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An Algebraic Basis for the Standard-Model Gauge Group

by

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Abstract

A geometric approach to the standard model in terms of the Clifford algebra $C\ell_7$ is advanced. The gauge symmetries and charge assignments of the fundamental fermions and the Higgs boson are seen to arise uniquely from a simple geometric model involving only four extra space-like dimensions. A key feature of the model is its use of double-sided operations on the algebraic spinors. Transformations separate naturally into left-sided or "exterior", and right-sided or "interior" types. The exterior transformations include those of the Poincaré and $SU(2)_L$ groups, the interior ones include those of $SU(3)_C$, and a unique double-sided form of transformation constitutes the $U(1)_Y$ group . The separation allows a nontrivial coupling of Poincaré and isotopic symmetries within the restrictions of the Coleman–Mandula theorem. The four extra dimensions are also shown to form a natural basis for the Higgs isodoublet field.

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Chapter 1

Introduction

In many physics equations of a fundamental nature, Clifford (geometric) algebras may be employed to recast conventional expressions into more holistic and aesthetically pleasing forms. This approach often reveals insights that were previously obscured by an inappropriate choice of mathematical architecture. The result may allow the consolidation of incongruous terms, suggest missing pieces, or reveal new restrictions imposed by identifying privileged subspaces within the chosen algebra. Moreover, the inclusion of higher dimensions into a model in a geometrically coherent manner requires that careful attention be paid to the algebraic structure of the equations. It is principally in this regard that conventional expressions often prove deficient.

The present dissertation introduces a geometric approach to the minimal standard model in terms of Clifford's geometric algebra[1] $\mathcal{C}\ell_7$. The aim here is to demonstrate how the seemingly disparate gauge symmetries $U(1)_Y \otimes SU(2)_L \otimes SU(3)_C$ arise in a simple algebraic model involving only four extra space-like dimensions. This is far fewer than the minimum of seven extra dimensions required in the usual approach[2] to using the isometries of a higher-dimensional manifold to carry the gauge symmetries, and stems from the availability of double-sided transformations on algebraic spinor elements. The gauge symmetries are also given a firm geometric foundation and are no longer relegated to abstract spaces as in the conventional approach. The model may thus lead to a better understanding of the overall geometry underlying the standard model.

There have been numerous attempts in the past to combine the existing symmetries into an encompassing structure. Many of these have fallen victim to the Coleman-Mandula theorem[3] that disallow most except trivial (i.e., direct-product) couplings of internal and space-time symmetries of the S matrix. One of the motivations of supersymmetric models has been to evade the restrictions of such theorems[4]. The algebraic approach proposed here, which builds on a previous formulation[5] in geometric algebra of the Dirac theory, is distinct in that the spinor, representing a single generation of fermions of the standard model, connects "interior" and "exterior" symmetries (transformations operating from the right and left side of the algebraic spinor respectively), but its two-sided transformations maintain a direct-product group structure. Similar attempts based on Clifford algebra[6] invariably treat time as a vector component of the particular algebra being considered and have not been successful in describing the full gauge group of the standard model. The paravector approach adopted here, where time is assigned to the scalar element of the algebra, has been previously overlooked as a possible candidate for generating the gauge structure from geometric transformations.

In chapter 2, it is useful to first briefly state the conventions adopted, since notations vary considerably even within the geometric-algebra community. An 8×8-matrix representation of $C\ell_7$ is also provided in order to make contact with more conventional expressions. Chapter 3 proceeds to develop the notion of spinors in $C\ell_7$ and shows how a number of different spinor representations contained within the algebra may be ascribed to distinct fermionic particles and currents. In chapter 4, the algebraic interrelationships between these spinors are shown to yield the gauge symmetries of the standard model as geometric symmetries of the generalized current. The gauge symmetries are seen to arise naturally from the algebra itself rather than being imposed on some abstract space. In chapter 5, the various basic terms used in the standard-model Lagrangian density are constructed algebraically, with emphasis placed on the holistic nature of the constructions. Of particular interest is how the four extra spatial dimensions and their transformation properties are precisely what is required for the four components of a minimal Higgs field.

The presentation is somewhat pedagogical with numerous explicit examples, as much of the algebraic formalism may be unfamiliar to many physicists. It is assumed that the reader is familiar with the traditional standard model and its Lagrangian formulation. Comparisons with conventional pieces of the standard model are made throughout the work as they arise, with emphasis placed on some of the deficiencies addressed by the geometric approach. As in the conventional notation, the natural units $c = \hbar = 1$ are used throughout this work.

Chapter 2

Algebraic Foundations

2.1 Geometric Algebra

Clifford algebras are associative algebras of vectors whose product is generally noncommutative. They are extensions of vector spaces, complex numbers, quaternions and Grassmann exterior algebras, and are particularly well suited to modeling the geometry of metric vector spaces in relativity and quantum mechanics. Their introduction by William Kingdon Clifford (1845-1879) predates these applications and was in part an effort to unite the discovery of quaternions by William Rowan Hamilton (1805-1865) and the work of Herman Grassmann (1809-1879) dealing with anticommuting variables. In the conventional vector notation, originated in part by J. Willard Gibbs (1839-1903) and Oliver Heaviside (1850-1925), vector manipulations involving the geometry of a given model are accomplished through a collection of disjoint definitions for operations such as the dot and cross product. It is largely a matter of historical consequence that these formalisms have not been displaced by the more efficient algebraic methods, where all that is required is a defining anticommutator for the base vector elements. The terms arising from the traditional vector operations are naturally contained within the products of algebraic elements. In addition, the algebraic method often provides a clearer geometric rationale for interpreting many equations, as there is a closer contact with the underlying geometry through the defining anticommutator. In the same manner that the various vector manipulations may be encompassed by simple algebraic products, it will be shown how the gauge symmetries of the standard model arise from the underlying algebraic structure of the equations and need not be artificially constructed in purely abstract spaces .

In the real Clifford algebra $C\ell_7$, the unit vector elements $e_1, e_2, ..., e_7$ are chosen to represent space-like directions, with e_1, e_2, e_3 allotted to the three observed (physical) directions. The product of any number of vectors is completely determined by the defining anticommutator

$$e_j e_k + e_k e_j = 2\delta_{jk}, \quad j, k = 1, \ldots, 7, \tag{2.1}$$

or equivalently,

$$\begin{aligned} e_j e_k &= -e_k e_j, \quad j \neq k, \\ e_j^2 &= 1. \end{aligned} \tag{2.2}$$

Although this completely characterizes the algebra and is the only tool required for carrying out algebraic manipulations, it will be useful at times to develop further particular theorems or appeal to a matrix representation.

Through the anticommutator (2.1), all higher-order products of the vectors can be reduced to linear combinations of $2^7 = 128$ basis forms. For example, there is one scalar, 7 vectors, 21 bivectors $e_j e_k$ $(j < k)$, and in general $\binom{7}{r}$ distinct basis forms built from r vectors. Each one of these forms is used to represent a geometric object. For example, $e_1 e_4$ represents the plane spanned by the directions e_1 and e_4, and $e_1e_2e_3$ represents the physical volume element. The defining anticommutator naturally embodies what one would expect geometrically from reversing the directions or the interchanging of any vectors. For example, $e_4e_1 = -e_1e_4$, since interchanging the vectors is equivalent to negating the plane.

Two basic conjugations are used in this work. The reversion of $K \in \mathcal{C}\ell_7$, denoted $K^\dagger$, reverses the order of appearance of all vector elements within K. For example,

$$(e_1e_2e_3)^\dagger = e_3e_2e_1 = -e_1e_2e_3. \tag{2.3}$$

Clifford conjugation, denoted by $\bar{K}$, both reverses the order and negates all vector elements of K. For example,

$$\overline{(e_1e_2)} = (-e_2)(-e_1) = -e_1e_2. \tag{2.4}$$

Both of these operations are antiautomorphic involutions, which can be expressed through the general rules

$$\begin{aligned} (AB)^\dagger &= B^\dagger A^\dagger, \\ \overline{(AB)} &= \overline{B}\,\overline{A}. \end{aligned} \tag{2.5}$$

The sign change induced by basic operations on elements composed of n distinct vectors may be summarized in table 2.1.

In the algebras $\mathcal{C}\ell_n$, the basis vectors e_j can all be taken to be Hermitian, and then reversion is equivalent to Hermitian conjugation. The algebra $\mathcal{C}\ell_7$ is appealing in that the maximal volume element of the algebra (i.e., the hyper-volume of seven spatial dimensions) commutes with all elements and squares to -1. Both of these statements follow from repeated application of Eq. (2.1), and noting that an element that commutes with all of the

Table 2.1. Sign changes in basic algebraic operations.

n	0	1	2	3	4	5	6	7
$e^\dagger = \pm e$	+	+	−	−	+	+	−	−
$\bar{e} = \pm e$	+	−	−	+	+	−	−	+
$\bar{e}^\dagger = \pm e$	+	−	+	−	+	−	+	−
$e^2 = \pm 1$	+	+	−	−	+	+	−	−

vector elements also commutes with any basis form in general. The maximal volume element can therefore be associated identically with the unit imaginary

$$i \equiv e_1 e_2 e_3 e_4 e_5 e_6 e_7, \tag{2.6}$$

where $i^\dagger = -i$. This definition can be used to reduce products of real vectors to elements of a complex space with 64 basis forms. For example,

$$\begin{aligned} e_1 e_2 e_3 e_7 &= -e_1 e_2 e_3 (e_4 e_5 e_6)^2 e_7 \\ &= (e_1 e_2 e_3 e_4 e_5 e_6 e_7)(e_4 e_5 e_6) \\ &= i e_4 e_5 e_6. \end{aligned} \tag{2.7}$$

This fortuitous circumstance occurs for every $\mathcal{C}\ell_{3+4n}$ with non-negative integer n, and $\mathcal{C}\ell_7$ is the smallest of the series that contains the Dirac algebra as a subalgebra. The choice of adding exactly four extra dimensions to physical space will also be further justified below in that they arise naturally from a metric-free approach to physical space and also form a natural basis for the four components of the minimal Higgs field. The presence of an imaginary unit arising from the geometry is an important point as complex vector spaces are central to the tenets of quantum mechanics but seldom is any attention brought to their fundamental origin. Using the algebra $\mathcal{C}\ell_7$ is in fact an extension of enlarging the real numbers to complex numbers, providing an architecture sufficiently elaborate to encompass the standard model.

2.2 Paravectors

The formalism used here builds on the physical applications of $C\ell_3$ (the Pauli algebra), in particular the use of *paravectors*[7, 8] to model space-time vectors. Paravectors are sums of scalars and vectors such as

$$V = V^0 + V^1 e_1 + V^2 e_2 + V^3 e_3 \equiv V^\mu e_\mu, \tag{2.8}$$

where for notational convenience we denote the unit scalar by e_0, and the scalar V^0 is the time component in the observer frame, that is, the frame with proper velocity $e_0 = \bar{e}_0 = 1$. The Minkowski space-time metric $\eta_{\mu\nu}$ with signature $(1,3)$ arises naturally through the square norm of paravectors

$$V\bar{V} = \left\langle V\bar{V}\right\rangle_S = V^\mu V^\nu \left\langle e_\mu \bar{e}_\nu\right\rangle_S = V^\mu V_\mu \tag{2.9}$$

as $\eta_{\mu\nu} = \left\langle e_\mu \bar{e}_\nu\right\rangle_S$. Here, $\langle \cdots \rangle_S$ means the scalar part of the enclosed expression, and we adopt the summation convention for repeated indices, with lower-case Greek indices taking integer values $0 \ldots 3$. The algebra generated by products of paravectors is just $C\ell_3$, which is isomorphic to quaternions over the complex numbers, and it admits a covariant formulation of relativity. It has also been shown to provide a natural formulation of the single-particle Dirac theory[5].

