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A Link-Based Day-to-Day Traffic Assignment Model

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Abstract: Existing day-to-day traffic assignment models are all built upon path flow variables. This paper demonstrates two essential shortcomings of these path-based models. One is that their application requires a given initial path flow pattern, which is typically unidentifiable, i.e. mathematically nonunique and practically unobservable. In particular, we show that, for the path-based models, different initial path flow patterns constituting the same link flow pattern generally gives different day-to-day link flow evolutions. The other shortcoming of the path-based models is the path overlapping problem. That is, the path-based models ignore the interdependence among paths and thus can give very unreasonable results for networks with paths overlapping with each other. These two path-based problems exist for most (if not all) deterministic day-to-day dynamics whose fixed points are the classic Wardrop user equilibrium. To avoid the two path-based problems, we propose a day-to-day traffic assignment model that directly deals with link flow variables. Our link-based model captures travelers' cost-minimization behavior in their path finding as well as their inertia. The fixed point of our link-based dynamical system is the classic Wardrop user equilibrium.

Key Words: Day-to-day traffic assignment model, link-based, Wardrop user equilibrium.

1. Introduction

Day-to-day (or inter-periodic) traffic modeling methods are believed to be most appropriate for analyzing traffic equilibration processes. With increasing applications of Intelligent Transport Systems (including traveler information and control systems), these day-to-day models aim to capture day-to-day traffic fluctuations and focus more on the evolution process itself, rather than the final (static) equilibrium state, which is the center point of traditional (deterministic and stochastic) static traffic assignment models. As mentioned by Watling and Hazelton (2003), the most appealing feature to researchers and practitioners is the great flexibility of day-to-day approaches, which allows a wide range of behavior rules, levels of aggregation, and traffic modes to be synthesized into a uniform framework. This equilibration paradigm helps transportation planning and management in modeling evolution of traffic states and the trajectories of evolution.

Day-to-day traffic dynamics can be established on continuous temporal spaces when time steps are “sufficiently” small. Existing continuous time day-to-day dynamics employ differential equations to describe traffic evolution. In this category, Smith (1984), Friesz et al. (1994), and Zhang and Nagurney (1996) proposed three dynamical systems. These three systems adopted the assumption of perfect perception of travel cost and developed deterministic traffic assignment processes over a continuous temporal dimension. Specifically, Smith (1984) assumed that travelers on higher travel cost routes will proportionally switch to those routes with lower travel costs. Friesz et al. (1994) proposed a day-to-day model that captures both the dynamics of route flows and origin-destination demands. Zhang and Nagurney (1996) modeled a projected dynamical system, which adjusts day-to-day route flows with a minimum norm projection operator, rather than the proportional switching in Smith’s work.

Although continuous time approaches have good mathematical properties in traffic evolution, Watling and Hazelton (2003) summarized two major limitations suffered by continuous day-to-day approaches: (1) continuous-time trip adjustment is not plausible in reality; (2) homogeneous population assumptions in these approaches require additional dispersion modules. Therefore, discrete versions of day-to-day traffic equilibration models are more suited to day-to-day fluctuation. In discrete time day-to-day traffic dynamical systems, travelers’ route choice behavior is assumed to be repeated daily, in accordance with daily changes in traffic flows. Specifically, Friesz et al. (1994) employed a projection-type discretization algorithm, given by Bertsekas and Gafni (1982), to approximate the continuous traffic trajectories in the dynamical system developed therein. Nagurney and Zhang (1997) specified their continuous model in a discrete temporal space with fixed demand and applied Euler’s method to solve the projected dynamical system.

Due to the random nature of transportation systems, stochasticity has been introduced into day-to-day approaches. Most existing stochastic day-to-day assignment models follow Markov processes; examples include models proposed in Cascetta (1989) and Hazelton and Watling (2004), where the computation of transition matrices depends only on the previous traffic state. The transition probability matrix could be specified and leads to approximations of system mean dynamics, as shown by Daganzo and Sheffi

(1977), Davis and Nihan (1993) and Yang and Liu (2007). Other stochastic approaches, e.g., Horowitz (1984), Canterella and Cascetta (1995), Watling (1999), adopted the assumption of memory length, assuming that route choice probabilities depend on weighted averages of experienced travel times. To solve these models, Davis and Nihan (1993) provided a particular Gaussian multivariate autoregressive process and Hazelton et al. (1996) proposed a Markov Chain Monte Carlo method.

Existing day-to-day traffic models are all built upon path-based variables, many of which have two essential shortcomings, as to be demonstrated in this paper. One shortcoming is that their application requires a given initial path flow pattern, which is typically unidentifiable, i.e. mathematically nonunique and practically unobservable (exceptions include some logit assignment based models and the stochastic models which focus on the probability distribution of flow states and/or the expected flow state rather than the day-to-day flow evolution trajectory, as to be discussed later). In particular, we show that, for the path-based models, different initial path flow patterns constituting the same link flow pattern generally gives different day-to-day link flow evolutions. Thus the difficulty of identifying the initial path flow pattern is indeed a problem for the path-based models. The other shortcoming of the path-based models is the path overlapping problem. That is, the path-based models ignore the interdependence among paths and thus can give very unreasonable results for networks with paths overlapping with each other. To avoid the two path-based problems, we propose a day-to-day traffic assignment model that directly deals with link flow variables. Our link-based model captures travelers' cost-minimization behavior as well as their inertia. The fixed point of our link-based dynamical system is the classic user equilibrium (UE) flow.

The remainder of the paper is organized as follows. Section 2 gives the preliminaries of day-to-day traffic assignment models. In Section 3, we demonstrate two shortcomings of many existing path-based day-to-day dynamics and provide detailed discussions. A link-based day-to-day model is proposed in Section 4, which effectively avoids the two path-based problems. Section 5 applies the proposed link-based model to a small test network and provides discussions on the model parameters based on the numerical results. Concluding remarks are given in Section 6.

2. Preliminaries of day-to-day traffic dynamics

Let a transportation network be a fully-connected directed graph denoted as $G(N,L)$, consisting of a set of nodes N and a set of links L. Let W be the set of OD pairs, d_w be the fixed travel demand between OD pair $w \in W$, P_w be the set of paths connecting OD pair $w \in W$, f_{pw}^t be the path flow on path $p \in P_w$ at day t, x_a^t be the link flow on link $a \in L$ at day t. Denote demand, path flow and link flow vectors as $\mathbf{d}$, $\mathbf{f}^t$, and $\mathbf{x}^t$, respectively. Let $\mathbf{A}$ be the link-path incidence matrix, then $\mathbf{x}^t = \mathbf{A}\mathbf{f}^t$. Let $\mathbf{\Phi}$ be the OD-path incidence matrix, then $\mathbf{d} = \mathbf{\Phi}\mathbf{f}^t$. Let $c_a(\mathbf{x})$ be the link cost function of link $a \in L$, then $c_a(\mathbf{x}^t)$ is the link cost of link $a \in L$ at day t, and we denote $\mathbf{c}(\mathbf{x}^t)$ as the corresponding link cost vector. Let $\mathbf{F}^t$ denote the path cost vector at day t, with individual path cost F_{pw}^t, then it holds $\mathbf{F}^t = \mathbf{A}'\mathbf{c}(\mathbf{x}^t) = \mathbf{A}'\mathbf{c}(\mathbf{A}\mathbf{f}^t)$, where $\mathbf{A}'$ is the transpose of $\mathbf{A}$.

