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Xiaolei Guo
University of Windsor

Hai Yang

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User Heterogeneity and Bi-criteria System Optimum

Xiaolei Guo and Hai Yang*

Department of Civil Engineering, The Hong Kong University of Science & Technology

Clear Water Bay, Kowloon, Hong Kong, P.R. China

Abstract

For a traffic network with fixed demand of heterogeneous users in terms of their different values of time (VOT), the system performance can be measured either in time unit by the total system travel time (in short, system time), or in monetary unit by the total system travel cost (in short, system cost). Thus we have two different objectives for network optimization, i.e. to minimize system time and to minimize system cost, which naturally gives rise to a bi-objective minimization problem. A Pareto optimum of this bi-objective optimization problem represents a bi-criteria system optimum for network optimization in the sense that, at each Pareto optimum, neither system time nor system cost can be further reduced without increasing the other one. In this paper we prove that any Pareto optimum can be decentralized into multi-class user equilibrium by positive anonymous link tolls. We then bound the system performance gap when optimized by the two different criteria. Specifically, we provide answers to the following questions: When system time is minimized, how far could the corresponding system cost deviate from its minimum value? Conversely, when system cost is minimized, how far could the corresponding system time deviate from its minimum value? More generally, how far can the system time and system cost at a given bi-criteria Pareto optimum deviate from their respective single-criterion based system optimum?

Keywords: User heterogeneity, system optimum, Pareto optimum, traffic equilibrium, networks

* Corresponding author. e-mail: cehyang@ust.hk; tel: (852) 2358-7178; Fax: (852) 2358-1534

1. Introduction

The concept of value of time (VOT) plays a pivot role in road pricing analysis as it describes how users make tradeoffs between money and time in response to road toll charges. Conventional network user equilibrium (UE) model typically assumes that users' VOT are identical, i.e. homogeneous users. With homogeneous users, the first-best congestion pricing theory, namely the theory of marginal cost pricing is well established in general networks for fixed and elastic travel demand. However, it is well known that travelers may value travel time differently, depending on their income levels or travel purposes, and thus heterogeneous users with different VOT have to be considered. In the presence of user heterogeneity, various network equilibrium models are developed by assuming either a discrete set of VOT for several distinct user classes or a continuously distributed VOT across the whole population (e.g. Leurent, 1993; Marcotte and Zhu, 2000; Mayet and Hansen, 2000; Nagurney, 2000a). Also, the optimal pricing problems for heterogeneous users with discrete or continuous VOT distributions are investigated extensively (e.g. Dial, 1999a,b; Yang and Zhang, 2002; Yang and Huang, 2004).

In traffic networks with users having different VOT, the system performance can be measured either in time unit or in cost (monetary) unit, and the two measures are in general mutually inconsistent with each other. In particular, for the case of fixed travel demand, the system performance can be measured either in time unit by the total system travel time (in short, system time), or in monetary unit by the total system travel cost (in short, system cost). From an economic viewpoint, system cost is a more appropriate system disutility measure when users value travel time differently. Nonetheless, system time is a traditional measure of system performance in transportation engineering context, and in the presence of multiple user classes, system time is still a proper measure of the environmental effect of traffic, i.e. the total system travel time roughly determines the total gas emission of vehicles (in this respect, one has to acknowledge the paradox demonstrated by Nagurney (2000b) and Yin and Lawphongpanich (2006) that an improvement in travel time might paradoxically increase traffic emission). Thus both system time and system cost are meaningful criteria for evaluating system performance. Yang and Huang (2004) examined the multi-class multi-criteria (cost versus time) network equilibrium and system optimum problem in a network with a discrete VOT distribution and fixed travel demand. They studied the cost-based and the time-based system optimization (SO) problems separately. That is, they studied the minimization of system cost without consideration of system time, and then studied the minimization of system time but ignored system cost.

In this paper, we propose to consider the two criteria for system optimization simultaneously. Specifically, we adopt a discrete VOT distribution model with fixed travel demand, and introduce a bi-objective minimization problem of system time and system cost. Using the theories of multi-objective optimization, we identify a Pareto optimal solution set and the corresponding objective value set (the Pareto optimal frontier) in our bi-objective optimization problem. Each point on the Pareto optimal frontier is a Pareto optimum where a bi-criteria system optimum is established in the sense that neither system time nor system cost can be further reduced without increasing the other one. After establishing the bi-criteria Pareto optimum, we then examine the question of whether or not we can always find positive anonymous link tolls (the same amount of toll levied on each link for all user classes) to decentralize a Pareto system optimum. We provide a positive answer to this question by rigorous analysis. Note that, in our study context we have to consider a pricing scheme with anonymous link tolls, because users differ from one another in VOT only, which is *observationally indistinguishable,* and thus toll differentiation across user classes is unrealistic and difficult to implement in reality.

Our subsequent concern in this paper is the system performance discrepancy between the time-based and the cost-based SO cases. System time and system cost are both weighted summations of the travel times of all user classes, and thus differ from each other only to a bounded extent. We explore the extent to which the system performance measures differ between the time-based and the cost-based SO cases. Specifically, we answer the following questions. When system time is minimized at the time-based SO, how far could the corresponding system cost deviate from its minimum value achieved at the cost-based SO? Conversely, when system cost is minimized at the cost-based SO, how far could the corresponding system time deviate from its minimum value achieved at the time-based SO?

More generally, how far can the system time and system cost at a given bi-criteria Pareto optimum deviate from their respective single-criterion based system optimum? We show that the Pareto optimal system time and cost achieve their maximum deviations from optimum when the other objective is optimized. We also establish a meaningful upper bound of the product of the two maximum deviations, which can be solely determined by users' VOT distribution.

The paper is organized as follows. Section 2 provides preliminaries on the multi-class network optimization and pricing problems. Section 3 introduces the bi-criteria Pareto optimum concept, and proves that any Pareto optimum can be supported as multi-class UE by positive anonymous link tolls. Section 4 bounds the system performance discrepancy between the time-based and the cost-based SO cases. Finally, some concluding remarks are offered in Section 5.

2. Preliminaries on multi-class network optimization and pricing problems

Let $G(N,A)$ denote a transportation network, with a set of nodes N and a set of links A, together with a set of origin-destination (OD) pairs W. We consider separable link travel time function $t_a(v_a)$, $a \in A$, i.e. travel time on a link depends on the flow on that link only. The link travel time function $t_a(v_a)$, $a \in A$, is assumed to be monotonically increasing with v_a. We consider a discrete set of user classes corresponding to the groups of users with different socio-economic characteristics, such as income level. Let M denote the set of such user classes, and β_m $\beta_m > 0$, be the average VOT for users of class $m \in M$. Let d_w^m. $d_w^m > 0$, be the travel demand of user class $m \in M$ between OD pair $w \in W$, R_w the set of all simple paths connecting OD pair $w \in W$, and f_{rw}^m the flow of user class m on path $r \in R_w$. The flow v_a^m by user class $m \in M$ and the total aggregate flow v_a on link $a \in A$

can be expressed in terms of path flows as follows:

$$v_a^m = \sum_{w \in W} \sum_{r \in R_w} f_{rw}^m \delta_{ar}, \; a \in A, \; m \in M \tag{1}$$

$$v_a = \sum_{m \in M} v_a^m, \quad a \in A \tag{2}$$

where $\delta_{ar} = 1$ if route r uses link a and 0 otherwise. For simplicity, we denote vectors as $\mathbf{f} = \left(f_{rw}^m, r \in R_w, w \in W, m \in M \right)$, $\mathbf{v} = (v_a, a \in A)$, and $\mathbf{v}^{\mathbf{M}} = (v_a^m, a \in A, m \in M)$.

In the presence of multiple user classes with different VOT, the system travel disutility can be measured either in time unit (time-based disutility or total system travel time) by

$$T = \sum_{a \in A} t_a(v_a) v_a = \sum_{a \in A} \sum_{m \in M} t_a(v_a) v_a^m \tag{3}$$

or in cost or monetary unit (cost-based disutility or total system monetary cost) by

$$C = \sum_{a \in A} \sum_{m \in M} \beta_m t_a(v_a) v_a^m \tag{4}$$

Clearly, both the time-based and the cost-based system disutilities can be regarded as weighted sums of the travel times of all user classes in the network. The former has a uniform weighting factor equal to unity, while the latter has non-uniform weighting factors equal to the VOT of respective user classes. As said earlier, system cost C is more meaningful for economists and system time T is usually used by engineers; both T and C are meaningful criteria for network optimization.

