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Abstract

Private provision of public roads through build-operate-transfer (BOT) contracts is increasing around the world. This paper investigates the properties of Pareto-efficient BOT contracts using a bi-objective programming approach under perfect information. Under certain conventional assumptions, we find that for any Pareto-efficient BOT contract: (1) the concession period should be set to be the whole road life; (2) the volume-capacity ratio (or the road service quality) and the average social cost per trip are constantly equal to those at the social optimum whenever there are constant returns to scale in road construction. Extensions are made to the cases with increasing (decreasing) returns to scale in road construction. A variety of regulatory regimes are investigated to analyze the behavior of the profit-maximizing private firm, and efficient regulations, including demand and markup charge regulations, are elucidated for both the public and private sectors to achieve a predetermined Pareto-optimal outcome.

*Keyword*s: Private road; Road franchising; Pareto efficiency; Regulation.

1. Introduction

Private-sector participation in road construction and operations has the advantages of efficiency gains, private financing, and better identification of attractive investment projects. Such participation is generally implemented through a build-operate-transfer (BOT) contract, under which a private firm builds and operates roads in a road network at its own expense, and in return receives the revenue from road tolls for a number of years, and then these roads are transferred to the government. Such commercial and private provision of public roads has attracted growing interest in

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recent years and such plans are being used to finance modern road systems worldwide (Roth, 1996). Private-sector participation in the form of BOT franchises has worked well in a number of projects such as road tunnels in Hong Kong. In mainland China, many local, mainly municipally affiliated companies have undertaken the development of toll roads in recent years, often in joint ventures with Hong Kong investors (Tam, 1998). Once road provision is market driven, many issues must be carefully addressed, because the interests of the private sector are different from those of the public sector. From the viewpoint of private investors, the profitability of a project is of great concern because private firms are put at risk. From the viewpoint of the public sector, it is meaningful to assess whether the construction of a road will lead to a positive welfare gain and also be profitable so that private provision is worthwhile.

Most previous analyses of road investment have focused on capacity choices and setting tolls and the resulting profitability and social welfare gain. The concession period and/or road life is usually assumed to be given; the prorated unit cost of capacity per unit period is thus also given (either a constant or increasing or decreasing with capacity). For comprehensive reviews, the reader may refer to Lindsey and Verhoef (2001), Yang and Huang (2005), and Lindsey (2006). An important result in the early literature is the self-financing theorem for congestion pricing and capacity choice of a single road in a first-best environment, in which the toll is set equal to the difference between the marginal social cost and the marginal private cost of a trip (Mohring and Harwitz, 1962; Keeler and Small, 1977). In a general traffic equilibrium context, Yang and Meng (2000) looked into the profitability and social welfare gain of a single BOT road in a network; various economic regimes were examined, including the regime with profitable and positive welfare increment, the regime with unprofitable but positive welfare increment, and the regime with unprofitable and negative welfare increment. Yang and Meng (2002) further showed that, under essentially the same conditions as in Mohring and Harwitz's (1962) model of a single link, the self-financing result still holds for each road individually in a full new network and consequently for the network in aggregate, provided that each link is optimally priced and all capacities are optimized. If one or more new roads are introduced to an existing network, the self-financing result also holds for each new link individually, even if the existing links do not have optimal capacities, as long as all existing and new links are optimally priced and the capacities of the new links are optimally selected (Proost et al., 2004; Verhoef and Rouwendal, 2004; Yang and Huang, 2005). Verhoef and Rouwendal (2004) addressed some implications of second-best congestion pricing on the applicability of the self-financing theorem using a numerical experiment approach. Verhoef (2007) and Ubbels and Verhoef (2008) analyzed capacity choice and toll setting by private

investors in a competitive bidding framework organized by the government. They considered capacity and toll selection based on various criteria (maximizing capacities or patronage, minimizing tolls or toll revenues) and compared the resulting welfare gains (or losses) from each criterion.

Engel *et al.* (1997, 2001) suggested that the fixed-term contract suffers certain pitfalls with traffic and revenue uncertainties and proposed a flexible term contract for road franchising. In the line of Engel *et al.* (1997), Nombela and de Rus (2004) discussed a new franchising mechanism based on a flexible-term contract and bi-dimensional bids for total net revenue. Recently, Guo and Yang (2008) conducted a preliminary study on the selection of the concession period with deterministic demand and homogeneous users. They incorporated all three essential variables (concession period, road capacity and toll charge) and explicitly considered traffic congestion and demand elasticity for unconstrained and profit-constrained, welfare-maximizing BOT contracts.

In view of the different interests between the public and private sectors, we consider a bi-objective optimization problem for maximizing social welfare and private profit, with respect to the three primary variables of concession period, road capacity and toll charge, for a given toll road. By assuming that the government and the private firm both have perfect information on the project cost and future traffic demand, we examine the properties of the Pareto-optimal solution set. Each Pareto-optimal solution dictates a Pareto-efficient BOT contract that leads to an efficient outcome in the sense that neither social welfare nor private profit can be further enhanced without reducing the other. Moreover, a variety of regulatory mechanisms are investigated to analyze the behavior of the profit-maximizing private firm, and efficient regulations including demand and mark-up charge regulations are elucidated for both the public and private sectors to achieve a predetermined Pareto-optimal outcome.

The paper is organized as follows. Section 2 introduces our bi-objective programming formulation of a BOT toll road scheme and the definition of Pareto-efficient contracts. Some important properties of the set of Pareto-efficient contracts are explored in Section 3. A further analysis of the efficiency of a given Pareto-optimal contract in comparison with the social optimum is conducted in Section 4. Section 5 extends the analysis to the general cases of decreasing and increasing returns to scale in road construction. Section 6 investigates a variety of regulatory regimes for the government to achieve a predetermined Pareto-efficient contract. Numerical examples are used to elucidate our results in Section 7, and, finally, conclusions are presented in Section 8.

2. Basic Definition and Assumption

Assume that the government wants to get a private firm to build a new highway whose technical characteristics are exogenous. Let $y \geq 0$ be the capacity of the new road, $q \geq 0$ be the travel demand and $B(q)$ be the inverse demand function (or the marginal benefit function), and $t(q, y)$ be the link travel time function. Note that q and y are measured in the number of vehicles per unit period. The following demand-supply equilibrium condition always holds:

$$B(q) = p + \beta t(q, y), \tag{1}$$

where p is the toll charged to each user of the road and β is the value of time to convert time into an equivalent monetary cost (we consider homogeneous users only). Condition (1) simply means that travel demand for the new road is determined by the full price for a trip. Let $I(y)$ be the construction cost of the highway as a function of capacity. The following assumption is made about $B(q)$, $I(y)$ and $t(q, y)$ throughout the paper.

Assumption 1.
(a) *The inverse demand function, $B(q)$, is a strictly decreasing and differentiable function of q for $q \geq 0$; $qB(q)$ is a strictly concave function of q for $q \geq 0$.*
(b) *The road construction cost function, $I(y)$, is a continuously increasing and differentiable function of y for $y \geq 0$.*
(c) *The travel time function, $t(q, y)$, is a continuously differentiable function of (q, y) for $q \geq 0$ and $y \geq 0$; for any $q > 0$, $t(q, y)$ decreases with y; for any $y > 0$, $t(q, y)$ is a convex and increasing function of q.*

From equilibrium condition (1), the toll, p, can be viewed as the following function of the demand, q, and the capacity, y:

$$p(q, y) = B(q) - \beta t(q, y). \tag{2}$$

For a given y, the toll, p, is uniquely determined by the demand, q. Therefore, determining the variables p and y is essentially equivalent to selecting the variables q and y. Hereafter, the demand, q, is substituted for the toll, p, for convenience of exposition.

We first consider the private firm's problem. Let $\hat{T}$, $\hat{T} > 0$, be the life of the road under consideration. The private firm must choose a combination of the BOT variables, including the concession period, T, with $0 \leq T \leq \hat{T}$, the travel demand q (or equivalently, the toll charge, p) and the road capacity, y, to maximize its profit, $P(T, q, y)$, during the concession period T:

$$P(T, q, y) = Tqp - I(y), \tag{3}$$

where the travel demand, q, is determined by condition (1), the first term of eqn. (3) is the total toll revenue collected by the private firm during the concession period and the second term is the construction cost, which is fully borne by the private firm. With eqn. (2), problem (3) can be rewritten as:

$$P(T, q, y) = TqB(q) - \beta Tqt(q, y) - I(y) \tag{4}$$

From Part (c) of Assumption 1 that $t(q, y)$ is convex, $qt(q, y)$ is convex in q for any given $y > 0$. From the strict concavity of $qB(q)$, the profit function, $P(T, q, y)$, is strictly concave in q for any given y and T. For simplicity, we do not adopt an interest rate to discount future revenues to their equivalent present values. In fact, the use of a discounting rate does not alter our results since both social welfare and profit in this study are invariant with the calendar time (see Appendix 1) for the same reason given by Guo and Yang (2008).

Next, we consider the government's problem of choosing the best combination of the BOT variables (T, q, y) to maximize the social welfare during the whole road life, $\hat{T}$:

$$W(T, q, y) = TS(q, y) + (\hat{T} - T)\tilde{S}(y) - I(y), \tag{5}$$

where $I(y)$ is again the construction cost, $S(q, y)$ and $\tilde{S}(y)$ are, respectively, the unit-time social welfare during the concession and post-concession periods and determined below.

$$S(q, y) = \int_0^q B(w) \, dw - \beta qt(q, y); \tag{6}$$

$$\tilde{S}(y) = \max_{q_1 \geq 0} S(q_1, y) = \max_{q_1 \geq 0} \int_0^{q_1} B(w) \, dw - \beta q_1 t(q_1, y). \tag{7}$$

Equation (7) implies that, during the post-concession period, the road capacity is given and fixed and the government can select the optimal traffic volume to maximize the unit-time social welfare only.

Therefore, the BOT problem can be defined as selecting simultaneously the combination of the three variables (T, q, y) to maximize the total social welfare and

the private firm's profit, which can be formulated as the following bi-objective programming problem:

$$\max_{(T,q,y)\in\Omega}\begin{pmatrix} W(T,q,y) \\ P(T,q,y) \end{pmatrix}, \tag{8}$$

where $\Omega=\{(T,q,y):0\le T\le\hat{T},q\ge0,y\ge0\}$ and social welfare, $W(T,q,y)$, and profit, $P(T,q,y)$, are defined by (5) and (4), respectively.

Our task here is to seek the set of the Pareto-efficient solutions of the bi-objective optimization problem (8). Since a BOT contract is essentially an outcome of negotiation between the government and the private firm and can be characterized by a combination of (T,q,y), we are now ready to define a Pareto-efficient contract for the BOT problem (8) as follows.

Definition (Pareto-efficient BOT Contract). *A BOT triple $(T^*,q^*,y^*)\in\Omega$ is called a Pareto-efficient contract if there is no other feasible BOT triple $(T,q,y)\in\Omega$ such that $W(T,q,y)\ge W(T^*,q^*,y^*)$ and $P(T,q,y)\ge P(T^*,q^*,y^*)$ with at least one strict inequality.*

The Pareto-efficient BOT contract is an important and meaningful concept that represents the situation in which no party can be made better off without making the other one worse off.

3. Properties of Pareto-efficient BOT Contracts

In this section, we examine the properties of Pareto-efficient contracts in the BOT problem (8) under perfect information, namely, the demand and construction costs are common knowledge to both the public and the private sectors. We begin with the following proposition (a rigorous proof is given in Appendix 2)

Proposition 1. *Under Assumption 1, if a triple $(T^*,q^*,y^*)\in\Omega$ is a Pareto-efficient BOT contract, then $T^*=\hat{T}$.*

Proposition 1 states that any Pareto-efficient BOT contract requires a whole road life concession period. This "lifetime concession period" result seems to be realistic because several BOT contracts around the world have been awarded for 99 years,

including Highway 407 in Toronto, the Chicago Skyway and the Pocahontas Parkway (Virginia Route 495) in Richmond, Virginia.

The economic logic of Proposition 1 becomes clear in the cases of monopoly and socially optimal solutions corresponding to the two polar points of the Pareto-efficient frontier of the bi-objective programming problem (8). First, maximizing the profit for a monopoly solution clearly requires as long a concession period as possible since profit in each operating period is positive (only the initial road construction costs are considered and maintenance and operating costs are ignored). Second, we note that the social welfare, given by eqn. (5), comes in the concession and the post-concession periods. If the concession period is less than the lifetime of the road, and the capacity and price of the road are fixed at their optimal (welfare-maximizing) values, it is a matter of indifference for the government how the transfer time is determined. However, the firm can reach a higher profit or Pareto improvement can be made when the transfer time is extended. In this case, the socially optimal and Pareto-efficient contracts must extend for the full lifetime of the road. Next, we consider a concession period less than the road life for any Pareto-efficient solution other than the monopoly and social optimum. Since the franchising firm realizes a positive contribution to its profit in each period, extending the concession period will therefore certainly increase profits at the prevailing price/capacity that differs from the socially optimal value. Such an extension therefore provides room for price and/or capacity changes that are more in the interest of social welfare without lowering private profits during the concession period. However, there may still be a loss associated with this change. Extending the concession period means that the government is no longer able to set a welfare-maximizing price during the extension of the concession period. In this case, it is not intuitively clear if extending the concession period will result in a net social welfare gain, although this is proved mathematically in Appendix 2.