Proper and orthochronous Lorentz transformations of space-time vectors are effected by double-sided transformations of the form[9]

$$V \to LVL^\dagger, \tag{2.10}$$

where L is any unimodular element: $L\bar{L} = 1$. Every such L can be expressed as the product

$$L = \exp(w/2)\exp(\theta/2) \tag{2.11}$$

of a spatial rotation $L_R = \exp(\theta/2)$ in the plane of the bivector $\theta = \frac{1}{2}\theta^{jk}e_j\bar{e}_k$ and a pure boost $L_B = \exp(w/2)$ in the direction of the rapidity $w = w^j e_j$ (or, equivalently, as a hyperbolic rotation in the space-time plane of $w^j e_j \bar{e}_0$). The scalar coefficients satisfy $\theta^{jk} = -\theta^{kj}$ and $w^j = 0$ for $j > 3$.

A key advantage of the formalism is that the generators of the transformations have direct physical significance. For example, the generator $e_1\bar{e}_2$ induces a rotation in the e_1e_2 plane. Explicitly, using the vector (2.8), we have

$$\begin{aligned} V &\rightarrow \exp(-\frac{1}{2}\theta e_1 e_2)V\exp(\frac{1}{2}\theta e_1 e_2) \\ &\rightarrow [\cos(\frac{\theta}{2}) - \sin(\frac{\theta}{2})e_1e_2]V[\cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})e_1e_2] \\ &\rightarrow V^0 + [\cos(\theta)V^1 - \sin(\theta)V^2]e_1 \\ &\quad +[\cos(\theta)V^2 + \sin(\theta)V^1]e_2 + V^3e_3. \end{aligned} \tag{2.12}$$

As a further explicit example, a boost in the e_1 direction would be given by

$$\begin{aligned} V &\rightarrow \exp(\frac{1}{2}\theta e_1)V\exp(\frac{1}{2}\theta e_1) \\ &\rightarrow [\cosh(\frac{\theta}{2}) + \sinh(\frac{\theta}{2})e_1]V[\cosh(\frac{\theta}{2}) + \sinh(\frac{\theta}{2})e_1] \\ &\rightarrow [\cosh(\theta)V^0 + \sinh(\theta)V^1] \\ &\quad +[\cosh(\theta)V^1 + \sinh(\theta)V^0]e_1 + V^2e_2 + V^3e_3. \end{aligned} \tag{2.13}$$

This lucidity will be of particular use when the formalism is extended to higher-dimensional spaces.

Note that a scalar is not necessarily the time component of some space-time vector. The mass m of a particle, for example, may be the time component of the momentum p (in units with $c = 1$) in the rest frame, or it may be the invariant norm of p. The two possibilities are distinguished by how they transform. In particular, the square norm of p transforms as

$$m^2 = p\bar{p} \rightarrow \left(LpL^\dagger\right)\left(\bar{L}^\dagger\bar{p}\bar{L}\right) = p\bar{p} \tag{2.14}$$

whereas the rest-frame momentum becomes

$$me_0 \rightarrow Lme_0L^\dagger = mLL^\dagger. \tag{2.15}$$

In general, the same algebra may be used to accommodate a number of different tensorial objects that are characterized by how they transform.

2.3 Extensions to Higher Dimensions

The extension from $\mathcal{C}\ell_3$ to $\mathcal{C}\ell_7$ requires four additional basis vectors, e_4, e_5, e_6, e_7, that are orthogonal to physical space, namely the span of $\{e_1, e_2, e_3\}$, which generates $\mathcal{C}\ell_3$. If z is any linear combination of e_4, e_5, e_6, e_7, its product with any $K \in \mathcal{C}\ell_3$ satisfies

$$zK = \bar{K}^\dagger z\,. \tag{2.16}$$

This relation follows from simply using the two basic involutions to summarize the sign changes effected by carrying z through to the opposite side. It then follows that z is invariant under any Lorentz transformation (2.10) with

$$L \in \mathcal{C}\ell_3 : z \rightarrow LzL^\dagger = L\bar{L}z = z\,. \tag{2.17}$$

More general rotations in $\mathcal{C}\ell_7$ have the form of Eq. (2.10) but are generated by bivectors that are not restricted to the three spatial planes of $\mathcal{C}\ell_3$. For example, the generator $e_1\bar{e}_4$ induces transformations in the e_1e_4 plane. Such rotations are the only nontrivial linear transformations that leave scalars invariant and transform vectors into vectors in $C\ell_7$.

It is natural to question the meaning, motivation, and apparent absence of the extra dimensions. Of course it may be that the additional dimensions are Kaluza-Klein[10, 11] in nature, but then one can still perform rotations in the tangent space, for example in the e_1e_4 plane or in other planes involving the extra dimensions. Alternatively, the extra dimensions may simply represent degrees of freedom that arise from the algebra but are not associated with a spatial direction. One way to arrive at $C\ell_7$ from $C\ell_3$ is to seek a metric-free foundation for $C\ell_3$. The anticommutation relation (2.1) implies a Euclidean spatial metric, but we may instead start with a three-dimensional metric-free Witt basis[12, 13] of null vectors $\{\alpha_1, \alpha_2, \alpha_3\}$ satisfying

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 0, \quad j,k = 1,2,3. \tag{2.18}$$

A dual space can then be defined as the span of $\{\alpha_1^*, \alpha_2^*, \alpha_3^*\}$, where

$$\alpha_j^*\alpha_k + \alpha_k\alpha_j^* = \delta_{jk}. \tag{2.19}$$

Making the identification

$$e_j = \alpha_j + \alpha_j^*, \tag{2.20}$$

we then have, using (2.18) and (2.19),

$$\begin{aligned} e_je_k + e_ke_j &= (\alpha_j + \alpha_j^*)(\alpha_k + \alpha_k^*) + (\alpha_k + \alpha_k^*)(\alpha_j + \alpha_j^*) \\ &= (\alpha_j^*\alpha_k + \alpha_k\alpha_j^*) + (\alpha_j^*\alpha_k + \alpha_k\alpha_j^*)^* \\ &= 2\delta_{jk}, \end{aligned} \tag{2.21}$$

and the anticommutation relation (2.1) for $C\ell_3$ follows directly. However, there are now three extra linearly independent vectors that we can label

$e_{-j} = \alpha_j - \alpha_j^*$. It is easily verified that the six basis vectors $e_{\pm k}$ anticommute and square to ± 1. The span of $\{e_{\pm k}\}_{1 \leq k \leq 3}$ is a six-dimensional space with the metric signature $(3,3)$. It generates the Clifford algebra $\mathcal{C}\ell_{3,3}$, and its volume element $e_4 \equiv e_{-3}e_{-2}e_{-1}e_1e_2e_3$ squares to $+1$ and anticommutes with the six $e_{\pm k}$. As in the familiar Dirac algebra, the volume element in $\mathcal{C}\ell_{3,3}$ acts as an additional spatial dimension. It can be added to the basis to form a seven-dimensional space with the corresponding universal Clifford algebra $\mathcal{C}\ell_{4,3}$. The algebra $\mathcal{C}\ell_{4,3}$ can be mapped to $\mathcal{C}\ell_7$ if we assume the existence of a scalar unit imaginary element i. We replace the three e_{-j} by elements $e_{4+j} \equiv ie_{-j}$ that square to $+1$. The elements e_k, with $k = 1, 2, \ldots, 7$, then satisfy Eqs. (2.1) and (2.6) and span a seven-dimensional Euclidean space such as used here. The Witt basis elements can now be written

$$\alpha_k = \frac{1}{2}(e_k - ie_{4+k}), \quad k = 1, 2, 3, \tag{2.22}$$

and if we take the e_k to be Hermitian, the dual elements are their Hermitian conjugates: $\alpha_k^* = \alpha_k^\dagger$. The anticommutation relations (2.18,2.19) are just those of fermion annihilation and creation operators, whose products, together with other constructions analogous to Eq. (2.22), can generate the isotopic groups used below.

Another motivation for using $\mathcal{C}\ell_7$ is that the Dirac algebra seems somewhat ill suited to clean geometric interpretations when higher dimensions are added. One sees hints of modeling problems in details such as a γ_5 that anticommutes with all the other gammas and thus acts like an extra dimension, the introduction of i as a distinct algebraic object, and the inclusion of γ_0 in the adjoint spinor $\bar{\psi} \equiv \psi^* \gamma^0$. Such behavior has become so routine that one seldom stops to ponder whether by absorbing the Dirac algebra into a larger, more mathematically uniform algebra we might gain insight.

2.4 Matrix Representation for $C\ell_7$

Building upon any 4×4-matrix representation of the Dirac algebra

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}, \tag{2.23}$$

a faithful 8×8 matrix representation of $C\ell_7$ may be constructed in the block-matrix form

$$\begin{aligned} 1 &\leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad & e_k &\leftrightarrow \begin{pmatrix} -\gamma_0\gamma_k & 0 \\ 0 & -\gamma_0\gamma_k \end{pmatrix}, \\ e_4 &\leftrightarrow \begin{pmatrix} i\gamma_0\gamma_5 & 0 \\ 0 & i\gamma_0\gamma_5 \end{pmatrix}, \quad & e_5 &\leftrightarrow \begin{pmatrix} \gamma_0 & 0 \\ 0 & -\gamma_0 \end{pmatrix}, \\ e_6 &\leftrightarrow \begin{pmatrix} 0 & \gamma_0 \\ \gamma_0 & 0 \end{pmatrix}, \quad & e_7 &\leftrightarrow \begin{pmatrix} 0 & -i\gamma_0 \\ i\gamma_0 & 0 \end{pmatrix}, \end{aligned} \tag{2.24}$$

with $k = 1, 2, 3$. Each basis vector e_k is thus represented by a Hermitian matrix. It can be seen that this representation absorbs γ_0 into the definition of a spatial direction, thus relegating time to the scalar part of the algebra, and then introduces four extra space-like dimensions in accordance with the defining anticommutator (2.1) so that $[i]_{8\times 8}$ arises naturally through the full volume element (2.6). Operations involving these higher dimensions may now be stated and executed cleanly in terms of the basis vectors e_k without having to appeal to confusing products of gamma matrices. It should be noted that Eq. (2.24) is only one of many possible representations of $C\ell_7$ that build upon the Dirac algebra, but it is sufficient as a tool for providing explicit examples of higher-dimensional spinors. Although all that follows may be done without reference to specific matrix representations, the representation (2.24) is useful for making comparisons to conventional expressions. Appendix A includes an explicit representation of $\mathcal{C}\ell_7$ using the Weyl representation of the Dirac algebra.

Chapter 3

Algebraic Spinors and Currents

3.1 Spinors

Algebraic spinors may be defined as entities that transform under the restricted Lorentz group not as vectors, as in Eq. (2.10), but according to the rule

$$\Psi \rightarrow L\Psi. \tag{3.1}$$

In the $\mathcal{C}\ell_3$ version of the Dirac theory[5], the spinor field Ψ is represented by a 2×2 matrix, and it is an amplitude for the bilinear Lorentz transformation [see Eq. (2.10)] relating the rest and laboratory frames of the particle. The current, in particular, corresponds to the transformation of the rest-frame time axis: $J = \Psi\Psi^{\dagger}$.

To describe one generation of the standard model, we use $\mathcal{C}\ell_7$, where Ψ is represented by an 8×8 matrix. The conventional spinor of a massive particle requires four complex numbers. The 128 basis forms of $\mathcal{C}\ell_7$ is therefore able to accommodate a maximum of 16 such spinors. This is sufficient to contain column spinors for the leptons (ν, e) as well as for three colors of quarks (u, d) and all their antiparticles and presumably specifies not only the motion and orientation of the particles in space-time, but in the space spanned by the extra four dimensions as well.