For simplicity, let Ω be the feasible set of path flows defined as

$$\Omega = \left\{ \mathbf{f} : \sum_{r \in R_w} f_{rw}^m = d_w^m; \; f_{rw}^m \geq 0; w \in W, \; m \in M \right\} \tag{5}$$

With the definition of T and C, the time-based SO problem is formulated as

$$\min_{\mathbf{f} \in \Omega} \; T(\mathbf{f}) = \sum_{a \in A} t_a(v_a) v_a \tag{6}$$

and the cost-based SO problem is formulated as

$$\min_{\mathbf{f} \in \Omega} \; C(\mathbf{f}) = \sum_{a \in A} \sum_{m \in M} \beta_m v_a^m t_a(v_a) \tag{7}$$

The system optimal flows given by the above SO problems are generally different from the equilibrium flows in the absence of toll pricing, because each user tries to minimize her own travel time. Thus congestion pricing has to be introduced to decentralize an SO or other second-best target flow pattern into (or support it as) a multi-class UE flow pattern. That is, a tolling system can alter the generalized travel disutility of links and paths faced by each class of users, and thereby induce new multi-class UE flow patterns, which are exactly the target (SO or other second-best) flow patterns.

Let u_a denote the toll charged on link $a \in A$ for all user classes and $\mathbf{u} = \left(u_a, a \in A\right)$ be the vector of all link toll charges. With a toll scheme $\mathbf{u}$ implemented, the generalized travel time $c_{rw}^{m,\mathrm{t}}$ for a user of class m traveling along route $r \in R_w$ between OD pair $w \in W$, is defined as (money is converted into equivalent travel time according to the user class-specific VOT, and the superscript 't' stands for time):

$$c_{rw}^{m,\mathrm{t}} = \sum_{a \in A} \left\{ t_a\left(v_a\right) + \frac{u_a}{\beta_m} \right\} \delta_{ar},\ r \in R_w,\ w \in W,\ m \in M \tag{8}$$

The corresponding generalized travel cost $c_{rw}^{m,\mathrm{c}}$ is defined as (time is converted into money, and the superscript 'c' stands for cost):

$$c_{rw}^{m,\mathrm{c}} = \sum_{a \in A} \left\{ \beta_m t_a\left(v_a\right) + u_a \right\} \delta_{ar},\ r \in R_w,\ w \in W,\ m \in M \tag{9}$$

With a toll scheme $\mathbf{u}$ implemented, the time-based UE conditions can be written (in time unit) as

$$\sum_{a \in A} t_a\left(v_a\right)\delta_{ar} + \sum_{a \in A} \frac{u_a}{\beta_m}\delta_{ar} = \mu_w^{m,\mathrm{t}},\ \text{ if } f_{rw}^m > 0,\quad r \in R_w,\ w \in W,\ m \in M \tag{10}$$

$$\sum_{a \in A} t_a\left(v_a\right)\delta_{ar} + \sum_{a \in A} \frac{u_a}{\beta_m}\delta_{ar} \geq \mu_w^{m,\mathrm{t}},\ \text{ if } f_{rw}^m = 0,\quad r \in R_w,\ w \in W,\ m \in M \tag{11}$$

and the cost-based UE conditions are (in monetary unit)

$$\sum_{a \in A} \beta_m t_a\left(v_a\right)\delta_{ar} + \sum_{a \in A} u_a \delta_{ar} = \mu_w^{m,\mathrm{c}},\ \text{ if } f_{rw}^m > 0,\ r \in R_w,\ w \in W,\ m \in M \tag{12}$$

$$\sum_{a \in A} \beta_m t_a\left(v_a\right)\delta_{ar} + \sum_{a \in A} u_a \delta_{ar} \geq \mu_w^{m,\mathrm{c}},\ \text{ if } f_{rw}^m = 0,\ r \in R_w,\ w \in W,\ m \in M \tag{13}$$

where $\mu_w^{m,\mathrm{t}}$ and $\mu_w^{m,\mathrm{c}}$ are the minimum travel disutility in time and monetary units, respectively, between OD pair $w \in W$ by users of class $m \in M$, i.e., $\mu_w^{m,\mathrm{t}} = \min_{r \in R_w} \left\{ c_{rw}^{m,\mathrm{t}} \right\}$, $\mu_w^{m,\mathrm{c}} = \min_{r \in R_w} \left\{ c_{rw}^{m,\mathrm{c}} \right\}$, and $\mu_w^{m,\mathrm{c}} = \beta_m \mu_w^{m,\mathrm{t}}$. Clearly, the disutility unit used (time or cost) does not affect the UE conditions or the equilibrium flows, i.e. the time-based and the cost-based UE conditions are equivalent to each other.

It is well known (e.g. Yang and Huang, 2004) that the multi-class UE problem can be formulated as the following equivalent minimization problem:

$$\min_{\mathbf{f} \in \Omega} \sum_{a \in A} \int_0^{v_a} t_a(\omega)\,\mathrm{d}\omega + \sum_{a \in A} \sum_{m \in M} \frac{1}{\beta_m} v_a^m u_a \tag{14}$$

Observe that objective function (14) is strictly convex in aggregate link flow $\mathbf{v}$ for monotonically increasing link travel time function $t_a(v_a)$, but linear in class-specific link flow $\mathbf{v}^{\mathbf{M}}$. Thus, under a given tolling system $\mathbf{u}$, the equilibrium link flow by user class $\mathbf{v}^{\mathbf{M}}$ is generally not unique (neither is the UE path flow $\mathbf{f}$), while the aggregate UE link flow $\mathbf{v}$ is unique. Since $\mathbf{v}$ is unique, the system time T at equilibrium given by (3) is unique. The uniqueness of system cost C at equilibrium is not obvious but can be shown below.

Multiplying both sides of (12) and (13) by equilibrium path flow f_{rw}^m and summing over all $r \in R_w,\ w \in W,\ m \in M,$ we have

$$\sum_{a \in A} \sum_{m \in M} \beta_m t_a(v_a) v_a^m + \Pi = \sum_{w \in W} \sum_{m \in M} \mu_w^{m,\mathrm{c}} d_w^m$$

or equivalently

$$C = \sum_{a \in A} \sum_{m \in M} \beta_m t_a(v_a) v_a^m = \sum_{w \in W} \sum_{m \in M} \mu_w^{m,\mathrm{c}} d_w^m - \Pi \tag{15}$$

where $\Pi = \sum_{a \in A} u_a v_a$ is the unique (due to unique $v_a,\ a \in A$) total toll revenue generated from all classes of user in the network, and the minimum travel cost $\mu_w^{m,\mathrm{c}}$ for each class between each OD pair is also unique. Since $\mu_w^{m,\mathrm{c}}$ and Π are unique at equilibrium, (15) means that system cost C is unique.

In summary, we have the following lemma.