With Proposition 1, it is sufficient to set $T = \hat{T}$ in our subsequent analysis of the bi-objective problem (8). To obtain useful insights into Pareto-efficient BOT contracts, the following two common assumptions in the literature are introduced and used hereafter unless otherwise explicitly noted.

Assumption 2. *Link travel time function, t, is homogeneous of degree zero in the link flow, q, and the link capacity, y, i.e., $t(\alpha q, \alpha y) = t(q, y)$ for any $\alpha > 0$.*

Clearly, the widely used BPR (Bureau of Public Roads) link travel time functions satisfy this assumption. Note that, with this assumption, for any $y > 0$, $t(q, y) = t(q/y, 1)$. With a slight abuse of the notation, we denote $t(q, y)$ as $t(\gamma)$

for convenience, where γ, $\gamma = q/y$, is the volume-capacity (v/c) ratio. The v/c ratio, γ, is an important index to capture the service quality of the highway: the larger the value of γ, the longer the travel time and thus the worse the service quality and vice versa. Since the concession period extends to the whole road life in the static environment considered here, the v/c ratio under a given BOT contract is unchanged over time.

Assumption 3. *Constant return to scale in road construction, namely, $I(y) = ky$, where k denotes the constant cost per unit of capacity.*

Let $\left(\hat{T}, \tilde{q}, \tilde{y}\right)$ and $\left(\hat{T}, \overline{q}, \overline{y}\right)$ be the socially optimal (SO) and monopoly optimal (MO) solutions, which maximize social welfare, $W\left(\hat{T}, q, y\right)$, and profit, $P\left(\hat{T}, q, y\right)$, respectively, or they meet the following first-order conditions, respectively:

$$\frac{\partial W}{\partial q} = B(\tilde{q}) - \beta t(\tilde{\gamma}) - \beta \tilde{\gamma} t'(\tilde{\gamma}) = 0 \tag{9}$$

$$\frac{\partial W}{\partial y} = \hat{T}\beta(\tilde{\gamma})^2 t'(\tilde{\gamma}) - k = 0 \tag{10}$$

and

$$\frac{\partial P}{\partial q} = B(\overline{q}) + \overline{q}B'(\overline{q}) - \beta t(\overline{\gamma}) - \beta \overline{\gamma} t'(\overline{\gamma}) = 0 \tag{11}$$

$$\frac{\partial P}{\partial y} = \hat{T}\beta(\overline{\gamma})^2 t'(\overline{\gamma}) - k = 0, \tag{12}$$

where $\tilde{\gamma}$ and $\overline{\gamma}$ denote the SO and MO v/c ratios, respectively, namely, $\tilde{\gamma} = \tilde{q}/\tilde{y}$ and $\overline{\gamma} = \overline{q}/\overline{y}$. By comparing conditions (10) and (12), it is readily seen that $\tilde{\gamma} = \overline{\gamma}$ from the fact that $\gamma^2 t'(\gamma)$ is strictly increasing in γ and thus both equations have the same unique solution of γ. More generally, we draw the following conclusion on the v/c ratio for any Pareto-efficient BOT contract (refer to Appendix 3 for the proof).

Proposition 2. *Under Assumptions 1-3, the v/c ratio, γ^*, for any Pareto-efficient BOT contract $\left(\hat{T}, q^*, y^*\right)$ solves*

$$\hat{T}\beta(\gamma^*)^2 t'(\gamma^*) = k. \tag{13}$$

Thus, it is constant along the Pareto-optimal frontier and equals the socially optimal v/c ratio, $\tilde{\gamma}$.

The v/c ratio, γ, governs the travel time or delay of road users. Xiao *et al.* (2007)

compared the service quality levels of a congested highway offered by a profit-maximizing monopoly and by the public sector. If users value product quality equally (corresponding to homogeneous users with identical values of time), Xiao et al. (2007) proved that the monopoly firm would offer the same service quality as the public sector, which is in line with the well-known economic findings (Spence, 1975). Proposition 2 further reveals that the service quality will coincide with that preferred by the public sector whenever the BOT contract is Pareto-efficient.

The structure of Pareto-efficient BOT contracts turns out to be very simple from Propositions 1 and 2: it includes a whole road-life concession period, $\hat{T}$, and a constant v/c ratio, $\tilde{\gamma}$. We define the contract curve in the demand-capacity space or the Pareto-optimal solution set of problem (8) (with concession period $T = \hat{T}$) as:

$$\Theta = \left\{ \left(q^*, y^*\right) \middle| \left(\hat{T}, q^*, y^*\right) \text{ is a Pareto optimal solution of problem (8)} \right\} \qquad (14)$$

Any efficient bargaining between the public and the private sectors should result in an agreement on the contract curve. Any feasible BOT contract off the contract curve would be inefficient.

From the assumption that $B(q)$ is strictly decreasing and $\overline{\gamma} = \tilde{\gamma}$, we immediately obtain $\tilde{q} \geq \overline{q}$ by comparing eqns. (9) and (11). Note that under Assumptions 1-3, both $W\left(\hat{T}, q, y\right)$ and $P\left(\hat{T}, q, y\right)$ given by the bi-objective programming problem (8) are jointly concave in (q, y), and, thus, the Pareto-optimal solution set is connected (Warburton, 1983). Therefore, the contract curve defined by (14) is the portion of the line connecting $\left(\overline{q}, \overline{y}\right)$ and $\left(\tilde{q}, \tilde{y}\right)$ with slope $\tilde{\gamma}$. Figure 1 shows the contract curve and the corresponding Pareto-optimal frontier in the decision space (q, y) and the objective space (P, W) as bold curves. The arrows indicate increasing social welfare.

For any BOT contract, (T, q, y), the average social cost (ASC) (per user per unit time or per trip during the concession period) is defined as:

$$\text{ASC} = \frac{\beta T q t(q/y) + I(y)}{Tq} = \beta t\left(\frac{q}{y}\right) + \frac{I(y)}{Tq}. \qquad (15)$$

The ASC does not change over calendar time in the static case considered; it equals the sum of two terms: the average travel time (in monetary units) and the construction cost allocation per trip. For any Pareto-efficient BOT contract, $\left(\hat{T}, q^*, y^*\right)$, by Assumption 3 and condition (13), the ASC given by (15) for $T = \hat{T}$ can be

calculated as:

$$\text{ASC} = \beta t\left(\gamma^*\right) + \beta \gamma^* t'\left(\gamma^*\right).\tag{16}$$

From Proposition 2, we readily obtain the following result.

Proposition 3. *Under Assumptions 1-3, the average social cost defined by* (15) *in any Pareto-efficient BOT contract* $\left(\hat{T}, q^*, y^*\right)$ *is the same as the socially optimal ASC.*

Like the v/c ratio in Pareto-efficient BOT contracts, the ASC is constant along the Pareto-efficient frontier and equals the socially optimal ASC. Let C_0 denote the constant ASC under Pareto-efficient BOT contracts. From eqn. (16) and condition (9), we have $C_0 = \beta t\left(\tilde{\gamma}\right) + \beta \tilde{\gamma} t'\left(\tilde{\gamma}\right) = B\left(\tilde{q}\right)$. Also, from eqn. (2), condition (9), Assumption 3 and the definition of ASC in (15), we readily know that in a socially optimal BOT contract $\left(\hat{T}, \tilde{q}, \tilde{y}\right)$, the toll charge equals the allocation of the construction cost per trip:

$$\tilde{p} = B\left(\tilde{q}\right) - \beta t\left(\tilde{\gamma}\right) = \frac{I\left(\tilde{y}\right)}{\hat{T}\tilde{q}}.$$

The toll revenue just covers the construction cost of the road, which is the classical self-financing result (Mohring and Harwitz, 1962). However, for any other Pareto-efficient BOT contract, $\left(\hat{T}, q^*, y^*\right)$, other than the socially optimal contract with $q^* < \tilde{q}$,

$$B\left(q^*\right) > B\left(\tilde{q}\right) = C_0,\tag{17}$$

which means that the average generalized travel cost would exceed the average social cost. By subtracting the average travel time (in monetary units) on both sides of inequality (17), we have:

$$p^* > B\left(\tilde{q}\right) - \beta t\left(\gamma^*\right) = B\left(\tilde{q}\right) - \beta t\left(\tilde{\gamma}\right) = \frac{I\left(\tilde{y}\right)}{\hat{T}\tilde{q}} = \frac{I\left(y^*\right)}{\hat{T}q^*}.\tag{18}$$

The last two equalities follow the result from Proposition 2. Equation (18) reveals that the Pareto-efficient BOT contract, $\left(\hat{T}, q^*, y^*\right)$, with $q^* < \tilde{q}$ will be strictly profitable. Hence, we can view the corresponding toll charge, p^*, as the sum of two distinct parts: one for recovering the road construction cost, denoted as p_1, and the other for gaining profits on the investment, denoted as p_2 and called the markup charge. From condition (13), p_1 can be expressed as

$$p_1 = \frac{I\left(y^*\right)}{\hat{T}q^*} = \frac{\beta \hat{T}\left(\gamma^*\right)^2 t'\left(\gamma^*\right) y^*}{\hat{T}q^*} = \beta \gamma^* t'\left(\gamma^*\right) = \beta q^* \frac{\partial t\left(q^*, y^*\right)}{\partial q}.\tag{19}$$

Thus, for any Pareto-efficient BOT contract, $\left(\hat{T}, q^*, y^*\right)$, p_1 is constantly equal to the socially optimal toll since $\gamma^* \equiv \tilde{\gamma}$ and it exactly equals the congestion externality. The markup charge imposed on each trip, p_2, can be calculated as

$$p_2 = p^* - p_1 = B\left(q^*\right) - \beta t\left(\gamma^*\right) - \beta\gamma^* t'\left(\gamma^*\right) = B\left(q^*\right) - C_0, \tag{20}$$

where the last equality follows from Proposition 3.

From the above analysis, we have clear knowledge of the profitability of a private road when a Pareto-efficient BOT contract, $\left(\hat{T}, q^*, y^*\right)$, is awarded to a private firm: the private firm invests $\hat{T}q^* p_1$ and earns a profit equal to $\hat{T}q^* p_2$. In particular, with a socially optimal BOT contract, $\left(\hat{T}, \tilde{q}, \tilde{y}\right)$, $p_2 = 0$ and the profit is nil. These observations are illustrated geometrically in Figure 2.

4. The Efficiency of Pareto-efficient BOT Contracts

In this section, we devote our analysis to the divergence between the socially optimal BOT contract and the other Pareto-efficient BOT contracts in terms of the realized social welfare. The degree of divergence is measured by the following ratio of social welfare:

$$\rho = \frac{W^*}{\tilde{W}} \leq 1.0, \tag{21}$$

where W^* is the total social welfare realized under a Pareto-efficient BOT contract, $\left(\hat{T}, q^*, y^*\right)$, and $\tilde{W} = W\left(\hat{T}, \tilde{q}, \tilde{y}\right)$ is the maximized social welfare in a socially optimal solution.

