Fall 2011

A mathematical programming approach to multi-attribute decision making with interval-valued intuitionistic fuzzy assessment information

Zhou-Jing Wang

Kevin W. Li Dr.
University of Windsor

Jianhui Xu

A Mathematical Programming Approach to Multi-Attribute Decision Making with Interval-Valued Intuitionistic Fuzzy Assessment Information

Zhoujing Wang [a,b*], Kevin W. Li[c], Jianhui Xu[b]

[a] *School of Computer Science and Engineering, Beihang University, Beijing 100083, China*

[b] *Department of Automation, Xiamen University, Xiamen, Fujian 361005, China*

[c] *Odette School of Business, University of Windsor, Windsor, Ontario N9B 3P4, Canada*

Abstract

This article proposes an approach to handle multi-attribute decision making (MADM) problems under the interval-valued intuitionistic fuzzy environment, in which both assessments of alternatives on attributes (hereafter, referred to as attribute values) and attribute weights are provided as interval-valued intuitionistic fuzzy numbers (IVIFNs). The notion of relative closeness is extended to interval values to accommodate IVIFN decision data, and fractional programming models are developed based on the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method to determine a relative closeness interval where attribute weights are independently determined for each alternative. By employing a series of optimization models, a quadratic program is established for obtaining a unified attribute weight vector, whereby the individual IVIFN attribute values are aggregated into relative closeness intervals to the ideal solution for final ranking. An illustrative supplier selection problem is employed to demonstrate how to apply the proposed procedure.

Keywords: Multi-attribute decision making (MADM), interval-valued intuitionistic fuzzy numbers (IVIFNs), fractional programming, quadratic programming

1. Introduction

Multi-attribute decision making (MADM) handles decision situations where a set of alternatives (usually discrete) has to be assessed against multiple attributes or criteria before a final choice is selected (Hwang and Yoon, 1981). MADM problems may arise

[*] Corresponding author. Telephone: +86 592 2580036; fax: +86 592 2180858.
Email: wangzj@xmu.edu.cn (Z. Wang), kwli@uwindsor.ca (K.W. Li).

from decisions in our daily life as well as complicated decisions in a host of fields such as economics, management and engineering. For instance, when deciding which car to buy, a customer may consider a number of cars by assessing their prices, security, driving experience, quality, and colour. It is understandable that the aforesaid five attributes in this decision problem are likely to play different roles in reaching a final purchase decision. These varying roles are typically reflected as different attribute weights in MADM. Eventually, the customer has to aggregate his/her individual assessments of different cars against each attribute into an overall evaluation and selects a car that yields the best overall value. This simple example reveals the three key components in a multi-attribute decision model: attribute values or performance measures, attribute weights, and a mechanism to aggregate this information into an aggregated value or assessment for each alternative.

Due to ambiguity and incomplete information in many decision problems, it is often difficult for a decision-maker (DM) to give his/her assessments on attribute values and weights in crisp values. Instead, it has become increasingly common that these assessments are provided as fuzzy numbers (FNs) or intuitionistic fuzzy numbers (IFNs), leading to a rapidly expanding body of literature on MADM under the fuzzy or intuitionistic fuzzy framework (Atanassov et al., 2005; Boran et al., 2009; Hong & Choi, 2000; Li, 2005; Li et al., 2009; Liu & Wang, 2007; Szmidt & Kacprzyk, 2002; Szmidt & Kacprzyk, 2003; Tan & Chen, 2010; Wang et al., 2009; Wang & Qian, 2007; Xu, 2007a; Xu, 2007b; Xu & Yager, 2008; Zhang et al., 2009). The notion of intuitionistic fuzzy sets (IFSs) is proposed by Atanassov (1986) to generalize the concept of fuzzy sets. In a fuzzy set, the membership of an element to a particular set is defined as a continuous value between 0 and 1, thereby extending the traditional 0-1 crisp logic to fuzzy logic (Karray & de Silva, 2004). IFSs move one step further by considering not only the membership but also the nonmembership of an element to a given set.

In an IFS, the membership and nonmembership functions are defined as real values between 0 and 1. By allowing these real-valued membership and nonmembership functions to assume interval values, Atanassov and Gargov (1989) extend the notion of IFSs to interval-valued intuitionistic fuzzy sets (IVIFSs). In recent years, the academic community has witnessed growing research interests in IVIFSs, such as investigations on basic operations and relations of IVIFSs as well as their basic properties (Bustince & Burillo, 1995; Hong, 1998; Hung & Choi, 2002; Xu & Chen, 2008), topological properties (Mondal & Samanta, 2001), relationships between IFSs, *L*-fuzzy sets, interval-valued fuzzy sets and IVIFSs (Deschrijver , 2007; Deschrijver, 2008; Deschrijver & Kerre, 2007), the entropy and subsethood (Liu, Zheng & Xiong, 2005), and distance measures and similarity measures of IVIFSs (Xu & Chen, 2008). With this enhanced understanding of IVIFNs, researchers have turned their attention to decision problems where some raw decision data are provided as IVIFNs (Xu, 2007b; Xu and Yager 2008; Wang et al., 2009). In the existing research on MADM with IVIFN assessments, it is generally assumed that attribute values are given as IVIFNs, but attribute weights are either provided as crisp values or expressed as a set of linear constraints (Wang et al., 2009). In this research, the focus is to consider MADM situations where both attribute values and weights are furnished as IVIFNs.

The remainder of this paper is organized as follows. Section 2 provides some preliminary background on IFSs and IVIFSs. In Section 3, fractional programs and quadratic programs are derived from TOPSIS and a corresponding approach is designed to solve MADM problems with interval-valued intuitionistic fuzzy assessments. Section 4 presents a numerical example to demonstrate how to apply the proposed approach, followed by some concluding remarks in Section 5.

2. Preliminaries

This section reviews some basic concepts on IFSs and IVIFSs to make the article self-contained and facilitate the discussion of the proposed method.

Definition 2.1 (Atanassov, 1986). Let Z be a fixed nonempty universe set, an intuitionistic fuzzy set (IFS) A in Z is defined as

$$A = \{< z, \mu_A(z), \nu_A(z) > | \, z \in Z\}$$

where $\mu_A : Z \to [0,1]$ and $\nu_A : Z \to [0,1]$, satisfying $0 \le \mu_A(z) + \nu_A(z) \le 1$, $\forall z \in Z$.

$\mu_A(z)$ and $\nu_A(z)$ are called, respectively, the membership and nonmembership functions of IFS A. In addition, for each IFS A in Z, $\pi_A(z) = 1 - \mu_A(z) - \nu_A(z)$ is often referred to as its intuitionistic fuzzy index, representing the degree of indeterminacy or hesitation of z to A. It is obvious that $0 \le \pi_A(z) \le 1$ for every $z \in Z$.

When the range of the membership and nonmembership functions of an IFS is extended to interval values rather than exact numbers, IFSs become interval-valued intuitionistic fuzzy sets (IVIFSs) (Atanassov and Gargov, 1989).

Definition 2.2 (Atanassov and Gargov, 1989). Let Z be a non-empty set of the universe, and $D[0,1]$ be the set of all closed subintervals of [0, 1], an interval-valued intuitionistic fuzzy set (IVIFS) $\tilde{A}$ over Z is an object in the following form:

$$\tilde{A} = \{< z, \tilde{\mu}_{\tilde{A}}(z), \tilde{\nu}_{\tilde{A}}(z) > | \, z \in Z\}$$

where $\tilde{\mu}_{\tilde{A}} : Z \to D[0,1]$, $\tilde{\nu}_{\tilde{A}} : Z \to D[0,1]$, and $0 \le \sup(\tilde{\mu}_{\tilde{A}}(z)) + \sup(\tilde{\nu}_{\tilde{A}}(z)) \le 1$ for any $z \in Z$.

The intervals $\tilde{\mu}_{\tilde{A}}(z)$ and $\tilde{\nu}_{\tilde{A}}(z)$ denote, respectively, the degree of membership and nonmembership of z to *A*. For each $z \in Z$, $\tilde{\mu}_{\tilde{A}}(z)$ and $\tilde{\nu}_{\tilde{A}}(z)$ are closed intervals and their lower and upper boundaries are denoted by $\tilde{\mu}^L_{\tilde{A}}(z), \tilde{\mu}^U_{\tilde{A}}(z), \tilde{\nu}^L_{\tilde{A}}(z)$ and $\tilde{\nu}^U_{\tilde{A}}(z)$. Therefore, another equivalent way to express IVIFS $\tilde{A}$ is

$$\tilde{A} = \{< z, [\tilde{\mu}^L_{\tilde{A}}(z), \tilde{\mu}^U_{\tilde{A}}(z)], [\tilde{\nu}^L_{\tilde{A}}(z), \tilde{\nu}^U_{\tilde{A}}(z)] > | \, z \in Z\},$$

where $\tilde{\mu}^U_{\tilde{A}}(z) + \tilde{\nu}^U_{\tilde{A}}(z) \le 1, 0 \le \tilde{\mu}^L_{\tilde{A}}(z) \le \tilde{\mu}^U_{\tilde{A}}(z) \le 1, 0 \le \tilde{\nu}^L_{\tilde{A}}(z) \le \tilde{\nu}^U_{\tilde{A}}(z) \le 1$.