The transformations (3.1) are preserved by multiplication from the right by Lorentz-invariant factors, in particular by real idempotent elements (projectors) that project the spinor Ψ onto left ideals of $\mathcal{C}\ell_7$. In particular, there are eight primitive idempotents that, in terms of matrices, reduce Ψ to a single nonvanishing column. (Examples are given below.) Each such column of Ψ transforms as a spinor [Eq. (3.1)], and the Lorentz operator from the left does not mix components between spinors in different columns. Currents constructed from these spinors reveal symmetries that otherwise must be imposed over abstract spaces in the conventional formalism.

3.2 Currents

In constructing an algebraic expression for the particle current J, we clearly want a form that is bilinear in the spinors, transforms as a vector, and satisfies $J^\dagger = J$. The later requirement ensures that the physical components J^μ of J are real. The simplest solution to this, and the one that we adopt here, has the same form as found for the Dirac theory in $\mathcal{C}\ell_3$:

$$J = \Psi\Psi^\dagger \rightarrow L\Psi\Psi^\dagger L^\dagger. \tag{3.2}$$

One may then ask what higher-dimensional algebraic solutions present themselves if we match the space-time components to the conventional Dirac formalism through

$$J = \lceil\bar{\psi}\gamma^\mu\psi\rfloor e_\mu, \tag{3.3}$$

where the delimiters designate the prevailing non-algebraic notation. A specific component of J may be extracted by contracting it with its associated direction through

$$J_\mu = \langle \Psi \Psi^\dagger \overline{e}_\mu \rangle_S = \langle \Psi^\dagger \overline{e}_\mu \Psi \rangle_S. \tag{3.4}$$

Here, we have used the algebraic property

$$\langle AB \rangle_S = \langle BA \rangle_S \,, \tag{3.5}$$

whose matrix representation through

$$\langle \cdots \rangle_S \leftrightarrow \frac{1}{8} tr(\cdots) \tag{3.6}$$

is the familiar trace theorem

$$tr(AB) = tr(BA). \tag{3.7}$$

Note that if both Ψ and Clifford-conjugated basis element $\overline{e}_\mu$ are subjected to the same Lorentz transformation L, the component J_μ is invariant:

$$\langle \Psi^\dagger \overline{e}_\mu \Psi \rangle_S \rightarrow \langle \Psi^\dagger L^\dagger \bar{L}^\dagger \overline{e}_\mu \bar{L} L \Psi \rangle_S = \langle \Psi^\dagger \overline{e}_\mu \Psi \rangle_S. \tag{3.8}$$

This form of expression will be suitable for later discussions of the Lagrangian density. From the matrix representation for e_k, we see that the form of (3.4) is identical to the conventional expression $\lceil \bar{\psi} \gamma_\mu \psi \rfloor$, except that it is duplicated in the upper and lower portions of the 8×8 matrices, where we simply use the columns of an arbitrary 8×8 matrix for our spinors, admitting four chiral states per column.

It is useful to distinguish transformations from the left with others from the right. Those from the left include the Lorentz transformations as well as transformations in the space of the extra four dimensions that commute with the Lorentz transformations. They are applied to the spinor after the particles have been given the motion and orientation described by Ψ and will be called "exterior" transformations to represent their position, as in Eq. (3.2), in transformations of the current J. Transformations applied from the right will similarly be called "interior". They are applied to the particles in their reference frames, before they acquire the motion and orientation implied by the spinor. It should be noted that exterior transformations are not synonymous with *external* transformations, since the extra four dimensions may relate to properties that are commonly considered to be *internal*.

The primitive idempotents needed to isolate columns of Ψ can be constructed from interior products of three pairs of simple projectors

$$P_{\pm} = P_{\pm}^{\dagger} = P_{\pm}^{2} = \bar{P}_{\mp}, \tag{3.9}$$

where

$$P_{+} + P_{-} = 1 \tag{3.10}$$

and

$$P_{+}P_{-} = 0. \tag{3.11}$$

From among several equivalent choices, we use the three mutually commuting projector pairs

$$\begin{aligned} P_{\pm 3} &\equiv \frac{1}{2}(1 \pm e_3), \\ P_{\pm\alpha} &\equiv \frac{1}{2}(1 \pm i e_4 e_5), \\ P_{\pm\beta} &\equiv \frac{1}{2}(1 \pm i e_6 e_7). \end{aligned} \tag{3.12}$$

For convenience, we abbreviate the product $P_{\pm 3}P_{\pm\alpha}P_{\pm\beta}$ by $P_{\pm\pm\pm}$. One reason for this choice is that in the Weyl γ-matrix representation (appendix A), which we adopt here, the products are simply the eight diagonal matrices with a single unit element. For example,

$$P_{++-} = \mathrm{diag}[1,0,0,0,0,0,0,0], \tag{3.13}$$

and the first-column spinor may be written ΨP_{++-}. Each of the eight primitive projectors $P_{\pm\pm\pm}$, applied from the right, projects Ψ (or other elements) onto one of eight minimal left ideals of $\mathcal{C}\ell_7$. Each column $\Psi P_{\pm\pm\pm}$ is identified with a different type of particle and forms current elements in Eq. (3.2) only with itself.

One pair of simple projectors, applied from the right, can be taken to separate particles from antiparticles. We let this be $P_{\pm 3}$, although this choice will be generalized later. (Note that we could have used ie_1e_2 in place of e_3 in $P_{\pm 3}$, again forming a set of three commuting projectors and only introducing at most sign changes in the $P_{\pm\pm\pm}$ column designations. This alternate construction is again characterized by a vector, in this case the e_3 that is not used explicitly. It is the direction of such a vector, which is inherent in any choice of three commuting projectors, that will later be generalized). Thus, columns $1, 4, 5, 8$, selected by P_{+3} are designated for particles and the remaining columns, selected by P_{-3}, contain antiparticle spinors. Each column contains the spinors for a fermion doublet, and the projectors for the two isotopic-spin components are $P_{\pm\beta}$ applied from the left. In the Weyl γ-matrix representation (appendix A), each four-component spinor in Ψ is split into two-component spinors of right and left chirality. For example, the upper spinor of column one comprises the nonvanishing components of $P_{-\beta}\Psi P_{++-}$:

$$\begin{pmatrix} \Psi_{11} \\ \Psi_{21} \\ \Psi_{31} \\ \Psi_{41} \end{pmatrix} \equiv \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} R \\ L \end{pmatrix}. \tag{3.14}$$

The lower four components Ψ_{51} to Ψ_{81} and the other P_{+3} columns are labeled in a similar manner. The P_{+3} (particle) spinors can be factored explicitly

Table 3.1. The algebraic P_{+3} (particle) spinors for leptons (ℓ) and quarks (q).

fermion	lower spinor	upper spinor
ℓ	$\sqrt{8}(\psi_R e_6 e_5 + \psi_L e_1 e_6)P_{++-}$	$\sqrt{8}(\psi_R + \psi_L e_1 e_5)P_{++-}$
$q(R)$	$\sqrt{8}(\psi_R e_6 e_1 + \psi_L e_6 e_5)P_{+--}$	$\sqrt{8}(\psi_R e_5 e_1 + \psi_L)P_{+--}$
$q(G)$	$\sqrt{8}(\psi_R + \psi_L e_5 e_1)P_{+-+}$	$\sqrt{8}(\psi_R e_5 e_6 + \psi_L e_1 e_6)P_{+-+}$
$q(B)$	$\sqrt{8}(\psi_R e_1 e_5 + \psi_L)P_{+++}$	$\sqrt{8}(\psi_R e_6 e_1 + \psi_L e_5 e_6)P_{+++}$

as in table 3.1, where for the fermion in each half column, ψ_R and ψ_L are given by

$$\psi_R \equiv \psi_0 + \psi_1 e_1,$$
$$\psi_L \equiv \psi_3 - \psi_2 e_1. \tag{3.15}$$

For the sake of brevity, we take the liberty of labeling the spinors with the particle designations shown, although no gauge structure has been determined yet. This is one of many possible arrangements that will later be generalized. The P_{-3} spinors have a similar form but have been excluded for brevity. Indeed, one need only work out the algebraic equivalent of the first column, since the remaining P_{+3} columns are easily obtained by multiplying the first-column spinor from the right by the elements e_5e_1, e_5e_6, e_6e_1, which shifts it to columns 4, 5, 8 respectively. These algebraic spinors transform under $\Psi \to L\Psi$ in the same manner as in the conventional column representation, and they match the conventional current expression $\lceil J^\mu = \bar{\psi}\gamma^\mu\psi \rfloor$ in the algebraic current (3.2).

3.3 Antiparticle Currents

Using the matrix representation, one may verify that the chiral projector for all fermions in the Weyl representation is an *exterior* operator (operating from the *left-hand* side) given by

$$P_{R/L} = P_{\bar{L}/\bar{R}} = \frac{1}{2}(1 \pm e_4 e_5 e_6 e_7). \tag{3.16}$$

The chirality is flipped by the transformation

$$\Psi \rightarrow -e_1 e_2 e_3 e_4 \Psi. \tag{3.17}$$

This has the effect of reversing the vector components of the current (3.2) in the span of $\{e_1, e_2, e_3, e_4\}$ while leaving the components in the span of $\{e_0, e_5, e_6, e_7\}$ invariant.

Charge conjugation can be defined by the algebraic operation

$$\Psi_C = i e_4 \overline{\Psi}^{\dagger}. \tag{3.18}$$

The combination of the two antiautomorphic involutions obeys the rule

$$\overline{(AB)}^{\dagger} = \overline{A}^{\dagger} \overline{B}^{\dagger}. \tag{3.19}$$

The conjugate spinor transforms under a Lorentz transformation (3.1) in the same manner as the particle spinor. Using rule (2.17), we have

$$\Psi_C \rightarrow i e_4 \overline{L}^{\dagger} \overline{\Psi}^{\dagger} = L i e_4 \overline{\Psi}^{\dagger} = L \Psi_C \ . \tag{3.20}$$

The conjugate of the upper spinor of the first column (see table 3.1), for example, is

$$\Psi_C = i e_4 \sqrt{8} (\bar{\psi}_R^{\dagger} + \bar{\psi}_L^{\dagger} e_1 e_5) P_{--+} \tag{3.21}$$

where

$$\bar{\psi}_R^{\dagger} = \psi_0^* - \psi_1^* e_1. \tag{3.22}$$

Table 3.2. Column designations of the matrix representation of Ψ.

ℓ	$-\bar{q}(G)$	$\bar{q}(B)$	$q(R)$	$q(G)$	$\bar{\ell}$	$-\bar{q}(R)$	$q(B)$
ν_R	$\bar{d}_L$	$-\bar{d}_L$	u_R	u_R	$-\bar{e}_L$	$\bar{d}_L$	u_R
ν_L	$\bar{d}_R$	$-\bar{d}_R$	u_L	u_L	$-\bar{e}_R$	$\bar{d}_R$	u_L
e_R	$-\bar{u}_L$	$\bar{u}_L$	d_R	d_R	$\bar{\nu}_L$	$-\bar{u}_L$	d_R
e_L	$-\bar{u}_R$	$\bar{u}_R$	d_L	d_L	$\bar{\nu}_R$	$-\bar{u}_R$	d_L

In the matrix representation, this operation is equivalent to defining the conventional charge conjugates through

$$\lceil \psi_C = i\gamma^2\psi^* \rfloor \tag{3.23}$$

and exchanging the upper and lower components as shown in table 3.2. Note that charge conjugation reverses the sign of all the projectors and shifts the particle spinors to the P_{-3} columns. This will turn out to be structurally advantageous in the later gauge symmetries.