The above notations are sufficient for describing discrete-time day-to-day traffic dynamics. For continuous-time versions, we denote the day-to-day path flow dynamic as $\dot{\mathbf{f}}$, which is the derivative of path flow with respect to time, and denote the day-to-day link flow dynamic as $\dot{\mathbf{x}}$. It holds readily $\dot{\mathbf{x}} = \mathbf{A}\dot{\mathbf{f}}$.

As summarized by Yang and Zhang (2009), there are five major categories of deterministic day-to-day dynamical systems. They are the simplex gravity flow dynamics (Smith, 1983), the proportional-switch adjustment process (e.g., Smith, 1984; Smith and Wisten, 1995; Huang and Lam, 2002; Peeta and Yang, 2003), the network tatonnement process (e.g., Friesz et al., 1994), the projected dynamical system (e.g., Zhang and Nagurney, 1996; Nagurney and Zhang, 1997), and the evolutionary traffic dynamics (e.g., Sandholm, 2001; Yang, 2005). All these day-to-day dynamics are path-based models, i.e. they all explicitly use path flow variables and give explicit path flow evolution trajectories. A common property of these models is that the stationary point or fixed point is the classic UE flow pattern. There are also some deterministic day-to-day dynamics that do not have UE as the fixed point, e.g. the logit assignment based model of Watling (1999) has the logit-based stochastic user equilibrium (SUE) as its fixed point. Besides, there are stochastic day-to-day traffic assignment models which focus on the probability distribution of flow states and/or the expected flow state (e.g. Cascetta 1989) rather than the flow evolution trajectory. Our discussions in next section on the shortcomings of existing day-to-day traffic models are mainly related to the deterministic models with UE as the fixed point, i.e., those summarized by Yang and Zhang (2009). Nevertheless, comments are provided regarding other models wherever necessary.

Here, as an illustration and also for later reference, we give the formulation of the proportional-switch adjustment process (PAP) according to Smith (1984). The continuous-time version of PAP is

$$\dot{f}_{pw} = \sum_{q \in P_w} \left(f_{qw} \left[F_{qw} - F_{pw} \right]_+ - f_{pw} \left[F_{pw} - F_{qw} \right]_+ \right) \tag{1}$$

where $\left[\cdot\right]_+$ is a projection operator defined as $\left[x\right]_+ = \max\left\{x, 0\right\}$.

The discrete-time version of PAP is

$$f_{pw}^{t} - f_{pw}^{t-1} = \frac{1}{T_w} \sum_{q \in P_w} \left(f_{qw}^{t-1} \left[F_{qw}^{t-1} - F_{pw}^{t-1} \right]_+ - f_{pw}^{t-1} \left[F_{pw}^{t-1} - F_{qw}^{t-1} \right]_+ \right) \tag{2}$$

where

$$T_w = \sum_{p \in P_w} \sum_{q \in P_w} \left[F_{p,t-1}^{w} - F_{q,t-1}^{w} \right]_+ + M$$

T_w can be regarded as a step size to discretize the continuous-time version, and $M > 0$ is a reluctance parameter, i.e. more travelers prefer maintaining previous choices when a larger M appears.

3. Two shortcomings of the path-based day-to-day traffic dynamics

In this section we demonstrate two essential shortcomings of the path-based day-to-day traffic dynamics. The two shortcomings become most manifest when the models are applied (rather than just theoretically discussed), thus we shall apply a path-based dynamic to a simple yet illustrative network. We adopt a discrete-time approach in this section because it suits real application better than the continuous-time approach does.

Consider a simple network shown in Figure 1, consisting of 4 nodes and 5 links with shown node and link numbers. There is one OD pair from Node O to Node D connected by four paths numbered as below:

Path 1, link sequence $1 \to 3 \to 4$,
Path 2, link sequence $1 \to 3 \to 5$,
Path 3, link sequence $2 \to 3 \to 4$,
Path 4, link sequence $2 \to 3 \to 5$.

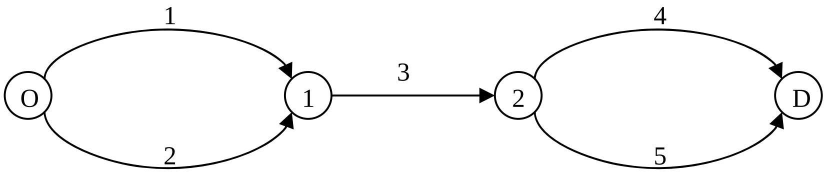

Figure 1. A network to demonstrate the shortcomings of path-based dynamics

Consider that the network flow in Figure 1 is initially at UE and a capacity reduction on Link 4 takes place at day 0. Let $\tilde{\mathbf{x}}$ be the initial UE link flow pattern, and consider that $\tilde{\mathbf{x}} > 0$, i.e. each link initially has some flow. Let $\tilde{F}$ be the initial UE travel cost, i.e. all 4 paths have the same travel cost $\tilde{F}$ before the capacity reduction takes place. When the capacity reduction happens at day 0, the link flow pattern is still $\tilde{\mathbf{x}}$, and thus it is clear that the path costs of Path 2 and Path 4 do not change (Link 4 not included in these paths), while the path costs of Path 1 and Path 3 increase to the same level (due to the same cost increase of Link 4) denoted as F^0 such that $F^0 > \tilde{F}$.

3.1 The path-flow-nonuniqueness problem

Now we have an observation of the network condition at day 0: link flow pattern $\mathbf{x}^0 = \tilde{\mathbf{x}}$, which gives link cost vector $\mathbf{c}(\tilde{\mathbf{x}})$ and path cost vector $\mathbf{F}^0 = \left(F^0, \tilde{F}, F^0, \tilde{F}\right)'$. Suppose we have at hand a well-established day-to-day traffic assignment model, then we should be able to predict the day-to-day traffic equilibration process after day 0. However, to apply any path-based model, we need a given initial path flow $\mathbf{f}^0$. Without further assumptions, all that we know about $\mathbf{f}^0$ is that

$$\mathbf{A}\mathbf{f}^0 = \mathbf{x}^0 \tag{3}$$

and it is clear that $\mathbf{f}^0$ satisfying (3) is not unique for the example here.

It should be mentioned that the path flow pattern corresponding to a given link flow pattern is generally nonunique and unobservable. In particular, the nonuniqueness of the UE path flow pattern is well known and is typically not considered as a problem of the UE solution because the UE link flow and thus the UE system performance are unique (under mild technical conditions). In the same spirit, neither should the nonuniqueness of $\mathbf{f}^0$ be regarded as a problem of the path-based day-to-day dynamics if different $\mathbf{f}^0$ (constituting the same $\mathbf{x}^0$) can give the same day-to-day evolution of the link flow pattern. Unfortunately, it generally does not hold such an ideal property, and thus the nonuniqueness of $\mathbf{f}^0$ is indeed a problem of the path-based day-to-day dynamics. We shall demonstrate this by our small example as well as some general derivations.