Similar to IFSs, for each element $z \in Z$, its hesitation interval relative to $\tilde{A}$ is given as:

$$\tilde{\pi}_{\tilde{A}}(z) = [\tilde{\pi}^L_{\tilde{A}}(z), \tilde{\pi}^U_{\tilde{A}}(z)] = [1 - \tilde{\mu}^U_{\tilde{A}}(z) - \tilde{\nu}^U_{\tilde{A}}(z), 1 - \tilde{\mu}^L_{\tilde{A}}(z) - \tilde{\nu}^L_{\tilde{A}}(z)]$$

Especially, for every $z \in Z$, if

$$\mu_{\tilde{A}}(z) = \tilde{\mu}^L_{\tilde{A}}(z) = \tilde{\mu}^U_{\tilde{A}}(z), \; \nu_{\tilde{A}}(z) = \tilde{\nu}^L_{\tilde{A}}(z) = \tilde{\nu}^U_{\tilde{A}}(z)$$

then, IVIFS $\tilde{A}$ reduces to an ordinary IFS.

For an IVIFS $\tilde{A}$ and a given *z*, the pair $(\tilde{\mu}_{\tilde{A}}(z), \tilde{\nu}_{\tilde{A}}(z))$ is called an interval-valued intuitionistic fuzzy number (IVIFN) [34,38]. For convenience, the pair $(\tilde{\mu}_{\tilde{A}}(z), \tilde{\nu}_{\tilde{A}}(z))$ is often denoted by $([a,b],[c,d])$, where $[a,b] \in D[0,1], [c,d] \in D[0,1]$ and $b + d \le 1$.

After the initial decision data in IVIFNs are processed, the proposed model will generate an aggregated relative closeness interval, expressed as an IVIFN, to the ideal solution for each alternative. To make a final choice based on the aggregated relative closeness intervals, it is necessary to examine how to rank IVIFNs. Xu (2007b) introduces the score and accuracy functions for IVIFNs and applies them to compare two IVIFNs. Wang et al. (2009) note that many distinct IVIFNs cannot be differentiated by these two functions. As such, two new functions, the membership uncertainty index and the hesitation uncertainty index, are defined therein. Along with the score and accuracy functions, Wang et al. (2009) devise a unique prioritized IVIFN comparison approach that is able to distinguish any two distinct IVIFNs. This same comparison approach will be adopted in this research for ranking alternatives based on IVIFNs. Next, these four functions are defined.

Definition 2.3 (Xu, 2007b). For an IVIFN $\tilde{\alpha} = ([a,b],[c,d])$, its score function is defined as $S(\tilde{\alpha}) = \dfrac{a+b-c-d}{2}$.

Definition 2.4 (Xu, 2007b). For an IVIFN $\tilde{\alpha} = ([a,b],[c,d])$, its accuracy function is defined as $H(\tilde{\alpha}) = \dfrac{a+b+c+d}{2}$.

Definition 2.5 (Wang et al., 2009). For an IVIFN $\tilde{\alpha} = ([a,b],[c,d])$, its membership uncertainty index is defined as $T(\tilde{\alpha}) = b+c-a-d$.

Definition 2.6 (Wang et al., 2009). For an IVIFN $\tilde{\alpha} = ([a,b],[c,d])$, its hesitation uncertainty index is defined as $G(\tilde{\alpha}) = b+d-a-c$.

For a discussion of these four functions and their properties, readers are referred to (Wang et al., 2009). Based on these functions, a prioritized comparison method is introduced as follows.

Definition 2.7 (Wang et al., 2009). For any two IVIFNs $\tilde{\alpha} = ([a_1,b_1],[c_1,d_1])$ and $\tilde{\beta} = ([a_2,b_2],[c_2,d_2])$,

If $S(\tilde{\alpha}) < S(\tilde{\beta})$, then $\tilde{\alpha}$ is smaller than $\tilde{\beta}$, denoted by $\tilde{\alpha} < \tilde{\beta}$;

If $S(\tilde{\alpha}) > S(\tilde{\beta})$, then $\tilde{\alpha}$ is greater than $\tilde{\beta}$, denoted by $\tilde{\alpha} > \tilde{\beta}$;

If $S(\tilde{\alpha}) = S(\tilde{\beta})$, then

1) If $H(\tilde{\alpha}) < H(\tilde{\beta})$, then $\tilde{\alpha}$ is smaller than $\tilde{\beta}$, denoted by $\tilde{\alpha} < \tilde{\beta}$;
2) If $H(\tilde{\alpha}) > H(\tilde{\beta})$, then $\tilde{\alpha}$ is greater than $\tilde{\beta}$, denoted by $\tilde{\alpha} > \tilde{\beta}$;
3) If $H(\tilde{\alpha}) = H(\tilde{\beta})$, then

i) If $T(\tilde{\alpha}) > T(\tilde{\beta})$, then $\tilde{\alpha}$ is smaller than $\tilde{\beta}$, denoted by $\tilde{\alpha} < \tilde{\beta}$;

ii) If $T(\tilde{\alpha}) < T(\tilde{\beta})$, then $\tilde{\alpha}$ is greater than $\tilde{\beta}$, denoted by $\tilde{\alpha} > \tilde{\beta}$;

iii) If $T(\tilde{\alpha}) = T(\tilde{\beta})$, then

a) If $G(\tilde{\alpha}) > G(\tilde{\beta})$, then $\tilde{\alpha}$ is smaller than $\tilde{\beta}$, denoted by $\tilde{\alpha} < \tilde{\beta}$;

b) If $G(\tilde{\alpha}) < G(\tilde{\beta})$, then $\tilde{\alpha}$ is greater than $\tilde{\beta}$, denoted by $\tilde{\alpha} > \tilde{\beta}$;

c) If $G(\tilde{\alpha}) = G(\tilde{\beta})$, then $\tilde{\alpha}$ and $\tilde{\beta}$ represent the same information, denoted by $\tilde{\alpha} = \tilde{\beta}$

For any two IVIFNs, $\tilde{\alpha}$ and $\tilde{\beta}$, denote $\tilde{\alpha} \le \tilde{\beta}$ iff $\tilde{\alpha} < \tilde{\beta}$ or $\tilde{\alpha} = \tilde{\beta}$.

Definition 2.8 (Wang et al., 2009). Let $[a_1, b_1], [a_2, b_2]$ be two interval numbers over [0, 1]. A relation "$\le$" in $D[0,1]$ is defined as: $[a_1, b_1] \le [a_2, b_2]$ iff $a_1 \le a_2$ and $b_1 \le b_2$.

If $\tilde{\alpha} = ([a,b],[c,d])$ is an IVIFN, from Definition 2.2 and 2.8, it may be rewritten as a pair of closed intervals $([a,b],[1-d,1-c])$ over [0, 1] with $[a,b] \le [1-d,1-c]$ and $b \le 1-d$. Conversely, given a pair of closed intervals $([a^-, a^+],[b^-, b^+])$ with $[a^-, a^+] \in D(0,1)$, $[b^-, b^+] \in D(0,1)$, $[a^-, a^+] \le [b^-, b^+]$ and $a^+ \le b^-$, then it can be expressed equivalently as an IVIFN $\tilde{\alpha} = ([a,b],[c,d])$, where $a = a^-$, $b = a^+$, $c = 1 - b^+$ and $d = 1 - b^-$. In Section 3, a pair of intervals will be adopted to represent the lower and upper bounds of satisfaction degrees or relative closeness, where the first interval indicates the lower bound and the second interval specifies the upper bound. The discussion here establishes the equivalence between an IVIFN and the representation of satisfaction degrees or relative closeness, and is of help to the development of the proposed decision model.

3. A mathematical programming approach to multi-attribute decision making under interval-valued intuitionistic fuzzy environments

This section puts forward a framework for MADM under the interval-valued intuitionistic environment, where both attribute values and weights are given as IVIFNs by the DM.