Geometrically, charge conjugation transforms the particle current as

$$J = \Psi\Psi^\dagger \rightarrow e_4\bar{\Psi}^\dagger\bar{\Psi}e_4 = e_4\bar{J}e_4 \tag{3.24}$$

and has the effect of negating the e_4 component while leaving all other directions invariant. Aside from the arbitrary direction chosen, this is a unique discrete symmetry of the higher-dimensional directions in that it is not accessible by a rotation. The negation of two or four directions can be achieved by rotations, and to negate three directions one simply reverses one direction followed by reversing another two or four. The choice of e_4 and the phase introduced in Eq. (3.18) is merely a convenient choice for the representation used. The effect of this transformation on charge currents will be apparent in the next section.

The total current obtained by simply adding all the left ideal doublets into a single element Ψ is then

$$J \equiv \Psi\Psi^\dagger = \sum_{a=1}^{16} \lceil J^\mu_{(a)} \rfloor e_\mu + (\text{higher-dim. terms}). \tag{3.25}$$

The sum here runs over the 16 four-component spinors assigned to the upper and lower halves of the eight minimal left ideals, each of which may be ascribed to distinct fermions. The residual part of the current involves cross-current terms between the upper and lower fermions of the same ideal, and mass-like terms of the form $\lceil \bar{\psi}\psi \rfloor$, all projected onto higher-dimensional elements.

It should be noted that a similar primitive idempotent structure for particle doublets has been proposed for the algebra $\mathcal{C}\ell_{1,6}$[6]. However, in spite of an isomorphism between $\mathcal{C}\ell_7$ and $\mathcal{C}\ell_{1,6}$, the use of paravectors here, specifically the relegation of time to the scalar component of the algebra, provides additional degrees of freedom. Indeed, it corresponds to a 2-to-1 mapping of the larger $\mathcal{C}\ell_{1,7}$ onto its even subalgebra $\mathcal{C}\ell^+_{1,7} \simeq \mathcal{C}\ell_7$.

The main idea of this section has simply been that, instead of writing a separate term for each particle current, they may be consolidated into a single expression that accommodates a number of spinorial representations. The advantage of using the algebraic formalism becomes evident when we enumerate all of the possible algebraic symmetries of this current. Although each of these representations transforms identically under a Lorentz transformation, gauge transformations may induce a different change on each due to their distinct projector structures. The charges of each of the fermions will be shown to result from this enriched architecture. The antiparticles occupy separate columns so that one encompassing gauge structure may be used to describe all the observed currents of one generation.

Chapter 4

Gauge Symmetries

4.1 Motivation

The current of Eq. (3.2) holds all the chiral currents of a single generation of the standard model, with distinct antiparticle currents. In this section, we show that within this framework, transformations that leave both the physical components of the current and right-chiral neutrino (and left-chiral antineutrino) invariant lead to symmetries of the currents arising naturally from the algebra itself. Specifically, we enumerate all possible continuous transformations on Ψ that transform vector components of the current into vectors, leave invariant the physical components of the current, and only mix the 15 remaining chiral spinors among themselves. This is analogous to the conventional case where one notices that $\lceil \psi \rightarrow \exp(i\theta)\psi \rfloor$ is a symmetry of the current, but now we consider all algebraic generators in general. This involves investigating various generators acting from both the left and right of the algebraic spinor, as these generators do not necessarily commute with Ψ. Furthermore, by combining the currents into the single form of Eq. (3.2), we uncover relationships among the fermions that are otherwise imposed over abstract spaces. The underlying idea is essentially complete at this point, and the rest of this chapter is devoted to showing in detail how it results in the gauge group of the standard model. While it will turn out that the generators of these symmetries are almost exclusively bivectors, we keep the construction as general as possible by considering all algebraic generators.

4.2 Exterior $SU(2)$ Transformations

We begin by considering exterior transformations

$$\Psi \rightarrow G\Psi = \exp(\theta T)\Psi \tag{4.1}$$

that leave the physical components of the form $\Psi\Psi^\dagger$ invariant, where T is some candidate algebraic generator. From the infinitesimal form

$$J \rightarrow (1 + \theta T)\Psi\Psi^\dagger(1 + \theta T^\dagger), \tag{4.2}$$

it is clear that $T = -T^\dagger$ in order to leave the scalar time component (e_0 component) of J invariant. This limits the choice to basis forms built from 2, 3, 6 or 7 vectors. We also have the restriction that the spatial components e_1, e_2, e_3 of the current should be invariant, so T must also commute with each of these elements. This further reduces the choices down to any linear combination of i, $e_1e_2e_3$, and the six bivector generators

$$e_je_k : (j, k) \in \{4, 5, 6, 7\},\ j < k. \tag{4.3}$$

We can eliminate both i and $e_1e_2e_3$ since it can be shown that every linear combination of them will change the phase of a conjugate pair of Weyl spinors by the same nonvanishing amount and therefore conflict with the definition (3.18) of charge conjugation. Note that all of the remaining generators are linear combinations of bivectors that rotate the higher-dimensional vector components of the current among themselves. There is some physics here in that these directions are presumed to be unobservable directly, so this is a statement that the orientation of the components should not matter at this point. As seen above, generators comprising products of e_4, e_5, e_6, e_7 are also invariant under Lorentz transformations and therefore generate intrinsic (rest-frame) transformations. The higher-dimensional bivector generators commute with any Lorentz generator, so a direct product structure is maintained between the Lorentz group and any group formed using these new generators.

Exclusion of the right-chiral neutrino (and the left-chiral antineutrino) imposes restrictions on possible rotations to ensure its Weyl spinor does not participate in any of the symmetry transformations. Simple rotations in any single plane in the span of $\{e_4, e_5, e_6, e_7\}$ transform both ν_R and $\bar{\nu}_L$. Only by combining rotations in orthogonal planes can we satisfy the exclusion. We therefore combine the six higher-dimensional bivectors into three pairs

$$\begin{aligned} T_1 &= \frac{1}{4}(e_6e_4 + e_5e_7), \\ T_2 &= \frac{1}{4}(e_7e_4 + e_6e_5), \\ T_3 &= \frac{1}{4}(e_5e_4 + e_7e_6). \end{aligned} \tag{4.4}$$

These combinations implicitly contain the left-chiral projector (3.16), for example

$$2T_1 = e_6e_4P_L, \tag{4.5}$$

and therefore act only on left-chiral particles and right-chiral antiparticles. The three generators (4.4), which induce simultaneous rotations in a pair of commuting planes, satisfy

$$[T_a, T_b] = f_{abc} T_c, \tag{4.6}$$

with the fully antisymmetric structure constants f_{abc} where $f_{123} = 1$. The conventional presence of the unit imaginary in front of T_c has been absorbed into the properties of bivectors.

The effect of the transformation

$$\Psi \rightarrow \exp(\theta_a T_a)\Psi \tag{4.7}$$

on the total spinor set is identical to that of the prevailing $SU(2)$ prescriptions

$$\begin{aligned} &\begin{pmatrix} \nu_L & u_L & -\bar{e}_R & -\bar{d}_R \\ e_L & d_L & \bar{\nu}_R & \bar{u}_R \end{pmatrix} \\ \rightarrow \quad & \exp(-i\theta_a \sigma_a/2) \begin{pmatrix} \nu_L & u_L & -\bar{e}_R & -\bar{d}_R \\ e_L & d_L & \bar{\nu}_R & \bar{u}_R \end{pmatrix} \end{aligned} \tag{4.8}$$

where σ_a are the Pauli spin matrices. Although the defining anticommutator of the algebra is the only tool required to verify these comparisons, this is an exercise where appealing to the matrix representations of the generators is easier: operations from the left shuffle entire rows about in the matrix representation but do not shift columns, so the assignment of doublets to columns is still arbitrary. Appendix B shows the computed matrix representations of the generators (4.4), where one can see that the conventional Pauli matrices are contained as submatrices operating on the left-chiral states. The three linearly independent generators formed by replacing the + signs in Eq. (4.4) by − signs, and indeed any linear combination of them, all have the form bP_R, where b is a bivector. They thus couple with ν_R and its conjugate and are therefore eliminated.

4.3 Interior $SU(3)$ Transformations

Now let us look at the possible ***interior*** transformations

$$\Psi \rightarrow \Psi G' = \Psi \exp(\theta T'). \tag{4.9}$$

To emphasize the fact that they act on the ***right*** side of Ψ, all interior operators and generators are denoted with a prime. Any such unitary transformation will leave $J = \Psi\Psi^\dagger$ invariant, but we want to ensure a stronger condition: we demand that the particle and antiparticle currents are separately invariant. Mathematically, we split the current in two using the $\Psi P_{\pm 3}$ spinors

$$\begin{aligned} J &= \frac{1}{2}\Psi(1+e_3)\Psi^\dagger + \frac{1}{2}\Psi(1-e_3)\Psi^\dagger \\ &\equiv J_{+3} + J_{-3} \end{aligned} \tag{4.10}$$

and require each part to be invariant. Recall that the P_{+3} and P_{-3} spinors are allotted to particles and antiparticles respectively, as in table 3.1. It should be noted here that the interior projectors do not Lorentz transform; they represent a choice in the intrinsic or rest-frame structure of the particles and are not altered by a Lorentz transformation operating from the opposite side of the spinors. This also applies to generators used in between Ψ and $\Psi^\dagger$. Thus, we may involve the elements e_1, e_2, e_3 in the interior symmetries while evading the Coleman–Mandula theorem[3] that prohibits any non-trivial combination of the Poincaré and isotopic groups. Considering the infinitesimal transformation

$$\Psi \rightarrow \Psi(1+\theta T'), \tag{4.11}$$

we have

$$J_{\pm 3} \rightarrow \frac{1}{2}\Psi(1+\theta T')(1 \pm e_3)(1+\theta T'^\dagger)\Psi^\dagger, \tag{4.12}$$

which may be viewed as a transformation of the central $P_{\pm 3}$ projector. We can see that the space of available bivector generators that leave e_3 invariant is now spanned by the larger set of 15 bivectors

$$e_j e_k : (j,k) \in \{1,2,4,5,6,7\},\ j < k. \tag{4.13}$$

Insulating the right-chiral neutrino from interior transformations in a similar manner as before now requires that the lepton columns (1 and 6 in the representation adopted) be avoided. This determines the surviving terms

$$\begin{aligned}
T_1' &= \frac{1}{4}(e_1 e_7 + e_6 e_2), \\
T_2' &= \frac{1}{4}(e_1 e_6 + e_2 e_7), \\
T_3' &= \frac{1}{4}(e_1 e_2 + e_7 e_6), \\
T_4' &= \frac{1}{4}(e_6 e_4 + e_5 e_7), \\
T_5' &= \frac{1}{4}(e_4 e_7 + e_5 e_6), \\
T_6' &= \frac{1}{4}(e_4 e_1 + e_2 e_5), \\
T_7' &= \frac{1}{4}(e_1 e_5 + e_2 e_4), \\
T_8' &= \frac{1}{4\sqrt{3}}(e_2 e_1 + 2 e_5 e_4 + e_7 e_6),
\end{aligned} \tag{4.14}$$

arranged to display the resulting $SU(3)$ symmetry satisfying

$$[T_a', T_b'] = -f_{abc} T_c', \tag{4.15}$$

with the antisymmetric structure constants f_{abc} where

$$\begin{aligned}
f_{123} &= 1, \\
f_{147} &= f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}, \\
f_{458} &= f_{678} = \frac{\sqrt{3}}{2}.
\end{aligned} \tag{4.16}$$

Computing the matrix representation for each of these generators, as shown in Appendix B, the transformation

$$\Psi \rightarrow \Psi \exp(\theta_a T_a') \tag{4.17}$$

can be seen to be identical in its effect on the P_{+3} spinor components to

$$(R, G, B) \rightarrow (R, G, B) \exp(-i\theta_a \lambda_a^*/2), \tag{4.18}$$

where λ_a are the Gell-Mann matrices. This is equivalent to the more familiar.

$$\begin{pmatrix} R \\ G \\ B \end{pmatrix} \rightarrow \exp(-i\theta_a \lambda_a/2) \begin{pmatrix} R \\ G \\ B \end{pmatrix}. \tag{4.19}$$

It must be emphasized here that operations from the left shuffle rows whereas operations from the right shuffle columns. These two operations commute with each other so it is of no consequence that the generators from the left do not necessarily commute with the generators acting from the right. They act on independent structural elements (rows and columns) of Ψ and thus effect transformations as if they were two commuting symmetries in an abstract space. This is basically how these gauge groups arise from only four extra dimensions.