From Proposition 3 and after simple manipulation, we have

$$W^* = W\left(q^*, y^*\right) = \hat{T}\left(\int_0^{q^*}\left(B(w) - C_0\right)\mathrm{d}w\right)$$

and

$$\tilde{W} = W\left(\tilde{q}, \tilde{y}\right) = \hat{T}\left(\int_0^{\tilde{q}}\left(B(w) - C_0\right)\mathrm{d}w\right).$$

Thus, the efficiency ratio, ρ, can be expressed as:

$$\rho = \frac{\int_0^{q^*}\left(B(w) - C_0\right)\mathrm{d}w}{\int_0^{\tilde{q}}\left(B(w) - C_0\right)\mathrm{d}w}. \tag{22}$$

It is noted that any Pareto-efficient BOT contract, $\left(\hat{T}, q^*, y^*\right)$, of problem (8) must uniquely solve the following scalar programming problem (Geoffrion, 1967):

$$\max_{(T,q,y)\in\Omega, T=\hat{T}} (1-\lambda)W(T,q,y)+\lambda P(T,q,y),$$ (23)

where $\lambda \in [0,1]$ is a weighting parameter of social welfare and profit. From Propositions 2 and 3, we readily obtain

$$B(q^*)+\lambda q^* B'(q^*)=C_0,$$ (24)

which can be rearranged as

$$B(q^*)=\frac{C_0 E_{q^*}^{B^*}}{E_{q^*}^{B^*}+\lambda},$$ (25)

where $E_{q^*}^{B^*}$ is the point price elasticity of demand at $\left(B(q^*), q^*\right)$, defined by

$$E_{q^*}^{B^*} = \frac{B(q^*)}{q^* B'(q^*)} \quad (<0).$$ (26)

On the other hand, function $\int_0^q (B(w)-C_0)\mathrm{d}w$ is strictly concave in q (because $B(\cdot)$ is a decreasing function) and thus has a unique maximum at $q=\tilde{q}$. Note that $q^* \in [0,\tilde{q}]$, and thus q^*, can be expressed as a convex combination of 0 and $\tilde{q}$:

$$q^* = \left(1-\frac{q^*}{\tilde{q}}\right)\cdot 0 + \frac{q^*}{\tilde{q}}\cdot\tilde{q}.$$

From the concavity of $\int_0^q (B(w)-C_0)\mathrm{d}w$, we readily have

$$\int_0^{q^*} (B(w)-C_0)\mathrm{d}w > \frac{q^*}{\tilde{q}}\int_0^{\tilde{q}} (B(w)-C_0)\mathrm{d}w,$$

which implies that

$$\rho = \frac{\int_0^{q^*} (B(w)-C_0)\mathrm{d}w}{\int_0^{\tilde{q}} (B(w)-C_0)\mathrm{d}w} > \frac{q^*}{\tilde{q}}.$$ (27)

We rewrite the last term of eqn. (27) as

$$\frac{q^*}{\tilde{q}} = \frac{1}{1-\dfrac{q^*-\tilde{q}}{B(q^*)-B(\tilde{q})}\dfrac{B(q^*)}{q^*}\left(1-\dfrac{B(\tilde{q})}{B(q^*)}\right)}.$$ (28)

Substituting eqn. (25) and $B(\tilde{q})=C_0$ into eqn. (28) gives rise to

$$\rho > \frac{q^*}{\tilde{q}} = \frac{E_{q^*}^{B^*}}{E_{q^*}^{B^*}+\lambda E_{sh_{q^*}^{B^*}}},$$ (29)

where $Esh_{q^*}^{B^*}$ is the price elasticity of demand measured by a shrinkage ratio at $\left(B(q^*),q^*\right)$ and $\left(B(\tilde{q}),\tilde{q}\right)$ and defined by

$$Esh_{q^*}^{B^*} = \frac{q^*-\tilde{q}}{B(q^*)-B(\tilde{q})}\frac{B(q^*)}{q^*} \quad (<0). \tag{30}$$

The bound of the efficiency ratio, ρ, associated with a Pareto-efficient BOT contract can be stated in the following proposition.

Proposition 4. *Under Assumptions 1-3, for any Pareto-efficient BOT contract,* $\left(\hat{T},q^*,y^*\right)$*, the efficiency ratio defined by* (21) *is bounded by*

$$\frac{1}{1+\lambda\xi} < \rho \leq 1.0 \tag{31}$$

where $\lambda \in [0,1]$ satisfies condition (24) and $\xi = Esh_{q^*}^{B^*}\big/E_{q^*}^{B^*} > 0$.

From this proposition, the bound of the efficiency ratio for a Pareto-efficient BOT contract, $\left(\hat{T},q^*,y^*\right)$, can be determined by the weighting parameter, λ, of social welfare and profit associated with $\left(\hat{T},q^*,y^*\right)$ and parameter ξ, which depends on the convexity of the benefit function.

With a convex benefit function $B(q)$ or $B'' \geq 0$, we can further tighten the lower bound explicitly. In this case, we have the following two inequalities (a graphical illustration can be found in Xiao et al., 2007):

$$\int_0^{q^*}\left(B(w)-C_0\right)\mathrm{d}w \geq \frac{1}{2}q^*\left(-q^*B'(q^*)\right)+\left(B(q^*)-C_0\right)q^* > 0$$

and

$$0 \leq \int_{q^*}^{\tilde{q}}\left(B(w)-C_0\right)\mathrm{d}w \leq \frac{1}{2}\left(\tilde{q}-q^*\right)\left(B(q^*)-C_0\right).$$

Based on the relationship between the above two inequalities,

$$a \geq \bar{a} > 0 \ \ \& \ \ 0 \leq b \leq \bar{b} \ \Rightarrow \ \frac{a}{a+b} \geq \frac{\bar{a}}{\bar{a}+\bar{b}}$$

and in view of definition (22), we immediately obtain

$$\rho \geq \frac{\frac{1}{2}q^*\left(-q^*B'(q^*)\right)+\left(B(q^*)-C_0\right)q^*}{\frac{1}{2}q^*\left(-q^*B'(q^*)\right)+\left(B(q^*)-C_0\right)q^*+\frac{1}{2}\left(\tilde{q}-q^*\right)\left(B(q^*)-C_0\right)}. \tag{32}$$

By rewriting (24) as

$$B(q^*) - C_0 = -\lambda q^* B'(q^*)$$

and substituting it into (32), we have

$$\frac{\frac{1}{2} + \lambda}{\frac{1}{2} + \lambda + \frac{\lambda^2}{2}\xi} \leq \rho \leq 1.0 .$$

which is a bound that is tighter than that given by (31).

With a linear benefit function, $\xi = 1$ and (32) is strictly equal. Thus, the bound of the efficiency ratio can be simply reduced to

$$\frac{3}{4} \leq \frac{\frac{1}{2} + \lambda}{\frac{1}{2} + \lambda + \frac{1}{2}\lambda^2} = \rho \leq 1.0 .$$

The lower bound $3/4$ corresponds to the monopoly situation with $\lambda = 1$, which is consistent with that derived in Xiao *et al.* (2007); the upper bound 1.0 corresponds to the socially optimal situation with $\lambda = 0$. When a BOT toll road is faced with a trade-off parameter of $\lambda = 0.5$, the efficiency ratio can be calculated as $\rho = 8/9$.

5. Effects of Returns to Scale in Road Construction

So far, we have examined the properties and the efficiency of Pareto-efficient contracts under constant returns to scale in road construction by assuming that $I(y) = ky$. To look into the effects of decreasing and increasing returns to scale in road construction, we now relax Assumption 3 to consider the following specific construction cost function

$$I(y) = ky^\alpha, \alpha > 0 . \tag{33}$$

Road construction exhibits decreasing returns to scale when $\alpha > 1$ and increasing returns to scale when $0 < \alpha < 1$.

5.1 Properties of Pareto-efficient BOT contracts with returns to scale

We now examine how the returns to scale in road construction affect the properties of the Pareto-efficient BOT contract set. From Proposition 1, we know that $T = \hat{T}$ for any Pareto-efficient BOT contract. Let $(\hat{T}, \tilde{q}, \tilde{y})$ and $(\hat{T}, \overline{q}, \overline{y})$ be the SO and MO solutions, which maximize social welfare, $W(T, q, y)$, and profit, $P(T, q, y)$,

respectively. By subtracting $P(\hat{T}, q, y)$ from $W(\hat{T}, q, y)$, we readily obtain

$$W(\hat{T}, q, y) - P(\hat{T}, q, y) = \hat{T}\left(\int_0^q B(w)\,dw - qB(q)\right). \tag{34}$$

Note that the term on the right-hand-side of eqn. (34) is strictly increasing in q, which implies that $\tilde{q} \geq \overline{q}$, because, if $\tilde{q} < \overline{q}$, then

$$W(\hat{T}, \overline{q}, \overline{y}) = P(\hat{T}, \overline{q}, \overline{y}) + \hat{T}\left(\int_0^{\overline{q}} B(w)\,dw - \overline{q}B(\overline{q})\right) > W(\hat{T}, \tilde{q}, \tilde{y}), \tag{35}$$

where the last inequality follows from the fact that $(\hat{T}, \overline{q}, \overline{y})$ maximizes profit, $P(T, q, y)$. Equation (35) contradicts the assumption that $(\hat{T}, \tilde{q}, \tilde{y})$ is the welfare-maximizing solution.

Suppose that $(\hat{T}, q^*, y^*)$ is a Pareto-efficient BOT contract. Like the constant returns to scale case, (q^*, y^*) solves the Lagrange problem (55) (shown in Appendix 3) for a certain Lagrange multiplier, λ, and taking the derivative in y yields the following first-order condition:

$$\hat{T}\beta(\gamma^*)^2 t'(\gamma^*) = k\alpha(y^*)^{\alpha-1}, \tag{36}$$

where $\gamma^* = q^*/y^*$ is the v/c ratio under the Pareto-efficient BOT contract $(\hat{T}, q^*, y^*)$. Equation (36) under assumption (33) is the counterpart of the Pareto-efficiency condition (13) associated with $I(y) = ky$.

The following proposition reveals a few important properties of Pareto-efficient BOT contracts (a rigorous proof is given in Appendix 4). Their departure from the case with constant returns to scale in road construction is self-evident.

Proposition 5. *Under Assumptions 1 and 2, for any two distinct Pareto-efficient BOT contracts, $(\hat{T}, q^*, y^*)$ and $(\hat{T}, q^{**}, y^{**})$, with $q^* < q^{**}$, we must have*

(a) $y^* < y^{**}$, $p^* > p^{**}$;

(b) $\gamma^* < \gamma^{**}$ and $ASC^* < ASC^{**}$ for $\alpha > 1$; $\gamma^* > \gamma^{**}$ and $ASC^* > ASC^{**}$ for $0 < \alpha < 1$, where ASC is the average social cost defined by (15);

(c) $P^* > P^{**}$, $W^* < W^{**}$;

(d) $ROR^* > ROR^{**}$, where ROR is the rate of return on investment defined as the ratio of the profit to the construction cost

$$ROR = \frac{P}{I} \ (\times 100\%). \tag{37}$$

The properties of the contract curve, Θ, defined by (14) are explained geometrically in Figure 3. Along the direction shown in the figure, the travel demand, road capacity and social welfare increase, while the profit decreases. However, the v/c ratio increases when $\alpha > 1$ (decreasing returns to scale) and decreases when $0 < \alpha < 1$ (increasing returns to scale). As a result, the service quality decreases or increases along the Pareto-efficient frontier from monopoly to the social optimum, namely, the private firm tends to offer higher or lower service quality than the socially optimal level.

5.2 The profit properties at the social optimum

We go further to investigate the profit properties of the socially optimal BOT contract, $\left(\hat{T}, \tilde{q}, \tilde{y}\right)$. In this case, the toll charge is exactly equal to the congestion externality,

$$\tilde{p} = B(\tilde{q}) - \beta t(\tilde{\gamma}) = \beta\tilde{\gamma}t'(\tilde{\gamma}).$$

From eqn. (36), the profit of the private sector can be expressed as

$$P = \left(1 - \frac{1}{\alpha}\right)\hat{T}\tilde{q}\tilde{p}, \tag{38}$$

which reveals that, with as socially optimal BOT contract, $\left(\hat{T}, \tilde{q}, \tilde{y}\right)$, the private firm would earn a positive profit with decreasing returns to scale in road construction ($\alpha > 1$); zero profit with constant returns to scale ($\alpha = 1$); and negative profit with increasing returns to scale ($0 < \alpha < 1$). The profit (positive or negative) is proportional to the total toll revenue with a proportional constant, $(1 - 1/\alpha)$. Equation (38) also implies that the cost recovery ratio (total toll revenue relative to the initial capacity cost) is exactly equal to α, which is consistent with the general rules derived by Mohring and Harwitz (1962).

5.3 The zero-profit constrained Pareto-efficient BOT contract

From the above analysis, the private sector would encounter a negative profit with a socially optimal BOT contract $\left(\hat{T}, \tilde{q}, \tilde{y}\right)$, when there are increasing returns to scale in road construction ($0 < \alpha < 1$). In this case it is of interest to look into the zero-profit-constrained Pareto-efficient BOT contract, $\left(\hat{T}, q_{zp}^{*}, y_{zp}^{*}\right)$, where "zp" stands for "zero-profit".

Before we seek such a zero-profit contract, $\left(\hat{T}, q_{zp}^{*}, y_{zp}^{*}\right)$, we provide the conditions

for its existence. Clearly, this requires that the profit of the private sector with a monopoly optimal BOT contract, $\left(\hat{T}, \overline{q}, \overline{y}\right)$, be positive. In this case, by combining eqn. (2) and eqn. (36), the monopoly profit can be calculated as:

$$P\left(\hat{T}, \overline{q}, \overline{y}\right) = \hat{T}\overline{q}\left(B\left(\overline{q}\right) - \beta t\left(\overline{\gamma}\right) - \frac{\beta}{\alpha}\overline{\gamma}t'\left(\overline{\gamma}\right)\right). \tag{39}$$

Note that, for any road capacity, $y > 0$, when $q \to 0$, the average travel time approaches the free-flow travel time, $t(0)$, and the congestion externality approaches zero. Therefore, $P\left(\hat{T}, \overline{q}, \overline{y}\right) \geq 0$ is guaranteed under the following condition:

$$B(0) - \beta t(0) > 0. \tag{40}$$

Intuitively, condition (40) is not practically restrictive, because we can reasonably expect that there is a positive potential traffic demand with a free-flow travel time. Otherwise, it is meaningless to build a new highway.