3.1 Problem formulation

Given a discrete alternative set $X = \{X_1, X_2, \cdots, X_n\}$, consisting of n non-inferior decision alternatives from which the most preferred alternative is to be selected or a ranking of all alternatives is to be obtained, and an attribute set $A = (A_1, A_2, \cdots A_m)$. Each alternative is assessed on each of the m attributes and each assessment is expressed as an IVIFN, describing the satisfaction and non-satisfaction ranges of the alternative to a fuzzy concept of "excellence" with respect to a particular attribute. More specifically, assume that a DM provides an IVIFN assessment $\tilde{r}_{ij} = ([a_{ij}, b_{ij}],[c_{ij}, d_{ij}])$ for alternative X_i with respect to attribute A_j, where $[a_{ij}, b_{ij}]$ and $[c_{ij}, d_{ij}]$ are the degree of membership (or satisfaction) and non-membership (or dissatisfaction) intervals relative to the fuzzy concept "excellence", respectively, and $[a_{ij}, b_{ij}] \in D[0,1]$, $[c_{ij}, d_{ij}] \in D[0,1]$, and $b_{ij} + d_{ij} \le 1$. Thus an MADM problem with interval-valued intuitionistic fuzzy attribute values can be expressed concisely in the matrix format as $\tilde{R} = (([a_{ij}, b_{ij}],[c_{ij}, d_{ij}]))_{n\times m}$.

It is clear that the lowest satisfaction degree of X_i with respect to A_j is $[a_{ij}, b_{ij}]$, as given in the membership function, and the highest satisfaction degree of X_i with respect to A_j is $[1-d_{ij}, 1-c_{ij}]$, when all hesitation is treated as membership or satisfaction. Therefore, the satisfaction degree interval of alternative X_i with respect to attribute A_j, denoted by $[\xi_{ij}, \eta_{ij}]$, should lie between $[a_{ij}, b_{ij}]$ and $[1-d_{ij}, 1-c_{ij}]$, and the matrix $\tilde{R} = (([a_{ij}, b_{ij}],[c_{ij}, d_{ij}]))_{n\times m}$ can be written in the satisfaction degree interval format as $\tilde{R}' = (([a_{ij}, b_{ij}],[1-d_{ij}, 1-c_{ij}]))_{n\times m}$.

Similarly, assume that the DM assesses the importance of each attribute as an IVIFN $([\omega_j^a, \omega_j^b],[\omega_j^c, \omega_j^d])$, where $[\omega_j^a, \omega_j^b]$ and $[\omega_j^c, \omega_j^d]$ are the degrees of membership and nonmembership of attribute A_j as per a fuzzy concept "importance", respectively, and $[\omega_j^a, \omega_j^b] \in D[0,1]$, $[\omega_j^c, \omega_j^d] \in D[0,1]$ and $\omega_j^b + \omega_j^d \le 1$. It is obvious that the lowest and highest weight intervals for attribute A_j are $[\omega_j^a, \omega_j^b]$ and $[1-\omega_j^d, 1-\omega_j^c]$, respectively. As such, the weight interval of A_j should lie between $[\omega_j^a, \omega_j^b]$ and $[1-\omega_j^d, 1-\omega_j^c]$.

3.2 Mathematical programming models for solving MADM problems

As mentioned in section 3.1, the satisfaction degree interval of alternative X_i with respect to attribute A_j, given by $[\xi_{ij}, \eta_{ij}]$, should lie between $[a_{ij}, b_{ij}]$ and $[1-d_{ij}, 1-c_{ij}]$, i.e., $[a_{ij}, b_{ij}] \le [\xi_{ij}, \eta_{ij}] \le [1-d_{ij}, 1-c_{ij}]$. According to Definition 2.8, ξ_{ij} and η_{ij} should satisfy $a_{ij} \le \xi_{ij} \le 1-d_{ij}$ and $b_{ij} \le \eta_{ij} \le 1-c_{ij}$.

As $a_{ij} \le b_{ij}$, $c_{ij} \le d_{ij}$ and $b_{ij} + d_{ij} \le 1$, we have $a_{ij} \le b_{ij} \le 1-d_{ij} \le 1-c_{ij}$.

In a similar way, the weight interval of attribute A_j, denoted by $[\omega_j^-, \omega_j^+]$, should lie between $[\omega_j^a, \omega_j^b]$ and $[1-\omega_j^d, 1-\omega_j^c]$, i.e., $[\omega_j^a, \omega_j^b] \le [\omega_j^-, \omega_j^+] \le [1-\omega_j^d, 1-\omega_j^c]$. According to Definition 2.8, ω_j^- and ω_j^+ should satisfy $\omega_j^a \le \omega_j^- \le 1-\omega_j^d$ and $\omega_j^b \le \omega_j^+ \le 1-\omega_j^c$.

As per Definition 2.7, we know that $([1,1],[0,0])$ and $([0,0],[1,1])$ are the largest and smallest IVIFNs, respectively. Therefore, the interval-valued intuitionistic fuzzy ideal solution X^+ can be specified as the largest IVIFN $([1,1],[0,0])$, where its satisfaction and dissatisfaction degrees on attribute A_j are $[1,1]$ and $[0,0]$, respectively. This ideal solution can be rewritten in the satisfaction degree interval format as $([1,1],[1,1])$, or equivalently, $[1,1]$.

As $[\xi_{ij}, \eta_{ij}]$ is the satisfaction degree interval of alternative X_i with respect to attribute A_j, the normalized Euclidean distance interval of alternative X_i from the ideal solution X^+, denoted by $[d_i^{+-}, d_i^{++}]$, can be calculated as follows:

$$d_i^{+-} = \sqrt{\sum_{j=1}^{m} \left[\omega_j (1-\eta_{ij})\right]^2} \tag{3.1}$$

$$d_i^{++} = \sqrt{\sum_{j=1}^{m} \left[\omega_j (1-\xi_{ij})\right]^2} \tag{3.2}$$

where $a_{ij} \le \xi_{ij} \le 1-d_{ij}$, $b_{ij} \le \eta_{ij} \le 1-c_{ij}$, $\omega_j^- \le \omega_j \le \omega_j^+$ and $\sum_{j=1}^{m} \omega_j = 1$ for each $i = 1, 2, \cdots, n$.

Similarly, the satisfaction and dissatisfaction degree of the anti-ideal solution X^- on attribute A_j are $[0,0]$ and $[1,1]$, respectively, which can be written in the satisfaction degree interval format as $([0,0],[0,0])$, equivalent to $[0,0]$. The separation interval of alternative X_i from the anti-ideal solution X^- is given by $[d_i^{--}, d_i^{-+}]$, where

$$d_i^{--} = \sqrt{\sum_{j=1}^{m} (\omega_j \xi_{ij})^2} \tag{3.3}$$

$$d_i^{-+} = \sqrt{\sum_{j=1}^{m} (\omega_j \eta_{ij})^2} \tag{3.4}$$

Equations (3.1)-(3.4) are employed to determine the distance from ideal and anti-ideal alternatives in interval values. While the individual attribute values are processed, this proposed approach works with interval values directly and the conversion to crisp values is delayed until the final aggregation process. This treatment helps to reduce the loss of information due to early conversion.

TOPSIS is a popular MADM approach proposed by Hwang and Yoon (1981) and has been widely used to handle diverse MADM problems (Boran et al., 2009; Celik et al., 2009; Chen & Tzeng, 2004; Dağdeviren et al., 2009; Fu, 2008; Shih, 2008; İÇ & Yurdakul, 2010). Recently, this method has been extended to address decision situations with fuzzy assessment data (Chen & Lee, 2009; Chen & Tsao, 2008; Li et al., 2009; Wang & Elhag, 2005; Xu & Yager, 2008). The basic principle is that the selected alternative should be as close as possible to the ideal solution and as far away as possible from the anti-ideal solution. Based on the TOPSIS method, a relative closeness interval for each $X_i \in X$ with respect to X^+, denoted by $[c_i^L, c_i^U]$, is defined as follows:

$$c_i^L = \frac{\sqrt{\sum_{j=1}^{m} (\omega_j \xi_{ij})^2}}{\sqrt{\sum_{j=1}^{m} (\omega_j \xi_{ij})^2} + \sqrt{\sum_{j=1}^{m} \left[\omega_j (1-\xi_{ij})\right]^2}} \tag{3.5}$$

and

$$c_i^U = \frac{\sqrt{\sum_{j=1}^{m} (\omega_j \eta_{ij})^2}}{\sqrt{\sum_{j=1}^{m} (\omega_j \eta_{ij})^2} + \sqrt{\sum_{j=1}^{m} \left[\omega_j (1-\eta_{ij})\right]^2}}. \tag{3.6}$$

where $a_{ij} \le \xi_{ij} \le 1 - d_{ij}$, $b_{ij} \le \eta_{ij} \le 1 - c_{ij}$, $\omega_j^- \le \omega_j \le \omega_j^+$ and $\sum_{j=1}^{m} \omega_j = 1$ for each $i = 1, 2, \cdots, n$.