To show that different $\mathbf{f}^0$ will give different link flow evolution, it suffices to calculate the link flow pattern at day 1, i.e. to show that $\mathbf{x}^1$ takes different values for different $\mathbf{f}^0$. Applying the discrete-time PAP dynamic (2) to our small example, we obtain $\mathbf{f}^1$ as follows

$$\mathbf{f}^1 = \begin{pmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ f_4^1 \end{pmatrix} = \begin{pmatrix} (1-\rho) f_1^0 \\ f_2^0 + 0.5\rho x_4^0 \\ (1-\rho) f_3^0 \\ f_4^0 + 0.5\rho x_4^0 \end{pmatrix}$$

where ρ is a coefficient given by

$$\rho = \frac{2\left(F^0 - \tilde{F}\right)}{4\left(F^0 - \tilde{F}\right) + M}$$

Then, from $\mathbf{x}^1 = \mathbf{A}\mathbf{f}^1$ we have

$$\mathbf{x}^1 = \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_4^1 \\ x_5^1 \end{pmatrix} = \begin{pmatrix} x_1^0 + 0.5\rho x_4^0 - \rho f_1^0 \\ x_2^0 + 0.5\rho x_4^0 - \rho f_3^0 \\ x_3^0 \\ (1-\rho) x_4^0 \\ x_5^0 + \rho x_4^0 \end{pmatrix} \tag{4}$$

It can be seen clearly from (4) that different $\mathbf{f}^0$ will give different $\mathbf{x}^1$, which means that the link flow day-to-day evolution depends on the (nonunique and unobservable) initial path flow pattern. This problem actually exists generally for many path-based day-to-day dynamics, not just for the PAP dynamic applied to our small example. To see this, we look into the general case.

For a general network, assuming an observed initial link flow pattern $\mathbf{x}^0$, then the initial path flow pattern $\mathbf{f}^0$ solves the linear equation system (3). Let n_P be the number of paths, when $n_p > rank(\mathbf{A})$, the $\mathbf{f}^0$ solution has a degree of freedom $DF = n_p - rank(\mathbf{A})$ and can be written in form of

$$\mathbf{f}^0 = \mathbf{Z}\boldsymbol{\theta} + \mathbf{f}^* \tag{5}$$

where $\mathbf{Z}$ is a constant $\left(n_p \times DF\right)$ matrix with full column rank satisfying $\mathbf{AZ}=0$, $\mathbf{f}^*$ is a constant vector, and $\boldsymbol{\theta}$ is an *indeterminate* $\left(DF \times 1\right)$ vector. It is clear that different values of $\boldsymbol{\theta}$ represent different $\mathbf{f}^0$. Observe that $\mathbf{AZ}=0$ guarantees that $\mathbf{x}^0=\mathbf{Af}^0$ does not depend on $\boldsymbol{\theta}$, i.e. different $\mathbf{f}^0$ constitute the same $\mathbf{x}^0$.

Consider a path-based day-to-day dynamic such that the path flow at day 1 is given by

$$\mathbf{f}^1=\mathbf{Uf}^0=\mathbf{UZ\theta}+\mathbf{Uf}^* \tag{6}$$

where $\mathbf{U}$ is the path flow updating matrix determined by the specific day-to-day dynamic. Then the link flow at day 1 is given by

$$\mathbf{x}^1=\mathbf{Af}^1=\mathbf{AUZ\theta}+\mathbf{AUf}^* \tag{7}$$

Note that it generally holds $\mathbf{AUZ}\neq 0$, because $\mathbf{U}$ is essentially given by the day-to-day dynamic, typically related to cost and flow, not determined by the link-path incidence matrix $\mathbf{A}$. Then it can be seem from (7) that $\mathbf{x}^1$ depends on $\boldsymbol{\theta}$, which simply means that $\mathbf{x}^1$ depends on $\mathbf{f}^0$. Thus it is demonstrated that different initial path flow patterns generally give different link flow evolutions.

To sum up, we formally give the following observation.

Observation 1. *For a path-based day-to-day dynamic, different initial path flow patterns constituting the same initial link flow pattern generally lead to different link flow day-to-day evolutions*.

From Observation 1, if a path-based day-to-day dynamic is to be applied, it is important to first identify the initial path flow pattern, which, however, is typically nonunique and unobservable. This shortcoming of many path-based models is referred to as the *path-flow-nonuniqueness problem* in this paper.

Here several comments are ready for the path-flow-nonuniqueness problem. First, a possible solution to this problem is to use some kind of estimation method to obtain an estimation of the initial path flow pattern. For example, the most likely path flow estimation method (e.g. Larsson et al. 1998; Bar-Gera 2006) could be adopted. In this paper we limit our attention on this direction because there is another problem associated with the path-based day-to-day dynamics which could not be solved by estimation methodologies, as we will discuss in the next section. Second, because the path-flow-nonuniqueness problem is actually caused by the difficulty of identifying the initial path flow pattern, arguably it is not a theoretical shortcoming of the path-based models. That is to say, if the initial path flow pattern can be somehow identified, then the path-based models can be simply applied. Indeed, it is theoretically possible to identify the path flow pattern of a network if all vehicle paths can be traced practically. In this sense, the path-flow-nonuniqueness problem could be viewed as a technology or cost problem rather than a modeling problem.

Last but not least, although existing day-to-day traffic assignment models are all built upon path flow variables, not all of them have this path-flow-nonuniqueness problem. For example, the models that essentially conduct a logit SUE assignment for each day (e.g. Watling, 1999) do not have this problem, because under the SUE setting, path flow pattern is uniquely determined once path costs are given. Also, for the stochastic models that focus on the probability distribution of flow states and/or the expected flow state (e.g. Cascetta 1989), the specific flow evolution trajectory is not a concern (as the trajectory is essentially a realization of a stochastic process). Such models can initialize their simulation processes by drawing a random flow pattern from the stationary distribution, and thus the nonuniqueness of path flow is irrelevant. In general, the path-flow-nonuniqueness problem exists for most (if not all) existing deterministic day-to-day dynamics whose fixed points are the classic Wardrop UE, i.e., those summarized by Yang and Zhang (2009).

3.2 The path-overlapping problem

While the path-flow-nonuniqueness problem is in some sense not a modeling problem but only a cost or technology problem for the path-based day-to-day dynamics, there does exist another problem which is rooted inherently in the path-based methodology. To see this problem, let us revisit the small example we have presented. By examining the network shown in Figure 1, we can see that the network is "separable": the subnetwork from Node O to Node 1 and the subnetwork from Node 2 to Node D are totally independent of each other. As a result, a capacity reduction on Link 4, as we have studied in the example, should not affect the flow split between Link 1 and Link 2. More rigorously, it could be stated as below:

For the network shown in Figure 1, assuming a fixed travel demand and separable link cost functions (i.e. no spillback effect), and consider that the network flow is originally at stable equilibrium, then, if a capacity reduction on Link 4 takes place, the flow split between Link 1 and Link 2 should remain stable and unchanged.