Lemma 1. *At traffic equilibrium under a toll scheme* $\mathbf{u}$*, the path flow* $\mathbf{f}$ *and the class-specific link flow* $\mathbf{v}^{\mathbf{M}}$ *are generally not unique, while the aggregate link flow* $\mathbf{v}$ *and the system disutilities,* T *and* C*, are all unique.*

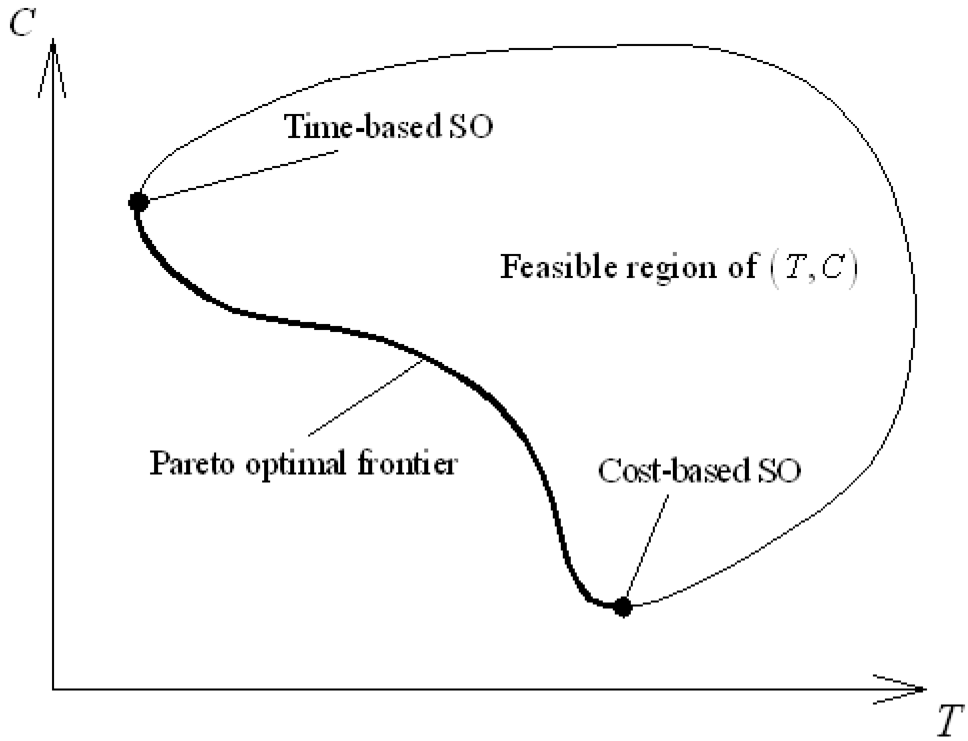

Figure 1. Pareto optimal frontier of nonconvex bi-objective problem (16)

3. Bi-criteria Pareto system optimum and Pareto optimal link toll

As mentioned before, both system time T and system cost C are meaningful measure of system performance. Nevertheless, previous researches on network pricing optimization typically consider only minimization of either T or C (Yang and Huang, 2004). Here we shall consider minimization of T and C simultaneously. Thus we have the following bi-objective minimization problem, which combines the two SO problems (6) and (7).

$$\min_{\mathbf{f}\in\Omega} \begin{pmatrix} T(\mathbf{f}) \\ C(\mathbf{f}) \end{pmatrix} = \begin{pmatrix} \sum_{a\in A} t_a(v_a)v_a \\ \sum_{a\in A}\sum_{m\in M} \beta_m t_a(v_a)v_a^m \end{pmatrix} \tag{16}$$

Instead of seeking for optimal flow patterns of minimizing T or C, this bi-objective problem (16) is to seek for a Pareto optimal solution set, which determines a Pareto optimal frontier of the two objective values. As an illustration, Figure 1 plots a typical Pareto optimal frontier (also referred to as "efficient frontier") for problem (16) in the (T,C) two-dimensional space. It should be mentioned that the feasible region of the objective value (T,C) is generally nonconvex due to the nonconvexity of $C(\mathbf{f})$ (Yang and Huang, 2005; Engelson and Lindberg, 2006).

As seen in Figure 1, the Pareto optimal frontier of problem (16) is the left-lower part of the boundary of the feasible region of (T,C), with the time-based SO and the cost-based SO being the two endpoints of the frontier. At each point on the Pareto optimal frontier (including the two SO points), neither T nor C can be further reduced without increasing the other one. Thus each point on the Pareto optimal frontier represents a bi-criteria system optimum, and is referred to as a Pareto system optimum (or shortly, Pareto optimum). A feasible flow pattern that gives a Pareto optimum is said to be a Pareto optimal flow pattern, and its mathematical definition is given as follows.

Definition 1. *A flow pattern* $\mathbf{f}_P \in \Omega$ *is said to be Pareto optimal if there does not exist another flow pattern* $\mathbf{f} \in \Omega$ *such that* $T(\mathbf{f}) \le T(\mathbf{f}_P)$, $C(\mathbf{f}) \le C(\mathbf{f}_P)$, *and at least one of* $T(\mathbf{f}) < T(\mathbf{f}_P)$ *and* $C(\mathbf{f}) < C(\mathbf{f}_P)$ *holds.*

The simplest way to solve bi-objective problem (16) is the so called weighted sum method, i.e. to solve a single objective problem as follows.

$$\min_{\mathbf{f}\in\Omega} \eta_1 T(\mathbf{f}) + \eta_2 C(\mathbf{f}) \tag{17}$$

With different nonnegative weights η_1 and η_2, problem (17) leads to different Pareto optima. Nonetheless, it is well known (e.g. Miettinen, 1999) that the weighted sum method may not be able to solve for every Pareto optimum when the multi-objective problem is nonconvex, as is the case of our bi-objective problem (16). As illustrated in Figure 2, the objective contour lines of the weighted-sum problem (17) are straight lines in the (T,C) two-dimensional space. Setting different values to η_1 and η_2 only changes the slope of the contour line. Whatever combination of (η_1,η_2) or whatever slope of contour line is adopted, some points on the Pareto optimal frontier (the concave part, as shown in Figure 2) can never be optimal to the weighted-sum minimization problem (17). This means that some Pareto optima can never be obtained by the weighted sum method.

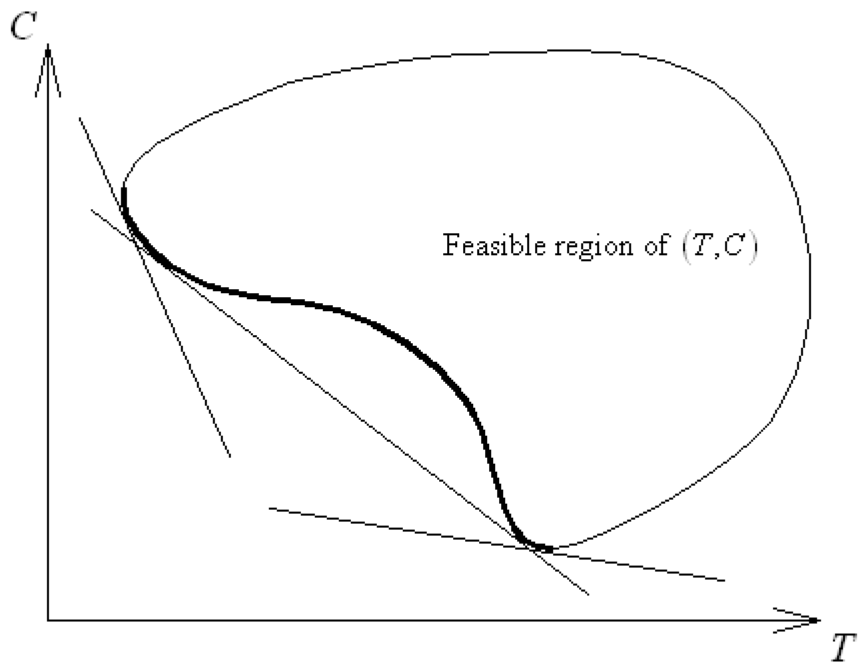

Figure 2. Weighted sum method for nonconvex bi-objective problem (16)

For a nonconvex multi-objective problem, several methods are available for finding every Pareto optimum (Miettinen, 1999). Here we shall use the so called ε-constraint method to solve our nonconvex bi-objective minimization problem (16). In this method, one objective function is selected to be optimized and the other objective function is converted into a constraint by setting an upper bound. Specifically, we shall solve the following problem.

$$\min_{\mathbf{f}\in\Omega} C(\mathbf{f}) \tag{18}$$

subject to

$$T(\mathbf{f}) \le \varepsilon_T \tag{19}$$

where ε_T is a predefined upper bound of the system time. As illustrated in Figure 3, by setting different values to ε_T, each point on the Pareto optimal frontier can be obtained.

