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What Constitutes a Formal Analogy?

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To the ancient Greeks, *analogia* meant proportion, that is, an equality of the form A:B = C:D between ratios. Our word 'analogy' retains the sense of an equivalence between relations and not just between objects. For example, the planetary model of the atom was based on the idea that the relation of electrons to the nucleus was similar to that of planets to the sun. To distinguish this sense from looser ones, we will use the term 'formal analogy', by which we mean to suggest an equivalence of form construed as the arrangement of parts.

Our practical interest in this question of what constitutes a formal analogy is based on a desire to have more analogical material on the Law School Admission Test (LSAT). This examination is used as an aid in making admission decisions in almost all English-speaking law schools in North America. There is ample justification for having analogical material in standardized tests of reasoning and reading skills, and perhaps especially in those that figure in law school admission since drawing inferences from the comparison of cases belongs to much of the practice of law. It has been argued that "analogical reasoning lies at the heart of legal thinking" (Sunstein 1996, 99; for more discussion, see Plumer 2000, 4-7). It has even been argued that "analogy presents the isomorphism which makes the application of our logic possible" (Sacksteder 1979, 32), where this isomorphism is between, roughly, the logical form and its material content or realization (31). If this is right, it would appear that any assessment of reasoning ability should incorporate significant assessment of analogical thinking.

On the LSAT, in the mostly informal logic sections called "Logical Reasoning," analogical arguments already do appear, and steps are being taken to increase their frequency. The typical general format is for the credited response of the question to be, for example, a conclusion, assumption, or analysis of a traditional argument from analogy. There is also a format that attempts to assess perception of refutation by logical analogy. In another format the task is to recognize the technique, structure, or reasoning pattern exhibited in different arguments, or the principles applied in different arguments or states of affairs, and compare them, evaluating which are most similar (e.g., 'Of the following, which one best illustrates the principle that the passage illustrates?', or 'The pattern of [flawed] reasoning in the argument above is most similar to that in which one of the following?'). Relatively infrequently, in "Reading Comprehension" sections analogical material appears as well, such as with the recent question "Which one of the following is most closely analogous to the process of classification in insurance, as it is discussed in the passage?" But is there room for more analogical material on the LSAT, perhaps of a different kind?

At least two major standardized tests for North American graduate school admission—the LSAT and the Graduate Record Examinations (GRE)—use a question type called "Analytical Reasoning" (AR). In AR a small system of objects and relations is described along with a set of constraints on that system. Usually, the information given is partial. Then a set of test questions ask what could or must be the case—by strict deduction—given the constraints or conditions and the limited information (see Example 1 below). This format would appear to be fertile ground for test questions that hinge on a *formal analogy* between two such systems of objects and relations. If so, two important advantages should be gained. First, the questions should be more defensible (as clearly having one and only one correct answer) than questions that hinge on looser conceptions of analogy. Second, AR sets are often called "Logic Games" by examinees and derided as irrelevant to law school or graduate school. Having formal analogy questions in AR should help to make AR sets less artificial in appearance.

Consider the following, relatively simple, AR passage—what is called a "stimulus"—and a sample traditional question from the set, which was on the GRE (ETS 1996, 128):

Example 1

> The manager of a commercial printing firm is scheduling exactly six jobs—P, Q, S, T, W, and X—for a particular week, Monday through Saturday. Each job can be completed in one full day, and exactly one job will be scheduled for each day. The jobs must be scheduled according to the following conditions:
>
> P must be printed sometime before S is printed.
>
> T must be printed on the day immediately before or the day immediately after the day on which X is printed.
>
> W must be printed on Thursday.
>
> Any of the following could be printed on Saturday EXCEPT
>
> (A) P
>
> (B) Q
>
> (C) S
>
> (D) T
>
> (E) X

(The credited response is (A).) Simplifying further, consider just what might be called the *scenario*, that is, the part in the stimulus before the sentence that introduces the "conditions" (what is called the "bridge" sentence, i.e., "The jobs must be scheduled according to the following conditions"). What would be a reasonable representation of the abstract structure of this scenario? One requirement would seem to be that basic set-theoretic relations be preserved. In this scenario there are six jobs and six days. Moreover, each job is scheduled for a different day. So the scenario gives us two six-element sets and stipulates a one-to-one map of one set onto the other. It is assumed that there will be a definite schedule, to which the map will correspond, but this map is underdetermined by the conditions. Prior to the conditions, there are no constraints on this map; we have no other information that determines what the particular map will be. Hence, the abstract space of possibilities determined by the scenario consists of $6! = 6x5x4x3x2x1 = 720$ maps. The days of the week have a standard ordering that the further conditions make use of. Hence, the second set must have a simple order—that is, a binary relation that is irreflexive, asymmetrical, transitive, and connected—defined on it. This requirement by itself does not affect the number of maps. (But the further conditions stated in terms of the ordering relation successively eliminate whole sets of them. For example, the condition that P be printed before S cuts the set in half by eliminating the 360 maps that assign S to an earlier day than P.) Alternatively, the abstract structure of the scenario could be described as consisting of 720 functions, for each of which the domain is six objects and the range six locations, such that the operation of the function puts the objects in a simple ordering relation.