When condition (40) is met, there exists a Pareto-efficient contract, $\left(\hat{T}, q_{zp}^{*}, y_{zp}^{*}\right)$, with zero profit for $0 < \alpha < 1$. In this case, the solution, $\left(q_{zp}^{*}, y_{zp}^{*}\right)$, or equivalently $\left(q_{zp}^{*}, \gamma_{zp}^{*}\right)$ can be determined by eqn. (36) and the following zero-profit condition:

$$P\left(q_{zp}^{*}, y_{zp}^{*}\right) = \hat{T}q_{zp}^{*}\left(B\left(q_{zp}^{*}\right) - \beta t\left(\gamma_{zp}^{*}\right)\right) - I\left(y_{zp}^{*}\right) = 0.$$

The corresponding toll charge is given by

$$p_{zp}^{*} = B\left(q_{zp}^{*}\right) - \beta t\left(\gamma_{zp}^{*}\right) = \frac{1}{\alpha}\beta\gamma_{zp}^{*}t'\left(\gamma_{zp}^{*}\right) > \beta\gamma_{zp}^{*}t'\left(\gamma_{zp}^{*}\right). \tag{41}$$

The charge is thus higher than the corresponding congestion externality for cost recovery. With increasing returns to scale in road construction, the zero-profit Pareto-efficient contract, $\left(\hat{T}, q_{zp}^{*}, y_{zp}^{*}\right)$, with projection $\left(q_{zp}^{*}, y_{zp}^{*}\right)$ in the (q, y) space shown in Figure 3(b), is thus a critical point; any Pareto-efficient contract, $\left(\hat{T}, q^{*}, y^{*}\right)$, with $q^{*} < q_{zp}^{*}$ would result in a positive profit for the private firm. Otherwise, the profit becomes negative.

6. Governmental Regulations

So far, we have examined the basic properties of Pareto-efficient BOT contracts for a road project. In this section, we focus on the regulatory mechanism that induces the private firm to choose a predetermined Pareto-optimal solution voluntarily, and we identify the regulatory regime that establishes a situation in which the regulatory

outcomes are efficient. In all regulations considered below, we assume that the government already predetermines a targeted Pareto-efficient BOT contract, $\left(\hat{T}, q^*, y^*\right)$ or $\left(\hat{T}, p^*, y^*\right)$, where

$$p^* = B\left(q^*\right) - \beta t\left(q^*/y^*\right) = B\left(q^*\right) - \beta t\left(\gamma^*\right). \tag{42}$$

6.1 Rate-of-return regulation

We first investigate a rate-of-return (ROR) regulatory mechanism, under which the private firm is allowed to earn no more than a "fair" rate of return on its investment. The private firm is free to choose a combination of the BOT variables as long as its profits do not exceed this fair rate.

Let s denote the ROR on the firm's investment and s^*, $s^* \geq 0$, be the ROR determined by (37) for a given Pareto-efficient BOT contract, $\left(\hat{T}, q^*, y^*\right)$. Under the ROR regulation, the government restricts the ROR on the investment of the private firm as follows:

$$s = \frac{Tqp - I\left(y\right)}{I\left(y\right)} \leq s^*. \tag{43}$$

Under the above ROR regulation, the problem of the profit-maximizing private firm can be expressed as:

$$\max_{0 \leq T \leq \hat{T}, p \geq 0, y \geq 0} Tpq - I\left(y\right) \tag{44}$$

subject to condition (43). Now we have the following proposition disclosing the behavior of the private sector under the ROR regulation (43) (the proof is relegated to Appendix 5).

Proposition 6. *Let* $\left(\hat{T}, q^*, y^*\right)$ *be a non-monopoly Pareto-efficient solution with a corresponding ROR* s^*. *Then, under the ROR regulation (43), the private sector would choose a non-Pareto-efficient BOT contract,* $\left(\hat{T}, q, y\right)$, *with* $q < q^*$ *and* $y > y^*$.

Proposition 6 states that the private sector's choice deviates from the Pareto-efficient outcome. Meanwhile, the resulting v/c ratio, $\gamma = q/y < \gamma^*$, or the service quality of the highway under the ROR regulation (43) will be higher than the Pareto-efficient level. Since the private sector makes more profit than at the Pareto-efficient level, we have

$$\hat{T}pq - I(y) > \hat{T}p^*q^* - I(y^*).\tag{45}$$

Equation (45) implies that $p > p^*$ from the strictly increasing assumption of $I(\cdot)$.

In summary, the ROR regulation is inefficient since the private firm would select a higher road capacity and a higher toll level than the targeted Pareto-efficient BOT solution, $(\hat{T}, q^*, y^*)$. It is worth noting that the ROR regulation creates an incentive for the private sector to choose an inefficiently high capacity. This over-investment is, in fact, an instance of the Averch-Johnson effect (Averch and Johnson, 1962).

6.2 Price-cap regulation

The price-cap regulation allows the private sector to set a price below or equal to a price ceiling set by the government. For a targeted Pareto-efficient BOT solution, $(\hat{T}, q^*, y^*)$, the toll ceiling, p^*, is given by (42). In this case one can easily see that the private firm will choose the concession period to be $\hat{T}$ whenever the solution, $(\hat{T}, q^*, y^*)$, is profitable and selects a toll equal to p^*. Therefore, under the price-cap regulation constraint, the private firm's problem is to maximize its profit as given below:

$$\max_{q \geq 0, y \geq 0} P = \hat{T}p^*q - I(y)$$

subject to $B(q) - \beta t(q/y) = p^*$. By viewing y as a function of q and taking the derivative of P in q at $q = q^*$, we have

$$\left.\frac{\mathrm{d}P}{\mathrm{d}q}\right|_{q=q^*} = B(q^*) + q^*B'(q^*) - \beta t\left(\frac{q^*}{y^*}\right) - \beta\frac{q^*}{y^*}t'\left(\frac{q^*}{y^*}\right).$$

If the targeted Pareto-efficient solution, $(\hat{T}, q^*, y^*)$, is the MO solution, then the private firm will choose $(\hat{T}, q^*, y^*)$ straightforwardly to maximize its profits.

If $(\hat{T}, q^*, y^*)$ is a non-MO Pareto-efficient solution, we must have $\mathrm{d}P/\mathrm{d}q < 0$ at $q = q^*$ for the following reason. First, note that $\mathrm{d}P/\mathrm{d}q = 0$ at $q = q^*$ is excluded since it is a non-MO solution. If, however, $\mathrm{d}P/\mathrm{d}q > 0$ at $q = q^*$, then increasing q from q^* will increase profits and social welfare from (34). Therefore, the private sector would select $(\hat{T}, q, y)$ with $q < q^*$ to earn more profits under the price cap

regulation of $p \leq p^*$. In addition, from the binding price-cap constraint, it is clear to see that $\gamma > \gamma^*$, and, as a result, that $y < y^*$.

In summary, the price-cap regulation is also inefficient and cannot induce the private firm to choose a Pareto-efficient BOT solution unless the targeted Pareto-efficient contract is the MO solution. Generally, under the price-cap regulation, the private sector would offer lower road capacity and lower service quality than would the Pareto-efficient solution, $\left(\hat{T}, q^*, y^* \right)$.

6.3 Capacity regulation

For any Pareto-efficient BOT solution, $\left(\hat{T}, q^*, y^* \right)$, from Proposition 5, $\bar{y} \leq y^*$, which means that it is pointless to regulate $y \leq y^*$ because the private firm would surely choose the monopoly optimal $\left(\hat{T}, \bar{q}, \bar{y} \right)$. Thus, in what follows, we examine the behavior of the private firm under the capacity regulation of $y \geq y^*$ $\left(> \bar{y} \right)$ only.

To be practically sensible, we suppose that the government predetermines a profitable target solution, $\left(\hat{T}, q^*, y^* \right)$, in setting up the capacity regulation. In this case, the private sector will certainly choose the concession period to be $\hat{T}$. Clearly, at any capacity level, the best response of the private firm is to set a monopoly price; that is, for any $y > 0$, the private firm will choose demand q such that the first-order condition (11) is satisfied. Viewing q as a function of y given by (11) and taking the derivative of profit, P, $P = \hat{T} q \left(B(q) - \beta t(q/y) \right) - I(y)$, in y gives rise to

$$\frac{\mathrm{d}P}{\mathrm{d}y} = \frac{\partial P}{\partial q} \frac{\mathrm{d}q}{\mathrm{d}y} + \frac{\partial P}{\partial y} = \frac{\partial P}{\partial y} = \hat{T} \beta \left(\frac{q}{y} \right)^2 t' \left(\frac{q}{y} \right) - I'(y), \tag{46}$$

where $\partial P / \partial q = 0$ from the first-order condition (11). If the private sector chooses $\left(\hat{T}, q, y \right)$ with $y > y^*$ to maximize its profits, then $\mathrm{d}P / \mathrm{d}y = 0$ in (46), or the Pareto-efficiency condition (36) is satisfied. With eqn. (11), we conclude that $\left(\hat{T}, q, y \right)$ is the MO Pareto-efficient contract with $y > y^*$, which conflicts with Part (a) of Proposition 5. Therefore, under the capacity regulation of $y \geq y^*$, the private sector must choose $y = y^*$.

Conditional on $y = y^*$, if the private firm can increase profits by lowering the toll,

the demand will increase and social welfare will increase as well, contradicting the Pareto efficiency of $(\hat{T}, q^*, y^*)$. The private firm would not choose $p = p^*$ either, because, otherwise, $(\hat{T}, q^*, y^*)$ must be the MO Pareto-efficient contract for the same reason above. Therefore, the private sector will choose a profit-maximizing toll that is certainly higher than p^* conditional on $y = y^*$. Meanwhile, the service quality is made higher than the Pareto-efficient level.

6.4 Demand regulation

Under the demand regulation by the government for a targeted Pareto-efficient solution, $(\hat{T}, q^*, y^*)$, the private firm is allowed to make choices subject to the resulting realized traffic volume level, $q \geq q^*$.

Like before, we consider the practically meaningful case that the solution $(\hat{T}, q^*, y^*)$ is profitable. The private firm will choose the concession period to be $\hat{T}$. From eqn. (34), we can see that there is no strictly profitable deviation from the Pareto-efficient solution, $(\hat{T}, q^*, y^*)$, for the private firm to choose $(\hat{T}, q, y)$ with $q \geq q^*$, because the social welfare must be improved under constraint $q \geq q^*$ whenever the profit is increased, which contradicts the Pareto-optimality of $(\hat{T}, q^*, y^*)$.

We further confirm that the Pareto-efficient BOT solution, $(\hat{T}, q^*, y^*)$, is unique and maximizes private profits under regulation $q \geq q^*$. Since there is no profitable deviation, any choice of $(\hat{T}, q, y)$ by the private sector must realize the same profit and social welfare as $(\hat{T}, q^*, y^*)$, simultaneously, namely,

$$\hat{T}\left(\int_0^q B(w)\,dw - \beta q t\left(\frac{q}{y}\right) \right) - I(y) = W(q^*, y^*) \tag{47}$$

and

$$\hat{T}\left(qB(q) - \beta q t\left(\frac{q}{y}\right) \right) - I(y) = P(q^*, y^*). \tag{48}$$

From Definition 1, we know that $(\hat{T}, q, y)$ is Pareto-efficient. Subtracting eqn. (48) from eqn. (47) on each side gives rise to:

$$\int_0^q \left(B(w) - wB(q) \right) \mathrm{d}w = \int_0^{q^*} \left(B(w) - wB(q^*) \right) \mathrm{d}w. \tag{49}$$

Note that function $\int_0^q B(w)\mathrm{d}w - qB(q)$ is strictly increasing in q. Thus, eqn. (49) implies that $q = q^*$. By substituting $y = q^*/\gamma$ into condition (36), we readily obtain $\gamma = \gamma^*$, and thus $y = y^*$. Therefore, under demand regulation $q \geq q^*$, the Pareto-efficient solution, $\left(\hat{T}, q^*, y^* \right)$, is the unique profit-maximizing BOT solution to be selected by the private firm.

Proposition 7. *Under Assumptions 1 and 2, the resulting choice by the private firm under the demand regulation, $q \geq q^*$, is $\left(\hat{T}, q^*, y^* \right)$ or Pareto-efficient.*

Proposition 7 shows that the demand regulation is an appealing regulatory choice. The government needs only to set a minimum level of demand and to let the private firm freely choose a preferable combination of road capacity, toll charge and concession period for profit maximization.