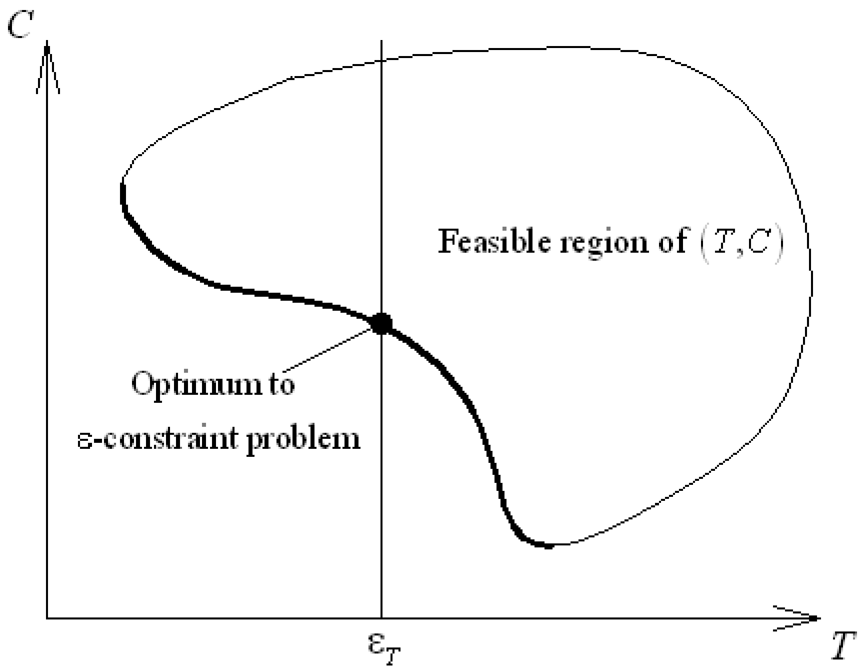

Figure 3. ε-constraint method for nonconvex bi-objective problem (16)

After introducing the bi-criteria Pareto system optimum, we are now interested in the following question: can we always find positive anonymous link tolls to induce a Pareto system optimum? For the two special cases of Pareto optimum, i.e. the respective time-based SO and cost-based SO, the answer is positive as shown in Yang and Huang (2004). In particular, Yang and Huang (2004) constructed a primal-dual pair of linear programming problems to obtain the set of nonnegative anonymous link toll patterns which can induce the time-based SO. In this section we first generalize their method of “primal-dual LP” to any feasible target link flow pattern and then apply the method to our Pareto system optimum.

3.1 Decentralization of a given feasible target link flow pattern

For a given feasible target link flow $\bar{\mathbf{v}} = (\bar{v}_a, a \in A)$, we consider the following linear programming (LP) problem:

$$\min_{\mathbf{f}\in\Omega} \sum_{a\in A}\sum_{m\in M} \beta_m t_a\left(\bar{v}_a\right) v_a^m \tag{20}$$

subject to

$$\sum_{w\in W}\sum_{r\in R_w}\sum_{m\in M} f_{rw}^m \delta_{ar} = \bar{v}_a\,,\ \ a\in A \tag{21}$$

In words, constraint (21) means that the aggregate link flow should be $\bar{\mathbf{v}}$. Thus LP (20)-(21) is to minimize the system cost with respect to class-specific path and link flows subject to the aggregate link flow being equal to $\bar{\mathbf{v}}$.

In view of the constraint set $\mathbf{f}\in\Omega$ defined in (5), the dual formulation of LP (20)-(21) is:

$$\max_{\boldsymbol{\lambda},\boldsymbol{\mu}} \sum_{a\in A}\bar{v}_a\lambda_a + \sum_{w\in W}\sum_{m\in M} d_w^m \mu_w^m \tag{22}$$

subject to

$$\mu_w^m + \sum_{a\in A}\lambda_a\delta_{ar} \le \sum_{a\in A}\beta_m t_a\left(\bar{v}_a\right)\delta_{ar},\ r\in R_w,\ w\in W,\ m\in M \tag{23}$$

where the dual variables $\boldsymbol{\lambda} = \left(\lambda_a, a\in A\right)$ and $\boldsymbol{\mu} = \left(\mu_w^m, w\in W, m\in M\right)$ are associated with the equality constraints (21) and the OD flow conservation constraint in (5), respectively, and λ_a and μ_w^m are unrestricted in sign. Note that the set of feasible solutions to LP (20)-(21) is not empty because $\bar{\mathbf{v}}$ is a feasible link flow pattern. Thus both the primal and dual LP have solutions.

Let u_a replace $(-\lambda_a)$, $a\in A$ and rewrite (23) as

$$\sum_{a\in A}\left\{\beta_m t_a\left(\bar{v}_a\right)+u_a\right\}\delta_{ar} \ge \mu_w^m,\ r\in R_w,\ w\in W,\ m\in M \tag{24}$$

According to the duality theory, at the optimal points of the primal and dual LP, the following complementary slackness conditions hold:

$$\left\{\sum_{a\in A}\left(\beta_m t_a\left(\bar{v}_a\right)+u_a\right)\delta_{ar} - \mu_w^m\right\} f_{rw}^m = 0,\ r\in R_w,\ w\in W,\ m\in M \tag{25}$$

$$\sum_{a\in A}\left(\beta_m t_a\left(\overline{v}_a\right)+u_a\right)\delta_{ar}-\mu_w^m\geq 0,\ r\in R_w,\ w\in W,\ m\in M \tag{26}$$

Observe that the optimality conditions (25)-(26) are simply the cost-based UE conditions (12) -(13), with μ_w^m rewritten as $\mu_w^{m,c}$. This equivalence between the LP optimality conditions and the UE conditions immediately leads to the following theorem.

Theorem 1. *Any feasible target link flow pattern can be supported as a multi-class UE link flow pattern by anonymous link tolls which can be positive or negative.*

Proof: For any feasible link flow $\overline{\mathbf{v}}=(\overline{v}_a, a\in A)$, we can construct the primal-dual pair of LP (20)-(21) and (22)-(23), both having solutions and thus having optimal solutions. Consider a pair of primal-dual optimal solutions $\mathbf{f}$ and $\left(\boldsymbol{\lambda},\boldsymbol{\mu}\right)$, and let $\mathbf{u}=-\boldsymbol{\lambda}$ be a toll scheme. Then $\mathbf{f}$, $\mathbf{u}$ and $\boldsymbol{\mu}$ satisfy the optimality conditions (25)-(26), which are just the UE conditions under toll scheme $\mathbf{u}=-\boldsymbol{\lambda}$, thus $\mathbf{f}$ is a UE path flow and $\overline{\mathbf{v}}$ is the UE link flow under toll scheme $\mathbf{u}=-\boldsymbol{\lambda}$. Therefore, $\overline{\mathbf{v}}$ is supported as a UE link flow by toll scheme $\mathbf{u}=-\boldsymbol{\lambda}$. Because λ_a is unrestricted in sign, toll scheme $\mathbf{u}=-\boldsymbol{\lambda}$ can have positive and negative link tolls. This completes the proof. ♦

Theorem 1 requires that link tolls can be "positive or negative". Yang and Huang (2004) showed the existence of positive link tolls to decentralize a cost or time-based SO link flow pattern into a multi-class UE, but this conclusion can not be generalized to any feasible link flow pattern that may have to resort to negative link tolls. Even in the case of homogeneous users, negative link toll is sometimes necessary to induce a feasible link flow pattern, as shown in Bai et al. (2006) by a simple example.

Once an aggregate link flow pattern is given, the system time is uniquely determined, nevertheless, the system cost is undetermined but dependent on the class-specific link flows that constitute the given aggregate link flows. Fortunately, if an anonymous tolling scheme is implemented to support a target link flow $\overline{\mathbf{v}}$ as a multi-class UE flow, the resulting UE class-specific flow $\mathbf{f}$ solves LP (20)-(21) for given $\overline{\mathbf{v}}$ (as the UE conditions are equivalent to the LP optimality conditions), which means that $\mathbf{f}$ minimizes the system cost under $\bar{\mathbf{v}}$. To sum up, we have

Lemma 2. *Under any anonymous link toll scheme that decentralizes a given feasible aggregate link flow pattern, the resulting class-specific UE flows minimize the system cost associated with the given aggregate link flow pattern.*

The intuition behind this lemma is that, under any anonymous link toll scheme, users with higher VOT tend to choose routes with lower travel times by paying higher toll charges, thus the system cost, C, which is exclusive of toll charge and given as the sum of the travel times of all user classes weighted by their VOT, is naturally minimized. The implication of the lemma is that, when analyzing congestion charge in the presence of user heterogeneity, we shall concentrate on the aggregate link flow rather than the class-specific flow, because the latter is self-optimized by the anonymous link tolls.

