2003

An harmonic analysis for operators on homogeneous Banach spaces.

Olusakin Joshua. Emmanuel
University of Windsor

An Harmonic Analysis for Operators on Homogeneous Banach Spaces

by

Olusakin Emmanuel

A Thesis
Submitted to the Faculty of Graduate Studies and Research
through the Department of Mathematics and Statistics
in Partial Fulfillment of the Requirements for
the Degree of Master of Science at the
University of Windsor

Windsor, Ontario, Canada

Your file Votre référence
ISBN: 0-612-82864-6
Our file Notre référence
ISBN: 0-612-82864-6

Abstract

In this thesis, we undertake an harmonic analysis of the Banach algebra $\mathcal{L}(B)$ of bounded linear operators on a homogeneous Banach space B of functions on a topological abelian group G. Our analysis is divided into two major parts. In the first, we examine the case where G is compact, particularly $G = \mathbb{T}$ (the circle group), and in the second G is locally compact. In both cases, we define the classes of invariant and almost invariant operators in $\mathcal{L}(B)$ and investigate their properties. With each $T \in \mathcal{L}(B)$, we associate a Fourier series and show that this series converges to T in a certain specified sense. For $G = \mathbb{T}$, we show that formal properties of the usual Fourier series hold and also obtain a generalization of the classical F. and M. Riesz theorem for $B = \mathfrak{C}(\mathbb{T})$. For locally compact G, we investigate a subspace of the class of almost invariant operators, namely the almost periodic operators.

Dedication

To Him Who has superabundantly supplied above my highest requests and has exceedingly manifested beyond my wildest imaginations.

Acknowledgements

I would like to gratefully acknowledge the enthusiastic supervision of Dr. Zhiguo Hu and Dr. Tim Traynor during this work. I deeply appreciate their efforts and their untiring patience expressed in explaining mathematical concepts to me in simplest terms. I thank Dr. Hu for her care, her generosity, and for *pushing* me. Thank you Dr. Traynor for your ready open arms.

I thank Dr. Harold Atkinson for offering to serve as the departmental reader for my thesis, Dr. Jessica Chen for serving as the external reader, and Dr. S. Ejaz Ahmed for offering to chair my thesis committee.

Thanks to Lisa Mac Donald for Winedt. The contributions of Soon Hee Mun, Guoha Yan, Fengling Shi, and Hussein Elsmaili cannot be forgotten in a hurry. Thank you Mostafa Zamani for your time and for those useful hints.

My profound gratitude goes to my brother, Segun Emmanuel, for expanding himself to the limit in making this dream a reality. I am also grateful to Yejide, Sogo, Bro. Leonard, Mr. and Mrs. Folami, Pastor Bowo, Mo, and other TOG folks for their continued moral support and for being the surrogate family during my stay in Windsor. They've really helped getting the going on.

Above all, I give praise to the Almighty God for being the very present Help in the crucial times of real need.

Contents

CHAPTER 1

Introduction

Just as the study of Fourier series involves investigating periodic functions which are translation invariant under the additive integer group translations, Abstract Harmonic Analysis involves studying objects (functions, measures, etc.) defined on locally compact groups which are invariant under the group translations.

In this thesis, we engage the study of a harmonic analysis for bounded linear operators on homogeneous Banach spaces (including the classical L_p-spaces with $1 \leq p < \infty$) defined on a compact abelian group or a locally compact abelian group. This particular study was initiated by Karel DeLeeuw [**6**] in 1975 when he investigated homogeneous Banach spaces of functions on the compact abelian group $\mathbb{T}$. In 1992, U. B. Tewari and S. Somasundaram [**15**] extended this study to general locally compact abelian groups. Our attempt in this thesis is to study and relate the results presented in these two papers.

The body of this thesis is composed of three chapters. In the first, we introduce fundamental concepts relevant to our study in set theory, measure theory, topological spaces, linear spaces and algebraic spaces of operators. This is followed by the chapter that gives an exposition of [**6**], focusing on compact abelian groups. The final chapter, based on [**15**], deals with locally compact abelian groups.

Indeed in Chapter 3, following DeLeeuw [**6**], we define the concept of a homogeneous Banach space B on $\mathbb{T}$ and denote by $\mathcal{L}(B)$ the Banach algebra of bounded linear operators on B. We then define some interesting subalgebras of $\mathcal{L}(B)$, consisting of the space $\mathcal{L}_0$ of invariant operators, the space $\mathcal{L}_*$ of almost invariant operators, and the space $\mathcal{L}_n$ of simple operators, and demonstrate their individual and related properties. We define the projection operator π_n on $\mathcal{L}(B)$ with which we define the Fourier series of an operator $T \in \mathcal{L}(B)$ as $\sum_{-\infty}^{\infty} \pi_n(T)$. The Fourier series of any operator $T \in \mathcal{L}(B)$ is C-1 summable to T in the strong operator topology of $\mathcal{L}(B)$. But if $T \in \mathcal{L}_*$, the Fourier series of T is C-1 summable to T in the operator norm. To prove this, we employ the the properties of summability kernels as presented by Katznelson ([**11**], (2.2)). We define the Fourier transform of T as $\widehat{T} = \pi_n(T)$ and demonstrate that it takes operator multiplication into convolution. We also define a convolution $\mu * T$ of a finite Borel measure μ with an operator $T \in \mathcal{L}(B)$ and show that the Fourier transform takes this convolution into $\hat{\mu} \cdot \widehat{T}$. To conclude the chapter, we give a generalization of the F. and M. Riesz theorem in terms of compact operators and show its connection with the classical one.

In Chapter 4, the homogeneous Banach space is defined on any locally compact abelian group G. In this case, a subspace $\mathcal{L}_A$ of $\mathcal{L}_*$, called the space of almost periodic operators, is studied. When G is compact, we show that $\mathcal{L}_A = \mathcal{L}_*$. The central theorem of the chapter is the approximation theorem (Theorem 3.4) which shows that $\mathcal{L}_A$ can be approximated by finite sums of operators of the form $M_\gamma U$, where M_γ is multiplication by the character γ of G and $U \in \mathcal{L}_0$. An invariant mean π_0

is defined and studied on $\mathcal{L}_A$. Properties of Banach $*$-algebras are employed. Finally, we define the Fourier series and transform of an operator $T \in \mathcal{L}_A$ and show that the Fourier series converges to T in a norm constructed by Arveson in [**3**].

CHAPTER 2

Preliminaries

Abstract Harmonic Analysis requires acquaintance with several concepts from the fields of *Topology, Group Theory, Linear Spaces, Algebraic Spaces, Measure Theory, and Abstract Integration.* This chapter is intended to give an overview of notions from these fields that serve as underlying concepts in subsequent chapters. We discuss each of these areas briefly.

1. Sets and Topological Concepts

Let X be any set, and $\mathcal{M}$ a nonempty collection of subsets of X. $\mathcal{M}$ is called an *algebra of sets* in X if it satisfies (1) $\emptyset \in \mathcal{M}$; (2) if $A, B \in \mathcal{M}$, then $A \cup B \in \mathcal{M}$; (3) if $A \in \mathcal{M}$, then the complement A^c of A in X is in $\mathcal{M}$. If $\mathcal{M}$ is also closed under *countable* unions of its members, then $\mathcal{M}$ is called a *σ-algebra* (in X).

Let X be a non-empty set. A *topology* on X is a collection $\mathcal{T}$ of subsets of X satisfying (1) $\emptyset$, $X \in \mathcal{T}$; (2) if $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is a collection of elements of $\mathcal{T}$, then $\bigcup_{\alpha \in \mathcal{A}} \in \mathcal{T}$; (3) if $U_1, \cdots U_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$. The members of the topology $\mathcal{T}$ are called *open sets* and their complements are *closed.* The pair $(X, \mathcal{T})$ is called a *topological space.* The largest open subset of a set $A \subseteq X$ is called the *interior* of A. Its elements are the *interior points* of A. If x is an interior point of A, then A is a *neighborhood* of x. The *closure* $\bar{A}$ of A is the smallest closed set containing A. A is

dense in X if $\bar{A} = X$. If to every *distinct* pair x_1, x_2 in X, there are corresponding neighborhoods U_1 of x_1 and U_2 of x_2 with empty intersection, then X is called a *Hausdorff space.*

A set Λ is a *directed set* if there is a relation $\preceq$ on Λ satisfying:

(1) $\alpha \preceq \alpha$ for each $\alpha \in \Lambda$;

(2) if $\alpha_1 \preceq \alpha_2$ and $\alpha_2 \preceq \alpha_3$, then $\alpha_1 \preceq \alpha_3$;

(3) if $\alpha_1, \alpha_2 \in \Lambda$, then there exists $\alpha_3 \in \Lambda$ such that $\alpha_1 \preceq \alpha_3$ and $\alpha_2 \preceq \alpha_3$.

The relation $\preceq$ is referred to as the *direction* on Λ.

A *net* in a set X is a function $\phi : \Lambda \to X$ from some directed set Λ to X. The point $\phi(\alpha)$ in the net is denoted by x_α and the net is written as $\{x_\alpha\}_{\alpha \in \Lambda}$, or simply $\{x_\alpha\}$. A net $\{x_\alpha\}$ in a topological space X is said to *converge* to $x \in X$ (written $x_\alpha \to x$) if for every neighborhood U of x, there exists $\alpha_0 \in \Lambda$ such that $x_\alpha \in U$ whenever $\alpha \succeq \alpha_0$.

If X and Y are topological spaces, then a function $f : X \to Y$ is *continuous* if for each open $V \subseteq Y$, $f^{-1}(V)$ is open in X. A subset K of X is called compact if every family of open sets covering K has a *finite* subfamily covering K. If each $x \in X$ has a compact neighborhood, then X is *locally compact.* The continuous image of a compact set is compact. A subset of a topological space is said to be *relatively compact* if its *closure* is compact.

Let X be a set and let $d : X \times X \to \mathbb{R}$. d is called a *metric* and (X, d) a *metric space* if the following conditions are satisfied.

(1) $d(x,y) \geq 0$ for all $x, y \in X$.

(2) $d(x,y) = 0$ if and only if $x = y$.

(3) $d(x,y) = d(y,x)$ for all $x, y \in X$.

(4) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

For $x_0 \in X$ and $r > 0$, the set $\mathbf{B}(x_0, r) = \{x \in X : d(x_0, x) < r\}$ is called the *open ball* of X with centre x_0 and radius r. A sequence $\{x_n\}$ in the metric space X is called a *Cauchy sequence* if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, if $m \geq N$ and $n \geq N$, then $d(x_m, x_n) < \varepsilon$. (X, d) is *complete* if every Cauchy sequence in X converges. X is *compact* if every sequence in X has a convergent subsequence. A subset S of a metric space is *totally bounded* if for any given $\varepsilon > 0$, S can be covered by a finite number of open balls with radius ε.

A subset S of a metric space which can be covered by a finite number of open balls is said to be *totally bounded.* A metric space is compact if and only if it is complete and totally bounded. A metric space is said to be *separable* if it contains a countable dense subset.

Let X be a topological space. We denote by $\mathfrak{C}(X)$ the set of all *bounded complex-valued continuous functions* on X and by $\mathfrak{C}_0(X)$ the set of all $f \in \mathfrak{C}(X)$ such that for every $\varepsilon > 0$, there exists a compact subset K of X such that $|f(x)| < \varepsilon$ for all $x \in K^c$. Let $\mathfrak{C}_{00}(X)$ denote the set of all $f \in \mathfrak{C}(X)$ such that there exists a compact subset K of X such that for all $x \in K^c$, $f(x) = 0$. For $f \in \mathfrak{C}(X)$, we define $\|f\| = \sup\{|f(x)| : x \in X\}$. If X is compact, then $\mathfrak{C}(X) = \mathfrak{C}_0(X) = \mathfrak{C}_{00}(X)$.

References: [**4**], [**8**], [**17**].

2. Linear Spaces

Let F be a field and X an additive abelian group (see §4 for definition) such that for each $\alpha \in F$ and $x \in X$, the product $\alpha x \in X$ is defined and the following conditions hold: $\alpha(x+y) = \alpha x + \alpha y$; $(\alpha+\beta)x = \alpha x + \beta x$; $(\alpha\beta)x = \alpha(\beta x)$; where $x, y \in X$ and $\alpha, \beta \in F$. Then X is said to be a *linear space* (or *vector space*) over F.

Let X be a linear space over $\mathbb{C}$. A *norm* on X is a non-negative real-valued function whose value at an $x \in X$ is denoted as $\|x\|$ with the following properties:

(1) $\|x\| = 0$ if and only if $x = 0$;

(2) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{C}$;

(3) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in E$.

The pair $(X, \|\cdot\|)$ is called a *normed space*. For simplicity, we write X for $(X, \|\cdot\|)$. If X is complete with respect to the metric $d(x,y) = \|x-y\|$, then X is called a *Banach space*. The set $\{x \in X : \|x\| \leq 1\}$ is called the *unit ball* of X.

Let X and Y be normed spaces over the same field. A map T from X to Y is a *linear operator* if $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$ for all $x_1, x_2 \in X$, $\alpha, \beta \in \mathbb{C}$. A linear operator T is *bounded* on X if there is an $M > 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in X$. A linear operator is continuous if and only if it is bounded. The *operator norm* of a linear operator $T : X \to Y$ is given by

$$\|T\| = \inf\{M > 0 : \|Tx\| \leq M\|x\| \text{ for all } x \in X\} = \sup\{\|Tx\| : \|x\| \leq 1\}.$$

If X and Y are normed spaces, we denote by $\mathcal{L}(X,Y)$ the linear space of all continuous linear operators from X into Y. It becomes a normed space under the

operator norm. If Y is complete, then so is $\mathcal{L}(X,Y)$. $\mathcal{L}(X)$ will be written for $\mathcal{L}(X,X)$ and X^* for the space $\mathcal{L}(X,\mathbb{C})$ of continuous linear functionals on X, which is called the *dual* of X.

An operator $T \in \mathcal{L}(X,Y)$ is *compact* if for every bounded subset of E of X, the image $T(E)$ is relatively compact, that is, $\overline{T(E)}$ is compact. $T \in \mathcal{L}(X,Y)$ is of *finite rank* if its range is of finite dimension. Every finite rank operator is compact. If $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $T \in \mathcal{L}(X,Y)$ is compact, then $TA : X \to Y$ and $BT : X \to Y$ are compact. If $\{T_n\}$ is a sequence of compact operators converging to T in the norm topology, then T is also compact. Thus, the family of compact operators is closed under the operator norm.

A scalar function $\langle \cdot\,, \cdot \rangle : X \times X \to \mathbb{C}$ is said to be an *inner product* on X if it satisfies the following conditions:

(1) $\langle x, x \rangle \geq 0$ for all $x \in X$ with equality if $x = \mathbf{0}$;

(2) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in X$;

(3) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for all $x, y \in X$ and $\alpha \in \mathbb{C}$;

(4) $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$ for all $x_1, x_2, y \in X$.

We say that X is an inner *product space* if X is a linear space equipped with an inner product. By setting $\|x\| = \langle x, x \rangle^{1/2}$, X becomes a normed linear space. If X is complete, then X is called a *Hilbert space.* The *Cauchy-Schwartz inequality* $|\langle x, y \rangle| \leq \|x\| \|y\|$ holds for every $x, y \in X$. To each $T \in X^*$, there corresponds a unique $y \in X$ such that $Tx = \langle x, y \rangle$ for all $x \in X$. A family of elements $\{x_\alpha\}$ in X is said to be *orthogonal* if $\langle x_\alpha, x_\beta \rangle = 0$ whenever $\alpha \neq \beta$. An orthogonal family in a

Hilbert space is *orthonormal* if $\langle x_\alpha, x_\alpha \rangle = 1$. Every Hilbert space contain orthonormal families $\{x_\alpha\}$ such that $x = \sum_\alpha \langle x, x_\alpha \rangle x_\alpha$ for each $x \in X$. An orthonormal family that satisfies this property is called an *orthonormal basis.*

For further details and proofs, see [4] or [5].

3. Measure Theory

We restrict our discussion here to measures and integrals on locally compact Hausdorff spaces X.

Let $\mathcal{B}$ be the σ-algebra generated by the family of open subsets of X. The members of $\mathcal{B}$ are called the *Borel sets* in X. Let E be the union of any countable family $\{E_i\}$ of pairwise disjoint Borel sets of X. A *(Borel) measure* on X is a set function μ, defined for all Borel sets in X, which is *countably additive* (i.e., $\mu(\bigcup_{i=1} E_i) = \sum_i \mu(E_i)$), and for which $\mu(E)$ is finite if the closure of E is compact. The *total variation* $|\mu|$ of μ on X is defined by $|\mu|(E) = \sup \sum_i |\mu(E_i)|$, where the supremum is taken over all partitions $\{E_i\}$ of E. Then $|\mu|$ is also a measure on X. If

$$|\mu|(E) = \sup_K \{|\mu|(K) : K \subseteq E,\ K \text{ is compact}\} = \inf_V \{|\mu|(V) : E \subseteq V,\ V \text{ is open}\},$$

then μ is called *regular.* We put $\|\mu\| = |\mu|(X)$ and define $M(X)$ to be the set of all *complex-valued regular measures* on X for which $\|\mu\|$ is *finite.* Let $\mu \in M(X)$ and λ a non-negative measure on X. If $\mu(E) = 0$ whenever $\lambda(E) = 0$, then μ is said to be *absolutely continuous* with respect to λ.

A complex function f defined on X is called a *Borel function* if $f^{-1}(V)$ is a Borel set for every open set V in the complex plane. If $\mu \in M(X)$, all bounded Borel

functions on X are integrable with respect to μ, and the inequality $|\int_X f\,d\mu| \leq \|\mu\| \cdot \sup_{x\in X} |f(x)|$ holds.

If λ is a non-negative measure on X and if $0 < p < \infty$, we denote simply by $L_p(X)$, the set $L_p(X,\lambda)$ of all Borel functions f on X for which the norm $\|f\|_{L_p} = \left(\int_X |f|^p\,d\lambda\right)^{1/p}$ $(1 \leq p < \infty)$ is finite. $L_\infty(X)$ is the space of all essentially bounded Borel functins on X equipped with the norm $\|f\|_\infty = \operatorname{ess\,sup} |f(x)| = \inf\{t > 0 : \mu(\{x : |f(x)| > t\}) = 0\}$, referred to as the essential supremum of $|f|$. If we identify functions which differ only on a set $E \subset X$ with $\lambda(E) = 0$, $L_p(X)$ $(1 \leq p \leq \infty)$ becomes a Banach space under the defined norms. And $L_2(X)$ is a Hilbert space the with inner product $\langle f, g\rangle = \int f\bar{g}\,d\lambda$.

THE RADON-NIKODYM THEOREM. Let $\mu \in M(X)$, and λ a non-negative measure on X. If μ is absolutely continuous with respect to λ, then there exists $f \in L_1(G)$ such that $\mu(E) = \int_E f\,d\lambda$ for all Borel sets E in X. In addition, $\|\mu\| = \int_X |f|\,d\lambda = \|f\|_{L_1}$.

Suppose $\lambda \geq 0$, $1 < p < \infty$, and $1/p + 1/q = 1$. The bounded linear functionals T on $L_p(X)$ are in one-to-one correspondence with the members g of $L_q(X)$. That is, each $T \in (L_p)^*$ is of the form $Tf = \int fg\,d\lambda$ $(f \in L_p(X))$. Moreover, $\|T\| = \|g\|_{L_q}$. Thus $L_q = (L_p)^*$.

References: [**4**], [**8**], [**14**].

4. Topological Groups

In this section we focus our discussion on topological abelian (commutative) groups as these are the groups discussed throughout this thesis. Expositions of the concepts presented here can be found in Hewitt and Ross [**8**] and Rudin [**14**].

An *abelian group* is a set G in which a binary operation, $+$, is defined with the following properties:

(1) $x + y = y + x$ for all $x, y \in G$;

(2) $x + (y + z) = (x + y) + z$ for all $x, y, z \in G$;

(3) there is an element $0 \in G$ such that $x + 0 = x$ for all $x \in G$;

(4) to every $x \in G$, there corresponds an element $-x \in G$ such that $x+(-x) = 0$.

If $A, B \subseteq G$, $A + B$ denotes all the elements of the form $a + b$ with $a \in A$, $b \in B$ and $-A$ denotes all $-a$ with $a \in A$. We call $A + x$ the *translate* of A by x.

A *topological abelian group* is a Hausdorff space $(G, +, \mathcal{T})$ in which $(G, +)$ is an abelian group and $(G, \mathcal{T})$ a topological space such that (i) the mapping $(x, y) \to x+y$ is a *continuous* map from the product space $G \times G$ onto G; (ii) the mapping $x \to -x$ of G onto G is *continuous*. The latter condition asserts that for every neighborhood U of $-x$, there is a neighborhood V of x such that $-V \subset U$. A neighborhood V in G is said to be *symmetric* if $-V = V$. On every locally compact abelian group G, there exists a non-negative regular measure λ, called the *Haar measure* of G, which is not identically 0 and satisfies the following:

(1) λ is *translation invariant*, i.e., for every $x \in G$ and every Borel set E in G, $\lambda(E + x) = \lambda(E)$.

(2) If V is a non-empty open subset of G, then $\lambda(V) > 0$.

(3) If K is a compact subset of G, then $\lambda(K) < \infty$.

(4) For every Borel set $E \in G$, $\lambda(-E) = \lambda(E)$.

λ is unique up to a multiplicative constant, i.e., if λ_1 and λ_2 are two Haar measures on G, then there exists $k > 0$ such that $\lambda_1 = k\lambda_2$.

