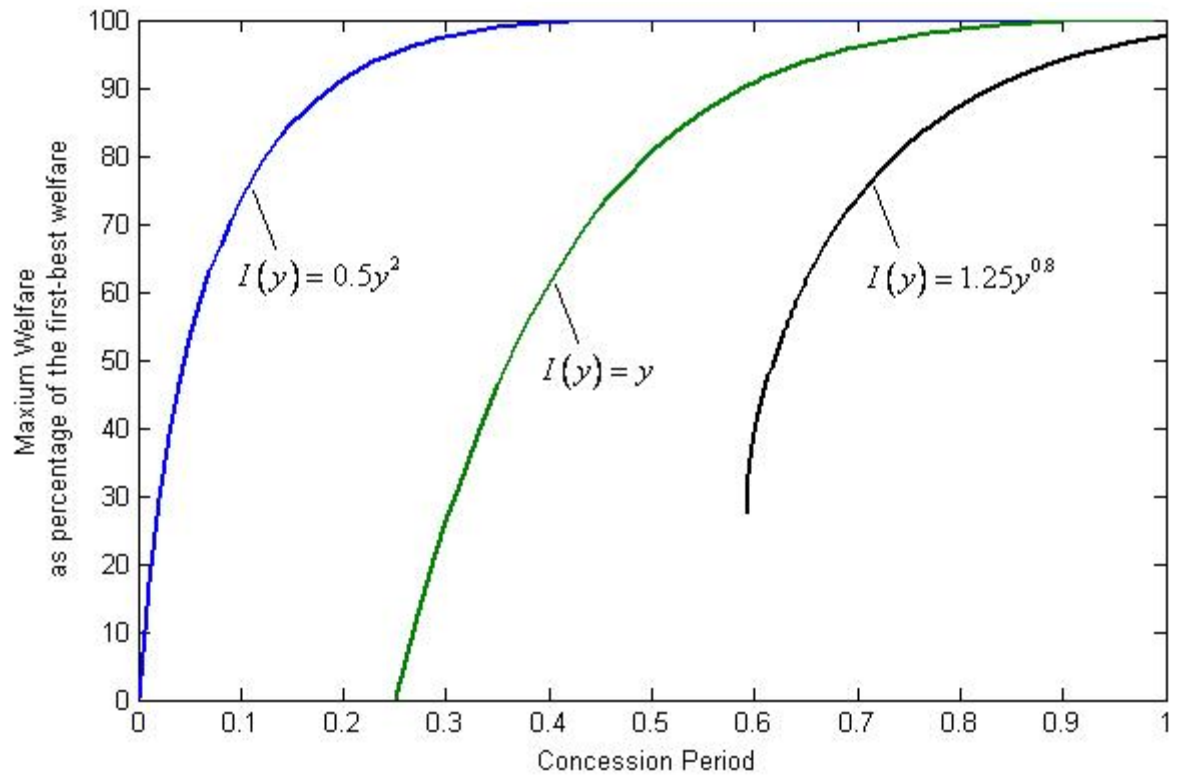**Figure 3. Maximum welfare function (as percentage of first-best welfare) for three cases**

FIGURES

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