One can see that this could be the abstract structure of any number of scenarios. For instance,

Example 2

> Exactly six different products are to be placed in the display window of a vending machine. The products are to be placed in six compartments that are arranged in a row from left to right, exactly one product to each compartment.

This could be the credited response in a formal analogy AR test question that asks, with respect to the scenario in Example 1, "Which one of the following scenarios exemplifies the abstract structure of objects and relations exemplified by the scenario above?"

It must be admitted that with schematic examples like these much of the work of discovering analogies has already been done. We have two simple models, each consisting of a set of elements with one or more properties or relations defined on it. To be precise, each model comprises a set of 12 elements, two properties (subsets), and a binary relation. This analysis into objects and relations is strongly suggested by the wording of the scenario. In the real world we typically have to abstract from a wealth of irrelevant detail in order to get at an underlying similarity of form, which may have significance only for the purpose at hand. The question above essentially asks for an isomorphism between models, that is, a one-one map of the first set of objects onto the second that preserves properties and relations. Isomorphic models are completely equivalent at a certain level of abstraction. A closer approximation to the real world problem would ask for a homomorphism, a map that preserves properties and relations but is not necessarily one-one or onto. We could modify the problem above by adding a second row of six products to the vending machine. The jobs could then be mapped into the products in a way that preserves properties and relations in two different ways, but the map would not be onto. Familiar puzzles like finding all of the triangles in a complex figure can sometimes be construed as finding all homomorphisms from a given triangle into the figure. A homomorphism of A into B is sometimes suggestively called an "A-shaped figure in B." But that description has to be taken with a grain of salt, not only because A and B may not possess shape in any literal sense, but because in some cases a homomorphism can send all of the points of A to a single point of B. It depends on the structure that is to be preserved. But everyday analogies do sometimes involve this kind of loss of detail, as when we see an animal in a constellation. It is easy to imagine modifying our original problem to accommodate this feature. A further level of realism would be reached by asking for a homomorphism from a *part* of A into B. To be sure, with each of these modifications, the risk of unintended analogies increases and our conception of analogy becomes looser, thereby defeating our ultimate purpose. It is for this reason that we generally confine ourselves to scenarios that have isomorphic models, thereby foregoing some realism.

It would be wrong to suppose that because isomorphism is logically more complex than homomorphism, it must also take more effort to recognize. The contrary would appear to be the case. An isomorphism always has an inverse that enables us to move back and forth between the two systems as if we are dealing with the same thing under two different names, which at a certain level of abstraction we are. However, there are ways to keep the task from becoming potentially too simple, and some of these ways will be explored below.

The idea that formal analogy rests on isomorphism has appeared in studies of *reasoning* by structural analogy. Such reasoning contrasts sharply with loose arguments from analogy (or what Sacksteder calls 'qualitative analogy'—1974, 240), as they are commonly understood, whose conclusions can be uncertain in the extreme, and whose usual textbook analysis is: *a* and *b* share properties *F, G, H. . .*; *a* has the further property *K*; so *b* has *K* too (e.g., Copi 1978, 380). In what Anderson calls 'conclusive analogical arguments' (1969), on the other hand, inferences are drawn that "follow apodictically" (Weitzenfeld 1984, 139; cf. Anderson, 55). Anderson discusses the fairly widely known analogy between electrical switching circuits and the propositional calculus. Within certain limitations, "for every well-formed function of the propositional calculus there is a corresponding circuit (and vice versa) in a corresponding state." The reasoning takes the form of an "analogical extension of either representation into an area already covered by the other" (50). Weitzenfeld sees formal analogical reasoning as drawing inferences based on "a genuine isomorphism" between two "structures," where a structure "consists of a set of elements and a set of relations defined on those elements." (A structure is thus what we called a 'model' above. Strictly speaking, a model is a structure regarded as satisfying some set of sentences. As discussed below, we prefer to use 'model' instead of 'structure' because, for the purposes of this paper, we want to be able to use 'structure' for something two systems of objects can share, not for the systems themselves.) Weitzenfeld thinks of the objects as organized into sets he calls "variables." For example, the set of players on a team constitutes a variable, as does the set of positions to be played (139). Isomorphism is defined, as above, as "a mapping of the elements of one structure onto the elements of another such that all of the relations are preserved." With this mapping, every element and relation in one structure "is assigned a counterpart in the other" (140; for similar analyses, see Sacksteder 1974, esp. 241; Weingartner 1979). Weitzenfeld distinguishes two kinds of formal analogues, "*homeomorphs*" and "*paramorphs*," where for the former the variables in the cases are the same and, strictly speaking, "there is a single determining structure," for example, "in the analogy between two games of chess." Paramorphs, on the other hand, have genuinely distinct variables and distinct (although isomorphic) structures (143).