The above statement is a reasonable and logical "expectation" about the network shown in Figure 1, and a model that violates this expectation is at least not amenable to this small network. Unfortunately, many existing path-based day-to-day dynamics violate this expectation, i.e. they unreasonably predict a flow fluctuation between Link 1 and Link 2 consequential to a capacity reduction on Link 4. As an illustration, equation (4) shows that the PAP dynamic generally gives $x_1^1 \neq x_1^0$ and $x_2^1 \neq x_2^0$, i.e. there is a change of the flow split between Link 1 and Link 2 when a capacity reduction on Link 4 happens. One may argue that if we set a particular value for $\mathbf{f}^0$, i.e. let $f_1^0 = f_3^0 = 0.5x_4^0$, then we have $x_1^1 = x_1^0$ and $x_2^1 = x_2^0$. However, $f_1^0 = f_3^0 = 0.5x_4^0$ could easily be infeasible, not to mention how arbitrary it is to set such a value. For example, if $x_1^0 = 1$ and $x_4^0 = 3$, then $f_1^0 = f_3^0 = 0.5x_4^0$ would give $f_1^0 = 1.5 > x_1^0$, which is obviously infeasible.

The path-based day-to-day dynamics do not apply to the small network shown in Figure 1 because these models do not consider path interdependence. That is, users of Path 1 are modeled to be indifferent to Path 2 and Path 4 when they consider route switching, because Path 2 and Path 4 have equal path costs. In reality, however, users of Path 1 probably prefer Path 2 to Path 4 because Path 2 overlaps more with their current path. Such a path overlapping effect is not taken into consideration by the existing path-based models, in which path cost is considered as the only driving factor of day-to-day traffic evolution. As shown by our small example, a path-based day-to-day dynamic ignoring the path-overlapping effect could give very unreasonable predictions on day-to-day traffic equilibration. This shortcoming of the path-based models is referred to as the *path-overlapping problem* in this paper.

It should be mentioned that our small example here is just a convenient demonstration of the path-overlapping problem, while the problem itself could be much more general. That is, even if a particular path-based day-to-day traffic model does not violate the intuition of our small network, it may still have the path-overlapping problem. In general, as long as a model (implicitly) assumes that users are indifferent to two paths with the same cost, then the path-overlapping problem is likely to exist, especially for deterministic models which provide explicit flow evolution trajectories. For the stochastic models dealing with the probability distribution of flow states and/or the expected flow state rather than the flow evolution trajectory, the path-overlapping effect may also exist, and may be related to the assumptions the models make regarding the randomness of the system. To keep this paper more focused, we limit our attention to the deterministic models.

The path-overlapping problem is actually a common problem for all path-based models dealing with travelers' route choice behaviors. One well known example is the logit-based SUE model in the context of static traffic assignment. In the literature many studies (e.g. Cascetta et al., 1996; Vovsha and Bekhor, 1998; Ben-Akiva and Bierlaire 1999; Frejinger and Bierlaire, 2007) have been devoted to overcoming the path-overlapping problem of the logit SUE model, typically by introducing some measure of path overlap or capturing the nested hierarchy of networks. Here, for the path-based day-to-day dynamics, it is possible to solve (or alleviate) the path-overlapping problem using similar methods. In this paper we do not take this effort because we have two problems with the path-based day-to-day dynamics, the path-flow-nonuniqueness problem and the path-overlapping problem. Although we have suggested possible solutions to each of the two problems, we conjecture that to handle the two problems simultaneously within the path-based methodology would be difficult and might end up with a rather complicated model. Therefore, we shall resort to a different approach, the link-based methodology.

4. A link-based day-to-day traffic model

In this section we propose a day-to-day traffic assignment model that directly deals with link flow variables. Such a link-based model effectively avoids the two path-based problems mentioned in last section.

The general form of our link-based day-to-day dynamic in continuous time is

$$\dot{\mathbf{x}} = \delta\left(\mathbf{y} - \mathbf{x}\right) \tag{8}$$

where δ is a positive constant parameter determining the flow changing rate, and $\left(\mathbf{y} - \mathbf{x}\right)$ provides a flow changing direction. In other words, dynamic (8) means that, at any day, the (link) flow pattern tends to move from the current flow pattern $\mathbf{x}$ towards a "target" flow pattern $\mathbf{y}$ based on the current day situation. Thus the model is essentially determined by how the "target" flow pattern $\mathbf{y}$ is defined. In this paper we let $\mathbf{y}$ solve the following problem given current link flow $\mathbf{x}$:

$$\min_{\mathbf{y} \in \Omega} \lambda \mathbf{c}\left(\mathbf{x}\right)' \mathbf{y} + \left(1 - \lambda\right) D\left(\mathbf{x}, \mathbf{y}\right) \tag{9}$$

where $\Omega = \left\{\mathbf{x} : \mathbf{x} = \mathbf{A}\mathbf{f}, \mathbf{d} = \mathbf{\Phi}\mathbf{f}, \mathbf{f} \geq 0\right\}$ is the feasible link flow set, λ is a positive scalar such that $0 < \lambda < 1$ (note that we do not allow $\lambda = 0$ or $\lambda = 1$), and $D\left(\mathbf{x}, \mathbf{y}\right)$ is a measure of the distance between the target flow $\mathbf{y}$ and the current flow $\mathbf{x}$. For example,

$D(\mathbf{x},\mathbf{y})$ could take the form of the Euclidean distance $D(\mathbf{x},\mathbf{y})=(\mathbf{x}-\mathbf{y})'(\mathbf{x}-\mathbf{y})$. We shall discuss the specification of $D(\mathbf{x},\mathbf{y})$ later.

It is clear that problem (9) is a weighted summation of two problems, namely the linear programming (LP) problem

$$\min_{\mathbf{y}\in\Omega} \mathbf{c}(\mathbf{x})' \mathbf{y} \tag{10}$$

and the following minimization problem

$$\min_{\mathbf{y}\in\Omega} D(\mathbf{x},\mathbf{y}) \tag{11}$$

We shall look into LP (10) and problem (11) to obtain a better understanding of the proposed day-to-day dynamic (8)-(9). LP (10) is to optimize the link flow pattern $\mathbf{y}$ in terms of cost minimization under a given fixed link cost vector $\mathbf{c}(\mathbf{x})$. Because link cost vector $\mathbf{c}(\mathbf{x})$ is given and fixed, LP (10) is essentially to find the shortest path for each OD pair. That is, the optimal solution to LP (10) corresponds to an all-or-nothing traffic assignment, i.e. all the travel demand will be assigned to the shortest path for each OD pair. Therefore, if we let the target flow $\mathbf{y}$ in dynamic (8) solve LP (10), then the dynamic would mean that, *at any day-to-day time, travelers tend to switch to the current shortest path.* This behavioral implication is reasonable: like most day-to-day dynamics, it just in some way captures travelers' cost-minimization behaviors.

