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BOUNDARY VALUE PROBLEMS IN CONTINUUM MECHANICS.

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BOUNDARY VALUE PROBLEMS
IN
CONTINUUM MECHANICS

by

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A Dissertation
Submitted to the Faculty of Graduate
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Dedicated with love to my wife

and to our children

ABSTRACT

This dissertation deals with some boundary value problems of continuum mechanics. The problems considered, in the main, deal with micropolar (or Cosserat) fluids subjected to creeping (Stokes') motions. Analytic solutions are developed for all of the problems considered.

The equations of motion for the Stokes' flows of micropolar fluids are uncoupled in a manner which is slightly different from that used by other authors.

In order to facilitate the study of the various problems considered, an alternate form for the general solutions of the creeping motion equations of a viscous fluid was developed based on the use of cartesian tensors. The accuracy and utility of these solutions are indicated through the use of a number of well known examples. The solutions are also extended to include micropolar fluid flows.

The solutions for the uniform creeping flow of a fluid past a solid sphere are determined for the cases where the fluid is either viscous or micropolar. The same problem is also solved for the fluid sphere case using the four possible combinations of viscous and Cosserat fluids.

To make comparison with known results direct and easy, the solutions for all of the above problems are generated by means of the stream function technique. The solutions for some of these problems by means of the method mentioned previously are also discussed, but in less detail.

For all of the problems mentioned above, the drag is determined and, where applicable, a comparison with the classical results is made with a subsequent discussion of the significance of some of the parameters involved. Additionally, a number of tables are also included which show how, for various streamlines, the angles are affected by changes in the various parameters.

Lastly, a variety of linear shear flow problems are solved using the alternate method mentioned previously. As before, the fluid flow is considered to be a Stokes' flow past a spherical body. The sphere under consideration was either solid or fluid, and the fluids considered were either viscous or micropolar. For all of these problems, the effective viscosity of the external fluid is calculated and, where applicable, comparisons are made.

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CHAPTER I

INTRODUCTION

Section 1. Background.

The equations of motion used to describe the motion of a viscous Newtonian fluid were first given by Navier (1822) and later by Stokes (1851). If the fluid motion is very slow (the Reynolds number is small)

then the creeping motion, or Stokes, equations are used. For motions where inertial forces are not negligible a perturbation scheme has been developed whereby Stokes type solutions are valid near an object immersed in the fluid, and Oseen type solutions are needed for flow far from the body. M. VanDyke (1975) gives an excellent discussion of perturbation theory.

A general solution to the creeping motion equations was put forward by Lamb (1911). Because of their great generality, these solutions have become the most popular method for dealing with such motions. However, the stream function technique, given by Lagrange (1781) and Stokes (1843), is widely used for two-dimensional problems and for those three-dimensional problems that exhibit some type of symmetry. More recently, the singularity method, first used by Lorentz (1897), has been revitalized by Chwang & Wu (1974,1975) as another method for solving these problems. For micropolar fluids the stream functions technique and the singularity method have been extended by Ramkissoon & Majumdar (1976,1976).

The creeping motion equations as described above were designed to consider the flow of a single phase fluid. In a way these are an idealization because particles are normally found suspended in almost every fluid. This generated an interest in what is now commonly called suspension rheology. The first major works in this area were done by Einstein (1906,1911), Hadamard (1911) and Rybczynski (1911), and a very great interest in the study of these types of fluids has persisted since that time, as seen in the review articles by Brenner (1972) and Leal (1980).

The original investigations of creeping flow problems mentioned above have dealt with viscous Newtonian fluids. It has been found, however, that not all fluids are Newtonian in nature. A number of theories have been put forward to describe various types of non-Newtonian fluids. The theory of microfluids was introduced by Eringen (1964) and later Eringen (1966) developed the theory of micropolar fluids. Micropolar fluids, a subclass of microfluids, are fluids that exhibit microinertial effects and can support stresses and body couples. Cowin (1968) has discussed polar fluid theory very completely, and expressed criticism of certain boundary conditions that have been used by some authors. Publications dealing with micropolar fluids are having the same genesis as those which have dealt with viscous Newtonian fluids, but at a much faster rate.

Section 2. Outline of present work.

Much work has been done on the low Reynolds number flow of both viscous Newtonian and micropolar fluids. Three basic methods of approach are used to determine the solutions to problems arising in these areas. However, each method suffers from various deficiencies. Lamb's general solutions though applicable to the widest range of problems are cumbersome and sometimes difficult to use because of their extreme generality, especially for considerations dealing with micropolar fluids. The stream function technique, though relatively easy to use is useful only for strictly two-dimensional problems or for those three-dimensional problems which display some form of symmetry. The singularity method is also easy to apply but requires much guesswork to determine the appropriate singularities necessary to solve any particular problem.

In this work, a new form of general solution, having its foundations in tensors, will be developed. This solution will be given in sufficient generality to solve a great number of problems taken from both viscous Newtonian and micropolar fluid theories.

In the subsequent section the basic equations for a micropolar fluid, following Eringen (1966), will be given and reworked to a form more suited to this work along with constraints on the constants appearing in the equations. The creeping motion equations are reworked so that the velocity and microrotation vectors are uncoupled.

In chapter II section 2, the new form for the general solution to the creeping motion equations of a viscous Newtonian fluid will be developed. Section 3 will give the solution for the uniform flow of a viscous fluid past a stationary fluid sphere by using both the stream function technique and the new form as developed in section 2. Also, a number of singular flow problems and the standard problems dealing with flow past a solid spherical body for viscous Newtonian fluids will be solved using this new form. Section 4 develops the solutions to the creeping flow equations for micropolar fluids by extending the results in section 2.

Chapter III discusses the problem of a uniform fluid flow past a fluid sphere. It is one of those rare problems which can be handled by both the stream function technique and the present method. Hence, the solution is given by both methods. However, in order to compare results directly and easily, without going over a lot of old ground, with the work of previous authors (viscous flow past a solid sphere Stokes (1851), viscous flow past a viscous sphere Hadamard (1911) & Rybczynski (1911), micropolar flow past a solid sphere Ramkissoon & Majumdar (1976)) the stream function technique is discussed in greater detail. All possible combinations of viscous Newtonian and micropolar fluids are considered.

Chapter IV deals with a variety of linear shear flow problems, using the technique developed in chapter II. Some of the problems considered include the flow of a micropolar fluid past a solid sphere, the flow of a micropolar fluid past a viscous Newtonian fluid sphere and the flow of a micropolar fluid past a micropolar fluid sphere. The expression for the effective viscosity is obtained in most cases.

Section 3. Equations governing micropolar fluids.

The equations which need to be considered when discussing a micropolar fluid flow problem are:

(i) the constitutive equations

(ii) the restrictions on the viscosity coefficients

(iii) the fluid equations.

Eringen (1966) developed the theory and set forth the general equations which are reproduced here.

(i) Constitutive Equations. The constitutive equations relating the stress tensor and the couple stress tensor to the rate of deformation tensor and the microrotation vector were given by Eringen (1966) as

$$t_{k\ell} = (-p+\lambda u_{r,r})\delta_{k\ell} + \mu(u_{k,\ell} + u_{\ell,k}) + \kappa(u_{\ell,k} - \varepsilon_{k\ell r}\nu_r) \quad (1.1)$$

$$m_{k\ell} = \alpha\, \nu_{r,r}\delta_{k\ell} + \beta\, \nu_{k,\ell} + \gamma\, \nu_{\ell,k} \quad (1.2)$$

where $\delta_{k\ell}$ is the identity tensor, $\varepsilon_{k\ell r}$ is the alternating tensor, u_k is the k-th component of the velocity vector, ν_k is the k-th component of the microrotation vector, p is the thermodynamic pressure, and $\alpha,\beta,\gamma,\kappa,\lambda,\mu$ are constant viscosity coefficients.

(ii) Restrictions on Viscosity Coefficients. If the Clausius-Duhem inequality is satisfied locally for all independent processes, then Eringen (1966) determined that the viscosity coefficients listed above must satisfy the inequalities

$$(3\lambda + 2\mu + \kappa) \geq 0 \ , \ 2\mu + \kappa \geq 0 \ , \ \kappa \geq 0$$
$$(3\alpha + \beta + \gamma) \geq 0 \ , \ -\gamma \leq \beta \leq \gamma \ , \ \gamma \geq 0 \qquad (1.3)$$

(iii) Field Equations. The general field equations, neglecting thermal effects, for an incompressible micropolar fluid with isotropic microstructure were given by Eringen (1966) as

$$\nabla . \underset{\sim}{u} = 0 \qquad (1.4)$$

$$\rho \frac{D}{Dt} \underset{\sim}{u} = -\nabla p + \rho \underset{\sim}{F} + (\mu+\kappa) \nabla^2 \underset{\sim}{u} + \kappa(\nabla x \underset{\sim}{\nu}) \qquad (1.5)$$

$$\rho j \frac{D}{Dt} \underset{\sim}{\nu} = -2\kappa \underset{\sim}{\nu} + \rho \underset{\sim}{L} + (\alpha+\beta) \nabla(\nabla . \underset{\sim}{\nu}) + \gamma \nabla^2 \underset{\sim}{\nu} + \kappa(\nabla x \underset{\sim}{u}) \qquad (1.6)$$

where (i) $\underset{\sim}{u}$ velocity vector

(ii) $\underset{\sim}{\nu}$ microrotation vector

(iii) p thermodynamic pressure

(iv) ρ density

(v) j micro-inertia

(vi) $\underset{\sim}{F}$ body force per unit mass

(vii) $\underset{\sim}{L}$ body couple per unit mass

(viii) $\alpha, \beta, \gamma, \kappa, \mu$ viscosity coefficients

In the event that the motion is steady, that inertial effects are negligible, and that there are no body forces or body couples then these equations reduce to the following

$$\nabla . \underset{\sim}{u} = 0 \qquad (1.4)$$

$$\nabla p = (\mu+\kappa) \nabla^2 \underset{\sim}{u} + \kappa(\nabla x \underset{\sim}{\nu}) \qquad (1.7)$$

$$\kappa(\nabla\times \underset{\sim}{u}) = 2\kappa\underset{\sim}{\nu} - (\alpha+\beta)\ \nabla(\nabla.\underset{\sim}{\nu}) - \gamma\nabla^2\underset{\sim}{\nu} \tag{1.8}$$

From taking the divergence of (1.7) comes

$$\nabla^2 p = 0 \tag{1.9}$$

Taking the curl of (1.8) and simplifying with (1.4) yields

$$-\tfrac{1}{2}\nabla^2\underset{\sim}{u} = (1 - \frac{\gamma}{2\kappa}\nabla^2)\ (\nabla\times\underset{\sim}{\nu}) \tag{1.10}$$

Using now the operator $(1 - \frac{\gamma}{2\kappa}\nabla^2)$ on (1.7) and using (1.9) and (1.10) in the resultant gives

$$\nabla p = (\mu+\kappa)\ (1 - \frac{\gamma}{2\kappa}\nabla^2)\ \nabla^2\underset{\sim}{u} - \frac{\kappa}{2}\nabla^2\underset{\sim}{u}$$

Or in simplified form

$$(\nabla^2 - L^2)\ \nabla^2\underset{\sim}{u} = -\frac{2\kappa}{\gamma(\mu+\kappa)}\nabla p \tag{1.11}$$

$$L^2 = \frac{\kappa(2\mu+\kappa)}{\gamma(\mu+\kappa)} \tag{1.12}$$

The curl of (1.7) and the divergence of (1.8) give

$$\frac{(\mu+\kappa)}{\kappa}\nabla^2(\nabla\times\underset{\sim}{u}) = \nabla^2\underset{\sim}{\nu} - \nabla(\nabla.\underset{\sim}{\nu}) \tag{1.13}$$

$$2\kappa\ (\nabla.\underset{\sim}{\nu}) - (\alpha+\beta+\gamma)\ \nabla^2\ (\nabla.\underset{\sim}{\nu}) = 0 \tag{1.14}$$

or alternately with $\delta = \alpha+\beta+\gamma$

$$(\nabla^2 - \frac{2\kappa}{\delta})\ (\nabla.\underset{\sim}{\nu}) = 0 \tag{1.15}$$

Defining $\underset{\sim}{\omega} = \nabla\times\underset{\sim}{u}$ and using the operator $(\nabla^2 - \frac{2\kappa}{\delta})$ on (1.13) produces

$$(\nabla^2 - \frac{2\kappa}{\delta})\ \nabla^2\ [\underset{\sim}{\nu} - \frac{\mu+\kappa}{\kappa}\underset{\sim}{\omega}] = 0$$

Hence, it is seen that it is possible to take

$$\underset{\sim}{\nu} = \frac{\mu+\kappa}{\kappa}\underset{\sim}{\omega} + \underset{\sim}{H} \tag{1.16}$$

$$\underset{\sim}{H} = \nabla\phi + \nabla\times\underset{\sim}{B} \tag{1.17}$$

provided that the following is satisfied

$$(\nabla^2 - \frac{2\kappa}{\delta}) \nabla^2 \underset{\sim}{H} = 0 \tag{1.18}$$

By taking the divergence of (1.16) using (1.17) and (1.15) the following devolves

$$(\nabla^2 - \frac{2\kappa}{\delta}) \nabla^2 \phi = 0 \tag{1.19}$$

Substitution of (1.16) in (1.8) and simplification of the resultant gives

$$\frac{\gamma(\mu+\kappa)}{2\kappa 2} (\nabla^2 - L^2) \underset{\sim}{\omega} = - \frac{\gamma}{2\kappa} (\nabla^2 - \frac{2\kappa}{\gamma})(\nabla \times \underset{\sim}{B}) - \frac{\delta}{2\kappa}(\nabla^2 - \frac{2\kappa}{\delta})\nabla\phi \tag{1.20}$$

Combining the Laplacian of (1.20) and the curl of (1.11)

$$- \frac{\gamma}{2\kappa}(\nabla^2 - \frac{2\kappa}{\gamma})\nabla^2 (\nabla \times \underset{\sim}{B}) - \frac{\delta}{2\kappa}(\nabla^2 - \frac{2\kappa}{\delta}) \nabla^2 (\nabla\phi) = 0$$

which in conjunction with (1.19) indicates that

$$(\nabla^2 - \frac{2\kappa}{\gamma}) \nabla^2 (\nabla \times \underset{\sim}{B}) = \underset{\sim}{0}$$

From substitution of (1.17) in (1.18) and using (1.19)

$$(\nabla^2 - \frac{2\kappa}{\delta}) \nabla^2 (\nabla \times \underset{\sim}{B}) = \underset{\sim}{0}$$

The net result from these last two expressions is that

$$\nabla^2 (\nabla \times \underset{\sim}{B}) = \underset{\sim}{0}$$

Using this in (1.8) along with (1.16) yields, finally, the desired result

$$\underset{\sim}{\nu} = \tfrac{1}{2} [1 + \frac{\gamma(\mu+\kappa)}{\kappa^2} \nabla^2] \underset{\sim}{\omega} + \frac{\delta}{2\kappa} \nabla\Psi \tag{1.21}$$

where $\Psi = \nabla^2 \phi$ must satisfy the relation

$$\nabla^2 \Psi = \frac{2\kappa}{\delta} \Psi \tag{1.22}$$

Hence, the final form for the equations to be solved is

$$\nabla^2 p = 0 \tag{1.9}$$

$$(\nabla^2 - L^2)\,\nabla^2 \underset{\sim}{u} = -\frac{2\kappa}{\gamma(\mu+\kappa)}\,\nabla p \tag{1.11}$$

$$\left(\nabla^2 - \frac{2\kappa}{\delta}\right)\Psi = 0 \tag{1.22}$$

subject to the requirements

$$\nabla . \underset{\sim}{u} = 0 \tag{1.4}$$

$$\underset{\sim}{\nu} = \left[1 + \frac{\gamma(\mu+\kappa)}{\kappa^2}\,\nabla^2\right]\frac{(\nabla \times \underset{\sim}{u})}{2} + \frac{\delta}{2\kappa}\,\nabla\Psi \tag{1.21}$$

where the coefficients L, δ are defined as

$$L^2 = \frac{\kappa(2\mu+\kappa)}{\gamma(\mu+\kappa)} \tag{1.12}$$

$$\delta = (\alpha+\beta+\gamma) \tag{1.14}$$

Equations (1.21) and (1.22) are worthy of special mention because they give the actual form for the micro-rotation vector when dealing with the creeping motion of a micropolar fluid. Various special forms of the above results have been used by such authors as Avudainayagam (1976), Ramkissoon & Majumdar (1976), Rao & Rao (1971) and Cowin (1968).

Also, when $\kappa \to 0$ the equations (1.1) - (1.6) reduce to the equations governing the creeping flow of an incompressible, viscous, Newtonian fluid.

CHAPTER II

TENSOR SOLUTIONS AND SOME BOUNDARY VALUE PROBLEMS FOR VISCOUS NEWTONIAN FLUIDS.

Section 1. Introductory Comments.

When solving problems involving the Stokes, or creeping motion, equations, use is normally made of either the solutions derived by Lamb (1932) or the singularity method as described by Chwang & Wu (1974, 1975) or the stream function technique as described by Happel & Brenner (1965). Most fluid flow problems commonly encountered can be solved by at least one of these methods. In many instances, however, the boundary conditions are presented in a fashion that is not readily translated to a form that is directly applicable to any of these methods. In the literature the boundary conditions are often given in a cartesian tensor form which cannot always be easily put into a form that is usable for any of the forms mentioned above.

The objective in this chapter then is to develop, partially, another form for the general solution based on the use of cartesian tensors. The solution thus generated will be directly applicable to problems for which the boundary conditions are given in cartesian tensor form and to those that can easily be written in this manner. This form for the general solution has previously been used in a highly specialized form by Peery (1966) and Schowalter (1978). The solution presented here will generalize and extend the results obtained by these two authors.

In this chapter, use will be made of arbitrary, spatially constant, second and third order tensors to generate a partial general solution. It is found that the solution so obtained is sufficient enough to solve most of the boundary value problems normally encountered in the literature.

In section 2, the general solution will be developed starting with the pressure equation and thence proceeding to the equations of motion. Finally, the restrictions that arise out of the continuity equation are imposed on the solution.

Section 3 will consider the problem of the slow uniform flow of an incompressible viscous Newtonian fluid past a stationary Newtonian fluid sphere. The solution will be given in two forms, by using the stream function technique and by employing the form of the general solution developed in section 2. Both methods are given so that the results from the new form of the general solution can be compared to a known solution. The advantages for this new form will also be pointed out. Additionally in the following subsections a number of examples will be detailed to further illustrate the applicability of this method. The three standard examples of flow past a solid sphere will be detailed in part B. In part C a number of singular flows will be considered.

Section 4 extends the results for viscous Newtonian fluids to micropolar fluids, the main thrust of this present work. Examples of the application of these results will not, however, be included in this section but will be considered in the following chapters.

Section 5 will entail a discussion of the solutions generated in section 2 and a consideration of possible applications to other types of flow problems will be pointed out. Further, the advantages of this form for the general solution over each of the other three main types of solution will be mentioned.

Section 2. Development of the solution.

From chapter I, section 3 the equations to be solved for the slow steady flow of an incompressible, inertialess, viscous Newtonian fluid are

$$\nabla^2 \underset{\sim}{u} = \nabla p \qquad (2.1)$$

$$\nabla . \underset{\sim}{u} = 0 \qquad (2.2)$$

where $p = \mu^{-1} p^1$ and p^1 contains any conservative extraneous body forces. It is well known that the solutions for the above system are equivalent to finding the solutions of

$$\nabla^2 p = 0 \qquad (2.3)$$

$$\nabla^2 \underset{\sim}{u} = \nabla p \qquad (2.1)$$

subject to the constraint that $\underset{\sim}{u}$ once determined must be restricted so that the continuity condition (2.2) is satisfied.

In order to generate solutions which will involve cartesian tensors it is necessary to first of all, determine the scalar invariants involving these cartesian tensors in combination with the position vector $\underset{\sim}{r}$. For any spatially constant second and third order tensors a_{ij}, A_{ijk} respectively, the scalar invariants linear in a_{ij}, A_{ijk} are

$$\begin{aligned} &(i) \quad a_{ii} \\ &(ii) \quad \varepsilon_{ijk} a_{kj} X_i \\ &(iii) \quad a_{ij} X_i X_j \\ &(iv) \quad A_{imm} X_i \end{aligned} \qquad (2.4)$$

(v) $A_{mim}X_i$

(vi) $A_{mmi}X_i$ (2.4)

(viii $A_{ijk}X_iX_jX_k$

Hence, a general form for $p(\underset{\sim}{r})$ is assumed to be

$$p(\underset{\sim}{r}) = H^0(r)a_{ii} + H^1(r)\varepsilon_{ijk}a_{kj}X_i + H^2(r)A_{imm}X_i + H^3(r)A_{mim}X_i + H^4(r)A_{mmi}X_i + H^5(r)a_{ij}X_iX_j + H^6(r)A_{ijk}X_iX_jX_k \quad (2.5)$$

where $r = (X_iX_i)^{\frac{1}{2}}$, $i \in \{1,2,\ldots,n\}$ (2.6)

Since a_{ij}, A_{ijk} are general tensors, in order to satisfy (2.3), the functions $H^p(r)$ must satisfy equations of the form

$$\frac{d^2}{dr^2} H^p(r) + \frac{n+(2m-1)}{r} \frac{d}{dr} H^p(r) = G(r) \quad (2.7)$$

where n is the dimension of the Euclidean space with $n \leq 3$ and m is the order of the coefficient tensor and G(r) is either known or zero. For the actual specific forms for ∇p, $\nabla^2 p$, $U_{\ell,m}$, $U_{\ell,\ell}$, $U_{\ell,mm}$ see the appendix. Solutions will be given for n=3 throughout the main body of this section and at the end, mention will be made of the special changes necessary for the case n=2. Thus for the function $p(\underset{\sim}{r})$, it is found that

(i) $H^0(r) = -\frac{1}{n} \overset{5}{A_1} r^{-n} + \overset{0}{A_1} r^{-(n-2)} + \overset{0}{A_2} - \frac{1}{n} \overset{5}{A_2} r^2$

(ii) $H^1(r) = \overset{1}{A_1} r^{-n} + \overset{1}{A_2}$

(iii) $H^2(r) = -\frac{1}{n+2} \overset{6}{A_1} r^{-(n+2)} + \overset{2}{A_1} r^{-n} + \overset{2}{A_2} - \frac{1}{n+2} \overset{6}{A_2} r^2$

(iv) $H^3(r) = -\frac{1}{n+2} \overset{6}{A_1} r^{-(n+2)} + \overset{3}{A_1} r^{-n} + \overset{3}{A_2} - \frac{1}{n+2} \overset{6}{A_2} r^2$

$$\text{(v)}\ H^{4}(r) = -\frac{1}{n+2}\overset{6}{A_1}r^{-(n+2)} + \overset{4}{A_1}r^{-n} + \overset{4}{A_2} - \frac{1}{n+2}\overset{6}{A_2}r^2$$

$$\text{(vi)}\ H^{5}(r) = \overset{5}{A_1}r^{-(n+2)} + \overset{5}{A_2}$$

$$\text{(vii)}\ H^{6}(r) = \overset{6}{A_1}r^{-(n+4)} + \overset{6}{A_2} \qquad (2.8)$$

From equations (2.1) and (2.5) the general form for the velocity component in the direction X_p is given by

$$\begin{aligned} u_p(\underset{\sim}{r}) = {} & \overset{0}{h_1}(r)a_{ii}X_p + \overset{1}{h_1}(r)\varepsilon_{ijk}a_{kj}X_iX_p + \overset{1}{h_2}(r)\varepsilon_{pjk}a_{kj} \\ & + \overset{2}{h_1}(r)A_{imm}X_iX_p + \overset{2}{h_2}(r)A_{pmm} + \overset{3}{h_1}(r)A_{mim}X_iX_p \\ & + \overset{3}{h_2}(r)A_{mpm} + \overset{4}{h_1}(r)A_{mmi}X_iX_p + \overset{4}{h_2}(r)A_{mmp} \\ & + \overset{5}{h_1}(r)a_{ij}X_iX_jX_p + \overset{5}{h_2}(r)a_{pj}X_j + \overset{5}{h_3}(r)a_{jp}X_j \\ & + \overset{6}{h_1}(r)A_{ijk}X_iX_jX_kX_p + \overset{6}{h_2}(r)A_{pjk}X_jX_k \\ & + \overset{6}{h_3}(r)A_{jpk}X_jX_k + \overset{6}{h_4}(r)A_{jkp}X_jX_k \end{aligned} \qquad (2.9)$$

Substitution of (2.9) and (2.5) in (2.1) requires that the functions $h_q^p(r)$ satisfy the following differential equations

$$\text{(i)}\ \frac{d^2}{dr^2}h_1^p(r) + \frac{n+(2m+1)}{r}\frac{d}{dr}h_1^p(r) = g_1(r)$$

where $p \in \{0,1,\ldots,6\}$ (2.10)

$$\text{(ii)}\ \frac{d^2}{dr^2}h_q^p(r) + \frac{n+(2m-3)}{r}\frac{d}{dr}h_q^p(r) = g_2(r)$$

where $q \neq 1$, $p \in \{0,1,\ldots,6\}$

with n,m as before and $g_1(r)$, $g_2(r)$ are either known functions or zero. As before, see appendix for the specific forms for $u_{p,qq}$, $u_{p,q}$. The solutions for equations (2.10) for $n \geq 3$ are given by

$$\text{(i)}\ \overset{0}{h_1}(r) = \frac{-1}{n+2}\overset{5}{A_3}r^{-(n+2)} + \overset{0}{A_3}r^{-n} + \tfrac{1}{2}\overset{0}{A_1}r^{(2-n)} + \overset{0}{A_4} - \frac{(\overset{5}{A_2}+n\overset{5}{A_4})}{n(n+2)}r^2$$

(ii) $h_1^1(r)=A_3^1r^{-(n+2)}+\frac{1}{2}A_1^1r^{-n}+A_4^1$

(iii) $h_2^1(r)=-\frac{1}{n}A_3^1r^{-n}+A_5^1r^{(2-n)}+A_6^1+\frac{A_2^1-2A_4^1}{2n}r^2$

(iv) $h_1^2(r)=\frac{-1}{n+4}A_3^6r^{-(n+4)}+A_3^2r^{-(n+2)}+\frac{1}{2}A_1^2r^{-n}$ (2.11)

$+A_4^2-\frac{(A_2^6+(n+2)A_4^6)}{(n+2)(n+4)}r^2$

(v) $h_2^2(r)=\frac{1}{(n+2)(n+4)}A_3^6r^{-(n+2)}-\frac{(A_1^6+2(n+2)[A_3^2+A_5^6)}{2n(n+2)}r^{-n}$

$+A_5^2r^{(2-n)}+A_6^2+\frac{(A_2^2-2[A_4^2+A_8^6])}{2n}r^2-\frac{(A_2^6-2A_4^6)}{2(n+2)(n+4)}r^4$

(vi) $h_1^3(r)=\frac{-1}{n+4}A_3^6r^{-(n+4)}+A_3^3r^{-(n+2)}+\frac{1}{2}A_1^3r^{-n}$

$+A_4^3-\frac{(A_2^6+(n+2)A_4^6)}{(n+2)(n+4)}r^2$ (2.11)

(vii) $h_2^3(r)=\frac{1}{(n+2)(n+4)}A_3^6r^{-(n+2)}-\frac{(A_1^6+2(n+2)[A_3^3+A_7^6])}{2n(n+2)}r^{-n}$

$+A_5^3r^{(2-n)}+A_6^3+\frac{(A_2^3-2[A_4^3+A_8^6])}{2n}r^2-\frac{(A_2^6-2A_4^6)}{2(n+2)(n+4)}r^4$

(viii) $h_1^4(r)=\frac{-1}{(n+4)}A_3^6r^{-(n+4)}+A_3^4r^{-(n+2)}+\frac{1}{2}A_1^4r^{-n}$

$+A_4^4-\frac{(A_2^6+(n+2)A_4^6)}{(n+2)(n+4)}r^2$ (2.11)

(ix) $h_2^4(r)=\frac{1}{(n+2)(n+4)}A_3^6r^{-(n+2)}-\frac{(A_1^6+2(n+2)[A_3^4+A_9^6])}{2n(n+2)}r^{-n}$

$+A_5^4r^{(2-n)}+A_6^4+\frac{(A_2^4-2[A_4^4+A_{10}^6])}{2n}r^2-\frac{(A_2^6-2A_4^6)}{2(n+2)(n+4)}r^4$

(x) $h_1^5(r)=A_3^5r^{-(n+4)}+\frac{1}{2}A_1^5r^{-(n+2)}+A_4^5$

(xi) $h_2^5(r)=-\frac{1}{(n+2)}A_3^5r^{-(n+2)}+A_5^5r^{-n}+A_6^5+\frac{A_2^5-2A_4^5}{2(n+2)}r^2$

(2.11)

(xii) $h_3^5(r) = \frac{-1}{(n+2)} A_3^5 r^{-(n+2)} + A_7^5 r^{-n} + A_8^5 + \frac{A_2^5 - 2A_4^5 r^2}{2(n+2)}$

(xiii) $h_1^6(r) = A_3^6 r^{-(n+6)} + \frac{1}{2} A_1^6 r^{-(n+4)} + A_4^6$ (2.11)

(xiv) $h_2^6(r) = \frac{-1}{(n+4)} A_3^6 r^{-(n+4)} + A_5^6 r^{-(n+2)} + A_6^6$

$+ \frac{A_2^6 - 2A_4^6}{2(n+4)} r^2$

(xv) $h_3^6(r) = \frac{-1}{(n+4)} A_3^6 r^{-(n+4)} + A_7^6 r^{-(n+2)} + A_8^6 + \frac{A_2^6 - 2A_4^6}{2(n+4)} r^2$

(xvi) $h_4^6(r) = \frac{-1}{(n+4)} A_3^6 r^{-(n+4)} + A_9^6 r^{-(n+2)} + A_{10}^6 + \frac{A_2^6 - 2A_4^6 r^2}{2(n+4)}$

(2.11)

Substitution of (2.9) using (2.11) in (2.2) gives rise to a set of differential equations of the form

$$r \frac{d}{dr} h_1^p(r) + (n+m)\, h_1^p(r) = g_3(r) \tag{2.12}$$

where n,m are as before and $g_3(r)$ involves $h_q^p(r)$ for $q \neq 1$ and their derivatives. This set of equations when solved impose the following restrictions of the A_j^m

(i) $A_1^6 = -2(n+2)\,[A_5^6 + A_7^6 + A_9^6]$ (2.13)

(ii) $3A_2^6 = -(n+1)(n+6)\, A_4^6$

(iii) $2A_2^5 = -n(n+4)\, A_4^5$

(iv) $A_5^5 = -A_7^5$ (2.13)

(v) $A_1^4 = -2(2-n)\, A_5^4$

(vi) $A_2^4 = -(n-1)(n+2)A_4^4 + 2A_{10}^6 - n(A_6^6 + A_8^6)$

(vii) $A_1^3 = -2(2-n)\, A_5^3$ (2.13)

(viii) $\overset{3}{A}_2 = -(n-1)(n+2)\overset{3}{A}_4 + 2\overset{6}{A}_8 - n(\overset{6}{A}_{10}+\overset{6}{A}_6)$

(ix) $\overset{2}{A}_1 = -2(2-n)\ \overset{2}{A}_5$

(x) $\overset{2}{A}_2 = -(n-1)(n+2)\ \overset{2}{A}_4 + 2\overset{6}{A}_6 - n\ (\overset{6}{A}_8 + \overset{6}{A}_{10})$

(xi) $\overset{1}{A}_1 = -2\ (2-n)\ \overset{1}{A}_5$

(xii) $\overset{1}{A}_2 = -(n-1)(n+2)\ \overset{1}{A}_4$

(xiii) $\overset{0}{A}_1 = 0$

(xiv) $n\overset{0}{A}_4 = -\ [\overset{5}{A}_6 + \overset{5}{A}_8]$ (2.13)

When the changes indicated by (2.13) have been made in (2.8), (2.11) the functions $H^p(r)$, $h_q^p(r)$ for $n \geq 3$ are as follows

(i) $H^0(r) = -\frac{1}{n}\overset{5}{A}_1 r^{-n} + \overset{0}{A}_2 - \frac{1}{n}\overset{5}{A}_2 r^2$ (2.14)

(ii) $H^1(r) = \overset{1}{A}_1 r^{-n} + \overset{1}{A}_2$

(iii) $H^2(r) = 2[\overset{6}{A}_5+\overset{6}{A}_7+\overset{6}{A}_9]\,r^{-(n+2)} + \overset{2}{A}_1 r^{-n} + \overset{2}{A}_2 - \frac{1}{n+2}\overset{6}{A}_2 r^2$

(iv) $H^3(r) = 2[\overset{6}{A}_5+\overset{6}{A}_7+\overset{6}{A}_9]\,r^{-(n+2)} + \overset{3}{A}_1 r^{-n} + \overset{3}{A}_2 - \frac{1}{n+2}\overset{6}{A}_2 r^2$

(v) $H^4(r) = 2[\overset{6}{A}_5+\overset{6}{A}_7+\overset{6}{A}_9]\,r^{-(n+2)} + \overset{4}{A}_1 r^{-n} + \overset{4}{A}_2 - \frac{1}{n+2}\overset{6}{A}_2 r^2$