3.2 Existence of nonnegative anonymous link tolls for a given Pareto system optimum

After establishing Theorem 1 for decentralizing any feasible target link flow pattern with anonymous positive or negative link tolls, we now turn our attention to a Pareto optimum of the bi-objective system optimization problem. First note that, although applied for any feasible target link flow pattern, Theorem 1 can not be simply said to be valid for a Pareto system optimum, because the solution to bi-objective minimization problem (16) is defined and determined by the *class-specific* path and link flow rather than the *aggregate* link flow only. Thus we are left with the following question: when we use anonymous link tolls to decentralize the aggregate link flow of a Pareto system optimum, whether the class-specific UE path and link flow under the given toll scheme is Pareto optimal. The answer is positive as stated in the following lemma.

Lemma 3. *A Pareto system optimum is supported as a multi-class UE by an anonymous link toll scheme if the toll scheme induces the aggregate link flow of the Pareto system optimum.*

Proof: Consider a Pareto optimal flow $\bar{\mathbf{f}}$ and the corresponding aggregate link flow $\bar{\mathbf{v}}$. Let $\mathbf{u}$ be an anonymous link toll scheme under which the aggregate UE link flow is $\bar{\mathbf{v}}$. It suffices to prove that the class-specific UE path flow $\mathbf{f}$ (generally non-unique) under toll scheme $\mathbf{u}$ gives $T(\mathbf{f}) = T(\bar{\mathbf{f}})$ and $C(\mathbf{f}) = C(\bar{\mathbf{f}})$. Since both $\bar{\mathbf{f}}$ and $\mathbf{f}$ have the same aggregate link flow $\bar{\mathbf{v}}$, we have $T(\mathbf{f}) = T(\bar{\mathbf{f}})$. Because $\mathbf{f}$ is a UE flow, $C(\mathbf{f})$ is the minimum system cost under link flow $\bar{\mathbf{v}}$ from Lemma 2, thus we have $C(\mathbf{f}) \le C(\bar{\mathbf{f}})$, which simply means $C(\mathbf{f}) = C(\bar{\mathbf{f}})$, because $C(\mathbf{f}) < C(\bar{\mathbf{f}})$ contradicts that $\bar{\mathbf{f}}$ is Pareto optimal. This completes the proof. ♦

Note that Theorem 1 and Lemma 3 already ensure the existence of anonymous link tolls to induce a Pareto system optimum. We now further prove the existence of nonnegative anonymous link tolls. This is meaningful because negative tolls (subsidies) on road networks are difficult and seldom implemented by policy makers in reality.

Theorem 2. *A Pareto system optimum can be supported as a multi-class UE by nonnegative anonymous link tolls.*

Proof: Consider a Pareto optimal flow $\bar{\mathbf{f}}$, and let $\bar{\mathbf{v}}$ be the corresponding aggregate link flow. With Lemma 3, it suffices to prove that there exists a *nonnegative* anonymous link toll scheme $\mathbf{u}$ under which the UE link flow is $\bar{\mathbf{v}}$. To do this, for the given link flow $\bar{\mathbf{v}}$, we construct a modified version of the primal LP (20)-(21) with the equality constraint (21) replaced by the following inequality constraint

$$\sum_{w \in W} \sum_{r \in R_w} \sum_{m \in M} f_{rw}^m \delta_{ar} \le \bar{v}_a, \quad a \in A \tag{27}$$

Because constraint (27) is a simple relaxation of constraint (21) and other constraints remain unchanged, the feasible region of the new LP is expanded and hence nonempty. This means that the new LP and its dual LP have solutions. The new dual formulation is given by (22) -(23) adding an additional constraint

$$\lambda_a \le 0, \quad a \in A \tag{28}$$

where $\boldsymbol{\lambda} = (\lambda_a, a \in A)$ is associated with the inequality constraint (27). Let $\mathbf{f}$ and $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ be a pair of primal-dual optimal solutions to this new pair of primal-dual LP. Then $\mathbf{u} = -\boldsymbol{\lambda}$ is a nonnegative link toll scheme because of constraint (28), and it still holds that $\mathbf{f}$, $\mathbf{u}$ and $\boldsymbol{\mu}$ satisfy the optimality conditions (25)-(26). We only need to prove that the optimality conditions (25)-(26) are still equivalent to the UE conditions under toll scheme $\mathbf{u} = -\boldsymbol{\lambda}$. It suffices to prove that the aggregate link flow of $\mathbf{f}$ is $\overline{\mathbf{v}}$, i.e. constraint (27) is binding at the optimum solution $\mathbf{f}$.

Suppose constraint (27) is not binding at the optimum solution $\mathbf{f}$, i.e. the aggregate link flow of $\mathbf{f}$ is $\mathbf{v}$ such that $\mathbf{v} \le \overline{\mathbf{v}}$ and $v_b < \overline{v}_b$ for some $b \in A$. Then by the monotonicity of the link cost function we have $t_b(v_b) < t_b(\overline{v}_b)$, $t_b(v_b)v_b^m < t_b(\overline{v}_b)v_b^m$ for $m \in M$, and $t_b(v_b)v_b < t_b(\overline{v}_b)\overline{v}_b$. Then we have

$$\sum_{a \in A} t_a(v_a)v_a < \sum_{a \in A} t_a(\overline{v}_a)\overline{v}_a \tag{29}$$

$$\sum_{a \in A}\sum_{m \in M} \beta_m t_a(v_a)v_a^m < \sum_{a \in A}\sum_{m \in M} \beta_m t_a(\overline{v}_a)v_a^m \tag{30}$$

Note that (29) is simply $T(\mathbf{f}) < T(\overline{\mathbf{f}})$. Also, because $\mathbf{f}$ solves the new primal LP with objective function (20) subject to constraint (27), and $\overline{\mathbf{f}}$ is within the feasible region of this new LP, it holds readily that the objective value of $\mathbf{f}$ is not larger than that of $\overline{\mathbf{f}}$, namely

$$\sum_{a \in A}\sum_{m \in M} \beta_m t_a(\overline{v}_a)v_a^m \le \sum_{a \in A}\sum_{m \in M} \beta_m t_a(\overline{v}_a)\overline{v}_a^m \tag{31}$$

Combining (30) and (31) yields $\sum_{a \in A}\sum_{m \in M} \beta_m t_a(v_a)v_a^m < \sum_{a \in A}\sum_{m \in M} \beta_m t_a(\overline{v}_a)\overline{v}_a^m$, which is $C(\mathbf{f}) < C(\overline{\mathbf{f}})$. Thus we have both $T(\mathbf{f}) < T(\overline{\mathbf{f}})$ and $C(\mathbf{f}) < C(\overline{\mathbf{f}})$, which contradicts that $\overline{\mathbf{f}}$ is Pareto optimal. This completes the proof. ♦

4. Bounding system performance deviation

In this section we shall explore the maximum extent to which the system performances vary when alternative Pareto system optimum is considered. Specifically, let $\mathbf{f}_P$ denote a Pareto optimal flow, and $\mathbf{f}_T$ and $\mathbf{f}_C$ be the time-based and the cost-based SO flows, respectively. Further, let $T_{\min}$ and $C_{\min}$ denote the minimum possible system time and system cost that are realized, respectively, at time-based and cost-based SO, i.e. $T_{\min} = T(\mathbf{f}_T)$ and $C_{\min} = C(\mathbf{f}_C)$. For a given Pareto optimal flow $\mathbf{f}_P$, we are interested in knowing how far the system disutilities $C(\mathbf{f}_P)$ and $T(\mathbf{f}_P)$ could be deviated from their minima, $C_{\min}$ and $T_{\min}$, respectively.