The total measure $\lambda(G)$ is finite if and only if G is compact. If G is compact, we take λ to be normalized so that $\lambda(G) = 1$. If G is noncompact, then $\lambda(G)$ is infinite. Indeed, let U be any relatively compact neighborhood of 0 in G. Then no finite collection of sets $\{U + x_i\}_{i=1}^n$ covers G, for if so, G will be compact. Now choose an infinite sequence $\{x_n\}_{n=1}^\infty$ in G such that $x_{n+1} \notin \bigcup_{i=1}^n (U + x_j)$ for all $n \in \mathbb{N}$. Let V be any symmetric neighborhood of 0 such that $V + V \subset U$. Then the sets $V + x_1, V + x_2, \ldots, V + x_3, \ldots$ are pairwise disjoint. Hence $\lambda(G) \geq \sum_{j=1}^n \lambda(V + x_j) = n \cdot \lambda(V)$ for all n. That is, $\lambda(G) = +\infty$.

Let f and g be any pair of Borel functions on a locally compact abelian group G with the Haar measure. The *convolution* $f * g$ of f and g is defined on G by

$$(f * g)(x) = \int_G f(x - y)g(y)\, dy$$

provided

$$\int_G |f(x - y)g(y)|\, dy < \infty. \tag{4.1}$$

The following properties hold if $f * g$ is defined on G.

(1) $(f * g)(x) = (g * f)(x)$ for all $x \in G$.

(2) If $f, g \in L_1(G)$, then $(f * g) \in L_1(G)$ and $\|f * g\|_{L_1} \leq \|f\|_{L_1}\|g\|_{L_1}$.

(3) If $f, g, h \in L_1(G)$, then $(f * g) * h = f * (g * h)$ and $f * (g + h) = f * g + f * h$.

Thus $L_1(G)$ with convolution operation as multiplication is a *commutative Banach algebra* (see §5).

Similarly, for $\mu, \nu \in M(G)$, if *convolution* $\mu * \nu$ is defined by

$$\mu * \nu(E) = \int_G \mu(E - x)\, d\nu(x) \quad (E \subseteq G \text{ a Borel}),$$

then the following properties hold:

(1) $\mu * \nu \in M(G)$;

(2) the convolution $*$ is commutative and associative;

(3) $\|\mu * \nu\| \leq \|\mu\|\|\nu\|$.

Thus $M(G)$ is also a *commutative Banach algebra* with convolution as multiplication.

Let G be a locally compact abelian group. A complex-valued function $\gamma : G \to \mathbb{C}$ is called a *character* on G if

(1) $|\gamma(x)| = 1$ for all $x \in G$

(2) $\gamma(x + y) = \gamma(x)\gamma(y)$ for all $x, y \in G$.

The set of all *continuous* characters of G form a group $\widehat{G}$, called the *dual* of G, if the operation on $\widehat{G}$ is given by $(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x)$ $(x \in G,\ \gamma_1, \gamma_2 \in \widehat{G})$. We write (x, γ) in place of $\gamma(x)$ so that $(x + y, \gamma) = (x, \gamma)(y, \gamma)$ and $(x, \gamma_1 + \gamma_2) = (x, \gamma_1)(x, \gamma_2)$. Combining these, we have $(0, \gamma) = (x, 0) = 1$ and $(-x, \gamma) = (x, -\gamma) = (x, \gamma)^{-1} = \overline{(x, \gamma)}$.

For each $r > 0$, let $U_r = \{z \in \mathbb{C} : |1 - z| < r\}$. If K is a compact subset of G, then we set $N(K, r) = \{\gamma \in \widehat{G} : (x, \gamma) \in U_r \text{ for all } x \in K\}$. Then the family of sets

$\{N(K,r)\}$ and their translates form a basis for a topology on $\widehat{G}$. With this topology, $\widehat{G}$ becomes a locally compact abelian group.

If $f \in L_1(G)$ and $\mu \in M(G)$, then the Fourier transform $\hat{f}$ of f and the Fourier-Stieltjes transform $\hat{\mu}$ of μ are respectively defined as

$$\hat{f}(\gamma) = \int_G (-x,\gamma) f(x)\, dx \quad \text{and} \quad \hat{\mu}(\gamma) = \int_G (-x,\gamma)\, d\mu(x) \quad (\gamma \in \widehat{G}).$$

These transforms are complex homomorphisms in the sense that if $f, g \in L_1(G)$ and $\mu, \nu \in M(G)$, then$\widehat{(f * g)} = \hat{f}\hat{g}$ and $\widehat{(\mu * \nu)} = \hat{\mu}\hat{\nu}$.

Given a locally compact abelian group G with its dual $\widehat{G}$, let $\widehat{G}_d$ be the group $\widehat{G}$ equipped with the *discrete topology.* The *dual* group of $\widehat{G}_d$ is then a *compact* abelian group which we call the *Bohr compactification* of G and denote by $\bar{G}$. There exists a continuous isomorphism of G onto a dense subgroup of $\bar{G}$.

5. Algebraic Spaces

A Banach Algebra is a complex Banach space $\mathfrak{A}$ together with an associative and distributive multiplication such that

$$\alpha(AB) = (\alpha A)B = A(\alpha B) \quad \text{and} \quad \|AB\| \leq \|A\|\|B\|$$

for all $A, B \in \mathfrak{A}$, $\alpha \in \mathbb{C}$. An element $\mathbb{I} \in \mathfrak{A}$ is called a *unit* (or *identity*) of $\mathfrak{A}$ if $\mathbb{I}A = A$ for every $A \in \mathfrak{A}$. $\mathfrak{A}$ is said to be *unital* if it possesses a unit.

$\mathfrak{A}$ is a called a *Banach $*$-algebra* if it is equipped with an involution $* : \mathfrak{A} \to \mathfrak{A}$ satisfying, for all $A, B \in \mathfrak{A}$, $\alpha \in \mathbb{C}$,

(i) $(A + B)^* = A^* + B^*$,

(ii) $(\alpha A)^* = \bar{\alpha} A^*$,

(iii) $(AB)^* = B^* A^*$,

(iv) $(A^*)^* = A$.

A Banach $*$-algebra $\mathfrak{A}$ is called a *C^*-algebra* if, for all $A \in \mathfrak{A}$, $\|A^*A\| = \|A\|^2$. By applying (iv) we see that $\|A^*\| = \|A\|$ for all $A \in \mathfrak{A}$.

A *C^*-algebra of operators* is a subset of the algebra $\mathcal{L}(H)$ of all bounded operators on a Hilbert space H which is closed under algebraic operations on $\mathcal{L}(H)$, closed in the norm topology of $\mathcal{L}(H)$, and closed under the *adjoint* operation $T \mapsto T^*$ in $\mathcal{L}(H)$. Every C^*-algebra is isometrically $*$-isomorphic with a C^*-algebra of operators on a Hilbert space.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras, and let π be an homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ (i.e., $\pi(\alpha A) = \alpha\pi(A)$, $\pi(A + B) = \pi(A) + \pi(B)$, $\pi(AB) = \pi(A)\pi(B)$ for $\alpha \in \mathbb{C}$ and $A, B \in \mathfrak{A}$). π is called a *$*$-homomorphism* if $\pi(A^*) = \pi(A)^*$ for all $A \in \mathfrak{A}$. Every $*$-homomorphism is contractive and has a closed rage, and is isometric if it is injective.

For further details and proofs, see [**1**].

6. Vector Valued Integration

We shall dwell here on *vector-valued functions* $F(s)$ defined on some topological space S with values in a Banach space X. $F(s)$ is said to be *strongly continuous* at s_0 if $\lim_{s \to s_0} \|F(s) - F(s_0)\| = 0$; and $F(s)$ is *weakly continuous* at s_0 if $\lim_{s \to s_0} |f^*[F(s) - F(s_0)]| = 0$ for each $f^* \in X^*$. In the case where the Banach space is $\mathcal{L}(X, Y)$, we speak of the function as an *operator-valued function* and denote it

by $T(s)$. $T(s)$ is said to be continuous on S in the *strong operator topology* at s_0 if $\lim_{s\to s_0} \|[T(s) - T(s_0)](x)\| = 0$ for each $x \in X$, and continuous in the *uniform topology* at s_0 if $\lim_{s\to s_0} \|T(s) - T(s_0)\| = 0$. See Hille and Phillips [**9**] for further details.

Let $(P, \boldsymbol{\tau})$ be a tagged partition of an interval $[a, b]$ in $\mathbb{R}$; that is, let $P = \{s_0, \ldots, s_n\}$ be a finite collection of points such that $a = s_0 \leq s_1 \leq \cdots \leq s_n = b$ and $\boldsymbol{\tau}$, a set of points t_i satisfying $s_{i-1} \leq t_i \leq s_i$. For such P, set $|P| = \max_i(s_i - s_{i-1})$.

Let B be a Banach space and $F : [a, b] \to B$ be a vector-valued function. Define $R(F, P, \boldsymbol{\tau}) = \sum_{i=1}^{n} F(t_i)(s_i - s_{i-1})$. Then F is (Riemann) integrable if, in the norm topology on B, the limit $I = \lim_{|P|\to 0} R(F, P, \boldsymbol{\tau})$ exists. I, denoted as $\int_a^b F(t)dt$, is called the (Riemann) integral of F over $[a, b]$. The proof of the existence of I is patterned on the classical case. Suppose F is strongly continuous on the compact interval $[a, b]$. Then we have uniform strong continuity. That is, given $\varepsilon > 0$, there exists $\delta > 0$ such that for every $s_1, s_2 \in [a, b]$, $\|F(s_1) - F(s_2)\| < \varepsilon$ whenever $|s_1 - s_2| < \delta$. Now consider the tagged partitions $(P_1, \boldsymbol{\tau}_1)$ and $(P_2, \boldsymbol{\tau}_2)$ with $|P_1| < \delta/2$ and $|P_2| < \delta/2$ such that $R(F, P, \boldsymbol{\tau}_1) = \sum_{j=1}^{n} F(t_{1_j})(s_j^1 - s_{j-1}^1)$ and $R(F, P, \boldsymbol{\tau}_2) = \sum_{k=1}^{m} F(t_{2_k})(s_k^2 - s_{k-1}^2)$. Then by comparing either sum to the sum corresponding to a common refinement of $\{s_j^1\}$ and $\{s_k^2\}$, we have

$$\|R(F, P, \boldsymbol{\tau}_1) - R(F, P, \boldsymbol{\tau}_2)\| = \|\sum_{j=1}^{n} F(t_{1_j})(s_j^1 - s_{j-1}^1) - \sum_{k=1}^{m} F(t_{2_k})(s_k^2 - s_{k-1}^2)\| < \varepsilon(b-a).$$

So every strongly continuous function is Riemann integrable.

Let $T : [a,b] \to \mathcal{L}(X)$ be an operator-valued function that is continuous in the strong operator topology, i.e., $T_g : t \to T(t)g$ is continuous for all $g \in X$. (Note that $\{T(t) : t \in \boldsymbol{\tau}\}$ is bounded by the principle of uniform boundedness (see [**5**], 14.1).) Then, there exists and operator $W \in \mathcal{L}(X)$ such that $\int_a^b T(t)g dt = Wg$ for all $g \in X$. That W is bounded follows from

$$\|Wg\| = \| \int_a^b T(t)g \, dt\| \leq \int_a^b \|T(t)g\| \, dt \leq \int_a^b \|T(t)\|\|g\| \, dt.$$

We define the integral $\int_a^b T(t)dt$ to be that W. Hence $\lim_{|P|\to 0} R(T_g, P, \boldsymbol{\tau}) = \int_a^b T(t)gdt$.

Fix $V \in \mathcal{L}(X)$. Then $\int_a^b VT(t) \, dt = V \int_a^b T(t)dt$. Indeed,

$$R((VT)g, P, \boldsymbol{\tau}) = \sum_{i=1}^n VT(t_i)g(s_i - s_{i-1}) = V \sum_{i=1}^n T(t_i)g(s_i - s_{i-1}) \to V \int_a^b T(t)gdt.$$

Let F and H be strongly continuous Banach space valued functions on $[a,b]$. Then following properties of the Riemann integral are obvious.

(i) If $\alpha, \beta \in \mathbb{C}$, then

$$\int_a^b (\alpha F(s) + \beta H(s)) \, ds = \alpha \int_a^b F(s) \, ds + \beta \int_a^b H(s) \, ds.$$

(ii) If $a < c < b$, then

$$\int_a^b F(s) \, ds = \int_a^c F(s) \, ds + \int_c^b F(s) \, ds.$$

(iii)

$$\left\| \int_a^b F(s) \, ds \right\| \leq \int_a^b \|F(s)\| \, ds.$$

References: [**11**], [**16**].

CHAPTER 3

Harmonic Analysis on $\mathbb{T}$

1. Definitions and Basic Properties

Let $\mathbb{T}$ be the circle group, defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$ with characters $e^{in\cdot}$ $(n \in \mathbb{Z})$. $\mathbb{T}$ can also be thought of as the set $\{z \in \mathbb{C} : |z| = 1\}$ and functions on $\mathbb{T}$ are naturally identified with 2π-periodic functions on $\mathbb{R}$. We define the translation operator R_t on a function f defined on $\mathbb{T}$ by

$$(R_t f)(s) = f(s-t) \quad (s, t \in \mathbb{T})$$

and denote by $L_1(\mathbb{T})$ the space of all complex-valued Lebesgue integrable functions on $\mathbb{T}$ equipped with the norm $\|f\|_{L_1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|\, dt$.

The following definitions and results for $\mathbb{T}$ are valid for any compact abelian group.

DEFINITION 1.1. A *homogeneous Banach space* on $\mathbb{T}$ is a dense linear subspace B of $L_1(\mathbb{T})$ with a norm $\|\cdot\|_B$ satisfying the following:

(i) $(B, \|\cdot\|_B)$ is a Banach space and $\|f\|_{L_1} \leq \|f\|_B$ for all $f \in B$;

(ii) $(B, \|\cdot\|_B)$ is translation invariant, i.e., for all $f \in B$ and $t \in \mathbb{T}$, $R_t f \in B$ and $\|R_t f\|_B = \|f\|_B$;

(iii) functions in B translates continuously, i.e., for all $f \in B$, $\lim_{t\to 0} \|R_t f - f\|_B = 0$;

(iv) B is closed under multiplication by the characters of $\mathbb{T}$.

EXAMPLES. The following spaces are homogeneous Banach spaces on $\mathbb{T}$:

(i) The subspace $L_p(\mathbb{T})$, $1 \leq p < \infty$, of $L_1(\mathbb{T})$ comprising of all functions f such that $\int_{\mathbb{T}} |f(t)|^p \, dt < \infty$ with the usual $\|\cdot\|_{L_p}$-norm.

(ii) The space $\mathfrak{C}(\mathbb{T})$ of all continuous 2π-periodic functions with the norm $\|f\|_\infty = \max_t |f(t)|$.

(iii) The subspace $\mathfrak{C}^n(\mathbb{T})$ of $\mathfrak{C}(\mathbb{T})$ of all n-times continuously differentiable functions with the norm $\|f\|_{\mathfrak{C}^n} = \sum_{k=0}^n \frac{1}{k!} \max_t |f^k(t)|$.

Let $(\mathcal{L}(B), \|\cdot\|_{\mathcal{L}})$ denote the Banach algebra of bounded linear operators on B. We define invariant, almost invariant and simple operators in $\mathcal{L}(B)$ as follows:

DEFINITION 1.2.

(i) An operator $T \in \mathcal{L}(B)$ is called *invariant* if T commutes with translation. That is, $TR_t = R_t T$ for all $t \in \mathbb{T}$. We denote by $\mathcal{L}_0$ the set of invariant operators in $\mathcal{L}(B)$.

(ii) An operator $T \in \mathcal{L}(B)$ is said to be *almost invariant* if, for all $t \in \mathbb{T}$,

$$\lim_{t \to 0} \|TR_t - R_t T\|_{\mathcal{L}} = 0.$$

(Note that R_t is an isometry, i.e., $\|R_t T\|_{\mathcal{L}} = \|T\|_{\mathcal{L}}$ for all $t \in \mathbb{T}$.) We denote the set of almost invariant operators in $\mathcal{L}(B)$ by $\mathcal{L}_*$.

(iii) An operator $T \in \mathcal{L}(B)$ is called *simple* if there exists an integer n such that

$$TR_t = e^{int} R_t T \qquad \text{for all } t \in \mathbb{T}.$$

The set of simple operators in $\mathcal{L}(B)$ satisfying the above equation is denoted by $\mathcal{L}_n$.

PROPOSITION 1.1. *$\mathcal{L}_0$ and $\mathcal{L}_*$ are closed subalgebras of $\mathcal{L}(B)$.*

PROOF. Let $T_1, T_2 \in \mathcal{L}_0$. Then for scalars α, β,

$$(\alpha T_1 + \beta T_2)R_t = \alpha R_t T_1 + \beta R_t T_2 = R_t(\alpha T_1 + \beta T_2).$$

Thus $\alpha T_1 + \beta T_2 \in \mathcal{L}_0$. Also, $(T_1 T_2)R_t = T_1 R_t T_2 = R_t(T_1 T_2)$ shows that $T_1 T_2 \in \mathcal{L}_0$. Hence $\mathcal{L}_0$ is a subalgebra of $\mathcal{L}(B)$. To show that $\mathcal{L}_0$ is closed, we take $T_n \in \mathcal{L}_0$ such that $T_n \to T \in \mathcal{L}(B)$. Now $T_n \to T$ implies $T_n R_t \to T R_t$ and $R_t T_n \to R_t T$. Since $T_n R_t = R_t T_n$ for all n, we have, by uniqueness of limit, that $T R_t = R_t T$. Hence $T \in \mathcal{L}_0$ and $\mathcal{L}_0$ is closed.

Suppose $T_1, T_2 \in \mathcal{L}_*$. Then

$$\lim_{t\to 0} \|(T_1 + T_2)R_t - R_t(T_1 + T_2)\|_{\mathcal{L}} = \lim_{t\to 0} \|T_1 R_t - R_t T_1 + T_2 R_t - R_t T_2\|_{\mathcal{L}}$$

$$\leq \lim_{t\to 0} \|T_1 R_t - R_t T_1\|_{\mathcal{L}} + \lim_{t\to 0} \|T_2 R_t - R_t T_2\|_{\mathcal{L}} = 0.$$

Thus $T_1 + T_2 \in \mathcal{L}_*$. Similarly, $\alpha T_1 \in \mathcal{L}_*$ for all scalars α. Also, $T_1 T_2 \in \mathcal{L}_*$ since

$$\begin{aligned}
\lim_{t\to 0} \|T_1 T_2 R_t - R_t T_1 T_2\|_{\mathcal{L}} &= \lim_{t\to 0} \|T_1 T_2 R_t - T_1 R_t T_2 + T_1 R_t T_2 - R_t T_1 T_2\|_{\mathcal{L}} \\
&= \lim_{t\to 0} \|T_1(T_2 R_t - R_t T_2) + (T_1 R_t - R_t T_1)T_2\|_{\mathcal{L}} \\
&\leq \|T_1\|_{\mathcal{L}} \lim_{t\to 0} \|T_2 R_t - R_t T_2\|_{\mathcal{L}} + \|T_2\|_{\mathcal{L}} \lim_{t\to 0} \|T_1 R_t - R_t T_1\|_{\mathcal{L}} \\
&= 0.
\end{aligned} \tag{1.1}$$

Thus $T_1, T_2 \in \mathcal{L}_*$, and $\mathcal{L}_*$ is a subalgebra of $\mathcal{L}(B)$. We now show that $\mathcal{L}_*$ is closed. Let $T_n \in \mathcal{L}_*$ such that $T_n \to T \in \mathcal{L}(B)$. Now, for all $n \in \mathbb{N}$, we have

$$\|TR_t - R_tT\|_{\mathcal{L}} = \|(T - T_n)R_t - R_t(T - T_n) + T_nR_t - R_tT_n\|_{\mathcal{L}}$$

$$\leq 2\|T - T_n\|_{\mathcal{L}} + \|T_nR_t - R_tT_n\|_{\mathcal{L}}.$$

Since $T_n \to T$, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|T - T_N\|_{\mathcal{L}} < \varepsilon/4$. There is also a $\delta = \delta(\varepsilon, N) > 0$ such that for every $|t| < \delta$, $\|T_NR_t - R_tT_N\|_{\mathcal{L}} < \varepsilon/2$. Thus $\|TR_t - R_tT\|_{\mathcal{L}} < \varepsilon$ for all $\varepsilon > 0$ and $T \in \mathcal{L}_*$. □

LEMMA 1.2. *Let $T \in \mathcal{L}(B)$. Then the following are equivalent*:

(i) $\lim_{t\to 0} \|TR_t - R_tT\|_{\mathcal{L}} = 0$.

(ii) *The mapping $t \mapsto R_{-t}TR_t$ is continuous from $\mathbb{T}$ to the norm topology of $\mathcal{L}(B)$.*

PROOF. Let $s, t \in \mathbb{T}$. Then $R_{-s}TR_s - R_{-t}TR_t = R_{-s}(TR_{s-t} - R_{s-t}T)R_t$. Since R_{-s} and R_t are isometries, $\|R_{-s}TR_s - R_{-t}TR_t\|_{\mathcal{L}} = \|TR_{s-t} - R_{s-t}T\|_{\mathcal{L}}$ so that

$$\lim_{s\to t} \|R_{-s}TR_s - R_{-t}TR_t\|_{\mathcal{L}} = \lim_{s\to t} \|TR_{s-t} - R_{s-t}T\|_{\mathcal{L}}.$$

Hence (i) and (ii) are equivalent.