6.5 Markup charge regulation

Under a markup charge regulation, the firm is allowed to earn a certain amount of profit on each unit of output it sells in an economic setting. This is equivalent to the return-on-output (ROO) regulation (Train, 1991). In the earlier special case with constant returns to scale in road construction, the markup charge, p_2, is given by (20). In the general case, the markup charge can be defined as

$$p_2 = \frac{P(T, q, y)}{Tq},$$

which means that the markup charge is the amount of profit earned from each unit of realized demand (each trip) during the concession period. For a given Pareto-efficient BOT contract $\left(\hat{T}, q^*, y^* \right)$, the markup charge regulation refers to setting a ceiling of the markup charge during the concession period as follows:

$$p_2 \leq p_2^* = \frac{P\left(\hat{T}, q^*, y^* \right)}{\hat{T}q^*} \tag{50}$$

From Proposition 7, we know that $q \leq q^*$ for the private sector to earn more profit. In view of $T \leq \hat{T}$, it is impossible for the private firm to choose (T, q, y) to earn more profit under (50), namely,

$$P(T, q, y) = Tqp_2 \leq \hat{T}q^*p_2^* = P\left(\hat{T}, q^*, y^* \right)$$

for any (T, q, y) under the markup charge regulation condition (50). We thus have the following proposition.

Proposition 8. *Under Assumptions 1 and 2, the resulting choice by the private firm under the markup charge regulation, $p_2 \leq p_2^*$, is $\left(\hat{T}, q^*, y^*\right)$ or Pareto-efficient.*

Proposition 8 reveals an alternative regulatory regime for the government to induce Pareto-optimal outcomes just by restricting the markup charge and letting the private firm choose any combination of the BOT variables.

6.6 Summary of regulatory outcomes and alternative auction strategies

For a targeted Pareto-efficient solution, $\left(\hat{T}, q^*, y^*\right)$, the outcomes of the five alternative regulatory regimes examined so far are summarized in Table 1, where $p^* = B(q^*) - \beta t(q^*/y^*)$, $\text{ROR}^* = P\left(\hat{T}, q^*, y^*\right)/I(y^*)$ and $p_2^* = P\left(\hat{T}, q^*, y^*\right)/\hat{T}q^*$.
The results of ROR and price-cap regulations are actually instances of the general results in economic settings (Train, 1991).

Ubbels and Verhoef (2008) discussed the efficiencies of various auction mechanisms for a private road with an un-tolled alternative through numerical examples. Their simulation results revealed that the auction to minimize generalized travel costs and subsidies divided by total traffic demand would approach the social optimum (they called it the second-best outcome), and the auction to minimize generalized travel costs could result in the social optimum with the zero-profit constraint (the second-best outcome with the zero-profit constraint). For a single road with constant returns to scale in road construction, the social optimums with and without the zero-profit constraint are identical. Therefore, the auction to minimize generalized travel costs does approach the social optimum, which is equivalent to maximizing travel demands. Proposition 7 shows that, for a more general construction cost function, any Pareto optimum including the social optimum can be achieved via the demand regulation, and Proposition 8 is an alternative to the demand regulation.

7. Numerical Examples

In this section, three simple examples with different returns to scale in road construction are presented to demonstrate the results obtained so far. The following BPR link travel time function is used:

$$t(q,y) = t_0\left(1.0 + 0.15\left(\frac{q}{y}\right)^4\right),$$

where the free-flow travel time for the new highway is $t_0 = 0.5$ (h). Time is converted into money with a value of time of $\beta = 100$ (HK\$/h). The benefit function takes the following negative exponential form:

$$B(q) = -\frac{1}{b}\ln\left(\frac{q}{Q}\right), \quad b > 0,$$

where Q is the potential demand, $Q = 1.0 \times 10^4$ (veh/h), b is a scaling parameter reflecting the sensitivity of demand to the full trip price, $b = 0.04$. It is easy to check that the benefit function satisfies Assumption 1(a). The construction cost function for the highway is assumed to take the following power form of capacity:

$$I(y) = k_\alpha t_0 y^\alpha, \alpha > 0,$$

where parameter α captures the returns to scale in road construction: increasing returns to scale (IRS) with $0 < \alpha < 1$, constant returns to scale (CRS) with $\alpha = 1$ and decreasing returns to scale (DRS) with $\alpha > 1$, as mentioned before. The free-flow travel time, t_0, is proportional to the length of the road and k_α is the construction cost parameter corresponding to the returns to scale, α. The values of parameter α and k_α in Table 2 are used without necessarily representing a realistic setting. Finally, the road life, $\hat{T}$, is assumed to be 30 (years), or $\hat{T} = 1.314 \times 10^5$ (h) by assuming the number of operating hours per year to be $4380 = 12 \times 365$ (h).

For the three cases with increasing, constant and decreasing returns to scale (denoted as IRS, CRS and DRS, respectively) listed in Table 2, we first view the maximized social welfare with a certain profit constraint, $P(T,q,y) \geq \tilde{P}$, as a parametric function of the concession period, T. Namely, we look at the result of the following maximization problem for any predetermined concession period:

$$W_{\max}(T) = \max_{q \geq 0, y \geq 0}\left\{W(T,q,y) : P(T,q,y) \geq \tilde{P}\right\}.$$

Figure 4 shows how the maximized social welfare changes with the concession period, T, when $\tilde{P} = 0$ and $\tilde{P} = 0.8 \times 10^9$ (HK\$), respectively. In the figure, it is clear that $W_{\max}$ is increasing in T and it reaches the maximum at $T = \hat{T}$ with a binding profit constraint of $P = \tilde{P}$. The Pareto-efficient BOT contract does require the concession period to be exactly the whole road life.

Figures 5-7 report the Pareto-optimal solution sets or contract curves in the two-dimensional (γ, q) space, with bold curves connecting the MO and SO points

for the three Cases of IRS, CRS and DRS, respectively, where the thick and thin contours represent social welfare and profit, respectively. The corresponding representative numerical results for MO and SO are shown in Table 3. It is clear that, when moving from SO to MO, the service quality decreases (the v/c ratio increases) with increasing returns to scale in road construction; it remains unchanged with constant returns to scale; and it increases with decreasing returns to scale. For the IRS case, the zero-profit Pareto-efficient BOT contract, $\left(\hat{T}, q_{zp}^*, y_{zp}^*\right)$, can be readily obtained with $\hat{T} = 30$ (year), $q_{zp}^* = 1206$ (veh/h), $y_{zp}^* = 2399$ (veh/h), and the corresponding toll charge of $p_{zp}^* = 2.40$ (HK$).

Figure 8 compares the outcomes of the private firm's choices under various regulatory regimes in the case with constant returns to scale in road construction (similar results can be obtained for the other two cases). For a predetermined Pareto-efficient solution, $\left(\hat{T}, q^*, y^*\right) =$ (30 years, 849 veh/h, 1230 veh/h), denoted as point A on the contract curve, we obtain the corresponding toll charge of $p^* = 9.96$ (HK$), the rate of return of $s^* = 12.16\%$ and the markup charge of $\tau_2^* = 3.34$ (HK$). The five bold curves, $L_1 \sim L_5$, indicate the binding constraints in the (γ, q) space, associated with the five regulatory regimes: (1) $p \le p^* = 9.96$ (HK$); (2) $s \le s^* = 12.16$ (%); (3) $y \ge y^* = 1230$ (veh/h); (4) $q \ge q^* = 849$ (veh/h) and (5) $p_2 \le p_2^* = 3.34$ (HK$). The corresponding feasible domains are located above these curves in the (γ, q) space. As seen from the figure, each regulatory binding curve is tangent to a profit contour; the corresponding profit represents the maximum profit earned by the private firm under the given regulatory control. The choices made by the private firms under each regulatory regime are identified by the corresponding tangent point, denoted as $A_1 \sim A_5$ respectively. Specifically, under the price-cap regulation, the private firm chooses point A_1 (30 years, 723 veh/h, 769 veh/h), with a lower capacity and a lower service quality, producing a maximum profit of about 4.90×10^8 (HK$). Under the ROR regulation, the private firm chooses point A_2 (30 years, 502 veh/h, 1725 veh/h), with a higher capacity, a higher toll charge and a higher service quality, producing a maximum profit of about 5.80×10^8 (HK$). Under the capacity regulation, the private firm chooses point A_3 (30 years, 487 veh/h, 1230 veh/h), with a higher toll charge and a higher service quality, producing a maximum profit of about 8.90×10^8 (HK$). In contrast, under the demand and markup charge regulations, the tangent points, A_4 and A_5, chosen by the private firm, coincide with Point A, or the private firm chooses the predetermined Pareto-efficient BOT solution, which is the

unique choice to maximize its profits.

8. Conclusions

The concession period, capacity and toll charge are three primary variables for a BOT toll road project. They determine the social welfare for the whole society during the whole life of the road and the profit of the private firm during the concession period. We analyzed the properties of Pareto-efficient BOT contracts via a bi-objective programming approach and established several key results. First, any Pareto-efficient BOT contract requires that the concession period should be the whole life of the road. Secondly, with constant returns to scale, the volume-capacity ratio and thus the service quality at any Pareto-efficient BOT contract coincide with the socially optimal levels; and the average social cost per trip is also constant along the Pareto-optimal frontier. We further established the efficiency bound of any Pareto-efficient BOT contract in terms of social welfare in comparison with the perfect social optimum. With a simple extension of the construction cost function, we proved that the private firm prefers to offer lower (higher) service quality than the socially optimal level when there are increasing (decreasing) returns to scale in road construction.

A variety of government regulatory regimes were investigated. Generally, both price-cap and rate-of-return regulations result in inefficient outcomes: the private firm tends to offer a lower road capacity and a lower service quality under the price-cap regulation, while it chooses a higher service quality, a higher capacity and a higher toll charge under the rate-of-return regulation than under the corresponding Pareto-efficient solution. The road capacity regulation is also inefficient. In contrast, we proved that both the demand and markup charge regulations lead to Pareto-optimal outcomes.

Finally, we now point out that the Pareto-efficient BOT contracts are complete in an ideal world without transaction costs, in which future contingencies can be described explicitly. Therefore, one of the main avenues in our future research is to model unforeseen contingencies using the incomplete contracting approach. More meaningful extensions include developing a model to incorporate the effects of user heterogeneity and to identify and allocate the project's risks among the public and the private sectors.

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Appendix 1. Effect of discounting factor

Here we show that the use of a discounting factor on the stream of future revenue does not alter our results. Assume time τ is continuous and let r be an interest rate of reference used for discounting all monetary units to equivalent values at $\tau = 0$. The social welfare (5) and the profit (4) functions should be rewritten as

$$
\begin{aligned}
W(T,q,y) &= \int_0^T S(q,y)e^{-r\tau}\mathrm{d}\tau + \int_T^{\hat{T}} \tilde{S}(y)e^{-r\tau}\mathrm{d}\tau - I(y) \\
&= \frac{1-e^{-rT}}{r}S(q,y) + \left(\frac{1-e^{-r\hat{T}}}{r} - \frac{1-e^{-rT}}{r}\right)\tilde{S}(y) - I(y)
\end{aligned}
\tag{51}
$$

$$
P(T,q,y) = \int_0^T R(q,y)e^{-r\tau}\mathrm{d}\tau - I(y) = \frac{1-e^{-rT}}{r}R(q,y) - I(y).
\tag{52}
$$

Denote $L = \left(1-e^{-rT}\right)/r$ and $\hat{L} = \left(1-e^{-r\hat{T}}\right)/r$. Then, the bi-objective programming problem (8) would not change. It is clear that our results remain valid.

Appendix 2. Proof of Proposition 1

Suppose that $\left(T^*, q^*, y^*\right) \in \Omega$ is a Pareto-efficient BOT contract and $T^* < \hat{T}$. Denote q_1 as the maximizer of $S(q, y^*)$ given by (6). From the first-order condition, we have

$$
p = B(q_1) - \beta t(q_1, y^*) = \beta q_1 \frac{\partial t(q_1, y^*)}{\partial q} > 0,
$$

which implies that, if $q^* = q_1$, then $T^* = \hat{T}$, because if $T^* < \hat{T}$ then $\left(T^*, q^*, y^*\right)$ is strictly dominated by $\left(\hat{T}, q^*, y^*\right)$, or the private firm can increase its profits by prolonging the concession period without changing the interest of the public sector.