(vi) $H^5(r) = \overset{5}{A}_1 r^{-(n+2)} + \overset{5}{A}_2$

(vii) $H^6(r) = -2(n+2)[\overset{6}{A}_5+\overset{6}{A}_7+\overset{6}{A}_9]\,r^{-(n+4)} + \overset{6}{A}_2$

(viii) $\overset{0}{h}_1(r) = \frac{-1}{n+2}\overset{5}{A}_3 r^{-(n+2)} + \overset{0}{A}_3 r^{-n} - \frac{[\overset{5}{A}_6+\overset{5}{A}_8]}{n} - \frac{1}{n(n+4)}\overset{5}{A}_2 r^2$

(ix) $\overset{1}{h}_1(r) = \overset{1}{A}_3 r^{-(n+2)} + \frac{1}{2}\overset{1}{A}_1 r^{-n} - \frac{1}{(n-1)(n+2)}\overset{1}{A}_2$

(x) $\overset{1}{h}_2(r) = -\frac{1}{n}\overset{1}{A}_3 r^{-n} + \frac{1}{2(n-2)}\overset{1}{A}_1 r^{(2-n)} + \overset{1}{A}_6$

$+ \frac{n+1}{2(n-1)(n+2)}\overset{1}{A}_2 r^2$ (2.14)

$$\text{(xi)}\quad h_1^2(r) = \frac{-1}{n+4} A_3^6 r^{-(n+4)} + A_3^2 r^{-(n+2)} + \tfrac{1}{2} A_1^2 r^{-n} \qquad (2.14)$$

$$- \frac{[A_2^2 - 2A_8^6 + n(A_8^6 + A_{10}^6)]}{(n-1)(n+2)} - \frac{n\, A_2^6}{(n+1)(n+2)(n+6)} r^2$$

$$\text{(xii)}\quad h_2^2(r) = \frac{1}{(n+2)(n+4)} A_3^6 r^{-(n+2)} + \frac{1}{n}[A_7^6 + A_9^6 + A_3^2]\; r^{-n}$$

$$+ \frac{1}{2(n-2)} A_1^2 r^{(2-n)} + A_6^2 + \frac{(n+1)[A_2^2 - 2A_6^6] + 2[A_8^6 + A_{10}^6]}{2(n-1)(n+2)} r^2$$

$$- \frac{(n+3)}{2(n+1)(n+2)(n+6)} A_2^6 r^4$$

$$\text{(xiii)}\quad h_1^3(r) = \frac{-1}{n+4} A_3^6 r^{-(n+4)} + A_3^3 r^{-(n+2)} + \tfrac{1}{2} A_1^3 r^{-n}$$

$$- \frac{[A_2^3 - 2A_8^6 + n(A_6^6 + A_{10}^6)]}{(n-1)(n+2)} - \frac{n\, A_2^6}{(n+1)(n+2)(n+6)} r^2$$

$$\text{(xiv)}\quad h_2^3(r) = \frac{1}{(n+2)(n+4)} A_3^6 r^{-(n+2)} + \frac{1}{n}\,[A_5^6 + A_9^6 - A_3^3] r^{-n}$$

$$+ \frac{1}{2(n-2)} A_1^3 r^{(2-n)} + A_6^3$$

$$+ \frac{(n+1)[A_2^3 - 2A_8^6] + 2\,[A_6^6 + A_{10}^6]}{2(n-1)(n+2)}\; r^2$$

$$- \frac{(n+3)}{2(n+1)(n+2)(n+6)} A_2^6 r^4 \qquad (2.14)$$

$$\text{(xv)}\quad h_1^4(r) = \frac{-1}{n+4} A_3^6 r^{-(n+4)} + A_3^4 r^{-(n+2)} + \tfrac{1}{2} A_1^4 r^{-n}$$

$$- \frac{[A_2^4 - 2A_{10}^6 + n(A_6^6 + A_8^6)]}{(n-1)(n+2)} - \frac{n\, A_2^6}{(n+1)(n+2)(n+6)}\; r^2$$

$$\text{(xvi)}\quad h_2^4(r) = \frac{1}{(n+2)(n+4)} A_3^6 r^{-(n+2)} + \frac{1}{n}\,[A_5^6 + A_7^6 - A_3^4]\quad r^{-n}$$

$$+ \frac{1}{2(n-2)} A_1^4 r^{(2-n)} + A_6^4 \qquad (2.14)$$

$$+ \frac{(n+1)\ [\overset{4}{A}_2 - 2\overset{6}{A}_{10}] + 2[\overset{6}{A}_6 + \overset{6}{A}_8]}{2(n-1)(n+2)}\, r^2$$

$$- \frac{(n+3)}{2(n+1)(n+2)(n+6)}\, \overset{6}{A}_2 r^4 \tag{2.14}$$

$$\text{(xvii)}\quad \overset{5}{h}_1(r) = \overset{5}{A}_3 r^{-(n+4)} + \tfrac{1}{2}\overset{5}{A}_1 r^{-(n+2)} - \frac{2}{n(n+4)}\overset{5}{A}_2$$

$$\text{(xviii)}\quad \overset{5}{h}_2(r) = \frac{-1}{n+2}\, \overset{5}{A}_3 r^{-(n+2)} + \overset{5}{A}_5 r^{-n} + \overset{5}{A}_6 + \frac{n+2}{2n(n+4)}\overset{5}{A}_2 r^2$$

$$\text{(xix)}\quad \overset{5}{h}_3(r) = \frac{-1}{n+2}\, \overset{5}{A}_3 r^{-(n+2)} - \overset{5}{A}_5 r^{-n} + \overset{5}{A}_8 + \frac{n+2}{2n(n+4)}\overset{5}{A}_2 r^2$$

$$\text{(xx)}\quad \overset{6}{h}_1(r) = \overset{6}{A}_3 r^{-(n+6)} - (n+2)\,[\overset{6}{A}_5 + \overset{6}{A}_7 + \overset{6}{A}_9]\, r^{-(n+4)}$$

$$- \frac{3}{(n+1)(n+6)}\overset{6}{A}_2$$

$$\text{(xxi)}\quad \overset{6}{h}_2(r) = \frac{-1}{n+4}\, \overset{6}{A}_3 r^{-(n+4)} + \overset{6}{A}_5 r^{-(n+2)} + \overset{6}{A}_6$$

$$+ \frac{n+3}{2(n+1)(n+6)}\, \overset{6}{A}_2 r^2$$

$$\text{(xxii)}\quad \overset{6}{h}_3(r) = \frac{-1}{n+4}\, \overset{6}{A}_3 r^{-(n+4)} + \overset{6}{A}_7 r^{-(n+2)} + \overset{6}{A}_8$$

$$+ \frac{n+3}{2(n+1)(n+6)}\, \overset{6}{A}_2 r^2$$

$$\text{(xxiii)}\quad \overset{6}{h}_4(r) = \frac{-1}{n+4}\, \overset{6}{A}_3 r^{-(n+4)} + \overset{6}{A}_9 r^{-(n+2)} + \overset{6}{A}_{10}$$

$$+ \frac{n+3}{2(n+1)(n+6)}\, \overset{6}{A}_2 r^2 \tag{2.14}$$

The functions $H^p(r)$ and $h_q^p(r)$, as given in equations (2.14), are valid in an n-dimensional Euclidean space where $n \geq 3$. However, for the special case of $n=2$, the functions presented above are still valid provided that certain minor changes are made. The necessary changes in equations (2.14) are

(i) $r^{2-n} \rightarrow \ln r$

(ii) $\frac{1}{2(n-2)} A_i^s \rightarrow -\frac{1}{2}A_i^s$

where $s \in \{1,2,3,4\}$.

Equations (2.5) & (2.9) together with (2.14) constitute the general solutions of the Stokes' equations (2.1) & (2.2). Once the boundary conditions for a specific problem are prescribed, the corresponding general solutios is easily generated from these equations. Since these solutions are not complete to the same degree as those of Lamb (1932), it is natural to illustrate some of the flow situations where they can be applied. In the next section, a number of examples are detailed.

Section 3. Examples.

A. Uniform Flow.

In this subsection the problem of the slow uniform flow of an incompressible viscous fluid past a fluid sphere will be considered. The solution will be generated using both the stream function technique and the new form of the general solution as given in section 2. One purpose for doing this is so that the similarities and differences in the final form can be considered almost side by side. Another reason for doing the work in such detail is that the solution of this problem is required in the subsequent chapter.

The equations to be considered and the boundary conditions for the uniform flow of an incompressible viscous fluid past a stationary fluid sphere are found to be

$$u_{i,i} = 0 \tag{2.15}$$

$$\mu u_{\ell,mm} = p_{,\ell} \tag{2.16}$$

(i) $\underset{\sim}{u}^{e}{}_{\infty} = U\hat{e}_z$

(ii) $|\underset{\sim}{u}^{i}| < \infty \quad @\ r = 0$ (2.17)

(iii) $\underset{\sim}{u}^{e}\cdot\underset{\sim}{r} = \underset{\sim}{u}^{i}\cdot\underset{\sim}{r} = 0 \quad @\ r = 1$

(iv) $\underset{\sim}{u}^{e} - (\underset{\sim}{u}^{e}\cdot\underset{\sim}{r})\underset{\sim}{r} = \underset{\sim}{u}^{i} - (\underset{\sim}{u}^{i}\cdot\underset{\sim}{r})\underset{\sim}{r} \quad @\ r = 1$

(v) $\underset{\approx}{T}^{e}\cdot\underset{\sim}{r} - (\underset{\sim}{r}\cdot\underset{\approx}{T}^{e}\cdot\underset{\sim}{r})\underset{\sim}{r} = \underset{\approx}{T}^{i}\cdot\underset{\sim}{r} - (\underset{\sim}{r}\cdot\underset{\approx}{T}^{i}\cdot\underset{\sim}{r})\underset{\sim}{r} \quad @\ r = 1$

(i) Stream Function Technique

$$\therefore \quad \underset{\sim}{u}^{e}{}_{\infty} = U\hat{e}_z = U\cos\theta\,\hat{e}_r - U\sin\theta\,\hat{e}_\theta$$

$\therefore U_r \to U\cos\theta,\ U_\theta \to -U\sin\theta \quad \text{as } r \to \infty$

Hence, it is assumed that

$$\underset{\sim}{U} = U_r(r,\theta)\,\hat{e}_r + U_\theta(r,\theta)\,\hat{e}_\theta$$

From (2.15) the restrictions on U_r, U_θ are

$$\frac{\partial}{\partial r}(r^2 \sin\theta\, U_r) + \frac{\partial}{\partial \theta}(r \sin\theta\, U_\theta) = 0$$

If $U_r = \dfrac{1}{r^2 \sin\theta}\dfrac{\partial \Psi}{\partial \theta}$, $U_\theta = \dfrac{-1}{r \sin\theta}\dfrac{\partial \Psi}{\partial r}$ (2.18)

then (2.15) is satisfied identically, and (2.17 (i)) becomes

$$\Psi \to \frac{U}{2} r^2 \sin^2\theta$$

By taking the curl of (2.16) the required equations of motion become

$$\nabla \times (\nabla \times \underset{\sim}{\theta}) = 0 = -\nabla^2 \underset{\sim}{\theta} \tag{2.19}$$

where $\underset{\sim}{\theta} \equiv \nabla \times \underset{\sim}{u} = 2\underset{\sim}{\omega}$

However, using (2.18) in the definition of $\underset{\sim}{\omega}$

ie $\underset{\sim}{\omega} = r^{-1}\hat{e}_\phi\{\frac{\partial}{\partial r}(r U_\theta) - \frac{\partial}{\partial \theta}(U_r)\}$

$$\therefore \quad \underset{\sim}{\omega} = r^{-1}\hat{e}\phi\left[-\frac{1}{\sin\theta}\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial \Psi}{\partial \theta}\right)\right]$$

$$= -\frac{1}{r\sin\theta}\left[\frac{\partial^2 \Psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial \Psi}{\partial\theta}\right)\right]\hat{e}\phi$$

and (2.19) thus requires solving

$$\left[\frac{\partial^2}{\partial^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \Psi = 0 \tag{2.20}$$

If the function $\Psi(r,\theta)$ is taken to be such that

$$\Psi(r,\theta) = f(r)\sin^2\theta \tag{2.21}$$

then equation (2.20) will be satisfied identically if

$$\left[\frac{d^2}{dr^2} - \frac{2}{r^2}\right]^2 f(r) = 0$$

Thus it is necessary to choose

$$f(r) = (Ar^{-1} + Br + Cr^2 + Dr^4) \qquad (2.22)$$

which when substituted in (2.21) and subsequently in (2.18) gives

$$\frac{U_r}{U} = \left(\frac{2A}{r^3} + \frac{2B}{r} + 2C + Dr^2 \right) \cos\theta$$

$$U_\theta = \left(\frac{A}{r^3} - \frac{B}{r} - 2C - 4Dr^2 \right) \sin\theta$$

and (2.17 (i)) requires that D=0, $C = \frac{u}{2}$.

For the internal flow to be finite @ r=0 from (2.17(ii)) it is necessary to choose A = B = 0. From the normal velocity conditions (2.17(iii)) come the set of equations

$$2A + 2B + U = 0$$

$$C + D = 0$$

and from (2.17(iv)) it is found that

$$A - B - U = -2C - 4D$$

and the continuity of tangential stress (2.17(v)) @ r=1 gives

$$6\mu^{(e)}A = 6\mu^{(i)}D \ .$$

In tabular form, the values for the coefficients for the interior and exterior flows are, with $\sigma = \frac{\mu^{(e)}}{\mu^{(i)}}$,

	$\underset{\sim}{u}^e$	$\underset{\sim}{u}^i$
A	$\frac{U}{4(1+\sigma)}$	0
B	$-\frac{U(2\sigma+3)}{4(1+\sigma)}$	0
C	$\frac{U}{2}$	$-\frac{\sigma U}{4(1+\sigma)}$
D	0	$\frac{\sigma U}{4(1+\sigma)}$

(2.23)

In order to determine the pressure, these expressions for $\underset{\sim}{u}^i$, $\underset{\sim}{u}^e$ must be substituted back into (2.16) which must then be solved.

(ii) New Form

Using now the method outlined in §(2.2) for the same problem written in tensor form

$$u_{i,i} = 0 \tag{2.24}$$

$$\mu u_{\ell,mm} = p,_{\ell} \tag{2.25}$$

(i) $u^e_{\ell\infty} = \varepsilon_{312}a_{21} = U\delta_{\ell 3}$

(ii) $|\underset{\sim}{u}^i| < \infty \quad @\ r=0$

(iii) $u^e_\ell X_\ell = u^i_\ell X_\ell = 0 \quad @\ r=1$ (2.26)

(iv) $u^e_\ell - u^e_j X_j X_\ell = u^i_\ell - u^i_j X_j X_\ell \quad @\ r=1$

(v) $t^e_{ij}X_j - t^e_{k\ell}X_k X_\ell X_i = t^i_{ij}X_j - t^i_{k\ell}X_k X_\ell X_i \quad @\ r=1$

From (2.26(i)), (2.5), (2.9) the appropriate choice in (2.14) is $A_j^m = 0$, $\forall_m \neq 1$, $a_{21} = U$, $a_{ij} = 0$ otherwise

$$\text{(i)} \quad p = (\overset{1}{A}_1 r^{-3} + \overset{1}{A}_2)\, UX_3$$

$$\text{(ii)} \quad u_\ell = (\overset{1}{A}_3 r^{-5} + \tfrac{1}{2}\overset{1}{A}_1 r^{-3} - \frac{1}{10}\overset{1}{A}_2) UX_3 X_\ell$$
$$+ (-\frac{1}{3}\overset{1}{A}_3 r^{-3} + \tfrac{1}{2}\overset{1}{A}_1 r^{-1} + \overset{1}{A}_6 + \frac{1}{5}\overset{1}{A}_2 r^2) U\delta_{\ell 3} \quad (2.27)$$

From boundary condition (2.26(i)) it is found necessary to choose $\overset{1}{A}_2 = 0$, $\overset{1}{A}_6 = 1$ and from (2.26(ii)) for the inner flow $\overset{1}{A}_3 = \overset{1}{A}_1 = 0$

For the normal velocities at the boundary the following set of equations are derived

$$\frac{2}{3}\overset{1}{A}_3 + \overset{1}{A}_1 + 1 = 0$$

$$\overset{1}{A}_6 + \frac{1}{10}\overset{1}{A}_2 = 0$$

and from (2.26(iv)) it is found that

$$\frac{1}{3}\overset{1}{A}_3 - \tfrac{1}{2}\overset{1}{A}_1 - 1 = -\overset{1}{A}_6 - \frac{1}{5}\overset{1}{A}_2$$

Lastly, from the continuity of tangential stress (2.26(v))

$$2\mu^e \overset{1}{A}_3 = \frac{3\mu^i}{10}\overset{1}{A}_2$$

In tabular form, when solved, the values for the coefficients for the interior and exterior flows are as follows, with

$$\sigma = \frac{\mu^{(e)}}{\mu^{(i)}},$$

	$\underset{\sim}{u}^e$	$\underset{\sim}{u}^i$
$\frac{1}{3} A_3^1$	$\frac{1}{4(1+\sigma)}$	0
$\frac{1}{2} A_1^1$	$-\frac{(2\sigma+3)}{4(1+\sigma)}$	0
$\frac{1}{2} A_6^1$	$\frac{1}{2}$	$-\frac{\sigma}{4(1+\sigma)}$
$\frac{1}{20} A_2^1$	0	$\frac{\sigma}{4(1+\sigma)}$

(2.28)

These coefficients also determine the pressure function for this problem.

Clearly, once the general solution is found by each method, the amount of work necessary to determine the numerical value of the coefficients for each is approximately the same. The new form for the general solution, however, already determines the value for $p(\underset{\sim}{r})$ both inside and outside the sphere while much work must still be done to determine $p(\underset{\sim}{r})$ for the stream function technique. This makes it desirable to use the new form for the general solution instead of the stream function technique whenever the value of $p(\underset{\sim}{r})$ is required.

In the next two subsections a number of examples will be considered, in less detail, to further illustrate the method. The boundary conditions will be written in cartesian tensor form using the simplest possible forms for the tensors a_{ij} and A_{ijk}. The structure of the general solution is deduced from the primary boundary conditions. It must be emphasized that the examples included, while not exhaustive, represent a sample of the kinds of problems that can be handled by this approach.

B. Flow past a Solid Spherical Body.

The three standard examples of flow past a solid spherical body will be discussed. For the uniform flow and quadratic shear flow examples, the standard solutions can be found in the book by J. Happel & H. Brenner (1965) while the linear shear flow example can be found in G.K. Batchelor (1967).

1. The boundary conditions for the uniform creeping flow of a fluid past a stationary solid spherical body are given by

$$\begin{aligned} &\text{(i)} \quad p_\infty = p_0 \\ &\text{(ii)} \quad \underset{\sim}{u}_\infty = U\hat{e}_z \\ &\text{(iii)} \quad \underset{\sim}{u} = 0 \quad @\ r = 1 \end{aligned} \tag{2.29}$$

The cartesian tensor forms for these boundary conditions are

$$\begin{aligned} &\text{(i)} \quad p_\infty = a_{ii} \\ &\text{(ii)} \quad u_{\ell\infty} = \varepsilon_{\ell jk} a_{kj} \\ &\text{(iii)} \quad u_\ell = 0 \quad @\ r = 1 \end{aligned} \tag{2.30}$$

The simplest cartesian tensor forms required for the matching of (2.29) and (2.30) are given as

$$\text{(i)}\quad a_{ii} = p_e\ ,\ a_{21} = U\ ,\ a_{ij} = 0 \quad \text{otherwise} \qquad (2.31)$$

$$\text{(ii)}\quad A_{ijk} = 0 \qquad \forall_{i,j,k}$$

From the cartesian tensor forms of the boundary conditions, the most natural choices for the coefficients in (2.14) are

$$\overset{m}{A_j} = 0 \qquad \forall_m \neq 0,1 \qquad (2.32)$$

Substitution of (2.14) using (2.32) in (2.5) & (2.9) gives

$$\text{(i)}\quad p = \overset{0}{A_2} a_{ii} + (\overset{1}{A_1} r^{-3} + \overset{1}{A_2})\varepsilon_{ijk} a_{kj} X_i$$

$$\text{(ii)}\quad u_\ell = \overset{0}{A_3} r^{-3} a_{ii} X_\ell + (\overset{1}{A_3} r^{-5} + \tfrac{1}{2}\overset{1}{A_1} r^{-3} - \frac{1}{10}\overset{1}{A_2})\varepsilon_{ijk} a_{kj} X_i$$

$$+ (-\frac{1}{3}\overset{1}{A_3} r^{-3} + \tfrac{1}{2}\overset{1}{A_1} r^{-1} + \overset{1}{A_6} + \frac{1}{5}\overset{1}{A_2} r^2)\varepsilon_{\ell jk} a_{kj}$$

(2.33)

In order for (2.33) to agree with the first two of (2.30), it is necessary to choose $\overset{0}{A_2} = 1$, $\overset{1}{A_2} = 0$, $\overset{1}{A_6} = 1$. Agreement with the surface boundary condition in (2.30) necessitates $\overset{0}{A_3} = 0$, $\overset{1}{A_3} = \frac{3}{4}$, $\overset{1}{A_1} = -\frac{3}{2}$. Hence the general solution (2.39) will describe the uniform creeping flow of a fluid past a fixed spherical solid provided that

$$\text{(i)}\quad \overset{0}{A_2} = 1$$

$$\text{(ii)}\quad \overset{1}{A_1} = -\frac{3}{2} \qquad (2.34)$$

$$\text{(iii)}\quad \overset{1}{A_3} = \frac{3}{4}$$

(iv) $A_6^1 = 1$

(v) $A_3^0 = A_2^1 = 0$ (2.34)

2. Any linear shearing flow for a fluid past a solid spherical body is described by the following set of boundary conditions

(i) $p_\infty = 0$

(ii) $u_{\ell\infty} = a_{\ell j} X_j$ (2.35)

(iii) $u_\ell = \frac{1}{2}\left[a_{\ell j} - a_{j\ell} X_j\right]$ @ $r = 1$

The proper choices for the tensors a_{ij} & A_{ijk} in this case are

(i) $a_{ii} = 0$ (2.36)

(ii) $A_{ijk} = 0 \quad \forall_{i,j,k}$

Consequently, from an examination of (2.35), the appropriate choice for the coefficients in (2.14) is

$$A_j^m = 0 \qquad \forall_m \neq 5 \tag{2.37}$$

With this choice, the general solution obtained by substituting (2.37) in (2.14) and subsequently in (2.5) and (2.9) is

(i) $p = (A_1^5 r^{-5} + A_2^5) a_{ij} X_i X_j$

$$\text{(ii)} \quad u_\ell = (\overset{5}{A}_3 r^{-7} + \tfrac{1}{5}\overset{5}{A}_1 r^{-5} - \tfrac{2}{21}\overset{5}{A}_2) a_{ij} X_i X_j X_\ell \qquad (2.38)$$

$$+ (-\tfrac{1}{5}\overset{5}{A}_3 r^{-5} + \overset{5}{A}_5 r^{-3} + \overset{5}{A}_6 + \tfrac{5}{42}\overset{5}{A}_2 r^2) a_{\ell j} X_j$$

$$+ (-\tfrac{1}{5}\overset{5}{A}_3 r^{-5} - \overset{5}{A}_5 r^{-3} + \overset{5}{A}_8 + \tfrac{5}{42}\overset{5}{A}_2 r^2) a_{j\ell} X_j$$

The general solution (2.38) will satisfy all of the boundary conditions of (2.35) if

$$\left.\begin{array}{ll} \text{(i)} & \overset{5}{A}_1 = -5 \\ \text{(ii)} & \overset{5}{A}_3 = \frac{5}{2} \\ \text{(iii)} & \overset{5}{A}_6 = 1 \\ \text{(iv)} & \overset{5}{A}_2 = \overset{5}{A}_5 = \overset{5}{A}_8 = 0 \end{array}\right\} \qquad (2.39)$$

3. The last example to be considered is the quadratic shearing flow of a fluid past a uniformly translating, axially located, solid spherical body. The boundary conditions for this simple example of a quadratic flow are given by

$$\begin{array}{ll} \text{(i)} & p_\infty = p_0 + 4b\underset{\sim}{z} \\ \text{(ii)} & \underset{\sim}{u}_\infty = \left[a + b(x^2 + y^2)\right]\hat{e}_z \qquad (2.40) \\ \text{(iii)} & \underset{\sim}{u} = U\hat{e}_z \quad @\ r = 1 \end{array}$$

These same boundary conditions written in a form suitable for the type of general solution found in this chapter can be expressed as

$$\text{(i)}\quad p_\infty = a_{ii} + 2A_{imm}X_i$$

$$\text{(ii)}\quad u_{\ell\infty} = A_{\ell jk}X_jX_k + \alpha A_{\ell mm} \qquad (2.41)$$

$$\text{(iii)}\quad u_\ell = U\,\delta_{\ell 3} \qquad @\ r = 1$$

where the appropriate choice for the tensors a_{ij} & A_{ijk} are found to be

$$\text{(i)}\quad a_{ii} = p_0\ ,\ a_{ij} = 0 \quad \text{otherwise} \qquad (2.42)$$

$$\text{(ii)}\quad A_{311} = A_{322} = b\ ,\ A_{ijk} = 0 \quad \text{otherwise}$$

and $\alpha = \frac{a}{2b}$. The tensor form for the boundary conditions (2.41) and the assumed form for the coefficient tensors suggest that the appropriate choice for the coefficients is

$$A_j^m = 0 \qquad \forall_m \neq 0,2,6 \qquad (2.43)$$

Substitution of (2.43) in (2.14) and thence in (2.5) and (2.9) yields

$$\text{(i)}\quad p = \overset{0}{A}_2 a_{ii} + \left[2(\overset{6}{A}_5 + \overset{6}{A}_7 + \overset{6}{A}_9)r^{-5} + \overset{2}{A}_1 r^{-3} + \overset{2}{A}_2\right] A_{imm}X_i + \left[-10(\overset{6}{A}_5 + \overset{6}{A}_7 + \overset{6}{A}_9)r^{-7} + \overset{6}{A}_2\right] A_{ijk}X_iX_jX_k$$

$$\text{(ii)}\quad u_\ell = \overset{0}{A}_3 r^{-3} a_{ii}X_\ell + \Big[-\tfrac{1}{7}\overset{6}{A}_3 r^{-7} + \overset{2}{A}_3 r^{-5} + \tfrac{1}{2}\overset{2}{A}_1 r^{-3} - \tfrac{1}{10}\{\overset{2}{A}_2 - 2\overset{6}{A}_6 + 3(\overset{6}{A}_8 + \overset{6}{A}_{10})\} - \tfrac{1}{60}\overset{6}{A}_2 r^2\Big] A_{imm}X_iX_\ell \qquad (2.44)$$

$$+ \Big[\tfrac{1}{35}\overset{6}{A}_3 r^{-5} + \tfrac{1}{3}(\overset{6}{A}_7 + \overset{6}{A}_9 - \overset{2}{A}_3)r^{-3} + \tfrac{1}{2}\overset{2}{A}_1 r^{-1} + \overset{2}{A}_6 + \tfrac{1}{20}\{4(\overset{2}{A}_2 - 2\overset{6}{A}_6) + 2(\overset{6}{A}_8 + \overset{6}{A}_{10})\}r^2 - \tfrac{1}{72}\overset{6}{A}_2 r^4\Big] A_{\ell mm}$$

$$+\left[A_3^6 r^{-9}-5(A_5^6+A_7^6+A_9^6)r^{-7}-\frac{1}{12}A_2^6\right]A_{ijk}X_iX_jX_kX_\ell$$

$$+\left[-\frac{1}{7}A_3^6 r^{-7}+A_5^6 r^{-5}+A_6^6+\frac{1}{12}A_2^6 r^2\right]A_{\ell jk}X_jX_k$$

$$+\left[-\frac{1}{7}A_3^6 r^{-7}+A_7^6 r^{-5}+A_8^6+\frac{1}{12}A_2^6 r^2\right]A_{j\ell k}X_jX_k$$

$$+\left[-\frac{1}{7}A_3^6 r^{-7}+A_9^6 r^{-5}+A_{10}^6+\frac{1}{12A_2^6}r^2\right]A_{jk\ell}X_jX_k$$

(2.44)

The general solution (2.44) will satisfy the first two boundary conditions of (2.41) provided that $A_2^0 = 1$, $A_2^2 = 2$, $A_6^2 = \frac{a}{2b}$, $A_2^6 = 0$, $A_6^6 + A_8^6 + A_{10}^6 = 1$. In order to satisfy the surface boundary condition in (2.41) the coefficients must be $A_3^0 = 0$, $A_1^2 = \frac{3}{4b}(U-a)-\frac{1}{2}$, $A_3^2 = \frac{-3}{8b}(U-a)+\frac{11}{8}-\frac{1}{2}A_6^6$, $A_3^6 = 5(A_5^6 + A_7^6 + A_9^6)$, $A_5^6 = \frac{5}{8} - A_6^6$, $A_7^6 + A_9^6 = \frac{1}{4} + A_6^6$, $A_6^2 = \frac{a}{2b}$. Thus the required values of the coefficients in (2.44) necessary to produce the flow given by (2.41) are

(i) $A_2^0 = 1$

(ii) $A_1^2 = \frac{3}{4b}(U-a)-\frac{1}{2}$

(iii) $A_2^2 = 2$ (2.45)

(iv) $A_3^2 = \frac{-3}{8b}(U-a)+\frac{11}{8}-\frac{1}{2}A_6^6$

(v) $A_6^2 = \frac{a}{2b}$

vi) $A_3^6 = \frac{35}{8}$ (2.45)

vii) $A_5^6 = \frac{5}{8} - A_6^6$

viii) $A_9^6 = \frac{1}{4} + A_6^6 - A_7^6$

ix) $A_{10}^6 = 1 - A_6^6 - A_8^6$

x) A_6^6, A_7^6, A_8^6 arbitrary

xi) $A_3^0 = A_2^6 = 0$

C. Singular Flows

I. A number of singular flows, some of which are detailed below, can also be deduced from the general solutions given in section 2.

One of the primary singular flows is the flow due to a source located at $r = 0$. For this flow

i) $\rho = a_{ii}$

ii) $u_\ell = \alpha\, a_{ii}\, X_\ell$ (2.46)

The simplest tensor forms for a_{ij}, A_{ijk} in equations (2.5) and (2.9) are

i) $a_{ii} = \rho_o$, $a_{ij} = o$ otherwise (2.47)

ii) $A_{ijk} = o \quad \forall\ i, j, k$

If $a_{11} = a_{22} = a_{33} = \frac{\rho_o}{3}$, then equations (2.5), (2.9) using (2.14) describe source flow if

i) $A_2^0 = 1$

ii) $A_3^0 + \frac{1}{6} A_1^5 = \alpha$

iii) $\overset{5}{A}_2, \overset{5}{A}_3, \overset{5}{A}_5, \overset{5}{A}_6, \overset{5}{A}_8$ are arbitrary (2.48)

If, however, not all of a_{11}, a_{22}, a_{33} are the same, then

i) $\overset{0}{A}_2 = 1$

ii) $\overset{0}{A}_3 = \alpha$

iii) $\overset{5}{A}_1 = \overset{5}{A}_2 = \overset{5}{A}_3 = \overset{5}{A}_6 = \overset{5}{A}_8 = 0$ (2.49)

iv) $\overset{5}{A}_5$ is arbitrary

2. Closely associated with the problem above is the flow due to a doublet located at $r = o$. In this example

i) $\rho = a_{ii}$ (2.50)

ii) $u_\ell = B_1 \varepsilon_{\ell jk} a_{kj} r^{-3} - 3 B_1 \varepsilon_{ijk} a_{kj} X_i X_\ell r^{-5}$

which dictates that the simplest tensor forms for a_{ij}, A_{ijk} are

i) $a_{ii} = \rho_o$, a_{ij} arbitrary for $i \neq j$ (2.51)

ii) $A_{ijk} = o \; \forall i, j, k$

Thus equations (2.5), (2.9) with (2.14) will describe doublet flow if

i) $\overset{0}{A}_2 = 1$

ii) $\overset{1}{A}_3 = -3B_1$

iii) $\overset{0}{A}_3 = \overset{1}{A}_1 = \overset{1}{A}_2 = \overset{1}{A}_6 = 0$, $\overset{5}{A}_j = 0$ (2.52)

3. Another primary singular flow is the flow due to what is commonly called a Stokeslet located at $r = o$. The pressure and velocity expressed in cartesian tensor form are

i) $\rho = 2\varepsilon_{ijk} a_{kj} X_i r^{-3}$ (2.53)

ii) $u_\ell = \varepsilon_{ijk} a_{kj} X_i X_\ell r^{-3} + \varepsilon_{\ell jk} a_{kj} r^{-1}$

Clearly, an appropriate choice for the coefficient tensors is

$$\text{i) } a_{ij} \text{ arbitrary } \forall i,j \qquad \text{ii) } A_{ijk} = o \quad \forall i,j,k \tag{2.54}$$

Equations (2.5), (2.9) along with (2.14) reproduce flow due to a Stokeslet located at r = o if

$$\text{i) } A_1^1 = 1 \qquad \text{ii) } A_2^1 = A_3^1 = A_6^1 = 0 \qquad \text{iii) } A_k^0 = A_j^5 = 0 \quad \forall k,j \tag{2.55}$$

4. As the doublet is associated with the source, so also is the rotelet associated with the Stokeslet. For a rotelet located at r=o

$$\text{i) } \rho = a_{ii} \qquad \text{ii) } u_\ell = a_{\ell j} X_j r^{-3} \tag{2.56}$$

whereby the simplest appropriate choices for the coefficient tensors are

$$\text{i) } a_{ii} = \rho o, \; a_{ij} = -a_{ji} \text{ for } i \neq j \qquad \text{ii) } A_{ijk} = o \quad \forall i,j,k \tag{2.57}$$

If $a_{11} = a_{22} = a_{33} = \frac{\rho o}{3}$ then equations (2.5), (2.9) with (2.14) describe rotelet flow if

$$\begin{aligned}
&\text{i)}\ \overset{0}{A}_2 = 1\\
&\text{ii)}\ \overset{5}{A}_6 = \overset{5}{A}_8\\
&\text{iii)}\ \overset{5}{A}_5 = \tfrac{1}{2}\\
&\text{iv)}\ \overset{0}{A}_3 = -\frac{1}{6}\ \overset{5}{A}_1\\
&\text{v)}\ \overset{5}{A}_2,\ \overset{5}{A}_3\ \text{arbitrary} \qquad \text{(vi)}\ \overset{1}{A}_j = 0\ \forall j
\end{aligned} \tag{2.58}$$

On the other hand if not all of a_{11}, a_{22}, a_{33} are the same then

$$\begin{aligned}
&\text{i)}\ \overset{0}{A}_2 = 1\\
&\text{ii)}\ \overset{5}{A}_5 = \tfrac{1}{2}\\
&\text{iii)}\ \overset{0}{A}_3 = \overset{5}{A}_1 = \overset{5}{A}_2 = \overset{5}{A}_3 = \overset{5}{A}_6 = \overset{5}{A}_8 = 0\\
&\text{iv)}\ \overset{1}{A}_j = 0 \quad \forall j
\end{aligned} \tag{2.59}$$

5. The final example known as Hill's spherical vortex is somewhat different in the sense that this flow possesses a singularity at infinity rather than at the origin. For this example

$$\begin{aligned}
&\text{i)}\ \rho = a_{ii} + A\varepsilon_{ijk}a_{kj}X_i\\
&\text{ii)}\ u_\ell = -\frac{A}{10}\left[\varepsilon_{ijk}a_{kj}X_iX_\ell + (a^2 - 2r^2)\ \varepsilon_{\ell jk}a_{kj}\right]
\end{aligned} \tag{2.60}$$

Hence, the coefficient tensors are

$$\begin{aligned}
&\text{i)}\ a_{ii} = \rho_o,\ a_{ij}\ \text{arbitrary for}\ i \neq j\\
&\text{ii)}\ A_{ijk} = 0 \quad \forall i,j,k
\end{aligned} \tag{2.61}$$

and the appropriate choices for the coefficients in equations (2.14) so that equations (2.5), (2.9) represent Hill's spherical vortex are

$$
\begin{aligned}
&\text{i)}\quad A_2^0 = 1\\
&\text{ii)}\quad A_2^1 = A\\
&\text{iii)}\quad A_6^1 = -\frac{a2}{10} A_2^1\\
&\text{iv)}\quad A_3^0 = A_1^1 = A_3^1 = 0\\
&\text{v)}\quad A_j^5 = 0 \quad \forall j
\end{aligned}
\tag{2.62}
$$

Section 4.