□

The following two lemmas demonstrate important properties of $\mathcal{L}_n$.

LEMMA 1.3. *Let M_n be the multiplication operator by the character $e^{in\cdot}$ of $\mathbb{T}$. Then $M_n \in \mathcal{L}_n$.*

PROOF. Let $f \in B$ and $s, t \in \mathbb{T}$. Then

$$[(M_n R_t)f](s) = [M_n(R_t f)](s) = e^{ins} f(s-t) = e^{int}[e^{in(s-t)} f(s-t)]$$

$$= e^{int} R_t[e^{ins} f(s)] = e^{int} R_t[M_n f](s).$$

Thus $M_n \in \mathcal{L}_n$. □

LEMMA 1.4.

(i) *Each* $\mathcal{L}_n$ *is a closed linear subspace of* $\mathcal{L}(B)$.

(ii) $\mathcal{L}_n \subseteq \mathcal{L}_*$ *for each* n.

(iii) $\mathcal{L}_m \cap \mathcal{L}_n = \{\mathbf{0}\}$ *for* $m \neq n$.

(iv) *If* $U \in \mathcal{L}_m$ *and* $V \in \mathcal{L}_n$, *then* $UV \in \mathcal{L}_{m+n}$.

PROOF.

(i) Suppose $U, V \in \mathcal{L}_n$, and α, β are scalars. Then

$$(\alpha U + \beta V)R_t = \alpha U R_t + \beta V R_t = \alpha e^{int} R_t U + \beta e^{int} R_t V = e^{int} R_t(\alpha U + \beta V).$$

Thus $(\alpha U + \beta V) \in \mathcal{L}_n$ and $\mathcal{L}_n$ is a linear subspace of $\mathcal{L}(B)$. To show that $\mathcal{L}_n$ is closed, let $T_k \in \mathcal{L}_n$ such that $T_k \to T \in \mathcal{L}(B)$. Now, $T_k \to T$ implies $T_k R_t \to T R_t$ and $T_k R_t = e^{int} R_t T_k \to e^{int} R_t T$. Therefore $T R_t = e^{int} R_t T$, and so $T \in \mathcal{L}_n$.

(ii) Suppose $T \in \mathcal{L}_n$, then

$$\lim_{t \to 0} \|T R_t - R_t T\|_{\mathcal{L}} = \lim_{t \to 0} \|e^{int} R_t T - R_t T\|_{\mathcal{L}} = \lim_{t \to 0} \|(e^{int} - 1) R_t T\|_{\mathcal{L}}$$

$$= \lim_{t \to 0} |e^{int} - 1| \|T\|_{\mathcal{L}} = 0.$$

Thus $T \in \mathcal{L}_*$ and $\mathcal{L}_n \subseteq \mathcal{L}_*$.

(iii) For $m \neq n$, suppose $T \in \mathcal{L}_m \cap \mathcal{L}_n$. Then $T \in \mathcal{L}_m$ and $T \in \mathcal{L}_n$. That is, $TR_t = e^{imt}R_tT$ and $TR_t = e^{int}R_tT$ so that $(e^{imt} - e^{int})R_tT = \mathbf{0}$. Hence $R_tT = \mathbf{0}$, i.e., $T = \mathbf{0}$.

(iv) Let $U \in \mathcal{L}_m$ and $V \in \mathcal{L}_n$. Then

$$R_{-t}UVR_t = (R_{-t}UR_t)(R_{-t}VR_t) = e^{imt}Ue^{int}V = e^{i(m+n)t}UV.$$

Thus $UV \in \mathcal{L}_{m+n}$.

□

COROLLARY 1.5. *Let $T \in \mathcal{L}(B)$. Then the following are equivalent:*

(i) $T \in \mathcal{L}_n$

(ii) *There exists $U \in \mathcal{L}_0$ such that $T = UM_n$.*

(iii) *There exists $V \in \mathcal{L}_0$ such that $T = M_nV$.*

PROOF. Let $T \in \mathcal{L}_n$. Then by (iv) of Lemma 1.4 and Lemma 1.3, $TM_{-n} \in \mathcal{L}_0$. Since $T = (TM_{-n})M_n$, we choose $U = TM_{-n}$ so that (i) implies (ii).

Now suppose $U \in \mathcal{L}_0$ such that $T = UM_n$. Taking $V = M_{-n}UM_n$, we have $V \in \mathcal{L}_0$ and $T = M_nV$. Thus (ii) implies (iii).

Finally, if $V \in \mathcal{L}_0$ and $T = M_nV$, then

$$TR_t = (M_nV)R_t = M_nR_tV = e^{int}R_t(M_nV) = e^{int}R_tT.$$

Thus $T \in \mathcal{L}_n$, showing (iii) implies (i).

□

2. The Fourier Series

In this section, we develop a Fourier series for operators in $\mathcal{L}(B)$ and show that the series thus obtained has properties analogous to that of the familiar Fourier series. We prove that if $T \in \mathcal{L}_*$, then its Fourier series is C-1 summable to T in the operator norm. By this we see that that $\mathcal{L}_*$ is the normed closed subalgebra of $\mathcal{L}(B)$ generated $\mathcal{L}_0$ and M_n. We also show that the Fourier series of an arbitrary operator $T \in \mathcal{L}(B)$ is C-1 summable to T in the strong operator topology of $\mathcal{L}(B)$.

The following lemma is useful for further results.

LEMMA 2.1. *Let $T \in \mathcal{L}(B)$ and $f \in B$. Then the mapping $s \mapsto R_{-s}TR_sf$ is continuous from $\mathbb{T}$ to the norm topology of B.*

PROOF. Let $t \in \mathbb{T}$. We show that the map $s \mapsto R_{-s}TR_sf$ is continuous at t. We have

$$\begin{aligned} R_{-s}TR_sf - R_{-t}TR_tf &= [R_{-s}TR_sf - R_{-s}TR_tf] + [R_{-s}TR_tf - R_{-t}TR_tf] \\ &= R_{-s}TR_s[f - R_{t-s}f] + R_{-s}[TR_tf - R_{s-t}TR_tf]. \end{aligned}$$

Hence, $\|R_{-s}TR_sf - R_{-t}TR_tf\|_B \leq \|T\|_{\mathcal{L}}\|f - R_{t-s}f\|_B + \|(TR_tf) - R_{s-t}(TR_tf)\|_B$.

Since $f \in B$ and $(TR_tf) \in B$, by applying Definition 1.1 (iii), we have

$$\lim_{s\to t} \|R_{-s}TR_sf - R_{-t}TR_tf\|_B = 0$$

and thus the mapping $s \mapsto R_{-s}TR_sf$ is continuous at t. □

Now for each $T \in \mathcal{L}(B)$, each integer n, and each $f \in B$, we define $\pi_n(T)(f) \in B$ by

$$\pi_n(T)(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} R_{-t} T R_t f \, dt.$$

This definition makes sense because of Lemma 2.1. We also see that if T is almost invariant, then the vector-valued integral

$$\pi_n(T) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} R_{-t} T R_t \, dt$$

exists since the integrand is continuous from $\mathbb{T}$ to the norm topology of $\mathcal{L}(B)$ by virtue of Lemma 1.2. The map $\pi_n(T) : B \to B$ is clearly linear. For boundedness, for all $f \in B$, we have

$$\|\pi_n(T)(f)\|_B \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \|e^{-int} R_{-t} T R_t f\|_B \, dt \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \|T\|_{\mathcal{L}} \|f\|_B \, dt \leq \|T\|_{\mathcal{L}} \|f\|_B. \tag{2.1}$$

That is,

$$\|\pi_n(T)\|_{\mathcal{L}} \leq \|T\|_{\mathcal{L}}. \tag{2.2}$$

Therefore $\pi_n(T) \in \mathcal{L}(B)$ and π_n:$\mathcal{L}(B) \to \mathcal{L}(B)$. Clearly, $\pi_n : \mathcal{L}(B) \to \mathcal{L}(B)$ is linear and (2.2) shows that π_n is bounded and $\|\pi_n\| \leq 1$.

PROPOSITION 2.2. *π_n is a projection of $\mathcal{L}(B)$ onto $\mathcal{L}_n$.*

PROOF. Since π_n is linear, we only need to show that

(a) π_n takes $\mathcal{L}(B)$ into $\mathcal{L}_n$;

(b) for every $T \in \mathcal{L}_n$, $\pi_n(T) = T$.

(a) Let s be any point of $\mathbb{T}$. For any $T \in \mathcal{L}(B)$ and $f \in B$, we have

$$[\pi_n(T)]R_s f = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} R_{-t} T R_t R_s f \, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} R_{-t} T R_{s+t} f \, dt$$

$$= \frac{e^{ins}}{2\pi} R_s \int_{-\pi}^{\pi} e^{-in(s+t)} R_{-(s+t)} T R_{(s+t)} f \, dt = e^{ins} R_s \pi_n(T) f.$$

Thus $[\pi_n(T)]R_s = e^{ins} R_s \pi_n(T)$, and $[\pi_n(T)] \in \mathcal{L}_n$.

(b) Suppose $T \in \mathcal{L}_n$ and $f \in B$. Then

$$[\pi_n(T)]f = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} R_{-t} T R_t f \, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} R_{-t} e^{int} R_t T f \, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} Tf \, dt = Tf.$$

Therefore $\pi_n(T) = T$.

Hence π_n is a projection of $\mathcal{L}(B)$ onto $\mathcal{L}_n$. □

To every $T \in \mathcal{L}(B)$, we assign the series

$$\sum_{-\infty}^{+\infty} \pi_n(T)$$

and call it the *Fourier series* of the operator T.

Using the assignment above, we demonstrate a fundamental property of Fourier series, showing that our definition gives an extension of the usual Fourier series concept.

PROPOSITION 2.3. *Suppose $B = \mathfrak{C}(\mathbb{T})$ and $\varphi \in B$. Let M_φ be the multiplication operator defined by $M_\varphi(f) = \varphi \cdot f, \quad f \in B$. Then the Fourier series of the operator M_φ is*

$$\sum_{-\infty}^{+\infty} \hat{\varphi}(n) M_n, \qquad \text{where} \quad \hat{\varphi}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}\varphi(t) \, dt.$$

PROOF. Let $s \in \mathbb{T}$ and $f \in B$. Then

$$\begin{aligned}\pi_n(M_\varphi)(f)(s) &= \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} R_{-t} M_\varphi R_t f \, dt\right](s) = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} R_{-t}(\varphi \cdot R_t f)\, dt\right](s) \\ &= \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}(R_{-t}\varphi)\cdot f \, dt\right](s) = \left[\frac{1}{2\pi} f \int_{-\pi}^{\pi} e^{-int} R_{-t}\varphi \, dt\right](s) \\ &= \frac{e^{ins}}{2\pi}\int_{-\pi}^{\pi} e^{-in(s+t)}\varphi(s+t)\, dt \cdot f(s) = e^{ins}\hat{\varphi}(n) f(s) \\ &= \hat{\varphi}(n)(M_n f)(s).\end{aligned} \tag{2.3}$$

Therefore, $\pi_n(M_\varphi) = \hat{\varphi}(n) M_n$. □

Before stating and proving some important features of the Fourier series, we give some definitions and useful results.

DEFINITION 2.1. Let X be a Banach space. Consider the series

$$\sum_{k=-\infty}^{\infty} a_k \qquad (a_k \in X) \tag{2.4}$$

and its partial sums

$$s_n = \sum_{k=-N}^{N} a_k \quad .$$

The average

$$\sigma_N = \frac{1}{N+1}\sum_{n=0}^{N} s_n = \sum_{k=-N}^{N}\left(1 - \frac{|k|}{N+1}\right) a_k \tag{2.5}$$

is called the Nth *Cesàro* (C-1) *sum* of the series (2.4).

The series (2.4) is said to be *Cesàro* (C-1) *summable* to L if the partial sums in (2.5) converge to L in the $\|\cdot\|_X$-norm.

DEFINITION 2.2. A *summability kernel* $\{k_n\}$ is a sequence of continuous 2π-periodic functions satisfying the following:

(i) For every integer n,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(t)\,dt = 1. \tag{2.6}$$

(ii) There is a constant C such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |k_n(t)|\,dt \le C \tag{2.7}$$

for all n.

(iii) For all $0 < \tau < \pi$,

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{\tau}^{2\pi-\tau} |k_n(t)|\,dt = 0. \tag{2.8}$$

The following lemma, proved by Katznelson ([**11**], 2.2) in terms of vector-valued integrals, is useful for further results.

LEMMA 2.4. *Let X be Banach space, φ a continuous X-valued function on $\mathbb{T}$, and $\{k_n\}$ a summability kernel. Then*

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(t)\varphi(t)\,dt = \varphi(0). \tag{2.9}$$

PROOF. For $0 < \delta < \pi$, we have

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(t)\varphi(t)\,dt - \varphi(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(t)[\varphi(t)-\varphi(0)]\,dt \qquad (2.10)\\ = \frac{1}{2\pi}\int_{-\delta}^{\delta} k_n(t)[\varphi(t)-\varphi(0)]\,dt &+ \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} k_n(t)[\varphi(t)-\varphi(0)]\,dt. \end{aligned}$$

By (2.7), there is a constant $C > 0$ such that $\|k_n\|_{L_1} \leq C$ for all n. And the continuity of φ at 0 implies for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $|t| < \delta, \|\varphi(t) - \varphi(0)\| < \varepsilon/C$. Consequently,

$$\begin{aligned}\left\| \frac{1}{2\pi} \int_{-\delta}^{\delta} k_n(t)[\varphi(t) - \varphi(0)]\, dt \right\| &\leq \frac{1}{2\pi} \int_{-\delta}^{\delta} |k_n(t)| \|\varphi(t) - \varphi(0)\|\, dt \\ &\leq \max_{|t|<\delta} \|\varphi(t) - \varphi(0)\|_X \|k_n\|_{L_1} \qquad (2.11) \\ &\leq \varepsilon.\end{aligned}$$

Since $\varphi : \mathbb{T} \to X$ is continuous and $\mathbb{T}$ is compact, there exists a constant $M > 0$ such that $\|\varphi(t)\| \leq M$ for all $t \in \mathbb{T}$. By (2.8), there is an $N \in \mathbb{N}$ such that $\frac{1}{2\pi} \int_{\delta}^{2\pi-\delta} |k_n(t)|\, dt < \varepsilon/2M$ for all $n \geq N$. So for $n \geq N$, we have

$$\left\| \frac{1}{2\pi} \int_{\delta}^{2\pi-\delta} k_n(t)[\varphi(t) - \varphi(0)]\, dt \right\| \leq \frac{1}{2\pi} \int_{\delta}^{2\pi-\delta} 2M|k_n(t)|\, dt \ < \varepsilon. \qquad (2.12)$$

Therefore (2.10) is bounded by 2ε for all $n \geq N$, and the proof follows. □

We denote by $\boldsymbol{K}_N$, the Nth order *Fejér's kernel*, defined by

$$\boldsymbol{K}_N(t) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{int}. \qquad (2.13)$$

The Fejér's kernel, $\{\boldsymbol{K}_N\}$, is a summability kernel. To prove this, we need the following.

LEMMA 2.5.

$$\boldsymbol{K}_N(t) \ = \ \frac{1}{N+1} \left[\frac{\sin[\frac{1}{2}(N+1)t]}{\sin \frac{t}{2}} \right]^2$$

PROOF. For $r \neq 1$,

$$\begin{aligned}\sum_{n=-N}^{N}(N-|n|+1)r^n &= r^{-N}\left(\sum_{n=-N}^{N}(N-|n|+1)r^{n+N}\right)\\ &= r^{-N}\left(\sum_{n=0}^{N}r^n\right)^2 = r^{-N}\left(\frac{1-r^{N+1}}{1-r}\right)^2.\end{aligned} \tag{2.14}$$

Hence if $t \neq 0$, then

$$\begin{aligned}\sum_{n=-N}^{N}(N-|n|+1)e^{int} &= e^{-iNt}\left[\frac{1-e^{i(N+1)t}}{1-e^{it}}\right]^2 = \left[\frac{e^{-i(N+1)t/2}(1-e^{i(N+1)t})}{e^{-it/2}(1-e^{it})}\right]^2\\ &= \left[\frac{e^{-i(N+1)t/2}-e^{i(N+1)t/2}}{e^{-it/2}-e^{it/2}}\right]^2 = \left[\frac{\sin[\frac{1}{2}(N+1)t]}{\sin\frac{t}{2}}\right]^2.\end{aligned} \tag{2.15}$$

So for $t \neq 0$,

$$\boldsymbol{K}_N(t) = \frac{1}{N+1}\sum_{n=-N}^{N}(N-|n|+1)e^{int} = \frac{1}{N+1}\left[\frac{\sin[\frac{1}{2}(N+1)t]}{\sin\frac{t}{2}}\right]^2.$$

For $t = 0$, we have

$$\boldsymbol{K}_N(t) = \frac{1}{N+1}\sum_{n=-N}^{N}(N-|n|+1)$$

$$= \frac{1}{N+1}[1+2+\cdots+(N+1)+\cdots+2+1] = N+1$$

and

$$\lim_{t\to 0}\frac{1}{N+1}\left[\frac{\sin[\frac{1}{2}(N+1)t]}{\sin\frac{t}{2}}\right]^2 = N+1.$$

Therefore

$$\boldsymbol{K}_N(t) = \frac{1}{N+1}\left[\frac{\sin[\frac{1}{2}(N+1)t]}{\sin\frac{t}{2}}\right]^2$$

for all t. □

LEMMA 2.6. *$\{\boldsymbol{K}_N\}$ is a summability kernel.*

PROOF. For each $N \in \mathbb{N}$, we have

$$\begin{aligned}\boldsymbol{K}_N(t) &= \sum_{n=-N}^{N}\left(1-\frac{|n|}{N+1}\right)e^{int} = 1+\sum_{n=1}^{N}\left(1-\frac{n}{N+1}\right)(e^{-int}+e^{int}) \\ &= 1+2\sum_{n=1}^{N}\left(1-\frac{n}{N+1}\right)\cos nt.\end{aligned} \tag{2.16}$$

Therefore,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\boldsymbol{K}_N(t)\,dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi}1\,dt+\frac{1}{\pi}\sum_{n=1}^{N}\left(1-\frac{n}{N+1}\right)\int_{-\pi}^{\pi}\cos nt\,dt \\ &= 1,\end{aligned}$$

showing (2.6) holds. By Lemma 2.5, $\boldsymbol{K}_N(t) \geq 0$ for every t, so (2.7) is satisfied. Now if $0 < \tau < \pi$, then, for $\tau < t < 2\pi - \tau$, we have

$$\boldsymbol{K}_N(t) \leq \frac{1}{(N+1)\sin^2(\tau/2)}.$$

So,

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_{\tau}^{2\pi-\tau}\boldsymbol{K}_N(t)\,dt = 0$$

for all $0 < \tau < \pi$. Thus (2.8) holds.

□

We now return to our discussion on Fourier series with

PROPOSITION 2.7. *Let $T \in \mathcal{L}_*$. Then the Fourier series of T is C-1 summable to T in the operator norm.*

PROOF. The Nth C-1 sum of the Fourier series of T is

$$\begin{aligned}\sum_{n=-N}^{N}\left(1-\frac{|n|}{N+1}\right)\pi_n(T) &= \sum_{n=-N}^{N}\left(1-\frac{|n|}{N+1}\right)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-int}R_{-t}TR_t\,dt\right]\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=-N}^{N}\left(1-\frac{|n|}{N+1}\right)e^{-int}\right]R_{-t}TR_t\,dt\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\boldsymbol{K}_N(t)R_{-t}TR_t\,dt, \qquad (2.17)\end{aligned}$$

where $\boldsymbol{K}_N(t)$ is the Nth order Fejér's kernel. Since $\boldsymbol{K}_N(t)$ is a summability kernel and $\varphi(t) = R_{-t}TR_t$ is continuous, we compare (2.17) with (2.9) of Lemma 2.4 to conclude that

$$\lim_{N\to\infty}\sum_{n=-N}^{N}\left(1-\frac{|n|}{N+1}\right)\pi_n(T) = \lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\boldsymbol{K}_N(t)\varphi(t)\,dt = \varphi(0) = T.$$

Therefore, the Fourier series $\sum_{-\infty}^{+\infty}\pi_n(T)$ is C-1 summable to T in the norm topology.

□

As an immediate consequence of Proposition 2.7, we have the following:

COROLLARY 2.8. *$\mathcal{L}_*$ is the normed closed subalgebra of $\mathcal{L}(B)$ generated by $\mathcal{L}_0$ and the M_n ($n \in \mathbb{N}$).*

PROOF. That $\mathcal{L}_*$ is a normed closed subalgebra of $\mathcal{L}(B)$ is seen in Proposition 1.1. Now let $\mathcal{K}$ be the normed closed subalgebra of $\mathcal{L}(B)$ generated by $\mathcal{L}_0$ and M_n. We show that $\mathcal{K} = \mathcal{L}_*$.

Obviously, $\mathcal{K} \subseteq \mathcal{L}_*$ by Lemma 1.3 and Lemma 1.4.

Conversely, let $T \in \mathcal{L}_*$. Then by Proposition 2.2 and Corollary 1.5,

$$\pi_n(T) \in \mathcal{L}_n = \mathcal{L}_0 M_n \subseteq \mathcal{K},$$

and by Proposition 2.7

$$\sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \pi_n(T) \longrightarrow T \qquad \text{as} \quad N \to \infty.$$

So $T \in \mathcal{K}$. Thus $\mathcal{L}_* \subseteq \mathcal{K}$. □

The following is an analogue of the Riemann-Lebesgue lemma (see [**11**], 2.8).