We now prove $T^* = \hat{T}$ if $q^* \neq q_1$. To see this, we show that any $\left(T^*, q^*, y^*\right)$ with $T^* < \hat{T}$ must not be Pareto-optimal, *i.e.,* it must be dominated by another feasible BOT triple. First, from Assumption 1, we know that, for any given $y > 0$, both the unit-time social welfare $S(q, y)$ given by (6) and the following unit-time toll revenue

$$
R(q,y) = q\big(B(q) - \beta t(q,y)\big)
$$

are strictly concave in q. Any convex combination of q^* and q_1 can be denoted as

$$
q^{**} = \eta q^* + (1-\eta)q_1, \quad \eta \in (0,1).
$$

From the concavities of $S(q, y)$ and $R(q, y)$, we have

$$W\left(T, q^{**}, y^*\right) > TS\left(q^*, y^*\right)\left(\eta - (1-\eta)\varepsilon_1\right) + \hat{T}S\left(q_1, y^*\right) - I\left(y^*\right) \tag{53}$$

and

$$P\left(T, q^{**}, y^*\right) > TR\left(q^*, y^*\right)\left(\eta + (1-\eta)\varepsilon_2\right) - I\left(y^*\right), \tag{54}$$

where $\varepsilon_1 = S\left(q_1, y^*\right) / S\left(q^*, y^*\right)$ and $\varepsilon_2 = R\left(q_1, y^*\right) / R\left(q^*, y^*\right)$. Consider functions

$$f(\eta) = \max\left\{\eta - (1-\eta)\varepsilon_1, \eta + (1-\eta)\varepsilon_2, 1\right\}$$

and

$$g(\eta) = \min\left\{\eta - (1-\eta)\varepsilon_1, \eta + (1-\eta)\varepsilon_2\right\}$$

Note that $f(\eta)$, $g(\eta)$ and $f(\eta)/g(\eta)$ all approach 1 when $\eta \to 1^-$, We can always choose a $\eta_0 \in (0,1)$ such that $f(\eta_0)/g(\eta_0) < \hat{T}/T^*$ and $g(\eta_0) > 0$ since $\hat{T}/T^* > 1$. Let

$$T^{**} = T^*/g(\eta_0)$$

and, at $T = T^{**}$, inequalities (53) and (54) can be changed to

$$W\left(T^{**}, q^{**}, y^*\right) > W\left(T^*, q^*, y^*\right),$$

$$P\left(T^{**}, q^{**}, y^*\right) > P\left(T^*, q^*, y^*\right),$$

which contradicts the Pareto-optimality of $\left(T^*, q^*, y^*\right)$. The proof is completed. □

Appendix 3. Proof of Proposition 2

Proof. Suppose that $\left(\hat{T}, q^*, y^*\right)$ is a Pareto-efficient contract. Then, $\left(q^*, y^*\right)$ solves the following Lagrange problem

$$L(q, y, \lambda) = \hat{T}\left(\int_0^q B(w)\,dw - \beta qt\left(\frac{q}{y}\right)\right) - I(y)$$

$$+ \eta\left(\left(\hat{T}q\left(B(q) - \beta t\left(\frac{q}{y}\right)\right) - I(y)\right) - P^*\right), \tag{55}$$

where $P^* = P\left(\hat{T}, q^*, y^*\right)$ and $\eta \geq 0$ is the Lagrange multiplier. We have the following first-order conditions:

$$\frac{\partial L}{\partial q} = (1+\eta)\hat{T}B\left(q^*\right) - (1+\eta)\hat{T}\left(\beta t\left(\frac{q^*}{y^*}\right) + \beta\frac{q^*}{y^*}\frac{\partial t\left(q^*/y^*\right)}{\partial q}\right)$$

$$+ \eta\hat{T}q^*B'\left(q^*\right) = 0 \tag{56}$$

and

$$\frac{\partial L}{\partial y} = (1+\eta)\left[\hat{T}\beta \left(\frac{q^*}{y^*}\right)^2 \frac{\partial t\left(q^*/y^*\right)}{\partial y} - k \right] = 0. \tag{57}$$

Denote $\gamma^* = q^*/y^*$ as the v/c ratio. Since $\eta \geq 0$, condition (57) can be reduced to

$$\hat{T}\beta \left(\gamma^*\right)^2 t'\left(\gamma^*\right) = k. \tag{58}$$

Since $t(\cdot)$ is strictly convex, eqn. (58) admits a unique solution, which implies that $\gamma^* = \tilde{\gamma}$. This completes the proof. □

Appendix 4. Proof of Proposition 5

By the definition of γ, the Pareto efficiency condition (36) can be rewritten as

$$\hat{T}\beta \left(\gamma^*\right)^{\alpha+1} t'\left(\gamma^*\right) = k\alpha \left(q^*\right)^{\alpha-1}$$

Note that the function $\hat{T}\beta (\gamma)^{\alpha+1} t'(\gamma)$ is strictly increasing in γ since $\alpha > 0$ and $t(\gamma)$ is increasing and convex. Therefore, for any two distinct Pareto-efficient BOT contracts, $\left(\hat{T}, q^*, y^*\right)$ and $\left(\hat{T}, q^{**}, y^{**}\right)$, when $\alpha > 1$, $q^* < q^{**}$ implies that $\gamma^* < \gamma^{**}$, which, from condition (36), implies that $y^* < y^{**}$; while when $0 < \alpha < 1$, $q^* < q^{**}$ implies that $\gamma^* > \gamma^{**}$, which yields $y^* < y^{**}$. Thus, $y^* < y^{**}$ for any $\alpha > 0$ in (a) is obtained. In addition, the average social cost defined by (15) for any a given $\left(\hat{T}, q^*, y^*\right)$ can be calculated as

$$\text{ASC}^* = \beta t\left(\gamma^*\right) + \frac{\beta}{\alpha} \gamma^* t'\left(\gamma^*\right)$$

Since ASC is a strictly increasing function of γ, we thus conclude that (b) is true.

From (34), if $P^* \leq P^{**}$, then $q^* < q^{**}$ must induce $W^* < W^{**}$, which contradicts the Pareto-optimality of $\left(\hat{T}, q^*, y^*\right)$. Thus $P^* > P^{**}$. Similarly, we also have $W^* < W^{**}$ whenever $q^* < q^{**}$. Thus, (c) is proved.

From Assumption 1, $B(q)$ is strictly decreasing in q and $t(\gamma)$ is strictly increasing in γ. For $\alpha > 1$, from eqn. (2), $q^* < q^{**}$ and $\gamma^* < \gamma^{**}$ directly derive $p^* > p^{**}$. To prove the case with $0 < \alpha < 1$, using eqn. (36), we first rewrite the profit function for a given $\left(\hat{T}, q^*, y^*\right)$ as

$$P^* = \hat{T}q^* \left(p^* - \frac{\beta}{\alpha} \gamma^* t'\left(\gamma^*\right) \right) \tag{59}$$

From (c), we know that $q^* < q^{**}$ implies that $P^* > P^{**}$. Thus,

$$p^* - \frac{\beta}{\alpha} \gamma^* t'\left(\gamma^*\right) > p^{**} - \frac{\beta}{\alpha} \gamma^{**} t'\left(\gamma^{**}\right), \tag{60}$$

Function $\gamma t'(\gamma)$ is strictly increasing and $\gamma^* > \gamma^{**}$. We obtain $p^* > p^{**}$. Therefore, $p^* > p^{**}$ in (a) for any $\alpha > 0$ is proved.

Finally, (a) and (c) together imply (d) since the construction cost, $I(y)$, strictly increases with y. The whole proof of Proposition 5 is completed. □

Appendix 5. Proof of Proposition 6

We first note that condition (43) is binding under the best response of the private firm since s^* is a realizable rate-of-return associated with the predetermined Pareto-efficient contract, $\left(\hat{T}, q^*, y^*\right)$. We thus have

$$Tqp - I(y) = s^* I(y). \tag{61}$$

Therefore, under the binding condition (43), the profit-maximizing problem (44) is equivalent to maximizing the investment, $I(y)$, or equivalently maximizing the capacity, y, since $I(y)$ is strictly increasing in y. Now we view y as a function of T and q as determined by (61) and p is a function of q given by (2). We take the derivatives of y with respect to T and q, yielding

$$\left(\left(s^* + 1\right) I'(y) - \beta T \frac{q^2}{y^2} t'\left(\frac{q}{y}\right) \right) \frac{\partial y}{\partial T} = q \left(B(q) - \beta t\left(\frac{q}{y}\right) \right) \tag{62}$$

$$\left(\left(s^* + 1\right) I'(y) - \beta T \frac{q^2}{y^2} t'\left(\frac{q}{y}\right) \right) \frac{\partial y}{\partial q} = T \left(B(q) + q B'(q) - \beta t\left(\frac{q}{y}\right) - \beta \frac{q}{y} t'\left(\frac{q}{y}\right) \right). \tag{63}$$

If the concession period, T, for maximizing the profit or equivalently for maximizing the capacity is an interior point or $T \in \left(0, \hat{T}\right)$, then $\partial y / \partial T = 0$. As a result, the right-hand side term of eqn. (62) must be zero or we must have $p(q, y) = 0$ from eqn. (2), which conflicts with the binding condition (43). This means that the private firm must choose the concession period to be $\hat{T}$ or zero under the ROR regulation (43). Clearly, the choice of no concession period is out of question. We thus conclude that the private sector will choose a whole road-life concession period, $T = \hat{T}$, under the ROR regulation (43).

If $\left(\hat{T}, q^*, y^*\right)$ happens to be a solution of problem (44), we have $\partial y / \partial q = 0$ and hence the term in the bracket on the left-hand side of eqn. (63) equals zero, which corresponds to the monopoly optimality conditions (11). In addition, $\left(\hat{T}, q^*, y^*\right)$ is a given Pareto-efficient solution and thus satisfies the Pareto-efficiency condition (36). These two observations allow us to conclude that $\left(\hat{T}, q^*, y^*\right)$ is the MO solution. This conclusion in turn implies that, if $\left(\hat{T}, q^*, y^*\right)$ is a non-MO Pareto-efficient solution, then it must not be the optimal solution of problem (44) or $\left(\hat{T}, q, y\right) \neq \left(\hat{T}, q^*, y^*\right)$ and $\partial y / \partial q \neq 0$.

If $\left(\hat{T}, q^*, y^*\right)$ is a non-MO Pareto-efficient solution, then we have $\partial y / \partial q < 0$ at $\left(\hat{T}, q^*, y^*\right)$ for the following reason. First, $\partial y / \partial q = 0$ is excluded from the above analysis. In addition, $\partial y / \partial q > 0$ at $\left(\hat{T}, q^*, y^*\right)$ implies that increasing the demand can increase the capacity under condition (61). As a result, the profit, $P\left(\hat{T}, q, y\right) = s^* I(y)$, will increase. Also, from eqn. (34), the social welfare must strictly increase from $W\left(\hat{T}, q^*, y^*\right)$ when both the demand and profit increase. These results contradict the fact that $\left(\hat{T}, q^*, y^*\right)$ is Pareto-optimal. Therefore, under the ROR $s \leq s^*$ associated with the given non-MO Pareto-efficient solution, $\left(\hat{T}, q^*, y^*\right)$, the private sector will choose a BOT contract, $\left(\hat{T}, q, y\right)$, with $q < q^*$ and $y > y^*$. Thus, Proposition 6 is proved. □

3. Table

Table 1. Summary of Regulatory Outcomes

Regulatory Regime	Choices of Private Firm	Pareto Efficiency
Price-Cap $\left(p \le p^*\right)$	$y < y^*, p = p^*, T = \hat{T}$	No
Rate of return $\left(s \le s^*\right)$	$y > y^*, p > p^*, T = \hat{T}$	No
Capacity $\left(y \ge y^*\right)$	$y = y^*, p > p^*, T = \hat{T}$	No
Demand $\left(q \ge q^*\right)$	$y = y^*, p = p^*, T = \hat{T}$	Yes
Markup charge $\left(p_2 \le p_2^*\right)$	$y = y^*, p = p^*, T = \hat{T}$	Yes

Table 2. Returns to scale in road construction and corresponding parameter values

Returns to scale in road construction	Increasing returns to scale (IRS)	Constant returns to scale (CRS)	Decreasing returns to scale (DRS)
α	0.80	1.00	1.20
$k_\alpha \left(10^6 \text{HK\$/h} \cdot (\text{veh/h})\right)$	1.50	1.20	0.25

Table 3. Returns to scale in road construction and numerical results for SO and MO