However, the optimal solution to LP (10) is generally not unique, which means that simply letting $\mathbf{y}$ solve LP (10) would make dynamic (8) indeterminate. This problem exists because there are generally multiple shortest paths between each OD pair. To see this problem, let us first revisit the example studied in last section. When a capacity reduction on Link 4 happens to the network shown in Figure 1, there are two equally shortest paths, namely Path 2 and Path 4. Then, any arbitrary demand split between Path 2 and Path 4 will give an optimal solution to LP (10) for the given current flow $\mathbf{x}=\tilde{\mathbf{x}}$. To see a more general example, which does not rely on specific network topology, let us consider that the current flow is at UE, i.e. $\mathbf{x}$ is the UE link flow. In this case, it is obvious that the UE shortest path between each OD pair is typically not unique, and neither is the optimal solution to LP (10). Observe that, in this case, $\mathbf{y}=\mathbf{x}$ is one of the optimal solutions to LP (10).

The fact that LP (10) has multiple optimal solutions implies that we need to pick one from its optimal solution set to be the target flow $\mathbf{y}$ in dynamic (8). For the example of Figure 1, we should pick the one such that the flow split between Link 1 and Link 2 remains unchanged, as we have discussed in last section. For the example that the current flow is at UE, obviously we should simply pick $\mathbf{y} = \mathbf{x}$, i.e. the UE flow pattern should not change. For both examples, we have picked a target flow $\mathbf{y}$ that is closest to the current flow $\mathbf{x}$, which, in behavioral sense, captures *travelers' inertia or reluctance to change, i.e. travelers do not make unnecessary changes when they seek to minimize their travel costs based on the current situation.* In the example of Figure 1, a change between Link 1 and Link 2 would be "unnecessary" because it would not reduce a traveler's travel cost based on the current day situation. In the same spirit, any change would be "unnecessary" when the current flow is already at UE. Mathematically, this inertia effect is best captured by problem (11), which is simply to minimize the distance $D(\mathbf{x},\mathbf{y})$ between the target flow $\mathbf{y}$ and the current flow $\mathbf{x}$.

Finally, we obtain problem (9) by combining LP (10) and problem (11), and the behavioral explanation of the proposed day-to-day dynamic (8)-(9) is clear, namely that the first term of (9) captures travelers' cost-minimization behaviors, and the second term reflects travelers' inertia.

Now we move on to look into the specification of the distance measure $D(\mathbf{x},\mathbf{y})$. The most natural specification of $D(\mathbf{x},\mathbf{y})$ is, as we mentioned earlier, the (square) Euclidean distance, i.e., $D(\mathbf{x},\mathbf{y}) = (\mathbf{x}-\mathbf{y})'(\mathbf{x}-\mathbf{y})$. With this specification, the objective function of problem (9) is a convex combination of a linear term and a quadratic term, and thus problem (9) is readily a strictly convex problem. This guarantees that the solution of problem (9) $\mathbf{y}$ is unique for any given $\mathbf{x}$, and thus the dynamic is well-defined.

Despite its perfect mathematical property and good intuition, the Euclidean distance specification of $D(\mathbf{x},\mathbf{y})$ has one deficiency that it is not robust to irrelevant changes to the network. To see this deficiency, let us consider a simple two-link network as shown in Figure 2(a), and then consider that a dummy node is added so that we obtain the network shown in Figure 2(b). The two networks are essentially the same network with link cost functions shown in the figure, because adding a dummy node is an irrelevant change to the network. A robust model should be independent of this kind of "dummy node" effect. Unfortunately, the Euclidean distance formulation $D(\mathbf{x},\mathbf{y}) = (\mathbf{x} - \mathbf{y})'(\mathbf{x} - \mathbf{y})$ is not robust in this respect: for the network of Figure 2(a) consisting of two links, we have

$$D(\mathbf{x},\mathbf{y}) = (x_1 - y_1)^2 + (x_2 - y_2)^2$$

while for the network of Figure 2(b) consisting of three links, we have

$$\begin{aligned} D(\mathbf{x},\mathbf{y}) &= (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_2 - y_2)^2 \\ &= (x_1 - y_1)^2 + 2(x_2 - y_2)^2 \end{aligned}$$

Clearly, adding a dummy node to link 2 makes the flow (change) on link 2 have more impact in the Euclidean distance formulation, which is a very undesirable property as the two networks are actually the same one.

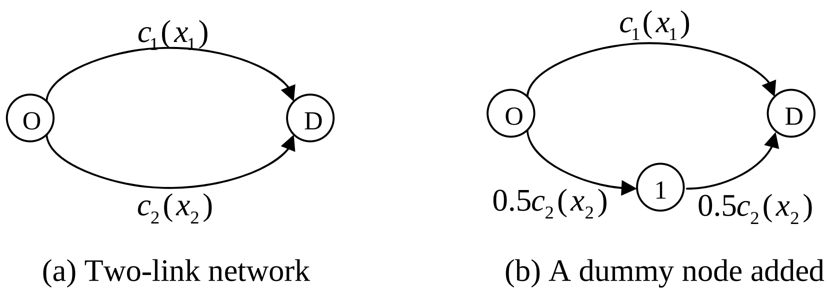

Figure 2. The "dummy node" effect

Another natural specification of $D(\mathbf{x},\mathbf{y})$ would be $D(\mathbf{x},\mathbf{y}) = (\mathbf{c}(\mathbf{x}) - \mathbf{c}(\mathbf{y}))'(\mathbf{c}(\mathbf{x}) - \mathbf{c}(\mathbf{y}))$, which is the (square) Euclidean distance between the link cost vectors. It can be easily verified that this specification also has good mathematical properties (e.g. strictly convex), but, again, the formulation is not robust to the "dummy node" effect: for the network of Figure 2(a) consisting of two links, we have

$$D(\mathbf{x},\mathbf{y}) = \left(c_1(x_1) - c_1(y_1)\right)^2 + \left(c_2(x_2) - c_2(y_2)\right)^2$$

while for the network of Figure 2(b) consisting of three links, we have

$$\begin{aligned} D(\mathbf{x},\mathbf{y}) &= \left(c_1(x_1) - c_1(y_1)\right)^2 + \left(0.5c_2(x_2) - 0.5c_2(y_2)\right)^2 + \left(0.5c_2(x_2) - 0.5c_2(y_2)\right)^2 \\ &= \left(c_1(x_1) - c_1(y_1)\right)^2 + 0.5\left(c_2(x_2) - c_2(y_2)\right)^2 \end{aligned}$$

Clearly, adding a dummy node to link 2 makes the flow (change) on link 2 have less impact in the formulation, which means that this specification of $D(\mathbf{x},\mathbf{y})$ is not robust to the "dummy node" effect.

Although the above two most natural specifications of $D(\mathbf{x},\mathbf{y})$ both have the same deficiency, i.e., not independent of irrelevant changes to the network, it is not difficult to construct other specifications of $D(\mathbf{x},\mathbf{y})$ such that our link-based model is robust to irrelevant network changes (or, more specifically, robust to the "dummy node" effect). For example, it can be easily verified that $D(\mathbf{x},\mathbf{y}) = \sum_{a\in L}\left(c_a(x_a) - c_a(y_a)\right)\left(x_a - y_a\right)$ and $D(\mathbf{x},\mathbf{y}) = \sum_{a\in L}\left|c_a(x_a) - c_a(y_a)\right|$ both measure the distance between $\mathbf{y}$ and $\mathbf{x}$ (i.e., their values increase as y_a moves away from x_a for each link), and are both robust to the "dummy node" effect. However, they are generally not strictly convex and thus do not guarantee the uniqueness of the solution to problem (9), which means that our day-to-day dynamic may be not well defined with these specifications of $D(\mathbf{x},\mathbf{y})$.