COROLLARY 2.9. *Let $T \in \mathcal{L}_*$. Then*

$$\lim_{|n| \to \infty} \|\pi_n(T)\|_{\mathcal{L}} = 0.$$

PROOF. Suppose $m \in \mathbb{N}$ and $T \in \mathcal{L}_m$. Then for $n > |m|$,

$$\pi_n(T) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} R_{-t} T R_t \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} e^{imt} T \, dt = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)t} \, dt\right) T = 0.$$

If T is a finite sum of a simple operators, say

$$T = T_{m_1} + T_{m_2} + \cdots + T_{m_k}, \quad \text{where} \quad T_{m_1} \in \mathcal{L}_{m_1}, \ldots, T_{m_k} \in \mathcal{L}_{m_k} \quad (m_i \in \mathbb{Z}).$$

Let $M = \max_i |m_i|$. Then for $n > M$, $\pi_n(T) = 0$.

Now suppose $T \in \mathcal{L}_*$ is an arbitrary operator and $\varepsilon > 0$. By Proposition 2.7, we can choose a finite sum S of simple operators such that $\|T - S\|_{\mathcal{L}} < \varepsilon$. Hence when $|n|$ is large enough, by applying (2.2), we have

$$\|\pi_n(T)\|_{\mathcal{L}} = \|\pi_n(T - S)\|_{\mathcal{L}} \leq \|T - S\|_{\mathcal{L}} < \varepsilon.$$

Therefore, $\lim_{|n|\to\infty} \|\pi_n(T)\|_{\mathcal{L}} = 0$. □

COROLLARY 2.10. *Let $T \in \mathcal{L}_*$. Then T is a compact operator if and only if $\pi_n(T)$ is a compact operator for each n.*

PROOF. Suppose $T \in \mathcal{L}_*$ is compact. Then $R_{-t}TR_t$ is also compact for all $t \in \mathbb{T}$. Let $\mathcal{K}$ be the normed closed linear subspace of $\mathcal{L}(B)$ spanned by $R_{-t}TR_t$. Then each operator in $\mathcal{K}$ is compact. Since

$$\pi_n(T) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}R_{-t}TR_t\,dt = \frac{1}{2\pi}\lim_{m\to\infty}\sum_{i=1}^{m}(e^{-int_i}R_{-t_i}TR_{t_i})(t_i - t_{i-1}), \qquad (2.18)$$

where $t_k = -\pi + \frac{2\pi k}{m}$ $(0 \le k \le m)$, $\pi_n(T) \in \mathcal{K}$ and hence $\pi_n(T)$ is compact.

Conversely let $\pi_n(T)$ be compact for each n. Then $\sum_{n=-N}^{N}(1 - \frac{|n|}{N+1})\pi_n(T)$ is compact and (by Proposition 2.7) converges to T in the operator norm. Therefore, T is compact. □

Generally, for an arbitrary $T \in \mathcal{L}(B)$, we have the following weaker version of Proposition 2.7.

PROPOSITION 2.11. *Let $T \in \mathcal{L}(B)$ and $f \in B$. Then the series $\sum_{n=-\infty}^{\infty}\pi_n(T)f$ is C-1 summable to Tf in the norm of B.*

PROOF. Analogous to the proof of Proposition 2.7. □

We also have a parallel of the Riemann-Lebesgue lemma in the strong operator topology of $\mathcal{L}(B)$.

COROLLARY 2.12. *Let $T \in \mathcal{L}(B)$ and $f \in B$. Then*

$$\lim_{|n|\to\infty} \|\boldsymbol{\pi}_n(T)(f)\|_B = 0.$$

PROOF. Analogous to the proof of Corollary 2.9. □

3. The Fourier Transform and Convolution

For every $T \in \mathcal{L}(B)$ and every integer n, we define $\widehat{T}(n)$ to be the operator $\boldsymbol{\pi}_n(T)$. We therefore have a map $\widehat{T} : \mathbb{Z} \to \mathcal{L}(B)$ which is called the Fourier transform of the operator T. As already seen, $\widehat{T}(n) \in \mathcal{L}_n$ for each n, and from (2.2),

$$\|\widehat{T}(n)\|_{\mathcal{L}} \leq \|T\|_{\mathcal{L}} \qquad \text{for each } n.$$

Also if $T \in \mathcal{L}_*$, by Corollary 2.9, we have

$$\lim_{|n|\to\infty} \|\widehat{T}(n)\|_{\mathcal{L}} = 0.$$

The following shows how the Fourier transform takes operator multiplication into convolution.

PROPOSITION 3.1. *Let $S, T \in \mathcal{L}_*$. Then the series*

$$\sum_{m=-\infty}^{\infty} \widehat{S}(n-m)\widehat{T}(m)$$

is C-1 summable to the operator $\widehat{ST}(n)$ in the operator norm.

PROOF. By definition, we have

$$(\widehat{ST})(n) = \boldsymbol{\pi}_n(ST) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} R_{-t} S T R_t \, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} R_{-t} S \left[\lim_{N\to\infty} \sum_{m=-N}^{N} \left(1 - \frac{|m|}{N+1}\right) \boldsymbol{\pi}_m(T)\right] R_t \, dt$$

(due to Proposition 2.7)

$$= \lim_{N\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} \left[\sum_{m=-N}^{N} \left(1 - \frac{|m|}{N+1}\right) R_{-t} S R_t R_{-t} \boldsymbol{\pi}_m(T) R_t\right] dt. \tag{3.1}$$

Since $\boldsymbol{\pi}_m(T) \in \mathcal{L}_m$, $R_{-t}\boldsymbol{\pi}_m(T)R_t = e^{imt}\boldsymbol{\pi}_m(T)$ so that (3.1) becomes

$$\lim_{N\to\infty} \sum_{m=-N}^{N} \left(1 - \frac{|m|}{N+1}\right) \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} R_{-t} S R_t e^{imt} \boldsymbol{\pi}_m(T) \, dt$$

$$= \lim_{N\to\infty} \sum_{m=-N}^{N} \left(1 - \frac{|m|}{N+1}\right) \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n-m)t} R_{-t} S R_t \, dt \, \boldsymbol{\pi}_m(T) \tag{3.2}$$

$$= \lim_{N\to\infty} \sum_{m=-N}^{N} \left(1 - \frac{|m|}{N+1}\right) \boldsymbol{\pi}_{n-m}(S)\boldsymbol{\pi}_m(T).$$

Therefore $\widehat{ST}(n)$ is the C-1 sum of

$$\sum_{m=-\infty}^{\infty} \widehat{S}(n-m)\widehat{T}(m).$$

□

This result, with Proposition 2.3, shows that the Fourier transform of the product of two functions is the convolution of their transforms. An analogue of Proposition 3.1 for an arbitrary operator in $\mathcal{L}(B)$ follows.

PROPOSITION 3.2. *Let $S, T \in \mathcal{L}(B)$ and $f \in B$. Then the series*

$$\sum_{m=-\infty}^{\infty} \widehat{S}(n-m)\widehat{T}(m)f$$

is C-1 summable to $[\widehat{ST}(n)]f$ in the norm topology of B.

PROOF. Analogous to the proof of Proposition 3.1. □

Let $M(\mathbb{T})$ be the Banach space of bounded complex valued regular Borel measures on $\mathbb{T}$ equipped with the total variation norm $\|\cdot\|_M$, i.e., for $\mu \in M(\mathbb{T})$,

$$\|\mu\|_M = \sup\left\{\left|\int_{-\pi}^{\pi} f(t)\, d\mu(t)\right| : f \in \mathfrak{C}(\mathbb{T}) \text{ and } \|f\|_\infty \leq 1\right\}. \tag{3.3}$$

For $\mu \in M(\mathbb{T})$, $T \in \mathcal{L}(B)$, and $f \in B$, we define the convolution $\mu * T$ by

$$(\mu * T)(f) = \int_{-\pi}^{\pi} R_t T R_{-t} f \, d\mu(t).$$

The integral is a well defined element of B since $t \mapsto R_t T R_{-t} f$ is continuous (by Lemma 2.1). $\mu * T{:}B \to B$ is clearly linear, and bounded since

$$\begin{aligned} \|(\mu * T)(f)\| &\leq \int_{-\pi}^{\pi} \|R_t T R_{-t} f\| \, d|\mu|(t) \leq \int_{-\pi}^{\pi} \|T\|\|f\| \, d|\mu|(t) \\ &= \left(\int_{-\pi}^{\pi} d|\mu(t)|\right)\|T\|\|f\| = \|\mu\|_M \|T\|_{\mathcal{L}} \|f\|_B. \end{aligned} \tag{3.4}$$

For $T \in \mathcal{L}_*$, $\mu * T$ is defined directly by the integral

$$\int_{-\pi}^{\pi} R_t T R_{-t} \, d\mu(t).$$

due to Lemma 1.2. Hence if $T \in \mathcal{L}_*$, then $\mu * T \in \mathcal{L}_*$ since $\mathcal{L}_*$ is normed closed in $\mathcal{L}(B)$.

In what follows, we present a justification of our definition of convolution.

LEMMA 3.3. *Let $B = \mathfrak{C}(\mathbb{T})$, $\varphi \in B$, and T the operator of multiplication by φ. Then $\mu * T$ is the operator of multiplication by $\mu * \varphi$, where $\mu * \varphi$ is the function defined by*

$$[\mu * \varphi](s) = \int_{-\pi}^{\pi} \varphi(s-t)\, d\mu(t).$$

PROOF. Let $f \in B$ and $s \in \mathbb{T}$. Then

$$\begin{aligned}
[(\mu * T)f](s) &= \left[\int_{-\pi}^{\pi} R_t T R_{-t} f \, d\mu(t)\right](s) = \int_{-\pi}^{\pi} R_t[TR_{-t}f](s)\, d\mu(t) \\
&= \int_{-\pi}^{\pi} [TR_{-t}f](s-t)\, d\mu(t) = \int_{-\pi}^{\pi} [\varphi \cdot R_{-t}f](s-t)\, d\mu(t) \\
&= \left(\int_{-\pi}^{\pi} \varphi(s-t)\, d\mu(t)\right) f(s) = [\mu * \varphi](s) \cdot f(s).
\end{aligned}$$

Thus $[\mu * T](f) = [\mu * \varphi] \cdot f$. □

We now show that the defined Fourier transform takes convolution into multiplication.

PROPOSITION 3.4. *Let $T \in \mathcal{L}(B)$ and $\mu \in M(\mathbb{T})$. Then $\widehat{[\mu * T]}(n) = \hat{\mu}(n)\widehat{T}(n)$ for all n, where $\hat{\mu}$ is the function defined by $\hat{\mu}(n) = \int_{-\pi}^{\pi} e^{-int}\, d\mu(t)$.*

PROOF. For $f \in B$, we have

$$\begin{aligned}
[\,\widehat{(\mu * T)}(n)\,](f) = [\,\boldsymbol{\pi}_n(\mu * T)\,](f) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ins} R_{-s}(\mu * T) R_s f \, ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ins} R_{-s}\left[\int_{-\pi}^{\pi} R_t T R_{-t} f \, d\mu(t)\right] R_s f \, ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ins}\left[\int_{-\pi}^{\pi} R_{-(s-t)} T R_{s-t} f \, d\mu(t)\right] ds \\
&= \int_{-\pi}^{\pi} e^{-int}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in(s-t)} R_{-(s-t)} T R_{s-t} f \, ds\right] d\mu(t) \\
&= \int_{-\pi}^{\pi} e^{-int}[\,\hat{T}(n) f\,]\, d\mu(t) = \left(\int_{-\pi}^{\pi} e^{-int}\, d\mu(t)\right)(\widehat{T}(n) f) \\
&= \hat{\mu}(n)\widehat{T}(n) f.
\end{aligned}$$

Thus $\widehat{\mu * T} = \hat{\mu} \cdot \widehat{T}$.

□

Indeed, the preceding result shows that for every $T \in \mathcal{L}(B)$ and $\mu \in M(\mathbb{T})$, we can find $S \in \mathcal{L}(B)$ such that $\widehat{S}(n) = \hat{\mu}(n)\widehat{T}(n)$ for every n.

4. A Generalization of F. and M. Riesz Theorem

In this section we will present a generalization of the classical F. and M. Riesz Theorem. To prove our result, we apply the characterization for compactness of almost invariant operators as given in Corollary 2.10 and the fact that compact invariant operators on $\mathfrak{C}(\mathbb{T})$ are given by convolution by $L_1(\mathbb{T})$ functions.

For convenience, we let $B = \mathfrak{C}(\mathbb{T})$ and define $\mathfrak{C}(\mathbb{T})_+$ and $\mathfrak{C}(\mathbb{T})_-$ as follows:

$$\mathfrak{C}(\mathbb{T})_+ = \{f : f \in \mathfrak{C}(\mathbb{T}), \hat{f}(n) = 0 \text{ if } n < 0\};$$

$$\mathfrak{C}(\mathbb{T})_- = \{f : f \in \mathfrak{C}(\mathbb{T}), \hat{f}(n) = 0 \text{ if } n > 0\}$$

where $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt$.

THEOREM 4.1 (F. and M. Riesz). *Let $\mu \in M(\mathbb{T})$ be such that*

$$\hat{\mu}(n) = \int_{-\pi}^{\pi} e^{-int}\,d\mu(t) = 0 \quad \textit{for all} \quad n < 0. \tag{4.1}$$

Then μ is absolutely continuous with respect to the Haar measure on $\mathbb{T}$, i.e., $\mu \in L_1(\mathbb{T})$.

PROOF. See Katznelson ([**11**], 3.13). □

We now introduce our generalization of Theorem 4.1.

THEOREM 4.2. *Let T be an almost invariant operator on $\mathfrak{C}(\mathbb{T})$ such that*

$$T[\,\mathfrak{C}(\mathbb{T})_-\,] \subseteq \mathfrak{C}(\mathbb{T})_+ \tag{4.2}$$

Then T is a compact operator.

Before we prove Theorem 4.2 and show that it is a generalization of Theorem 4.1, we present some definitions and relevant results.

For $\mu \in M(\mathbb{T}), s, t \in \mathbb{T}$, we define the convolution operator C_μ on $\mathfrak{C}(\mathbb{T})$ by

$$[\,C_\mu f\,](t) = (\mu * f)(t) = \int_{-\pi}^{\pi} f(t-s)\,d\mu(s), \tag{4.3}$$

and the translated measure μ_s by

$$\int_{-\pi}^{\pi} f(t)\, d\mu_s(t) = \int_{-\pi}^{\pi} f(t+s)\, d\mu(t) \qquad (f \in \mathfrak{C}(\mathbb{T})).$$

DEFINITION 4.1. Let $\mu \in M(\mathbb{T})$ and ν a non-negative measure on $\mathbb{T}$. We say that μ is *absolutely continuous* with respect to ν, and we write $\mu << \nu$, if for all $E \subseteq \mathbb{T}$, $\mu(E) = 0$ whenever $\nu(E) = 0$.

Equivalently, $\mu << \nu$ if and only if for each $\varepsilon > 0$, there exists $\delta > 0$, such that for all $E \subseteq \mathbb{T}$, $\nu(E) < \delta$ implies $\mu(E) < \varepsilon$.

We denote by $M_a(\mathbb{T})$ the Banach space of all $\mu \in M(\mathbb{T})$ such that μ is absolutely continuous with respect to the Haar measure λ on $\mathbb{T}$. Then $M_a(\mathbb{T})$ is a closed two-sided ideal in $M(\mathbb{T})$ (see [8], 19.18).

LEMMA 4.3. *Let $\mu \in M(\mathbb{T})$. Then $\mu \in M_a(\mathbb{T})$ if and only if $\mu_t \to \mu$ as $t \to 0$ in the $\|\cdot\|_M$-norm.*

PROOF. Suppose $\mu << \lambda$. Then by the Radon Nikodym Theorem ([8], Appendix E9), there is a $g \in L_1(\mathbb{T}) = L_1(\mathbb{T}, \lambda)$ such that

$$\mu(A) = \int_A g\, d\lambda \qquad \text{and} \qquad \|\mu\|_M = \int_{\mathbb{T}} |g|\, d\lambda$$

for all Borel sets A in $\mathbb{T}$. Now, for all $f \in \mathfrak{C}(\mathbb{T})$, $\int_{\mathbb{T}} f(x)\, d\mu(x) = \int_{\mathbb{T}} f(x)g(x)\, d\lambda(x)$, i.e., $d\mu = g d\lambda$. We call g the density function for μ.

For this g and $\phi \in \mathfrak{C}(\mathbb{T})$, we have

$$\begin{aligned}\int_{\mathbb{T}} \phi(x)\, d\mu_t(x) &= \int_{\mathbb{T}} \phi(x+t)\, d\mu(x) = \int_{\mathbb{T}} \phi(x+t)\, g(x)\, d\lambda(x) \\ &= \int_{\mathbb{T}} \phi(x)\, g(x-t)\, d\lambda(x).\end{aligned} \tag{4.4}$$

Hence the density of μ_t with respect to λ is $g_t(x) = g(x-t)$. So,

$$\|\mu_t - \mu\|_M = \int_{\mathbb{T}} |g_t - g|\, d\lambda. \tag{4.5}$$

Note that $\mathfrak{C}(\mathbb{T})$ is $\|\cdot\|_{L_1}$-dense in $L_1(\mathbb{T})$. So, for every $\varepsilon > 0$, there exists $f \in \mathfrak{C}(\mathbb{T})$ such that $\|g-f\|_{L_1} < \varepsilon/3$. Since $\mathbb{T}$ is compact, each $f \in \mathfrak{C}(\mathbb{T})$ is uniformly continuous. Hence there is a $\delta > 0$ such that $|f(x-t) - f(x)| < \varepsilon/3$ for all $x, t \in \mathbb{T}$ with $|t| < \delta$.

Thus for every $|t| < \delta$,

$$\|f_t - f\|_{L_1} = \int_{\mathbb{T}} |f_t - f|\, d\lambda = \int_{\mathbb{T}} |f(x-t) - f(x)|\, d\lambda(x) < \varepsilon/3.$$

Now

$$\begin{aligned}\int_{\mathbb{T}} |g_t - g|\, d\lambda &= \|g_t - g\|_{L_1} = \|g_t - f_t + f_t - f + f - g\|_{L_1} \\ &= \|g_t - f_t\|_{L_1} + \|f_t - f\|_{L_1} + \|f - g\|_{L_1} \\ &= 2\|f - g\|_{L_1} + \|f_t - f\|_{L_1} < \varepsilon.\end{aligned} \tag{4.6}$$

Thus by (4.5) and (4.6), $\|\mu_t - \mu\|_M < \varepsilon$. Therefore, $\mu_t \to \mu$ in the $\|\cdot\|_M$-norm as $t \to 0$.

Conversely, suppose $\mu_t \to \mu$ as $t \to 0$ and F is any compact subset of $\mathbb{T}$ such that $\lambda(F) = 0$. Then

$$\mu(F-t) \to \mu(F) \quad \text{as} \quad t \to 0. \tag{4.7}$$

Since $\lambda \in M_a(\mathbb{T})$ and $M_a(\mathbb{T})$ is a two-sided ideal of $M(\mathbb{T})$ under convolution,

$$0 = |\mu| * \lambda(F) = \int_{\mathbb{T}} |\mu|(F - t)\, dt \geq \int_{\mathbb{T}} |\mu(F - t)|\, dt.$$

Hence $\mu(F - t) = 0$ for almost all $t \in \mathbb{T}$. In particular, there exists a sequence $t_n \to 0$ such that $\mu(F - t_n) = 0$. So, by (4.7),

$$\mu(F) = \lim_{n \to \infty} \mu(F - t_n) = 0.$$

Since F is any compact set in $\mathbb{T}$ and μ is a regular Borel measure on $\mathbb{T}$, $\mu << \lambda$. □

LEMMA 4.4. *Let $\mu \in M(\mathbb{T})$. Then the following are equivalent:*

(i) *μ is absolutely continuous with respect to the Haar measure λ;*

(ii) *the convolution operator C_μ on $\mathfrak{C}(\mathbb{T})$ is compact.*

PROOF. Suppose $\mu << \lambda$. We show that C_μ takes the unit ball $\mathbf{B_1}$ of $\mathfrak{C}(\mathbb{T})$ into a bounded, equicontinuous subset of $\mathfrak{C}(\mathbb{T})$ so that (ii) follows from Arzelà-Ascoli Theorem ([**7**], IV.6.8).

Let $f \in \mathbf{B_1}$. Then

$$\|C_\mu f\|_{\mathfrak{C}(\mathbb{T})} = \|\mu * f\|_{\mathfrak{C}(\mathbb{T})} \leq \|\mu\|_M \|f\|_{\mathfrak{C}(\mathbb{T})} \leq \|\mu\|_M.$$

So $C_\mu(\mathbf{B_1})$ is bounded. The equicontinuity of $C_\mu(\mathbf{B_1})$ follows from

$$\begin{aligned} |(C_\mu f)(s) - (C_\mu f)(t)| &= |(\mu * f)(s) - (\mu * f)(t)| = |[(\mu_{s-t} - \mu) * f](t)| \\ &\leq \|\mu_{s-t} - \mu\|_M \|f\|_{\mathfrak{C}(\mathbb{T})} \leq \|\mu_{s-t} - \mu\|_M \qquad (f \in \mathbf{B_1}) \end{aligned} \tag{4.8}$$

by virtue of Lemma 4.3. Hence C_μ is compact.