Under the same algebraic operation, the effect of the remaining submatrices on the conjugate spinors $(-\bar{G}, \bar{B}, -\bar{R})$ is equivalent to

$$(\bar{R}, \bar{G}, \bar{B}) \rightarrow (\bar{R}, \bar{G}, \bar{B}) \exp(i\theta_a \lambda_a/2), \tag{4.20}$$

which is the correct transformation. Note that in the case of $SU(2)$, the fact that the doublets can be written in the same representation by using the column $(-\bar{d}, \bar{u})$ is a special property of $SU(2)$. Such a construction is not possible in the case of the $SU(3)$ triplet, but the geometric symmetries here provide a separate set of $SU(3)$ submatrices, one in terms of $-\lambda_a^*$ and the other λ_a, operating on the two carrier spaces. This is an advantage of having the conjugate spinors in a separate set of columns in that the same algebraic symmetry applies to both particles and antiparticles.

In considering the other possible interior generators, the invariance of the time components again restricts them to basis forms built from 2, 3, 6, or 7 vectors, and the condition $T' = \bar{T}'^{\dagger}$, required for consistency with charge conjugation, further restricts T' to be an even element. The remaining candidate ie_3, which is the only 6-vector that commutes with the $P_{\pm 3}$ projector, may be excluded since it does not avoid ν_R or $\bar{\nu}_L$.

4.4 Double-sided $U(1)$ Symmetry

There remains one potential symmetry that has not yet been exploited. We may consider a synchronized double-sided transformation that has the effect of various phase changes on the spinors and that conspires to cancel out in the case of the right-chiral neutrino. As this is to represent a distinct symmetry, the left- and right-side generators must commute with all $SU(2)$ and $SU(3)$ generators respectively. We may resurrect the previously discarded generators $(e_4e_5 + e_7e_6)$ acting from the left, and ie_3, $(e_1e_2 + e_5e_4 + e_6e_7)$ operating from the right. One may verify with the infinitesimal operator

$$\Psi \rightarrow (1 + \theta_0 T_0)\Psi(1 + \theta_0 T_0') \tag{4.21}$$

that the solution for which there is no phase change on the right-chiral neutrino may be normalized to

$$\begin{aligned} T_0 &= \frac{1}{2}(e_4e_5 + e_7e_6), \\ T_0' &= \beta(e_1e_2 + e_5e_4 + e_6e_7) + (1-3\beta)ie_3. \end{aligned} \qquad (4.22)$$

Applying this operation to each spinor in turn proves to be identical to the $U(1)_Y$ transformation

$$\psi_{(j)} \longrightarrow \exp(-i\theta_0 Y_{(j)})\psi_{(j)} \qquad (4.23)$$

with the weak hypercharge assignments

$$\begin{aligned} Y(\nu_R, \nu_L, e_R, e_L) &= (0, -1, -2, -1) \\ &= -Y(\bar{\nu}_L, \bar{\nu}_R, \bar{e}_L, \bar{e}_R), \\ Y(u_R, u_L, d_R, d_L) &= (4\beta, 4\beta - 1, 4\beta - 2, 4\beta - 1) \\ &= -Y(\overline{u}_L, \overline{u}_R, \overline{d}_L, \overline{d}_R) \end{aligned} \qquad (4.24)$$

This produces the conventional weak hypercharge assignments for the leptons. The ie_3 term is eliminated by restricting the set of generators to those with geometrical significance, namely bivectors. This implies $\beta = 1/3$, which gives the correct values for the quarks as well.

The previous transformations may now be combined into a single expression

$$\Psi \longrightarrow \exp(\theta_0 T_0 + \theta_a T_a)\Psi \exp(\theta_0 T_0' + \theta_b' T_b'), \qquad (4.25)$$

exhausting the algebraic gauge symmetries. The complete gauge group of the standard model is now seen to arise from a single encompassing transformation on the spinor set, including both particles and antiparticles, and describes the rotational invariances of the physical part of a generalized current expression. The use of the correct algebra is essential here, since even if one were to construct such a formalism by embedding the gauge generators into 8×8 matrices that commuted with the Lorentz group generators, the identification of these generators as a privileged subspace of bivectors would not be evident.

4.5 Electromagnetism

In the standard model, the electromagnetic charge is obtained by mixing the $U(1)$ symmetry with the T_3 component of the $SU(2)$ symmetry. The details of the gauge fields associated with these symmetries will be handled in chapter 5, but it is illustrative at this point to show how the algebraic formalism provides a simple derivation of why this mixing is necessary.

The laws of electromagnetism are invariant under a parity transformation of the spatial coordinates. Furthermore, electromagnetic interactions do not change the identity of particles, so only generators that induce a phase change need be considered. The generators T_3' and T_8' are excluded since no linear combination has a uniform effect on the three colours of quarks, which are postulated to all have the same electric charge for a given type. Therefore, in order to derive the electromagnetic charges we must look for linear combinations of the generators $\{T_0 \oplus T_0', T_3\}$ yielding assignments that are invariant under the parity transformation of Eq. (3.17). Since the generators from the left must commute with the parity operator $-e_1e_2e_3e_4$, the only possibility is to isolate the generator $e_7e_6 = (T_3 + \frac{1}{2}T_0)$ on the left. The $U(1)_{em}$ symmetry is then specified by the generators

$$\begin{aligned} T_{em} &= \frac{1}{2}e_7e_6, \\ T'_{em} &= \frac{1}{6}(e_1e_2 + e_5e_4 + e_6e_7) \end{aligned} \tag{4.26}$$

Table 4.1. Charge assignments of the fermions.

	T_3	Y	Q		T_3	Y	Q
ν_L	1/2	-1	0	$\bar{\nu}_L$	0	0	0
ν_R	0	0	0	$\bar{\nu}_R$	-1/2	1	0
e_L	-1/2	-1	-1	$\bar{e}_L$	0	2	1
e_R	0	-2	-1	$\bar{e}_R$	1/2	1	1
u_L	1/2	1/3	2/3	$\bar{u}_L$	0	-4/3	-2/3
u_R	0	4/3	2/3	$\bar{u}_R$	-1/2	-1/3	-2/3
d_L	-1/2	1/3	-1/3	$\bar{d}_L$	0	2/3	1/3
d_R	0	-2/3	-1/3	$\bar{d}_R$	1/2	-1/3	1/3

operating simultaneously from the left and right respectively. It is readily verified that these furnish the correct electromagnetic charge assignments for each of the fermions with

$$\Psi \longrightarrow \exp(\theta_{em} T_{em}) \Psi \exp(\theta_{em} T'_{em}) \tag{4.27}$$

being identical to

$$[\psi_{(j)} \longrightarrow \exp(-i\theta_{em} Q_{(j)}) \psi_{(j)}]. \tag{4.28}$$

This is essentially the basis of the Gell-Mann-Nishijima formula[14, 15]

$$Q = T_3 + \frac{1}{2} Y, \tag{4.29}$$

as this is the relation obeyed by the generators for each of the corresponding charges, as listed in table 4.1.

4.6 Alternate Representations

The $U(1)_Y \otimes SU(2)_L \otimes SU(3)_C$ result here is a general consequence of the algebra not specific to the ΨP_{+3} spinors. Any arbitrary fitting of the doublets into some orthogonal linear combination of the columns is accessible by shuffling the P_{+3} columns through a transformation $\Psi \rightarrow \Psi S$ where $SS^\dagger = 1$. The constraint that the transformations are consistent with charge conjugation demands that Ψ and $\bar{\Psi}^\dagger$ transform in the same way, and this implies that S is an even element, comprising only terms with products of an even number of vectors. The accompanying similarity transformations $T'_a \rightarrow S^\dagger T'_a S$ and $P_{+3} \rightarrow S^\dagger P_{+3} S$ maintain the results, with the former preserving the structure constants of the interior symmetries.

In any other set in which the interior generators are written solely as bivectors, the same weak hypercharge assignments are obtained. This can be shown by considering the transformations

$$\begin{aligned} e_1e_2 &\rightarrow S^\dagger e_1e_2S = b_1 + f, \\ e_4e_5 &\rightarrow S^\dagger e_4e_5S = b_2 - f, \\ e_6e_7 &\rightarrow S^\dagger e_6e_7S = b_3 + f, \end{aligned} \tag{4.30}$$

where b_1, b_2, b_3 are bivectors and f is some 6-vector. Note that a 4-vector result is ruled out by the reversion of the terms in (4.30). For example,

$$(S^\dagger e_1e_2S)^\dagger = -S^\dagger e_1e_2S. \tag{4.31}$$

The transformations (4.30) parameterize the most general case where T'_3 and T'_8 both transform to bivectors, and the bivector portion of T_0 in Eq. (4.22) is transformed to

$$\beta S^\dagger(e_1e_2 + e_5e_4 + e_6e_7)S = \beta(b_1 - b_2 + b_3 + 3f). \tag{4.32}$$

We must initially allow for the possibility of f arising since it may mix with the transformation of the remaining $(1 - 3\beta)ie_3$ portion of T_0 and alter the value of β determined by restricting the generators to bivectors. However, by considering contractions of the form

$$\begin{aligned} \langle S^\dagger e_1 e_2 S f \rangle_S &= \langle (e_1 e_2 S f S^\dagger \rangle_S \\ &= \langle (e_1 e_2 S (R i e_3 R^\dagger) S^\dagger \rangle_S, \end{aligned} \tag{4.33}$$

where R is some rotation operator, we can see that the term

$$SR(ie_3)R^\dagger S^\dagger = SR(e_1 e_2)(e_4 e_5)(e_6 e_7)R^\dagger S^\dagger \tag{4.34}$$

must contain the bivectors e_1e_2, e_5e_4, e_6e_7 in equal proportions. The element SR will also be even and satisfy the property $(SR)(R^\dagger S^\dagger) = 1$. It is not difficult to convince oneself that such an element does not exist, therefore it must be that f does not exist. The transformation S then preserves the value of $\beta = 1/3$, at most rotating e_3 to some new direction

$$ie_3 \to iS^\dagger e_3 S. \tag{4.35}$$

This framework then gives a geometric basis for the gauge group of the standard model, which arises unambiguously through the various rotational symmetries of the algebraic current in the seven-dimensional space of $C\ell_7$.

Chapter 5

Lagrangian Terms

5.1 Gauge Fields

In constructing the Lagrangian density (Lagrangian) of the standard model, the basic strategy is to consider all possible terms that are invariant under both Lorentz and local gauge transformations. The latter condition is accomplished by adding pieces to the basic Lorentz-invariant derivative terms that compensate for the derivatives arising from gauge transformations that may vary from point to point in space-time. The fundamental reason why gauge invariance should be a guiding principle of the universe is not at present understood, although it has been shown to be a necessary feature of any renormalizable theory[4]. Any formalism that illuminates a stronger connection between the Lorentz and gauge generators may prove useful in addressing this question.