Finally, in this paper, we propose the following specification of $D(\mathbf{x},\mathbf{y})$

$$D(\mathbf{x},\mathbf{y}) = \sum_{a\in L}\int_{x_a}^{y_a}\left(c_a(w) - c_a(x_a)\right)dw \tag{12}$$

and make some widely adopted assumptions on link cost functions as below.

Assumption 1. *The link cost functions are separable, i.e.,* $c_a(\mathbf{x}) = c_a(x_a)$, $a \in L$, *continuously differentiable and monotonically increasing.*

Intuitively, $D(\mathbf{x},\mathbf{y})$ given by (12) is a reasonable measure of the distance between the target flow $\mathbf{y}$ and the current flow $\mathbf{x}$, i.e. its value increases as y_a moves away from x_a for each link. With Assumption 1, formulation (12) is a strictly convex function of $\mathbf{y}$ for any given $\mathbf{x}$, and it can be easily verified that this formulation is robust to the "dummy node" effect. Thus, we finally have a "good" specification of the distance measure $D(\mathbf{x},\mathbf{y})$. It should be mentioned here that, more complicated "good" specifications of $D(\mathbf{x},\mathbf{y})$ certainly exist. For example, we can always change the specific formulation within the integral. From a more general viewpoint, how we specify $D(\mathbf{x},\mathbf{y})$ actually reflects how we are modeling travelers' route switching behaviors (e.g., inertia, habitude, consideration of the network hierarchy). At this stage, without further empirical evidence, we shall just use formulation (12) in this paper. Also note that, if the "dummy node" effect is not a big concern for some networks, then the Euclidean distance formulations are actually “good” specifications of $D(\mathbf{x},\mathbf{y})$, and they provide more direct intuitions than the formulation proposed here.

Finally, a specific version of our link-based day-to-day traffic model is given by (8)-(9) with the distance measure $D(\mathbf{x},\mathbf{y})$ specified as (12). Now we shall revisit the two path-based problems mentioned in last section and see how well they are addressed by our link-based model. It is obvious that the path-flow-nonuniqueness problem does not exist any more because the application of our link-based model only needs an initial link flow pattern, which is practically observable and mathematically unique (under mild technical conditions).

The path-overlapping problem is a bit more complicated. In the general sense, the path-overlapping problem implies that a perfect day-to-day dynamic model needs to capture how travelers take into consideration network hierarchy when making their route switching decisions. We can not claim that our model (8)-(9) achieves this goal without further empirical study. Nevertheless, in the narrow sense, how paths overlap with each other is an irrelevant question to our link-based model, and thus we do not have to make assumptions related to path overlapping. That is, unlike the path-based models, our link-based model does not have to (implicitly) assume that travelers are indifferent to two paths with equal costs. In this sense, the path-overlapping problem is indeed solved or avoided by our link-based model. This is best illustrated by the small network shown in Figure 1, which our model is amenable to while many existing path-based models are not. More specifically, applying our model to the network shown in Figure 1, the second term of problem (9) ensures that a capacity reduction on Link 4 will not cause flow fluctuation between Link 1 and Link 2, while many existing path-based models violate this reasonable and intuitive prediction, as discussed in last section.

In the following we will prove that the fixed point of the day-to-day dynamic (8)-(9) with $D(\mathbf{x},\mathbf{y})$ specified as (12) is the UE link flow. We shall start with several lemmas.

Lemma 1. *A link flow pattern* $\mathbf{y}$ *solves LP (10) for given* $\mathbf{x}$ *if and only if* $\mathbf{y}$ *assigns all travel demand to the shortest paths determined by link cost vector* $\mathbf{c}(\mathbf{x})$.

Lemma 1 is self-evident, and directly leads to the following lemma.

Lemma 2. *A link flow pattern* $\mathbf{x}$ *is the UE link flow if and only if* $\mathbf{y}=\mathbf{x}$ *solves LP (10) for given* $\mathbf{x}$.

By the definition of UE, a link flow pattern $\mathbf{x}$ is the UE link flow if and only if $\mathbf{x}$ assigns all travel demand to the shortest paths determined by link cost vector $\mathbf{c}(\mathbf{x})$. Then we simply have Lemma 2 from Lemma 1.

Lemma 3. *Let* $D(\mathbf{x},\mathbf{y})$ *have formulation* (12), *and suppose Assumption 1 holds, then, for a given link flow pattern* $\mathbf{x}$, $\mathbf{y}=\mathbf{x}$ *solves problem (9) if and only if* $\mathbf{y}=\mathbf{x}$ *solves LP (10).*

Proof: It is readily seen that $\mathbf{y}=\mathbf{x}$ always minimizes the second term of problem (9). When $\mathbf{y}=\mathbf{x}$ solves LP (10), it also minimizes the first term of problem (9). Thus, if $\mathbf{y}=\mathbf{x}$ solves LP (10), then $\mathbf{y}=\mathbf{x}$ minimizes both terms of problem (9) and thus solves problem (9). This completes the proof of the "if" part.

To prove the "only if" part, let us rewrite the objective function of problem (9) to be

$$Z(\mathbf{y})=\lambda Z_1(\mathbf{y})+(1-\lambda)Z_2(\mathbf{y})$$

where $0<\lambda<1$, $Z_1(\mathbf{y})=\mathbf{c}(\mathbf{x})'\,\mathbf{y}$, and $Z_2(\mathbf{y})=D(\mathbf{x},\mathbf{y})=\sum_{a\in L}\int_{x_a}^{y_a}\left(c_a(w)-c_a(x_a)\right)dw$. Suppose that $\mathbf{y}=\mathbf{x}$ solves problem (9) but does not solve LP (10), then there is a feasible direction $\mathbf{z}$ at $\mathbf{y}=\mathbf{x}$ such that

$$\nabla Z_1(\mathbf{y})'\,\mathbf{z}<0 \tag{13}$$

where $\nabla Z_1(\mathbf{y})$ is the gradient of $Z_1(\mathbf{y})$ at $\mathbf{y}=\mathbf{x}$. On the other hand, it holds readily $\nabla Z_2(\mathbf{y})=0$ at $\mathbf{y}=\mathbf{x}$, and thus we have

$$\nabla Z_2(\mathbf{y})'\,\mathbf{z}=0 \tag{14}$$

Combining (13) and (14) gives $\nabla Z(\mathbf{y})'\,\mathbf{z}<0$ at $\mathbf{y}=\mathbf{x}$, which contradicts that $\mathbf{y}=\mathbf{x}$ solves problem (9). This completes the proof. ♦

The fixed point of dynamic (8)-(9), i.e. the link flow $\mathbf{x}$ gives $\dot{\mathbf{x}}=0$, is clearly the $\mathbf{x}$ such that $\mathbf{y}=\mathbf{x}$ solves problem (9). Then, combining Lemma 2 and Lemma 3 immediately gives the following theorem.