Conversely suppose C_μ is a compact operator on $\mathfrak{C}(\mathbb{T})$. For each $f \in \mathfrak{C}(\mathbb{T})$, we define the function

$$\tilde{f}(t) = f(-t), \qquad t \in \mathbb{T}.$$

By Arzelà-Ascoli Theorem, we can find a $\delta > 0$ such that for any $\varepsilon > 0$ and $f \in \mathbf{B_1}$,

$$|(C_\mu \tilde{f})(s) - (C_\mu \tilde{f})(t)| < \varepsilon \quad \text{whenever } |s-t| < \delta.$$

Now suppose $s, t \in \mathbb{T}$ such that $|s-t| < \delta$ and $f \in \mathbf{B_1}$. Then

$$\begin{aligned} \left|(C_\mu \tilde{f})(s) - (C_\mu \tilde{f})(t)\right| &= \left|\int_{-\pi}^{\pi} \tilde{f}(s-x)\, d\mu(x) - \int_{-\pi}^{\pi} \tilde{f}(t-x)\, d\mu(x)\right| \\ &= \left|\int_{-\pi}^{\pi} f(x)\, d\mu_s(x) - \int_{-\pi}^{\pi} f(x)\, d\mu_t(x)\right| < \varepsilon. \end{aligned} \tag{4.9}$$

Since

$$\|\mu_s - \mu_t\|_M = \sup_{f \in \mathbf{B_1}} \left|\int_{-\pi}^{\pi} f(x)\, d\mu_s(x) - \int_{-\pi}^{\pi} f(x)\, d\mu_t(x)\right|,$$

$\mu << \lambda$ by Lemma 4.3. □

LEMMA 4.5. *Let E and F be closed translation invariant linear subspaces of $B(= \mathfrak{C}(\mathbb{T}))$ and $T \in \mathcal{L}(B)$. Then $\pi_n(T)(E) \subseteq F$ for all n if $T(E) \subseteq F$.*

PROOF. Suppose $T(E) \subseteq F$. Let $g \in E$ and $n \in \mathbb{N}$. Then

$$\pi_n(T)g = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} R_{-t} T R_t g\, dt. \tag{4.10}$$

Now since E and F are translation invariant subspaces of B, for each $t \in \mathbb{T}$, $R_t g \in E$, subsequently $R_{-t} T R_t g \in F$ for each $t \in \mathbb{T}$. Since F is a closed linear subspace of B, the integral (4.10) is in F. □

In the following theorem, we give some equivalent conditions for an operator on $L_1(G)$ to be invariant, where G is any locally compact abelian group, as presented by Larson ([**12**], 0.1). By this we see that invariant operators on $L_1(G)$ are basically the convolution operators C_μ. For our current application, however, $\mathfrak{C}(\mathbb{T})$ will replace $L_1(G)$ and $\mathbb{T}$ will serve in the place of G.

THEOREM 4.6. *Let G be a locally compact abelian group and T: $L_1(G) \to L_1(G)$ a continuous linear operator. Then the following are equivalent*:

(i) $TR_s = R_sT$ *for all* $s \in G$.

(ii) $T(f * g) = Tf * g$ *for all* $f, g \in L_1(G)$.

(iii) *There exists a unique function* φ *defined on* $\widehat{G}$ *such that* $\widehat{(Tf)} = \varphi\hat{f}$ *for all* $f \in L_1(G)$.

(iv) *There exists a unique measure* $\mu \in M(G)$ *such that* $\widehat{(Tf)} = \hat{\mu}\hat{f}$ *for all* $f \in L_1(G)$.

(v) *There exists a unique measure* $\mu \in M(G)$ *such that* $Tf = f * \mu$ *for all* $f \in L_1(G)$.

PROOF. (i) $\Rightarrow$ (ii). Suppose $TR_s = R_sT$ for all $s \in G$. Let $k \in L_\infty(G)$. Then the mapping $f \mapsto \int_G Tf(t)k(-t)\,d\lambda(t)$ defines a bounded linear functional on $L_1(G)$ since for all $f \in L_1(G)$,

$$\left|\int_G Tf(t)k(-t)d\lambda(t)\right| \leq \int_G |Tf(t)|\,|k(-t)|\,d\lambda(t) \leq \|Tf\|_1\,\|k\|_\infty \leq \|k\|_\infty\,\|T\|\,\|f\|_1\,,$$

where $\|T\|$ is the usual operator norm of T. By the isometric isomorphism between $L_1(G)^*$ and $L_\infty(G)$, we can find a function $h \in L_\infty(G)$ such that

$$\int_G Tf(t)k(-t)\,d\lambda(t) = \int_G f(t)h(-t)\,d\lambda(t) \qquad (f \in L_1(G)).$$

For $f, g \in L_1(G)$, we have

$$\begin{aligned}
\int_G Tf * g(t)k(-t)\,d\lambda(t) &= \int_G \left[\int_G Tf(t-s)g(s)\,d\lambda(s)\right] k(-t)\,d\lambda(t) \\
&= \int_G \left[\int_G (R_sT)f(t)g(s)\,d\lambda(s)\right] k(-t)\,d\lambda(t) \\
&= \int_G g(s)\left[\int_G TR_sf(t)k(-t)\,d\lambda(t)\right] d\lambda(s) \\
&= \int_G g(s)\left[\int_G R_sf(t)h(-t)\,d\lambda(t)\right] d\lambda(s) \qquad (4.11) \\
&= \int_G \left[\int_G f(t-s)g(s)\,d\lambda(s)\right] h(-t)\,d\lambda(t) \\
&= \int_G (f*g)(t)h(-t)\,d\lambda(t) \\
&= \int_G T(f*g)(t)k(-t)\,d\lambda(t).
\end{aligned}$$

Since (4.11) holds for all $k \in L_\infty(G)$, we conclude by Hahn Banach Theorem ([**7**], II.3.11) that $Tf * g = T(f * g)$ for all $f, g \in L_1(G)$.

(ii) $\Rightarrow$ (iii). Suppose that $Tf * g = T(f * g)$ for all $f, g \in L_1(G)$. Then by the commutativity of $L_1(G)$ with convolution operation, it follows that $Tf*g = T(g*f) = f*Tg$ for all $f, g \in L_1(G)$. Hence for all $f, g \in L_1(G)$, $\widehat{(Tf)}\hat{g} = \hat{f}\widehat{(Tg)}$. For each $\gamma \in \widehat{G}$, choose a $g \in L_1(G)$ such that $\hat{g}(\gamma) \neq 0$ ([**9**], 4.15) and let $\varphi(\gamma) = \widehat{(Tg)}(\gamma)/\hat{g}(\gamma)$. With φ so defined, the equation $\widehat{(Tf)}\hat{g} = \hat{f}\widehat{(Tg)}$ shows that $\widehat{(Tf)} = \varphi\hat{f}$ for each $f \in L_1(G)$. Thus φ is independent of the choice of g. Suppose now that ψ is another function on

$\widehat{G}$ such that $\widehat{(Tf)} = \psi\hat{f}$ for each $f \in L_1(G)$. Then $\varphi\hat{f} = \psi\hat{f}$ for all $f \in L_1(G)$ implies that $\varphi = \psi$.

(iii) $\Rightarrow$ (iv). Now let $\widehat{(Tf)} = \varphi\hat{f}$ for all $f \in L_1(G)$. Then $\varphi\hat{f} \in \widehat{L_1(G)}$ for each $\hat{f} \in \widehat{L_1(G)}$. We set $\|\hat{f}\| = \|f\|_1$ so that $\widehat{L_1(G)}$ becomes a Banach space and $S\hat{f} = \varphi\hat{f}$ defines a linear mapping from $\widehat{L_1(G)}$ to $\widehat{L_1(G)}$. Let (f_n) be a sequence and $f, g \in L_1(G)$ such that $\hat{f}_n \to \hat{f}$ and $\varphi\hat{f}_n \to \hat{g}$ in the norm. Since for each $f \in L_1(G)$, $\|\hat{f}\|_\infty \leq \|f\|_1$ ([**14**], 1.2.4(d)), we have, for each $\gamma \in \widehat{G}$, $\hat{g}(\gamma) = \lim_n \varphi(\gamma)\hat{f}_n(\gamma) = \varphi(\gamma)\hat{f}(\gamma)$. Thus S is a closed mapping. By the closed graph theorem ([**7**], II.2.4), we conclude that S is continuous and hence bounded. That is, we can find a constant $K > 0$ such that $\|\varphi\hat{f}\| = \|S\hat{f}\| \leq K\|\hat{f}\|$ for all $f \in L_1(G)$.

Suppose $\gamma_1, \gamma_2, \ldots, \gamma_n \in \widehat{G}$, $\varepsilon > 0$, and $f \in L_1(G)$ such that $\|\hat{f}\| = \|f\|_1 < 1 + \varepsilon$ and $\hat{f}(\gamma_i) = 1$, $i = 1, 2, \ldots, n$ ([**14**], 2.6.1). Then for all $c_1, c_2, \ldots c_n \in \mathbb{C}$ and $\hat{g} = \varphi\hat{f}$, we have

$$\begin{aligned}
\left|\sum_{i=1}^n c_i\varphi(\gamma_i)\right| &= \left|\sum_{i=1}^n c_i\varphi(\gamma_i)\hat{f}(\gamma_i)\right| = \left|\sum_{i=1}^n c_i\hat{g}(\gamma_i)\right| = \left|\int_G \left[\sum_{i=1}^n c_i\,(-t, \gamma_i)\right] g(t)\, d\lambda(t)\right| \\
&\leq \int_G \left|\sum_{i=1}^n c_i\,(t, -\gamma_i)\right| |g(t)|\, d\lambda(t) \\
&\leq \|g\|_1 \left\|\sum_{i=1}^n c_i\,(\,\cdot\,, -\gamma_i)\right\|_\infty = \|\varphi\hat{f}\| \left\|\sum_{i=1}^n c_i\,(\,\cdot\,, -\gamma_i)\right\|_\infty \\
&\leq K\|\hat{f}\| \left\|\sum_{i=1}^n c_i\,(\,\cdot\,, -\gamma_i)\right\|_\infty \\
&< K(1+\varepsilon) \left\|\sum_{i=1}^n c_i\,(\,\cdot\,, -\gamma_i)\right\|_\infty .
\end{aligned}$$

Since ε is arbitrary, it follows that $|\sum_{i=1}^{n} c_i\varphi(\gamma_i)| \leq K \|\sum_{i=1}^{n} c_i(\cdot, -\gamma_i)\|_\infty$ for any choices of $\gamma_i \in \widehat{G}$ and $c_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$. Note that, for each $f \in L_1(G)$, $S\hat{f}$ and $\hat{f}$ are continuous functions on $\widehat{G}$. And for every open subset U of $\widehat{G}$ with compact closure, there exists an $f \in L_1(G)$ such that $\hat{f}$ is $\mathbf{1}$ on U (see [**14**], 2.6). So, φ is continuous on $\widehat{G}$ and we conclude by the characterization of Fourier-Stieltjes transforms ([**14**], 1.9.1) that there exists a unique $\mu \in M(G)$ such that $\varphi = \hat{\mu}$.

Now suppose (iv) holds. Then $\widehat{(Tf)} = \hat{\mu}\hat{f} = \widehat{(\mu * f)}$ for all $f \in L_1(G)$ and hence $Tf = \mu * f = f * \mu$ for all $f \in L_1(G)$, establishing (v).

Finally, (v) implies (i) since if $Tf = f * \mu$ for all $f \in L_1(G)$, then for each $s \in G$, $(TR_s)f = T(R_s f) = R_s f * \mu = R_s(f * \mu) = R_s(Tf) = (R_s T)f$. □

The foregoing clearly shows that the convolution operator C_μ is an invariant operator,i.e., $C_\mu \in \mathcal{L}_0$. Finally, we need the following result to complete the proof of Theorem 4.2.

LEMMA 4.7. *Let T be a bounded linear on $\mathfrak{C}(\mathbb{T})$ such that*

$$T[\,\mathfrak{C}(\mathbb{T})_-\,] \subseteq \mathfrak{C}(\mathbb{T})_+.$$

Then for each integer n, $\pi_n(T)$ is an operator of the form $M_n C_\mu$ where M_n is multiplication by $e^{in\cdot}$ and C_μ is convolution by a measure μ which is absolutely continuous with respect to the Haar measure.

PROOF. Let n be an integer. Then $\pi_n(T) \in \mathcal{L}_n$ (see Proposition 2.2). By Lemma 1.5 and Theorem 4.6, $\pi_n(T) = M_n C_\mu$ for some $\mu \in M(\mathbb{T})$. From Lemma 4.5,

choosing $E = \mathfrak{C}(\mathbb{T})_-$ and $F = \mathfrak{C}(\mathbb{T})_+$, we have

$$[M_n C_\mu](\mathfrak{C}(\mathbb{T})_-) \subseteq \mathfrak{C}(\mathbb{T})_+. \tag{4.12}$$

Now for each n,

$$C_\mu(e^{im\cdot}) = \mu * e^{im\cdot} = \hat{\mu}(m)e^{im\cdot} \quad \text{so that} \quad M_n C_\mu(e^{im\cdot}) = \hat{\mu}(m)e^{i(m+n)\cdot}.$$

Therefore $\hat{\mu}(m) = 0$ for $m+n < 0$ because of (4.12). Thus μ is absolutely continuous with respect to the Haar measure by Theorem 4.1. □

As a direct consequence of the preceding result, we have the following:

COROLLARY 4.8. *The operator norm closure of the set of finite sums of the form* $\sum_{n=-N}^{N} M_n U_n$, *where* $U_n \in \mathcal{L}_0$, *is the class of almost invariant operators* $\mathcal{L}_*$.

We are now in a position to establish the main result of this section.

PROOF OF THEOREM 4.2. Given $T[\,\mathfrak{C}(\mathbb{T})_-\,] \subseteq \mathfrak{C}(\mathbb{T})_+$. Let n be any integer. By Lemma 4.7, $\pi_n(T) = M_n C_\mu$, where μ is absolutely continuous with respect to the Haar measure. By Lemma 4.4, C_μ is compact. So, $\pi_n(T)$ is compact for each n. Thus T is compact by Corollary 2.10. □

To end this section, we show that the classical F. and M. Riesz Theorem (Theorem 4.1) is a special case of Theorem 4.2. We first show that $C_\mu[\,\mathfrak{C}(\mathbb{T})_-\,] \subseteq \mathfrak{C}(\mathbb{T})_+$ if and only if $\hat{\mu}(n) = 0$ for all $n < 0$.

Suppose $f \in \mathfrak{C}(\mathbb{T})_-$. Then $\hat{f}(n) = 0$ for $n > 0$. Hence if $f \in \mathfrak{C}(\mathbb{T})_-$, then

$$\widehat{C_\mu(f)}(n) = \widehat{\mu * f}(n) = \hat{\mu}(n)\hat{f}(n) = 0 \quad \text{for } n > 0.$$

But $\mu * f \in \mathfrak{C}(\mathbb{T})_+$ implies $\hat{\mu}(n)\hat{f}(n) = 0$ for all $n < 0$. Hence $C_\mu[\,\mathfrak{C}(\mathbb{T})_-\,] \subseteq \mathfrak{C}(\mathbb{T})_+$ if and only if $\hat{\mu}(n) = 0$ for $n < 0$.

Now by Theorem 4.2, if $C_\mu[\,\mathfrak{C}(\mathbb{T})_-\,] \subseteq \mathfrak{C}(\mathbb{T})_+$, then C_μ is compact and so, by Lemma 4.4, μ is absolutely continuous with respect to the Haar measure.

CHAPTER 4

Almost Periodicity in Operator Algebras

In Chapter 3, our harmonic analysis was concentrated on the circle group $\mathbb{T}$, or any compact abelian group. In this chapter, we extend our study to any locally compact abelian group. In addition to the classes of operators in $\mathcal{L}(B)$ mentioned in Chapter 3, we introduce another class of the operators, called almost periodic operators, and investigate its properties. We prove an approximation theorem for these operators. Then we show the existence of an invariant mean for almost periodic operators and discuss some functional properties of the invariant mean. And we conclude the chapter by defining a Fourier series on almost periodic operators and studying its convergence.

In what follows, we present definitions parallel to those in Chapter 3 for the homogeneous Banach space B, but on a locally compact abelian group G.

1. Homogeneous Banach Spaces on G

DEFINITION 1.1. A *homogeneous Banach space* B on a locally compact abelian group G is a Banach space of functions or equivalence classes of functions on G satisfying the following:

(i) $(B, \|\cdot\|_B)$ is translation invariant, i.e, for all $f \in B$ and $x \in G, R_x f \in B$ and $\|R_x f\|_B = \|f\|_B$, where $(R_x f)(y) = f(y - x)$ for $y \in G$;

(ii) functions in B translate continuously, i.e., for all $f \in B$, $\lim_{x\to 0} \|R_x f - f\|_B = 0$;

(iii) B is closed under multiplication by the characters of G.

Examples of Homogeneous Banach spaces on G include the function spaces $L_p(G)$, $1 \leq p < \infty$, and the space $\mathfrak{C}_0(G)$ of continuous functions vanishing at infinity.

2. Almost Periodic Operators

Before we present our definition of almost periodic operators, we define almost periodic functions and give an illustrative theorem to express our idea.

DEFINITION 2.1. An almost periodic function on a topological group G is a bounded continuous complex valued function f whose set of translates $S_f = \{R_x f : x \in G\}$ is relatively compact under the uniform norm.

THEOREM 2.1. *A bounded continuous function f on $\mathbb{R}$ is almost periodic if and only if for every $\varepsilon > 0$ there exists a positive number L such that in every interval of length L, there is a number t such that $\|f - R_t f\| < \varepsilon$.*

PROOF. Let f be a bounded continuous function on $\mathbb{R}$. The theorem can be restated as: f is almost periodic if and only if for every $\varepsilon > 0$, there exists L such that, for every x, there exists $y \in [-L, L]$ such that $\|R_x f - R_y f\| < \varepsilon$.

Suppose f is almost periodic and $\{x_1, x_2, \ldots, x_n\} \in \mathbb{R}$ such that $\{R_{x_i} f\}_{i=1}^n$ is an ε-net in S_f. Let $[-L, L]$ be the smallest interval containing the points $\{x_i\}$. Then for every x, there is an x_i such that $\|R_x f - R_{x_i} f\| < \varepsilon$. And putting $y = x_i$ satisfies the stated condition.

Conversely, suppose for every $\varepsilon > 0$, there exists L such that, for every x, there is a $y \in [-L, L]$ such that $\|R_x f - R_y f\| < \varepsilon/3$. We first show that f is uniformly continuous. Let $\varepsilon > 0$. Then there exists $0 < \delta(\varepsilon) < 1$ such that for all $z_1, z_2 \in [-L-1, L+1]$, $|z_1 - z_2| < \delta$ implies that $|f(z_1) - f(z_2)| < \varepsilon/3$. Then for each a, we can choose $y \in [-L, L]$ such that $\|R_a f - R_y f\| < \varepsilon/3$. Hence, if $|t| < \delta$, then

$$|f(a+t) - f(a)| \leq |f(a+t) - f(y+t)| + |f(y+t) - f(y)| + |f(y) - f(a)| < \varepsilon. \quad (2.1)$$

That is $\|R_t f - f\| < \varepsilon$ whenever $|t| < \delta$. Now let $\{x_1, x_2, \ldots, x_n\}$ be a δ-net in $[-L, L]$. Given x, we choose $y \in [-L, L]$ such that $\|R_x f - R_y f\| < \varepsilon$ and x_i such that $|x_i - y| < \delta$, so that $\|R_{x_i} f - R_y f\| = \|R_{x-y_i} f - f\| < \varepsilon$. Thus $\|R_{x_i} f - R_x f\| \leq \|R_{x_i} f - R_y f\| + \|R_y f - R_x f\| < 2\varepsilon$, which shows that $\{R_{x_i} f\}_{i=1}^n$ is a 2ε-net in S_f and so f is almost periodic. □

In the following, let B be a homogeneous Banach space on a locally compact abelian group G and $\mathcal{L}(B)$ the Banach algebra of all bounded linear operators on B.

DEFINITION 2.2. An operator $T \in \mathcal{L}(B)$ is called *almost periodic* if $\{R_{-x} T R_x : x \in G\}$ is relatively compact in $\mathcal{L}(B)$ in the operator norm topology.

We denote by $\mathcal{L}_A$ the class of almost invariant operators in $\mathcal{L}(B)$. $\mathcal{L}_0$ and $\mathcal{L}_*$ retain their definitions as given in Chapter 3 except that they are now defined on a locally compact abelian group G.

We demonstrate some remarkable properties of $\mathcal{L}_A$ in the following results.

PROPOSITION 2.2. *$\mathcal{L}_A$ is a closed subalgebra of $\mathcal{L}(B)$.*

PROOF. Let $A_T = \overline{\{R_{-x}TR_x : x \in G\}}$ for each $T \in \mathcal{L}_A$. For $T_1, T_2 \in \mathcal{L}_A$, A_{T_1}, A_{T_2} are compact and so the images of $A_{T_1} \times A_{T_2}$ under the continuous mappings $(X, Y) \to X + Y$ and $(X, Y) \to XY$ are compact. Now

$$A_{T_1+T_2} = \overline{\{R_{-x}(T_1 + T_2)R_x\}} = \overline{\{R_{-x}T_1R_x + R_{-x}T_2R_x\}}$$

$$\subseteq \overline{\{R_{-x}T_1R_x\}} + \overline{\{R_{-x}T_2R_x\}} = A_{T_1} + A_{T_2} \quad \text{and}$$

$$A_{T_1T_2} = \overline{\{R_{-x}T_1T_2R_x\}} \subseteq \overline{\{R_{-x}T_1R_x\}}\,\overline{\{R_{-x}T_2R_x\}}.$$

Thus, $A_{T_1+T_2}, A_{T_1T_2}$, and obviously $A_{\alpha T_1} (\alpha \in \mathbb{C})$ are also compact so that $\mathcal{L}_A$ is a subalgebra of $\mathcal{L}(B)$.