It is obvious that $0 \le c_i^L \le 1$ and c_i^L is a function of $\xi_{ij} \in [a_{ij}, 1-d_{ij}]$ and $\omega_j \in [\omega_j^-, \omega_j^+]$. By varying ξ_{ij} and ω_j in the intervals $[a_{ij}, 1-d_{ij}]$ and $[\omega_j^-, \omega_j^+]$, respectively, c_i^L lies in a closeness interval, $[c_i^{LL}, c_i^{LU}]$. The lower bound c_i^{LL} and upper bound c_i^{LU} of c_i^L can be obtained by solving the following two fractional programming models:

$$\min \ c_i^{LL} = \frac{\sqrt{\sum_{j=1}^{m} (\omega_j \xi_{ij})^2}}{\sqrt{\sum_{j=1}^{m} (\omega_j \xi_{ij})^2} + \sqrt{\sum_{j=1}^{m} \left[\omega_j (1-\xi_{ij})\right]^2}} \tag{3.7}$$

$$s.t. \begin{cases} a_{ij} \le \xi_{ij} \le 1-d_{ij},\ j=1,2,\dots,m, \\ \omega_j^- \le \omega_j \le \omega_j^+,\ j=1,2,\dots,m, \\ \sum_{j=1}^{m} \omega_j = 1. \end{cases}$$

and

$$\max \ c_i^{LU} = \frac{\sqrt{\sum_{j=1}^{m} (\omega_j \xi_{ij})^2}}{\sqrt{\sum_{j=1}^{m} (\omega_j \xi_{ij})^2} + \sqrt{\sum_{j=1}^{m} \left[\omega_j (1-\xi_{ij})\right]^2}} \tag{3.8}$$

$$s.t. \begin{cases} a_{ij} \le \xi_{ij} \le 1-d_{ij},\ j=1,2,\dots,m, \\ \omega_j^- \le \omega_j \le \omega_j^+,\ j=1,2,\dots,m, \\ \sum_{j=1}^{m} \omega_j = 1. \end{cases}$$

for each i=1,2,…,n.

As

$$\frac{\partial c_i^L}{\partial \xi_{ij}} = \frac{(\omega_j)^2 \xi_{ij} \sqrt{\sum_{j=1}^{m} \left[\omega_j (1-\xi_{ij})\right]^2 \Big/ \sum_{j=1}^{m} (\omega_j \xi_{ij})^2} + (\omega_j)^2 (1-\xi_{ij}) \sqrt{\sum_{j=1}^{m} (\omega_j \xi_{ij})^2 \Big/ \sum_{j=1}^{m} \left[\omega_j (1-\xi_{ij})\right]^2}}{\left(\sqrt{\sum_{j=1}^{m} (\omega_j \xi_{ij})^2} + \sqrt{\sum_{j=1}^{m} \left[\omega_j (1-\xi_{ij})\right]^2} \right)^2} > 0$$

for $j = 1, 2, \dots m$, c_i^L is a monotonically increasing function in ξ_{ij}. Hence, c_i^L reaches its minimum at a_{ij} and arrives at its maximum at $1-d_{ij}$. Therefore, (3.7) and (3.8) can be converted to the following two fractional programs:

$$\min \ c_i^{LL} = \frac{\sqrt{\sum_{j=1}^{m} (\omega_j a_{ij})^2}}{\sqrt{\sum_{j=1}^{m} (\omega_j a_{ij})^2} + \sqrt{\sum_{j=1}^{m} \left[\omega_j (1-a_{ij})\right]^2}} \tag{3.9}$$

$$s.t. \begin{cases} \omega_j^- \le \omega_j \le \omega_j^+,\ j=1,2,\cdots,m, \\ \omega_j^a \le \omega_j^- \le 1-\omega_j^d, \omega_j^b \le \omega_j^+ \le 1-\omega_j^c, \\ \sum_{j=1}^{m} \omega_j = 1. \end{cases}$$

and

$$\max \quad c_i^{LU} = \frac{\sqrt{\sum_{j=1}^{m}\left[\omega_j(1-d_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m}\left[\omega_j(1-d_{ij})\right]^2}+\sqrt{\sum_{j=1}^{m}(\omega_j d_{ij})^2}} \tag{3.10}$$

$$s.t.\ \begin{cases}\omega_j^- \le \omega_j \le \omega_j^+,\ j=1,2,\cdots,m,\\ \omega_j^a \le \omega_j^- \le 1-\omega_j^d,\ \omega_j^b \le \omega_j^+ \le 1-\omega_j^c,\\ \sum_{j=1}^{m}\omega_j = 1.\end{cases}$$

for each *i*=1,2,…,*n*.

In the similar way, c_i^U is confined to a closeness interval $[c_i^{UL}, c_i^{UU}]$ after η_{ij} and ω_j assume all values in the intervals $[b_{ij}, 1-c_{ij}]$ and $[\omega_j^-, \omega_j^+]$, respectively. By following the same procedure, c_i^{UL} and c_i^{UU} can be derived by solving the following two fractional programming models:

$$\min \quad c_i^{UL} = \frac{\sqrt{\sum_{j=1}^{m}(\omega_j \cdot b_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(\omega_j b_{ij})^2}+\sqrt{\sum_{j=1}^{m}\left[\omega_j(1-b_{ij})\right]^2}} \tag{3.11}$$

$$s.t.\ \begin{cases}\omega_j^- \le \omega_j \le \omega_j^+,\ j=1,2,\cdots,m,\\ \omega_j^a \le \omega_j^- \le 1-\omega_j^d,\ \omega_j^b \le \omega_j^+ \le 1-\omega_j^c,\\ \sum_{j=1}^{m}\omega_j = 1.\end{cases}$$

and

$$\max \quad c_i^{UU} = \frac{\sqrt{\sum_{j=1}^{m}\left[\omega_j(1-c_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m}\left[\omega_j(1-c_{ij})\right]^2}+\sqrt{\sum_{j=1}^{m}(\omega_j c_{ij})^2}} \tag{3.12}$$

$$s.t.\ \begin{cases}\omega_j^- \le \omega_j \le \omega_j^+,\ j=1,2,\cdots,m,\\ \omega_j^a \le \omega_j^- \le 1-\omega_j^d,\ \omega_j^b \le \omega_j^+ \le 1-\omega_j^c,\\ \sum_{j=1}^{m}\omega_j = 1.\end{cases}$$

for each *i*=1,2,…,*n*.

Models (3.9)-(3.12) can be solved by using an appropriate optimization software package. Denote their optimal solutions by $\tilde{W}_i^{LL} = (\tilde{\omega}_{i1}^{LL}, \tilde{\omega}_{i2}^{LL}, \cdots, \tilde{\omega}_{im}^{LL})^T$, $\tilde{W}_i^{LU} = (\tilde{\omega}_{i1}^{LU}, \tilde{\omega}_{i2}^{LU}, \cdots, \tilde{\omega}_{im}^{LU})^T$, $\tilde{W}_i^{UL} = (\tilde{\omega}_{i1}^{UL}, \tilde{\omega}_{i2}^{UL}, \cdots, \tilde{\omega}_{im}^{UL})^T$ and $\tilde{W}_i^{UU} = (\tilde{\omega}_{i1}^{UU}, \tilde{\omega}_{i2}^{UU}, \cdots, \tilde{\omega}_{im}^{UU})^T$ (i = 1, 2, …, n), respectively, and let

$$\begin{aligned}
\tilde{c}_i^{LL} &\triangleq \frac{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{LL}a_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{LL}a_{ij})^2}+\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{LL}(1-a_{ij})\right]^2}} \\
\tilde{c}_i^{LU} &\triangleq \frac{\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{LU}(1-d_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{LU}(1-d_{ij})\right]^2}+\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{LU}d_{ij})^2}} \\
\tilde{c}_i^{UL} &\triangleq \frac{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{UL}b_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{UL}b_{ij})^2}+\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{UL}(1-b_{ij})\right]^2}} \\
\tilde{c}_i^{UU} &\triangleq \frac{\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{UU}(1-c_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{UU}(1-c_{ij})\right]^2}+\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{UU}c_{ij})^2}}
\end{aligned} \tag{3.13}$$

for each i=1,2,…,n. Then Theorem 3.1 follows.