Theorem 1. *Let* $D(\mathbf{x},\mathbf{y})$ *have formulation* (12), *and suppose Assumption 1 holds, then, a link flow pattern* $\mathbf{x}$ *is the fixed point of dynamic (8)-(9) if and only if* $\mathbf{x}$ *is the UE link flow.*

At the end of this section, we give the discrete-time version of dynamic (8)-(9) as below

$$\mathbf{x}^{t+1}-\mathbf{x}^{t}=\alpha\left(\mathbf{y}^{t}-\mathbf{x}^{t}\right) \tag{15}$$

where $0<\alpha\leq 1$ represents the step-size of this discrete-time version, and $\mathbf{y}^{t}$ solves problem (9) for given current link flow $\mathbf{x}^{t}$.

5. Numerical Example

In this section we apply our link-based day-to-day traffic assignment model to a test network, and provide some discussions on the model parameters based on the numerical results.

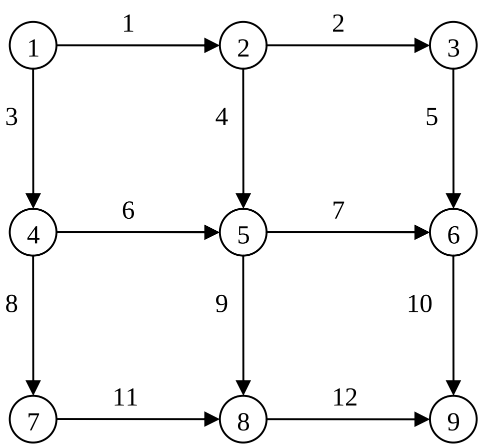

Figure 3. Test network of numerical example

The test network is a 3×3 grid network with 9 nodes, 12 links and 6 routes connecting one OD pair from Node 1 to Node 9. Node and link numbers are shown in Figure 3. The total OD demand is 2000. The link cost function is of BPR type

$$c_a=c_0\left[1+0.15\left(\frac{x_a}{C_a}\right)^4\right]$$

where c_0 is the free flow travel time, and C_a is the nominal link capacity. All 12 links have the same free flow travel time $c_0 = 1500$ and nominal capacity $C_a = 1000$. Consider that the initial network condition is at UE with the (unique) UE link flow

$$\tilde{\mathbf{x}} = (1000,\ 500,\ 1000,\ 500,\ 500,\ 500,\ 500,\ 500,\ 500,\ 1000,\ 500,\ 1000)'$$

Observe that the UE path flow constituting $\tilde{\mathbf{x}}$ is not unique in this example, which means that the path-flow-nonuniqueness problem exists for the path-based models.

The testing scenario is that a 50% capacity reduction on Link 1 takes place at day 0. Applying the discrete version of our link-based model, we have two parameters, the step-size α in (15) and the weight parameter λ in problem (9). We first set constant $\alpha = 0.7$ and $\lambda = 0.7$ to show an application of the model. Figure 4 shows the flow evolutions of five links (Links 1-3 and Links 10 and 12) for the testing scenario. To facilitate illustration, the day-to-day link flow of each link is normalized by its initial UE link flow, and thus all link flow evolutions start with value 1.

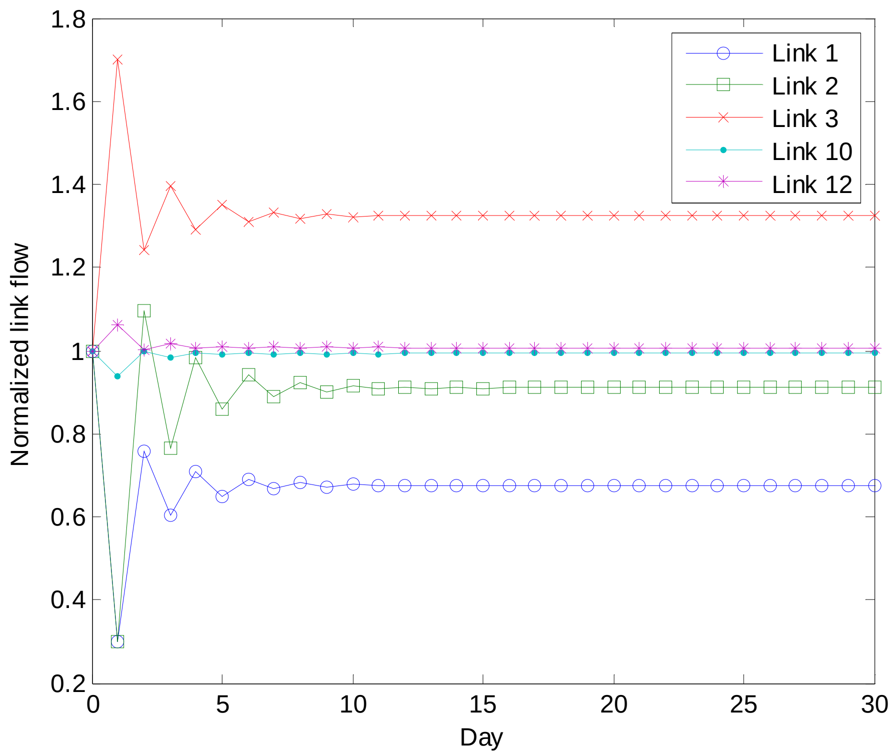

Figure 4. Link flow evolution after 50% capacity reduction on Link 1

As shown in Figure 4, all links have some flow fluctuations for the first several days after the capacity reduction happens, and finally the link flow pattern converges to a fixed point, which, as we have proved in last section, is the new UE link flow. It can also be seen that Links 1-3 have stronger flow fluctuations (larger percentage changes) and slower convergence rates as compared with Links 10 and12. This is consistent with our intuition about a gird network: Links 10 and12 are far from Link 1 and thus should be impacted to a less degree by an accident on Link 1, while Link 2 and 3 are adjacent to Link 1 and thus should be more severely affected.