To show $\mathcal{L}_A$ is closed in $\mathcal{L}(B)$, let $T_n \in \mathcal{L}_A$ such that $T_n \to T \in \mathcal{L}(B)$. Let $\varepsilon > 0$. Then there is an integer N such that $\|T - T_N\| < \varepsilon/3$. Let $\{R_{-x_i}T_NR_{x_i}\}_{i=1}^m$ be a $\varepsilon/3$-net in $\{R_{-x}T_NR_x : x \in G\}$. For each $x \in G$, there is a k, $1 \leq k \leq m$, such that $\|R_{-x}T_NR_x - R_{-x_k}T_NR_{x_k}\| < \varepsilon/3$. Therefore,

$$\begin{aligned}\|R_{-x}TR_x - R_{-x_k}TR_{x_k}\| &\leq \|R_{-x}TR_x - R_{-x}T_NR_x\| + \|R_{-x}T_NR_x - R_{-x_k}T_NR_{x_k}\| \\ &\quad + \|R_{-x_k}T_NR_{x_k} - R_{-x_k}TR_{x_k}\| \\ &\leq \|T - T_N\| + \|R_{-x}T_NR_x - R_{-x_k}T_NR_{x_k}\| + \|T_N - T\| \\ &< \varepsilon.\end{aligned}$$

Thus $T \in \mathcal{L}_A$ and $\mathcal{L}_A$ is closed. □

In what follows, we prove that every almost periodic operator is almost invariant.

PROPOSITION 2.3. $\mathcal{L}_A \subseteq \mathcal{L}_*$.

PROOF. Let $T \in \mathcal{L}_A$. Then $\{R_{-x}TR_x : x \in G\}$ is totally bounded. Let $\{R_{-x_k}TR_{x_k}\}_{k=1}^N$ be a $\varepsilon/2$-net for $\{R_{-x}TR_x : x \in G\}$. For $k = 1, \cdots, N$, let $A_k = \{x \in G : \|R_{-x}TR_x - R_{-x_k}TR_{x_k}\| \leq \varepsilon/2\}$. We show that A_k is closed.

Let $x_\alpha \in A_k$ and $x_\alpha \to x \in G$. For $f \in B$,

$$\begin{aligned}\|R_{-x_\alpha}TR_{x_\alpha}f - R_{-x}TR_xf\| &= \|R_{-x_\alpha}(TR_{x_\alpha}f - R_{(x_\alpha - x)}TR_xf)\| \\ &= \|TR_{x_\alpha}f - R_{(x_\alpha - x)}TR_xf\| \\ &\leq \|TR_{x_\alpha}f - TR_xf\| + \|TR_xf - R_{(x_\alpha - x)}TR_xf\| \\ &\leq \|T\|\|R_{(x_\alpha - x)}f - f\| + \|TR_xf - R_{(x_\alpha - x)}(TR_xf)\| \longrightarrow 0\end{aligned}$$

by (ii) of Definition 1.1.

For $\delta > 0$ and $\|f\| \leq 1$, choose α such that $\|R_{-x_\alpha}TR_{x_\alpha}f - R_{-x}TR_xf\| < \delta$. Then

$$\begin{aligned}\|R_{-x}TR_xf - R_{-x_k}TR_{x_k}f\| &\leq \|R_{-x}TR_xf - R_{-x_\alpha}TR_{x_\alpha}f\| + \|R_{-x_\alpha}TR_{x_\alpha}f - R_{-x_k}TR_{x_k}f\| \\ &\leq \delta + \varepsilon/2.\end{aligned}$$

Since δ is arbitrary, we have $\|R_{-x}TR_xf - R_{-x_k}TR_{x_k}f\| \leq \varepsilon/2$. Thus $x \in A_k$ and A_k is closed.

Now $G = \bigcup_{k=1}^N A_k$, where each A_k is closed. By Baire category theorem (see[**8**], (5.28)), there exists some A_{k_0}, $x_0 \in A_{k_0}$, and a neighborhood U of 0 in G such that

$x_0 + U \subseteq A_{k_0}$. Therefore, for $x \in U$,

$$\|R_{-x}TR_x - T\| = \|R_{-(x_0+x)}TR_{(x_0+x)} - R_{-x_0}TR_{x_0}\|$$

$$\leq \|R_{-(x_0+x)}TR_{(x_0+x)} - R_{-x_{k_0}}TR_{x_{k_0}}\| + \|R_{-x_{k_0}}TR_{x_{k_0}} - R_{-x_0}TR_{x_0}\| \leq \varepsilon$$

Hence $T \in \mathcal{L}_*$ and $\mathcal{L}_A \subseteq \mathcal{L}_*$. □

In the following, we show that if G is compact, then the almost periodic operators are precisely the almost invariant operators.

COROLLARY 2.4. *If G is compact, then $\mathcal{L}_A = \mathcal{L}_*$.*

PROOF. It suffices to show $\mathcal{L}_* \subseteq \mathcal{L}_A$. Let $T \in \mathcal{L}_*$. Then, by definition, the mapping $x \mapsto R_{-x}TR_x$ is continuous. Therefore, as a continuous image of the compact space G, $\{R_{-x}TR_x : x \in G\}$ is compact. Hence $T \in \mathcal{L}_A$. □

For a locally compact abelian group G, let $UC(G)$ be the space of all bounded uniformly continuous functions f on G (i.e., given $\varepsilon > 0$, there exists a neighborhood U of the unit 0 of G such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in G$ and $x - y \in U$).

LEMMA 2.5. *Let G be a locally compact abelian group and $\varphi \in L_\infty(G)$. Then the following are equivalent:*

(1) $\lim_{t\to 0} \|R_t\varphi - \varphi\|_\infty = 0$.

(2) *φ is equivalent in $L_\infty(G)$ to a function in $UC(G)$.*

PROOF. (2) $\Rightarrow$ (1). It is obvious.

(1) $\Rightarrow$ (2). Suppose $\lim_{t\to 0} \|R_t\varphi - \varphi\|_\infty = 0$. Then, given $\varepsilon > 0$, there exists a neighborhood U of 0 such that $\|R_t\varphi - \varphi\|_\infty < \varepsilon$ whenever $t \in U$.

Let $\{U_i\}_{i\in \mathrm{I}}$ be a basis at $0 \in G$ such that each $\overline{U_i}$ is compact. For every $i \in \mathrm{I}$, let

$$f_i = \frac{1}{\lambda(U_i)}\chi_{U_i},$$

where $\lambda(U_i)$ is the Haar measure of U_i and χ_{U_i} is the characteristic function of U_i. Then

$$(f_i * \varphi)(x) = \int_G f_i(t)\varphi(x-t)\,dt = \frac{1}{\lambda(U_i)}\int_{U_i} \varphi(x-t)\,dt$$

and hence

$$|(f_i*\varphi)(x) - \varphi(x)| = \frac{1}{\lambda(U_i)}\left|\int_{U_i}[\varphi(x-t)-\varphi(x)]\,dt\right| \le \frac{1}{\lambda(U_i)}\int_{U_i}|\varphi(x-t)-\varphi(x)|\,dt.$$

So, if $U_i \subseteq U$, then $|(f_i * \varphi)(x) - \varphi(x)| \le \varepsilon$ for all $x \in G$, i.e., $\|(f_i * \varphi) - \varphi\|_\infty \le \varepsilon$ whenever $U_i \subseteq U$. Thus $\lim_i f_i * \varphi = \varphi$ in the $\|\cdot\|_\infty$ -norm.

Note that $f_i * \varphi \in UC(G)$ (see [8], (20.19)) and $\{f_i * \varphi\}_{i\in \mathrm{I}}$ is Cauchy in $(UC(G), \|\cdot\|_\infty)$. Therefore, there exists a function $g \in UC(G)$ such that $g = \lim_i f_i * \varphi$ in the $\|\cdot\|_\infty$ -norm since $(UC(G), \|\cdot\|_\infty)$ is complete. It follows that $\varphi = g$ in $L_\infty(G)$. □

We denote by M_φ the operator of multiplication by an appropriate function φ.

PROPOSITION 2.6. *Suppose $B = L_p(G)$ $(1 \le p < \infty)$ and $\varphi \in L_\infty(G)$ or $B = \mathfrak{C}_0(G)$ and φ is a bounded continuous function on G. Then*

(1) *M_φ is almost invariant if and only if φ is equivalent in $L_\infty(G)$ to a uniformly continuous function on G;*

(2) *M_φ is almost periodic if and only if φ is equivalent to an almost periodic function.*

PROOF. For $f \in B$ and $x \in G$,

$$(R_x M_\varphi R_{-x})f = R_x M_\varphi (R_{-x} f) = R_x[\varphi \cdot (R_{-x} f)] = R_x \varphi \cdot f = M_{R_x \varphi} f.$$

Thus $R_x M_\varphi R_{-x} = M_{R_x \varphi}$.

(1) follows from the identity

$$\|R_x M_\varphi R_{-x} - M_\varphi\| = \|M_{R_x \varphi} - M_\varphi\| = \|M_{R_x \varphi - \varphi}\| = \|R_x \varphi - \varphi\|_\infty \text{ and Lemma 2.5.}$$

(2) follows from $\|R_x M_\varphi R_{-x} - R_y M_\varphi R_{-y}\| = \|M_{R_x \varphi} - M_{R_y \varphi}\| = \|R_x \varphi - R_y \varphi\|_\infty$. □

PROPOSITION 2.7. *Let G be compact and $B = L_p(G)$, $1 < p < \infty$. Then any compact operator on B is almost periodic.*

PROOF. Since G is compact, it suffices to show that any compact operator is almost invariant. We show that any rank one operator T on B is almost invariant.

Suppose $T \in \mathcal{L}(B)$, $g \in L_q(G)$ and $h \in L_p(G)$ such that $Tf = \langle f, g \rangle h$ for all $f \in L_p(G)$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $\langle f, g \rangle = \int_G f \bar{g}\, dx$. For $\|f\| \leq 1$,

$$\begin{aligned} \|R_x T f - T R_x f\| &= \|\langle f, g \rangle R_x h - \langle R_x f, g \rangle h\| \\ &\leq \|\langle f, g \rangle R_x h - \langle f, g \rangle h\| + \|\langle f, g \rangle h - \langle R_x f, g \rangle h\| \\ &\leq \|\langle f, g \rangle\| \|R_x h - h\| + \|\langle f, g \rangle - \langle f, R_{-x} g \rangle\| \|h\| \\ &\leq \|g\| \|R_x h - h\| + \|g - R_{-x} g\| \|h\|. \end{aligned}$$

So, $\|R_x T f - T R_x f\| \longrightarrow 0$ uniformly for $\|f\| \leq 1$ as $x \to 0$ by (ii) of Definition 1.1. Thus $\lim_{x \to 0} \|R_x T - T R_x\| = 0$ so that T is almost invariant, hence almost periodic.

Therefore every finite rank operator T on B (being linear combination of rank one operators) is almost periodic. Since compact operators are norm limits of finite rank operators and $\mathcal{L}_A$ is closed, compact operators on B are almost periodic. □

3. An Approximation Theorem

In Chapter 3 (Theorem 4.7), we showed that almost invariant operators on a Homogeneous Banach space defined on a compact group G can be approximated by finite sums of the form $\sum_{i=1}^{n} M_{\gamma_i} U_i$, where M_{γ_i} is multiplication by the character γ_i of G and U_i is an invariant operator for each i. In this section, we show that if G is not compact, then the norm closure of these finite sums yield the class of almost periodic operators.

At this point, we demonstrate two important properties of M_γ.

LEMMA 3.1. *Let $x \in G$ and $\gamma \in \widehat{G}$. Then $M_\gamma R_x = (x, \gamma) R_x M_\gamma$.*

PROOF. Let $f \in B$ and $y \in G$. Then

$$\begin{aligned}[M_\gamma R_x f](y) &= (y, \gamma) f(y - x) = (x, \gamma)(y - x, \gamma) f(y - x) \\ &= (x, \gamma)[R_x M_\gamma f](y).\end{aligned} \tag{3.1}$$

□

LEMMA 3.2. *If $V \in \mathcal{L}_0$ and $\gamma \in G$, then $M_\gamma V M_{-\gamma} \in \mathcal{L}_0$.*

PROOF. Let $x \in G$. Then, by Lemma 3.1,

$$R_x M_\gamma V M_{-\gamma} R_{-x} = (-x, \gamma) M_\gamma R_x V R_{-x} M_{-\gamma}(x, \gamma) = M_\gamma V M_{-\gamma} \tag{3.2}$$

so that $M_\gamma V M_{-\gamma} \in \mathcal{L}_0$. □

The following result is useful in the proof of our main theorem of this section.

LEMMA 3.3. *Let (X, ρ) be a compact metric space. Then the set of isometries on (X, ρ) defined by $I(X, \rho) = \{S{:}X \to X | \, \rho(S(x_1), S(x_2)) = \rho(x_1, x_2) \ \forall x_1, x_2 \in X\}$ is also compact in the topology given by the uniform metric*

$$d(S_1, S_2) = \sup_{x \in X} \rho(S_1(x), S_2(x)). \tag{3.3}$$

PROOF. Clearly, d is a metric on $I(X, \rho)$. Since X is a compact metric space, it is separable. So X has a countable dense subset, say $\{x_1, x_2, \cdots\}$.

Let $\{S_n\}$ be a sequence in $I(X, \rho)$. Since X is compact, using the "subsequence of a subsequence" argument and the *Cantor diagonalization* argument, we can get a subsequence $\{S_{n_i}\}_{i=1}^{\infty}$ of $\{S_n\}$ such that $\{S_{n_i}(x_k)\}_{i=1}^{\infty}$ is convergent for all k. Now, let $x \in X$. We claim that $\{S_{n_i}(x)\}_{i=1}^{\infty}$ is also convergent. In fact, given $\varepsilon > 0$, we can choose a k such that $\rho(x_k, x) < \varepsilon/3$. For this k, since $\{S_{n_i}(x_k)\}_{i=1}^{\infty}$ is convergent, there exists an i_0 such that $\rho(S_{n_i}(x_k), S_{n_j}(x_k)) < \varepsilon/3$ whenever $i, j \geq i_0$. It follows that

$$\begin{aligned} \rho(S_{n_i}(x), S_{n_j}(x)) &\leq \rho(S_{n_i}(x), S_{n_i}(x_k)) + \rho(S_{n_i}(x_k), S_{n_j}(x_k)) + \rho(S_{n_j}(x_k), S_{n_j}(x)) \\ &= \rho(x, x_k) + \rho(S_{n_i}(x_k), S_{n_j}(x_k)) + \rho(x_k, x) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Therefore, $\{S_{n_i}(x)\}_{i=1}^{\infty}$ is Cauchy and hence it is convergent since X is compact. Let $S(x) = \lim_{i \to \infty} S_{n_i}(x)$ $(x \in X)$. For all $x_1, x_2 \in X$, we have

$$\rho(S(x_1), S(x_2)) = \lim_{i \to \infty} \rho(S_{n_i}(x_1), S_{n_i}(x_2)) = \rho(x_1, x_2).$$

Therefore, $S{:}X \to X$ is an isometry, i.e., $S \in I(X,\rho)$. Since X is compact and $S_{n_i}(x) \to S(x)$ for all $x \in X$, we have $S_{n_i}(x) \to S(x)$ uniformly for $x \in X$, i.e., $d(S_{n_i}, S) \to 0$ (as $i \to \infty$). Consequently, $I(X,\rho)$ is a compact metric space. □

THEOREM 3.4. *Let G be a locally compact abelian group and B a Homogeneous Banach space on G. If $T \in \mathcal{L}_A$, then T is the norm limit of finite sums of the form $\sum_{i=1}^{n} M_{\gamma_i} U_i$ where $\gamma_i \in \widehat{G}$ and U_i are invariant operators on B.*

PROOF. We divide the proof into three segments: first, we develop the Bohr compactification $\bar{G}$ of G. Secondly, we show that for $\gamma \in \bar{G}$, $\gamma * T = M_\gamma U_\gamma$, where U_γ is invariant. We then conclude the proof by showing that T can be approximated in the norm by a finite linear combination of $\gamma_i * T$.

Let $T \in \mathcal{L}_A$ and $F_T = \overline{\{R_{-x} T R_x : x \in G\}}$. By definition, F_T is a compact metric space with the operator norm. For each $g \in G$, we define the translation operator $\boldsymbol{\rho}_g$ on $\mathcal{L}(B)$ by $\boldsymbol{\rho}_g(S) = R_{-g} S R_g$ and denote by $\boldsymbol{\rho}_g^T$ the restriction of $\boldsymbol{\rho}_g$ to F_T. Then $\boldsymbol{\rho}_g^T$ is an isometry on F_T. Let G_T be the uniform closure of $\{\boldsymbol{\rho}_g^T : g \in G\}$ in $I(F_T, \|\cdot\|_{\mathcal{L}})$. We show that the homomorphism $g \mapsto \boldsymbol{\rho}_g^T$ from G to G_T is continuous. In fact, for all $x, g \in G$,

$$\begin{aligned} \|R_{-x} T R_x - \boldsymbol{\rho}_g^T(R_{-x} T R_x)\| &= \|R_{-x} T R_x - R_{-(x+g)} T R_{(x+g)}\| \\ &= \|R_{-x}[T - R_{-g} T R_g] R_x\| = \|T - R_{-g} T R_g\|. \end{aligned} \tag{3.4}$$

Let $\varepsilon > 0$. Since T is almost invariant, there is a neighborhood V of 0 in G such that $\|T - R_{-g} T R_g\| < \varepsilon$ for $g \in V$. Thus for all $S \in F_T$ and $g \in V$, $\|S - \boldsymbol{\rho}_g^T(S)\| < \varepsilon$, i.e., $d(\boldsymbol{\rho}_0^T, \boldsymbol{\rho}_g^T) \leq \varepsilon$ for all $g \in V$. It follows that the map $g \mapsto \boldsymbol{\rho}_g^T$ is continuous at 0.

Therefore, the map $g \mapsto \boldsymbol{\rho}_g^T$ is continuous from G to G_T since it is a homomorphism. By Lemma 3.3, isometries on a compact metric space form a compact metric space under the uniform metric. Thus G_T is a compact group (with composition of operators as the group operation) and so is the cartesian product $\prod_{T\in\mathcal{L}_A} G_T$. Each $g \in G$ corresponds to a point in $\prod_{T\in\mathcal{L}_A} G_T$ whose Tth co-ordinate is $\boldsymbol{\rho}_g^T \in G_T$. Let $\bar{G}$ be the closure of G in $\prod_{T\in\mathcal{L}_A} G_T$. Since $g \mapsto \boldsymbol{\rho}_g^T$ is continuous from G to G_T for every $T \in \mathcal{L}_A$, the map $g \to (\boldsymbol{\rho}_g^T)_{T\in\mathcal{L}_A}$ from G to $\bar{G}$ is also continuous. In fact, $\bar{G}$ is the Bohr compactification of G (see [**14**], (1.8)).

Let $\gamma \in \widehat{G}$. Note that $\widehat{\bar{G}} = (\widehat{G})_d$, the group $\widehat{G}$ equipped with discrete topology. Then $\gamma \in L_1(\bar{G})$. For $\psi \in L_1(\bar{G})$, define $\psi * T$ by the integral $\int_{\bar{G}} \psi(-g)\boldsymbol{\varrho}_g(T)\,dg$, where $g \mapsto \boldsymbol{\varrho}_g(T)$ $(g \in \bar{G})$ is the extension of the map $g \mapsto R_{-g}TR_g$ $(g \in G)$ and dg is the normalized Haar measure on $\bar{G}$. We show that $\gamma * T = M_\gamma U_\gamma$, where U_γ is an invariant operator. Now,

$$\begin{aligned} R_{-x}(\gamma * T)R_x &= \int_{\bar{G}} R_{-x}\gamma(-g)\boldsymbol{\varrho}_g(T)R_x\,dg = \int_{\bar{G}} \gamma(-g)R_{-(x+g)}TR_{(x+g)}\,dg \\ &= \int_{\bar{G}} (-g,\gamma)\boldsymbol{\varrho}_{x+g}(T)\,dg = (x,\gamma)\int_{\bar{G}} (-x-g,\gamma)\boldsymbol{\varrho}_{x+g}(T)\,dg \qquad (3.5) \\ &= (x,\gamma)(\gamma * T) \quad \text{by the translation invariance of } dg. \end{aligned}$$

Also,

$$\begin{aligned} M_{-\gamma}(\gamma * T)R_x &= M_{-\gamma}R_xR_{-x}(\gamma * T)R_x = M_{-\gamma}R_x(x,\gamma)(\gamma * T) \quad \text{(by (3.5))} \\ &= R_xM_{-\gamma}(\gamma * T) \quad \text{(by (3.1))}. \end{aligned}$$

Hence $U_\gamma = M_{-\gamma}(\gamma * T)$ is invariant and $M_\gamma U_\gamma = \gamma * T$.