The transformations encompassed by Eq. (4.25) may be locally gauged by introducing vector gauge fields $\{B, W_a, G_a\} \in \mathcal{C}\ell_3$ that transform according to

$$\bar{B} \to \bar{B} + \frac{2}{g'}\bar{\partial}\theta_0,$$
$$\bar{W}_a \to \bar{W}_a + \frac{1}{g}\bar{\partial}\theta_a + f_{abc}\theta_b\bar{W}_c,$$
$$\bar{G}_a \to \bar{G}_a + \frac{1}{g_s}\bar{\partial}\theta'_a + f_{abc}\theta'_b\bar{G}_c \tag{5.1}$$

into the Lagrangian derivative terms

$$\begin{aligned}\mathcal{L}_\partial = &\langle\Psi^\dagger i\bar{\partial}\Psi\rangle_S \\ &- \frac{g'}{2}\langle\Psi^\dagger i\bar{B}(T_0\Psi + \Psi T'_0)\rangle_S \\ &- g\langle\Psi^\dagger i\bar{W}_a T_a\Psi\rangle_S \\ &- g_s\langle\Psi^\dagger i\bar{G}_a\Psi T'_a\rangle_S.\end{aligned} \tag{5.2}$$

The algebraic operator used in the derivative term is defined by

$$\bar{\partial} \equiv \partial_0 + \partial_1 e_1 + \partial_2 e_2 + \partial_3 e_3, \tag{5.3}$$

with the derivatives acting to the right. To avoid confusion when involutions are invoked, we suspend the often used bidirectional derivative form $\overrightarrow{\partial} - \overleftarrow{\partial}$ in favour of simply redefining the scalar delimiter to be the real scalar part of the algebra, although either definition is tenable. In either case, care must be taken when applying antiautomorphic involutions since they reverse the direction of operation of the derivatives. For example,

$$\partial \equiv \partial_0 - \partial_1 e_1 - \partial_2 e_2 - \partial_3 e_3$$

operates on the left with a positive sign.

To show gauge invariance in detail it is sufficient to look at the infinitesimal spinor transformations. Consider first the exterior $SU(2)$ generators.

Note that all bivector generators uniformly obey $T^\dagger = \bar{T} = -T$, and all exterior T commute with the physical gauge fields. The Lagrangian derivative term transforms to first order in θ_a as

$$
\begin{aligned}
&\langle \Psi^\dagger i\bar{\partial}\Psi\rangle_S \\
\rightarrow\ &\langle \Psi^\dagger (1-\theta_a T_a) i\bar{\partial}(1+\theta_b T_b)\Psi\rangle_S \\
\rightarrow\ &\langle \Psi^\dagger i\bar{\partial}\Psi\rangle_S + \langle \Psi^\dagger i(\bar{\partial}\theta_b)T_b\Psi\rangle_S . \qquad (5.4)
\end{aligned}
$$

The relevant gauge field term transforms to first order as

$$
\begin{aligned}
-g\langle \Psi^\dagger i\bar{W}_a T_a\Psi\rangle_S \quad \rightarrow \quad & -g\langle \Psi^\dagger i(\bar{W}_a + \frac{1}{g}\bar{\partial}\theta_a + f_{abc}\theta_b\bar{W}_c)T_a\Psi\rangle_S \\
& +g\langle \Psi^\dagger (\theta_b T_b) i\bar{W}_a T_a\Psi\rangle_S \\
& -g\langle \Psi^\dagger i\bar{W}_a T_a(\theta_b T_b)\Psi\rangle_S \\
\rightarrow \quad & -g\langle \Psi^\dagger i\bar{W}_a T_a\Psi\rangle_S \\
& -\langle \Psi^\dagger i(\bar{\partial}\theta_b)T_b\Psi\rangle_S, \qquad (5.5)
\end{aligned}
$$

where we have used $[T_a, T_b] = f_{abc}T_c$ and

$$
\begin{aligned}
f_{bac}T_c\theta_b W_a &= f_{bca}T_a\theta_b W_c \\
&= f_{abc}T_a\theta_b W_c . \qquad (5.6)
\end{aligned}
$$

Note that $SU(2)$ transformations on the spinor leave the remaining Lagrangian terms invariant since all T_a commute with T_0 and the interior generators are on the opposite side of the spinor. The latter term in (5.4) then cancels with that in (5.5), leaving the Lagrangian invariant.

Gauge invariance for the $SU(3)$ transformations follows in a similar manner, except that the analogy of trace theorems as in Eq. (3.5) is used to move generators across to the right side in the gauge-coupling term. In detail,

$$\begin{aligned}
&\langle\Psi^\dagger i\bar{\partial}\Psi\rangle_S \\
\rightarrow\ &\langle(1-\theta'_a T'_a)\Psi^\dagger i\bar{\partial}\Psi(1+\theta'_b T'_b)\rangle_S \\
\rightarrow\ &\langle\Psi^\dagger i\bar{\partial}\Psi\rangle_S + \langle\Psi^\dagger i(\bar{\partial}\theta'_b)\Psi T'_b\rangle_S
\end{aligned} \tag{5.7}$$

is compensated for by the term

$$\begin{aligned}
&-g_s\langle\Psi^\dagger i\bar{G}_a\Psi T'_a\rangle_S \\
\rightarrow\ &-g_s\langle\Psi^\dagger i(\bar{G}_a + \frac{1}{g_s}\bar{\partial}\theta'_a + f_{abc}\theta'_b\bar{G}_c)\Psi T'_a\rangle_S \\
&+g_s\langle\Psi^\dagger i\bar{G}_a\Psi T'_a(\theta'_b T'_b)\rangle_S \\
&-g_s\langle\Psi^\dagger i\bar{G}_a\Psi(\theta'_b T'_b)T'_a\rangle_S \\
\rightarrow\ &-g_s\langle\Psi^\dagger i\bar{G}_a\Psi T'_a\rangle_S - \langle\Psi^\dagger i(\bar{\partial}\theta'_b)\Psi T'_b\rangle_S.
\end{aligned} \tag{5.8}$$

Note that the choice of originally introducing a minus sign in $[T'_a, T'_b] = -f_{abc}T'_c$ is an artifact of the double-sided transformations and done merely to maintain all plus signs in Eq. (5.1).

The remaining $U(1)$ spinor transformation,

$$\begin{aligned}
&\langle\Psi^\dagger i\bar{\partial}\Psi\rangle_S \\
\rightarrow\ &\langle(1-\theta_0 T'_0)\Psi^\dagger(1-\theta_0 T_0)i\bar{\partial}(1+\theta_0 T_0)\Psi(1+\theta_0 T'_0)\rangle_S \\
\rightarrow\ &\langle\Psi^\dagger i\bar{\partial}\Psi\rangle_S + \langle\Psi^\dagger i(\bar{\partial}\theta_0)T_0\Psi\rangle_S + \langle\Psi^\dagger i(\bar{\partial}\theta_0)\Psi T'_0\rangle_S,
\end{aligned} \tag{5.9}$$

is compensated for by the term

$$\begin{aligned}
&-\frac{g'}{2}\langle\Psi^\dagger i\bar{B}(T_0\Psi + \Psi T'_0)\rangle_S \\
\rightarrow\ &-\frac{g'}{2}\langle\Psi^\dagger i(\bar{B} + \frac{2}{g'}\bar{\partial}\theta_0)(T_0\Psi + \Psi T'_0)\rangle_S.
\end{aligned} \tag{5.10}$$

Since T_0 and T'_0 commute with the other $SU(2)$ and $SU(3)$ generators respectively, there are no cross terms arising between the different symmetries. Note that T'_0 by itself would be the singlet generator associated with the often ruminated "ninth gluon", which has now been absorbed into the definition of the weak hypercharge.

All of the terms in Eq. (5.2) are Lorentz invariant since the generators of the Lorentz group commute with all exterior generators and we have, for example,

$$
\begin{aligned}
\Psi^{\dagger} i \bar{W}_a T_a \Psi &\rightarrow \Psi^{\dagger} L^{\dagger} i \bar{L}^{\dagger} \bar{W}_a \bar{L} T_a L \Psi \\
&\rightarrow \Psi^{\dagger} i (\bar{L} L)^{\dagger} \bar{W}_a T_a (\bar{L} L) \Psi \\
&\rightarrow \Psi^{\dagger} i \bar{W}_a T_a \Psi,
\end{aligned} \tag{5.11}
$$

where W is a vector field and therefore transforms according to Eq. (2.10). The derivative term does not pick up any extra Lorentz terms since these transformations are not local in flat space-time.

The electromagnetic field is traditionally incorporated by introducing linear combinations of the B and W_3 fields through

$$
\begin{aligned}
A &= B \cos\theta_w + W_3 \sin\theta_w, \\
Z &= -B \sin\theta_w + W_3 \cos\theta_w,
\end{aligned} \tag{5.12}
$$

or equivalently,

$$
\begin{aligned}
W_3 &= A \sin\theta_w + Z \cos\theta_w, \\
B &= A \cos\theta_w - Z \sin\theta_w,
\end{aligned} \tag{5.13}
$$

where θ_w is the weak-mixing (or Weinberg) angle[16] to be determined by experiment. This redefinition preserves the norm of the fields since

$$
A\overline{A} + Z\overline{Z} = B\overline{B} + W_3\overline{W}_3. \tag{5.14}
$$

The relevant Lagrangian terms become

$$
\begin{aligned}
& -\frac{g'}{2}\langle \Psi^\dagger i\bar{B}(T_0\Psi + \Psi T_0')\rangle_S - g\langle \Psi^\dagger i\bar{W}_3 T_3\Psi\rangle_S \\
= \quad & -g'\sin\theta_w\langle \Psi^\dagger i\bar{Z}(\frac{T_0}{2}\Psi + \Psi\frac{T_0'}{2})\rangle_S - g\cos\theta_w\langle \Psi^\dagger i\bar{Z}T_3\Psi\rangle_S \\
& -g'\cos\theta_w\langle \Psi^\dagger i\overline{A}(\frac{T_0}{2}\Psi + \Psi\frac{T_0'}{2})\rangle_S - g\sin\theta_w\langle \Psi^\dagger i\overline{A}T_3\Psi\rangle_S. \quad (5.15)
\end{aligned}
$$

Since the last two terms should be equivalent to the electromagnetic coupling term

$$-e\langle \Psi^\dagger i\overline{A}(T_{em}\Psi + \Psi T_{em}')\rangle_S \quad (5.16)$$

in order to gauge-compensate the transformation of Eq. (4.27), where e is the electromagnetic coupling constant, we must have the relation

$$e = g\sin\theta_w = g'\cos\theta_w. \quad (5.17)$$

The value of θ_w measured in experiments depends upon the energy regime under investigation, as the running coupling constants are subject to renormalization effects.

5.2 Free-Field Terms

Although the core argument of this work is centered around terms involving the gauge transformations of the algebraic spinor, the free-field terms, which describe the coupling of the fields to themselves, may also be formulated within the $C\ell_7$ algebra. This approach suggests an interesting high-energy limit to the Weinberg angle.