In section 2, solutions were developed which were suitable for problems dealing with the steady creeping flow of a viscous Newtonian fluid. Much of the emphasis in this work, however, is on the steady creeping flow of micropolar fluids. The form assumed for the velocity and pressure functions in section 2 is found to be suitable for use in the equations governing the flow of micropolar fluids also. As a result, the technique is extended to obtain the general solutions for such fluids, whose governing equations are

$$\nabla^2 p = 0 \tag{2.63}$$

$$\nabla^2 \underset{\sim}{u} = \frac{-2L^2}{\eta} \nabla p \tag{2.64}$$

$$(\nabla^2 - L^2)\underset{\sim}{v} = \underset{\sim}{u} \tag{2.65}$$

$$(\nabla^2 - \frac{2\kappa}{\delta})\phi = 0 \tag{2.66}$$

The solution of equation (2.63) is given by equations (2.5) & (2.8), and the solutions of equations (2.64) with suitable modifications can be deduced from equations (2.9) & (2.11). The functions ψ , $\underset{\sim}{v}$, in equations (2.66), (2.65), will be assumed to have the same structural form as the functions p , $\underset{\sim}{u}$ as expressed by equations (2.5), (2.9). The general forms for $\phi(\underset{\sim}{r})$ and $v(\underset{\sim}{r})$ are, thus, taken to be

$$\psi(\underset{\sim}{r}) = F^{0}(r)a_{ii}+F^{1}(r)\varepsilon_{ijk}a_{kj}X_i+F^{2}(r)A_{imm}X_i +$$

$$+F^{3}(r)A_{mim}X_i+F^{4}(r)A_{mmi}X_i+F^{5}(r)a_{ij}X_iX_j +$$

$$+ F^{6}(r)A_{ijk}X_iX_jX_k \qquad (2.67)$$

$$v_\ell(\underset{\sim}{r}) = f_1^{0}(r)a_{ii}X_\ell+f_1^{1}(r)\varepsilon_{ijk}a_{kj}X_iX_\ell+f_2^{1}(r)\varepsilon_{\ell jk}a_{kj} +$$

$$+f_1^{2}(r)A_{imm}X_iX_\ell+f_2^{2}(r)A_{\ell mm}+f_1^{3}(r)A_{mim}X_iX_\ell +$$

$$+f_2^{3}(r)A_{m\ell m}+f_1^{4}(r)A_{mmi}X_iX_\ell+f_2^{4}(r)A_{mm\ell} +$$

$$+f_1^{5}(r)a_{ij}X_iX_jX_\ell+f_2^{5}(r)a_{\ell j}X_j+f_3^{5}(r)a_{j\ell}X_j +$$

$$+f_1^{6}(r)A_{ijk}X_iX_jX_kX_\ell+f_2^{6}(r)A_{\ell jk}X_jX_k+$$

$$+f_3^{6}(r)A_{j\ell k}X_jX_k + f_4^{6}(r)A_{jk\ell}X_jX_k \qquad (2.68)$$

In general, the differential equations satisfied by the functions $F^p(r)$ and $f_q^p(r)$ are the homogeneous and inhomogeneous Bessel differential equations

$$\frac{d^2}{dr^2} F^p(r)+ \frac{n+(2m-1)}{r} \frac{d}{dr} F^p(r)- \frac{2\kappa}{\delta}F^p(r) = K(r) \qquad (2.69)$$

$$\frac{d^2}{dr^2} f_1^p(r)+ \frac{n+(2m+1)}{r} \frac{d}{dr}f_1^p(r)-L^2f_1^p(r) = k_1(r) \qquad (2.70)$$

$$\frac{d}{dr^2} f_q^p(r) + \frac{n+(2m-3)}{r} \frac{d}{dr} f_q^p(r) - L^2 f_q^p(r) = k_q(r), \quad q \neq 1 \tag{2.71}$$

where $n \geq 2$ is the dimension of the Euclidean space, m is the order of the coefficient tensor and the functions $K(r)$, $k_1(r)$, ..$k_q(r)$ are either known or zero. The actual specific forms of the above differential equations are given in the appendix. Further, since the flow problems considered in this work deal only with $n = 3$ and also only with linear shearing flow and uniform flow, the solutions are developed only for those functions which involve the coefficient tensor a_{ij}. Hence, it is found that

$$F^5(r) = y^{-\frac{5}{2}} \left[D_1^5 I_{-\frac{5}{2}}(y) + D_2^5 I_{\frac{5}{2}}(y) \right]$$

$$F^1(r) = y^{-\frac{3}{2}} \left[D_1^1 I_{-\frac{3}{2}}(y) + D_2^1 I_{\frac{3}{2}}(y) \right]$$

$$F^0(r) = y^{-\frac{1}{2}} \left[D_1^0 I_{-\frac{1}{2}}(y) + D_1^1 I_{\frac{1}{2}}(y) - \frac{D_1^5}{M^2} I_{-\frac{5}{2}}(y) - \frac{D_2^5}{M^2} I_{\frac{5}{2}}(y) \right] \tag{2.72}$$

where $y = Mr$, $M^2 = \frac{2\kappa}{\delta}$.

The solutions for the functions $f_q^p(r)$ associated with the tensor a_{ij} are

$$\text{(i)}\quad f_1^0(r) = \frac{1}{5L^2}(A_3^5 + \frac{10}{\eta}A_1^5)r^{-5} - \frac{1}{L^2}(A_3^0 + \frac{2}{\eta}A_1^0)r^{-3} + \tag{2.73}$$

$$+ \frac{1}{\eta}A_1^0 r^{-1} - \frac{1}{L^2}(A_4^0 + \frac{4L^2}{3\eta}A_2^5) +$$

$$+ \frac{1}{5L^2}(A_4^5 - \frac{2L^2}{3\eta}A_2^5)r^2 + x^{-\frac{3}{2}}\Big[B_1^0 I_{-\frac{3}{2}}(x) +$$

$$+ B_2^0 I_{\frac{3}{2}}(x) - \frac{B_1^5}{5L^2} I_{-\frac{7}{2}}(x) - \frac{B_2^5}{5L^2} I_{\frac{7}{2}}(x)\Big]$$

$$\text{(ii)}\quad f_1^1(r) = \frac{-1}{L^2}(A_3^1 + \frac{6}{\eta}A_1^1)r^{-5} + \frac{1}{\eta}A_1^1 r^{-3} - \frac{1}{L^2}A_4^1 +$$

$$+ x^{-\frac{5}{2}}\Big[B_1^1 I_{-\frac{5}{2}}(x) + B_2^1 I_{\frac{5}{2}}(x)\Big]$$

$$\text{(iii)}\quad f_2^1(r) = \frac{1}{3L^2}(A_3^1 + \frac{6}{\eta}A_1^1)r^{-3} - \frac{1}{L^2}A_5 r^{-1} +$$

$$+ \frac{1}{L^2}(\frac{2}{\eta}A_2^1 - A_6^1) + \frac{1}{3L^2}(A_4^1 + \frac{L^2}{\eta}A_2^1)r^2 + \tag{2.73}$$

$$+ x^{-\frac{1}{2}}\Big[B_3^1 I_{-\frac{1}{2}}(x) + B_4^1 I_{\frac{1}{2}}(x) -$$

$$- \frac{B_1^1}{3L^2} I_{-\frac{5}{2}}(x) - \frac{B_2^1}{3L^2} I_{\frac{5}{2}}(x)\Big]$$

$$\text{(iv)}\quad f_1^5(r) = \frac{-1}{L^2}(A_3^5 + \frac{10}{\eta}A_1^5)r^{-7} + \frac{1}{\eta}A_1^5 r^{-5} - \frac{1}{L^2}A_4^5 +$$

$$+ x^{-\frac{7}{2}}\Big[B_1^5 I_{-\frac{7}{2}}(x) + B_2^5 I_{\frac{7}{2}}(x)\Big] \tag{2.73}$$

$$\text{(v)} \quad f_2^5(r) = \frac{1}{5L^2}(A_3^5 + \frac{10}{\eta}A_1^5)r^{-5} - \frac{1}{L^2}A_5^5 r^{-3} - \frac{1}{L^2}(A_6^5 - \frac{2}{\eta}A_2^5) +$$

$$+ \frac{1}{5L^2}(A_4^5 + \frac{L^2}{\eta}A_2^5)r^2 + x^{-\frac{1}{2}}\Big[B_3^5 I_{-\frac{3}{2}}(x) +$$

$$+ B_4^5 I_{\frac{3}{2}}(x) - \frac{B_1^5}{5L^2} I_{-\frac{7}{2}}(x) - \frac{B_2^5}{5L^2} I_{\frac{7}{2}}(x)\Big]$$

$$\text{(vi)} \quad f_3^5(r) = \frac{1}{5L^2}(A_3^5 + \frac{10}{\eta}A_1^5)r^{-5} - \frac{1}{L^2}A_7^5 r^{-3} - \frac{1}{L^2}(A_8^5 - \frac{2}{\eta}A_2^5) +$$

$$+ \frac{1}{5L^2}(A_4^5 + \frac{L^2}{\eta}A_2^5)r^2 + x^{-\frac{1}{2}}\Big[B_3^5 I_{-\frac{3}{2}}(x) +$$

$$+ B_4^5 I_{\frac{3}{2}}(x) - \frac{B_1^5}{5L^2} I_{-\frac{7}{2}}(x) - \frac{B_2^5}{5L^2} I_{\frac{7}{2}}(x)\Big]$$

(2.73)

where $x = Lr$, $L^2 = \frac{\kappa\eta}{\gamma(\mu+\kappa)} = \frac{\kappa(2\mu+\kappa)}{\gamma(\mu+\kappa)}$

Application of the continuity condition, equation (1.4), to the functions $f_q^p(r)$, in equation (2.73), imposes the following restrictions on the coefficients A_j^m , B_j^m

(i) $A_1^0 = 0$

(ii) $A_4^0 = -\frac{1}{3}(A_6^5 + A_8^5)$

(iii) $\frac{1}{L^2}A_4^1 = \frac{1}{5\eta}A_2^1$

(2.74)

(iv) $\frac{1}{L^2} A_5^1 = -\frac{1}{\eta} A_1^1$

(v) $\frac{1}{L^2} A_4^5 = \frac{4}{21\eta} A_2^5$

(vi) $A_7^5 = -A_5^5$

(vii) $B_1^0 = -\frac{1}{3}(B_3^5 + B_5^5)$

(viii) $B_2^0 = -\frac{1}{3}(B_4^5 + B_6^5)$

(ix) $B_1^1 = -\frac{3L^2}{2} B_3^1$

(x) $B_2^1 = -\frac{3L^2}{2} B_4^1$

(xi) $B_1^5 = -\frac{5L^2}{3}(B_3^5 + B_5^5)$

(xii) $B_2^5 = -\frac{5L^2}{3}(B_3^5 + B_5^5)$

(2.74)

Hence, the general solutions for the velocity in a micropolar fluid flow are

$$\text{(i)}\quad f_1^0(r) = \frac{1}{5L^2}(A_3^5 + \frac{10}{\eta}A_1^5)r^{-5} - \frac{1}{L^2}A_3^0 r^{-3} - \frac{1}{3L^2}(\frac{4}{\eta}A_2^5 - \{A_6^5 + A_8^5\}) - \frac{2}{21\eta}A_2^5 r^2 - \frac{(B_3^5+B_5^5)}{3} x^{-\frac{1}{2}}\left[I_{-\frac{3}{2}}(x) - I_{-\frac{7}{2}}(x)\right] - \frac{(B_4^5+B_6^5)}{3} x^{-\frac{1}{2}}\left[I_{\frac{3}{2}}(x) - I_{\frac{7}{2}}(x)\right]$$

(ii) $$f_1^1(r) = \frac{-1}{L^2}(A_3^1 + \frac{6}{\eta}A_1^1)r^{-5} + \frac{1}{\eta}A_1^1 r^{-3} - \frac{1}{5\eta}A_2^1 - $$

$$- \frac{3L^2}{2} x^{-\frac{5}{2}} \left[B_3^1 I_{-\frac{5}{2}}(x) + B_4^1 I_{\frac{5}{2}}(x)\right]$$

(iii) $$f_2^1(r) = \frac{1}{3L^2}(A_3^1 + \frac{6}{\eta}A_1^1)r^{-5} + \frac{1}{\eta}A_1^1 r^{-1} + \frac{1}{L^2}(\frac{2}{\eta}A_2^1 - A_6^1) +$$

$$+ \frac{2}{5\eta}A_2^1 r^2 + x^{-\frac{1}{2}}B_3^1 \left[I_{-\frac{1}{2}}(x) + \tfrac{1}{2}I_{-\frac{5}{2}}(x)\right] +$$

$$+ x^{-\frac{1}{2}} B_4^1 \left[I_{\frac{1}{2}}(x) + \tfrac{1}{2}I_{\frac{5}{2}}(x)\right]$$

(iv) $$f_1^5(r) = \frac{-1}{L^2}(A_3^5 + \frac{10}{\eta}A_1^5)r^{-7} + \frac{1}{\eta}A_1^5 r^{-5} - \frac{4}{21\eta}A_2^5 -$$

$$- \frac{5L^2}{3} x^{-\frac{7}{2}} \left[(B_3^5 + B_5^5)I_{-\frac{7}{2}}(x) + (B_4^5 + B_6^5)I_{\frac{7}{2}}(x)\right]$$

(v) $$f_2^5(r) = \frac{1}{5L^2}(A_3^5 + \frac{10}{\eta}A_1^5)r^{-5} - \frac{1}{L^2}A_5^5 r^{-3} - \frac{1}{L^2}(A_6^5 - \frac{2}{\eta}A_2^5) +$$

$$+ \frac{5}{21\eta}A_2^5 r^2 + x^{-\frac{3}{2}}\left[B_3^5 I_{-\frac{3}{2}}(x) + \tfrac{1}{3}(B_3^5 + B_5^5)I_{-\frac{7}{2}}(x) +\right.$$

$$\left. + B_4^5 I_{\frac{3}{2}}(x) + \tfrac{1}{3}(B_4^5 + B_6^5)\ I_{\frac{7}{2}}(x)\right]$$

(vi) $$f_3^5(r) = \frac{1}{5L^2}(A_3^5 + \frac{10}{\eta}A_1^5)r^{-5} + \frac{1}{L^2}A_5^5 r^{-3} - \frac{1}{L^2}(A_8^5 - \frac{2}{\eta}A_2^5) +$$

$$+ \frac{5}{21\eta}A_2^5 r^2 + x^{-\frac{3}{2}}\left[B_5^5 I_{-\frac{3}{2}}(x) + \tfrac{1}{3}(B_3^5 + B_5^5)I_{-\frac{7}{2}}(x) +\right.$$

$$\left. + B_6^5 I_{\frac{3}{2}}(x) + \tfrac{1}{3}(B_4^5 + B_6^5)\ I_{\frac{7}{2}}(x)\right] \qquad (2.75)$$

Section 5. Discussion.

This chapter has presented a new form for the general solution to the steady creeping motion equations for a viscous Newtonian fluid. This general form can be applied to a great variety of examples in creeping motion because of the general nature of the tensors a_{ij} and A_{ijk}.

This form for the general solution displays advantages over the other methods of determining solutions for particular problems. When a comparison to Lamb's general solution is made, it is found that, even though Lamb's solution can solve a great many more problems, the new form presented in this chapter is much easier to apply because so few coefficients are left to determine once the boundary conditions are written down and this new form also indicates which powers of r can be combined for any particular type of flow problem. The singularity method is as easy to apply as this new form of solution once the singularities have been determined. The requisite singularities are, however, normally very difficult to determine for a particular problem; it is basically a trial and error method, which makes the new form in this chapter more easily applicable. The stream function technique, as found in section 3, is no more difficult to use than the new form found in this chapter, but its chief disadvantages are that it is useful only for strictly two-dimensional flow or for three-dimensional flows exhibiting some sort of symmetry and that the function p(r) is still not known explicitly once the stream function is known.

This new form can also be used for problems other than the steady creeping flow of a viscous Newtonian fluid. Some of the other areas where this new form can be used include the quasi-steady and unsteady creeping motion of a viscous Newtonian fluid, when the time dependence is separable, and for the steady, quasi-steady and unsteady creeping motion of a micropolar fluid. In fact, Lamb's solutions and the singularity method are not applicable to micropolar fluid flows and the stream function technique has the same limitations for micropolar fluids as for viscous Newtonian fluids. This form for the general solution is also ideally suited to problems concerning a fluid to fluid interface and for Stokes' solutions in perturbation theory. Other areas for application might also arise from the fact that this general solution is valid in an n-dimensional Euclidean space where $n \leq 3$.

CHAPTER III

A FLUID SPHERE IN A UNIFORM FLOW FIELD

Section 1. Introduction.

In recent years considerable attention has been given to fluid mechanical theories in which the couple stress, in addition to the normal Cauchy stress, and the spin of the fluid particle, over and above the usual velocity vector, play a significant role. These fluids have been suggested to describe the complex behaviour of such materials as liquid crystals, fluid suspensions and the flow of blood. Eringen (1966) has introduced a theory for such fluids, which he calls micropolar fluids, in which the fluid can support stress and body couples and possess a rotation field which is independent of the velocity field. The theory, thus, has two independent kinematical variables: the velocity vector u_i and the spin or microrotation vector ν_i. The linear constitutive equation for the stress contains an additional material coefficient, which describes the coupling between u_i and ν_i. The linear equation for the couple stress contains an additional three viscosity coefficients. These equations along with constraints on the viscosity coefficients and the field equations have already been detailed in chapter I section 3.

The standard solution for the flow of a viscous fluid past a viscous fluid drop can be found in Happel and Brenner (1965). Because of uncertainty of the appropriate relation between the microrotation and the vorticity at the interface of the two fluids a parameter S has been introduced which turns out to have a significant effect on the drag along with the viscosity ratio. The viscous fluid drop in a viscous fluid was first considered by Rybczynski (1911) and Hadamard (1911). Ramkissoon and Majumdar (1976) calculated the drag for the uniform flow of a micropolar fluid past a solid sphere and it is found that the solution presented in section 4 reduces to their solution if the viscous fluid sphere is assumed to tend to a solid sphere.

In section 2 the basic equations along with their solutions in terms of the stream function will be presented. The stream function will be used because this is one of the few micropolar fluid flow problems for which it will work. The solutions will be for both viscous Newtonian and micropolar fluids.

In section 3 the flow of a viscous Newtonian fluid past a micropolar fluid drop will be discussed. The complete solution in terms of velocity and microrotation components, the stream function, the stress and the drag will be computed. Additionally, a comparison of the drag for this problem will be made to the drag for the flow of a viscous fluid past a viscous fluid drop.

In sections 4 and 5 the identical procedure will be carried out as far as possible. The flows in these two sections, however, will be the flow of a micropolar fluid past a viscous Newtonian fluid drop and the flow of a micropolar fluid past a micropolar fluid drop.

Section 6 considers the problem of the uniform flow of a micropolar fluid past a viscous Newtonian fluid sphere and the problem of the uniform flow of a viscous Newtonian fluid past a micropolar fluid sphere. The method used is that developed in Chapter II.

Section 2. Basic equations and their solutions.

The basic equations, neglecting thermal effects for an incompressible steady micropolar fluid where it is assumed that there are no body forces or body couples and that inertial effects are negligible were presented in section 3 of chapter I. They will be given again here along with the boundary conditions and other simplifying assumptions.

The constitutive equations for the stress tensor $t_{k\ell}$ and the couple stress tensor $m_{k\ell}$ are

$$t_{k\ell} = (-p + \lambda u_{r,r})\delta_{k\ell} + \mu(u_{k,\ell} + u_{\ell,k}) + + \kappa(u_{\ell,k} - \varepsilon_{k\ell r}\nu_r) \qquad (3.1)$$

$$m_{k\ell} = \alpha\nu_{r,r}\delta_{k\ell} + \beta\nu_{k,\ell} + \gamma\nu_{\ell,k} \qquad (3.2)$$

The restrictions on the viscosity coefficients necessary to maintain non-negative energy dissipation are

$$(3\lambda + 2\mu + \kappa) \geq 0 \;,\; 2\mu + \kappa \geq 0 \;,\; \kappa \geq 0$$
$$(3\alpha + \beta + \gamma) \geq 0 \;,\; -\gamma \leq \beta \leq \gamma \;,\; \gamma \geq 0 \qquad (3.3)$$

The field equations under the assumptions outlined above are from chapter I, section 3 given by

$$\nabla^2 p = 0 \qquad (3.4)$$

$$(\nabla^2 - L^2)\nabla^2 \underset{\sim}{u} = \frac{-2\kappa}{\gamma(\mu+\kappa)} \nabla p \qquad (3.5)$$

$$(\nabla^2 - \frac{2\kappa}{\delta})\phi = 0 \qquad (3.6)$$

where the following must also hold

$$\nabla \cdot \underset{\sim}{u} = 0 \tag{3.7}$$

$$\underset{\sim}{\nu} = \left[1 + \frac{\gamma(\mu+\kappa)}{\kappa^2} \nabla^2\right] \left(\nabla \times \frac{\underset{\sim}{u}}{2}\right) + \frac{\delta}{2\kappa} \nabla\phi \tag{3.8}$$

In the above equations, the various quantities have the following meanings: $\underset{\sim}{u}$ is the velocity vector, $\underset{\sim}{\nu}$ is the microrotation vector, p is the thermodynamic pressure, $\alpha, \beta, \gamma, \kappa, \lambda, \mu$ are constant viscosity coefficients, $t_{k\ell}$ is the Cauchy stress tensor, $m_{k\ell}$ is the couple stress tensor, $\delta_{k\ell}$ is the identity tensor, $\varepsilon_{k\ell r}$ is the alternating tensor, $\delta = \alpha+\beta+\gamma$, $L^2 = \frac{\kappa(2\mu+\kappa)}{\gamma(\mu+\kappa)}$.

The boundary conditions for the various problems to be considered are

$$
\begin{aligned}
&\text{(i)} \quad \underset{\sim}{u}^{(e)}_{\infty} = U\hat{e}_z \ , \ \underset{\sim}{\nu}^{(e)} \text{ finite} \qquad \text{as } r \to \infty \\
&\text{(ii)} \quad \underset{\sim}{u}^{(i)} \ , \ \underset{\sim}{\nu}^{(i)} \text{ finite} \qquad @\ r = 0 \\
&\text{(iii)} \quad \underset{\sim}{u}^{(e)} \cdot \underset{\sim}{r} = \underset{\sim}{u}^{(i)} \cdot \underset{\sim}{r} = 0 \qquad @\ r = a \\
&\text{(iv)} \quad \underset{\sim}{u}^{(e)} - (\underset{\sim}{u}^{(e)} \cdot \underset{\sim}{r})\underset{\sim}{r} = \underset{\sim}{u}^{(i)} - (\underset{\sim}{u}^{(i)} \cdot \underset{\sim}{r})\underset{\sim}{r} \qquad @\ r = a \\
&\text{(v)} \quad t^{(e)}_{ij}X_j - t^{(e)}_{k\ell}X_kX_\ell X_i = t^{(i)}_{ij}X_j - t^{(i)}_{k\ell}X_kX_\ell X_i \qquad @\ r = a \\
&\text{(vi)} \quad \underset{\sim}{\nu} = S\left(\nabla \times \frac{\underset{\sim}{u}}{2}\right) \ , \ S \in [0,1] \qquad @\ r = a
\end{aligned}
\tag{3.9}
$$

Since the flow far upstream from the body is uniform, it is possible to introduce the Stokes' stream function as shown in chapter II section 3. That is, if

$$\text{(i)} \quad u_r = \frac{1}{r^2 \sin\theta} \frac{\partial \Psi}{\partial \theta}$$

$$\text{(ii)} \quad u_\theta = \frac{-1}{r \sin\theta} \frac{\partial \Psi}{\partial r} \qquad (3.10)$$

$$\text{(iii)} \quad \nabla \times \frac{\underset{\sim}{u}}{2} = \omega_\phi \hat{e}_\phi = \left(\frac{1}{2r \sin\theta} E^2 \Psi\right) \hat{e}_\phi$$

$$\text{(iv)} \quad E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\right)$$

From the boundary conditions (3.9) and the field equations (3.4) - (3.8) it is easy to deduce that $\nabla . \underset{\sim}{\nu} = 0$. Hence it is necessary in (3.6) and (3.8) to choose

$$\phi = 0 \qquad (3.11)$$

Also equation (3.7) is satisfied identically and taking the curl of (3.5) the following is obtained

$$(E^2 - L^2) E^4 \Psi = 0 \qquad (3.12)$$

and using (3.11) and (3.10) in (3.8) yields

$$\nu\phi = \frac{1}{2r \sin\theta} \left[1 + \frac{\gamma(\mu+\kappa)}{\kappa^2} E^2\right] E^2 \Psi \qquad (3.13)$$

Since the operator in (3.12) is linear and commutative, it is convenient to write

$$\Psi = \Psi_1 + \Psi_2 \qquad (3.14)$$

where

$$E^4 \Psi_1 = 0 \qquad (3.15)$$

$$(E^2 - L^2) \Psi_2 = 0 \qquad (3.16)$$

Following Happel & Brenner (1973) it is assumed

$$\Psi = \left[f_1(r) + f_2(r)\right] \sin^2\theta \tag{3.17}$$

With this form for Ψ the differential equations to be solved are

$$\begin{aligned} &\text{(i)} \quad \left[\frac{d^2}{dr^2} - \frac{2}{r^2}\right]^2 f_1(r) = 0 \\ &\text{(ii)} \quad \left[\frac{d^2}{dr^2} - \left(\frac{2}{r^2} + L^2\right)\right] f_2(r) = 0 \end{aligned} \tag{3.18}$$

The solutions to (3.18) are

$$\begin{aligned} &\text{(i)} \quad f_1(r) = (A_1 r^{-1} + B_1 r + C_1 r^2 + D_1 r^4) \\ &\text{(ii)} \quad f_2(r) = (Lr)^{\frac{1}{2}} \left[E_1 I_{\frac{3}{2}}(Lr) + F_1 I_{-\frac{3}{2}}(Lr)\right] \end{aligned} \tag{3.19}$$

However, since

$$I_{\frac{3}{2}}(Lr) = \sqrt{\frac{2}{\pi Lr}} \left[\cosh Lr - (Lr)^{-1} \sinh Lr\right]$$

$$I_{-\frac{3}{2}}(Lr) = \sqrt{\frac{2}{\pi Lr}} \left[\sinh Lr - (Lr)^{-1} \cosh Lr\right]$$

the solution $f_2(r)$ can be rewritten

$$\text{(i)} \quad f_2(r) = \sqrt{\frac{2}{\pi}} \left[\left(E_1 - \frac{F_1}{Lr}\right) \cosh Lr + \left(F_1 - \frac{E_1}{Lr}\right) \sinh Lr\right]$$

or

$$\text{(ii)} \quad f_2(r) = \left[E_2\left(1 - \frac{1}{Lr}\right)e^{Lr} + F_2\left(1 + \frac{1}{Lr}\right)e^{-Lr}\right] \tag{3.20}$$

Hence the expressions for the stream function, the velocity components, the microrotation vector and the tangential Cauchy stress are

$$\Psi = \left[A_1 r^{-1} + B_1 r + C_1 r^2 + D_1 r^4 + E_2 \left(\frac{Lr-1}{Lr}\right) e^{Lr} + \right.$$

$$\left. + F_2 \left(\frac{Lr+1}{Lr}\right) e^{-Lr} \right] \sin^2\theta$$

$$u_r = \left[2A_1 r^{-3} + 2B_1 r^{-1} + 2C_1 + 2D_1 r^2 + 2L^2 E_2 \left(\frac{Lr-1}{L^3 r^3}\right) e^{Lr} + \right.$$

$$\left. + 2L^2 F_2 \left(\frac{Lr+1}{L^3 r^3}\right) e^{-Lr} \right] \cos\theta$$

$$u_\theta = \left[A_1 r^{-3} - B_1 r^{-1} - 2C_1 - 4D_1 r^2 - L^2 E_2 \left(\frac{L^2 r^2 - Lr + 1}{L^3 r^3}\right) e^{Lr} + \right.$$

$$\left. + L^2 F_2 \left(\frac{L^2 r^2 + Lr + 1}{L^3 r^3}\right) e^{-Lr} \right] \sin\theta$$

$$\nu_\phi = \left[-B_1 r^{-2} + 5D_1 r + \frac{\mu+\kappa}{\kappa} L^2 E_2 \left(\frac{Lr-1}{Lr^2}\right) e^{Lr} + \right.$$

$$\left. + \frac{\mu+\kappa}{\kappa} L^2 F_2 \left(\frac{Lr+1}{Lr^2}\right) e^{-Lr} \right] \sin\theta$$

$$t_{r\theta} = -(2\mu+\kappa) \left[3A_1 r^{-4} + 3D_1 r + L^3 E_2 \left(\frac{-L^2 r^2 + 3Lr - 3}{L^4 r^4}\right) e^{Lr} + \right.$$

$$\left. + L^3 F_2 \left(\frac{L^2 r^2 + 3Lr + 3}{L^4 r^4}\right) e^{-Lr} \right] \sin\theta$$

(3.21)

From Happel & Brenner (1973) the solutions, in terms of the stream function, for a viscous Newtonian Fluid are

$$\Psi = \left[A_2 r^{-1} + B_2 r + C_2 r^2 + D_2 r^4\right] \sin^2\theta$$

$$u_r = \left[2A_2 r^{-3} + 2B_2 r^{-1} + 2C_2 + 2D_2 r^2\right] \cos\theta \qquad (3.22)$$

$$u_\theta = \left[A_2 r^{-3} - B_2 r^{-1} - 2C_2 - 4D_2 r^2\right] \sin\theta$$

$$\omega_\phi = \left[-B_2 r^{-2} + 5D_2 r\right] \sin\theta$$

$$t_{r\theta} = -2\mu \left[3A_2 r^{-4} + 3D_2 r\right] \sin\theta \qquad (3.22)$$

Finally, the formula for calculating the drag on the sphere is given by

$$\bar{D} = 2\pi a^2 \int_0^\pi \left(t_{rr}\cos\theta - t_{rr}\sin\theta\right)\Big|_{r=a} \sin\theta \, d\theta \qquad (3.23)$$

Section 3. Micropolar drop in a viscous fluid.