Theorem 3.1 For $X_i \in X, i = 1, 2, ..., n$, assume that $\tilde{c}_i^{LL}, \tilde{c}_i^{LU}, \tilde{c}_i^{UL}$, and $\tilde{c}_i^{UU}$ are defined by (3.13), then $\tilde{c}_i^{LL} \le \tilde{c}_i^{UL} \le \tilde{c}_i^{LU} \le \tilde{c}_i^{UU}$.

Proof. Since $\tilde{W}_i^{UL} = (\tilde{\omega}_{i1}^{UL}, \tilde{\omega}_{i2}^{UL}, \cdots, \tilde{\omega}_{im}^{UL})^T$ is an optimal solution of (3.11), it is also a feasible solution of (3.9) as they share the same constraints. Notice that $\tilde{W}_i^{LL} = (\tilde{\omega}_{i1}^{LL}, \tilde{\omega}_{i2}^{LL}, \cdots, \tilde{\omega}_{im}^{LL})^T$ is an optimal solution of the minimization problem (3.9), therefore,

$$\tilde{c}_i^{LL} \triangleq \frac{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{LL}a_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{LL}a_{ij})^2}+\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{LL}(1-a_{ij})\right]^2}} \le \frac{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{UL}a_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{UL}a_{ij})^2}+\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{UL}(1-a_{ij})\right]^2}}$$

Note that c_i^L is a monotonically increasing function in ξ_{ij} and $a_{ij} \le b_{ij}$, it follows that

$$\frac{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{UL}a_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{UL}a_{ij})^2}+\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{UL}(1-a_{ij})\right]^2}} \le \frac{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{UL}\cdot b_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{UL}b_{ij})^2}+\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{UL}(1-b_{ij})\right]^2}} \triangleq \tilde{c}_i^{UL}.$$

Thus, we have $\tilde{c}_i^{LL} \le \tilde{c}_i^{UL}$.

Similarly, from (3.12), one can obtain

$$\tilde{c}_i^{LU} \triangleq \frac{\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{LU}(1-d_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{LU}(1-d_{ij})\right]^2}+\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{LU}d_{ij})^2}} \le \frac{\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{LU}(1-c_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{LU}(1-c_{ij})\right]^2}+\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{LU}c_{ij})^2}}$$
$$\le \frac{\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{UU}(1-c_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{UU}(1-c_{ij})\right]^2}+\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{UU}c_{ij})^2}} \triangleq \tilde{c}_i^{UU}$$

where the first inequality holds true because c_i^L is monotonically increasing in ξ_{ij} and $c_{ij} \le d_{ij}$, or equivalently, $1-d_{ij} \le 1-c_{ij}$, and the second inequality is due to the fact that $\tilde{\omega}_{ij}^{UU}$ is an optimal solution of the maximization model (3.12) and $\tilde{\omega}_{ij}^{LU}$ is its feasible solution.

Furthermore, since $b_{ij}+d_{ij} \le 1$, or equivalently, $b_{ij} \le 1-d_{ij}$, we have

$$\tilde{c}_i^{UL} \triangleq \frac{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{UL}b_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{UL}b_{ij})^2}+\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{UL}(1-b_{ij})\right]^2}} \le \frac{\sqrt{\sum_{j=1}^{m}[\tilde{\omega}_{ij}^{UL}(1-d_{ij})]^2}}{\sqrt{\sum_{j=1}^{m}[\tilde{\omega}_{ij}^{UL}(1-d_{ij})]^2}+\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{UL}d_{ij})^2}}$$
$$\le \frac{\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{LU}(1-d_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{LU}(1-d_{ij})\right]^2}+\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{LU}d_{ij})^2}} \triangleq \tilde{c}_i^{LU}$$

Once again, the first inequality is confirmed since c_i^U is a monotonically increasing function in η_{ij} and $b_{ij} \le 1-d_{ij}$, and the second inequality follows from the fact that $\tilde{\omega}_{ij}^{LU}$ is an optimal solution of the maximization problem in (3.10) and $\tilde{\omega}_{ij}^{UL}$ is its feasible solution. The proof is thus completed. Q.E.D.

Theorem 3.1 indicates that the optimal relative closeness interval of $X_i \in X$ can be characterized by a pair of intervals: $[\tilde{c}_i^{LL}, \tilde{c}_i^{UL}]$ and $[\tilde{c}_i^{LU}, \tilde{c}_i^{UU}]$. As $[\tilde{c}_i^{LL}, \tilde{c}_i^{UL}] \le [\tilde{c}_i^{LU}, \tilde{c}_i^{UU}]$ and $\tilde{c}_i^{UL} \le \tilde{c}_i^{LU}$, based on the argument in the last paragraph in Section 2, the optimal relative closeness interval can be expressed as an equivalent IVIFN:

$$\tilde{c}_i = \left(\left[\tilde{c}_i^{LL}, \tilde{c}_i^{UL}\right], \left[1-\tilde{c}_i^{UU}, 1-\tilde{c}_i^{LU}\right]\right)$$
$$= \left(\begin{array}{l} \left[\frac{\sqrt{\sum_{j=1}^{m} (\tilde{\omega}_{ij}^{LL} a_{ij})^2}}{\sqrt{\sum_{j=1}^{m} (\tilde{\omega}_{ij}^{LL} a_{ij})^2} + \sqrt{\sum_{j=1}^{m} \left[\tilde{\omega}_{ij}^{LL}(1-a_{ij})\right]^2}}, \frac{\sqrt{\sum_{j=1}^{m} (\tilde{\omega}_{ij}^{UL} b_{ij})^2}}{\sqrt{\sum_{j=1}^{m} (\tilde{\omega}_{ij}^{UL} b_{ij})^2} + \sqrt{\sum_{j=1}^{m} \left[\tilde{\omega}_{ij}^{UL}(1-b_{ij})\right]^2}} \right], \\ \left[1 - \frac{\sqrt{\sum_{j=1}^{m} \left[\tilde{\omega}_{ij}^{UU}(1-c_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m} \left[\tilde{\omega}_{ij}^{UU}(1-c_{ij})\right]^2} + \sqrt{\sum_{j=1}^{m} (\tilde{\omega}_{ij}^{UU} c_{ij})^2}}, 1 - \frac{\sqrt{\sum_{j=1}^{m} \left[\tilde{\omega}_{ij}^{LU}(1-d_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m} \left[\tilde{\omega}_{ij}^{LU}(1-d_{ij})\right]^2} + \sqrt{\sum_{j=1}^{m} (\tilde{\omega}_{ij}^{LU} d_{ij})^2}} \right] \end{array} \right) \quad (3.14)$$

As the weight vectors $\tilde{W}_i^{LL}, \tilde{W}_i^{LU}, \tilde{W}_i^{UL}$, and $\tilde{W}_i^{UU}$ are independently determined by the four fractional programs (3.9), (3.10), (3.11) and (3.12), they are generally different, i.e., $\tilde{W}_i^{LL} \neq \tilde{W}_i^{LU} \neq \tilde{W}_i^{UL} \neq \tilde{W}_i^{UU}$ for $X_i \in X$, or $\tilde{\omega}_{ij}^{LL} \neq \tilde{\omega}_{ij}^{LU} \neq \tilde{\omega}_{ij}^{UL} \neq \tilde{\omega}_{ij}^{UU}$ for i = 1, 2, …, n and j = 1, 2, …, m. In order to compare the relative closeness intervals across different alternatives, it is necessary to obtain an integrated common weight vector for all alternatives. Next, a procedure will be introduced to derive such a weight vector.

As

$$c_i^{LL} = \frac{\sqrt{\sum_{j=1}^{m} (\omega_j a_{ij})^2}}{\sqrt{\sum_{j=1}^{m} (\omega_j a_{ij})^2} + \sqrt{\sum_{j=1}^{m} \left[\omega_j(1-a_{ij})\right]^2}} = \frac{1}{1 + \sqrt{\sum_{j=1}^{m} \left[\omega_j(1-a_{ij})\right]^2} / \sqrt{\sum_{j=1}^{m} (\omega_j a_{ij})^2}}$$

and (3.9) is a minimization fractional programming problem, the objective function of (3.9) is equivalent to maximize

$$\sqrt{\sum_{j=1}^{m} \left[\omega_j(1-a_{ij})\right]^2} / \sqrt{\sum_{j=1}^{m} (\omega_j a_{ij})^2}$$

This maximization problem can then be approximated by the following quadratic programming model:

$$\max \quad z_i^1 = \sum_{j=1}^{m} \left[\omega_j(1-a_{ij})\right]^2 - \sum_{j=1}^{m} (\omega_j a_{ij})^2 \qquad (3.15)$$

$$s.t. \quad \begin{cases} \omega_j^- \leq \omega_j \leq \omega_j^+, \; j = 1, 2, \cdots, m, \\ \omega_j^a \leq \omega_j^- \leq 1 - \omega_j^d, \omega_j^b \leq \omega_j^+ \leq 1 - \omega_j^c, \\ \sum_{j=1}^{m} \omega_j = 1. \end{cases}$$

for each i=1,2,…,n.