This differs from the approach to analogy that we are inclined to take only in that, first, we do not think that homeomorphs should count as analogues since it seems basic to the concept of analogy that different kinds of things are compared. Second, we see going to a higher level of abstraction as more useful. So for the scenarios in Examples 1 and 2, instead of casting these, as Weitzenfeld would, as exhibiting two structures, the one involving the relation of temporal succession between jobs on days of the week, and the other as involving a linear left-right relation between products in compartments, we see the two examples exhibiting a *single* structure of objects and locations in a simple ordering relation, even though different kinds of things are compared. What allows this ascent to greater abstraction, from the point of view of Weitzenfeld's analysis, is the isomorphism between the two structures, which is a systematic way in which they are the *same*.

'Structure' is used, in both ordinary and technical contexts, in these two ways, for the thing that has a certain form, and for the form itself. Hence, at one time we speak of a building as an "imposing structure" and at another we say that two buildings have the same structure. The set-theoretic notion of a relational structure employed by Weitzenfeld conforms to the former usage.

We have found it convenient to emphasize the latter usage, which has affinities with Mac Lane's in his discussion of "Structure in Mathematics" (1996). For instance, one mathematical structure, a "metric space," is exhibited by 1-, 2-, and 3-dimensional Euclidean spaces, as well as by time:

> A metric space M is a set together with a 'distance' function d which assigns to any two points s and t of M a non-negative real number $d(s,t)$. . .[such that] $d(s,t) = 0$ if and only if $s = t$; $d(s,t) = d(t,s)$ for all s and t, and, for any three points r, s, and t,
>
> $d(r,t) \leq d(r,s) + d(s,t)$. [175]

While isomorphism is a structural notion, systems or objects having the same structure in Mac Lane's sense need not be isomorphic to each other. A structure is given by a set of axioms, such as the axioms for a metric space or for an algebraic group; it is what all models of the set have in common, in the case of algebraic groups, the group structure. But, of course, algebraic groups differ widely from one another. Like metric spaces, some are finite, while some are infinite, and they are certainly not all isomorphic to each other. (Mac Lane asks the interesting questions: "What is it about the nature of the world which makes it possible for the same simple structural form to have so many and such varied realizations?. . .why are groups, spaces and rings present in so many widely different realizations?" (183).) On our view, an analogy can be drawn between *objects of different kinds* whenever they have *any* structure in common. But for purposes of testing the ability to draw formal analogies, we confine our attention here to isomorphic cases, for reasons adumbrated above.

What could be a noncredited response in the formal analogy AR test question that we are constructing, when the scenario in Example 1 is the stimulus and Example 2 is the credited response? One possibility is this (adapted from the GRE, ETS 1996, 261):

Example 3

> The members of the Public Service Commission and the members of the Rent Control Commission are to be selected from exactly six qualified candidates: U, V, W, X, Y, and Z. Each commission is to have exactly three members.

The abstract structure of this response differs from that of the stimulus and credited response essentially in that there is no one-to-one mapping of objects to locations in a simple ordering relation. The scenario requires more than one object (candidate) be assigned to each location (commission), and it allows that some of the objects might not be assigned at all and some might be assigned to both locations. The number of possible assignments to each commission is 6x5x4 = 120, and so the total number of possible assignments is 120x120 = 14400, much more than the 720 of the preceding problem, which itself shows that the abstract structure is different on at least some level. In a major research project that attempts the automated generation of the 'logical skeletons' of traditional AR sets with questions, and attempts to calculate predictions of question difficulty (for any specific scenario or verbal 'clothing' for the 'skeleton'), the researchers see the feature of whether or not "more than one X element may be mapped into one Y slot" as the fundamental difference between two kinds of AR sets: "Assign" and "Order" sets, respectively (Brandon, et al. 2000, 5; the researchers employ model-theoretic semantics to determine the 'logical skeletons').