In this section the uniform flow of a viscous Newtonian fluid past a micropolar fluid drop will be considered. The boundary conditions (3.9) once the stream function has been assumed to exist become

$$\begin{aligned}
&\text{(i)} \quad \Psi^{(e)} \to \frac{U}{2} r^2 \sin^2\theta \qquad \text{as } r \to \infty \\
&\text{(ii)} \quad \Psi^{i} \text{ is finite} \qquad @\, r = 0 \\
&\text{(iii)} \quad u_r^{(e)} = u_r^{(i)} = 0 \qquad @\, r = a \\
&\text{(iv)} \quad u_\theta^{(e)} = u_\theta^{(i)} \qquad @\, r = a \\
&\text{(v)} \quad t_{r\theta}^{(e)} = t_{r\theta}^{(i)} \qquad @\, r = a \\
&\text{(vi)} \quad \nu_\phi^{(i)} = S\omega_\phi^{(e)} \qquad @\, r = a
\end{aligned} \tag{3.24}$$

From equations (3.22) the exterior viscous flow gives

$$\begin{aligned}
\Psi(e) &= \left[A_2 r^{-1} + B_2 r + \frac{U}{2} r^2\right] \sin^2\theta \\
u_r^{(e)} &= \left[2A_2 r^{-3} + 2B_2 r^{-1} + U\right] \cos\theta \\
u_\theta^{(e)} &= \left[A_2 r^{-3} - B_2 r^{-1} - U\right] \sin\theta \\
\omega_\phi^{(e)} &= -B_2 r^{-2} \sin\theta \\
t_{r\theta}^{(e)} &= -6\mu^e A_2 r^{-4} \sin\theta
\end{aligned} \tag{3.25}$$

In a similar fashion from (3.21) for the interior micropolar flow

$$\psi^{(i)} = \left[C_1 r^2 + D_1 r^4 + 2E_2\left(\cosh Lr - \frac{\sinh Lr}{Lr}\right)\right]\sin^2\theta$$

$$u_r^{(i)} = \left[2C_1 + 2D_1 r^2 + \frac{4E_2}{r^2}\left(\cosh Lr - \frac{\sinh Lr}{Lr}\right)\right]\cos\theta$$

$$u_\theta^{(i)} = \left[-2C_1 - 4D_1 r^2 + \frac{2E_2}{r^2}\left(\cosh Lr - \frac{(1+L^2r^2)}{Lr}\sinh Lr\right)\right]\sin\theta \qquad (3.26)$$

$$\nu_\theta^{(i)} = \left[5D_1 r + \frac{2(\mu+\kappa)}{\kappa}\frac{L^2E_2}{r}\left(\cosh Lr - \frac{\sinh Lr}{Lr}\right)\right]\sin\theta$$

$$t_{r\theta}^{(i)} = -(2\mu+\kappa)\left[3D_1 r + 2E_2\left(\frac{3}{r^3}\cosh Lr - \left(\frac{3+L^2r^2}{r^3}\right)\frac{\sinh Lr}{Lr}\right)\right]\sin\theta$$

From the velocity and tangential stress boundary conditions in (3.24) come the following

$$a^{-1}A_2 + aB_2 = -\frac{Ua^2}{2}$$

$$C_1 + a^2D_1 = -2a^{-2}E_2\left[\cosh La - \frac{\sinh La}{La}\right]$$

$$a^{-1}A_2 - aB_2 + 2a^2C_1 + 4a^4D_1 =$$

$$Ua^2 + 2E_2\left[\cosh La - (1+L^2a^7)\frac{\sinh La}{La}\right]$$

$$\frac{2\mu_e}{2\mu_i+\kappa_i}\ 3a^{-4}A_2 - 3aD_1 = 2a^{-3}E_2\left[3\cosh La - (3+L^7a^2)\frac{\sinh La}{La}\right]$$

The final boundary condition in (3.24) gives

$$SB_2 + 5a^3D_1 + \frac{2(\mu+\kappa)}{\kappa}L^2E_2\left[a\cosh La - \frac{\sinh La}{L}\right] = 0$$

The solution to this system of algebraic equations with the changes $M = a \cosh La$, $N = \frac{\sinh La}{L}$, $\sigma_1 = \frac{2\mu_e}{2\mu_i + \kappa_i}$ is

$$A_2 = \frac{Ua^3}{4(1+\sigma_1)} + \frac{5}{2(1+\sigma_1)} \{(M-N) - \frac{L^2a^2}{3} N\} \bar{E}_2$$

$$a^2B_2 = -\frac{(3+2\sigma_1)Ua^3}{4(1+\sigma_1)} - \frac{5}{2(1+\sigma_1)} \{(M-N) - \frac{L\ a}{3} N\} \bar{E}_2$$

$$a^3C_1 = \frac{-\sigma_1 Ua^3}{4(1+\sigma_1)} - \frac{1}{2(1+\sigma_1)} \{5\sigma_1(M-N) - \frac{(3\sigma_1-2)}{3} L^2a^2N\} \bar{E}_2$$

$$a^5D_1 = \frac{\sigma_1 Ua^3}{4(1+\sigma_1)} + \frac{3\sigma_1-2}{2(1+\sigma_1)} \{(M-N) - \frac{L^2a^2}{3} N\} \bar{E}_2$$

$$\bar{E}_2 = 2E_2 = \tag{3.27}$$

$$= \frac{3\kappa\left[3S + 2(S-5)\sigma_1\right]L}{5\kappa\left[3\sigma_1 - (2+S)\right]\{3(M-N) - L^2a^2N\} + \left[6(1+\sigma_1)(\mu+\kappa)(M-N)L^2a^2\right]}$$

When the solutions (3.27) are compared with the solutions (2.23) in chapter II section 3 it is seen that the first term in each of A_2, a^2B_2, a^3C_1, a^5D_1 are essentially the same as the solutions given there. The other terms related to $\bar{E}_2$ are due to the presence of the spin in the micropolar fluid. As is to be expected, these terms systematically disappear in the limit as κ and/or γ tend to zero. Because of the axisymmetrical nature of the flow assumed and because of the similar dependence on the θ co-ordinate terms in both the classical (viscous fluid drop in a viscous fluid) and in the present case, the overall pattern of the stream-lines appears similar in both cases. In particular, the circulation within the droplet, as is observed in the classical case, is also predicted in the present situation. The difference in the present case is, however, that stream-lines are displaced towards or away from the origin depending on the magnitudes of S and σ_1. Similar remarks apply for the stream-lines outside of the sphere.

The drag can be calculated using (3.23) or if it is calculated for the undetermined coefficients it is found that

$$\bar{D} = 4\pi\varepsilon B \tag{3.28}$$

where $\varepsilon = 2\mu^e$ or $\varepsilon = 2\mu^e + \kappa^e$ depending on whether the flow is viscous Newtonian or micropolar. For the classical case of a viscous drop in a viscous fluid

$$D_1 = 4\pi(2\mu^e)\left[\frac{-(3+2\sigma)}{1+\sigma}Ua\right] = \frac{-6\pi\mu^e(1+\frac{2}{3}\sigma)Ua}{1+\sigma}\,, \tag{3.29}$$

where $\sigma = \dfrac{\mu^e}{\mu^I}$

For the case of a micropolar fluid drop in a viscous Newtonian fluid, the calculated drag is

$$\bar{D} = 4\pi(2\mu^e)\left[\frac{-(3+2\sigma_1)Ua}{4(1+\sigma_1)} - \frac{5}{2(1+\sigma_1)}\{M-N-\frac{L^2a^2}{3}N\}\frac{\bar{E}_2}{a^2}\right]$$

When the value for $\bar{E}_2$ in (3.27) is substituted and the resulting expression is appropriately regrouped the expression for the drag becomes

$$\bar{D} = -\left[\frac{6\pi\mu^{e}(1+\frac{2}{3}\sigma_1)Ua}{(1+\sigma_1)}\right] X$$

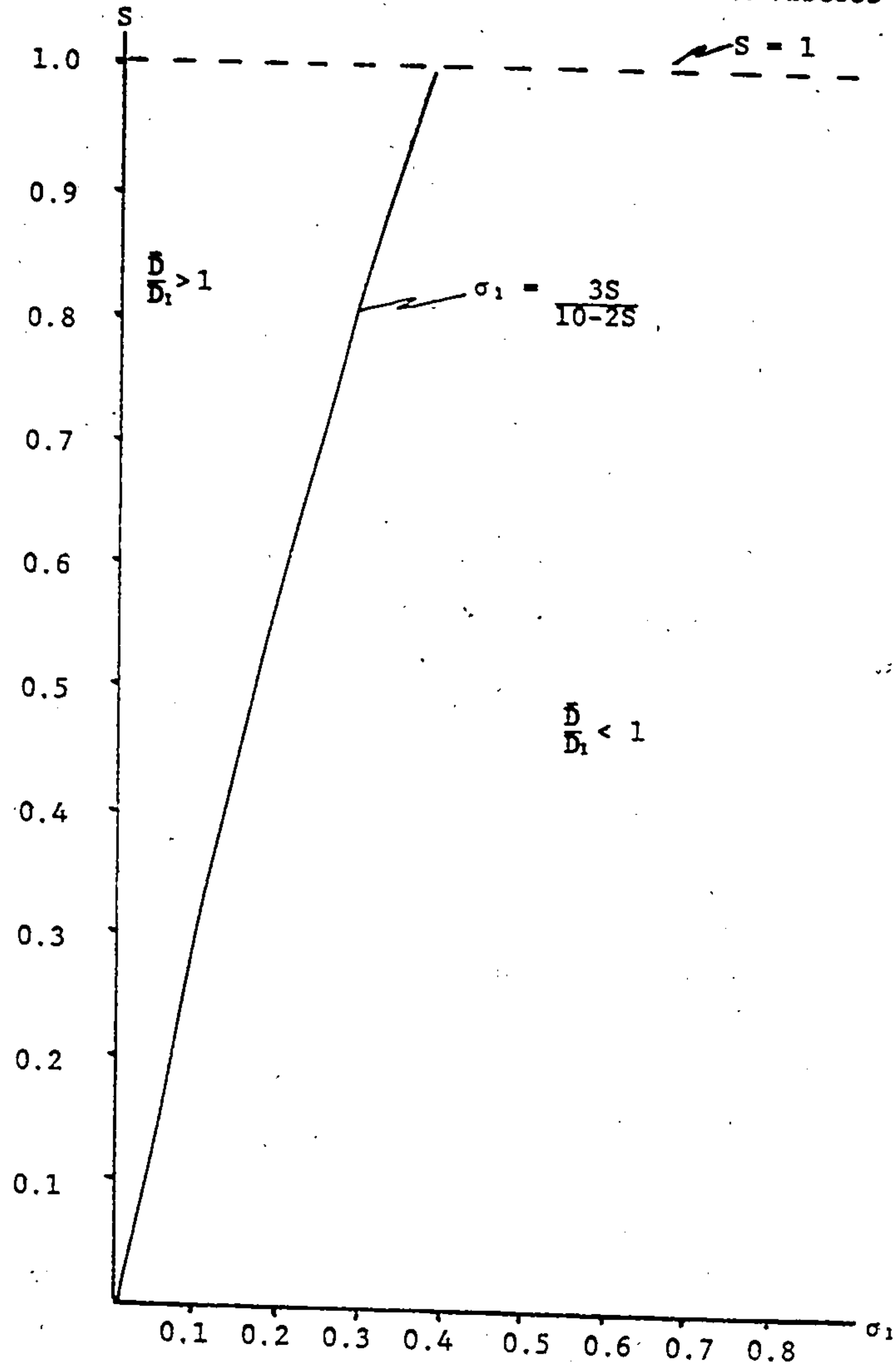
$$X\left[\frac{5\kappa^{i}\left[\frac{(3\sigma_1+2)(2\sigma_1-3)}{2\sigma_1+3}\right]\{3(M-N)-L^2a^2N\}+6(1+\sigma_1)(\mu^{i}+\kappa^{i})L^2a^2(M-N)}{5\kappa^{i}\left[3\sigma_1-(2+S)\right]\{3(M-N)-L^2a^2N\} + 6(1+\sigma_1)(\mu^{i}+\kappa^{i})L^2a^2(M-N)}\right]$$

or since $\sigma_1 = \frac{2\mu^{e}}{2\mu^{i}+\kappa^{i}} \doteq \sigma = \frac{\mu^{e}}{\mu^{i}}$ it is possible to write

$$\frac{\bar{D}}{\bar{D}_1} = \left[\frac{5\kappa^{i}\left[\frac{(3\sigma_1+2)(2\sigma_1-3)}{2\sigma_1+3}\right]\{3(M-N)-L^2a^2N\}+6(1+\sigma_1)(\mu^{i}+\kappa^{i})L^2a^2(M-N)}{5\kappa^{i}\left[3\sigma_1-(2+S)\right]\{3(M-N)-L^2a^2N\}+6(1+\sigma_1)(\mu^{i}+\kappa^{i})L^2a^2(M-N)}\right] \tag{3.30}$$

A graphical representation of this relationship is given on the next page. From equation (3.30) it is easy to notice that $\frac{\bar{D}}{\bar{D}_1}$, apart from other quantities, depends significantly upon the values of σ_1 and S. It is seen that for large values of σ_1 i.e. when $\sigma_1 \geq 0.375$ and for all $S \in [0,1]$, $\frac{\bar{D}}{\bar{D}_1} < 1$. However, when $\sigma_1 < 0.375$ there is a small region where $\frac{\bar{D}}{\bar{D}_1} > 1$ is possible for different values of S. The obvious conclusion reached here is that when fluid inside of the sphere is a micropolar fluid, the drag on the sphere will, in general, be smaller that when the fluid inside of the sphere is a viscous Newtonian fluid. However, it is possible to have an increase in drag through the appropriate choices of σ_1 and S.

DRAG COMPARISON FOR A MICROPOLAR SPHERE IN A VISCOUS FLUID.



On the next few pages a number of tables are given to show how variations in the parameters, which arise for micropolar fluid flows, affect the angle of intersection between various streamlines and lines of constant radius. In all of the tables that follow

$\kappa \varepsilon [0.1\eta, 10\eta]$	is the coupling viscosity
$\ell \varepsilon [0.1, 10]$	is the characteristic length
$s \varepsilon [0, 1]$	is the parameter appearing in the compromise boundary condition
$\sigma \varepsilon [0.1, 10]$	is the ratio of external to internal viscosity
$H = \frac{4(1+\sigma)}{\sigma}\psi$	can be determined from (2.21) and (2.23) (Viscous) and (3.26) and (3.27) (Micropolar)

TABLE 1. Changes in the value of theta for variable s.

$H = -0.125 \quad \kappa = 0.1 \quad \ell = 0.1 \quad \sigma = 4.6$

r \ s	0.1	0.4	0.7	1.0	Viscous
.4	46.7	47.8	49.0	50.3	74.7
.45	43.8	44.7	45.6	46.7	61.6
.5	41.4	42.1	42.9	43.8	54.7
.55	39.6	40.2	40.9	41.6	50.3
.6	38.3	38.8	39.4	40.0	47.4
.65	37.6	38.1	38.6	39.2	45.7
.7	37.5	37.9	38.4	39.0	45.0
.75	38.2	38.6	39.1	39.6	45.5
.8	40.0	40.5	41.0	41.5	47.4
.85	43.8	44.4	44.9	45.5	52.1
.9	52.6	53.3	54.0	54.8	64.3

TABLE 2. Changes in the value of theta for variable σ

$H = -0.125$ $\kappa = 1.0$ $\ell = 0.1$ $s = 1.0$

r \ σ	0.1	0.7	4.6	10.0	Viscous
.4	40.5	83.8	54.1	49.7	74.7
.45	36.1	65.3	47.8	44.3	61.6
.5	33.0	57.7	43.6	40.6	54.7
.55	30.8	52.9	40.8	37.9	50.3
.6	29.2	49.9	38.3	36.2	47.4
.65	28.1	48.2	37.7	35.2	45.7
.7	27.6	47.6	37.4	34.9	45.0
.75	27.6	48.3	38.0	35.4	45.5
.8	28.4	50.8	39.8	37.1	47.4
.85	30.3	56.6	43.8	40.6	52.1
.9	34.8	73.3	52.8	48.6	64.3

TABLE 3. Changes in the value of theta for variable κ

$H = -0.125 \qquad \ell = 0.1 \qquad s = 1.0 \qquad \sigma = 0.4$

$r \setminus \kappa$	0.1	1.0	10.0	Viscous
.4	72.8	76.5	77.0	74.7
.45	60.9	62.5	62.8	61.6
.5	54.4	55.4	55.6	54.7
.55	50.1	50.9	51.1	50.3
.6	47.3	48.0	48.2	47.4
.65	45.7	46.3	46.4	45.7
.7	45.0	45.6	45.7	45.0
.75	45.5	46.1	46.2	45.5
.8	47.5	48.1	48.3	47.8
.85	52.2	53.0	53.3	52.1
.9	64.5	65.8	66.3	64.3

TABLE 4. Changes in the value of theta for variable ℓ

$H = -0.125$ $\kappa = 0.1$ $s = 1.0$ $\sigma = 0.4$

$r \backslash \ell$	0.1	1.0	10.0	Viscous
.4	72.8	2.9	0.0	74.7
.45	60.9	3.2	0.0	61.6
.5	54.4	3.6	0.0	54.7
.55	50.1	4.0	0.0	50.3
.6	47.3	4.4	0.0	47.4
.65	45.7	4.8	0.0	45.7
.7	45.0	5.4	0.0	45.0
.75	45.5	6.1	0.0	45.5
.8	47.5	7.1	0.1	47.4
.85	52.2	8.4	0.1	52.1
.9	64.5	10.5	0.1	64.3

Section 4. Viscous drop in a micropolar fluid.

This section will deal with the uniform flow of a micropolar fluid past a viscous Newtonian fluid drop. A similar investigation for a solid sphere has been carried out by Ramkissoon & Majumdar (1976). The value determined for the drag in this section will be compared to the drag for the flow of a micropolar fluid past a solid sphere. The boundary conditions will be identical to those given in (3.24). The values for the stream function, velocity components, vorticity, microrotation and tangential stress for both the internal and external flows differ from those given by (3.25) and (3.26). For this problem, they are given by

From equations (3.21) for the exterior micropolar fluid flow the following hold

$$\psi^{(e)} = \left[A_1 r^{-1} + B_1 r + \frac{U}{2} r^2 + F_2 \left(\frac{Lr+1}{Lr}\right) e^{-Lr}\right] \sin^2\theta$$

$$u_r^{(e)} = \left[2A_1 r^{-3} + 2B_1 r^{-1} + U + 2L^2 F_2 \left(\frac{1+Lr}{L^3 r^3}\right) e^{-Lr}\right] \cos\theta$$

$$u_\theta^{(e)} = \left[A_1 r^{-3} - B_1 r^{-1} - U + L^2 F_2 \left(\frac{1+Lr+L^2 r^2}{L^3 r^3}\right) e^{-Lr}\right] \sin\theta$$

$$\nu_\phi^{(e)} = \left[-B_1 r^{-2} + \frac{\mu^e+\kappa^e}{\kappa^e} L^2 F_2 \left(\frac{1+Lr}{Lr^2}\right) e^{-Lr}\right] \sin\theta$$

$$t_{r\theta}^{(e)} = -(2\mu^e+\kappa^e)\left[3A_1 r^{-4} + L^3 F_2 \left(\frac{3+3Lr+L^2 r^2}{L^4 r^4}\right) e^{-Lr}\right] \sin\theta \qquad (3.31)$$

For the flow of the interior viscous fluid (3.22) gives

$$\begin{aligned}
\psi^{(i)} &= \left[C_2 r^2 + D_2 r^4\right] \sin^2\theta \\
u_r^{(i)} &= \left[2C_2 + 2D_2 r^2\right] \cos\theta \\
u_\theta^{(i)} &= \left[-2C_2 - 4D_2 r^2\right] \sin\theta \qquad (3.32) \\
\omega_\phi^{(i)} &= 5D_2 r \sin\theta \\
t_{r\theta}^{(i)} &= -(2\mu^i) \left[3D_2 r\right] \sin\theta
\end{aligned}$$

From the boundary conditions for the velocity and tangential stress in (3.24) come the following

$$C_2 + a^2 D_2 = 0$$

$$A_1 + a^2 B_1 = \frac{-Ua^3}{2} - F_2\left(\frac{1+La}{L}\right)e^{-La}$$

$$A_1 - a^2 B_1 + 2a^3 C_2 + 4a^5 D_2 = Ua^3 - F_2 \left(\frac{1+La+L^2a^2}{L}\right)e^{-La}$$

$$3A_1 - \frac{2\mu^i}{2\mu^e+\kappa^e} 3a^5 D_2 = -F_2 \left(\frac{3+3La+L^2a^2}{L}\right) e^{-La}$$

The final boundary condition of (3.24) now gives

$$B_1 + 5a^3 5D_2 = \frac{u^e+\kappa^e}{\kappa^e} L^2 F_2 \left(\frac{1+La}{L}\right) e^{-La}$$

The solution of this system of equations produce the following values for the coefficients

$$A_1 = \frac{Ua^3}{4(1+\sigma_2)} - \left[1+La+ \frac{3+2\sigma_2}{6(1+\sigma_2)} L^2 a^2\right] \frac{F_2}{Le^{La}} \qquad (3.33)$$

$$a^2 B_1 = \frac{-(3+2\sigma_2)Ua^3}{4(1+\sigma_2)} + \frac{3+2\sigma_2}{6(1+\sigma_2)} L^2 a^2 \frac{F_2}{Le^{La}}$$

$$a^3 C_2 = \frac{-\sigma_2 Ua^3}{4(1+\sigma_2)} + \frac{\sigma_2}{6(1+\sigma_2)} L^2 a^2 \frac{F_2}{Le^{La}}$$

$$a^5 D_2 = \frac{\sigma_2 Ua^3}{4(1+\sigma_2)} - \frac{\sigma_2}{6(1+\sigma_2)} L^2 a^2 \frac{F_2}{Le^{La}}$$

$$L^2 a^2 \frac{F_2}{Le^{La}} = \frac{3\kappa^e\{3+(2-5s)\sigma_2\}}{\{3+(2-5s)\sigma_2\}\kappa^e - 6(1+\sigma_2)(\mu^e+\kappa^e)(1+La)} \frac{Ua^3}{2} \tag{3.33}$$

In the present case, as in the previous example, the general features of the stream-lines, both inside and outside the drop, are similar to those observed in the classical case. The presence of the micropolar fluid outside the drop changes the streamlines inside the drop slightly. However, circulation inside the droplet is still predicted. When $\kappa \to 0$, the above results reduce to the classical case as given in chapter II section 3, while when $\sigma_2 \to 0$ (i.e. when the sphere is considered solid) the above expressions reduce to those given by Ramkissoon and Majumdar (1976).

The drag is again calculated by using equation (3.28), which after some simplification is

$$\bar{D} = \frac{-6\pi(\mu^e + \frac{\kappa^e}{2})(1+\frac{2}{3}\sigma_2)\, Ua}{1+\sigma_2} \quad X$$

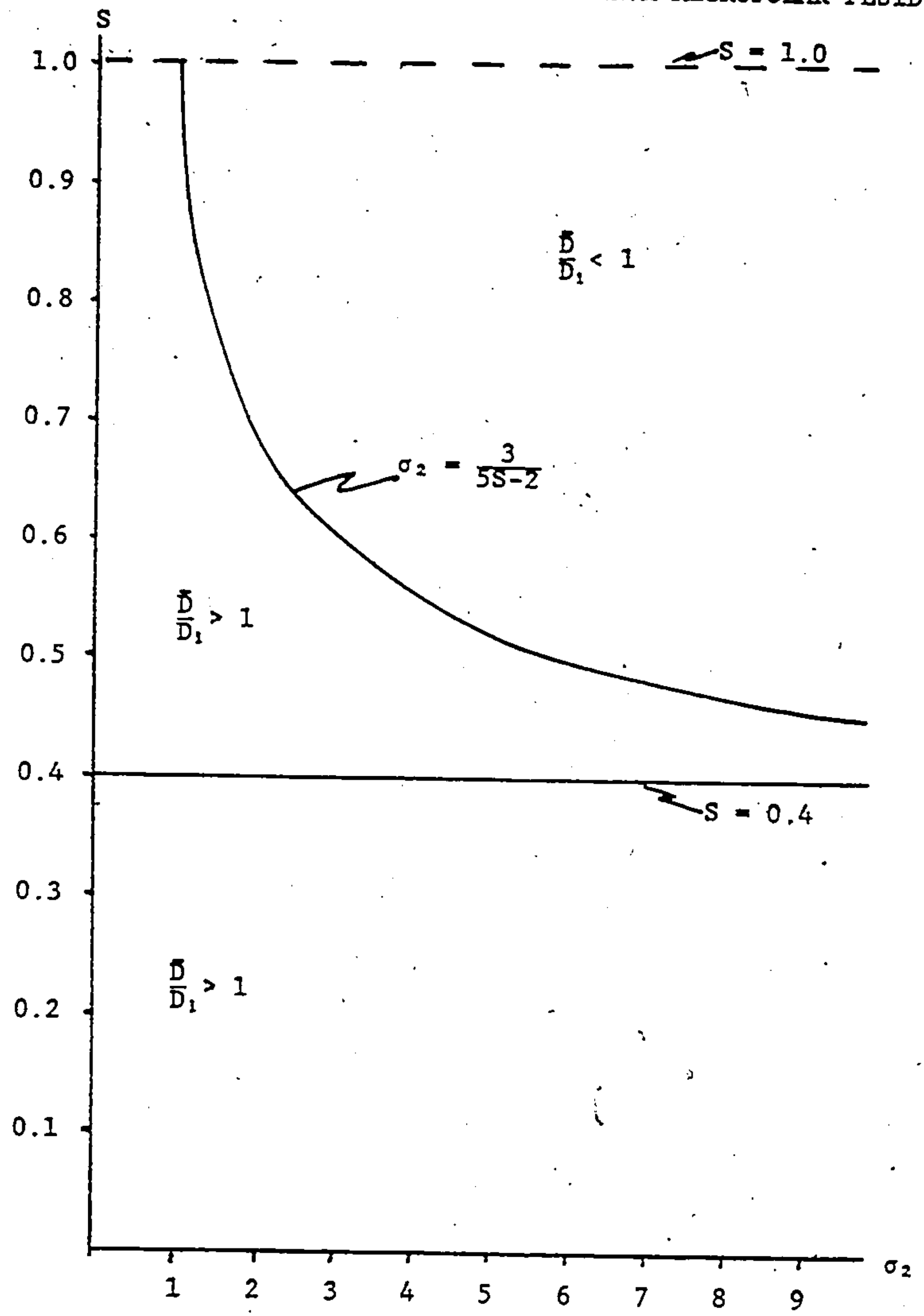
$$X \left[\frac{6(1+\sigma_2)(\mu^e+\kappa^e)(1+La)}{6(1+\sigma_2)(\mu^e+\kappa^e)(1+La) - \kappa^e\{3+(2-5s)\sigma_2\}}\right]$$

Using the fact that $\sigma_2 = \frac{2\mu^e+\kappa^e}{2\mu^I} \doteq \sigma = \frac{\mu^e}{\mu^I}$, and the definition of D_1 in equation (3.29) it is possible, as before, to write

$$\frac{\bar{D}}{D} \doteq \left[\frac{6(1+\sigma_2)(\mu^e+\kappa^e)(1+La)}{6(1+\sigma_2)(\mu^e+\kappa^e)(1+La) - \kappa^e\{3+(2-5s)\sigma_2\}}\right] \qquad (3.34)$$

A graphical interpretation of this result appears on the next page. Also from (3.34) it is possible to observe that $\bar{D} > D_1$ when s is small i.e. when $s \leq 0.5$, for all values of σ_2. However, for values of s closer to one and $\sigma_2 > 1$ there is a region where $\bar{D} < D_1$. Moreover, for $\sigma_2 < 1$ and for all values of s it is found that $\bar{D} > D_1$. It is thus possible to conclude that the drag on a viscous sphere moving in a micropolar fluid is, generally speaking, greater than the drag for a viscous sphere in a viscous fluid. This result is just the opposite of the result in the previous section. The drag found in this section depends strongly on the choices of σ_2 and s and expresses a result which is slightly different from the result for a solid sphere in a micropolar fluid which Ramkissoon and Majumdar (1976) conclude is always greater that the classical drag.

DRAG COMPARISON FOR A VISCOUS SPHERE IN A MICROPOLAR FLUID.



On the next few pages a number of tables are given to illustrate how changes in the various parameters, which appear in the study of micropolar fluids, affect the angle of intersection of various streamlines and lines of constant radius. In all of the tables that follow

$\kappa \varepsilon \left[0.1\eta, 10\eta\right]$	is the coupling vis cosity
$\ell \varepsilon \left[0.1, 10\right]$	is the characteristic length
$s \varepsilon \left[0, 1\right]$	is the parameter that appears in the compromise boundary condition
$\sigma \varepsilon \left[0.1, 10\right]$	is the ratio of external to internal viscosities
$H = \frac{4(1+\sigma)\Psi}{\sigma}$	can be determined from (2.21) and (2.23) (Viscous) and (3.31) and (3.33) (Micropolar)

TABLE 5a. Change in the value of theta for variable s.

H = 1 κ = 1.0 ℓ = 10.0 σ = 7.6

r \ s	0.1	0.4	0.7	1.0	Viscous
1.1	41.7	48.0	57.7	83.1	50.1
1.6	13.9	14.5	15.2	15.9	14.7
2.1	9.1	9.3	9.4	9.6	9.3
2.6	6.8	6.9	6.9	7.0	6.9
3.1	5.5	5.5	5.5	5.5	5.5
3.6	4.6	4.6	4.6	4.6	4.6
4.1	3.9	3.9	3.9	3.9	3.9
4.6	3.4	3.4	3.4	3.4	3.4
5.1	3.0	3.0	3.0	3.0	3.0
5.6	2.7	2.7	2.7	2.7	2.7

TABLE 5b. Change in the value of theta for variable s.

H = 1 κ = 1.0 ℓ = 10.0 σ = 4.6

r \ s	0.1	0.4	0.7	1.0	Viscous
1.1	56.9	68.6			78.5
1.6	17.5	18.3	19.1	20.0	18.7
2.1	11.4	11.6	11.8	12.0	11.7
2.6	8.5	8.6	8.7	8.7	8.6
3.1	6.8	6.8	6.9	6.9	6.9
3.6	5.7	5.7	5.7	5.7	5.7
4.1	4.9	4.9	4.9	4.9	4.9
4.6	4.3	4.3	4.3	4.3	4.3
5.1	3.8	3.9	3.9	3.8	3.9
5.6	3.4	3.4	3.4	3.4	3.4

TABLE 6a. Change in the value of theta for variable σ (micropolar).

H = 1 κ = 1.0 ℓ = 10.0 s = 1.0

r \ σ	0.1	2.5	4.6	7.6
1.1				83.1
1.6	56.9	25.9	20.0	15.9
2.1	31.3	15.5	12.0	9.6
2.6	22.1	11.2	8.7	7.0
3.1	17.1	8.8	6.9	5.5
3.6	14.0	7.3	5.7	4.6
4.1	11.8	6.2	4.9	3.9
4.6	10.2	5.4	4.3	3.4
5.1	9.0	4.8	3.8	3.0
5.6	8.1	4.3	3.4	2.7

TABLE 6b. Change in the value of theta for variable σ (viscous).

H = 1 κ = 1.0 ℓ = 10.0 s = 1.0

r \ σ	0.1	2.5	4.6	7.6
1.1				50.1
1.6	69.7	24.7	18.7	14.7
2.1	32.5	15.2	11.7	9.3
2.6	22.3	11.1	8.6	6.9
3.1	17.1	8.8	6.9	5.5
3.6	13.9	7.3	5.7	4.6
4.1	11.8	6.2	4.9	3.9
4.6	10.2	5.4	4.3	3.4
5.1	9.0	4.8	3.9	3.0
5.6	8.0	4.3	3.4	2.7

TABLE 7. Change in the value of theta for variable κ

H = 1.0 ℓ = 10.0 s = 1.0 σ = 7.6

r \ κ	0.1	1.0	10.0	Viscous
1.1	83.1	83.1	83.1	50.1
1.6	15.9	15.9	15.9	14.7
2.1	9.6	9.6	9.6	9.3
2.6	7.0	7.0	7.0	6.9
3.1	5.5	5.5	5.5	5.5
3.6	4.6	4.6	4.6	4.6
4.1	3.9	3.9	3.9	3.9
4.6	3.4	3.4	3.4	3.4
5.1	3.0	3.0	3.0	3.0
5.6	2.7	2.7	2.7	2.7

TABLE 8. Change in the value of theta for variable ℓ

$H = 1.0$ $\kappa = 0.1$ $s = 1.0$ $\sigma = 7.6$

r \ ℓ	0.1	1.0	10.0	Viscous
1.1	51.6	75.4	83.1	50.1
1.6	14.8	15.8	15.9	14.7
2.1	9.4	9.6	9.6	9.3
2.6	6.9	7.0	7.0	6.9
3.1	5.5	5.5	5.5	5.5
3.6	4.6	4.6	4.6	4.6
4.1	3.9	3.9	3.9	3.9
4.6	3.4	3.4	3.4	3.4
5.1	3.0	3.0	3.0	3.0
5.6	2.7	2.7	2.7	2.7

Section 5. Micropolar drop in a micropolar fluid.