The internal and external generators, when taken by themselves and out of context with no spinor to separate them, do not necessarily commute with each other. The design in the algebraic free-field expressions is then to divide the internal and external transformations into two separate terms. The physical part of the tensor associated with each generator T_c occupies the six vectors and bivectors of $C\ell_3$ and may be written in the form

$$F_c = (\bar{\partial} W_c - \bar{W}_c \partial) - g \bar{W}_a W_b f_{abc}. \tag{5.18}$$

The gauge transformations of Eq. (5.1) induce the transformation

$$\begin{aligned}
F_a &\rightarrow F_a + f_{abc}\theta_b(\bar{\partial} W_c - \bar{W}_c \partial) \\
&\quad -g\theta_j \bar{W}_k W_c (f_{bjk} f_{bca} + f_{bjc} f_{kba}) \\
&\rightarrow F_a + f_{abc}\theta_b(\bar{\partial} W_c - \bar{W}_c \partial) \\
&\quad +g\theta_b \bar{W}_j W_k f_{cjk} f_{cba} \\
&\rightarrow F_a + f_{abc}\theta_b F_c.
\end{aligned} \tag{5.19}$$

Using the B field as an example that is unhindered by structure constants, the components of the first term are

$$\begin{aligned}
\bar{\partial} B - \bar{B}\partial &= 2[(\partial_0 B^1 + \partial_1 B^0) + j(\partial_2 B^3 + \partial_3 B^2)]e_1 \\
&\quad +2[(\partial_0 B^2 + \partial_2 B^0) + j(\partial_3 B^1 + \partial_1 B^3)]e_2 \\
&\quad +2[(\partial_0 B^3 + \partial_3 B^0) + j(\partial_1 B^2 + \partial_2 B^1)]e_3 \\
&= -2[(\partial_0 B_1 - \partial_1 B_0) + j(\partial_2 B_3 - \partial_3 B_2)]e_1 \\
&\quad + \text{(permutations)}.
\end{aligned} \tag{5.20}$$

where we have defined $j \equiv e_1 e_2 e_3$ for convenience. Within $C\ell_3$, the element j has the properties of the unit imaginary[17], but in the extension to $C\ell_7$ one must take caution since j anticommutes with higher-dimensional vector elements. The six components in the span of $\{e_1, e_2, e_3, je_1, je_2, je_3\}$ are the same as those arising from the conventional six-component antisymmetric tensor

$$\lceil B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu \rfloor. \tag{5.21}$$

By squaring Eq. (5.20) we have

$$-\frac{1}{8}\langle(\bar{\partial}B-\bar{B}\partial)^2\rangle_S=\lceil\frac{1}{4}B_{\mu\nu}B^{\mu\nu}\rfloor, \tag{5.22}$$

which is the conventional Lagrangian term for a free field. Terms such as this are Lorentz invariant since, for example,

$$\begin{aligned}\bar{\partial}B-\bar{B}\partial &\rightarrow \bar{L}^{\dagger}\bar{\partial}\bar{L}LBL^{\dagger}-\bar{L}^{\dagger}\bar{B}\bar{L}L\partial L^{\dagger}\\ &\rightarrow \bar{L}^{\dagger}(\bar{\partial}B-\bar{B}\partial)L^{\dagger}\end{aligned} \tag{5.23}$$

transforms as the components of a six-tensor, and

$$\begin{aligned}\langle(\bar{\partial}B-\bar{B}\partial)^2\rangle_S &\rightarrow \langle\bar{L}^{\dagger}(\bar{\partial}B-\bar{B}\partial)L^{\dagger}\bar{L}^{\dagger}(\bar{\partial}B-\bar{B}\partial)L^{\dagger}\rangle_S\\ &\rightarrow \langle(\bar{\partial}B-\bar{B}\partial)(\bar{L}L)^{\dagger}(\bar{\partial}B-\bar{B}\partial)(\bar{L}L)^{\dagger}\rangle_S\\ &\rightarrow \langle(\bar{\partial}B-\bar{B}\partial)^2\rangle_S.\end{aligned} \tag{5.24}$$

For self-interacting fields (those with structure constants), the conventional approach is to construct a tensor

$$\lceil W^{(a)}_{\mu\nu}\equiv\partial_\mu W^{(a)}_\nu-\partial_\nu W^{(a)}_\mu-gf_{abc}W^{(b)}_\mu W^{(c)}_\nu\rfloor \tag{5.25}$$

using a somewhat cumbersome mix of tensor components $\mu\nu$ and fields (a) associated with each generator. The gauge-invariant Lagrangian term is then

$$\lceil-\frac{1}{4}W_{\mu\nu}\cdot W^{\mu\nu}\equiv-\frac{1}{4}\sum_a W^{(a)}_{\mu\nu}W^{(a)\mu\nu}\rfloor. \tag{5.26}$$

In the algebraic form, the tensor and gauge generator contractions are both handled within the algebra. The full Lagrangian term is

$$\mathcal{L}_F=-\frac{1}{2}\langle F'_aF'_b\{T'_a,T'_b\}+F_aF_b\{T_a,T_b\}\rangle_S. \tag{5.27}$$

where $a=0$ and $b=0$ are now included in the sum, since T_0 and T'_0 commute with all other generators on their respective sides and also do not contract any scalar elements with them. The anticommutator in Eq. (5.27) is evidently necessary in the case of internal rotations since products such as $T_4T_5 = \frac{1}{8}(e_7e_6 + e_2e_1)$ would otherwise introduce spurious terms through the presence of a physical plane, which would contract between the e_1 component of F_4 and the e_2 component of F_5, for example. This also ensures that under Eq. (5.19), the scalar part in (5.27) remains invariant since the only purely physical element contracted by the generators is then the identity element. Eq. (5.27) is infinitesimally gauge invariant since

$$\begin{aligned} F_aF_a &\rightarrow (F_a + f_{abc}\theta_bF_c)(F_a + f_{ajk}\theta_jF_k) \\ &\rightarrow F_aF_a + F_cF_af_{abc}\theta_b + F_cF_af_{cba}\theta_b \\ &\rightarrow F_aF_a. \end{aligned} \tag{5.28}$$

One of the reasons for displaying the free-field terms in this manner is to emphasize a key point concerning the field normalizations. The W and G fields may be entered into Eq. (5.27) directly, since they share a common factor of

$$\langle T_W^2\rangle_S = \langle T_G^2\rangle_S = -\frac{1}{8}. \tag{5.29}$$

In the case of B we have

$$\langle T_0^2\rangle_S = -\frac{1}{2}, \quad \langle T_0'^2\rangle_S = -\frac{1}{3}, \tag{5.30}$$

and are obliged to insert

$$W_0 = G_0 = \sqrt{3/20}B \tag{5.31}$$

in order to recover the conventional expression

$$\mathcal{L}_F = -\frac{1}{4}\lceil B_{\mu\nu}B^{\mu\nu} + W_{\mu\nu}\cdot W^{\mu\nu} + G_{\mu\nu}\cdot G^{\mu\nu}\rfloor. \tag{5.32}$$

At unification energies, where one would expect pure geometry to dominate, the gauge transformations of W_a and G_a via Eq. (5.1) should share a conjoint coupling constant relating the geometric rotation angle to the field strengths. We would then expect to have

$$\bar{W}_0 \rightarrow \bar{W}_0 + \frac{1}{g}\bar{\partial}\theta_0, \tag{5.33}$$

which is equivalent to

$$\bar{B} \rightarrow \bar{B} + \frac{1}{g}\sqrt{\frac{20}{3}}\bar{\partial}\theta_0. \tag{5.34}$$

Comparing both Eq. (5.34) and the similar transformation for G_0 with Eq. (5.1) immediately implies that the coupling constants at this high-energy limit should obey

$$\frac{g}{g_s} = 1, \quad \tan\theta_w \equiv \frac{g'}{g} = \sqrt{\frac{3}{5}}, \tag{5.35}$$

which results in a Weinberg angle of

$$\sin^2\theta_w = \frac{3}{8}. \tag{5.36}$$

Radiative corrections are assumed to lower the weak mixing angle to the observed value of $\sin^2\theta_w \simeq 0.23$ at accelerator energies. It has not been examined as yet what role the extra space-like dimensions might play in such a renormalization procedure carried out within this framework.

5.3 Higgs Field

In the conventional standard model, it is observed that the $W^\pm$ and Z fields must be massive in order to explain their low-energy absence. Introducing a mass term such as

$$[\mathcal{L} = \frac{1}{2}m^2 Z^\mu Z_\mu] \tag{5.37}$$

directly into the Lagrangian is not permitted since this form is not gauge-invariant. In order to produce gauge-invariant mass terms for the vector bosons, the standard model appeals to the Higgs mechanism[18], whereby a Lorentz-invariant scalar field that transforms as an $SU(2)$ isodoublet is postulated in order to yield a gauge-invariant mass term as a by-product of the coupling of the Higgs field to the vector fields. The same mechanism is also invoked to add gauge invariant mass terms for the fermions. In the conventional approach, this is achieved through a decidedly artificial construction that is one of the least palatable aspects of the standard model. It will be shown here how the Higgs field arises as a natural extension of the previous terms to include higher-dimensional components.

When looking at the exterior invariances of the current, we previously disregarded the higher-dimensional vector components and allowed them to freely rotate among each other. This Lorentz-invariant vector space is then a carrier space for the set of exterior gauge transformations and affords a natural inclusion of the minimal Higgs field. With the help of the matrix representation (2.24), one can verify that by formulating the complex scalar isodoublet H and conjugate Higgs $H_c = -\tilde{H}^\dagger$ as

$$\begin{aligned} H &= -(\phi_1 e_6 + \phi_2 e_7)P_{+\alpha} - (\phi_3 e_5 + \phi_4 e_4)P_{-\beta} \\ &\sim \left[\begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}\right] \\ H_c &= -(\phi_1 e_6 + \phi_2 e_7)P_{-\alpha} - (\phi_3 e_5 + \phi_4 e_4)P_{+\beta} \\ &\sim \left[\begin{pmatrix} -\phi_3 + i\phi_4 \\ \phi_1 - i\phi_2 \end{pmatrix}\right] , \end{aligned} \tag{5.38}$$

the expression

$$\mathcal{L}_M = \frac{1}{\sqrt{2}}\langle \Psi^\dagger G_e H \Psi P_l + \Psi^\dagger (G_d H + G_u H_c)\Psi P_q \rangle_S \tag{5.39}$$

proves to be identical in form to the conventional Higgs-coupling Lagrangian term with coupling strengths $G_{e,d,u}$. The projectors $P_l = P_{++-}$ and $P_q = P_{+--} + P_{+-+} + P_{+++}$ are used to separate the quark and lepton currents. The transformation required for gauge invariance,

$$H \to \exp(\theta_0 T_0 + \theta_a T_a) H \exp(-\theta_0 T_0 - \theta_b T_b), \tag{5.40}$$

is equivalent to the conventional notation

$$\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \to \exp(-iY\theta_0 - i\theta_a\sigma_a/2) \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \tag{5.41}$$

where the weak hypercharge assignment of $Y = 1$ ($Y = -1$) for the Higgs field (conjugate field) is recovered naturally from the double-sided algebraic transformation.

Note that $H + H_c$ consists only of elements in the span of $\{e_4, e_5, e_6, e_7\}$ and exhaust the couplings between R and L leptons and between the R and L quarks. Application of the gauge transformation (5.40) naturally separates the higher-dimensional vector space into the two carrier spaces of H and H_c using the same projectors as previously defined in Eq. (3.12). Ignoring the various weighted projectors used in both the Higgs field (5.38) and Lagrangian term (5.39) used to give distinct masses to the different fermions, the form of Eq. (5.39) is the same as that of the current, Eq. (3.4), where the components of the current being extracted are from the set $\{e_4, e_5, e_6, e_7\}$. The Higgs field—one of the least understood aspects of the standard model—then has the interpretation of a coupling to the higher-dimensional vector components of the current. Here again, the Higgs field is no longer an artificial appendage cast in some abstract space, but emerges readily from the higher-dimensional geometry.