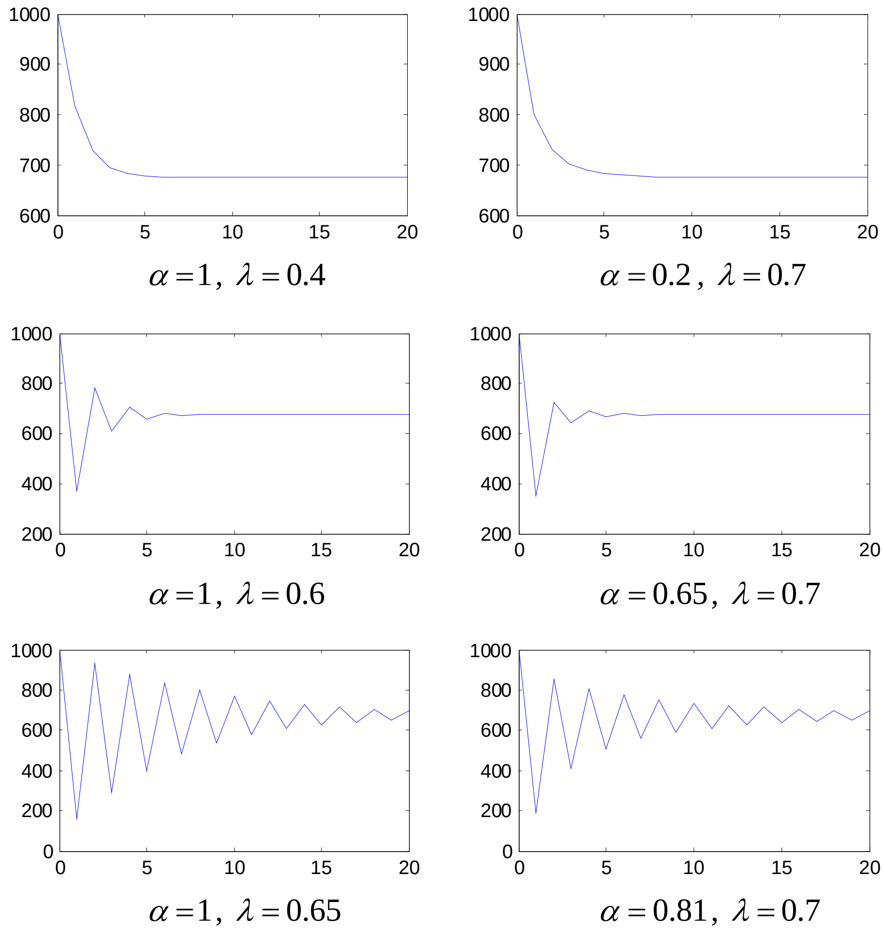

Figure 5. Flow evolution of Link 1 with different parameter values

Now we set different values to the two parameters α and λ to see their impacts on the day-to-day traffic pattern. For graph simplicity, we only show the flow evolution of Link 1. In each subfigure of Figure 5, the x-axis value is the time "day", and the y-axis value is the flow on Link 1. Thus each subfigure of Figure 5 gives a day-to-day flow evolution of Link 1 with specific α and λ values.

To see the impact of parameter λ, we fix $\alpha = 1$ and increase the value of λ from 0.4 to 0.6 and 0.65. The corresponding day-to-day dynamics are shown by the three subfigures in the left column of Figure 5. It can be seen that, as λ increases, the fluctuation of the day-to-day flow evolution increases. This is consistent with the physical meaning of λ: the larger λ is, the less weight is put on the inertia term of problem (9), which means that travelers tend to change routes more drastically from day to day. Similarly, to see the impact of the step-size parameter α, we fix $\lambda = 0.7$ and change the value of α, as shown by the three subfigures in the right column of Figure 5. We can see clearly that the fluctuation of the day-to-day dynamic increases with step-size α. This is an expected result because step-size naturally reflects how drastically travelers change their routes.

To sum up, we have observed that the two parameters α and λ both represent travelers' inertia (larger parameter value means less inertia) and thus impact the fluctuation and convergence of the day-to-day dynamic in a similar manner. As a result, there may be some redundancy in the parameter pair α and λ, i.e. we can fix one and just vary the other to obtain a range of system dynamics. Indeed, comparing the subfigures in the left and the right columns of Figure 5, we can see that fixing either one of α and λ at a relatively large value ($\alpha = 1$ and $\lambda = 0.7$ are both large values within their respective feasible regions) and changing the other parameter alone can generate a range of link flow evolution patterns, from very smooth patterns (the first row of Figure 5) to very fluctuated ones (the third row of Figure 5). This observation gives an implication on model calibration. That is, it may be unnecessary to calibrate the two parameters simultaneously, because calibrating one parameter only (with the other predetermined) may give a model that works as well. This conjecture needs to be verified by future empirical studies.

Note that if one of α and λ is set to be a very small value, then the value of the other does not matter much, and the flow evolution is going to be in a smooth pattern. Mathematically, if the step-size is very small (α very small), then the distance between the target flow and the current flow is not important (λ not important) because a bounded distance multiplied by a small step-size is always small; reversely, if the target flow is very close to the current flow (λ very small), then the step-size α is not important because even if the step-size is one the flow change (a full step) is still small. In both cases, the flow change from day to day is going to be very small and the flow evolution will be smooth. This explains why the two subfigures in the first row of Figure 5 have very smooth flow evolutions despite each has one large-valued parameter (with $\alpha = 1$ and $\lambda = 0.7$ respectively). It is simply because the other parameter is chosen to be small (with $\lambda = 0.4$ and $\alpha = 0.2$ respectively). This observation implies that, if we are calibrating one parameter with the other predetermined, then the predetermined parameter value should not be too small, otherwise the to-be-calibrated parameter does not play an important role in the model. The observation also indicates that, only when both α and λ are large enough can the flow evolution be fluctuated, as shown by the two subfigures in the third row of Figure 5. Actually, when both parameters are large valued, the day-to-day dynamic may not converge, as shown in Figure 6.

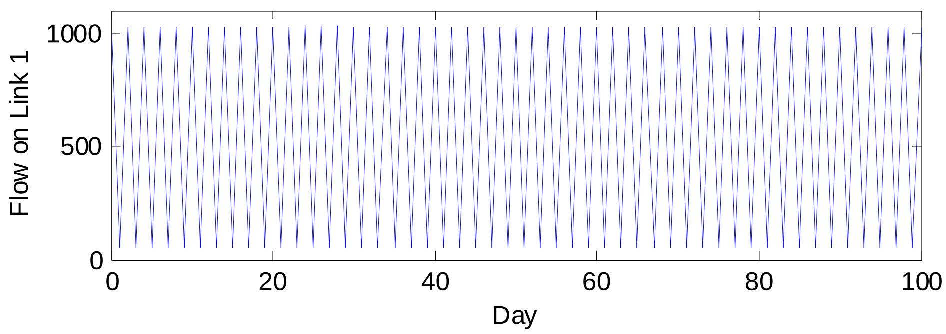

Figure 6. Flow evolution without convergence when $\alpha = 0.95$, $\lambda = 0.7$

6. Conclusions

In this paper we have demonstrated two shortcomings of many existing path-based day-to-day traffic dynamics, namely the path-flow-nonuniqueness problem and the path-overlapping problem. The first problem exists because the application of the path-based models need a given initial path flow pattern, which is typically unidentifiable and thus make their application problematic. The second problem arises because the path-based models ignore the interdependence among paths and thus can provide very unreasonable results for networks with paths overlapping with each other.

In view of the difficulty of solving the two problems within the path-based methodology, we proposed a link-based day-to-day traffic assignment model. Our link-based model captures travelers' cost-minimization behavior as well as their inertia, and has the classic UE link flow as the fixed point. The two path-based problems are effectively avoided by our link-based model. We also provided discussions on the model parameters based on some preliminary numerical results.

Because our link-based day-to-day dynamic is a relatively new model, there are many possible future researches, including both theoretical and empirical ones. Perhaps empirical studies or real applications are of more urgency, because so far no day-to-day dynamics have been applied in real networks.

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