Solution Variable	IRS		CRS		DRS	
	SO	MO	SO	MO	SO	MO
Demand (veh/h)	1230	445	970	357	955	368
Capacity (veh/h)	2448	851	1414	520	1375	549
Toll (HK$)	1.90	27.25	6.65	31.65	6.97	31.05
Social welfare (10^9 HK$)	3.96	2.89	3.19	2.34	3.28	2.47
Profit (10^9 HK$)	-0.07	1.43	0	1.18	0.15	1.26
Volume to capacity ratio	0.50	0.52	0.68	0.68	0.69	0.67
ASC (HK$)	52.9	53.4	58.5	58.5	60.5	59.1
ROR (%)	-0.8	57.6	0	108.3	20.3	519.2
Markup charge (HK$)	-0.45	24.41	0	25.00	1.19	26.03
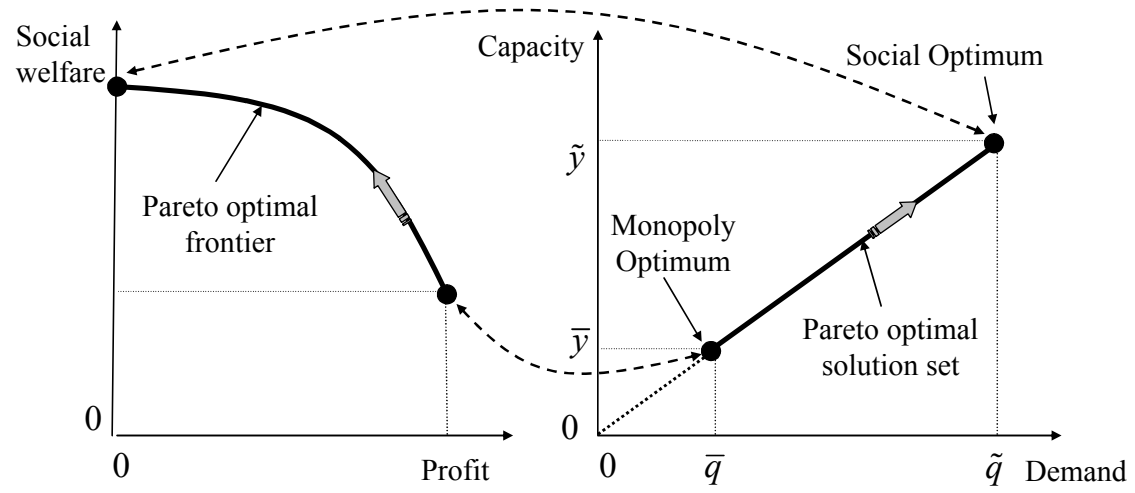
Figure 1. The contract curve and Pareto-optimal frontier.
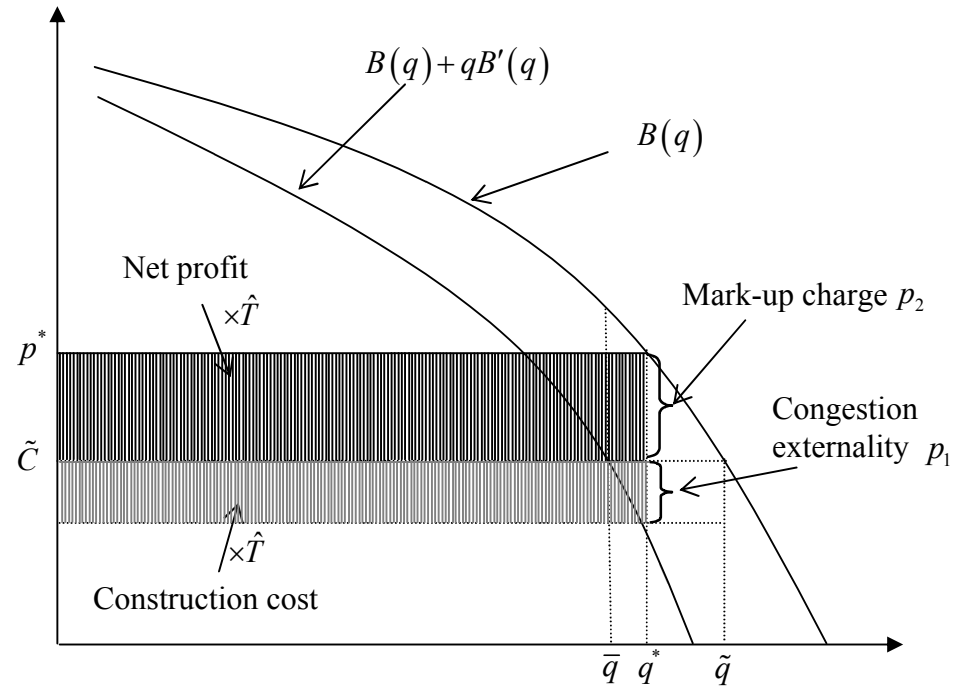
Figure 2. A geometric illustration of the Pareto-efficient toll charge, demand and profit
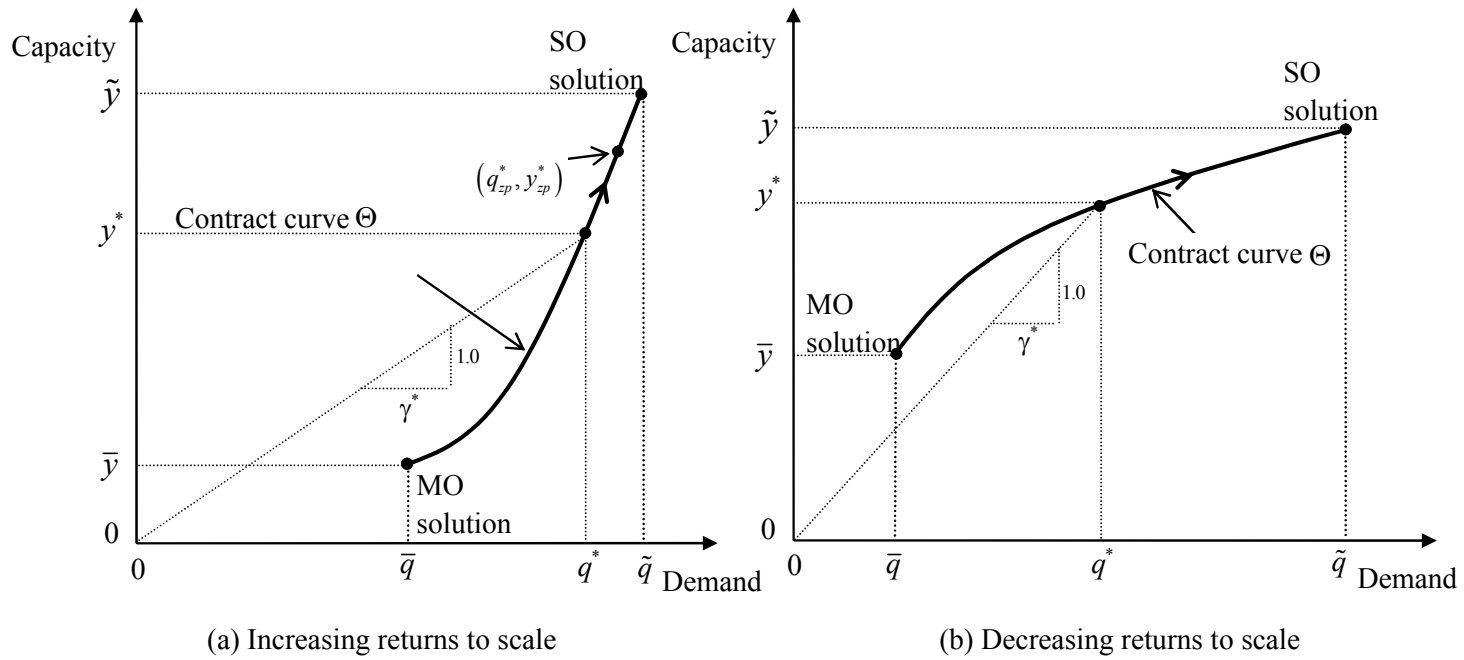
Figure 3 Contract curves with increasing and decreasing
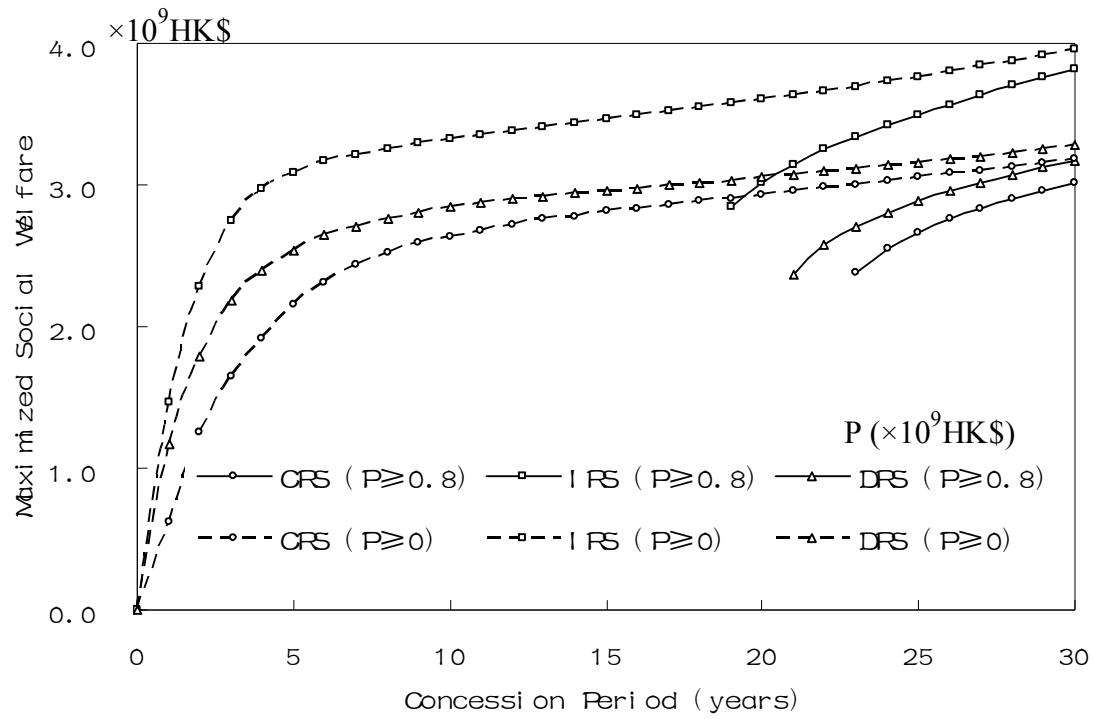returns to scale in road construction

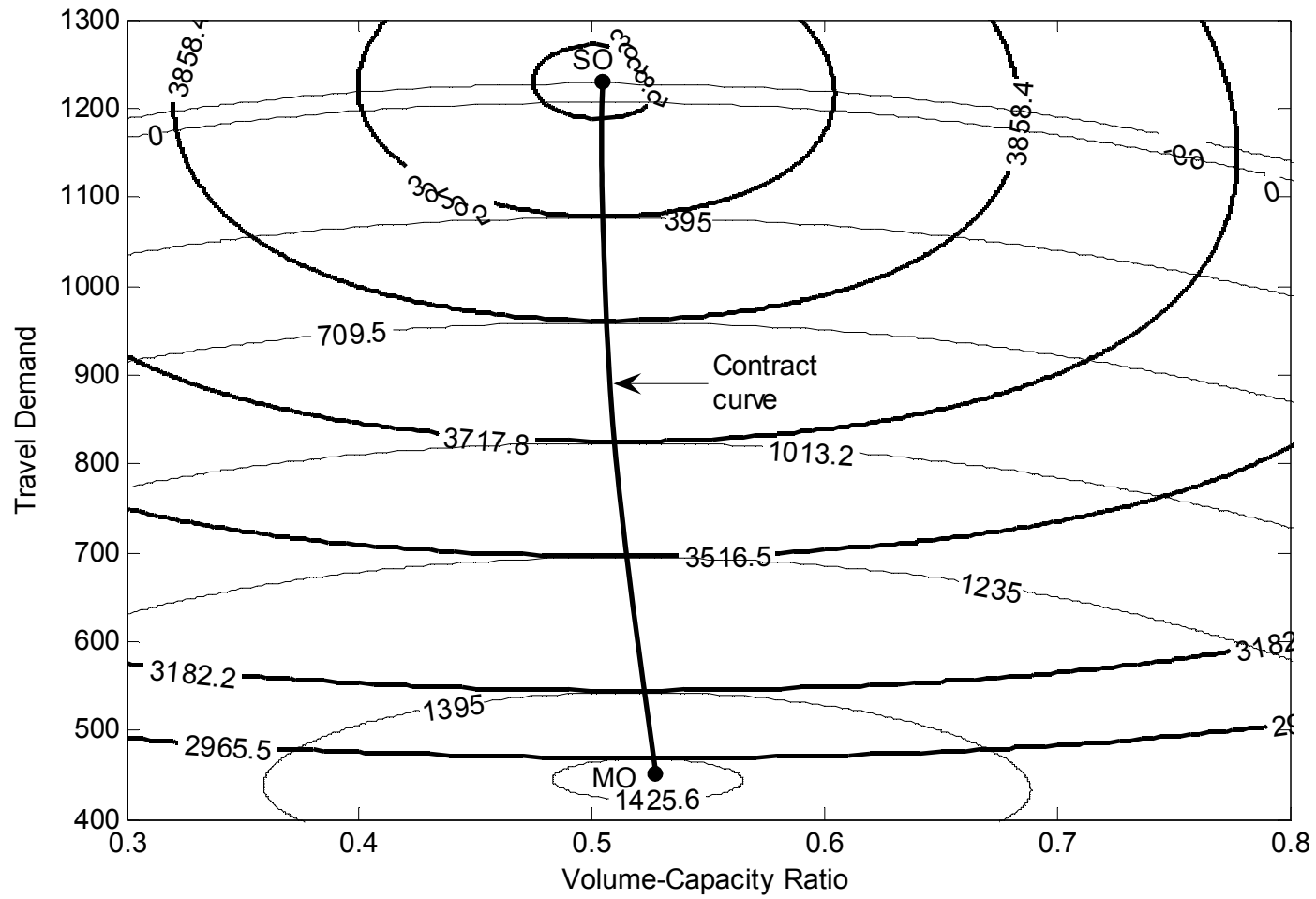Figure 4 Maximized social welfare versus the concession period with profit constraints