For completeness, the remaining parts of the minimal Higgs sector may also be written algebraically. The gauge-invariant free-field term is given by

$$\mathcal{L}_H = \langle(\bar{\partial} H - \frac{g'}{2}\bar{B}[T_0, H] - g\bar{W}_a[T_a, H])^2\rangle_S, \tag{5.42}$$

where Eq. (2.16) provides for the Minkowski contraction of the physical components. In detail, the first term transforms as

$$\begin{aligned}\bar{\partial} H \quad \rightarrow \quad & (1 + \theta_0 T_0 + \theta_a T_a)(\bar{\partial} H)(1 - \theta_0 T_0 + \theta_b T_b) \\ & +(\bar{\partial}\theta_0)[T_0, H] + (\bar{\partial}\theta_a)[T_a, H].\end{aligned} \tag{5.43}$$

The incongruous latter two terms in (5.43) are cancelled out by adding

$$\begin{aligned}& -\frac{g'}{2}\bar{B}[T_0, H] - g\bar{W}_a[T_a, H] \\ \rightarrow \quad & -\frac{g'}{2}(\bar{B} + \frac{2}{g}\bar{\partial}\theta_0)[T_0, (1 + \theta_0 T_0 + \theta_a T_a)H(1 - \theta_0 T_0 + \theta_b T_b)] \\ & -g(\bar{W}_a + \frac{1}{g}\bar{\partial}\theta_a + f_{abc}\theta_b W_c)[T_a, (1 + \theta_0 T_0 + \theta_d T_d)H(1 - \theta_0 T_0 + \theta_f T_f)] \\ \rightarrow \quad & (1 + \theta_0 T_0 + \theta_a T_a)(-\frac{g'}{2}\bar{B}[T_0, H] - g\bar{W}_b[T_b, H])(1 - \theta_0 T_0 + \theta_c T_c) \\ & -(\bar{\partial}\theta_0)[T_0, H] - (\bar{\partial}\theta_a)[T_a, H].\end{aligned} \tag{5.44}$$

The exterior transformations on the left and right then cancel by use of Eq. (3.5). The Lagrangian term is squared here since the original $\bar{\partial} H$ term does not contain any scalar elements.

The gauge symmetry may be broken by choosing a vacuum expectation value

$$H_0 = -v e_5 P_{-\beta}, \quad v \equiv \frac{\mu}{\sqrt{\lambda}}, \quad \mu^2, \lambda > 0, \tag{5.45}$$

and inserting this into the gauge-invariant potential term

$$\mathcal{L}_V = \langle \mu H^2 - \frac{\lambda}{2} H^4 + \cdots \rangle_S, \tag{5.46}$$

analogous to the conventional choice. (Note that the factor of 1/2 in (5.46) is due to the fact that $\langle P_{-\beta}^4 \rangle_S = 1/2$ and not 1/4). This leads directly to the vector-boson mass relations of the Weinberg-Salam model, the relevant initial term being

$$\mathcal{L}_{H_0} = \frac{v^2}{8}[g^2\bar{W}_j W_j - gg'(\bar{W}_3 B + \bar{B}W_3) + g'^2\bar{B}B] + \cdots . \tag{5.47}$$

Through the field redefinition (5.13) and the relation (5.17), this term may be rewritten as

$$\mathcal{L}_{H_0} = \frac{v^2}{8}[(g+g')\bar{Z}Z + (0)(\bar{A}A + \bar{A}Z + \bar{Z}A) + g^2(\bar{W}^+W^+ + \bar{W}^-W^-)] + \cdots , \tag{5.48}$$

where

$$W^{\pm} \equiv \frac{1}{\sqrt{2}}(W^1 \mp iW^2). \tag{5.49}$$

This produces the conventional mass terms for the $W^{\pm}$ and Z gauge bosons in a gauge-invariant manner with

$$\begin{aligned} m_z &= \frac{1}{2}v\sqrt{g^2 + g'^2}, \\ m_w &= \frac{1}{2}vg, \\ \frac{m_w}{m_z} &= \cos\theta_w . \end{aligned} \tag{5.50}$$

The principle of gauge invariance used to deduce the form of the various Lagrangian couplings is the same as that of the conventional formalism. When used with the interior and exterior generators found above, the expression (5.2) yields all the usual particle and antiparticle charge currents. However, although the above terms look similar to the conventional forms, it should be emphasized that all of the currents are now simultaneously handled in the same expression using one algebraic spinor set, and the gauge symmetries are no longer relegated to abstract spaces but arise naturally from the algebra itself. As the Lorentz and gauge generators now have a common geometric basis, and both are compensated for by the transformation properties of the gauge fields, it is speculated that this formalism may be of great use when incorporating gravity into the standard model. Although it is beyond of the scope of this dissertation, the core components of general relativity involve extending the Poincaré group to a more general set of transformations set upon curved spaces that may be modeled using higher-dimensional Euclidean spaces. The algebraic method shown here seems ideally suited to further advances in this direction.

Chapter 6

Conclusion

We began by formulating a generalized current expression in the Clifford algebra $\mathcal{C}\ell_7$. The new formulation made it easier to enumerate geometric symmetries in a higher-dimensional space where four extra space-like dimensions were added in accordance with the defining geometric anticommutator. The extended algebra provided exactly the number of basis forms required to represent all the observed fermions (particles and antiparticles) of one generation (including a right-chiral neutrino) combined into a single spinorial element.

By examining the algebraic invariances of the extended current, considering both *exterior* (left-sided) and *interior* (right-sided) operations on the combined algebraic spinor Ψ, and through the local isomorphisms $SO(4) \sim SU(2) \otimes SU(2)$ and $SO(6) \sim SU(4)$, exclusion of the right-chiral neutrino led directly to

$$SU(2) \otimes SU(2) \otimes SU(4) \rightarrow U(1)_Y \otimes SU(2)_L \otimes SU(3). \tag{6.1}$$

We thus found the remarkable result that the gauge group of the standard model arises naturally and uniquely from local geometric rotations in the tangent space of a manifold that requires only four extra spatial dimensions. The isotopic groups are no longer relegated to an independent abstract space but are tied to local rotational invariances. Furthermore, the $SU(2)$ transformations arise as exterior transformations whereas the $SU(3)$ are interiors ones, a fact that may suggest a relation to quark confinement. The $U(1)_Y$ transformation arises as a unique double-sided transformation and returns the correct weak hypercharge assignments of the fermions. As a bonus, the four extra dimensions added to accomplish this, together with their exterior transformation properties, turned out to be precisely what is needed for the four components of a minimal scalar Higgs field along with the correct weak hypercharge assignment.

Note that the value of $\sin^2\theta_w = \frac{3}{8}$ at unification energies is identical to the often touted result from minimal $SU(5)$ grand unification[19]. This is not entirely surprising, as both originate from the normalization of the weak hypercharge operator. However, the notion of embedding the gauge symmetries of the standard model into some master group such as $SU(5)$ to accomplish this adds extra gauge bosons and the hierarchy problem associated with their apparent absence. The traditional prescription for handling the gauge fields needed to compensate additional generators is to have them acquire tremendous mass, beyond the range of current experiments. This incurs further problems in explaining why there should exist such disjoint mass scales and in preventing these scales from mixing through higher-order perturbative corrections (the so-called fine-tuning problem[4]). One of the motivations for supersymmetry, for which there does not exist a single piece of experimental evidence even in high-energy cosmic ray detectors, is to accommodate this situation. This can all be avoided if one abandons the notion of an abstract master group right from the start, instead having the gauge group result naturally from the underlying geometry. Furthermore, there is seldom any physical justification for a particular abstract master group, other than its suiting a selection process under various constraints. This has been done here to some extent in that the choice of $\mathcal{C}\ell_7$ was in part to accommodate the number of observed fermions. However, the notion that higher dimensions are directly responsible for the presence of gauge groups is more fundamental, as the physical basis for those symmetries is apparent.

A clear deficiency of this model is the need to suppress interactions with the right-chiral neutrino, as this does not appear to have an obvious geometric basis. We must have a total exclusion of this sector in the model presented, otherwise there would be extra massless gluons mediating the additional interior transformations, which is ruled out by experiment. It is speculated that this restriction may be related to the incorporation of gravity into a more elaborate version of the model. Also lacking is a geometric rationale for the inclusion of three generations of fermions and for some indication as to the origin of their disjointed masses. It is hoped that this framework may provide new insights into these questions.

APPENDIX A

Conventional Matrix Representations

The 2×2 Pauli spin matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

There are various conventions for the 4×4 Weyl representation. To avoid confusion, the choice here is explicitly stated as

$$\gamma_0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \ \gamma_k = -\gamma^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix},$$

$$\gamma_5 \equiv -i\gamma_0\gamma_1\gamma_2\gamma_3, \ \psi^{Weyl} = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}.$$

This is consistent with Kaku[4].

The 3×3 Gell-Mann matrices are

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \ \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \ \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned}$$

APPENDIX B

Selected 8×8 Matrix Representations

Using the matrix representation of appendix A, the explicit 8×8 matrix representations of the vector elements of $C\ell_7$ are as follows. Zeros have been denoted by dots for clarity.

$$e_1 = \begin{pmatrix} . & 1 & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . & . \\ . & . & . & -1 & . & . & . & . \\ . & . & -1 & . & . & . & . & . \\ . & . & . & . & . & 1 & . & . \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & . & -1 \\ . & . & . & . & . & . & -1 & . \end{pmatrix},$$

$$e_2 = \begin{pmatrix} . & -i & . & . & . & . & . & . \\ i & . & . & . & . & . & . & . \\ . & . & . & i & . & . & . & . \\ . & . & -i & . & . & . & . & . \\ . & . & . & . & . & -i & . & . \\ . & . & . & . & i & . & . & . \\ . & . & . & . & . & . & . & i \\ . & . & . & . & . & . & -i & . \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ . & -1 & . & . & . & . & . & . \\ . & . & -1 & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & -1 & . & . \\ . & . & . & . & . & . & -1 & . \\ . & . & . & . & . & . & . & 1 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} . & . & i & . & . & . & . & . \\ . & . & . & i & . & . & . & . \\ -i & . & . & . & . & . & . & . \\ . & -i & . & . & . & . & . & . \\ . & . & . & . & . & . & i & . \\ . & . & . & . & . & . & . & i \\ . & . & . & . & -i & . & . & . \\ . & . & . & . & . & -i & . & . \end{pmatrix},$$

$$e_5 = \begin{pmatrix} . & . & -1 & . & . & . & . & . \\ . & . & . & -1 & . & . & . & . \\ -1 & . & . & . & . & . & . & . \\ . & -1 & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & 1 \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & 1 & . & . \end{pmatrix},$$

$$e_6 = \begin{pmatrix} . & . & . & . & . & . & -1 & . \\ . & . & . & . & . & . & . & -1 \\ . & . & . & . & -1 & . & . & . \\ . & . & . & . & . & -1 & . & . \\ . & . & -1 & . & . & . & . & . \\ . & . & . & -1 & . & . & . & . \\ -1 & . & . & . & . & . & . & . \\ . & -1 & . & . & . & . & . & . \end{pmatrix},$$

$$e_7 = \begin{pmatrix} . & . & . & . & . & . & i & . \\ . & . & . & . & . & . & . & i \\ . & . & . & . & i & . & . & . \\ . & . & . & . & . & i & . & . \\ . & . & -i & . & . & . & . & . \\ . & . & . & -i & . & . & . & . \\ -i & . & . & . & . & . & . & . \\ . & -i & . & . & . & . & . & . \end{pmatrix}.$$

The parity operator is given by

$$-e_1e_2e_3e_4 = \begin{pmatrix} . & . & 1 & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . \\ 1 & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & 1 \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & 1 & . & . \end{pmatrix}.$$

The explicit matrix representations of the $SU(2)$ gauge-group generators are

$$2T_1 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -i \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$2T_2 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$2T_3 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & i \end{pmatrix}.$$

The explicit matrix representations of the $SU(3)$ gauge-group generators are

$$2T_1' = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$2T_2' = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$2T_3' = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$2T_4' = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -i \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$2T_5' = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$2T_6' = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -i \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -i & \cdot & \cdot & \cdot \end{pmatrix},$$

$$2T_7' \;=\; \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \end{pmatrix},$$

$$2\sqrt{3}T_8' \;=\; \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -2i & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2i \end{pmatrix}.$$

The explicit matrix representations of the $U(1)$ double-sided group generators are

$$T_0 = \begin{pmatrix} -i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & i & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$3T_0' = \begin{pmatrix} 3i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & i & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -3i & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -i \end{pmatrix}.$$

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