Finally, we show that T can be approximated in the norm by finite sums of the form $\sum_{i=1}^{n} a_i(\gamma_i * T)$ where a_i are scalars. Let $\varepsilon > 0$ and choose a symmetric neighborhood V of 0 in $\bar{G}$ such that for $g \in V$, $\|\boldsymbol{\varrho}_g(T) - T\| < \varepsilon/2$. Choose $\psi \in L_1(\bar{G})$ such that $\psi \geq 0$, ψ is supported in V, and $\int_V \psi = 1$. Note that $T = \int_{\bar{G}} \psi(g)T\,dg = \int_{\bar{G}} \psi(-g)T\,dg$ since $\bar{G}$ is compact. Then

$$\|\psi * T - T\| = \left\| \int_{\bar{G}} \psi(-g)(\boldsymbol{\varrho}_g(T) - T)\,dg \right\| \leq \int_{\bar{G}} \psi(-g)\|\boldsymbol{\varrho}_g(T) - T\|\,dg$$
$$\leq \frac{\varepsilon}{2}\int_{\bar{G}} \psi(-g)\,dg \leq \frac{\varepsilon}{2}\int_{\bar{G}} \psi(g)\,dg = \frac{\varepsilon}{2}$$

By the Stone-Weierstrass Theorem and using the fact that $\mathfrak{C}(\bar{G})$ is $\|\cdot\|_{L_1}$-dense in $L_1(\bar{G})$, we can take an $h = \sum_{i=1}^{n} a_i\gamma_i$ such that $\|h - \psi\|_{L_1} < \varepsilon/2(\|T\| + 1)$. Then $\|h * T - \psi * T\| \leq \|h - \psi\|_{L_1}\|T\| < \varepsilon/2$. Thus $\|h * T - T\| < \varepsilon$. Since $h * T = \sum_{i=1}^{n} a_i(\gamma_i * T)$, the proof is complete. □

4. The Invariant Mean

Let $T \in \mathcal{L}_A$. Define $\boldsymbol{\pi}_\gamma : \mathcal{L}_A \to \mathcal{L}_A$ by $\boldsymbol{\pi}_\gamma(T) = \gamma * T = \int_{\bar{G}}(-g, \gamma)\boldsymbol{\rho}_g(T)\,dg$, where dg is the normalized Haar measure on $\bar{G}$, $\gamma \in \widehat{G}$, and $\boldsymbol{\rho}_g$ is as defined in the foregoing proof. It is interesting to note that when $G = \mathbb{T}$ and $\gamma(t) = e^{int}$, $\boldsymbol{\pi}_\gamma(T)$ can be identified with $\boldsymbol{\pi}_n(T)$ defined in §2 of Chapter 3. For $\gamma = 0$, $\boldsymbol{\pi}_0(T) \in \mathcal{L}_0$ since

$$R_{-x}\boldsymbol{\pi}_0(T)R_x = \int_{\bar{G}} R_{-x}(-g, 0)\boldsymbol{\rho}_g(T)R_x\,dg = \int_{\bar{G}} \boldsymbol{\rho}_{x+g}(T)\,dg = \int_{\bar{G}} \boldsymbol{\rho}_g(T)\,dg = \boldsymbol{\pi}_0(T)$$

for all $x \in G$. We call the map $\boldsymbol{\pi}_0(T) : \mathcal{L}_A \to \mathcal{L}_0$ the *invariant mean* of the operator T. In what follows, we obtain $\boldsymbol{\pi}_0(T)$ by a standard limiting process. To do this, we

present the following three lemmas, the first one being a special case of Lemma 18.12 of [8].

LEMMA 4.1. *Let G be a locally compact abelian group with Haar measure λ. Let V be a neighborhood of 0 with compact closure in G, and $\varepsilon > 0$. Then there is a relatively compact open subset H of G such that $V \subseteq H$ and $\frac{\lambda[(H+V)\cap H^c]}{\lambda(H)} < \varepsilon$, where H^c is the complement of H in G.*

PROOF. See page 254 of [8]. □

LEMMA 4.2. *Let G be a non-compact locally compact abelian group with Haar measure λ. There are relatively compact neighborhoods H_α of 0 in G such that*

(i) $\bigcup_{\alpha\in \mathrm{I}} H_\alpha = G$;

(ii) *the index set* I *can be ordered so that*

$$\lim_{\alpha} \frac{\lambda[(H_\alpha + x) \cap H_\alpha^c]}{\lambda(H_\alpha)} = 0 \quad \textit{for every } x \in G.$$

PROOF. (i) Let $\{V_\alpha : \alpha \in \mathrm{I}\}$ be the set of all relatively compact open neighborhoods of 0 in G. Then $\bigcup_{\alpha\in \mathrm{I}} V_\alpha = G$ (Indeed, for any open neighborhood U of 0 in G, $(U + x) \cup U$ is an open neighborhood of 0 and of x.). Make I into a directed set by setting $\alpha \geq \beta$ if and only if $V_\alpha \supseteq V_\beta$. By Lemma 4.1, for every $\alpha \in \mathrm{I}$, there is a relatively compact open subset $H_\alpha \supseteq V_\alpha$ such that

$$\frac{\lambda[(H_\alpha + V_\alpha) \cap H_\alpha^c]}{\lambda(H_\alpha)} < \frac{1}{\lambda(V_\alpha)}.$$

Since $\bigcup_{\alpha\in G} V_\alpha = G$, $\bigcup_{\alpha\in G} H_\alpha = G$ is also true.

(ii) Let $x \in G$. Choose $\beta \in \mathrm{I}$ such that $x \in V_\beta$. Then for all $\alpha \geq \beta$, $x \in V_\alpha \subseteq H_\alpha$ so that

$$\frac{\lambda[(H_\alpha + x) \cap H_\alpha^c]}{\lambda(H_\alpha)} \leq \frac{\lambda[(H_\alpha + V_\alpha) \cap H_\alpha^c]}{\lambda(H_\alpha)} < \frac{1}{\lambda(V_\alpha)}. \tag{4.1}$$

By the regularity of λ, for every $N \in \mathbb{N}$, there is a compact subset K of G such that $\lambda(K) > N$. By the compactness of K, there exist $\alpha_1, \ldots, \alpha_n \in \mathrm{I}$ with $\bigcup_{i=1}^n V_{\alpha_i} \supseteq K$. Choose $\alpha \in \mathrm{I}$ such that $\alpha \geq \alpha_i$ for $i = 1, \ldots, n$. Then for every $\kappa \geq \alpha$, $V_\kappa \supseteq V_\alpha \supseteq \bigcup_{i=1}^n V_{\alpha_i} \supseteq K$, so, $\lambda(V_\kappa) \geq \lambda(K) > N$. Therefore, $\lambda(V_\alpha) \to \infty$ and so $\lim_\alpha \frac{\lambda[(H_\alpha + x) \cap H_\alpha^c]}{\lambda(H_\alpha)} = 0$. □

LEMMA 4.3. *Let G be a locally compact abelian group with Haar measure λ. Let $\{H_\alpha\}_{\alpha \in \mathrm{I}}$ be as in Lemma 4.2. Then*

$$\lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} (x, \gamma)\, dx = \begin{cases} 1 & \text{if } \gamma = 0 \\ 0 & \text{if } \gamma \neq 0 \end{cases}. \tag{4.2}$$

PROOF. For $\gamma = 0$, we have

$$\lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} 1\, dx = \lim_\alpha \frac{1}{\lambda(H_\alpha)} \cdot \lambda(H_\alpha) = 1.$$

For $\gamma \neq 0$, choose $x_0 \neq 0$ so that $(x_0, \gamma) \neq 1$. Then

$$\begin{aligned}
&\left|\frac{1}{\lambda(H_\alpha)}\int_{H_\alpha}[(x+x_0,\gamma)-(x,\gamma)]\,dx\right| = \frac{1}{\lambda(H_\alpha)}\left|\int_{H_\alpha+x_0}(x,\gamma)\,dx - \int_{H_\alpha}(x,\gamma)\,dx\right| \\
&= \frac{1}{\lambda(H_\alpha)}\left|\int_{(H_\alpha+x_0)\cap H_\alpha^c}(x,\gamma)\,dx - \int_{(H_\alpha+x_0)^c\cap H_\alpha}(x,\gamma)\,dx\right| \\
&\leq \frac{1}{\lambda(H_\alpha)}\int_{(H_\alpha+x_0)\cap H_\alpha^c}|(x,\gamma)|\,dx + \frac{1}{\lambda(H_\alpha)}\int_{(H_\alpha+x_0)^c\cap H_\alpha}|(x,\gamma)|\,dx \\
&= \frac{1}{\lambda(H_\alpha)}\int_{(H_\alpha+x_0)\cap H_\alpha^c}1\,dx + \frac{1}{\lambda(H_\alpha)}\int_{(H_\alpha+x_0)^c\cap H_\alpha}1\,dx \\
&= \frac{1}{\lambda(H_\alpha)}\Big(\lambda[(H_\alpha+x_0)\cap H_\alpha^c] + \lambda[(H_\alpha+x_0)^c\cap H_\alpha]\Big). \qquad (4.3)
\end{aligned}$$

Now, $\lambda[(H_\alpha+x_0)^c\cap H_\alpha] = \lambda[[(H_\alpha+x_0)^c\cap H_\alpha]-x_0] = \lambda[H_\alpha^c\cap(H_\alpha-x_0)]$ so that (4.3) becomes

$$\frac{1}{\lambda(H_\alpha)}\Big(\lambda[(H_\alpha+x_0)\cap H_\alpha^c] + \lambda[H_\alpha^c\cap(H_\alpha-x_0)]\Big) \to 0$$

by Lemma 4.2. Hence $\lim_\alpha \frac{1}{\lambda(H_\alpha)}\int_{H_\alpha}(x,\gamma)\,dx[(x_0,\gamma)-1] = 0$. Since $(x_0,\gamma)\neq 1$, the result follows. □

For easy application in further results, we present the limit form of the invariant mean in the following.

PROPOSITION 4.4. *Let* $\Phi(\alpha,T) = \frac{1}{\lambda(H_\alpha)}\int_{H_\alpha} R_{-x}TR_x\,dx$ *for every* $T\in\mathcal{L}_A$ *and* $\alpha\in \mathrm{I}$. *Then* $\lim_\alpha \Phi(\alpha,T)$ *exists and is equal to* $\pi_0(T) = \int_{\tilde{G}}\rho_g(T)\,dg$.

PROOF. For $U \in \mathcal{L}_0$, $\gamma \in \widehat{G}$, let $T = M_\gamma U$. Then

$$\begin{aligned}\lim_\alpha \Phi(\alpha, T) &= \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} R_{-x}(M_\gamma U)R_x \, dx = \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} (R_{-x}M_\gamma R_x)U \, dx \\ &= \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} (x,\gamma) \, dx \, M_\gamma U \quad \text{(by Lemma 3.1)} \\ &= \begin{cases} U & \text{if } \gamma = 0 \\ 0 & \text{if } \gamma \neq 0 \end{cases} \quad \text{(by Lemma 4.2).}\end{aligned} \tag{4.4}$$

Thus $\lim_\alpha \Phi(\alpha, T)$ exists for $T = M_\gamma U$ and so $\lim_\alpha \Phi(\alpha, P)$ exists for every finite sum of the form $P = \sum_{i=1}^n M_{\gamma_i} U_i$. Let $T \in \mathcal{L}_A$ and $\varepsilon > 0$. Then by Theorem 3.4 we can find a corresponding finite sum P_ε such that $\|T - P_\varepsilon\| < \varepsilon/3$. Since $\lim_\alpha \Phi(\alpha, P_\varepsilon)$ exists, there exists $\alpha_0 \in \mathrm{I}$ such that if $\alpha, \beta \geq \alpha_0$, then $\|\Phi(\alpha, P_\varepsilon) - \Phi(\beta, P_\varepsilon)\| \leq \varepsilon/3$. Note also that $\|\Phi(\alpha, W)\| \leq \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} \|R_{-x}WR_x\| \, dx = \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} \|W\| \, dx = \|W\|$ for all $W \in \mathcal{L}_A$. Hence

$$\begin{aligned}\|\Phi(\alpha, T) - \Phi(\beta, T)\| &\leq \|\Phi(\alpha, T - P_\varepsilon)\| + \|\Phi(\alpha, P_\varepsilon) - \Phi(\beta, P_\varepsilon)\| + \|\Phi(\beta, T - P_\varepsilon)\| \\ &\leq \|T - P_\varepsilon\| + \|\Phi(\alpha, P_\varepsilon) - \Phi(\beta, P_\varepsilon)\| + \|T - P_\varepsilon\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \, .\end{aligned}$$

Thus $\{\Phi(\alpha, T)\}$ is a Cauchy net and converges since $\mathcal{L}_A$ is a complete normed algebra. So $\lim_\alpha \Phi(\alpha, T)$ exists for all $T \in \mathcal{L}_A$.

We now show that $\lim_\alpha \Phi(\alpha, T) = \boldsymbol{\pi}_0(T)$ for every $T \in \mathcal{L}_A$. For $x \in G$, $\boldsymbol{\rho}_x(M_\gamma U) = R_{-x}M_\gamma UR_x = R_{-x}M_\gamma R_x U = (x, \gamma)M_\gamma U$ by Lemma 3.1. Consequently, since G is

dense in $\bar{G}$ and $g \mapsto \boldsymbol{\rho}_g$ is continuous, $\boldsymbol{\rho}_g(M_\gamma U) = (g,\gamma)M_\gamma U$ for all $g \in \bar{G}$. Therefore,

$$\pi_0(M_\gamma U) = \int_{\bar{G}} \boldsymbol{\rho}_g(M_\gamma U)\, dg = \int_{\bar{G}} (g,\gamma)\, dg M_\gamma U = \begin{cases} U & \text{if } \gamma = 0 \\ 0 & \text{if } \gamma \neq 0 \end{cases} . \tag{4.5}$$

By the previous paragraph, $\pi_0(T) = \lim_\alpha \Phi(\alpha, T)$ for $T = M_\gamma U$. Hence, it holds for every T of the form $T = \sum_{i=1}^n M_{\gamma_i} U_i$, and therefore for every $T \in \mathcal{L}_A$ by Theorem 3.4.

□

PROPOSITION 4.5. *Let $B = L_2(G)$. Then $\mathcal{L}_0, \mathcal{L}_A$ and $\mathcal{L}_*$ are $*$-subalgebras of $\mathcal{L}(B)$.*

PROOF. We only have to show that $\mathcal{L}_0, \mathcal{L}_A$ and $\mathcal{L}_*$ are closed under the involution $*$. Let $x \in G$ and $f, g \in L_2(G)$. Then

$$\langle R_x f, g\rangle = \int_G f(t-x)\overline{g(t)}\, dt = \int_G f(t)\overline{g(t+x)}\, dt = \int_G f(t)\overline{(R_{-x}g)}(t)\, dt = \langle f, R_{-x}g\rangle$$

Thus $\langle f, (R_x)^* g\rangle = \langle R_x f, g\rangle = \langle f, R_{-x}g\rangle$. So, $(R_x)^* = R_{-x}$ for all $x \in G$.

If $T \in \mathcal{L}_0$, then $T^* R_x = T^*(R_{-x})^* = (R_{-x}T)^* = (TR_{-x})^* = R_x T^*$ for all $x \in G$, i.e., $T^* \in \mathcal{L}_0$.

Note that for all $T \in \mathcal{L}(B)$, $\|T^* R_x - R_x T^*\| = \|(R_{-x}T)^* - (TR_{-x})^*\| = \|(R_{-x}T - TR_{-x})^*\| = \|R_{-x}T - TR_{-x}\|$. Therefore, if $T \in \mathcal{L}_*$, then $T^* \in \mathcal{L}_*$.

Also note that if $\mathfrak{A} \subseteq \mathcal{L}(B)$ is relatively compact, since the involution $* : \mathcal{L}(B) \to \mathcal{L}(B)$ is an isometry and hence continuous, then $\mathfrak{A}^* = \{T^* | T \in \mathfrak{A}\} \subseteq \mathcal{L}(B)$ is also relatively compact. It follows that $\mathcal{L}_A$ is closed under the involution $*$. □

Before proceeding to the next result, we give some key definitions as follows:

DEFINITION 4.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be $*$-algebras of bounded linear operators on some Hilbert spaces and let $\omega : \mathfrak{A} \to \mathfrak{B}$ be linear a map. ω is said to be *positive* if $\omega(A^*A) \geq 0$ for all $A \in \mathfrak{A}$; ω is said to be *faithful* if for all $A \in \mathfrak{A}$, $\omega(A^*A) = 0$ implies that $A = 0$. If ω is a positive linear functional on $\mathfrak{A}$ with norm 1, then ω is called a *state* on $\mathfrak{A}$. If $\mathfrak{A}$ has identity $\mathbb{I}$, then ω is a *state* if it is a positive linear functional such that $\omega(\mathbb{I}) = 1$.

Before we prove the next result, we collect a few generalities. Let Y be a Banach space. Let $CB(G,Y) = \{F{:}G \to Y | F \text{ is bounded and continuous}\}$. For $F \in CB(G,Y)$, let $\|F\| = \sup_{x \in G} \|F(x)\|$. Then under the pointwise operations and the norm $\|\cdot\|$, $CB(G,Y)$ is a Banach space.

For $F \in CB(G,Y)$ and $x \in G$, let $(R_xF)(g) = F(g-x)$ $(g \in G)$. Then $R_xF \in CB(G,Y)$ and $\|R_xF\| = \|F\|$. A function $F \in CB(G,Y)$ is called *almost periodic* if the orbit $O(F) = \{R_xF | x \in G\}$ of F is relatively compact in $(CB(G,Y), \|\cdot\|)$. Let $AP(G,Y)$ be the set of all almost periodic functions in $CB(G,Y)$. Then $AP(G,Y)$ is a closed linear subspace of $CB(G,Y)$.

If Y is a C^*-algebra, for $F_1, F_2 \in CB(G,Y)$, let $(F_1F_2)(x) = F_1(x)F_2(x)$ and $F_1^*(x) = F_1(x)^*$ $(x \in G)$. Then $(CB(G,Y), \|\cdot\|)$ is a C^*-algebra and $AP(G,Y)$ is a C^*-subalgebra of $CB(G,Y)$.

PROPOSITION 4.6. *Let $B = L_2(G)$. Then $\pi_0 : \mathcal{L}_A \to \mathcal{L}_0$ is a positive faithful $*$-map.*

PROOF. Recall that an operator A on a Hilbert space H is positive if $\langle Af, f\rangle \geq 0$ for all $f \in H$. For $T \in \mathcal{L}_A$ and $f \in L_2(G)$, we have

$$\langle \pi_0(T)f, f\rangle = \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} \langle R_{-x}TR_x f, f\rangle \, dx = \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} \langle TR_x f, R_x f\rangle \, dx \, .$$

Thus if $T \geq 0$, then $\pi_0(T) \geq 0$ and so π_0 is positive. That π_0 is a $*$-map follows from

$$\begin{aligned}\langle \pi_0(T)f, g\rangle &= \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} \langle R_{-x}TR_x f, g\rangle \, dx = \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} \langle f, R_{-x}T^*R_x g\rangle \, dx \\ &= \langle f, \pi_0(T^*)g\rangle \, .\end{aligned}$$

To prove π_0 is faithful, we follow a method used by Arveson in [**3**] to prove faithfulness. To apply apply this method, we need a C^*-algebra $\mathcal{C}$, a positive faithful linear map $\omega : \mathcal{C} \to \mathcal{L}_0$ and a $*$-homomorphism π of $\mathcal{C}$ onto $\mathcal{L}_A$ such that $\pi_0 \circ \pi = \omega$. Indeed, suppose $T \in \mathcal{L}_A$ such that $\pi_0(T^*T) = 0$ and $S \in \mathcal{C}$ such that $\pi(S) = T$. Then $\omega(S^*S) = (\pi_0 \circ \pi)(S^*S) = \pi_0(\pi(S^*)\pi(S)) = \pi_0(T^*T) = 0$ because π is a $*$-homomorphism. And since ω is faithful, $S = 0$, so that $T = \pi(S) = 0$. Now we prove the existence of such C^*-algebra $\mathcal{C}$.

Let $\mathcal{C}_1 = AP(G, \mathcal{L}(L_2(G)))$ with the norm $\|F\| = \sup_{x\in G} \|F(x)\|$ for $F \in \mathcal{C}_1$. Since $\mathcal{L}(L_2(G))$ is a C^*-algebra, then $\mathcal{C}_1$ is also a C^*-algebra. It is obvious that π, defined on $\mathcal{C}_1$ by $\pi(F) = F(0)$, is a $*$-homomorphism of $\mathcal{C}_1$ into $\mathcal{L}(L_2(G))$. Now define $\omega(F) = \int_{\bar{G}} F(x)\, dx$ $(F \in \mathcal{C}_1)$, i.e., $\omega(F)$ is the operator in $\mathcal{L}(L_2(G))$ satisfying

$$\langle \omega(F)f, g\rangle = \int_{\bar{G}} \langle F(x)f, g\rangle \, dx \quad \text{for all } f, g \in L_2(G).$$

Note here that, for all $f, g \in L_2(G)$, $\langle F(\cdot)f, g\rangle$ is an almost periodic function on G and hence it can be extended uniquely to a continuous function on $\bar{G}$ which is still

denoted as $\langle F(\cdot)f, g\rangle$. For every $f \in L_2(G)$, the fact that

$$\langle \omega(F^*F)f, f\rangle = \int_{\bar{G}} \langle F^*(x)F(x)f, f\rangle\, dx = \int_{\bar{G}} \langle F(x)f, F(x)f\rangle\, dx = \int_{\bar{G}} \|F(x)f\|^2\, dx$$

shows that ω is a positive linear map of $\mathcal{C}_1$ into $\mathcal{L}(L_2(G))$. And if $\omega(F^*F) = 0$, then $F(x)f = 0$ for all $f \in L_2(G)$ and $x \in G$. That is, $F = 0$ and so ω is faithful.