Similarly, (3.10), (3.11) and (3.12) can be converted to quadratic programming models with the same constraint conditions as follows:

$$\max \quad z_i^2 = \sum_{j=1}^{m} \left[\omega_j(1-d_{ij})\right]^2 - \sum_{j=1}^{m} (\omega_j d_{ij})^2 \qquad (3.16)$$

$$\max \quad z_i^3 = \sum_{j=1}^{m}\left[\omega_j(1-b_{ij})\right]^2 - \sum_{j=1}^{m}(\omega_j \cdot b_{ij})^2 \tag{3.17}$$

$$\max \quad z_i^4 = \sum_{j=1}^{m}\left[\omega_j(1-c_{ij})\right]^2 - \sum_{j=1}^{m}(\omega_j c_{ij})^2 \tag{3.18}$$

$$s.t. \quad \begin{cases} \omega_j^- \le \omega_j \le \omega_j^+,\ j=1,2,\cdots,m, \\ \omega_j^a \le \omega_j^- \le 1-\omega_j^d,\ \omega_j^b \le \omega_j^+ \le 1-\omega_j^c, \\ \sum_{j=1}^{m}\omega_j = 1. \end{cases}$$

for each i=1,2,…,n.

Since (3.15)-(3.18) are all maximization models with the same constraints, we may combine the four quadratic problems into a single model if the four objectives are equally weighted:

$$\max \quad z_i = (z_i^1 + z_i^2 + z_i^3 + z_i^4)/4 = \frac{1}{2}\sum_{j=1}^{m}(2-a_{ij}-b_{ij}-c_{ij}-d_{ij})\omega_j^2 \tag{3.19}$$

$$s.t. \quad \begin{cases} \omega_j^- \le \omega_j \le \omega_j^+,\ j=1,2,\cdots,m, \\ \omega_j^a \le \omega_j^- \le 1-\omega_j^d,\ \omega_j^b \le \omega_j^+ \le 1-\omega_j^c, \\ \sum_{j=1}^{m}\omega_j = 1. \end{cases}$$

for each i=1,2,…,n.

Since X is a non-inferior alternative set, no alternative dominates or is dominated by any other alternative. (3.19) considers one alternative at a time. If all n alternatives are taken into account simultaneously, the contribution from each individual alternative should be treated with an equal weight of $1/n$. Therefore, we have the following aggregated quadratic programming model.

$$\max \quad z = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}(2-a_{ij}-b_{ij}-c_{ij}-d_{ij})\omega_j^2}{2n} \tag{3.20}$$

$$s.t. \quad \begin{cases} \omega_j^- \le \omega_j \le \omega_j^+,\ j=1,2,\cdots,m, \\ \omega_j^a \le \omega_j^- \le 1-\omega_j^d,\ \omega_j^b \le \omega_j^+ \le 1-\omega_j^c, \\ \sum_{j=1}^{m}\omega_j = 1. \end{cases}$$

(3.20) is a standard quadratic program that can be solved by using an appropriate optimization package. Denote its optimal solution by $w^0 = (\omega_1^0, \omega_2^0, \cdots, \omega_m^0)^T$, and use similar notation as (3.13) to define:

$$
\begin{aligned}
c_i^{0LL} &\triangleq \frac{\sqrt{\sum_{j=1}^{m}(\omega_j^0 a_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(\omega_j^0 a_{ij})^2}+\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-a_{ij})\right]^2}} \\
c_i^{0LU} &\triangleq \frac{\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-d_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-d_{ij})\right]^2}+\sqrt{\sum_{j=1}^{m}(\omega_j^0 d_{ij})^2}} \\
c_i^{0UL} &\triangleq \frac{\sqrt{\sum_{j=1}^{m}(\omega_j^0 b_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(\omega_j^0 b_{ij})^2}+\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-b_{ij})\right]^2}} \\
c_i^{0UU} &\triangleq \frac{\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_j^0(1-c_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-c_{ij})\right]^2}+\sqrt{\sum_{j=1}^{m}(\omega_j^0 c_{ij})^2}}
\end{aligned}
\tag{3.21}
$$

Since c_i^L and c_i^U are monotonically increasing in ξ_{ij} and η_{ij}, respectively, and $a_{ij} \le b_{ij}$, $c_{ij} \le d_{ij}$ and $b_{ij} + d_{ij} \le 1$, it is easy to verify that $c_i^{0LL} \le c_i^{0UL} \le c_i^{0LU} \le c_i^{0UU}$. Therefore, the optimal relative closeness interval of alternative X_i based on the unified weight vector w^0 can be determined by a pair of closed intervals, $[c_i^{0LL}, c_i^{0UL}]$ and $[c_i^{0LU}, c_i^{0UU}]$. Equivalently, this interval can be expressed as an IVIFN:

$$
\begin{aligned}
c_i^0 &= \left(\left[c_i^{0LL}, c_i^{0UL}\right], \left[1-c_i^{0UU}, 1-c_i^{0LU}\right]\right) \\
&= \left(\begin{array}{l}
\left[\frac{\sqrt{\sum_{j=1}^{m}(\omega_j^0 a_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(\omega_j^0 a_{ij})^2}+\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-a_{ij})\right]^2}}, \frac{\sqrt{\sum_{j=1}^{m}(\omega_j^0 b_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(\omega_j^0 b_{ij})^2}+\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-b_{ij})\right]^2}}\right], \\
\left[1-\frac{\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-c_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-c_{ij})\right]^2}+\sqrt{\sum_{j=1}^{m}(\omega_j^0 c_{ij})^2}}, 1-\frac{\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-d_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-d_{ij})\right]^2}+\sqrt{\sum_{j=1}^{m}(\omega_j^0 d_{ij})^2}}\right]
\end{array}\right)
\end{aligned}
\tag{3.22}
$$

for each i = 1, 2, …, n.

Theorem 3.2 Assume that IVIFNs $\tilde{c}_i$ and c_i^0 are respectively defined by (3.14) and (3.22), then for $X_i \in X, i = 1, 2, ..., n$,

$$
[\tilde{c}_i^{LL}, \tilde{c}_i^{UL}] \le [c_i^{0LL}, c_i^{0UL}] \le [c_i^{0LU}, c_i^{0UU}] \le [\tilde{c}_i^{LU}, \tilde{c}_i^{UU}]
$$

Proof. Since $w^0 = (\omega_1^0, \omega_2^0, \cdots, \omega_m^0)^T$ is an optimal solution of (3.20), it is automatically a feasible solution of (3.9), (3.10), (3.11) and (3.12) due to the fact that these models all have the same constraints. Furthermore, because c_i^L and c_i^U are monotonically increasing in ξ_{ij} and η_{ij}, respectively, and $\tilde{W}_i^{LL}=(\tilde{\omega}_{i1}^{LL},\tilde{\omega}_{i2}^{LL},\cdots,\tilde{\omega}_{im}^{LL})^T$ and $\tilde{W}_i^{LU}=(\tilde{\omega}_{i1}^{LU},\tilde{\omega}_{i2}^{LU},\cdots,\tilde{\omega}_{im}^{LU})^T$ are, respectively, an optimal solution of (3.9) and (3.10), and $a_{ij}\le b_{ij}$ and $b_{ij}+d_{ij}\le 1$, it follows that

$$\begin{aligned}
\tilde{c}_i^{LL} &\triangleq \frac{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{LL}a_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{LL}a_{ij})^2}+\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{LL}(1-a_{ij})\right]^2}} \le \frac{\sqrt{\sum_{j=1}^{m}(\omega_j^0 a_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(\omega_j^0 a_{ij})^2}+\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-a_{ij})\right]^2}} \triangleq c_i^{0LL} \\
&\le \frac{\sqrt{\sum_{j=1}^{m}(\omega_j^0 b_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(\omega_j^0 b_{ij})^2}+\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-b_{ij})\right]^2}} \le \frac{\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-d_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-d_{ij})\right]^2}+\sqrt{\sum_{j=1}^{m}(\omega_j^0 d_{ij})^2}} \triangleq c_i^{0LU} \\
&\le \frac{\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{LU}(1-d_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{LU}(1-d_{ij})\right]^2}+\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{LU}d_{ij})^2}} \triangleq \tilde{c}_i^{LU}
\end{aligned}$$

Here the first inequality is derived as $\tilde{\omega}_{ij}^{LL}$ is an optimal solution of the minimization model (3.9) and ω_j^0 is its feasible solution. The 2nd and 3rd inequalities hold true because c_i^L is monotonically increasing in ξ_{ij} and $a_{ij}\le b_{ij}\le 1-d_{ij}$. The last inequality is due to the fact that a feasible solution ω_j^0 always yields an objective function value that is less than or equal to that of an optimal solution $\tilde{\omega}_{ij}^{LU}$ for the maximization problem (3.10). Therefore, we have $\tilde{c}_i^{LL}\le c_i^{0LL}\le c_i^{0LU}\le \tilde{c}_i^{LU}$.