IN order to complete the study of uniform flow in this chapter the work in this section will deal with the uniform flow of a micropolar fluid past a micropolar fluid sphere. The final boundary condition of (3.24) will be changed and another boundary condition will be added. These two new boundary conditions will be taken to be

$$\begin{aligned} \nu_\phi^{(e)} &= \nu_\phi^{(i)} \quad @\ r = a \\ M_{r\phi}^{(e)} &= M_{r\phi}^{(i)} \quad @\ r = a \end{aligned} \qquad (3.35)$$

Using equations (3.21) for the external flow it is found

$$\psi^{(e)} = \left[A_1 r^{-1} + B_1 r + \frac{U}{2} r^2 + F_2 \left(\frac{1+Lr}{Lr}\right) e^{-Lr}\right] \sin^2\theta$$

$$U_r^{(e)} = \left[2A_1 r^{-3} + 2B_1 r^{-1} + U + 2L^2 F_2 \left(\frac{1+Lr}{L^3 r^3}\right) e^{-Lr}\right] \cos\theta$$

$$u_\theta^{(e)} = \left[A_1 r^{-3} - B_1 r^{-1} - U + L^2 F_2 \left(\frac{1+Lr+L^2 r^2}{L^3 r^3}\right) e^{-Lr}\right] \sin\theta$$

$$\nu_\phi^{(e)} = \left[-B_1 r^{-2} + \frac{\mu^e + \kappa^e}{\kappa^e} L^2 F_2 \left(\frac{1+Lr}{Lr^2}\right) e^{-Lr}\right] \sin\theta$$

$$t_{r\theta}^{(e)} = -(2\mu^e + \kappa^e) \left[3A_1 r^{-4} + L^3 F_2 \left(\frac{3+3Lr+L^2 r^2}{L^4 r^4}\right) e^{-Lr}\right] \sin\theta$$

$$m_{r\phi}^{(e)} = \left[2(\gamma^e - \beta^e) \frac{B_1}{r^3} - \frac{2\mu^e + \kappa^e}{\gamma^e} \frac{\left[(2\gamma^e + \beta^e)(1+La) + \gamma^e L^2 a^2\right]}{Lr^3} F_2 e^{-La}\right] \sin\theta$$

(3.36)

In a similar fashion from (3.21) for the internal flow

$$\psi^{(i)}=\left[C_1r^2+D_1r^4+2E_2\left(\cosh L^ir-\frac{\sinh L^ir}{L^ir}\right)\right]\sin^2\theta$$

$$u_r^{(i)}=\left[2C_1+2D_1r^2+\frac{4E_2}{r^2}\left(\cosh L^ir-\frac{\sinh L^ir}{L^ir}\right)\right]\cos\theta$$

$$u_\theta^{(i)}=\left[-2C_1-4D_1r^2+\frac{2E_2}{r^2}\left(\cosh L^ir-\frac{1+(L^ir)^2}{L^ir}\sinh L^ir\right)\right]\sin\theta$$

$$\nu_\phi^{(i)}=\left[5D_1r+\frac{2(\mu^i+\kappa^i)}{\kappa^i}\;\frac{(L^i)^2E_2}{r}\left(\cosh L^ir-\frac{\sinh L^ir}{L^ir}\right)\right]\sin\theta$$

$$t_{r\theta}^{(i)}=-(2\mu^i+\kappa^i)\left[3D_1r+2E_2\left(\frac{3}{r^3}\cosh L^ir-\frac{3+(L^ir)^2}{r^3}\frac{\sinh L^ir}{L^ir}\right)\right]\sin\theta$$

$$m_{r\phi}^{(i)}=\left[5(\gamma^i-\beta^i)D_1-\frac{2\mu^i+\kappa^i}{\gamma^i}\left\{(2\gamma^i+\beta^i)\left[\frac{\cosh L^ir}{r^2}-\frac{\sinh L^ir}{L^ir^3}\right]-\right.\right.$$

$$\left.\left.-\gamma^i(L^i)^2\frac{\sinh L^ir}{L^ir}\right\}E_2\right]\sin\theta \tag{3.37}$$

From the boundary conditions remaining in equation (3.24) come the following with $M = a\cosh L^ia$, $N = \frac{\sinh L^ia}{L^i}$

$$A_1+a^2B_1+F_2\left(\frac{1+La}{L}\right)e^{-La}=\frac{-Ua^3}{2}$$

$$a^3C_1+a^5D_1+2E_2(M-N)=0$$

$$A_1-a^2B_1+F_2\frac{(1+La+L^2a^2)}{L}e^{-La}-Ua^3=-2a^3C_1-4a^5D_1+2E_2\left[(M-N)-(L^ia)^2N\right]$$

$$\frac{2\mu^e+\kappa^e}{2\mu^i+\kappa^i}\left[3A_1+F_2\frac{(3+3La+L^2a^2)}{L}e^{-La}\right]=3a^5D_1+2E_2\left[3(M-N)-(L^ia)^2N\right]$$

The final two boundary conditions as expressed in equations (3.35) give

$$-a^2B_1+\frac{\mu^e+\kappa^e}{\kappa^e}L^2a^2F_2\left(\frac{1+La}{L}\right)e^{-La}=5a^5D_1+\frac{2(\mu^i+\kappa^i)}{\kappa^i}(L^ia)^2E_2\left[M-N\right]$$

$$2(\gamma^e-\beta^e)a^2B_1-\frac{2\mu^e+\kappa^e}{\gamma^e}\left[\frac{(2\gamma^e+\beta^e)(1+La)+\gamma^eL^2a^2}{L}\right]a^2F_2e^{-La}=$$

$$=5(\gamma^i-\beta^i)a^5D_1-\frac{2\mu^i+\kappa^i}{\gamma^i}\left[(2\gamma^i+\beta^i)(M-N)-\gamma^i(L^ia)^2N\right]a^2E_2$$

The solution of this set of equations produces for the coefficients the following values with $\sigma_4=\frac{2\mu^e+\kappa^e}{2\mu^i+\kappa^i}$

$$A_1=\frac{Ua^3}{4(1+\sigma_4)}-\left[1+La+\frac{3+2\sigma_4}{6(1+\sigma_4)}L^2a^2\right]\frac{F_2e^{-La}}{L}+ \qquad (3.38)$$

$$+\frac{5}{6(1+\sigma_4)}\{3\left[M-N\right]-(L^ia)^2N\}\,E_2$$

$$a^2B_1=\frac{-(3+2\sigma_4)Ua^3}{4(1+\sigma_4)}+\frac{3+2\sigma_4}{6(1+\sigma_4)}L^2a^2\,\frac{F_2e^{-La}}{L}-$$

$$-\frac{5}{6(1+\sigma_4)}\{3\left[M-N\right]-(L^ia)^2N\}\,E_2$$

$$a^3C_1=\frac{-\sigma_4Ua^3}{4(1+\sigma_4)}+\frac{\sigma_4}{6(1+\sigma_4)}L^2a^2\,\frac{F_2e^{-La}}{L}-$$

$$-\left\{\frac{15\sigma_4\left[M-N\right]+(2-3\sigma_4)(L^ia)^2N}{6(1+\sigma_4)}\right\}E_2$$

$$a^5D_1=\frac{\sigma_4Ua^3}{4(1+\sigma_4)}-\frac{\sigma_4}{6(1+\sigma_4)}L^2a^2\,\frac{F_2e^{-La}}{L}-$$

$$-\frac{(2-3\sigma_4)}{6(1+\sigma_4)}\{3\left[M-N\right]-(L^ia)^2N\}\,E_2$$

(3.38)

$$L^2a^2\,\frac{F_2e^{-La}}{L} = \frac{3\kappa^e(1-\sigma_*)}{\kappa^e(1-\sigma_*)-2(\mu^e+\kappa^e)(1+\sigma_*)(1+La)}\,\frac{Ua^3}{2} +$$

$$+\frac{5\kappa^i(1-\sigma_*)\{3[M-N]-(L^ia)^2N\}-2(L^ia)^2(\mu^i+\kappa^i)(1+\sigma_*)[M-N]}{\kappa^e(1-\sigma_*)-2(\mu^e+\kappa^e)(1+\sigma_*)(1+La)}\,\frac{\kappa^e}{\kappa^i}\,\bar{E}_2$$

$$\Big[\Big\{6(1+\sigma_*)(\mu^e+\kappa^e)\left[(2\gamma^e+\beta^e)(1+La)+\gamma^eL^2a^2\right]$$

$$-\kappa^e\left[5(\gamma^i-\beta^i)\sigma_*+2(\gamma^e-\beta^e)(3+2\sigma_*)\right]\Big\}\ X$$

$$X\ \frac{5\kappa^i(1-\sigma_*)\{3[M-N]-(L^ia)^2N\}-2(L^ia)^2(\mu^i+\kappa^i)(1+\sigma_*)[M-N]}{6\kappa^i(1+\sigma_*)\left[\kappa^e(1-\sigma_*)-2(\mu^e+\kappa^e)(1+\sigma_*)(1+La)\right]} +$$

$$+\frac{(3\sigma_*-2)+10(\gamma^e-\beta^e)\{3[M-N]-(L^ia)^2N\}}{6(1+\sigma_*)} -$$

$$-\frac{(2\mu^i+\kappa^i)\{(2\gamma^i+\beta^i)[M-N]-(L^ia)^2N\}}{\gamma^i}\Big]\bar{E}_2 .$$

$$= \Big[\{6(\gamma^e+\beta^e)(1+\sigma_*)-\left[5(\gamma^i-\beta^i)\sigma_*+2(3+2\sigma_*)(\gamma^e-\beta^e)\right](\mu^e+\kappa^e)\}$$

$$(1+La)+2\gamma^e(1-\sigma_*)L^2a^2 \quad \div$$

$$\div\ \kappa^e(1-\sigma_*)-2(\mu^e+\kappa^e)(1+\sigma_*)(1+La)\Big]\ X\ \frac{Ua^3}{2}$$

As in the previous examples the drag is still to be calculated using either (3.23) or (3.28). The latter will again be used here to give

$$\bar{D} = 4\pi(2\mu^e+\kappa^e)\left[\frac{(3+2\sigma_*)}{4(1+\sigma_*)}\{Ua-\frac{2}{3}L^2\frac{F_2e^{-La}}{L}\} - \right.$$

$$\left. - \frac{5}{6(1+\sigma_*)}\{3[M-N] - (L^i a)^2 N\}\frac{\bar{E}_2}{a^2}\right]$$

$$= \frac{-6\pi(\mu^e+\frac{\kappa^e}{2})(1+\frac{2}{3}\sigma_*)Ua}{1+\sigma_*} \quad X$$

$$X\left[1 - \frac{2L}{3e^{La}Ua}F_2 - \frac{10}{(3+2\sigma_*)}\{[M-N] - \frac{(L^i a)^2}{3}N\}\frac{\bar{E}_2}{Ua^3}\right]$$

Since $\mu^e + \frac{\kappa^e}{2} \doteq \mu^e$ and $\sigma_* = \frac{2\mu^e+\kappa^e}{2\mu^i+\kappa^i} \doteq \sigma = \frac{\mu^e}{\mu^i}$ this can be rewritten in the form

$$\bar{D} \doteq D_1\left[1-\frac{2}{3Ua^3}\left(L^2a^2\frac{F_2e^{-La}}{L}\right) - \frac{10}{(3+2\sigma_*)Ua^3}\{[M-N] - \frac{(L^i a)^2}{3}N\}\bar{E}_2\right] \tag{3.39}$$

where $L^2a^2\frac{F_2e^{-La}}{L}$ and $\bar{E}_2$ are given in (3.38). The value for the drag as determined in equation (3.39) will reduce to that in equation (3.34) if $\bar{E}_2 = 0$ and if $s = 1$ in (3.34). In an analogous fashion, equation (3.39) with $F_2 = 0$ is the same as equation (3.30) with $s = 1$.

Section 6.

In sections 3, 4 the problems of the uniform flow of a viscous Newtonian fluid past a micropolar fluid drop and of a micropolar fluid past a viscous Newtonian fluid drop were considered in detail by using the stream function approach. In this section, the solutions to both of these problems will be sketched using the technique developed in chapter II.

For a viscous fluid subjected to a uniform flow the general solutions to the equations of motion are

$$p(\underset{\sim}{r}) = (\overset{1}{A}_1 r^{-3} + \overset{1}{A}_2)\varepsilon_{ijk}a_{kj}X_i \tag{3.40}$$

$$u_\ell(\underset{\sim}{r}) = (\overset{1}{A}_3 r^{-5} + \frac{1}{\eta}\overset{1}{A}_1 r^{-3} - \frac{1}{5\eta}\overset{1}{A}_2)\varepsilon_{ijk}a_{kj}X_i X_\ell + $$
$$+ (-\tfrac{1}{3}\overset{1}{A}_3 r^{-3} + \frac{1}{\eta}\overset{1}{A}_1 r^{-1} + \overset{1}{A}_6 + \frac{2}{5\eta}\overset{1}{A}_2 r^2)\varepsilon_{\ell jk}a_{kj} \tag{3.41}$$

$$\omega_\ell(\underset{\sim}{r}) = (\frac{1}{\eta}\overset{1}{A}_1 r^{-3} - \frac{1}{2\eta}\overset{1}{A}_2)\ (a_{\ell q} - a_{q\ell})X_q \tag{3.42}$$

$$t_{\ell m}\frac{X_\ell}{r}\frac{X_m}{r} = -\frac{\eta}{2}(4\overset{1}{A}_3 r^{-5} + \frac{6}{\eta}\overset{1}{A}_1 r^{-3} + \frac{6}{5\eta}\overset{1}{A}_2)\varepsilon_{ijk}a_{kj}X_i \tag{3.43}$$

$$t_{\ell m}\frac{X_\ell}{r} - t_{pq}\frac{X_p X_q}{r^2}\frac{X_m}{r} = -\frac{\eta}{2}(2\overset{1}{A}_3 r^{-6} + \frac{3}{5\eta}\overset{1}{A}_2 r^{-1})\varepsilon_{mjk}(a_{jq} - a_{qj})X_q X_k \tag{3.44}$$

If the viscous fluid is outside of a spherical body, then

$$\text{(i)}\quad A_2^1 = 0$$
$$\text{(ii)}\quad A_6^1 = 1 \qquad (3.45)$$

On the other hand, for a viscous fluid inside of a spherical body

$$\text{(i)}\quad A_1^1 = 0$$
$$\text{(ii)}\quad A_3^1 = 0 \qquad (3.46)$$

For a micropolar fluid subjected to a uniform flow, the general solutions to the equations of motion are

$$p(\underset{\sim}{r}) = (A_1^1 r^{-3} + A_2^1)\varepsilon_{ijk}a_{kj}X_i \qquad (3.47)$$

$$v_\ell(\underset{\sim}{r}) = (A_3^1 r^{-5} - \frac{L^2}{\eta} A_1^1 r^{-3} + \frac{L^2}{5\eta} A_2^1)\varepsilon_{ijk}a_{kj}X_iX_\ell +$$
$$+(-\tfrac{1}{3}A_3^1 r^{-3} - \frac{L^2}{\eta} A_1^1 r^{-1} + A_6^1 - \frac{2L^2}{5\eta} A_2^1 r^2)\varepsilon_{\ell jk}a_{kj} \qquad (3.48)$$

$$u_\ell(\underset{\sim}{r}) = \Big[-\frac{1}{L^2}(A_3^1 + \frac{6}{\eta} A_1^1)r^{-5} + \frac{1}{\eta} A_1^1 r^{-3} - \frac{1}{5\eta} A_2^1 -$$
$$- \frac{3L^2}{2} C_3^1 d_7(x)e^x + \frac{3L^2}{2} C_4^1 d_8(x)e^{-x}\Big]\varepsilon_{ijk}a_{kj}X_iX_\ell +$$
$$+\Big[\frac{1}{3L^2}(A_3^1 + \frac{6}{\eta} A_1^1)r^{-3} + \frac{1}{\eta} A_1^1 r^{-1} + \frac{1}{L^2}(\frac{2}{\eta} A_2^1 - A_6^1) +$$
$$+ \frac{2}{5\eta} A_2^1 r^2 + C_3^1(\frac{x^2}{2} d_7(x) + d_5(x))e^x -$$
$$- C_4^1(\frac{x^2}{2} d_8(x) + d_6(x))e^{-x}\Big]\varepsilon_{\ell jk}a_{kj}$$

$$(3.49)$$

$$\nu_\ell(\underset{\sim}{r}) = \left[\frac{1}{\eta} A_1^1 r^{-3} - \frac{1}{2\eta} A_2^1 - \frac{3\eta}{2\gamma} C_3^1 \left(\frac{d_5' + d_5}{x}\right) e^{x} + \right.$$

$$\left. + \frac{3\eta}{2\gamma} \left(\frac{d_6' - d_6}{x}\right) e^{-x} \right] (a_{\ell q} - a_{q\ell}) X_q \tag{3.50}$$

$$t_{\ell m} \frac{X_\ell}{r} \frac{X_m}{r} = -\frac{\eta}{2}\left(\frac{-4}{L^2}(A_3^1 + \frac{6}{\eta} A_1^1) r^{-5} + \frac{6}{\eta} A_1^1 r^{-3} + \frac{6}{5\eta} A_2^1 - \right.$$

$$\left. -6L^2 C_3^1 d_7(x) e^{x} + 6L^2 C_4^1 d_8(x) e^{-x} \right) \varepsilon_{ijk} a_{kj} X_i \tag{3.51}$$

$$t_{\ell m} \frac{X_\ell}{r} - t_{pq} \frac{X_p X_q}{r^2} \frac{X_m}{r} = -\frac{\eta}{2}\left(-\frac{2}{L^2}(A_3^1 + \frac{6}{\eta} A_1^1) r^{-6} + \right.$$

$$+ \frac{3}{5\eta} A_2^1 r^{-1} - 3L^3 C_3^1 \frac{d_7(x)}{x} e^{x} +$$

$$\left. + 3L^3 C_4^1 d_8(x) e^{-x} \right) \varepsilon_{mjk} (a_{jq} - a_{qj}) X_q X_k \tag{3.52}$$

If the micropolar fluid is outside of a spherical body, then

$$\begin{aligned} &\text{(i)} \quad A_2^1 = 0 \\ &\text{(ii)} \quad A_6^1 = -L^2 \\ &\text{(iii)} \quad C_3^1 = 0 \end{aligned} \tag{3.53}$$

If, however, the micropolar fluid is on the interior of a spherical body, then

$$\begin{aligned} &\text{(i)} \quad A_1^1 = 0 \\ &\text{(ii)} \quad A_3^1 = 0 \\ &\text{(iii)} \quad C_3^1 = C_4^1 \end{aligned} \tag{3.54}$$

The boundary conditions used in the fluid sphere case when a micropolar fluid is involved are

$$\text{(i)} \quad u^e_\ell X_\ell = u^i_\ell X_\ell = 0 \qquad @\ r = a$$

$$\text{(ii)} \quad u^e_\ell - u^e_p \frac{X_p}{r}\frac{X_\ell}{r} = u^i_\ell - u^i_p \frac{X_p}{r}\frac{X_\ell}{r} \qquad @\ r = a$$

$$\text{(iii)} \quad t^e_{\ell m} X_\ell - t^e_{pq} \frac{X_p X_q}{r^2} X_m = t^i_{\ell m} X_\ell - t^i_{pq} \frac{X_p X_q}{r^2} X_m \qquad @\ r = a \tag{3.55}$$

$$\text{(iv)} \quad \underset{\sim}{\nu} = s\underset{\sim}{\omega} \qquad @\ r = a$$

In the final boundary condition $\underset{\sim}{\omega}$ is associated with the viscous Newtonian fluid while $\underset{\sim}{\nu}$ is coupled with the micropolar fluid.

For a viscous Newtonian sphere in a micropolar fluid subjected to uniform flow by using equation (3.46) in equations (3.40) - (3.44) and (3.53) in equations (3.47) - (3.52) in the boundary conditions (3.55)

$$\text{(i)} \quad \frac{-2}{3L^2}\left(A^1_3 + \frac{6}{\eta^e} A^1_1\right)a^{-3} + \frac{2}{\eta} A^1_1 a^{-1} + 1 + + \left(L^2 a^2 d_8(La) - d_6(La)\right)e^{-La}\, C^1_4 = 0$$

$$\text{(ii)} \quad A^1_6 + \frac{1}{5\eta} A^1_2 a^2 = 0 \tag{3.56}$$

$$\text{(iii)}\quad \frac{1}{3L^2}(A_3^1 + \frac{6}{\eta}A_1^1)a^{-3} + \frac{1}{\eta}A_1^1 a^{-1} + 1 -$$

$$-(\frac{L^2a^2}{2}d_8(La) + d_6(La))e^{-La}C_4^1 = A_6^1 + \frac{2}{5\eta}A_2^1 a^2$$

$$\text{(iv)}\quad \eta^e\left[\frac{-2}{L^2}(A_3^1 + \frac{6}{\eta}A_1^1)a^{-5} + 3L^2 d_8(La)e^{-La}C_4^1\right] = \eta^i\left[\frac{3}{5\eta}A_2^1\right]$$

$$\text{(v)}\quad \frac{1}{\eta}A_1^1 a^{-3} + \frac{3\eta}{2\gamma}\left(\frac{d_6'(La) - d_6(La)}{La}\right)e^{-La}C_4^1 = \frac{-s}{2\eta}A_2^1$$

(3.56)

From equation (3.56) with $\sigma = \frac{\eta^e}{\eta^i} = \frac{(2\mu+\kappa)^e}{2\mu^i}$

$$\text{(i)}\quad \frac{1}{10\eta}A_2^1 a^2 = \frac{\sigma}{4(1+\sigma)} - \frac{\sigma}{4(1+\sigma)}d_6(La)e^{-La}C_4^1$$

$$\text{(ii)}\quad \tfrac{1}{2}A_6^1 = \frac{-\sigma}{4(1+\sigma)} + \frac{\sigma}{4(1+\sigma)}d_6(La)e^{-La}C_4^1$$

$$\text{(iii)}\quad \frac{1}{\eta}A_1^1 a^{-1} = \frac{-(2\sigma+3)}{4(1+\sigma)} + \frac{2\sigma+3}{4(1+\sigma)}d_6(La)e^{-La}C_4^1$$

$$\text{(iv)}\quad -\frac{1}{3L^2}(A_3^1 + \frac{6}{\eta}A_1^1)a^{-3} = \frac{1}{4(1+\sigma)} -$$

$$-\frac{1}{4(1+\sigma)}(d_6(La) + 2(1+\sigma)L^2a^2 d_8(La))e^{-La}C_4^1$$

$$\text{(v)}\quad e^{-La}C_4^1 =$$

$$= \frac{\kappa^e\{3+(2-5s)\sigma\}\,La}{\kappa^e\{3+(2-5s)\sigma\} - 6(\mu+\kappa)^e(1+\sigma)(1+La)}$$

(3.57)

For a micropolar sphere in a viscous Newtonian fluid subjected to uniform flow by using equation (3.45) in equations (3.40) - (3.44) and (3.54) in equations (3.47) - (3.52) in the boundary conditions (3.55)

$$\text{(i)}\quad \frac{2}{3}A_3^1 a^{-3} + \frac{2}{\eta} A_1^1 a^{-1} + 1 = 0$$

$$\text{(ii)}\quad \frac{1}{L^2}\left(\frac{2}{\eta} A_2^1 - A_6^1\right) + \frac{1}{5\eta} A_2^1 a^2 + C_4^1 \Big[-(L^2 a^2 d_7(La) - d_5(La))e^{La} + (L^2 a^2 d_8(La) - d_6(La))e^{-La}\Big] = 0.$$

$$\text{(iii)}\quad \frac{1}{L^2}\left(\frac{2}{\eta} A_2^1 - A_6^1\right) + \frac{2}{5\eta} A_2^1 a^2 + C_4^1 \Big[\left(\frac{L^2 a^2}{2} d_7(La) + d_5(La)\right)e^{La} - \left(\frac{L^2 a^2}{2} d_8(La) + d_6(La)\right)e^{-La}\Big] = -\frac{1}{3}A_3^1 a^{-3} + \frac{1}{\eta} A_1^1 a^{-1} + 1$$

$$\text{(iv)}\quad \eta^i \left[\frac{3}{5\eta} A_2^1 - 3L^2 (d_7(La)e^{La} - d_8(La)e^{-La})C_4^1\right] = \eta^e \left[2A_3^1 a^{-5}\right]$$

$$\text{(v)}\quad -\frac{1}{2\eta} A_2^1 - \frac{3\eta}{2\gamma}\left[\frac{d_5'(La) + d_5(La)}{La} e^{La} - \frac{d_6'(La) - d_6(La)}{La} e^{-La}\right] C_4^1 = \frac{s}{\eta} A_1^1 a^{-3}$$

(3.58)

From equations (3.58) with $\sigma = \frac{\eta^e}{\eta^i} = \frac{2\mu^e}{(2\mu+\kappa)^i}$

(i) $$\frac{1}{10\eta} A_2^1 a^2 = \frac{\sigma}{4(1+\sigma)} + \frac{2-3\sigma}{4(1+\sigma)} \left[L^2a^2d_7(La)e^{La} - L^2a^2d_8(La)e^{-La}\right] C_4^1$$

(ii) $$\frac{1}{2L^2}\left(\frac{2}{\eta} A_2^1 - A_6^1\right) = \frac{-\sigma}{4(1+\sigma)} + \frac{5}{4(1+\sigma)} \left[L^2a^2d_7(La)e^{La} - L^2a^2d_8(La)e^{-La}\right] C_4^1 - \frac{1}{2}\left[d_5(La)e^{La} - d_6(La)e^{-La}\right] C_4^1$$

(iii) $$\frac{1}{\eta} A_1^1 a^{-1} = - \frac{-(2\sigma+3)}{4(1+\sigma)} + \frac{5}{4(1+\sigma)} \left[L^2a^2d_7(La)e^{La} - L^2a^2d_8(La)e^{-La}\right] C_4^1$$

(iv) $$\frac{1}{3} A_3^1 a^{-3} = \frac{1}{4(1+\sigma)} - \frac{5}{4(1+\sigma)} \left[L^2a^2d_7(La)e^{La} - L^2a^2d_8(La)e^{-La}\right] C_4^1$$

(v) $$C_4^1 = \kappa^{i}\{3s+(2s-5)\sigma\} \div 5\kappa^{i}\{(s+2)-3\sigma\}\{L^2a^2d_7(La)e^{La} - L^2a^2d_8(La)e^{-La}\} + 6(\mu+\kappa)^{i}(1+\sigma) La\{(d_5'(La)+d_5(La))e^{La} - (d_6'(La)-d_6(La))e^{-La}\}$$

(3.58)

CHAPTER IV

SOLUTIONS FOR A FLUID SPHERE IN A LINEAR SHEARING FLOW FIELD.

Section 1. Introduction.

The linear shearing flow of a viscous Newtonian fluid past a spherical body was first considered by Einstein (1906, 1911) when he found an expression for the effective viscosity if a solid sphere is present in the fluid. A consideration of the case of a viscous Newtonian sphere in the linear shearing flow of a viscous Newtonian fluid was carried out by Taylor (1932). The effective viscosity is the measurable value of the viscosity of a fluid containing a number of particles when treated as a homogeneous medium.

Any linear shearing flow is worthy of study not only because a determination of the effective viscosity can be made but also because this is the most easily studied of all flows for which a stream function does not exist. Hence, in this chapter an exhaustive study of the linear shearing flow of a fluid past a sphere will be carried out. The external fluid will be either a viscous Newtonian fluid or a micropolar fluid and the sphere will be either solid or liquid, in which case it will be either a viscous Newtonian fluid or a micropolar fluid.

Section 2 will present, again, the basic equations for a micropolar fluid flow and will also give the solutions using the method as outlined in chapter II and the Appendix. The Bessel functions appearing in these general solutions will be expressed either in terms of hyperbolic functions or exponential functions, whichever is better suited to a particular boundary value problem.

In section 3 the values of the undetermined coefficients and the effective viscosity will be calculated for the flow of a viscous Newtonian fluid past a solid sphere and also for the flow of a micropolar fluid past a solid sphere.

In sections 4, 5, 6, 7 the same calculations will be made for the flow of a viscous Newtonian fluid past a viscous Newtonian sphere, for the flow of a viscous Newtonian fluid past a micropolar fluid sphere, for the flow of a micropolar fluid past a viscous Newtonian fluid sphere and for the flow of a micropolar fluid past a micropolar fluid sphere respectively. The results obtained will be compared to the results for the solid sphere in each example as far as is possible.

Section 2. Basic equations and their solutions.

In this section the basic equations for the flow of a micropolar fluid as developed in chapter I section 3 will be restated and their solutions, using the form found in chapter II will be given. In addition to the Laplacian, the inhomogeneous HelmHoltz operator is also needed to generate the general solutions. The solutions appropriate to the linear shear flow problem will be found in the appendix.