To measure the deviation, we define the following two ratios, called system performance deviation factors:

$$\alpha_T(\mathbf{f}) = \frac{T(\mathbf{f})}{T_{\min}} = \frac{T(\mathbf{f})}{T(\mathbf{f}_T)},\quad \alpha_C(\mathbf{f}) = \frac{C(\mathbf{f})}{C_{\min}} = \frac{C(\mathbf{f})}{C(\mathbf{f}_C)} \tag{32}$$

Clearly, $\alpha_T(\mathbf{f}) \geq 1$ and $\alpha_C(\mathbf{f}) \geq 1$ for any feasible $\mathbf{f}$. Particularly, let α_T^* and α_C^* denote the values of the two factors when $\mathbf{f} = \mathbf{f}_C$ and $\mathbf{f} = \mathbf{f}_T$, respectively:

$$\alpha_T^* = \frac{T(\mathbf{f}_C)}{T_{\min}} = \frac{T(\mathbf{f}_C)}{T(\mathbf{f}_T)},\quad \alpha_C^* = \frac{C(\mathbf{f}_T)}{C_{\min}} = \frac{C(\mathbf{f}_T)}{C(\mathbf{f}_C)} \tag{33}$$

α_T^* measures the deviation of the system time at a cost-based SO from its minimum achieved at a time-based SO; conversely, α_C^* measures the deviation of the system cost at a time-based SO from its minimum achieved at a cost-based SO. Furthermore, α_T^* and α_C^* are the upper bounds of $\alpha_T(\mathbf{f}_P)$ and $\alpha_C(\mathbf{f}_P)$ for any Pareto optimal flow $\mathbf{f}_P$, as given in the following lemma.

Lemma 4. For any Pareto optimal flow $\mathbf{f}_P$, we have $\alpha_T(\mathbf{f}_P) \leq \alpha_T^*$ and $\alpha_C(\mathbf{f}_P) \leq \alpha_C^*$.

Proof: By definitions (32)-(33), $\alpha_T(\mathbf{f}_P) \le \alpha_T^*$ is equivalent to $T(\mathbf{f}_P) \le T(\mathbf{f}_C)$, which holds for a simple reason: if $T(\mathbf{f}_P) > T(\mathbf{f}_C)$, then we have $T(\mathbf{f}_C) < T(\mathbf{f}_P)$ and $C(\mathbf{f}_C) \le C(\mathbf{f}_P)$, which contradicts that $\mathbf{f}_P$ is Pareto optimal. Similarly, $\alpha_C(\mathbf{f}_P) \le \alpha_C^*$ holds. ♦

For illustration of the two factors α_T^* and α_C^*, Figure 4 plots the Pareto optimal frontier in the rescaled (T, C) two-dimensional space, where the x-axis value is $\alpha_T = T/T_{\min}$ and the y-axis value is $\alpha_C = C/C_{\min}$. The origin of this rescaled (T, C) space is set to point $(1,1)$, which represents the *ideal case* that both T and C are minimized simultaneously. The ideal case can be realized when users are homogeneous. In this rescaled (T, C) space, each Pareto optimum $(T(\mathbf{f}_P), C(\mathbf{f}_P))$ is represented by the point $(\alpha_T(\mathbf{f}_P), \alpha_C(\mathbf{f}_P))$, and in particular, the two points $(1, \alpha_C^*)$ and $(\alpha_T^*, 1)$ represent the time-based and the cost-based SO, respectively. From Lemma 4, each Pareto optimum is upper bounded by point (α_T^*, α_C^*), thus the Pareto optimal frontier as a whole is upper bounded by point (α_T^*, α_C^*) and confined within a rectangular, as shown in Figure 4.

Now we move on to look into the bound properties of the two factors α_T^* and α_C^*. We first define the following terms and introduce a lemma.

$\beta_{\min} = \min\{\beta_m, m \in M\}$, the minimum VOT among all user classes;

$\overline{\beta}_w = \sum_{m\in M} \beta_m d_w^m / \sum_{m\in M} d_w^m$, the average VOT of all users between OD pair $w \in W$;

$\overline{\beta}_{\max} = \max\{\overline{\beta}_w, w \in W\}$, the maximum OD-average VOT among all OD pairs;

$\beta_{\max} = \max\{\beta_m, m \in M\}$, the minimum VOT among all user classes.

Evidently, $\beta_{\min} \le \overline{\beta}_{\max} \le \beta_{\max}$.

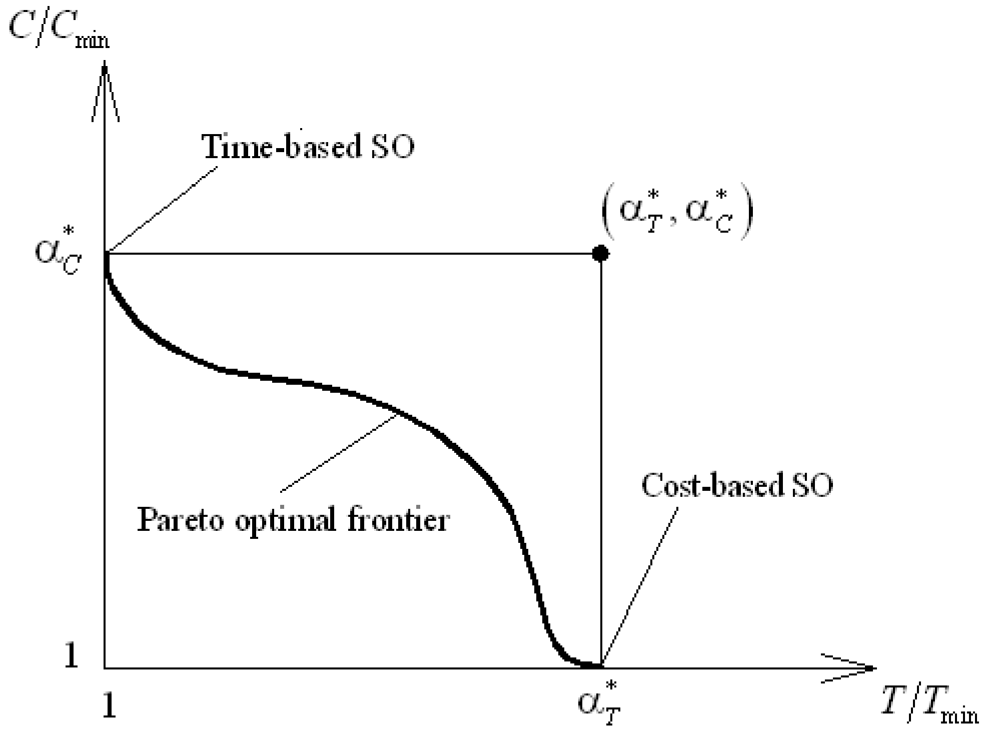

Figure 4. Pareto optimal frontier in rescaled (T, C) space

Lemma 5. Let $\mathbf{f}$ be a UE flow under an anonymous link toll scheme, then it holds

$$\beta_{\min} T(\mathbf{f}) \le C(\mathbf{f}) \le \overline{\beta}_{\max} T(\mathbf{f}) \tag{34}$$

Proof: Because $\beta_{\min} T(\mathbf{f}) \le C(\mathbf{f})$ holds trivially, here we only prove $C(\mathbf{f}) \le \overline{\beta}_{\max} T(\mathbf{f})$. Let $\mathbf{f}$ be a UE flow under anonymous link toll scheme $\mathbf{u}$, then the UE conditions hold, and it follows from the cost-based UE conditions (12)-(13) that

$$\mu_w^{m,c} - \sum_{a \in A} u_a \delta_{ar} \le \beta_m \sum_{a \in A} t_a(v_a) \delta_{ar}, \quad r \in R_w, \quad w \in W, \quad m \in M \tag{35}$$

Multiplying both sides of (35) by equilibrium path flow $f_{rw} = \sum_{m \in M} f_{rw}^m$ aggregated over all user classes and summing over all $r \in R_w$, we have

$$\mu_w^{m,c} \sum_{r \in R_w} f_{rw} - \sum_{a \in A} u_a \sum_{r \in R_w} f_{rw} \delta_{ar} \le \beta_m \sum_{a \in A} t_a(v_a) \sum_{r \in R_w} f_{rw} \delta_{ar}, \quad w \in W, \quad m \in M \tag{36}$$