Figure 5 A contract curve with increasing returns to scale in road construction
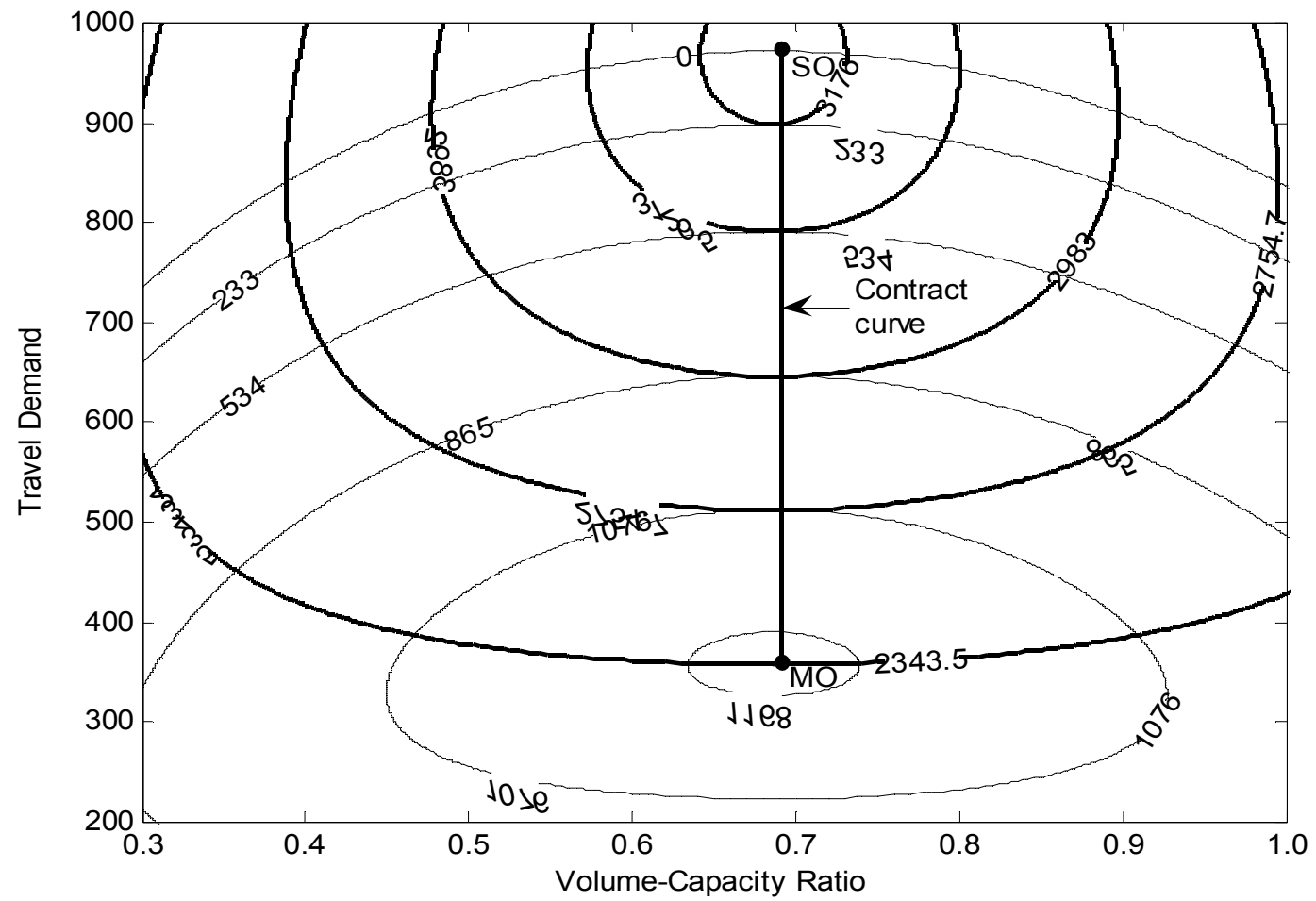
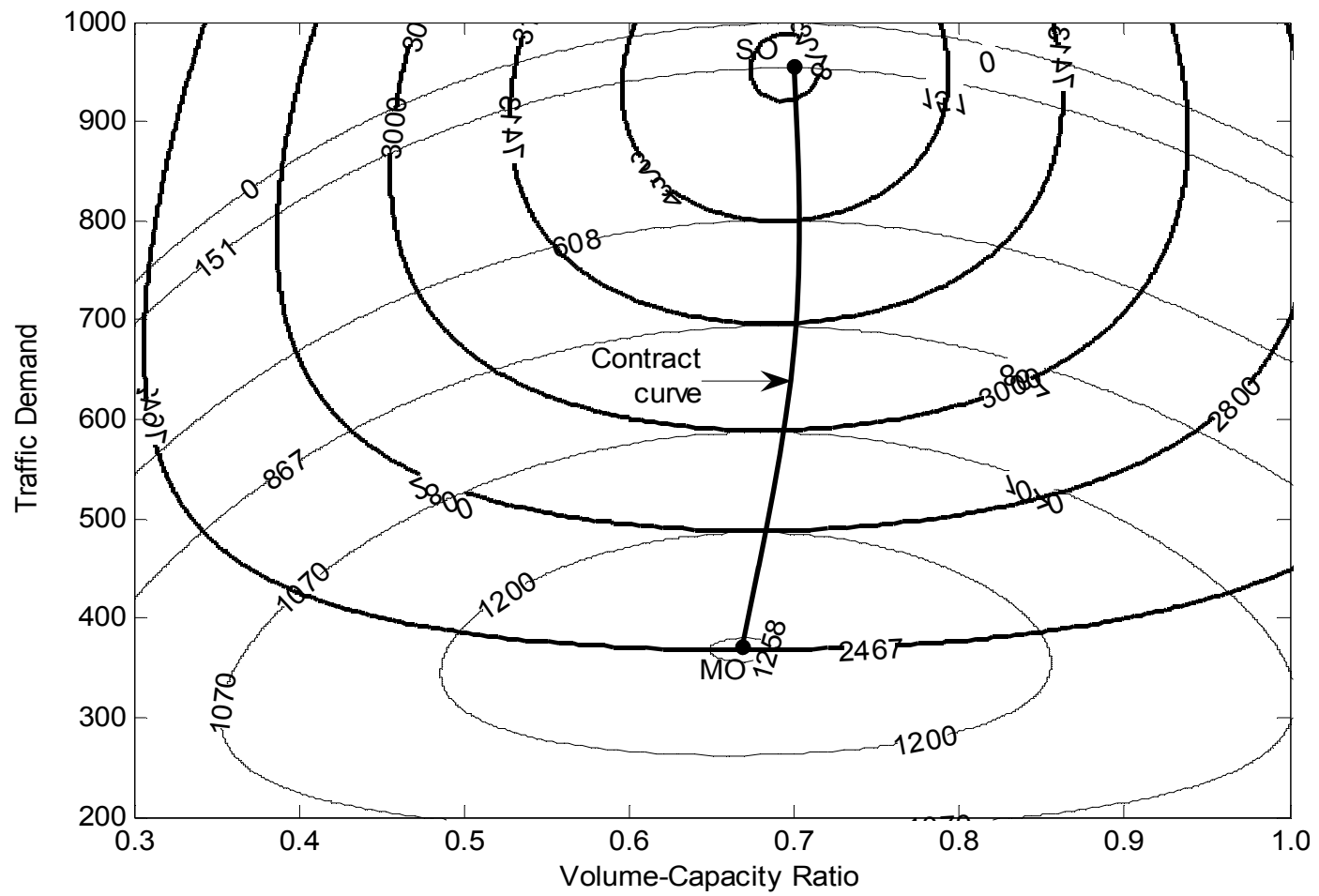Figure 6 A contract curve with constant returns to scale in road construction

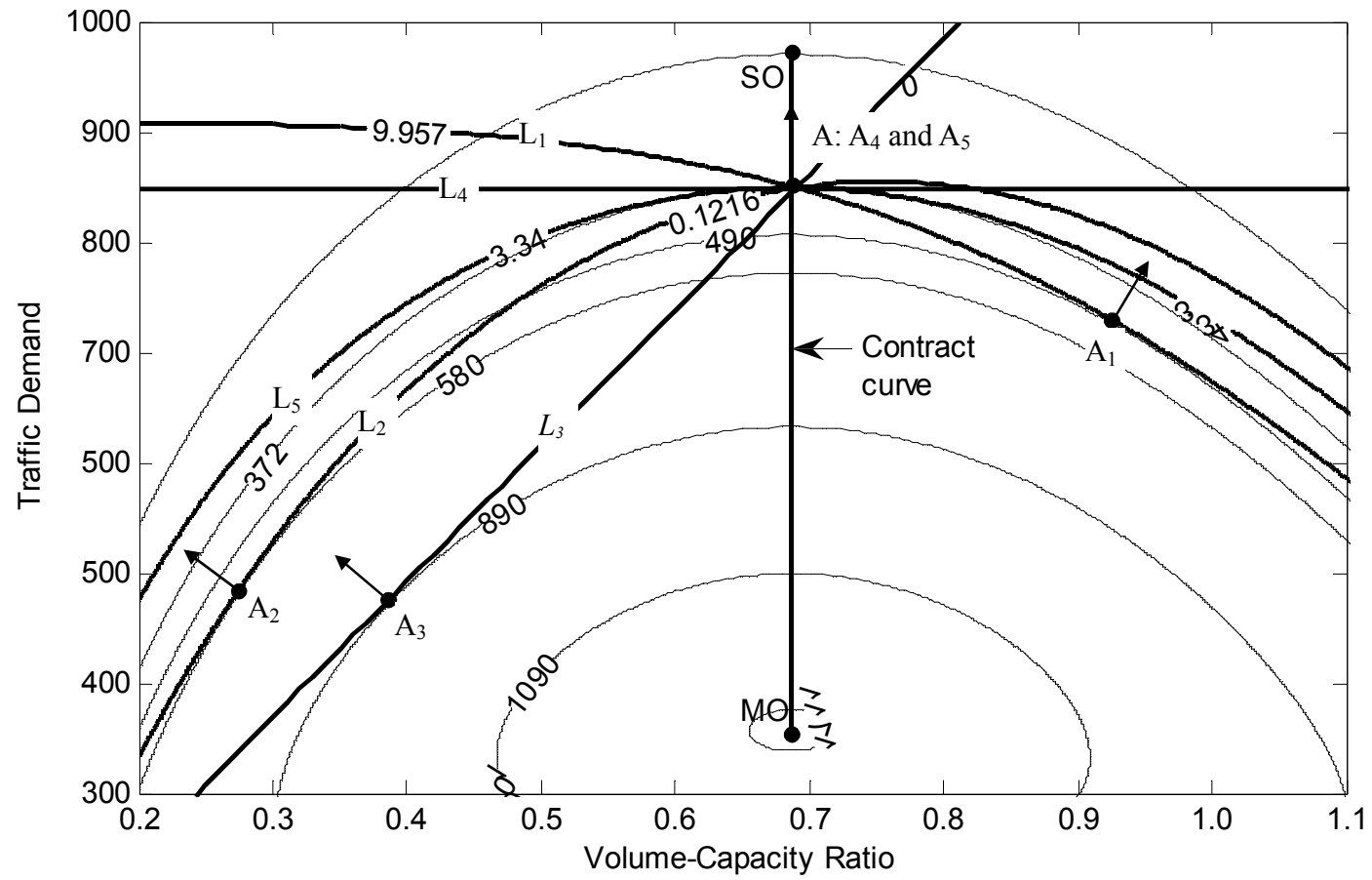Figure 7 A contract curve with decreasing returns to scale in road construction

Figure 8 Regulations and the corresponding outcomes