A. Basic equations with $\eta = 2\mu+\kappa$, $L^2 = \frac{\kappa\eta}{\gamma(\mu+\kappa)}$

$$\nabla^2 p = 0 \tag{4.1}$$

$$(\nabla^2 - L^2)\nabla^2 \underset{\sim}{u} = \frac{-2\kappa}{\gamma(\mu+\kappa)} \nabla p = \frac{-2L^2}{\eta} \nabla p \tag{4.2}$$

$$\nabla^2 \Psi_\nu = \frac{2\kappa}{\delta} \Psi_\nu \tag{4.3}$$

$$\underset{\sim}{\nu} = \tfrac{1}{2}\left[1+\frac{\gamma(\mu+\kappa)}{\kappa^2} \nabla^2\right](\nabla \times \underset{\sim}{u}) + \frac{\delta}{2\kappa} \nabla \Psi_\nu \tag{4.4}$$

$$\nabla . \underset{\sim}{u} = 0 \tag{4.5}$$

Because of the linearity of the operators involved in equation (4.2) it shall be assumed that

$$(\nabla^2 - L^2)\underset{\sim}{u} = \underset{\sim}{V} \tag{4.6}$$

and $$\nabla^2 \underset{\sim}{V} = \frac{-2\kappa}{\gamma(\mu+\kappa)} \nabla p \tag{4.7}$$

B. <u>Solutions</u>.

Also for any linear shearing flow the upstream boundary condition $\underset{\sim}{u}_\infty = \underset{\sim}{a}.\underset{\sim}{r}$ indicates that the appropriate choices for p, $\underset{\sim}{V}$, $\underset{\sim}{u}$ are

$$p = (A_1^{5} r^{-5} + A_2^{5}) a_{ij} X_i X_j \tag{4.8}$$

$$\begin{aligned} V_p = {} & (A_3^{5} r^{-7} - \frac{L^2}{\eta} A_1^{5} r^{-5} + A_4^{5}) a_{ij} X_i X_j X_p \; + \\ & + (-\tfrac{1}{5} A_3^{5} r^{-5} + A_5^{5} r^{-3} + A_6^{5} - \tfrac{1}{5} (\frac{L^2}{\eta} A_2^{5} + A_4^{5}) r^2) a_{pj} X_j \; + \\ & + (-\tfrac{1}{5} A_3^{5} r^{-5} + A_7^{5} r^{-3} + A_8^{5} - \tfrac{1}{5} (\frac{L^2}{\eta} A_2^{5} + A_4^{5}) r^2) a_{jp} X_j \end{aligned} \tag{4.9}$$

$$\begin{aligned} u_p = {} & \Big[\frac{-1}{L^2} (A_3^{5} + \frac{10}{\eta} A_1^{5}) r^{-7} + \frac{1}{\eta} A_1^{5} r^{-5} - \frac{1}{L^2} A_4^{5} + C_1^{5} d_1 (Lr) e^{Lr} - \\ & - C_2^{5} d_2 (Lr) e^{-Lr} \Big] a_{ij} X_i X_j X_p \; + \\ & + \Big[\frac{1}{5L^2} (A_3^{5} + \frac{10}{\eta} A_1^{5}) r^{-5} - \frac{1}{L^2} A_5^{5} r^{-3} + \frac{1}{L^2} (\frac{2}{\eta} A_2^{5} - A_6^{5}) + \frac{1}{5L^2} (A_4^{5} + \frac{L^2}{\eta} A_2^{5}) r^2 - \\ & - \frac{C_1^{5}}{5L^2} (Lr)^2 d_1 (Lr) e^{Lr} + \frac{C_2^{5}}{5L^2} (Lr)^2 d_2 (Lr) e^{-Lr} + C_3^{5} d_3 (Lr) e^{Lr} - \\ & - C_4^{5} d_4 (Lr) e^{-Lr} \Big] a_{pj} X_j \; + \; \Big[\frac{1}{5L^2} (A_3^{5} + \frac{10}{\eta} A_1^{5}) r^{-5} - \frac{1}{L^2} A_7^{5} r^{-3} + \end{aligned}$$

$$+\frac{1}{L^2}\left(\frac{2}{\eta}A_2^5-A_8^5\right)+\frac{1}{5L^2}\left(A_4^5+\frac{L^2}{\eta}A_2^5\right)r^2-\frac{1}{5L^2}C_1^5(Lr)^2d_1(Lr)e^{Lr}-$$

$$-\frac{1}{5L^2}C_2^5(Lr)^2d_2(Lr)e^{-Lr}+C_5^5d_3(Lr)e^{Lr}-C_6^5d_4(Lr)e^{-Lr}\Big]a_{jp}X_j \tag{4.10}$$

By the continuity condition in (4.5) it is found that the coefficients must satisfy the following relations

$$\begin{aligned} &A_5^5+A_7^5=0\\ &A_4^5-\frac{4L^2}{21\eta}A_2^5=0\\ &C_1^5=\frac{-5L^2}{3}(C_3^5+C_5^5)\\ &C_2^5=\frac{-5L^2}{3}(C_4^5+C_6^5)\end{aligned} \tag{4.11}$$

Hence the general solutions to (4.1), (4.6), (4.7) using the results in equation (4.11) are

$$p=(A_1^5r^{-5}+A_2^5)a_{ij}X_iX_j \tag{4.12}$$

$$u_p=\Big[-\frac{1}{L^2}\left(A_3^5+\frac{10}{\eta}A_1^5\right)r^{-7}+\frac{1}{\eta}A_1^5r^{-5}-\frac{4}{21\eta}A_2^5-$$

$$-\frac{5L^2(C_3^5+C_5^5)}{3}d_1(Lr)e^{Lr}+\frac{5L^2}{3}(C_4^5+C_6^5)d_2(Lr)e^{-Lr}\Big]a_{ij}X_iX_jX_p+$$

$$+\Big[\frac{1}{5L^2}\left(A_3^5+\frac{10}{\eta}A_1^5\right)r^{-5}-\frac{1}{L^2}A_5^5r^{-3}+\frac{1}{L^2}\left(\frac{2}{\eta}A_2^5-A_8^5\right)+\frac{5}{21\eta}A_2^5r^2+$$

$$+\frac{1}{3}C_5^5\left[(Lr)^2 d_1(Lr)+3d_3(Lr)\right]e^{Lr}-\frac{1}{3}C_6^5\left[(Lr)^2 d_2(Lr)+3d_4(Lr)\right]e^{-Lr}$$

$$+\frac{1}{3}C_5^5(Lr)^2 d_1(Lr)e^{Lr}-\frac{1}{3}C_6^5(Lr)^2 d_2(Lr)e^{-Lr}\Big]a_{pj}X_j$$

$$+\Big[\frac{1}{5L^2}(A_3^5+\frac{10}{\eta}A_1^5)r^{-5}+\frac{1}{L^2}A_3^5 r^{-3}+\frac{1}{L^2}(\frac{2}{\eta}A_2^5-A_8^5)+\frac{5}{21\eta}A_2^5 r^2$$

$$+\frac{1}{3}C_5^5\left[(Lr)^2 d_1(Lr)+3d_3(Lr)\right]e^{Lr}-\frac{1}{3}C_6^5\left[(Lr)^2 d_2(Lr)+3d_4(Lr)\right]e^{-Lr}$$

$$+\frac{1}{3}C_5^5(Lr)^2 d_1(Lr)e^{Lr}-\frac{1}{3}C_4^5(Lr)^2 d_2(Lr)e^{-Lr}\Big]a_{jp}X_j \qquad (4.13)$$

From the general form for the vorticity vector as deduced from equation (4.13) it is seen that in order for the microrotation vector to have the same functional structure as the vorticity vector, it is essential to choose the general form for the function Ψ_ν to be

$$\Psi_\nu = g(r)\varepsilon_{ijk}a_{kj}X_i \qquad (4.14)$$

From equation (4.3) and the appendix it is found that

$$\Psi_\nu=\left[C_7^5 d_3(Mr)e^{Mr}-C_8^5 d_4(Mr)e^{-Mr}\right]\varepsilon_{ijk}a_{kj}X_i \qquad (4.15)$$

where $M^2 = \frac{2\kappa}{\delta}$

Using now equations (4.13), (4.9), (4.6), (4.15) in the defining equation for the microrotation vector as given by equation (4.4) it is found that

$$\nu_p = \Bigg[\tfrac{1}{2}\Big(\frac{A_1^5}{\eta} - \frac{3A_5^5}{L^2}\Big)r^{-5} - \frac{A_2^5}{3\eta} - \frac{\eta}{3\gamma}(4C_3^5+C_5^5)\Big[\frac{d_3'(Lr)+d_3(Lr)}{Lr}\Big]e^{Lr} +$$

$$+ \frac{\eta}{3\gamma}(4C_4^5+C_6^5)\Big[\frac{d_4'(Lr)-d_4(Lr)}{Lr}\Big]e^{-Lr} +$$

$$+ \frac{\kappa}{\mu+\kappa} C_7^5\Big[\frac{d_3'(Mr)+d_3(Mr)}{Mr}\Big]e^{Mr} -$$

$$- \frac{\kappa}{\mu+\kappa} C_8^5\Big[\frac{d_4'(Mr)-d_4(Mr)}{Mr}\Big]e^{-Mr}\Bigg]\varepsilon_{p\ell m}a_{\ell j}X_j X_m +$$

$$+ \Bigg[\tfrac{1}{2}\Big(\frac{A_1^5}{\eta} + \frac{3A_5^5}{L^2}\Big)r^{-5} - \frac{A_2^5}{3\eta} - \frac{\eta}{3\gamma}(C_3^5+4C_5^5)\Big[\frac{d_3'(Lr)+d_3(Lr)}{Lr}\Big]e^{Lr} +$$

$$+ \frac{\eta}{3\gamma}(C_4^5+4C_6^5)\Big[\frac{d_4'(Lr)-d_4(Lr)}{Lr}\Big]e^{-Lr} -$$

$$- \frac{\kappa}{\mu+\kappa} C_7^5\Big[\frac{d_3'(Mr)+d_3(Mr)}{Mr}\Big]e^{Mr} +$$

$$+ \frac{\kappa}{\mu+\kappa} C_8^5\Big[\frac{d_4'(Mr)-d_4(Mr)}{Mr}\Big]e^{-Mr}\Bigg]\varepsilon_{p\ell m}a_{j\ell}X_j X_m +$$

$$+ \Bigg[\frac{-A_5^5}{L^2} r^{-3} - \frac{(A_6^5-A_8^5)}{2L^2} + \frac{\mu+\kappa}{\kappa}(C_3^5-C_5^5)d_3(Lr)e^{Lr} -$$

$$- \frac{\mu+\kappa}{\kappa}(C_4^5-C_6^5)d_4(Lr)e^{-Lr} +$$

$$+ \frac{\delta}{2\kappa} C_7^5\Big(d_3(Mr)+(Mr)\big[d_3'(Mr)+d_3(Mr)\big]\Big)e^{Mr} -$$

$$- \frac{\delta}{2\kappa} C_8^5\Big(d_4(Mr)+(Mr)\big[d_4'(Mr)-d_4(Mr)\big]\Big)e^{-Mr}\Bigg]\varepsilon_{pmn}a_{nm}$$

(4.16)

C. <u>Viscous fluid case.</u>

In the event that one is dealing with a viscous Newtonian fluid rather than a micropolar fluid, then the basic equations as given by equations (4.1) → (4.5) reduce to

$$\nabla^2 p = 0 \tag{4.17}$$

$$\nabla^2 \underset{\sim}{u} = \frac{1}{\mu} \nabla p \tag{4.18}$$

$$\nabla . \underset{\sim}{u} = 0 \tag{4.19}$$

The solutions to this system of equations, using the form as outlined in chapter II are

$$p = (\overset{5}{A}_1 r^{-5} + \overset{5}{A}_2) a_{ij} X_i X_j \tag{4.20}$$

$$\begin{aligned} u_\ell = {} & (\overset{5}{A}_3 r^{-5} + \frac{1}{2\mu} \overset{5}{A}_1 r^{-5} - \frac{2}{21\mu} \overset{5}{A}_2) a_{ij} X_i X_j X_\ell + \\ & + (-\tfrac{1}{5}\overset{5}{A}_3 r^{-5} + \overset{5}{A}_5 r^{-3} + \overset{5}{A}_6 + \frac{5}{42\mu} \overset{5}{A}_2 r^2) a_{\ell j} X_j + \\ & + (-\tfrac{1}{5}\overset{5}{A}_3 r^{-5} - \overset{5}{A}_5 r^{-3} + \overset{5}{A}_8 + \frac{5}{42\mu} \overset{5}{A}_2 r^2) a_{j\ell} X_j \end{aligned} \tag{4.21}$$

Section 3. Flow past a solid sphere.

In this section two problems will be solved. The two problems dealt with will be the flow of a viscous Newtonian fluid past a solid sphere and the flow of a micropolar fluid past a solid sphere. The solutions to each of these problems will be complete in the sense that the boundary conditions will be stated, the values for all of the coefficients will be calculated, and the effective viscosity will be determined for each flow and a comparison of the two values will be made.

A. Viscous Newtonian fluid.

For the linear shearing flow of a viscous Newtonian fluid past a solid sphere, the general solutions are given by equations (4.20) & (4.21). For this type of flow situation, the boundary conditions in tensorial form appear as

$$p \to 0 \qquad \text{as } r \to \infty \tag{4.22}$$

$$u_\ell \to a_{\ell j} X_j \qquad \text{as } r \to \infty \tag{4.23}$$

$$u_\ell = \tfrac{1}{2}\left[a_{\ell j} X_j - a_{j \ell} X_j\right] \qquad @\ r = a \tag{4.24}$$

The boundary conditions (4.22) & (4.23) when applied to the general solutions (4.20) & (4.21) require that

$A_2^s = 0$, $A_6^s = 1$, $A_8^s = 0$. From the final boundary condition (4.24), when applied to equation (4.21), comes the following set of linear equations

$$A_3^s a^{-5} + \frac{1}{2\mu} A_1^s a^{-3} = 0$$

$$-\tfrac{1}{5} A_3^s a^{-5} + A_5^s a^{-3} = -\tfrac{1}{2} \qquad (4.25)$$

$$-\tfrac{1}{5} A_3^s a^{-5} - A_5^s a^{-3} = -\tfrac{1}{2}$$

The solution of the system (4.25) along with the previous constraints on the coefficients require that, in order for equations (4.20) & (4.21) to represent the linear shearing flow of a viscous Newtonian fluid past a solid sphere, it is necessary to choose

$$A_1^s = -\frac{5\eta}{2} a^3 \qquad ; \ \eta = 2\mu$$

$$A_3^s = \frac{5}{2} a^5$$

$$A_6^s = 1 \qquad (4.26)$$

$$A_2^s = A_5^s = A_8^s = 0$$

In his book, Batchelor (1967) has given a general method for the determination of the effective viscosity for a dilute suspension of particles in a linear shearing flow. He has determined that the effective viscosity, η^*, satisfies the relation

$$\eta^* = \eta - \frac{4\pi}{3V_1} \Sigma A_1^5 \qquad (4.27)$$

where η is the viscosity of the ambient fluid, V_1 is the volume of the region under consideration and ΣA_1^5 is the sum of the values A_1^5 for all particles involved. For a single particle having coefficients as given by (4.26) it is found that

$$\frac{\eta^*}{\eta} = 1 + \frac{5\phi}{2} \qquad (4.28)$$

where ϕ is the volume fraction $\frac{4\pi a^3}{3V_1}$. This is exactly the value determined by Einstein (1906, 1911).

B. <u>Micropolar fluid</u>.

The solutions for the linear shearing flow of a micropolar fluid past a solid spherical object are given by equations (4.12), (4.13), & (4.16). The tensorial form of the boundary conditions is

$$p \to 0 \qquad \text{as } r \to \infty \qquad (4.29)$$

$$u_\ell \to a_{\ell j} X_j \qquad \text{as } r \to \infty \qquad (4.30)$$

$$\nu_\ell \text{ finite} \qquad \text{as } r \to \infty \qquad (4.31)$$

$$u_\ell = \tfrac{1}{2}(a_{\ell j} X_j - a_{j \ell} X_j) \qquad \text{at } r = a \qquad (4.32)$$

$$\nu_\ell = \frac{S}{2} \varepsilon_{\ell jk} a_{kj} \qquad \text{at } r = a \qquad (4.33)$$

The boundary conditions (4.29), (4.30), (4.31) when applied to the general solutions (4.12), (4.13), & (4.16) require that the coefficients $\overset{5}{A}_2 = \overset{5}{A}_8 = \overset{5}{C}_3 = \overset{5}{C}_5 = \overset{5}{C}_7 = 0$ and that $-\frac{\overset{5}{A}_6}{L^2} = 1$. The final six boundary conditions as found in equations (4.32), (4.33) when applied to the solutions (4.13), (4.16) result in the set of linear equations

$$\frac{-1}{L^2}(\overset{5}{A}_3 + \frac{10}{\eta}\overset{5}{A}_1)a^{-7} + \frac{1}{\eta}\overset{5}{A}_1 a^{-5} = \frac{-5L^2}{3}(\overset{5}{C}_4 + \overset{5}{C}_6)\, d_2(La)e^{-La}$$

$$\frac{1}{5L^2}(\overset{5}{A}_3 + \frac{10}{\eta}\overset{5}{A}_1)a^{-5} - \frac{1}{L^2}\overset{5}{A}_5 a^{-3} = -\tfrac{1}{2}+\tfrac{1}{3}\overset{5}{C}_4\left[(La)^2 d_2(La) + 3d_4(La)\right]e^{-La} +$$

$$+ \tfrac{1}{3}\overset{5}{C}_6 (La)^2 d_2(La)e^{-La}$$

$$\frac{1}{5L^2}(\overset{5}{A}_3 + \frac{10}{\eta}\overset{5}{A}_1)a^{-5} + \frac{1}{L^2}\overset{5}{A}_5 a^{-3} = -\tfrac{1}{2}+\tfrac{1}{3}\overset{5}{C}_4 (La)^2 d_2(La)e^{-La} +$$

$$+ \tfrac{1}{3}\overset{5}{C}_6\left[(La)^2 d_2(La) + 3d_4(La)\right]e^{-La}$$

$$\left(\frac{\overset{5}{A}_1}{2\eta} - \frac{3\overset{5}{A}_5}{2L^2}\right)a^{-5} = \frac{-\eta}{3\gamma}(4\overset{5}{C}_4 + \overset{5}{C}_6)\left(\frac{d_4'(La) - d_4(La)}{La}\right)e^{-La} +$$

$$+ \frac{\kappa}{\mu+\kappa}\,\overset{5}{C}_8\left(\frac{d_4'(Ma) - d_4(Ma)}{Ma}\right)e^{-Ma}$$

$$\left(\frac{\overset{5}{A}_1}{2\eta} + \frac{3\overset{5}{A}_5}{2L^2}\right)a^{-5} = \frac{-\eta}{3\gamma}(\overset{5}{C}_4 + 4\overset{5}{C}_6)\left(\frac{d_4'(La) - d_4(La)}{La}\right)e^{-La} -$$

$$- \frac{\kappa}{\mu+\kappa}\,\overset{5}{C}_8\left(\frac{d_4'(Ma) - d_4(Ma)}{Ma}\right)e^{-Ma}$$

$$-\frac{1}{L^2}\overset{5}{A}_5 a^{-3} = \frac{S-1}{2} + \frac{\mu+\kappa}{\kappa}\, d_4(La)(\overset{5}{C}_4 - \overset{5}{C}_6)e^{-La}$$

$$+ \frac{\delta}{2\kappa}\left[d_4(Ma) + (Ma)\left[d_4'(Ma) - d_4(Ma)\right]\right]\overset{5}{C}_8 e^{-Ma} \tag{4.34}$$

The solutions to this set of six linear algebraic equations with $\eta = 2\mu+\kappa$ are

$$\frac{-1}{L^2}(A_3^5 + \frac{10}{\eta}A_1^5) = \frac{5a^5}{2}\left[1+\left(\frac{2L^2a^2}{3}d_2(La)+d_4(La)\right)(C_4^5+C_6^5)e^{-La}\right]$$

$$\frac{1}{\eta}A_1^5 = -\frac{5a^3}{2}\left[1-d_4(La)(C_4^5+C_6^5)e^{-La}\right]$$

$$-\frac{1}{L^2}A_5^5 = \frac{a^3}{2}d_4(La)(C_4^5-C_6^5)e^{-La}$$

$$(C_4^5+C_6^5)e^{-La} = \frac{-3\kappa L^3a^3}{\kappa L^2a^2 + \eta\left[3 + 3La + L^2a^2\right]}$$

$$-\eta\left(\frac{1+La}{L^3}\right)(C_4^5-C_6^5)e^{-La}+\delta\left(\frac{2+2Ma+M^2a^2}{M^3}\right)C_8^5e^{-Ma}=(S-1)\kappa a^3$$

$$(\mu+\kappa)\left[\frac{(3+3LA+L^2a^2)+\frac{\kappa}{2(\mu+\kappa)}(1+La)}{L^3}\right](C_4^5-C_6^5)e^{-La} +$$

$$+\kappa\delta\left(\frac{3+3Ma+M^2a^2}{M^3}\right)C_8^5e^{-Ma} = 0 \qquad (4.35)$$

Again, using equation (4.27) it is found that the effective viscosity for the linear shearing flow of a micropolar fluid past a solid spherical body is given by

$$\frac{\eta^*}{\eta} = 1-\frac{4\pi}{3V_I}\sum\left(\frac{-5a^3}{2} + \frac{5a^3}{2}\left(\frac{1+La}{L^3a^3}\right)\left[\frac{-3\kappa L^3a^3}{\kappa(L^2a^2)+\eta(3+3La+L^2a^2)}\right]\right)$$

$$= 1+\frac{5\phi}{2}\left\{\frac{\eta(3+3LA+L^2a^2)+\kappa(3+3La+L^2a^2)}{\eta(3+3La+L^2a^2)+\kappa(L^2a^2)}\right\} \qquad (4.36)$$

A close examination of the value of the effective viscosity as expressed by equation (4.36) shows that the ratio of effective viscosity to external viscosity is independent of the choice of parameter S used in the compromise boundary condition (4.32). Also, this effective viscosity ratio is always going to be greater than the effective viscosity ratio for the linear shearing flow of a viscous fluid past a solid sphere as expressed in equation (4.28). As before ϕ is the volume fraction $\frac{4\pi a^3}{3V_1}$ and it is noted that (4.36) reduces to (4.28) if $\kappa = 0$. Also it is noted that the first portion of the value for each of the coefficients in this solution are essentially the same as the value in the corresponding power of r for the viscous Newtonian flow case. The second portion is due to the presence of the microrotation in the flow field.

In the work which appears in the next sections, we follow the assumption of Taylor and others when dealing with fluid sphere problems. The assumption made is that the drops are small enough or the flow conditions slow enough so that interfacial tension keeps the drop nearly spherical. In this way, the normal stress balance, to this order of approximation, can be neglected. However, the shear stresses are balanced and a condition of zero net force and torque is assumed to hold.

Section 4. Viscous sphere in a viscous fluid.

The solutions for a viscous Newtonian fluid subjected to a very slow linear shear flow past a viscous Newtonian fluid sphere were first obtained by Taylor (1932). The method used by Taylor, however, employed Lamb's general solutions. For completeness in this work these solutions will again be developed using the method developed in Chapter II and the appendix so that a ready reference is available for the following sections. The general solutions are given by equations (4.20) & (4.21). In this section the boundary conditions will first be stated and then the general solutions for both the internal and external flows will be listed and used in the boundary conditions to determine the remaining coefficients and the effective viscosity. A comparison of this value for the effective viscosity is made with the value for the solid sphere (equation 4.28).

The boundary conditions for the linear shearing flow of a viscous Newtonian fluid past a viscous Newtonian fluid sphere are expressed in tensor form by

$$p^{(e)} \to 0 \qquad \text{as} \quad r \to \infty \tag{4.37}$$

$$u_\ell^{(e)} \to a_{\ell j} X_j \qquad \text{as} \quad r \to \infty \tag{4.38}$$

$$u_\ell^{(i)} \quad \text{finite} \qquad \text{at} \quad r = 0 \tag{4.39}$$

$$u_\ell^{(e)} X_\ell = u_\ell^{(i)} X_\ell = 0 \qquad \text{at } r = a \qquad (4.40)$$

$$u_\ell^{(e)} - u_p^{(e)} \frac{X_p X_\ell}{r^2} = u_\ell^{(i)} - u_p^{(i)} \frac{X_p X_\ell}{r^2} \qquad \text{at } r = a \qquad (4.41)$$

$$t_{ij}^{(e)} X_j - t_{mn}^{(e)} \frac{X_m X_n}{r^2} X_i = t_{ij}^{(i)} X_j - t_{mn}^{(i)} \frac{X_m X_n}{r^2} X_i$$

$$\text{at } r = a \qquad (4.42)$$

Using the conditions (4.37) & (4.38) the pressure and velocity distribution for the external flow are, in general with $\eta^{(e)} = 2\mu^{(e)}$, found to be given by

$$p^{(e)} = A_1^5 r^{-5} a_{ij} X_i X_j \qquad (4.43)$$

$$u_\ell^{(e)} = (A_3^5 r^{-7} + \frac{1}{\eta^{(e)}} A_1^5 r^{-5}) a_{ij} X_i X_j X_\ell +$$

$$+(-\tfrac{1}{5} A_3^5 r^{-5} + A_5^5 r^{-3} + 1) a_{\ell j} X_j + (-\tfrac{1}{5} A_3^5 r^{-5} - A_5^5 r^{-3}) a_{j\ell} X_j$$

$$(4.44)$$

In a similar fashion using the condition (4.39) with $\eta^{(i)} = 2\mu^{(i)}$ the pressure and velocity distribution for the internal flow are described by

$$p^{(i)} = A_2^5 a_{ij} X_i X_j \qquad (4.45)$$

$$u_\ell^{(i)} = \left(-\frac{4}{21\eta} A_2^5\right) a_{ij} X_i X_j X_\ell +$$

$$+ (A_6^5 + \frac{5}{21\eta} A_2^5 r^2) a_{\ell j} X_j + (A_8^5 + \frac{5}{21\eta} A_2^5 r^2) a_{j\ell} X_j$$

$$(4.46)$$

Applying the boundary conditions (4.40) - (4.42) to the internal and external flows results in the following set of linear algebraic equations for the undetermined coefficients

$$\frac{1}{5}A_3^5 a^{-5} + \frac{1}{\eta^{(e)}} A_1^5 a^{-3} + 1 = 0$$

$$\frac{2}{7\eta} A_2^5 a^2 + A_6^5 + A_8^5 = 0$$

$$-\frac{1}{5}A_3^5 a^{-5} + A_5^5 a^{-3} + 1 = \frac{5}{21\eta} A_2^5 a^2 + A_6^5$$

$$-\frac{1}{5}A_3^5 a^{-5} - A_5^5 a^{-3} = \frac{5}{21\eta} A_2^5 a^2 + A_8^5$$

$$\eta^{(e)}\left[\frac{8}{5}A_3^5 a^{-7} + \frac{1}{\eta^{(e)}} A_1^5 a^{-5} + a^{-2}\right] = \eta^{(i)}\left[\frac{16}{21\eta^{(i)}} A_2^5 + \frac{A_6^5 + A_8^5}{a^2}\right]$$

$$\eta^{(e)}\left[\frac{8}{5}A_3^5 a^{-5} + \frac{1}{\eta^{(e)}} A_1^5 a^{-3} + 3A_5^5 a^{-3} + 1\right] = \eta^{(i)}\left[\frac{16}{21\eta^{(i)}} A_2^5 a^2 + A_6^5 + A_8^5\right]$$

(4.47)

Defining $\sigma = \frac{\eta^{(i)}}{\eta^{(e)}}$ the solutions to the system of equations (4.47) are found to be

$$\frac{1}{\eta^{(e)} a^3} A_1^5 = \frac{-(5\sigma+2)}{2(1+\sigma)}$$

$$\frac{4a^2}{21\eta^{(i)}} A_2^5 = \frac{1}{1+\sigma}$$

$$\frac{1}{a^5} A_3^5 = \frac{5\sigma}{2(1+\sigma)}$$

(4.48)

$$\frac{1}{a^3} A_5^5 = 0$$

$$A_6^5 = \frac{2\sigma-1}{4(1+\sigma)}$$

$$A_8^5 = -\frac{(2\sigma+5)}{4(1+\sigma)} \tag{4.48}$$

In order to determine the value of the effective viscosity it is necessary to use the value of the coefficient A_1^5 as given in equation (4.48) in the formula for the effective viscosity as expressed by (4.27). Hence,

$$\frac{\eta^*}{\eta^{(e)}} = 1 + \frac{5\phi}{2}\left[\frac{\sigma+\frac{2}{5}}{1+\sigma}\right] \tag{4.49}$$

This result is in total agreement with that expressed by Taylor (1932) and also reduces to (4.28) if $\eta^{(i)} \to \infty$ (i.e. the solid sphere case).

Peery (1966) considered a similar problem in his thesis but no consideration of the effective viscosity was included.

Section 5. Micropolar sphere in a viscous fluid.

A natural extension of the results in the previous section is the consideration of a linear shearing flow of an inertialess viscous Newtonian fluid past a micropolar fluid sphere. The general solutions for both the external (equations (4.20) & (4.21)) and the internal (equations (4.12), (4.13), & (4.16)) flow will be restated after boundary conditions are detailed. The boundary conditions will then be applied to the general solutions and the resulting set of linear algebraic equations will be solved. From the calculated value of A_1^5 the effective viscosity will be determined and a comparison will be made with the values expressed in (4.27) & (4.49).

The boundary conditions for the linear shearing flow of a viscous Newtonian fluid past a micropolar fluid sphere in tensor form are found to be

$$\left.\begin{aligned} p^{(e)} &\to 0 \\ u_\ell^{(e)} &\to a_{\ell j} X_j \end{aligned}\right\} \quad \text{as } r \to \infty \qquad (4.50)$$

$$\left.\begin{aligned} u_\ell^{(i)} &\quad \text{finite} \\ \nu_\ell^{(i)} &\quad \text{finite} \end{aligned}\right\} \quad \text{at } r = 0 \qquad (4.51)$$

$$u_\ell^{(e)}X_\ell = u_\ell^{(i)}X_\ell = 0 \qquad \text{at } r = a \qquad (4.52)$$

$$u_\ell^{(e)} - u_p^{(e)} \frac{X_p X_\ell}{r^2} = u_\ell^{(i)} - u_p^{(i)} \frac{X_p X_\ell}{r^2} \qquad \text{at } r = a \qquad (4.53)$$

$$t_{ij}^{(e)}X_j - t_{mn}^{(e)} \frac{X_m X_n}{r^2} X_i = t_{ij}^{(i)}X_j - t_{mn}^{(i)} \frac{X_m X_n}{r^2} X_i \qquad (4.54)$$

at $r = a$

$$\nu_\ell^{(i)} = \frac{S}{2} \varepsilon_{\ell jk} u_{k,j}^{(e)}, \qquad 0 \le S \le 1 \qquad \text{at } r = a \qquad (4.55)$$

Using conditions (4.50) in the general solutions as given by equations (4.20) & (4.21) the pressure and velocity distrubution are given by

$$p^{(e)} = A_1^5 r^{-5} a_{ij} X_i X_j \qquad (4.56)$$

$$u_\ell^{(e)} = (A_3^5 r^{-7} + \frac{1}{\eta^e} A_1^5 r^{-5}) a_{ij} X_i X_j X_\ell +$$

$$+(-\tfrac{1}{5}A_3^5 r^{-5} + A_5^5 r^{-3} + 1) a_{\ell j} X_j + (-\tfrac{1}{5}A_3^5 r^{-5} - A_5^5 r^{-3}) a_{j\ell} X_j$$

where $\eta^e = 2\mu^e$

In a similar fashion applying the conditions (4.51) to the general solutions for the interior flow as expressed by equations (4.12), (4.13) & (4.16) it is found that the appropriate solutions are

$$p^{(i)} = A_2^5 a_{ij} X_i X_j \qquad (4.58)$$

$$u_\ell^{(i)}=\left[\frac{-4}{21\eta^i}A_2^5+\frac{5L^2(C_4^5+C_6^5)}{3}(d_1(Lr)e^{Lr}+d_2(Lr)e^{-Lr})\right]a_{ij}X_iX_jX_\ell+$$

$$+\left[\frac{1}{L^2}\left(\frac{2}{\eta^i}A_2^5-A_6^5\right)+\frac{5}{21\eta^i}A_2^5r^2-\frac{L^2r^2}{3}(C_4^5+C_6^5)(d_1(Lr)e^{Lr}+d_2(Lr)e^{-Lr})\right.$$

$$\left.-C_4^5(d_3(Lr)e^{Lr}+d_4(Lr)e^{-Lr})\right]a_{\ell j}X_j\quad+$$

$$+\left[\frac{1}{L^2}\left(\frac{2}{\eta^i}A_2^5-A_8^5\right)+\frac{5}{21\eta^i}A_2^5r^2-\frac{L^2r^2}{3}(C_4^5+C_6^5)(d_1(Lr)e^{Lr}+d_2(Lr)e^{-Lr})\right.$$

$$\left.-C_6^5(d_3(Lr)e^{Lr}+d_4(Lr)e^{-Lr})\right]a_{j\ell}X_j$$

(4.59)

$$\nu_p^{(i)}=\left[\frac{-A_2^5}{3\eta^i}+\frac{\eta^i}{3\gamma}(4C_4^5+C_6^5)\left(\frac{d_3'(Lr)+d_3(Lr)}{Lr}e^{Lr}+\frac{d_4'(Lr)-d_4(Lr)}{Lr}e^{-Lr}\right)\right.$$

$$\left.-C_8^5\left(\frac{d_3'(Mr)+d_3(Mr)}{Mr}e^{Mr}+\frac{d_4'(Mr)-d_4(Mr)}{Mr}e^{-Mr}\right)\right]\varepsilon_{p\ell m}a_{\ell j}X_jX_m$$

$$+\left[\frac{-A_2^5}{3\eta^i}+\frac{\eta^i}{3\gamma}(C_4^5+4C_6^5)\left(\frac{d_3'(Lr)+d_3(Lr)}{Lr}e^{Lr}+\frac{d_4'(Lr)-d_4(Lr)}{Lr}e^{-Lr}\right)\right.$$

$$\left.+C_8^5\left(\frac{d_3'(Mr)+d_3(Mr)}{Mr}e^{Mr}+\frac{d_4'(Mr)-d_4(Mr)}{Mr}e^{-Mr}\right)\right]\varepsilon_{p\ell m}a_{j\ell}X_jX_m$$

$$+\left[\frac{-(A_6^5-A_8^5)}{2L^2}-\frac{\mu+\kappa}{\kappa}(C_4^5-C_6^5)(d_3(Lr)e^{Lr}+d_4(Lr)e^{-Lr})\;-\right.$$

$$-\frac{\delta}{2\kappa}C_8^5\left(\left[d_3(Mr)+(Mr)(d_3'(Mr)+d_3(Mr))\right]e^{Mr}+\right.$$

$$\left.\left.+\left[d_4(Mr)+(Mr)(d_4'(Mr)-d_4(Mr))\right]e^{-Mr}\right)\right]\varepsilon_{p\ell m}a_{m\ell}$$

(4.60)