Similarly, as $\tilde{W}_i^{UL}=(\tilde{\omega}_{i1}^{UL},\tilde{\omega}_{i2}^{UL},\cdots,\tilde{\omega}_{im}^{UL})^T$ and $\tilde{W}_i^{UU}=(\tilde{\omega}_{i1}^{UU},\tilde{\omega}_{i2}^{UU},\cdots,\tilde{\omega}_{im}^{UU})^T$ are an optimal solution of (3.11) and (3.12), respectively, c_i^U is monotonically increasing in η_{ij}, and $c_{ij}\le d_{ij}$ and $b_{ij}+d_{ij}\le 1$, following the same argument, one can have

$$\tilde{c}_i^{UL} \triangleq \frac{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{UL} b_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{UL} b_{ij})^2}+\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{UL}(1-b_{ij})\right]^2}} \le \frac{\sqrt{\sum_{j=1}^{m}(\omega_j^0 b_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(\omega_j^0 b_{ij})^2}+\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-b_{ij})\right]^2}} \triangleq c_i^{0UL}$$

$$\le \frac{\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-d_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-d_{ij})\right]^2}+\sqrt{\sum_{j=1}^{m}(\omega_j^0 d_{ij})^2}} \le \frac{\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-c_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m}\left[\omega_j^0(1-c_{ij})\right]^2}+\sqrt{\sum_{j=1}^{m}(\omega_j^0 c_{ij})^2}} \triangleq c_i^{0UU}$$

$$\le \frac{\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{UU}(1-c_{ij})\right]^2}}{\sqrt{\sum_{j=1}^{m}\left[\tilde{\omega}_{ij}^{UU}(1-c_{ij})\right]^2}+\sqrt{\sum_{j=1}^{m}(\tilde{\omega}_{ij}^{UU} c_{ij})^2}} \triangleq \tilde{c}_i^{UU}$$

i.e., $\tilde{c}_i^{UL} \le c_i^{0UL} \le c_i^{0UU} \le \tilde{c}_i^{UU}$.

By Definition 2.8, the proof of Theorem 3.2 is completed. Q.E.D.

Theorem 3.2 demonstrates that the relative closeness interval derived from the aggregated model (3.20) for each alternative X_i is always bounded by that obtained from individual models (3.9) – (3.12) in the sense of Definition 2.8.

The aforesaid derivation process can be summarized in the following steps to handle MADM problems where both attribute values and weights are given as IVIFNs.

Step 1. Utilize the model (3.20) to obtain an optimal aggregated weight vector $w^0 = (\omega_1^0, \omega_2^0, \cdots, \omega_m^0)^T$.

Step 2. Determine the optimal relative closeness interval c_i^0 for all alternatives $X_i \in X$, $i = 1, 2, \cdots, n$, by plugging w^0 into (3.22).

Step 3. Rank all alternatives according to the decreasing order of their relative closeness intervals as per Definition 2.7. The best alternative is the one with the largest relative closeness interval.

4 An illustrative example

This section adapts a global supplier selection problem in (Chan & Kumar, 2007) to demonstrate how to apply the proposed approach.

Supplier selection is a fundamental issue for an organization. The continuing globalization has extended the supplier selection to an international arena and makes it a complex and difficult MADM task. Decisions on choosing appropriate suppliers for a firm typically have long-term impact on its performance, and poor decisions could cause significant damage to a firm's competitive advantage and profitability. Therefore, the supplier selection problem has been traditionally treated as one of the most important activities in the purchase department. To address the selection issue, difficult comparison and tradeoff among diverse factors have to be considered within the MADM framework. Due to business confidentiality and other reasons, the evaluation of global suppliers has to be conducted with uncertainty. As such, it could well be the case that both weights among different attributes and individual assessments are provided IVIFNs, and the manager has to make his/her final selection by aggregating these IVIFN data.

In the following example, assume that a manufacturing firm desires to select a suitable supplier for a key component in producing its new product. After preliminary screening, five potential global suppliers ($X = \{X_1, X_2, X_3, X_4, X_5\}$) remain as viable choices. The company requires that the purchasing manager come up with a final recommendation after evaluating each supplier against five attributes: supplier's profile (A_1), overall cost of the component (A_2), quality of the component (A_3), service performance of the supplier (A_4), as well as the risk factor (A_5). Assume that the assessments of each supplier against the five attributes are provided as IVIFNs as shown in the following interval-valued intuitionistic fuzzy matrix $\tilde{R} = (\tilde{r}_{ij})_{5\times 5}$.

Table 1. Interval-valued intuitionistic fuzzy matrix $\tilde{R}$

	A_1	A_2	A_3	A_4	A_5
X_1	([0.40, 0.50],[0.32, 0.40])	([0.67, 0.78],[0.14, 0.20])	([0.50, 0.65],[0.13, 0.22])	([0.45, 0.60],[0.30, 0.35])	([0.60, 0.65],[0.18, 0.30])
X_2	([0.52, 0.60],[0.10, 0.17])	([0.56, 0.68],[0.23, 0.28])	([0.65, 0.70],[0.20, 0.25])	([0.56, 0.62],[0.20, 0.28])	([0.55, 0.68],[0.15, 0.19])
X_3	([0.62, 0.72],[0.20, 0.25])	([0.35, 0.45],[0.33, 0.43])	([0.55, 0.63],[0.28, 0.32])	([0.45, 0.62],[0.19, 0.30])	([0.63, 0.67],[0.16, 0.20])
X_4	([0.42, 0.48],[0.40, 0.50])	([0.40, 0.50],[0.20, 0.50])	([0.50, 0.80],[0.10, 0.20])	([0.55, 0.75],[0.15, 0.25])	([0.45, 0.65],[0.25, 0.35])
X_5	([0.40, 0.50],[0.40, 0.50])	([0.30, 0.60],[0.30, 0.40])	([0.60, 0.70],[0.05, 0.25])	([0.60, 0.70],[0.10, 0.30])	([0.50, 0.60],[0.20, 0.40])

Each cell of the matrix gives the purchasing manager's IVIFN assessment of an alternative against an attribute. For instance, the top-left cell, ([0.40, 0.50], [0.32, 0.40]), reflects the purchasing manager's belief that alternative X_1 is an excellent supplier from the supplier's profile (A_1) with a margin of 40% to 50% and X_1 is not an excellent choice given its supplier's profile (A_1) with a chance between 32% and 40%.

Assume further that the purchasing manager provides his/her assessments on importance degree of the five attributes as the following IVIFNs:

$$\omega = \begin{pmatrix} ([0.12, 0.19],[0.55, 0.69]), ([0.09, 0.14],[0.62, 0.75]), ([0.08, 0.15],[0.68, 0.78]), \\ ([0.20, 0.30],[0.42, 0.58]), ([0.13, 0.20],[0.60, 0.72]) \end{pmatrix}$$

Based on the procedure established in Section 3, we first obtain the following quadratic programming model as per (3.20).

$$\max \quad z = \frac{1.60\omega_1^2 + 1.70\omega_2^2 + 1.72\omega_3^2 + 1.68\omega_4^2 + 1.64\omega_5^2}{5}$$

$$s.t. \begin{cases} \omega_1^- \le \omega_1 \le \omega_1^+, 0.12 \le \omega_1^- \le 0.31, 0.19 \le \omega_1^+ \le 0.45, \\ \omega_2^- \le \omega_2 \le \omega_2^+, 0.09 \le \omega_2^- \le 0.25, 0.14 \le \omega_2^+ \le 0.38, \\ \omega_3^- \le \omega_3 \le \omega_3^+, 0.08 \le \omega_3^- \le 0.22, 0.15 \le \omega_3^+ \le 0.32, \\ \omega_4^- \le \omega_4 \le \omega_4^+, 0.20 \le \omega_4^- \le 0.42, 0.30 \le \omega_4^+ \le 0.58, \\ \omega_5^- \le \omega_5 \le \omega_5^+, 0.13 \le \omega_5^- \le 0.28, 0.20 \le \omega_5^+ \le 0.40, \\ \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 = 1. \end{cases}$$