Denote $d_w = \sum_{r \in R_w} f_{rw} = \sum_{m \in M} d_w^m$ and $v_{a,w} = \sum_{r \in R_w} f_{rw} \delta_{ar}$, then (36) becomes

$$\mu_w^{m,c} d_w - \sum_{a \in A} u_a v_{a,w} \le \beta_m \sum_{a \in A} t_a \left(v_a\right) v_{a,w}, \quad w \in W, \quad m \in M \tag{37}$$

Multiplying both sides of (37) by d_w^m / d_w and summing over all $m \in M$, we have

$$\sum_{m \in M} \mu_w^{m,c} d_w^m - \sum_{a \in A} u_a v_{a,w} \le \overline{\beta}_w \sum_{a \in A} t_a \left(v_a\right) v_{a,w}, \quad w \in W \tag{38}$$

Since $\overline{\beta}_{\max} = \max \left\{ \overline{\beta}_w, w \in W \right\}$, (38) can be relaxed to

$$\sum_{m \in M} \mu_w^{m,c} d_w^m - \sum_{a \in A} u_a v_{a,w} \le \overline{\beta}_{\max} \sum_{a \in A} t_a \left(v_a\right) v_{a,w}, \quad w \in W \tag{39}$$

Taking summation of (39) over all $w \in W$, we obtain

$$\sum_{w \in W} \sum_{m \in M} \mu_w^{m,c} d_w^m - \sum_{a \in A} u_a v_a \le \overline{\beta}_{\max} \sum_{a \in A} t_a \left(v_a\right) v_a$$

which is simply

$$\sum_{w \in W} \sum_{m \in M} \mu_w^{m,c} d_w^m - \Pi \le \overline{\beta}_{\max} T\left(\mathbf{f}\right) \tag{40}$$

The left-hand side of (40) is equal to $C\left(\mathbf{f}\right)$ as mentioned in (15), thus we have $C\left(\mathbf{f}\right) \le \overline{\beta}_{\max} T\left(\mathbf{f}\right)$. This completes the proof. ♦

The following theorem gives an upper bound of the product of the two factors α_T^* and α_C^*.

Theorem 3. *It holds that*

$$\alpha_T^* \alpha_C^* \le \frac{\overline{\beta}_{\max}}{\beta_{\min}} \le \frac{\beta_{\max}}{\beta_{\min}} \tag{41}$$

Proof: The second inequality follows from the definitions of $\overline{\beta}_{\max}$ and $\beta_{\max}$. By definition (33), we have

$$\alpha_T^* \alpha_C^* = \frac{T\left(\mathbf{f}_C\right)}{T\left(\mathbf{f}_T\right)} \frac{C\left(\mathbf{f}_T\right)}{C\left(\mathbf{f}_C\right)} = \left(\frac{T\left(\mathbf{f}_C\right)}{C\left(\mathbf{f}_C\right)} \right) \left(\frac{C\left(\mathbf{f}_T\right)}{T\left(\mathbf{f}_T\right)} \right) \tag{42}$$

Lemma 5 applies to $\mathbf{f}_T$ and $\mathbf{f}_C$, respectively, because there exist anonymous link toll schemes that can support the time-based and the cost-based SO as UE. Then, by applying Lemma 5 to $\mathbf{f}_C$, we have $\beta_{\min} T(\mathbf{f}_C) \le C(\mathbf{f}_C)$, which gives

$$\frac{T(\mathbf{f}_C)}{C(\mathbf{f}_C)} \le \frac{1}{\beta_{\min}} \tag{43}$$

Apply Lemma 5 to $\mathbf{f}_T$, we have $C(\mathbf{f}_T) \le \overline{\beta}_{\max} T(\mathbf{f}_T)$, which gives

$$\frac{C(\mathbf{f}_T)}{T(\mathbf{f}_T)} \le \overline{\beta}_{\max} \tag{44}$$

Substituting (43) and (44) into (42) gives (41). ♦

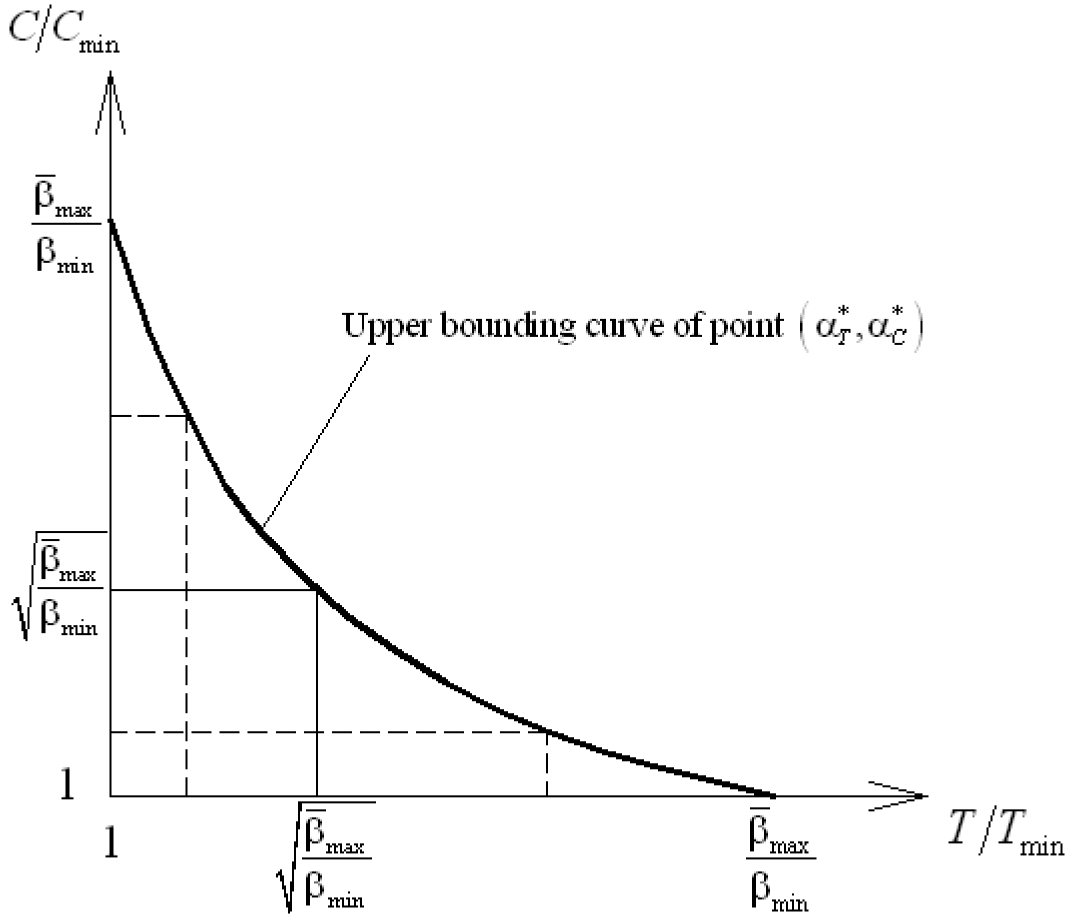

Figure 5. Graphical representation of Theorem 3 in rescaled (T, C) space

In view of $\alpha_T^* \geq 1$ and $\alpha_C^* \geq 1$, Theorem 3 gives rise to $\alpha_T^* \leq \bar{\beta}_{\max}/\beta_{\min}$, $\alpha_C^* \leq \bar{\beta}_{\max}/\beta_{\min}$ and $\min\left\{\alpha_T^*, \alpha_C^*\right\} \leq \sqrt{\bar{\beta}_{\max}/\beta_{\min}}$. Thus $\bar{\beta}_{\max}/\beta_{\min}$ or $\sqrt{\bar{\beta}_{\max}/\beta_{\min}}$ can be regarded as an upper bound on the system performance discrepancy when optimization is based on different (time vs cost) criteria. Figure 5 gives a graphical representation of Theorem 3 in the rescaled (T, C) two-dimensional space. The bold line ($\alpha_T \alpha_C = \bar{\beta}_{\max}/\beta_{\min}$) together with the two axes constitutes a bounding region of point $\left(\alpha_T^*, \alpha_C^*\right)$, which in turn gives a bounding rectangular of the Pareto optimal frontier (see Figure 4).