The application of the boundary conditions as given by equations (4.52) - (4.54) to the internal and external flows results in the following set of linear algebraic equations for the coefficients

$$\tfrac{3}{5}\overset{5}{A}_3 a^{-5}+ \frac{1}{\eta^e} \overset{5}{A}_1 a^{-3}+1 = 0$$

$$\frac{1}{L^2}\left(\frac{4}{\eta^i} \overset{5}{A}_2-(\overset{5}{A}_6+\overset{5}{A}_8)\right) +\frac{2}{7\eta^i} \overset{5}{A}_2 a^2+(\overset{5}{C}_4+\overset{5}{C}_6)\left[\left(L^2a^2d_1(La)-d_3(La)\right) e^{La}+ \right.$$

$$\left. +\left(L^2a^2d_2(La)-d_4(La)\right) e^{-La} \right]= 0$$

$$-\tfrac{1}{5}\overset{5}{A}_3 a^{-5}+\overset{5}{A}_5 a^{-3}+1 = \frac{1}{L^2}(\frac{2}{\eta^i} \overset{5}{A}_2-\overset{5}{A}_6)+ \frac{5}{21\eta^i} \overset{5}{A}_2 a^2 -$$

$$-(\overset{5}{C}_4+\overset{5}{C}_6)\left[\frac{(L^2a^2)}{3} \left(d_1(La) e^{La}+d_2(La) e^{La}\right)\right] -$$

$$-\overset{5}{C}_4 \left[d_3(La) e^{La}+d_4(La) e^{-La}\right]$$

$$-\tfrac{1}{5}\overset{5}{A}_3 a^{-5}-\overset{5}{A}_5 a^{-3} = \frac{1}{L^2}(\frac{2}{\eta^i} \overset{5}{A}_2-\overset{5}{A}_8)+\frac{5}{21\eta^i} \overset{5}{A}_2 a^2 -$$

$$-(\overset{5}{C}_4+\overset{5}{C}_6)\left[\frac{(L^2a^2)}{3}\left(d_1(La) e^{La}+d_2(La) e^{-La}\right)\right] -$$

$$-\overset{5}{C}_6 \left[d_3(La) e^{La}+d_4(La) e^{-La}\right]$$

$$\frac{\eta^e}{2}\left[\tfrac{4}{5}\overset{5}{A}_3 a^{-5}+ \frac{1}{\eta^e} \overset{5}{A}_1 a^{-3}-3\overset{5}{A}_5 a^{-3}+1\right] = \frac{\eta^i}{2} \left\{ \frac{1}{L^2}\left(\frac{4}{\eta^i} \overset{5}{A}_2-(\overset{5}{A}_6+\overset{5}{A}_8)\right) + \right.$$

$$+\frac{16}{21\eta^i} \overset{5}{A}_2 a^2- \frac{2(\overset{5}{C}_4+\overset{5}{C}_6)L^2a^2}{3} \left[\left(Lad_1'(La)+3d_1(La)+Lad_1(La)\right) e^{La}+ \right.$$

$$\left. +\left(Lad_2'(La)+d_2(La)-Lad_2(La)\right)e^{-La} \right] - \qquad (4.61)$$

$$-2C_4^5\left[\left(Lad_3'(La)+d_3(La)+Lad_3(La)\right)e^{La}+\right.$$

$$\left.+\left(Lad_4'(La)+d_4(La)-Lad_4(La)\right)e^{-La}\right]-$$

$$\left.-\frac{\delta C_8^5}{\eta}\left(d_3(Ma)e^{Ma}+d_4(Ma)e^{-Ma}\right)\right\}$$

$$\frac{\eta^e}{2}\left[-\frac{16}{5}A_3^5a^{-7}-\frac{2}{\eta^e}A_1^5a^{-5}-\frac{2}{a^2}\right]=\frac{\eta^i}{2}\left\{\frac{-2}{L^2a^2}\left(\frac{4}{\eta^i}A_2^5-(A_6^5+A_8^5)\right)-\right.$$

$$-\frac{32}{21\eta^i}A_2^5+\frac{4L^2}{3}(C_4^5+C_6^5)\left[\left(Lad_1'(La)+3d_1(La)+Lad_1(La)\right)e^{La}+\right.$$

$$\left.+\left(Lad_2'(La)+3d_2(La)-Lad_2(La)\right)e^{-La}\right]+$$

$$+2L^2(C_4^5+C_6^5)\left[\left(\frac{Lad_3'(La)+d_3(La)+Lad_3(La)}{L^2a^2}\right)e^{La}+\right.$$

$$\left.\left.+\left(\frac{Lad_4'(La)+d_4(La)-Lad_4(La)}{L^2a^2}\right)e^{-La}\right]\right\}$$

$$(4.61)$$

The final boundary condition as expressed by equation (4.55) leads to the following

$$\frac{-A_2^5}{3\eta^i}+\frac{\eta^i}{3\gamma}(4C_4^5+C_6^5)\left(\frac{d_3'(La)+d_3(La)}{La}e^{La}+\frac{d_4'(La)-d_4(La)}{La}e^{-La}\right)-$$

$$-C_8^5\left(\frac{d_3'(Ma)+d_3(Ma)}{Ma}e^{Ma}+\frac{d_4'(Ma)-d_4(Ma)}{Ma}e^{-Ma}\right)=$$

$$=\frac{S}{2}\left(\frac{1}{\eta^e}A_1^5+3A_5^5\right)a^{-5}$$

$$\frac{-A_2^5}{3\eta^i}+\frac{\eta^i}{3\gamma}(C_4^5+4C_6^5)\left(\frac{d_3'(La)+d_3(La)}{La}e^{La}+\frac{d_4'(La)-d_4(La)}{La}e^{-La}\right)+$$

$$+C_8^5\left(\frac{d_3'(Ma)+d_3(Ma)}{Ma}e^{Ma}+\frac{d_4'(Ma)-d_4(Ma)}{Ma}e^{-Ma}\right)=\frac{S}{2}\left(\frac{1}{\eta^e}A_1^5-3A_5^5\right)a^{-5}$$

$$\frac{-(A_6^5-A_8^5)}{2L^2} - \frac{\mu+\kappa}{\kappa}(C_4^5-C_6^5)\left(d_3(La)e^{La}+d_4(La)e^{-La}\right) -$$

$$-\frac{\delta}{2\kappa}C_8^5\Big[\Big(Mad_3'(Ma)+d_3(Ma)+Mad_3(Ma)\Big)e^{Ma} +$$

$$+\Big(Mad_4'(Ma)+d_4(Ma)-Mad_4(Ma)\Big)e^{-Ma}\Big] = \frac{S}{2}\left(2A_3^5a^{-3}+1\right) \tag{4.62}$$

When the system of equations as given by the equations (4.61) & (4.62) is solved, it is found that the values of the coefficients, with $\sigma = \frac{\eta^i}{\eta^e}$, are

$$A_3^5a^{-5} = \frac{5\sigma}{2(1+\sigma)} + \frac{5\sigma}{2(1+\sigma)}\Bigg[\left(\frac{7L^2a^2d_1(La)-2La(d_3'(La)+d_3(La))}{3}e^{La}\right)+$$

$$+\left(\frac{7L^2a^2d_2(La)-2La(d_4'(La)-d_4(La))}{3}e^{-La}\right)\Bigg](C_4^5+C_6^5)$$

$$\frac{A_1^5}{\eta^e a^3} = \frac{-(5\sigma+2)}{2(1+\sigma)} - \frac{\sigma}{2(1+\sigma)}\Bigg[\Big[7L^2a^2d_1(La)-2La(d_3'(La)+d_3(La))\Big]e^{La}+$$

$$+\Big[7L^2a^2d_2(La)-2La(d_4'(La)-d_4(La))\Big]e^{-La}\Bigg](C_4^5+C_6^5)$$

$$\frac{1}{L^2}\left(\frac{2}{\eta^i}A_2^5-A_6^5\right) = \frac{2\sigma-1}{4(1+\sigma)} - \frac{(C_4^5+C_6^5)}{4(1+\sigma)}\Bigg[\Big[7L^2a^2d_1(La)+2La(d_3'(La)+d_3(La))-$$

$$-(2\sigma+2)d_3(La)\Big]e^{La} + \Big[7L^2a^2d_2(La)+2La(d_4'(La)-d_4(La))-$$

$$-(2\sigma+2)d_4(La)\Big]e^{-La}\Bigg] + \frac{1}{6}\Bigg[\Big[2\sigma La(d_3'(La)+d_3(La))+$$

$$+(2\sigma+3)d_3(La)\Big]e^{La} + \Big[2\sigma La(d_4'(La)-d_4(La)) +$$

$$+(2\sigma+3)d_4(La)\Big]e^{-La}\Bigg](C_4^5-C_6^5) +$$

$$+\frac{\delta\sigma}{3\eta^i}\Big[d_3(Ma)e^{Ma}+d_4(Ma)e^{-Ma}\Big]C_8^5$$

$$\frac{1}{L^2}\left(\frac{2}{\eta^I}A_2^5-A_8^5\right) = \frac{-(2\sigma+5)}{4(1+\sigma)} - \frac{(C_4^5+C_6^5)}{4(1+\sigma)}\{\Big[7L^2a^2d_1(La)+2La(d_3'(La) + $$

$$+d_3(La))-(2\sigma+2)d_3(La)\Big]e^{La}+\Big[7L^2a^2d_2(La)+2La(d_4'(La)-$$

$$-d_4(La))-(2\sigma+2)d_4(La)\Big]e^{-La}\}-\tfrac{1}{4}\{\Big[2\sigma La(d_3'(La)+d_3(La))+$$

$$+(2\sigma+3)d_3(La)\Big]e^{La}+\Big[2\sigma La(d_4'(La)-d_4(La)+(2\sigma+3)d_4(La)\Big]e^{-La}\}\cdot$$

$$(C_4^5-C_6^5) - \frac{\delta\sigma}{3\eta^I}\Big[d_3(Ma)e^{Ma}+d_4(Ma)e^{-Ma}\Big]C_8^5$$

$$\frac{4a^2}{21\eta^I}A_2'^5 = \frac{1}{1+\sigma} + \frac{1}{3(1+\sigma)}\{\Big[2\sigma La(d_3'(La)+d_3(La))-$$

$$-(2\sigma-5)L^2a^2d_1(La)\Big]e^{La} + \Big[2\sigma La(d_4'(La)-d_4(La)) -$$

$$-(2\sigma-5)L^2a^2d_2(La)\Big]e^{-La}\}(C_4^5+C_6^5)$$

$$\frac{A_5^5}{a^3} = \frac{\sigma}{3}\{\Big[La(d_3'(La)+d_3(La))+d_3(La)\Big]e^{La} +$$

$$+\Big[La(d_4'(La)-d_4(La))+d_4(La)\Big]e^{-La}\}(C_4^5-C_6^5) +$$

$$+\frac{\delta\sigma}{3\eta^I}\Big[d_3(Ma)e^{Ma}+d_4(Ma)e^{-Ma}\Big]C_8^5$$

(4.63)

where the values for the coefficients C_4^5, C_6^5, C_8^5 as calculated from (4.62) are

$$(C_4^5+C_6^5) = 3\kappa\Big[5S\sigma+(2S-7)\Big] \div \Big[6\kappa S\sigma^2+(6\kappa-8\beta)\sigma-8(\mu+\kappa)\Big]$$

$$\{\Big[La(d_3'(La)+d_3(La))\Big]e^{La}+\Big[La(d_4'(La)-d_4(La))\Big]e^{-La}\}-$$

$$-7\kappa\Big[(2+3S)\sigma-5\Big]\{L^2a^2d_1(La)e^{La}+L^2a^2d_2(La)e^{-La}\}$$

$$\Big\{\frac{\mu^i+\kappa^i-S\kappa^i\sigma}{\kappa^i}\Big[La(d_3'(La)+d_3(La))e^{La} + La(d_4'(La)-d_4(La))e^{-La}\Big] -$$

$$-S\sigma\Big[d_3(La)e^{La}+d_4(La)e^{-La}\Big]\Big\}\ (C_4^5-C_6^5) -$$

$$-\frac{\delta^i}{\eta^i}\Big\{\frac{\gamma^i L^2}{\kappa^i}\Big[Ma(d_3'(Ma)+d_3(Ma))e^{Ma}+Ma(d_4'(Ma)-d_4(Ma))e^{-Ma}\Big] +$$

$$+S\sigma\Big[d_3(Ma)e^{Ma}+d_4(Ma)e^{-Ma}\Big]\Big\}C_8^5 = 0$$

$$\Big\{\frac{(S-1)\sigma}{3}\Big[La(d_3'(La)+d_3(La))e^{La}+La(d_4'(La)-d_4(La))e^{-La}\Big] +$$

$$+\left(\frac{(S-1)\sigma}{3}+\frac{\eta^i}{2\kappa^i}\right)\Big[d_3(La)e^{La}+d_4(La)e^{-La}\Big]\Big\}(C_4^5-C_6^5) +$$

$$+\frac{\delta^i}{\eta^i}\Big\{\frac{\eta^i}{2\kappa^i}\Big[Ma(d_3'(Ma)+d_3(Ma))e^{Ma}+Ma(d_4'(Ma)-d_4(Ma))e^{-Ma}\Big] +$$

$$+\left(\frac{(S-1)\sigma}{3}+\frac{\eta^i}{2\kappa^i}\right)\Big[d_3(Ma)e^{Ma}+d_4(Ma)e^{-Ma}\Big]\Big\}C_8^5 = \frac{1-S}{2}$$

(4.64)

As in previous sections, the effective viscosity for a micropolar fluid drop in a viscous Newtonian fluid is determined by the substitution of A_1^5, as given in (4.63), into the general equation (4.27). When this is done, it is found that

$$\frac{\eta^*}{\eta^e} = 1 + \frac{5\phi}{2}\left(\frac{\sigma+\frac{2}{5}}{\sigma+1}\right)\Big[1+\frac{\sigma}{5\sigma+2}\{\big[7L^2a^2d_1(La)-2\sigma La(d_3'(La) +$$

$$+d_3(La))\big]e^{La} + \big[7L^2a^2d_2(La)-2\sigma La(d_4'(La) +$$

$$+d_4(La))\big]e^{-La}\}\ (C_4^5+C_6^5)\Big] \tag{4.65}$$

where

$$(C_4^5+C_6^5) = 3\kappa\big[5S\sigma+(2S-7)\big]\div\big[6\kappa 5\sigma^2+(6\kappa-8\mu)\sigma-8(\mu+\kappa)\big]\{\big[La(d_3'(La)+$$

$$+d_3(La))\big]e^{La} + \big[La(d_4'(La)-d_4(La))\big]e^{-La}\}-$$

$$-7\kappa\big[(2+3S)\sigma-5\big]\{L^2a^2d_1(La)e^{La}+L^2a^2d_2(La)e^{-La}\}$$

A comparison of equations (4.49) & (4.65) makes evident the similarities and differences in the two values. A relatively simple calculation will show that (4.65) reduces to (4.49) if the coupling parameter κ approaches zero. And it has been explained previously that if the interior viscosity becomes infinite (a solid sphere) then (4.49) reduces to (4.28). Thus (4.65) also reduces to (4.28) when the interior viscosity becomes infinite. A problem similar to the one discussed immediately above has been discussed by Avudainayagam (1976). It is, however, felt that some doubt is cast on the validity of his results because of the nature of the boundary conditions assumed at the interface and also because of the technique employed.

The example considered in this section leads, naturally, to a consideration of the reverse phenomenon (i.e. - a study of the linear shearing flow of a micropolar fluid past a viscous Newtonian fluid sphere). This example is dealt with in some detail in the next section.

Section 6. Viscous sphere in a micropolar fluid.

Continuing the progression of the work in this chapter, the next problem to be considered is the linear shearing flow of a micropolar fluid past a viscous Newtonian fluid sphere. The general solutions for the external and internal flows are given by equations (4.12), (4.13), (4.16) and (4.20), (4.21).

The boundary conditions for the linear shearing flow of a micropolar fluid, where inertial terms are neglected, past a viscous Newtonian fluid sphere, in tensor form, are given by

$$\left.\begin{aligned} & p^{(e)} \to 0 && \text{as } r \to \infty \\ & u_\ell^{(e)} \to a_{\ell j} X_j && \text{as } r \to \infty \\ & \nu_\ell^{(e)} \quad \text{finite} && \text{as } r \to \infty \end{aligned}\right\} \quad (4.66)$$

$$u_\ell^{(i)}\,,\ p^{(i)} \quad \text{finite} \qquad \text{at } r = 0 \qquad (4.67)$$

$$u_\ell^{(e)} X_\ell = u_\ell^{(i)} X_\ell = 0 \qquad \text{at } r = a \qquad (4.68)$$

$$u_\ell^{(e)} - u_p^{(e)} \frac{X_p X_\ell}{r^2} = u_\ell^{(i)} - u_p^{(i)} \frac{X_p X_\ell}{r^2} \qquad \text{at } r = a \qquad (4.69)$$

$$t_{ij}^{(e)} X_j - t_{mn}^{(e)} \frac{X_m X_n}{r^2} X_i = t_{ij}^{(i)} X_j - t_{mn}^{(i)} \frac{X_m X_n}{r^2} X_i$$

$$\text{at } r = a \qquad (4.70)$$

$$\nu_\ell^{(e)} = \frac{S}{2}\, \varepsilon_{\ell jk}\, u_{k,j}^{(i)}, \qquad 0 \le S \le 1 \tag{4.71}$$

The conditions (4.67) when used in the general solutions for the interior flow as given by equations (4.20), (4.21), with $\eta^i = 2\mu^i$, give

$$p^{(i)} = \overset{5}{A}_2 a_{ij} X_i X_j \tag{4.72}$$

$$u_\ell^{(i)} = \left(-\frac{4}{21\eta^i}\overset{5}{A}_2\right) a_{ij} X_i X_j X_\ell + \left(\overset{5}{A}_6 + \frac{5}{21\eta^i}\overset{5}{A}_2 r^2\right) a_{\ell j} X_j +$$

$$+ \left(\overset{5}{A}_8 + \frac{5}{21\eta^i}\overset{5}{A}_2 r^2\right) a_{j\ell} X_j \tag{4.73}$$

In an analogous manner the conditions in (4.66) when applied to the general solutions for the external flow as given by (4.12), (4.13), (4.16), with $\eta^e = (2\mu+\kappa)^e$, give

$$p^{(e)} = (\overset{5}{A}_1 r^{-5}) a_{ij} X_i X_j \tag{4.74}$$

$$u_\ell^{(e)} = \Bigg(-\frac{1}{L^2}\left(\overset{5}{A}_3 + \frac{10}{\eta^e}\overset{5}{A}_1\right) r^{-7} + \frac{1}{\eta^e}\overset{5}{A}_1 r^{-5} +$$

$$+ \frac{5L^2}{3}(\overset{5}{C}_4 + \overset{5}{C}_6) d_2(Lr) e^{-Lr} \Bigg) a_{ij} X_i X_j X_\ell +$$

$$+ \Bigg(\frac{1}{5L^2}\left(\overset{5}{A}_3 + \frac{10}{\eta^e}\overset{5}{A}_1\right) r^{-5} - \frac{1}{L^2}\overset{5}{A}_5 r^{-3} + 1 -$$

$$- \frac{1}{3}\overset{5}{C}_4 (L^2 r^2 d_2(Lr) + 3 d_4(Lr)) e^{-Lr} -$$

$$- \frac{1}{3}\overset{5}{C}_8 L^2 r^2 d_2(Lr) e^{-Lr} \Bigg) a_{\ell j} X_j \;\; +$$

$$+ \left(\frac{1}{5L^2}\left(A_3^5 + \frac{10}{\eta^e}A_1^5\right) r^{-5} + \frac{1}{L^2} A_5^5 r^{-3} - \right.$$

$$- \frac{1}{3} C_6^5 \left(L^2 r^2 d_2(Lr) + 3 d_4(Lr)\right) e^{-Lr} -$$

$$\left. - \frac{1}{3} C_4^5 L^2 r^2 d_2(Lr) e^{-Lr} \right) a_{j\ell} X_j$$

$$\nu_\ell^{(e)} = \left(\frac{1}{2}\left(\frac{A_1^5}{\eta^e} - \frac{3A_5^5}{L^2}\right) r^{-5} + \frac{\eta^e}{3\gamma^e}(4C_4^5 + C_6^5)\left(\frac{d_4'(Lr) - d_4(Lr)}{Lr}\right) e^{-Lr} - \right.$$

$$\left. - \frac{\kappa^e}{(\mu+\kappa)^e} C_8^5 \left(\frac{d_4'(Mr) - d_4(Mr)}{Mr}\right) e^{-Mr} \right) \varepsilon_{\ell mn} a_{mj} X_j X_n \quad +$$

$$+ \left(\frac{1}{2}\left(\frac{A_1^5}{\eta^e} + \frac{3A_5^5}{L^2}\right) r^{-5} + \frac{\eta^e}{3\gamma^e}(C_4^5 + 4C_6^5)\left(\frac{d_4'(Lr) - d_4(Lr)}{Lr}\right) e^{-Lr} + \right.$$

$$\left. + \frac{\kappa^e}{(\mu+\kappa)^e} C_8^5 \left(\frac{d_4'(Mr) - d_4(Mr)}{Mr}\right) e^{-Mr} \right) \varepsilon_{\ell mn} a_{jm} X_j X_n \quad +$$

$$+ \left(- \frac{A_5^5}{L^2} r^{-3} + \frac{1}{2} - \frac{(\mu+\kappa)^e}{\kappa^e} (C_4^5 - C_6^5) d_4(Lr) e^{-Lr} - \right.$$

$$\left. - \frac{\delta^e}{2\kappa^e} C_8^5 \left(Mr\left(d_4'(Mr) - d_4(Mr)\right) + d_4(Mr)\right) e^{-Mr} \right) \varepsilon_{\ell mn} a_{nm} \tag{4.76}$$

Application of the boundary conditions (4.68), (4.69) & (4.70) to the solutions for the internal and external flows as given by (4.72) - (4.76) results in the following set of linear algebraic equations

$$\tfrac{1}{2}\left(\frac{4}{21\eta^{I}} A_2^5 a^2\right) + A_6^5 + A_8^5 = 0$$

$$\frac{-3}{5L^2}\left(A_3^5 + \frac{10}{\eta^e} A_1^5\right)a^{-5} + \frac{1}{\eta^e} A_1^5 a^{-3} + 1 + \left(L^2a^2 d_2(La) - d_4(La)\right)e^{-La}(C_4^5 + C_6^5)$$
$$= 0$$

$$\tfrac{5}{4}\left(\frac{4}{21\eta^{I}} A_2^5 a^2\right) + A_6^5 = \frac{1}{5L^2}\left(A_3^5 + \frac{10}{\eta^e} A_1^5\right)a^{-5} - \frac{1}{L^2} A_5^5 a^{-3} + 1 -$$
$$-d_4(La)e^{-La}C_4^5 - \tfrac{1}{3}\left(L^2a^2 d_2(La)\right)e^{-La}(C_4^5 + C_6^5)$$

$$\tfrac{5}{4}\left(\frac{4}{21\eta^{I}} A_2^5 a^2\right) + A_8^5 = \frac{1}{5L^2}\left(A_3^5 + \frac{10}{\eta^e} A_1^5\right)a^{-5} + \frac{1}{L^2} A_5^5 a^{-3} - d_4(La)e^{-La}C_6^5 -$$
$$- \tfrac{1}{3}\left(L^2a^2 d_2(La)\right)e^{-La}(C_4^5 + C_6^5)$$

$$\eta^{i}\left(4\left(\frac{4}{21\eta^{I}} A_2^5\right) + \left(\frac{A_6^5 + A_8^5}{a^2}\right)\right) = \eta^{e}\left(\frac{-8}{5L^2}\left(A_3^5 + \frac{10}{\eta^e} A_1^5\right)a^{-7} + \frac{1}{\eta^e} A_1^5 a^{-5} \pm\right.$$
$$+ a^{-2} - \frac{L^2}{3}\left[-8d_2(La) + \frac{5(d_4'(La) - d_4(La))}{La} +\right.$$
$$\left.\left. + \frac{3d_4(La)}{L^2a^2}\right] e^{-La}\,(C_4^5 + C_6^5)\right)$$

$$\eta^{i}\left(4\left(\frac{4}{21\eta^{I}} A_2^5 a^2\right) + A_6^5 + A_8^5\right) = \eta^{e}\left(\frac{-8}{5L^2}\left(A_3^5 + \frac{10}{\eta^e} A_1^5\right)a^{-5} + \frac{1}{\eta^e} A_1^5 a^{-3} +\right.$$
$$+ \frac{3}{L^2} A_5^5 a^{-3} + 1 - \frac{C_4^5}{3}\left[-8L^2a^2 d_2(La) +\right.$$
$$\left. + 8La(d_4'(La) - d_4(La)) + 6d_4(La)\right] e^{-La} -$$
$$- \frac{C_6^5}{3}\left[-8L^2a^2 d_2(La) + 2La(d_4'(La) - d_4(La))\right] e^{-La} -$$
$$\left. - \frac{\delta^e}{\eta^e} d_4(Ma)e^{-Ma}\, C_8^5\right) \qquad (4.77)$$

and the final boundary conditions, equation (4.71), result in the following set of algebraic equations

$$\frac{1}{2}\left(\frac{1}{\eta^e}A_1^5 - \frac{3}{L^2}A_5^5\right)a^{-5} + \frac{\eta^e}{3\gamma^e}\left(\frac{d_4'(La)-d_4(La)}{La}\right)(4C_4^5+C_6^5)e^{-La} -$$

$$- \frac{\kappa^e}{(\mu+\kappa)^e}\left(\frac{d_4'(Ma)-d_4(Ma)}{Ma}\right)C_8^5 e^{-Ma} = \frac{7S}{4}\left(\frac{4}{21\eta^i}A_2^5\right)$$

$$\frac{1}{2}\left(\frac{1}{\eta^e}A_1^5 + \frac{3}{L^2}A_5^5\right) + \frac{\eta^e}{3\gamma^e}\left(\frac{d_4'(La)-d_4(La)}{La}\right)(C_4^5+4C_6^5)e^{-La} +$$

$$+ \frac{\kappa^e}{(\mu+\kappa)^e}\left(\frac{d_4'(Ma)-d_4(Ma)}{Ma)}\right)C_8^5 e^{-Ma} = \frac{-7S}{4}\left(\frac{4}{21\eta^i}A_2^5\right)$$

$$\frac{-1}{L^2}A_5^5 a^{-3} + \frac{1}{2} - \frac{(\mu+\kappa)^e}{\kappa^e}\, d_4(La)(C_4^5-C_6^5)e^{-La} -$$

$$-\frac{\delta^e}{2\kappa^e}\left[Ma(d_4'(Ma)-d_4(Ma))+d_4(Ma)\right]C_8^5 e^{-Ma} = \frac{S}{2}(A_6^5-A_8^5)$$

(4.78)

With $\sigma = \frac{\eta^i}{\eta^e}$ the values of the coefficients in equations (4.77) & (4.78) are found to be

$$\frac{-1}{L^2}\left(A_3^5 + \frac{10}{\eta^e}A_1^5\right)a^{-5} = \frac{5\sigma}{2(1+\sigma)} - \frac{5}{2(1+\sigma)}\Big[2L^2a^2d_2(La) - 2(La)(d_4'(La)-d_4(La)) +$$

$$+ 3\sigma d_4(La) + 3\Big](C_4^5+C_6^5)e^{-La}$$

$$\frac{1}{\eta^e} A_1^5 a^{-3} = -\left(\frac{5\sigma+2}{2(1+\sigma)}\right) -$$

$$-\left(\frac{5\sigma+2}{2(1+\sigma)}\right)\left[\frac{2La(d_4'(La)-d_4(La))}{5\sigma+2} - d_4(La)\right] (C_4^5+C_6^5)e^{-La}$$

$$\frac{1}{L^2}A_3^5 a^{-3} = \left[\frac{La(d_4'(La)-d_4'(La))+d_4(La)}{3}\right](C_4^5-C_6^5)e^{-La} +$$

$$+ \frac{\delta^e}{3\eta^e} d_4(Ma)\ C_8^5 e^{-Ma}$$

$$\frac{4a^2}{21\eta^1} A_2^5 = \frac{1}{1+\sigma} - \frac{1}{1+\sigma}\left[\frac{2La(d_4'(La)-d_4(La))+3d_4(La)}{3}\right](C_4^5+C_6^5)e^{-La}$$

$$A_6^5 = \frac{2\sigma-1}{4(1+\sigma)} + \left[\frac{2La(d_4'(La)-d_4(La))+3d_4(La)}{4(1+\sigma)}\right](C_4^5+C_6^5)e^{-La} -$$

$$-\left[\frac{2La(d_4'(La)-d_4(La))+5d_4(La)}{6}\right](C_4^5-C_6^5)e^{-La} +$$

$$+ \frac{\delta^e}{3\eta^e} d_4(Ma)C_8^5 e^{-Ma}$$

$$A_8^5 = -\left(\frac{2\sigma+5}{4(1+\sigma)}\right) + \left[\frac{2La(d_4'(La)-d_4(La))+3d_4(La)}{4(1+\sigma)}\right](C_4^5+C_6^5)e^{-La} +$$

$$+\left[\frac{2La(d_4'(La)-d_4(La))+5d_4(La)}{6}\right](C_4^5-C_6^5)e^{-La} -$$

$$- \frac{\delta^e}{3\eta^e} d_4(Ma)C_8^5 e^{-Ma} \tag{4.79}$$

and the values of the coefficients C_4^5, C_6^5 and C_8^5 as determined from (4.78) using (4.79) are

$$(C_4^5+C_6^5)e^{-La} = 3\kappa^e\{5\sigma+(2-7S)\} \div 3\kappa^e\{5\sigma+(2-7S)\}d_4(La) +$$

$$+\{2\kappa^e(5\sigma+(2-7S))+10\mu^e(1+\sigma)\}La(d_4'(La)-d_4(La))$$

$$\eta^e(\mu+\kappa^e)\{\kappa^e d_4(La)-\mu^e La(d_4'(La)-d_4(La))\}(C_4^5-C_6^5)e^{-La} +$$

$$+\delta^e\kappa^e\{(\mu+\kappa)^e d_4(Ma)+\eta^e Ma(d_4'(Ma)-d_4(Ma))\}C_8^5 e^{-Ma} = 0$$

$$\eta^e\{2\kappa^e(1-S)La(d_4'(La)-d_4(La))+(6\mu^e+(8-5S)\kappa^e)d_4(La)\}(C_4^5-C_6^5)e^{-La}+$$

$$+\delta^e\{3\eta^e Ma(d_4'(Ma)-d_4(Ma))+(6\mu^e+(5-2S)\kappa^e)d_4(Ma)\}C_8^5 e^{-Ma} =$$

$$= 3\kappa^e\eta^e\ (1-S)$$

(4.80)

Using now equation (4.27) and the values of A_1^5 and $(C_4^5+C_6^5)e^{-La}$ as given in equations (4.79) & (4.80) it is found that the effective viscosity for a micropolar fluid having suspended in it a sphere of a viscous Newtonian fluid is given by

$$\frac{\eta^*}{\eta} = 1+ \frac{5\phi}{2}\left[\frac{\sigma+\frac{2}{5}}{1+\sigma} + \left(\frac{\frac{2}{5}La(d_4'(La)-d_4(La))-(\sigma+\frac{2}{5})d_4(La)}{1+\sigma}\right)(C_4^5+C_6^5)e^{-La}\right]$$

$$(C_4^5+C_6^5)e^{-La} = 3\kappa^e\{5\sigma+(2-7S)\} \div 3\kappa^e\{5\sigma+(2-7S)\}d_4(La) +$$

$$+ \{2\kappa^e(5\sigma+(2-7S))+10\mu^e(1+\sigma)\}La(d_4'(La)-d_4(La)$$

(4.81)

If, in equations (4.81) it is assumed that $\sigma \to \infty$, the solid sphere case, then the result in (4.81) reduces to that given by (4.36). If instead, it is assumed that the coupling viscosity coefficient $\kappa \to 0$, then it is found that equation (4.81), as expected, reduces to the value given by equation (4.49).

The linear shearing flow of a micropolar fluid past a micropolar fluid sphere is a natural extension of the preceding sections and will be considered inthe next section.

Section 7. Micropolar sphere in a micropolar fluid.

The last problem to be dealt with in this chapter considers the linear shearing flow of a micropolar fluid past a micropolar fluid sphere. The boundary conditions, the pressure distributions and the components of the velocity and the microrotation vectors both internal and external are presented as are the solutions for the coefficients. Lastly, the effective viscosity for a suspension of this type is calculated by using equation (4.27).

An examination of the solutions for the internal and external flows of a micropolar fluid, as presented in previous sections, reveals that, in the present case, a total of twelve boundary conditions are needed to evaluate all of the coefficients. Since the boundary conditions for the linear shearing flow of a viscous Newtonian fluid past a viscous Newtonian fluid sphere form a total of six boundary conditions, it is necessary to prescribe another six boundary conditions. Three of these conditions are obtained by assuming that the microrotation vector, $\underset{\sim}{\nu}$, is continuous at the interface. For the final three boundary conditions an assumption made by Avudainayagam (1976) is used, although the form in which it appears is different. The assumption used here is that the couple stress vector is continuous at the interface. This vector form parallels the assumption that the tangential Cauchy stress vector is continuous at the interface. Additionally, this boundary condition also agrees with the boundary condition used in chapter III section 5. The boundary conditions in tensor form for this problem are

$$\left.\begin{aligned} &p^e \to 0 && \text{as} \quad r \to \infty \\ &u^e_\ell \to a_{\ell j}X_j && \text{as} \quad r \to \infty \\ &v^e_\ell \quad \text{finite} && \text{as} \quad r \to \infty \end{aligned}\right\} \qquad (4.82)$$

$$p^i \ , \ u^i_\ell \ , \ v^i_\ell \quad \text{finite} \qquad \text{as} \quad r = 0 \qquad (4.83)$$

$$u^e_\ell X_\ell = u^i_\ell X_\ell = 0 \qquad \text{at} \quad r = a \qquad (4.84)$$

$$u^e_\ell - u^e_p \frac{X_p X_\ell}{r^2} = u^i_\ell - u^i_p \frac{X_p X_\ell}{r^2} \qquad \text{at} \quad r = a \qquad (4.85)$$

$$t^e_{ij}X_j - t^e_{pq} \frac{X_p X_q}{r^2} X_i = t^i_{ij}X_j - t^i_{pq} \frac{X_p X_q}{r^2} X_i$$

$$\text{at} \quad r = a \qquad (4.86)$$

$$v^e_\ell = v^i_\ell \qquad \text{at} \quad r = a \qquad (4.87)$$

$$M^e_{ij}X_j = M^i_{ij}X_j \qquad \text{at} \quad r = a \qquad (4.88)$$

where t_{ij} , M_{ij} are as given by equations (1.1) & (1.2).