As alluded to earlier, a potential problem with the formal analogy AR test question that we are constructing is that the structure might possibly be *too* simple and transparent, and hence the question *too* easy for the advanced undergraduate target population (if it were just easy, that would be all right, since there is a need for easy items too). One possible solution to this is to base a question on a more complex stimulus (from the February 1990 LSAT):

Example 4

Burroughs and Yoder are the opposing litigants in a case that will be decided in a judicial system that has five levels of courts—local, township, county, state, and federal—in order of increasing authority. Each court has three justices, all of whom vote on each case that comes before their court. In each court above the local level, at least one justice votes in favor of the litigant who won in the preceding court. The system operates in accordance with the following principles:

> The litigants must begin at the local level; they cannot skip levels; and they can use only one court at each level.
>
> At each level a litigant wins if and only if at least two of the three justices vote in the litigant's favor.
>
> At each level, only the litigant who loses can appeal the decision.
>
> A litigant cannot appeal after losing at two consecutive levels.

However, here the scenario is probably too intricate to base a formal analogy question on it; relative to traditional AR questions, the question would be disproportionately, hence overly, long. But a formal analogy question could be based on a smaller part of the stimulus, viz., one of the conditions:

Example 5

Which one of the following principles exemplifies the abstract structure of objects and relations conveyed by the principle above that the litigants must begin at the local level; they cannot skip levels; and they can use only one court at each level?

(A) A request for a computer upgrade must specify just one model; employees cannot request a model that is not in stock; and they can request a computer upgrade only in their first year of employment.

(B) Employees can request a computer upgrade if they do so in the first year of employment or if they are requesting the next higher model; each request must specify just one model.

(C) Employees must complete their first year of employment before requesting a computer upgrade; they can request only one computer upgrade each year; and requests must be made in the first quarter of each year.

(D) Employees who start out with the lowest-grade computer model can upgrade only to the next highest model, and they have the opportunity to do so only once each year.

(E) Computer upgrades are given just once a year; upgrades are allowed only to the next highest model; and employees must start out with the lowest-grade model.

[We are grateful to Victoria Tredinnick, who wrote the original draft of this example.]

Here the intended correct response is (E). Adapting Weitzenfeld's "counterpart" method, we have:

	Original	Counterpart
objects	litigants	employees
relation	'uses'	'uses' (in a different sense?)
objects/location	courts, each at a level	computer models, each for a year
relation (simple ordering)	increasing authority	temporal succession

Notice that even though the stimulus specifies how many litigants there are in the case, as well as how many courts/levels there are, the principle itself leaves this open, as do the responses in the question.

There is a sense in which distinct models can be equivalent, though not *qua* models. For example, the same arrangement of objects from left to right can be represented using either a simple ordering relation, or a mathematical successor operation, or a betweenness relation plus an assignment of one object as 'first'. The resulting models are not isomorphic, because they contain relations with different formal properties (e.g., how many 'places' the relations are). We could imagine test items that would attempt to assess the ability to perceive this kind of equivalence between structures, with a question stem on the order of 'Which one of the following conditions gives the same arrangement of objects as the condition above that. . .?' A related possibility is to test for the ability to perceive disanalogies as follows. Suppose we are given two (sets of) conditions of the sort that appear in AR stimuli and we are asked for a further condition that, if added to the second condition, would force the second situation to be disanalogous to the first.

Example 6

Consider two situations, one for which 1 holds and another for which 2 holds:

1) Each of the five cars in the lot is of a different color.
2) Each of the five members of the team has a different father.

Suppose Brian and Cathy are team members. Which one of the following conditions, if added to 2, would force the second situation to have a significantly different abstract structure than that of the first?

(A) Brian's father is younger than Cathy's father.

(B) Cathy is a better player than Brian.

(C) Brian's father is the brother of Cathy's father.

(D) Brian is Cathy's father.

(E) Brian's mother is also Cathy's mother.

The intended correct response is (D). Almost all of the six or so people we informally tried this question out on eventually gave the correct answer. When asked to explain their answer, at least two of them replied "A car can't be a color," which is a correct and pithy explanation.

One advantage of this kind of item is that it does not rely on a privileged analysis of the situations into objects and relations. We are free to regard *having a different color than* and *having a different father from* as binary relations. Alternatively, imagine that several colors, say, white, brown, and green, and an equal number of fathers, say, Mr. White, Mr. Brown, and Mr. Green, have been named, and construe *being green* and *having Mr. Green as a father* as properties. A third possibility is to treat *having as color* and *having as a father* as binary relations between cars and colors and team members and fathers. What we cannot do is construe a car as a color, given what we know about cars and colors, although we can construe a team member as a father.

A second advantage of this kind of item is that it is difficult, since it requires one to make a distinction between those additional facts that are compatible with a structural similarity between two situations and those that are not.

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