Solving this quadratic programming, one can get its optimal solution as:

$$w^0 = (\omega_1^0, \omega_2^0, \omega_3^0, \omega_4^0, \omega_5^0)^T = (0.12, 0.23, 0.32, 0.20, 0.13)^T$$

Plugging the weight vector w^0 and individual assessments in the decision matrix $\tilde{R}$ into (3.22), the optimal relative closeness intervals for the five alternatives are determined.

$$c_1^0 = \left([0.5310, 0.6580],[0.1891, 0.2611]\right),$$

$$c_2^0 = \left([0.5964, 0.6724][0.1989, 0.2541]\right),$$

$$c_3^0 = \left([0.4962, 0.5922],[0.2656, 0.3319]\right),$$

$$c_4^0 = \left([0.4769, 0.6755],[0.1768, 0.3230]\right),$$

$$c_5^0 = \left([0.5092, 0.6539],[0.1833, 0.3259]\right).$$

Next, the score function is calculated for each c_i^0 as

$$S(c_1^0) = 0.3694, S(c_2^0) = 0.4080, S(c_3^0) = 0.2455, \quad S(c_4^0) = 0.3263 \quad S(c_5^0) = 0.3270$$

As $S(c_2^0) > S(c_1^0) > S(c_5^0) > S(c_4^0) > S(c_3^0)$, by Definition 2.7 we have a full ranking of all five alternatives as

$$X_2 \succ X_1 \succ X_5 \succ X_4 \succ X_3.$$

5 CONCLUSIONS

In this article, a procedure is proposed to tackle multi-attribute decision making problems with both attribute weights and attributes values being provided as IVIFNs. Fractional programming models based on the TOPSIS method are established to obtain a relative closeness interval where attribute weights are independently determined for each alternative. The proposed approach employs a series of optimization models to deduce a quadratic programming model for obtaining a unified attribute weight vector, which is subsequently used to synthesize individual IVIFN assessments into an optimal relative closeness interval for each alternative. A global supplier selection problem is adapted to demonstrate how the proposed procedure can be applied in practice.

REFERENCES

Atanassov, K. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems, 20*, 87-96.

Atanassov, K. (1994). Operators over interval-valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems, 64,* 159-174.

Atanassov, K., & Gargov, G. (1989). Interval-valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems 31*, 343-349.

Atanassov, K., Pasi, G., & Yager, R. R. (2005). Intuitionistic fuzzy interpretations of multi-criteria multiperson and multi-measurement tool decision making. *International Journal of Systems Science, 36,* 859–868.

Boran, F. E., Genc, S., Kurt, M., & Akay, D. (2009). A multi-criteria intuitionistic fuzzy group decision making for supplier selection with TOPSIS method. *Expert Systems with Applications, 36*, 11363–11368

Bustince, H., & Burillo, P. (1995). Correlation of interval-valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems, 74,* 237-244.

Celik, M., Cebi, S., Kahraman, C., & Er, I. D. (2009). Application of axiomatic design and TOPSIS methodologies under fuzzy environment for proposing competitive strategies on Turkish container ports in maritime transportation network. *Expert Systems with Applications,36*, 4541–4557.

Chan, F. T. S., & Kumar N. (2007). Global supplier development considering risk factors using fuzzy extended AHP-based approach. *Omega, 35*, 417 – 431.

Chen, M. F., & Tzeng, G. H. (2004). Combining grey relation and TOPSIS concepts for selecting an expatriate host country. *Mathematical and Computer Modeling, 40*, 1473–1490.

Chen, S. M., & Lee, L. W. (2009). Fuzzy multiple attributes group decision-making based on the interval type-2 TOPSIS method. *Expert Systems with Applications,* doi:10.1016/j.eswa.2009.09.012.

Chen, T. Y., & Tsao, C. Y. (2008). The interval-valued fuzzy TOPSIS method and experimental analysis. *Fuzzy Sets and Systems*, *159*, 1410-1428.

Dağdeviren, M., Yavuz, S., & Kılınç, N. (2009). Weapon selection using the AHP and TOPSIS methods under fuzzy environment. *Expert Systems with Applications, 36*, 8143-8151

Deschrijver, G. (2007). Arithmetic operators in interval-valued fuzzy set theory. *Information Sciences, 177*, 2906-2924.

Deschrijver, G. (2008). A representation of t-norms in interval-valued *L*-fuzzy set theory. *Fuzzy Sets and Systems, 159*, 1597-1618.

Deschrijver, G., & Kerre, E.E. (2007). On the position of intuitionistic fuzzy set theory in the framework of theories modelling imprecision. *Information Sciences, 177*, 1860 – 1866.

Fu, G. (2008). A fuzzy optimization method for multicriteria decision making:An application to reservoir flood control operation. *Expert Systems with Applications, 34*, 145–149.

Hong, D.H. (1998). A note on correlation of interval-valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems, 95*, 113-117.

Hong, D. H., & Choi, C. H. (2000). Multicriteria fuzzy decision-making problems based on vague set theory. *Fuzzy Sets and Systems*, *114*, 103–113.

Hung, W. L., & Wu, J. W. (2002). Correlation of intuitionistic fuzzy sets by centroid method. *Information Sciences, 144*, 219 – 225.

Hwang, C. L., & Yoon, K. (1981). *Multiple Attribute Decision Making: Methods and Applications*. Springer, Berlin, Heideberg, New York, 1981.

İÇ, Y. T., & Yurdakul, M. (2010).Development of a quick credibility scoring decision support system using fuzzy TOPSIS. *Expert Systems with Applications*, *37*, 567-574.

Karray, F. & de Silva C.W. (2004), Soft Computing and Intelligent Systems Design: Theory, Tools and Applications, Addison-Wesley.

Li, D. F. (2005). Multiattribute decision making models and methods using intuitionistic fuzzy sets. *Journal of Computer and System Sciences*, *70*, 73-85.

Li, D. F., Wang, Y. C., Liu, S., & Shan, F. (2009). Fractional programming methodology for multi-attribute group decision-making using IFS. *Applied Soft Computing, 9*, 219–225

Liu, H. W., & Wang, G.J. (2007). Multi-criteria decision-making methods based on intuitionistic fuzzy sets. *European Journal of Operational Research*, *179*, 220–233.

Liu, X. D., Zheng, S. H., & Xiong, F. L. (2005). Entropy and subsethood for general interval-valued intuitionistic fuzzy sets. Lecture Notes in Artificial Intelligence. vol. 3613, pp.42-52.

Mondal, T. K., & Samanta, S. K. (2001). Topology of interval-valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems, 119*, 483-494.

Shih, H. S. (2008). Incremental analysis for MCDM with an application to group TOPSIS. *European Journal of Operational Research, 186*, 720-734.

Szmidt, E., & Kacprzyk, J. (2002). Using intuitionistic fuzzy sets in group decision making. *Control and Cybernetics, 31*, 1037–1053.

Szmidt, E., & Kacprzyk, J. (2003). A consensus-reaching process under intuitionistic fuzzy preference relations. *International Journal of Intelligent Systems, 18*, 837–852.

Tan, C., & Chen, X. (2010). Intuitionistic fuzzy Choquet integral operator for multi-criteria decision making. *Expert Systems with Applications, 37*, 149–157.

Wang, Y. M. & Elhag, T. M. S. (2005). TOPSIS method based on alpha level sets with an application to bridge risk assessment. *Expert Systems with Applications, 31*, 309-319.

Wang, Z., & Qian, E. Y. (2007). A vague-set-based fuzzy multi-objective decision making model for bidding purchase. *Journal of Zhejiang University SCIENCE A, 8*, 644-650.

Wang, Z. Li, K. W., & Wang W. (2009). An approach to multiattribute decision making with interval-valued intuitionistic fuzzy assessments and incomplete weights. *Information Sciences, 179*, 3026-3040.

Xu, Z. (2007a). Intuitionistic preference relations and their application in group decision making. *Information Sciences, 177*, 2363-2379.

Xu, Z. (2007b). Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making. *Control and Decision, 22*, 215-219 (in Chinese).

Xu, Z., & Chen, J. (2008). An overview of distance and similarity measures of intuitionistic fuzzy sets. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 16*, 529–555.

Xu, Z., & Yager, R. R. (2008). Dynamic intuitionistic fuzzy multi-attribute making. *International Journal of Approximate Reasoning, 48*, 246-262.

Zhang, D., Zhang, J., Lai, K. K., & Lu, Y. (2009). An novel approach to supplier selection based on vague sets group decision. *Expert Systems with Applications, 36*, 9557–9563.