Now let $\mathcal{C}$ be the norm closure of all functions $F \in \mathcal{C}_1$ of the form $F(x) = \sum_{\gamma\in\widehat{G}}(x, \gamma)M_\gamma U_\gamma$, where $\gamma \in \widehat{G}$ and $U_\gamma \in \mathcal{L}_0$. We claim that $\mathcal{C}$ is a C^*-subalgebra of $\mathcal{C}_1$ containing the identity. Clearly, $\mathcal{C}$ is closed under linear operations and the involution $*$. For multiplication we have $M_{\gamma_1}U_1M_{\gamma_2}U_2 = M_{\gamma_1+\gamma_2}U$, where $U = M_{-\gamma_2}U_1M_{\gamma_2}U_2$. By Lemma 3.2, $M_{-\gamma_2}U_1M_{\gamma_2}U_2 \in \mathcal{L}_0$ and thus $U \in \mathcal{L}_0$. Thus $\mathcal{C}$ is a C^*-algebra. Now if $F(x) = \sum_{\gamma\in\widehat{G}}(x, \gamma)M_\gamma U_\gamma$, with finitely many non-zero γ's. Then, by (4.5),

$$\omega(F) = \int_{\bar{G}} F(x)\, dx = \sum_{\gamma\in\widehat{G}} \int_{\bar{G}} (x, \gamma)M_\gamma U_\gamma\, dx = U_0$$

and

$$\pi_0 \circ \pi(F) = \pi_0(F(0)) = \int_{\bar{G}} R_{-g}F(0)R_g\, dg = \sum_{\gamma\in\widehat{G}} \int_{\bar{G}} R_{-g}M_\gamma U_\gamma R_g\, dg$$

$$= \sum_{\gamma\in\widehat{G}} \int_{\bar{G}} (x, \gamma)M_\gamma U_\gamma\, dg = U_0,$$

i.e., $\omega(F) = U_0 = \pi_0 \circ \pi(F)$. By continuity, $\omega = \pi_0 \circ \pi$ on $\mathcal{C}$. We also see that $\omega(\mathcal{C}) \subseteq \mathcal{L}_0$ since π_0 maps into $\mathcal{L}_0$. Since $\pi : \mathcal{C} \to \mathcal{L}_A$ is a $*$-homomorphism of the C^*-algebra $\mathcal{C}$ into the C^*-algebra $\mathcal{L}_A$, $\pi(\mathcal{C})$ is closed in $\mathcal{L}_A$ and contains all finite sums of the form $\sum_{i=1}^n M_{\gamma_i}U_i$, $U_i \in \mathcal{L}_0$. Hence $\pi(\mathcal{C}) = \mathcal{L}_A$ by Theorem 3.4. □

To conclude this section, we present another property of the invariant mean that is required to prove the convergence result in the next section.

PROPOSITION 4.7. (i) $\pi_0 : \mathcal{L}_A \to \mathcal{L}_0$ *is idempotent, i.e.,* $\pi_0 \circ \pi_0 = \pi_0$*;*

(ii) *Let* $B = L_2(G)$. *Then for all* $T \in \mathcal{L}_A$, $\pi_0(T)^*\pi_0(T) \leq \pi_0(T^*T)$.

PROOF.

(i) By definition, $\pi_0(T) \in \mathcal{L}_0$ for every $T \in \mathcal{L}_A$. Also if $V \in \mathcal{L}_0$, then

$$\pi_0(V) = \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} R_{-x} V R_x \, dx = \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} V \, dx = V.$$

Hence $\pi_0(\pi_0(T)) = \pi_0(T)$ for all $T \in \mathcal{L}_A$.

(ii) For $T \in \mathcal{L}_A$ and $U \in \mathcal{L}_0$,

$$\begin{aligned} \pi_0(UT) &= \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} R_{-x} U T R_x \, dx = \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} U R_{-x} T R_x \, dx \\ &= U \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} R_{-x} T R_x \, dx = U \pi_0(T) \end{aligned} \tag{4.6}$$

and, in the same way, $\pi_0(TU) = \pi_0(T)U$. Since π_0 is positive, then for all $T \in \mathcal{L}_A$, $\pi_0[(T - \pi_0(T))^*(T - \pi_0(T))] \geq 0$. That is,

$$\pi_0(T^*T) - \pi_0(T^*\pi_0(T)) - \pi_0(\pi_0(T)^*T) + \pi_0(\pi_0(T)^*\pi_0(T)) \geq 0.$$

Since $\pi_0(T) \in \mathcal{L}_0$ for all $T \in \mathcal{L}_A$, and π_0 is an idempotent $*$-map, we have

$$\begin{aligned} &\pi_0(T^*T) - \pi_0(T^*\pi_0(T)) - \pi_0(\pi_0(T)^*T) + \pi_0(\pi_0(T)^*\pi_0(T)) \\ &= \pi_0(T^*T) - \pi_0(T^*)\pi_0(T) - \pi_0(T)^*\pi_0(T) + \pi_0(T)^*\pi_0(T) \\ &= \pi_0(T^*T) - \pi_0(T)^*\pi_0(T) \geq 0. \end{aligned}$$

Hence $\pi_0(T)^*\pi_0(T) \leq \pi_0(T^*T)$. □

5. Fourier Series on $\mathcal{L}_A$

In a similar manner as in Chapter 3, we define the Fourier series of an operator $T \in \mathcal{L}_A$ as the series $\sum_{\gamma \in \widehat{G}} \pi_\gamma(T)$. Throughout this section, we assume that $L_2(G)$ is separable. We now study the convergence of $\sum_{\gamma \in \widehat{G}} \pi_\gamma(T)$ on $L_2(G)$ in the norm constructed by Arveson [**2**]. To define this norm, we need a faithful state, say ρ, on $\mathcal{L}_A$ which preserves the invariant mean in the sense that $\rho \circ \pi_0 = \rho$. Since $L_2(G)$ is a separable Hilbert space, it admits a countable orthonormal basis, say $\{\xi_n\}$.

Define φ on $\mathcal{L}_0$ by $\varphi(T) = \sum_{n=1}^{\infty} \frac{1}{2^n} \langle T\xi_n, \xi_n \rangle$, $T \in \mathcal{L}_0$. Then φ is positive since $\varphi(T^*T) = \sum_{n=1}^{\infty} \frac{1}{2^n} \langle T\xi_n, T\xi_n \rangle = \sum_{n=1}^{\infty} \frac{1}{2^n} \|T\xi_n\|^2 \geq 0$; φ is faithful since if $\varphi(T^*T) = 0$, then $T\xi_n = 0$ for all n and hence $T = 0$; and φ is a state because $\varphi(\mathbb{I}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \langle \xi_n, \xi_n \rangle = 1$. Let $\rho = \varphi \circ \pi_0$. Then ρ is a faithful state on $\mathcal{L}_A$ since φ is a faithful state on $\mathcal{L}_0$ and $\pi_0 : \mathcal{L}_A \to \mathcal{L}_0$ is a positive faithful map with $\pi_0(\mathbb{I}) = \mathbb{I}$. We therefore make $\mathcal{L}_A$ an inner product space with the inner product defined by $\langle T, S \rangle_\rho = \rho(S^*T)$ for $T, S \in \mathcal{L}_A$ and denote the induced norm on $\mathcal{L}_A$ by $\| \cdot \|_\rho$. We now present the convergence result as follows:

PROPOSITION 5.1. *The Fourier series $\sum_{\gamma \in \widehat{G}} \pi_\gamma(T)$ of any $T \in \mathcal{L}_A$ converges to T in the $\| \cdot \|_\rho$-norm.*

PROOF. Let $\mathcal{H}$ be the Hilbert space completion of $\mathcal{L}_A$ with respect to the the inner product $\langle \cdot, \cdot \rangle_\rho$. For each $\gamma \in \widehat{G}$, let $\mathcal{H}_\gamma$ be the $\| \cdot \|_\rho$-closure of the subspace $\{M_\gamma U : U \in \mathcal{L}_0\}$. We first show that if $\gamma_1 \neq \gamma_2$, then $\mathcal{H}_{\gamma_1}$ and $\mathcal{H}_{\gamma_2}$ are orthogonal.

Suppose $U, V \in \mathcal{L}_0$ and $\gamma_1, \gamma_2 \in \widehat{G}$ such that $\gamma_1 \neq \gamma_2$. Then

$$\langle M_{\gamma_1} U, M_{\gamma_2} V \rangle_\rho = \rho(V^* M_{-\gamma_2} M_{\gamma_1} U) = \rho(V^* M_{\gamma_1 - \gamma_2} U) = \rho(V^* U_0 M_{\gamma_1 - \gamma_2}),$$

where $U_0 = M_{\gamma_1 - \gamma_2} U M_{\gamma_2 - \gamma_1} \in \mathcal{L}_0$ by Lemma 3.2. Now, using the fact that ρ preserves $\boldsymbol{\pi}_0$ and by applying (4.6), we have

$$\rho(V^* U_0 M_{\gamma_1 - \gamma_2}) = \rho \circ \boldsymbol{\pi}_0(V^* U_0 M_{\gamma_1 - \gamma_2}) = \rho(V^* U_0 \boldsymbol{\pi}_0(M_{\gamma_1 - \gamma_2})) = 0$$

since $\boldsymbol{\pi}_0(M_\gamma) = 0$ if $\gamma \neq 0$ by (4.5). Thus if $\gamma_1 \neq \gamma_2$, $\mathcal{H}_{\gamma_1}$ and $\mathcal{H}_{\gamma_2}$ are orthogonal. By Theorem 3.4, the linear space generated by $M_\gamma U$ is operator norm dense in $\mathcal{L}_A$. And for $T \in \mathcal{L}_A$, $\|T\|_\rho^2 = \langle T, T \rangle_\rho = \rho(T^* T) \leq \|\rho\| \|T^* T\| = \|T\|^2$. So the subspace generated by the $M_\gamma U$'s is $\|\cdot\|_\rho$-dense in $\mathcal{L}_A$. Thus the $\mathcal{H}_\gamma$'s span a dense subspace of $\mathcal{H}$. Hence, $\mathcal{H}$ can be decomposed into orthogonal subspaces $\mathcal{H}_\gamma$'s. For every $\gamma \in \widehat{G}$, let $P_\gamma : \mathcal{H} \to \mathcal{H}_\gamma$ be the orthogonal projection. We show that for $T \in \mathcal{L}_A$, $P_\gamma(T)$ is precisely $\boldsymbol{\pi}_\gamma(T)$. For this purpose, we make the following claims.

Claim 1. If $T \in \mathcal{L}_A$, then $\boldsymbol{\pi}_\gamma(T) = \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} (-x, \gamma) R_{-x} T R_x \, dx$:

Suppose $T = M_{\gamma_0} U$ for a fixed $\gamma_0 \in \widehat{G}$ and $U \in \mathcal{L}_0$. Then using the definition of $\boldsymbol{\pi}_\gamma$ given at the beginning of §4, along with the invariance of U and (4.5), we have

$$\begin{aligned}
\boldsymbol{\pi}_\gamma(T) &= \int_{\bar{G}} (-g, \gamma) \boldsymbol{\rho}_g(M_{\gamma_0} U) \, dg = \int_{\bar{G}} (-g, \gamma) R_{-x}(M_{\gamma_0} U) R_x \, dg \\
&= \int_{\bar{G}} (-g, \gamma)(g, \gamma_0) M_{\gamma_0} U \, dg = \left(\int_{\bar{G}} (g, \gamma_0 - \gamma) \, dg \right) M_{\gamma_0} U \\
&= \begin{cases} M_{\gamma_0} U & \text{if } \gamma = \gamma_0 \\ 0 & \text{if } \gamma \neq \gamma_0 \end{cases}
\end{aligned}$$

$$= \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} (x, \gamma_0 - \gamma)\, dx\, M_{\gamma_0} U \quad \text{(by (4.4))}$$

$$= \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} (-x, \gamma)[(x, \gamma_0) M_{\gamma_0} U]\, dx$$

$$= \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} (-x, \gamma)[R_{-x}(M_{\gamma_0} U) R_x]\, dx \quad \text{(by Lemma 3.1)}$$

$$= \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} (-x, \gamma) R_{-x} T R_x\, dx\,.$$

So the formula holds for $T = \sum_{i=1}^n M_{\gamma_i} U_i$ and hence for all $T \in \mathcal{L}_A$.

Claim 2. If $T \in \mathcal{L}_A$, then $\pi_\gamma(T) = M_\gamma \pi_0(M_{-\gamma} T)$:

By Claim 1 we have

$$\begin{aligned} \pi_\gamma(T) &= \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} (-x, \gamma) R_{-x} T R_x\, dx \\ &= M_\gamma \left(\lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} (-x, \gamma) M_{-\gamma} R_{-x} T R_x\, dx \right) \\ &= M_\gamma \left(\lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} R_{-x}(M_{-\gamma} T) R_x\, dx \right) \quad \text{(by Lemma 3.1)} \\ &= M_\gamma \pi_0(M_{-\gamma} T). \end{aligned} \tag{5.1}$$

Claim 3. π_γ is an idempotent linear transformation of $\mathcal{L}_A$ onto $\{M_\gamma U |\ U \in \mathcal{L}_0\}$:

Let $T \in \mathcal{L}_A$. Then by applying Claim 2, we have

$$(\pi_\gamma \circ \pi_\gamma)(T) = \pi_\gamma(M_\gamma \pi_0(M_{-\gamma} T)) = M_\gamma \pi_0(M_{-\gamma} M_\gamma \pi_0(M_{-\gamma} T))$$

$$= M_\gamma \pi_0(\pi_0(M_{-\gamma} T)) = M_\gamma \pi_0(M_{-\gamma} T) = \pi_\gamma(T)$$

by the idempotence of π_0. Hence π_γ is idempotent.

Continuing the proof, for $T \in \mathcal{L}_A$, we have

$$\|\pi_\gamma(T)\|_\rho^2 = \rho([\pi_\gamma(T)]^* \pi_\gamma(T)) = \rho([\pi_0(M_{-\gamma} T)]^* M_{-\gamma} M_\gamma \pi_0(M_{-\gamma} T)) \quad \text{(by Claim 2)}$$

$$= \rho([\pi_0(M_{-\gamma}T)]^*\pi_0(M_{-\gamma}T)) \leq \rho[\pi_0(T^*M_\gamma M_{-\gamma}T)] \quad \text{(by Proposition 4.7(ii))}$$

$$= \rho[\pi_0(T^*T)] = \rho(T^*T) = \|T\|_\rho^2 ,$$

which shows that $\|\pi_\gamma\|_\rho \leq 1$. Hence, by continuity, there is a unique extension $\overline{\pi}_\gamma$ of π_γ of $\mathcal{H}$ onto $\mathcal{H}_\gamma$ which is also idempotent. So, $\overline{\pi}_\gamma = P_\gamma$.

Now, for every $S \in \mathcal{H}$, the finite sum $\sum_{\gamma\in\widehat{G}} P_\gamma(S)$ converges to S in the $\|\cdot\|_\rho$-norm. In particular, for $T \in \mathcal{L}_A$, $\sum_{\gamma\in\widehat{G}} \pi_\gamma(T)$ converges to T in the $\|\cdot\|_\rho$-norm. □

For $T \in \mathcal{L}_A$, $\gamma \in \widehat{G}$, we define $\widehat{T}(\gamma)$ to be the operator $\pi_\gamma(T)$ and call it the Fourier transform of the operator T. Note that $\widehat{T}$ is bounded since for every $\gamma \in \widehat{G}$,

$$\|\widehat{T}(\gamma)\| \leq \limsup_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} \|(-x,\gamma)R_{-x}TR_x\|\, dx = \limsup_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} \|T\|\, dx = \|T\|.$$

In the following, we prove that the Fourier transform takes operator multiplication into convolution. Note that this is an analogue of Proposition 3.1 in Chapter 3.

PROPOSITION 5.2. *Let $S, T \in \mathcal{L}_A$ and $\gamma_0 \in \widehat{G}$. Then the series $\sum_{\gamma\in\widehat{G}} \widehat{S}(\gamma_0 - \gamma)\widehat{T}(\gamma)$ converges to $\widehat{ST}(\gamma_0)$ in the $\|\cdot\|_\rho$-norm.*

PROOF. We have

$$\begin{aligned}
\widehat{ST}(\gamma_0) = \pi_{\gamma_0}(ST) &= \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} (-x,\gamma_0)R_{-x}STR_x\, dx \\
&= \lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} (-x,\gamma_0)R_{-x}S\left(\sum_{\gamma\in\widehat{G}} \pi_\gamma(T)\right)R_x\, dx \qquad (5.2) \\
&= \sum_{\gamma\in\widehat{G}} \left(\lim_\alpha \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} (-x,\gamma_0)[R_{-x}SR_xR_{-x}\pi_\gamma(T)R_x]\, dx\right).
\end{aligned}$$

By Claim 2 of Proposition 5.1 and Lemma 3.1,

$$\begin{aligned} R_{-x}\boldsymbol{\pi}_\gamma(T)R_x &= R_{-x}(M_\gamma \boldsymbol{\pi}_0(M_{-\gamma}T)R_x = R_{-x}M_\gamma R_x(\boldsymbol{\pi}_0(M_{-\gamma}T)) \\ &= (x,\gamma)M_\gamma\boldsymbol{\pi}_0(M_{-\gamma}T) = (x,\gamma)\boldsymbol{\pi}_\gamma(T) \end{aligned} \tag{5.3}$$

so that (5.2) becomes

$$\begin{aligned} &\sum_{\gamma\in\widehat{G}}\left(\lim_\alpha \frac{1}{\lambda(H_\alpha)}\int_{H_\alpha}(-x,\gamma_0)R_{-x}SR_x(x,\gamma)\boldsymbol{\pi}_\gamma(T)\,dx\right) \\ &= \sum_{\gamma\in\widehat{G}}\left(\lim_\alpha \frac{1}{\lambda(H_\alpha)}\int_{H_\alpha}(-x,\gamma_0-\gamma)R_{-x}SR_x\,dx\right)\boldsymbol{\pi}_\gamma(T) \\ &= \sum_{\gamma\in\widehat{G}}\boldsymbol{\pi}_{\gamma_0-\gamma}(S)\boldsymbol{\pi}_\gamma(T). \end{aligned} \tag{5.4}$$

Hence the proof is complete. □

Let G be a locally compact abelian group and H_α's as in Lemma 4.2. Let $T \in \mathcal{L}_A$ and μ a finite Borel measure on the H_α's. For each $f \in B$, we define the convolution $\mu * T$ by the integral

$$\lim_\alpha \int_{H_\alpha} R_x T R_{-x}\, d\mu(t). \tag{5.5}$$

Note that $\mu * T$ is also almost periodic since R_xTR_{-x} is in $\mathcal{L}_A$ and $\mathcal{L}_A$ is operator norm closed in $\mathcal{L}(B)$. Just as in §3 of Chapter 3, the Fourier transform defined takes convolution $\mu * T$ into $\hat{\mu}\cdot\hat{T}$. Indeed, Proposition 3.4 of Chapter 3 holds for $T \in \mathcal{L}_A$ with $\mathbb{T}$ replaced by G, using our definition of convolution in (5.5).

Bibliography

[1] W. B. Arveson, *An Invitation to C^*-Algebra*, Springer-Verlag, New York (1976).

[2] W. B. Arveson, *Analyticity in Operator Algebras*, American J. of Math. **89** (1967), 578-642.

[3] W. B. Arveson, *Operator Algebras and Measure Preserving Automorhisms*, Acta Math. **118** (1967), 95-109.

[4] Bruckner, Bruckner, Thomson, *Real Analysis*, Prentice-Hall, (1997).

[5] J. Conway, *A Course In Functional Analysis*, Springer-Verlag, New York (1990).

[6] K. De Leeuw, *An Harmonic Analysis for Operators I*, Illinois J. of Math. **19** (1967), 593-606.

[7] N. Dunford and J. T. Schwartz, *Linear Operators; Part 1: General Theory*, Interscience, New York (1958).

[8] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Springer-Verlag, New York (1963).

[9] E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, American Mathematical Society (1957).

[10] Z. Hu, *Introduction to Harmonic Analysis*, University of Windsor Lecture Notes (2001).

[11] Y. Katznelson, *An Introduction to Harmonic Analysis*, Wiley, New York (1976).

[12] R. Larson, *An Introduction to the Theory of Multipliers*, Springer-Verlag, New York (1971).

[13] L.H. Loomis, *An introduction to Abstract Harmonic Analyis*, Van Nostrand, New York, (1953).

[14] W. Rudin, *Fourier Analysis on Groups*, Wiley (1962).

[15] U.B. Tewari and S. Somasundaram, *Almost Periodicity in Operator Algebras*, Rend. Istit. Mat. Univ. Trieste **24** (1992), no.1-2, 39-54 (1994).

[16] T. Traynor, *Lecture Notes*, University of Windsor (2002).

[17] S. Willard, *General Topology*, Addison-Wesley (1970).

Vita Auctoris

Mr. Olusakin Joshua Emmanuel was born on 24 April 1976, in Lagos, Nigeria. Following his passion for solving problems, he decided to pursue a degree in Mathematics at the University of Ilorin, Nigeria, where he earned his First Class Honors Bachelor of Science in Mathematics in February 1999. After graduation, he taught high school Mathematics, worked with a bank, and later with an audit firm.

In September 2001, Mr. Emmanuel moved to Canada to begin graduate study in Mathematics at the University of Windsor, Ontario. In the course of his study there, he was privileged to serve as a graduate teaching assistant in the Mathematics Department and as a Research Assistant with Dr. Zhiguo Hu, who was also his supervisor. He graduated with a Master of Science degree in Mathematics in Winter 2003. Mr. Emmanuel plans to pursue a Ph.D. in Management Science/Operations Research.