The bound information on the two factors α_T^* and α_C^* given in Theorem 3 is independent of network topology and link travel time functions, but solely dependent upon VOT distribution, or more specifically, on $\bar{\beta}_{\max}/\beta_{\min}$, the ratio of the highest OD-average VOT among all OD pairs to the lowest VOT among all user classes. Here two remarks are ready. We can use the above tighter upper bounds if the information of the average VOT of users between each OD pair is available, otherwise we can replace $\bar{\beta}_{\max}$ with $\beta_{\max}$. Second, when users are homogeneous, we readily have $\beta_{\max} = \bar{\beta}_{\max} = \beta_{\min}$, then Theorem 3 gives the expected result of $\alpha_T^* = \alpha_C^* = 1$, which means that the time-based and the cost-based system optimizations coincide with each other (independent of the unit for system performance measure).

Here, we give a numerical example to show the Pareto optimal frontier and demonstrate the difference between the time-based and the cost-based SO cases. Consider a simple network consisting of 4 nodes and 5 links, as illustrated in Figure 6. The link travel time functions are given in the figure: link $3 \rightarrow 4$ has an increasing travel time function, and the other 4 links have constant link travel times for simplicity. There are two OD pairs:

OD pair $1 \rightarrow 4$, demand $d_1 = 2.6$ with VOT $\beta_1 = 1$;

OD pair $2 \to 4$, demand $d_2 = 2.0$ with VOT $\beta_2 = 4$.

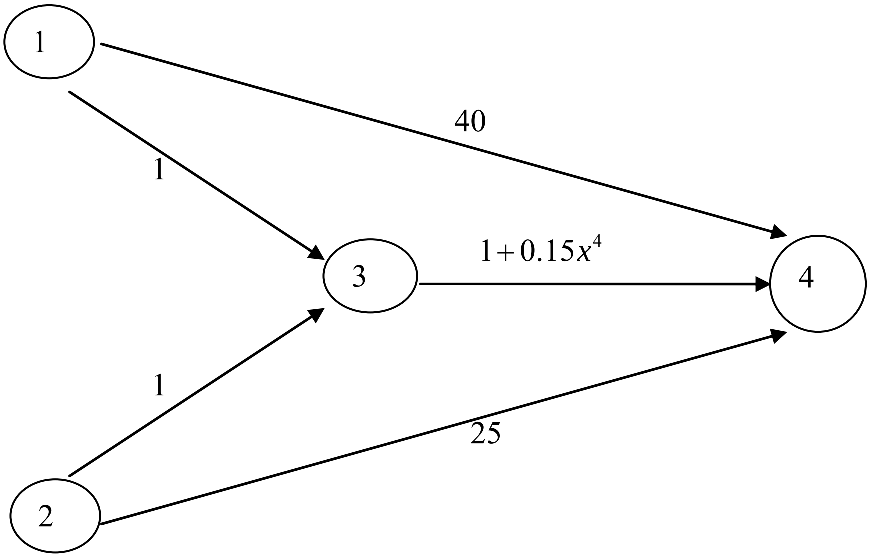

Figure 6. Network of numerical example

Let f_1 and f_2 be the flows on link $1 \to 3$ and link $2 \to 3$, respectively. Then, due to simplicity of the network, vector $\mathbf{f} = (f_1, f_2)$ effectively represents a class-specific traffic assignment, and the feasible flow set is $\Omega = \{\mathbf{f} : 0 \le f_1 \le d_1, 0 \le f_2 \le d_2\}$. Solving the time-based SO problem gives

$$\mathbf{f}_T = (d_1, 0) = (2.6, 0), \text{ with } T_{\min} = 73.0 \text{ and } C(\mathbf{f}_T) = 223.0$$

Solving the cost-based SO problem gives

$$\mathbf{f}_C = (0, d_2) = (0, 2), \text{ with } C_{\min} = 139.2 \text{ and } T(\mathbf{f}_C) = 112.8$$

Thus we have

$$\alpha_T^* = \frac{112.8}{73.0} = 1.55 \text{ and } \alpha_C^* = \frac{223.0}{139.2} = 1.60$$

Furthermore, solving the ε-constraint problem (18)-(19) with the ε_T value increasing from $T_{\min} = 73.0$ to $T(\mathbf{f}_C) = 112.8$, we can solve for every Pareto optimum and thereby obtain the Pareto optimal frontier. Figure 7 plots the Pareto optimal frontier in the rescaled (T, C)

two-dimensional space. The point $\left(\alpha_T^*, \alpha_C^*\right) = (1.55, 1.60)$ and its theoretical bounding curve ($\alpha_T \alpha_C = \beta_2/\beta_1 = 4.0$) based on Theorem 3 are also displayed in Figure 7 for comparison.

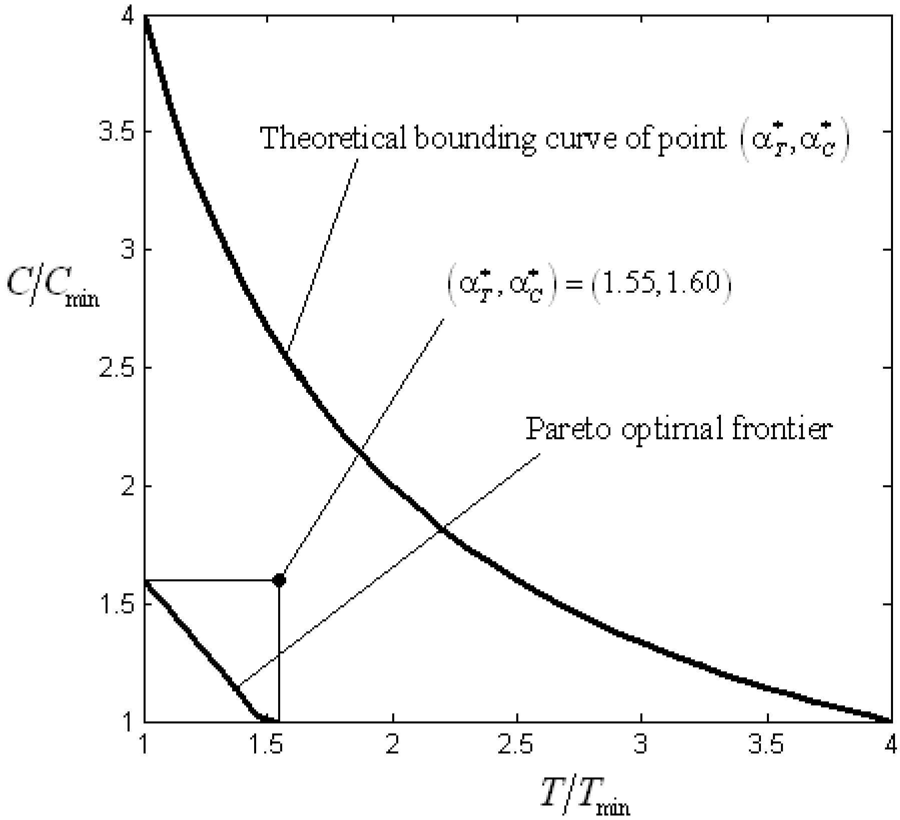

Figure 7. Numerical example results in rescaled (T, C) space

5. Conclusions

For a traffic network with heterogeneous users and fixed demand, system time and system cost are two meaningful measures of system performance. Thus, there are two objectives of network optimization, i.e. to minimize system time and to minimize system cost. This paper considered these two objectives simultaneously by introducing a bi-objective minimization problem and exploring bi-criteria Pareto optimum. We proved that a bi-criteria Pareto optimum can always be supported as a multi-class UE by positive anonymous link tolls. We also observe that: when dealing with pricing optimization with heterogeneous users, it suffices to concentrate on the aggregate link flows rather than the class-specific path or link flows, because the latter will be automatically optimized by the anonymous link toll schemes.

In an attempt to understand how system cost and system time vary with alternative Pareto system optimum including the time-based and the cost-based SO cases, we introduced two system performance deviation factors and established a meaningful upper bound of their product. The upper bound is solely determined by users' VOT distribution, i.e. the ratio of the highest to the lowest VOT among all user classes. It should be mentioned that these results are independent of network topology and travel time functions.

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