From equations (4.58) - (4.60) & (4.74) - (4.76), the general solutions for the internal and external flows are

$$p^i = A_2^5 a_{ij}X_iX_j \qquad (4.89)$$

$$u_\ell^i = \left[\frac{-4}{21\eta^i}A_2^5 + \frac{5L^2}{3}(C_4^5+C_6^5)\left(d_1(Lr)e^{Lr}+d_2(Lr)e^{-Lr}\right)\right]a_{ij}X_iX_jX_\ell +$$

$$+\left[\frac{1}{L^2}\left(\frac{2}{\eta^i}A_2^5-A_8^5\right)+\frac{5}{21\eta^i}A_2^5r^2-\frac{L^2r^2}{3}(C_4^5+C_6^5)\left(d_1(Lr)e^{Lr}+d_2(Lr)e^{-Lr}\right)-\right.$$

$$\left.-C_4^5\left(d_3(Lr)e^{Lr}+d_4(Lr)e^{-Lr}\right)\right]a_{\ell j}X_j +$$

$$+\left[\frac{1}{L^2}\left(\frac{2}{\eta^i}A_2^5-A_8^5\right)+\frac{5}{21\eta^i}A_2^5r^2-\frac{L^2r^2}{3}(C_4^5+C_6^5)\left(d_1(Lr)e^{Lr}+d_2(Lr)e^{-Lr}\right)-\right.$$

$$\left.-C_6^5\left(d_3(Lr)e^{Lr}+d_4(Lr)e^{-Lr}\right)\right]a_{j\ell}X_j \tag{4.90}$$

$$v_\ell^i = \left[\frac{-A_4^5}{3\eta^i}+\frac{\eta^i}{3\gamma^i}(4C_4^5+C_6^5)\left(\frac{d_3'(Lr)+d_3(Lr)}{Lr}e^{Lr}+\frac{d_4'(Lr)-d_4(Lr)}{Lr}e^{-Lr}\right)-\right.$$

$$\left.-C_8^5\left(\frac{d_3'(Mr)+d_3(Mr)}{Mr}e^{Mr}+\frac{d_4'(Mr)-d_4(Mr)}{Mr}e^{-Mr}\right)\right]\varepsilon_{\ell pm}a_{pj}X_jX_m +$$

$$+\left[\frac{-A_4^5}{3\eta^i}+\frac{\eta^i}{3\gamma^i}(C_4^5+4C_6^5)\left(\frac{d_3'(Mr)+d_3(Lr)}{Lr}e^{Lr}+\frac{d_4'(Lr)-d_4(Lr)}{Lr}e^{-Lr}\right)+\right.$$

$$\left.+C_8^5\left(\frac{d_3'(Mr)+d_3(Mr)}{Mr}e^{Mr}+\frac{d_4'(Mr)-d_4(Mr)}{Mr}e^{-Mr}\right)\right]\varepsilon_{\ell pm}a_{jp}X_jX_m +$$

$$+\left[\frac{-(\overset{5}{A_6}-\overset{5}{A_8})}{2L^2}-\frac{(\mu+\kappa)^i}{\kappa^i}(C_4^5-C_6^5)\left(d_3(Lr)e^{Lr}+d_4(Lr)e^{-Lr}\right)-\right.$$

$$-\frac{\delta^i}{2\kappa^i}C_8^5\left(Mr\left[(d_3'(Mr)+d_3(Mr))e^{Mr}+(d_4'(Mr)-d_4(Mr))e^{-Mr}\right]+\right.$$

$$\left.\left.+\,d_3(Mr)e^{Mr}+d_4(Mr)e^{-Mr}\right)\right]\varepsilon_{\ell pm}a_{mp} \tag{4.91}$$

$$p^e = (A_1^5 r^{-5}) a_{ij} X_i X_j \tag{4.92}$$

$$u_\ell^e = \left[\frac{-1}{L^2}(A_3^5 + \frac{10}{\eta^e}A_1^5) r^{-7} + \frac{1}{\eta^e}A_1^5 r^{-5} + \frac{5L^2}{3}(\bar{C}_4^5 + \bar{C}_6^5) d_2(Lr) e^{-Lr}\right] a_{ij} X_i X_j X_\ell +$$

$$+ \left[\frac{1}{5L^2}(A_3^5 + \frac{10}{\eta^e}A_1^5) r^{-5} - \frac{1}{L^2}A_3^5 r^{-3} + 1 - \frac{L^2 r^2}{3}(\bar{C}_4^5 + \bar{C}_6^5) d_2(Lr) e^{-Lr} - \right.$$

$$\left. - \bar{C}_4^5 \, d_4(Lr) e^{-Lr}\right] a_{\ell j} X_j$$

$$+ \left[\frac{1}{5L^2}(A_3^5 + \frac{10}{\eta^e}A_1^5) r^{-5} + \frac{1}{L^2}A_3^5 r^{-3} - \frac{L^2 r^2}{3}(\bar{C}_4^5 + \bar{C}_6^5) d_2(Lr) e^{-Lr} - \right.$$

$$\left. - \bar{C}_6^5 \, d_4(Lr) e^{-Lr}\right] a_{j\ell} X_j \tag{4.93}$$

$$\nu_\ell^e = \left[\tfrac{1}{2}(\frac{A_1^5}{\eta^e} - \frac{3A_3^5}{L^2}) r^{-5} + \frac{\eta^e}{3\gamma^e}(4\bar{C}_4^5 + \bar{C}_6^5)\left(\frac{d_4'(Lr) - d_4(Lr)}{Lr}\right) e^{-Lr} - \right.$$

$$\left. - \bar{C}_8^5 \left(\frac{d_4'(Mr) - d_4(Mr)}{Mr}\right) e^{-Mr}\right] \varepsilon_{\ell pm} a_{pj} X_j X_m +$$

$$+ \left[\tfrac{1}{2}(\frac{A_1^5}{\eta^e} + \frac{3A_3^5}{L^2}) r^{-5} + \frac{\eta^e}{3\gamma^e}(\bar{C}_4^5 + 4\bar{C}_6^5)\left(\frac{d_4'(Lr) - d_4(Lr)}{Lr}\right) e^{-Lr} + \right.$$

$$\left. + \bar{C}_8^5 \left(\frac{d_4'(Mr) - d_4(Mr)}{Mr}\right) e^{-Mr}\right] \varepsilon_{\ell pm} a_{jp} X_j X_m +$$

$$+ \left[- \frac{A_3^5}{L^2} r^{-3} + \tfrac{1}{2} - \frac{(\mu+\kappa)^e}{\kappa^e}(\bar{C}_4^5 - \bar{C}_6^5) d_4(Lr) e^{-Lr} - \right.$$

$$\left. - \frac{\delta^e}{2\kappa^e}\bar{C}_8^5 \Big(Mr(d_4'(Mr) - d_4(Mr)) + d_4(Mr)\Big) e^{-Mr}\right] \varepsilon_{\ell pm} a_{mp} \tag{4.94}$$

Application of the boundary conditions (4.84) - (4.86) to the solutions (4.89) - (4.94) yields

$$\frac{-3}{5L^2}(A_3^5+\frac{10}{\eta^e}A_1^5)a^{-5}+\frac{1}{\eta^e}A_1^5a^{-3}+1+(L^2a^2d_2(La)-d_4(La))e^{-La}(\tilde{C}_4^5+\tilde{C}_6^5)=0$$

$$0=\frac{1}{L^2}(\frac{4}{\eta^i}A_2^5-(A_6^5+A_8^5))+\frac{2}{7\eta^i}A_2^5a^2+(L^2a^2d_1(La)-d_3(La))e^{La}(C_4^5+C_6^5)+$$

$$+(L^2a^2d_2(La)-d_4(La))e^{-La}(C_4^5+C_6^5)$$

$$\frac{1}{5L^2}(A_3^5+\frac{10}{\eta^e}A_1^5)a^{-5}-\frac{1}{L^2}A_5^5a^{-3}+1-(\frac{L^2a^2d_2(La)}{3}+d_4(La))e^{-La}\tilde{C}_4^5-$$

$$-\frac{L^2a^2}{3}d_2(La)e^{-La}\tilde{C}_6^5=\frac{1}{L^2}(\frac{2}{\eta^i}A_2^5-A_6^5)+\frac{5}{21\eta^i}A_2^5a^2-$$

$$-\left[\frac{L^2a^2}{3}(d_1(La)e^{La}+d_2(La)e^{-La})+d_3(La)e^{La}+d_4(La)e^{-La}\right]C_4^5-$$

$$-\left[\frac{L^2a^2}{3}(d_1(La)e^{La}+d_2(La)e^{-La})\right]C_6^5 \qquad (4.95)$$

$$\frac{1}{5L^2}(A_3^5+\frac{10}{\eta^e}A_1^5)a^{-5}+\frac{1}{L^2}A_5^5a^{-3}-\frac{L^2a^2}{3}d_2(La)e^{-La}\tilde{C}_4^5-$$

$$-\left[\frac{L^2a^2d_2(La)}{3}+d_4(La)\right]e^{-La}\tilde{C}_6^5=\frac{1}{L^2}(\frac{2}{\eta^i}A_2^5-A_8^5)+$$

$$+\frac{5}{21\eta^i}A_2^5a^2-\left[\frac{L^2a^2}{3}(d_1(La)e^{La}+d_2(La)e^{-La})\right]C_4^5-$$

$$-\left[\frac{L^2a^2}{3}(d_1(La)e^{La}+d_2(La)e^{-La})+d_3(La)e^{La}+d_4(La)e^{-La}\right]C_6^5$$

$$\eta^{e}\left[\frac{8}{5L^2}\left(A_3^5+\frac{10}{\eta^{e}}A_1^5\right)a^{-5}-\frac{1}{\eta^{e}}A_1^5a^{-3}-1+\left[\frac{-8L^2a^2}{3}d_2(La)+\right.\right.$$

$$\left.\left.+\frac{5La}{3}\left(d_4'(La)-d_4(La)\right)+d_4(La)\right]e^{-La}\left(\bar{C}_4^5+\bar{C}_6^5\right)\right]= \tag{4.95}$$

$$=\eta^{i}\left[-\frac{1}{L^2}\left(\frac{4}{\eta^{i}}A_2^5-\left(A_6^5+A_8^5\right)\right)-\frac{6}{21\eta^{i}}A_2^5a^2+\right.$$

$$+\left[\frac{-8L^2a^2}{3}\left(d_1(La)e^{La}+d_2(La)e^{-La}\right)+\frac{5La}{3}\left(\left[d_3'(La)+d_3(La)\right]e^{La}+\right.\right.$$

$$\left.\left.\left.+\left[d_4'(La)-d_4(La)\right]e^{-La}\right)+\left(d_3(La)e^{La}+d_4(La)e^{-La}\right)\right]\left(C_4^5+C_6^5\right)\right]$$

$$\eta^{e}\left[\frac{-8}{5L^2}\left(A_3^5+\frac{10}{\eta^{e}}A_1^5\right)a^{-5}+\left(\frac{1}{\eta^{e}}A_1^5+\frac{3}{L^2}A_5^5\right)a^{-3}+1+\right.$$

$$+\frac{8L^2a^2}{3}d_2(La)e^{-La}\left(\bar{C}_4^5+\bar{C}_6^5\right)-\left[\frac{8L^2a^2}{3}\left(\frac{d_4'-d_4}{La}\right)e^{-La}+\right.$$

$$\left.+\frac{2L^2a^2d_4(La)e^{-La}}{L^2a^2}\right]\bar{C}_4^5-\frac{2L^2a^2}{3}\frac{d_4'(La)-d_4(La)}{La}e^{-La}\bar{C}_6^5-$$

$$\left.-\frac{\delta^{e}}{\eta^{e}}d_4(Ma)e^{-Ma}\bar{C}_8^5\right]=\eta^{i}\left[\frac{1}{L^2}\left(\frac{4}{\eta^{i}}A_2^5-\left(A_6^5+A_8^5\right)\right)+\frac{16}{21\eta^{i}}A_2^5a^2+\right.$$

$$+\frac{8L^2a^2}{3}\left(d_1(La)e^{La}+d_2(La)e^{-La}\right)\left(C_4^5+C_6^5\right)-$$

$$-L^2a^2\left\{\frac{8}{3}\left(\frac{d_3'(La)+d_3(La)}{La}e^{La}+\frac{d_4'(La)-d_4(La)}{La}e^{-La}\right)+\right.$$

$$\left.+2\left(\frac{d_3(La)e^{La}+d_4(La)e^{-La}}{L^2a^2}\right)\right\}C_4^5-\frac{2L^2a^2}{3}\left\{\frac{d_3'(La)+d_3(La)}{La}e^{La}+\right.$$

$$\left.\left.+\frac{d_4'(La)-d_4(La)}{La}e^{-La}\right\}C_6^5-\frac{\delta^{i}}{\eta^{i}}\left(d_3(Ma)e^{Ma}+d_4(Ma)e^{-Ma}\right)C_8^5\right] \tag{4.95}$$

The solution to this system of algebraic equations, with

$$\sigma = \frac{\eta^i}{\eta^e} \text{ is}$$

$$\frac{1}{L^2}(A_5^5 + \frac{10}{\eta^e}A_1^5)a^{-5} = \frac{-5\sigma}{2(1+\sigma)} + \frac{5}{3}\left[g_1(La) + \frac{(5\sigma+2)g_2(La) - 2Lag_2'(La)}{2(1+\sigma)}\right](\bar{C}_4^5 + \bar{C}_6^5) -$$

$$- \frac{5}{3}\left[\frac{7\sigma(h_1(La)+h_2(La)) - 2\sigma Lah_2'(La)}{2(1+\sigma)}\right](C_4^5 + C_6^5)$$

$$\frac{1}{\eta^e}A_1^5 a^{-5} = \frac{-(5\sigma+2)}{2(1+\sigma)} + \left[\frac{(5\sigma+2)g_2(La) - 2Lag_2'(La)}{2(1+\sigma)}\right](\bar{C}_4^5 + \bar{C}_6^5) -$$

$$- \left[\frac{7\sigma(h_1(La)+h_2(La)) - 2\sigma Lah_2'(La)}{2(1+\sigma)}\right](C_4^5 + C_6^5)$$

$$\frac{1}{L^2}A_5^5 a^{-5} = \frac{1}{3}\left[g_2(La) + Lag_2'(La)\right](\bar{C}_4^5 - \bar{C}_6^5) + \frac{1}{3}g_2(Ma)\,\frac{\delta^e}{\eta^e}\,\bar{C}_8^5 -$$

$$- \frac{\sigma}{3}\left[h_2(La) + Lah_2'(La)\right](C_4^5 - C_6^5) - \frac{\sigma}{3}h_2(Ma)\,\frac{\delta^i}{\eta^i}\,C_8^5$$

$$\frac{1}{L^2}(\frac{2}{\eta^i}A_2^5 - A_6^5) = \frac{(2\sigma-1)}{4(1+\sigma)} + \left[\frac{3g_2(La) + 2Lag_2'(La)}{4(1+\sigma)}\right](\bar{C}_4^5 + \bar{C}_6^5) -$$

$$- \left[\frac{5g_2(La) + 2Lag_2'(La)}{6}\right](\bar{C}_4^5 - \bar{C}_6^5) - \frac{1}{3}\,g_2(Ma)\,\frac{\delta^e}{\eta^e}\,\bar{C}_8^5 -$$

$$- \left[\frac{7h_1(La) + (5-2\sigma)h_2(La) + 2\sigma Lah_2'(La)}{4(1+\sigma)}\right](C_4^5 + C_6^5) +$$

$$+ \left[\frac{(2\sigma+3)h_2(La) + 2\sigma Lah_2'(La)}{6}\right](C_4^5 - C_6^5) + \frac{\sigma}{3}\,h_2(Ma)\,\frac{\delta^i}{\eta^i}\,C_8^5$$

(4.96)

$$\frac{1}{L^2}\left(\frac{2}{\eta^I}A_2^5 - A_8^5\right) = \frac{-(2\sigma+5)}{4(1+\sigma)} + \left[\frac{3g_2(La)+2Lag_2'(La)}{4(1+\sigma)}\right](\bar{C}_4^5+\bar{C}_6^5) +$$

$$+\left[\frac{5g_2(La)+2Lag_2'(La)}{6}\right](\bar{C}_4^5-\bar{C}_6^5) + \frac{1}{3}\, g_2(Ma)\, \frac{\delta^e}{\eta^e}\, \bar{C}_8^5 -$$

$$-\left[\frac{7h_1(La)+(5-2\sigma)h_2(La)+2\sigma Lah_2'(La)}{4(1+\sigma)}\right](C_4^5+C_6^5) -$$

$$-\left[\frac{(2\sigma+3)h_2(La)+2\sigma Lah_2'(La)}{6}\right](C_4^5-C_6^5) - \frac{\sigma}{3}\, h_2(Ma)\, \frac{\delta^i}{\eta^I}\, C_8^5$$

$$\frac{4}{21\eta^I}A_2^5 a^2 = \frac{1}{1+\sigma} - \left[\frac{3g_2(La)+2Lag_2'(La)}{3(1+\sigma)}\right](\bar{C}_4^5+\bar{C}_6^5) +$$

$$+\left[\frac{(5-2\sigma)(h_1(La)+h_2(La))+2\sigma Lah_2'(La)}{3(1+\sigma)}\right](C_4^5+C_6^5)$$

(4.96)

where $\sigma = \dfrac{\eta^i}{\eta^e} = \dfrac{(2\mu+\kappa)^i}{(2\mu+\kappa)^e}$

$$g_1(p) = \left[p^2 d_2(p) - d_4(p)\right]e^{-p}$$

$$g_2(p) = d_4(p)e^{-p}\ ,\quad g_2'(p) = \left[d_4'(p) - d_4(p)\right]e^{-p}$$

$$h_1(p) = \left[p^2 d_1(p) - d_3(p)\right]e^{p} + \left[p^2 d_2(p) - d_4(p)\right]e^{-p}$$

$$h_2(p) = d_3(p)e^{p} + d_4(p)e^{-p}\ ,$$

$$h_2'(p) = \left[d_3'(p) + d_3(p)\right]e^{p} + \left[d_4'(p) - d_4(p)\right]e^{-p}$$

The constant L appearing with the external coefficients $\overset{s}{A}_3$, $\overset{s}{A}_1$, $\overset{s}{A}_5$, $\overset{s}{\bar{C}}_4$, $\overset{s}{\bar{C}}_6$, $\overset{s}{\bar{C}}_8$ is defined by

$$L^2 = \frac{\kappa^e \eta^e}{\gamma^e (\mu+\kappa)^e}$$

Similarily, the L appearing with the inner coefficients $\overset{s}{A}_2$, $\overset{s}{A}_4$, $\overset{s}{A}_6$, $\overset{s}{C}_4$, $\overset{s}{C}_6$, $\overset{s}{C}_8$ is defined by

$$L^2 = \frac{\kappa^i \eta^i}{\gamma^i (\mu+\kappa)^i}$$

The coefficients $\overset{s}{\bar{C}}_4$, $\overset{s}{\bar{C}}_6$, $\overset{s}{\bar{C}}_8$, $\overset{s}{C}_4$, $\overset{s}{C}_6$, $\overset{s}{C}_8$ are determined by solving the system of equations that arises as a result of substituting (4.91) & (4.94) in (4.87) & (4.88). This system of six equations in six unknowns is

$$\frac{1}{2}\left(\frac{\overset{s}{A}_1}{\eta^e} - \frac{3\overset{s}{A}_5}{L^2}\right) a^{-s} + \frac{\eta^e}{3\gamma^e} \frac{g_2'(La)}{La} (4\overset{s}{\bar{C}}_4 + \overset{s}{\bar{C}}_6) - \frac{g_2'(Ma)}{Ma} \overset{s}{\bar{C}}_8 =$$

$$= \frac{-\overset{s}{A}_2}{3\eta^i} + \frac{\eta^i}{3\gamma^i} \frac{h_2'(La)}{La} (4\overset{s}{C}_4 + \overset{s}{C}_6) - \frac{h_2'(Ma)}{Ma} \overset{s}{C}_8$$

$$\frac{1}{2}\left(\frac{\overset{s}{A}_1}{\eta^e} + \frac{3\overset{s}{A}_5}{L^2}\right) a^{-s} + \frac{\eta^e}{3\gamma^e} \frac{g_2'(La)}{La} (\overset{s}{\bar{C}}_4 + 4\overset{s}{\bar{C}}_6) + \frac{g_2'(Ma)}{Ma} \overset{s}{\bar{C}}_8 =$$

$$= \frac{-\overset{s}{A}_2}{3\eta^i} + \frac{\eta^i}{3\gamma^i} \frac{h_2'(La)}{La} (\overset{s}{C}_4 + 4\overset{s}{C}_6) + \frac{h_2'(Ma)}{Ma} \overset{s}{C}_8 \qquad (4.97)$$

$$\frac{-A_5^5}{L^2}\, a^{-5} + \frac{1}{2}\left(\frac{\mu+\kappa}{\kappa}\right)^e g_2(La)\,(\bar{C}_4^5 - \bar{C}_6^5) - \frac{\delta^e}{2\kappa^e}\left[g_2(Ma) + Ma\, g_2'(Ma)\right]\bar{C}_8^5 =$$

$$= -\frac{(A_6^5 - A_8^5)}{2L^2} - \left(\frac{\mu+\kappa}{\kappa}\right)^i h_2(La)\,(C_4^5 - C_6^5) - \frac{\delta^i}{2\kappa^i}\left[h_2(Ma) + Ma\, h_2'(Ma)\right] C_8^5$$

$$\frac{9(\beta+\gamma)^e}{2L^2} A_5^5 a^{-5} - \frac{(3\beta+\gamma)^e}{2\eta^e} A_1^5 a^{-5} + \frac{\eta^e}{\gamma^e}\left[\frac{4\beta^e}{3}\left(g_2''(La) + \frac{g_2'(La)}{La}\right) - \right.$$

$$\left. - \frac{7\gamma^e}{3}\,\frac{g_2'(La)}{La}\right]\bar{C}_4^5 + \frac{\eta^e}{\gamma^e}\left[\frac{\beta^e}{3}\left(g_2''(La) + \frac{g_2'(La)}{La}\right) + \frac{2\gamma^e}{3}\, g_2'(La)\right]\bar{C}_6^5 -$$

$$-\left[\delta^e\left(g_2''(Ma) + \frac{g_2'(Ma)}{Ma}\right) + 3\alpha^e\,\frac{g_2'(Ma)}{Ma}\right]\bar{C}_8^5 =$$

$$= \frac{-(2\beta-\gamma)^i}{3\eta^i} A_2^5 + \frac{\eta^i}{\gamma^i}\left[\frac{4\beta^i}{3}\left(h_2''(La) + \frac{h_2'(La)}{La}\right) - \frac{7\gamma^i}{3}\,\frac{h_2'(La)}{La}\right] C_4^5 +$$

$$+ \frac{\eta^i}{\gamma^i}\left[\frac{\beta^i}{3}\left(h_2''(La) + \frac{h_2'(La)}{La}\right) + \frac{2\gamma^i}{3}\,\frac{h_2'(La)}{La}\right] C_6^5 -$$

$$- \left[\delta^i\left(h_2''(Ma) + \frac{h_2'(Ma)}{Ma}\right) + 3\alpha^i\,\frac{h_2'(Ma)}{Ma}\right] C_8^5 \qquad (4.97)$$

$$\frac{-9(\beta+\gamma)^e}{2L^2} A_5^5 a^{-5} - \frac{(3\beta+\gamma)^e}{2\eta^e} A_1^5 a^{-5} + \frac{\eta^e}{\gamma^e}\left[\frac{\beta^e}{3}\left(g_2''(La) + \frac{g_2'(La)}{La}\right) + \right.$$

$$\left. + \frac{2\gamma^e}{3}\,\frac{g_2'(La)}{La}\right]\bar{C}_4^5 + \frac{\eta^e}{\gamma^e}\left[\frac{4\beta^e}{3}\left(g_2''(La) + \frac{g_2'(La)}{La}\right) - \frac{7\gamma^e}{3}\,\frac{g_2'(La)}{La}\right]\bar{C}_6^5 +$$

$$+ \left[\delta^e\left(g_2''(Ma) + \frac{g_2'(Ma)}{Ma}\right) + 3\alpha^e\,\frac{g_2'(Ma)}{Ma}\right]\bar{C}_8^5 =$$

$$= \frac{-(2\beta-\gamma)^{i}}{3\eta^{i}} A_2^5 + \frac{\eta^{i}}{\gamma^{i}}\left[\frac{\beta^{i}}{3}(h_2''(La) + \frac{h_2'(La)}{La}) + \frac{2\gamma^{i}}{3}\frac{h_2'(La)}{La}\right]C_4^5 +$$

$$+ \frac{\eta^{i}}{\gamma^{i}}\left[\frac{4\beta^{i}}{3}(h_2''(La) + \frac{h_2'(La)}{La}) - \frac{7\gamma^{i}}{3}\frac{h_2'(La)}{La}\right]C_6^5 +$$

$$+ \left[\delta^{i}(h_2''(Ma) + \frac{h_2'(Ma)}{Ma}) + 3\alpha^{i}\frac{h_2'(Ma)}{Ma}\right]C_8^5$$

$$\frac{3(\beta+\gamma)^{e}}{L^2} A_5^5 a^{-5} + (\beta+\gamma)^{e}\frac{\eta^{e}}{\gamma^{e}}\frac{g_2'(La)}{La}(\bar{C}_4^5 - \bar{C}_6^5) -$$

$$- \left[\delta^{e}(g_2''(Ma) + \frac{g_2'(Ma)}{Ma}) + (\delta+2\alpha)^{e}\frac{g_2'(Ma)}{Ma}\right]\bar{C}_8^5 =$$

$$= (\beta+\gamma)^{i}\frac{\eta^{i}}{\gamma^{i}}\frac{h_2'(La)}{La}(C_4^5 - C_6^5) -$$

$$- \left[\delta^{i}(h_2''(Ma) + \frac{h_2'(Ma)}{Ma}) + (\delta+2\alpha)^{i}\frac{h_2'(Ma)}{Ma}\right]C_8^5 \tag{4.97}$$

Using equation (4.27) and the value A_1^5 given in (4.96), the effective viscosity for a micropolar fluid having a micropolar fluid sphere suspended in it is

$$\frac{\eta^*}{\eta} = 1 + \frac{5\phi}{2}\left[\frac{\sigma+\frac{2}{5}}{1+\sigma} + \left(\frac{\frac{2}{5}La g_2'(La) - (\sigma+\frac{2}{5})g_2(La)}{1+\sigma}\right)(\bar{C}_4^5+\bar{C}_6^5) - \right.$$

$$\left. - \left(\frac{\frac{2\sigma}{5}La h_2'(La) - \frac{7\sigma}{5}(h_1(La)+h_2(La))}{1+\sigma}\right)(C_4^5+C_6^5)\right] \tag{4.98}$$

where $(\bar{C}_4^5+\bar{C}_6^5)$ & $(C_4^5+C_6^5)$ are determined from (4.97).

APPENDICES

A. Derivation of results used in Chapter II

In chapter II, solutions to the equations of motion were presented which were based on the use of the scalar and vector invariants of spatially constant second and third order tensors in combination with the position vector $\underset{\sim}{r}$. The only assumptions made were that the invariants of the second order tensor were considered independent of the invariants of the third order tensor and that the n-dimensional space being considered was Euclidean in the sense of J. L. Synge and A. Schild (1949).

Here, a general scalar function and a general vector function will be considered and certain representative solutions involving these functions and their partial derivatives will be elucidated.

If H(x) is any scalar function of the position vector $\underset{\sim}{r}$, then with $H,_{\ell} = \frac{\partial H}{\partial x_{\ell}}(\underset{\sim}{r})$ and $H^{\alpha'} = \frac{dH}{d\rho}(\rho)$

$$H(\underset{\sim}{r}) = H^{0}(r)\, a_{ii} + H^{1}(r)\,\varepsilon_{ijk} a_{kj} X_i + H^{2}(r)\, A_{imm} X_i + H^{3}(r)\, A_{mim} X_i + H^{4}(r)\, A_{mmi} X_i + H^{5}(r)\, a_{ij} X_i X_j + H^{6}(r)\, A_{ijk} X_i X_j X_k \qquad \text{A.1}$$

$$H,_{\ell} = \frac{H^{0'}}{r} a_{ii} X_{\ell} + \frac{H^{1'}}{r} \varepsilon_{ijk} a_{kj} X_i X_{\ell} + H^{1} \varepsilon_{\ell jk} a_{kj} + \frac{H^{2'}}{r} A_{imm} X_i X_{\ell} + H^{2} A_{\ell mm} + \frac{H^{3'}}{r} A_{mim} X_i X_{\ell} + H^{3} A_{m\ell m} + \frac{H^{4'}}{r} A_{mmi} X_i X_{\ell} + H^{4} A_{mm\ell} + \qquad \text{A.2}$$

$$+ \frac{H^{5'}}{r} a_{ij} X_i X_j X_\ell + H^5 (a_{\ell j} + a_{j\ell}) X_j + \frac{H^{6'}}{r} A_{ijk} X_i X_j X_k X_\ell +$$

$$+ H^6 (A_{\ell jk} + A_{j\ell k} + A_{jk\ell}) X_j X_k$$

$$H_{,\ell n} = \left(\frac{H^{0''}}{r^2} - \frac{H^{0'}}{r^3}\right) a_{ii} X_\ell X_n + \frac{H^{0'}}{r} a_{ii} \delta_{\ell n} + \left(\frac{H^{1''}}{r^2} - \frac{H^{1'}}{r^3}\right)$$

$$\epsilon_{ijk} a_{kj} X_i X_\ell X_n + \frac{H^{1'}}{r} (\epsilon_{njk} a_{kj} X_\ell + \epsilon_{\ell jk} a_{kj} X_n +$$

$$+ \epsilon_{ijk} a_{kj} X_i \delta_{\ell n}) + \left(\frac{H^{2''}}{r^2} - \frac{H^{2'}}{r^3}\right) A_{imm} X_i X_\ell X_n +$$

$$+ \frac{H^{2'}}{r} (A_{nmm} X_\ell + A_{\ell mm} X_n + A_{imm} X_i \delta_{\ell n}) +$$

$$+ \left(\frac{H^{3''}}{r^2} - \frac{H^{3'}}{r^3}\right) A_{mim} X_i X_\ell X_n + \frac{H^{3'}}{r} (A_{mnm} X_\ell + A_{m\ell m} X_n +$$

$$+ A_{mim} X_i \delta_{\ell n}) + \left(\frac{H^{4''}}{r^2} - \frac{H^{4'}}{r^3}\right) A_{mmi} X_i X_\ell X_n + \qquad \text{A.3}$$

$$+ \frac{H^{4'}}{r} (A_{mmn} X_\ell + A_{mm\ell} X_n + A_{mm\ell} X_n + A_{mmi} X_i \delta_{\ell n}) +$$

$$+ \frac{H^{5''}}{r^2} - \frac{H^{5'}}{r^3}) a_{ij} X_i X_j X_\ell X_n + \frac{H^{5'}}{r} (a_{nj} X_j X_\ell + a_{jn} X_j X_\ell +$$

$$+ a_{\ell j} X_j X_n + a_{j\ell} X_j X_n + a_{ij} X_i X_j \delta_{\ell n}) + H^5 (a_{\ell n} + a_{n\ell}) +$$

$$+ \left(\frac{H^{6''}}{r^2} - \frac{H^{6'}}{r^3}\right) A_{ijk} X_i X_j X_k X_\ell X_n + \frac{H^{6'}}{r} (A_{njk} X_\ell + A_{jnk} X_\ell +$$

$$+ A_{jkn} X_\ell + A_{\ell jk} X_n + A_{j\ell k} X_n + A_{jk\ell} X_n + A_{ijk} X_i \delta_{\ell n}) X_j X_k +$$

$$+ H^6 (A_{\ell nk} + A_{\ell kn} + A_{n\ell k} + A_{k\ell n} + A_{nk\ell} + A_{kn\ell}) X_k$$

If N is the dimension of the space and $\ell = n$, then

$$H_{,\ell\ell} = \left(H^{0''} + \frac{N-1}{r} H^{0'} + 2H^5\right) a_{ii} + \left(H^{1''} + \frac{N+1}{r} H^{1'}\right) \qquad \text{A.4}$$

$$\varepsilon_{ijk} a_{kj} X_i + \left(H^{2''} + \frac{N+1}{r} H^{2'} + 2H^6\right) A_{imm} X_i + \left(H^{3''} + \right.$$
$$\left. + \frac{N+1}{r} H^{3'} + 2H^6\right) A_{mim} X_i + \left(H^{4''} + \frac{N+1}{r} H^{4'} + 2H^6\right) A_{mmi} X_i +$$
$$+ \left(H^{5''} + \frac{N+3}{r} H^{5'}\right) a_{ij} X_i X_j + \left(H^{6''} + \frac{N+5}{r} H^{6'}\right) A_{ijk} X_i X_j X_k$$

For any vector function, $\underset{\sim}{h}(\underset{\sim}{r})$, which is a function of the position vector $\underset{\sim}{r}$.

$$h_\ell(\underset{\sim}{r}) = h_1^0(r)\, a_{ii} X_\ell + h_1^1(r)\, \varepsilon_{ijk} a_{kj} X_i X_\ell + h_2^1(r)\, \varepsilon_{\ell jk} a_{kj} +$$
$$+ h_1^2(r)\, A_{imm} X_i X_\ell + h_2^2(r)\, A_{\ell mm} + h_1^3(r)\, A_{mim} X_i X_\ell +$$
$$+ h_2^3(r)\, A_{m\ell m} + h_1^4(r)\, A_{mmi} X_i X_\ell + h_2^4(r)\, A_{mm\ell} +$$
$$+ h_1^5(r)\, a_{ij} X_i X_j X_\ell + h_2^5(r)\, a_{\ell j} X_j + h_3^5(r)\, a_{j\ell} X_j + \qquad \text{A.5}$$
$$+ h_1^6(r)\, A_{ijk} X_i X_k X_\ell + h_2^6(r)\, A_{\ell jk} X_j X_k + h_3^6(r)\, A_{j\ell k} X_j X_k +$$
$$+ h_4^6(r)\, A_{jk\ell} X_j X_k$$

$$h_{\ell,\rho} = \frac{h_1^{0'}}{r} a_{ii} X_\ell X_\rho + h_1^0 a_{ii} \delta_{\ell\rho} + \frac{h_1^{1'}}{r} \varepsilon_{ijk} a_{kj} X_i X_\ell X_\rho +$$
$$+ h_1^1 \varepsilon_{ijk} a_{kj} (\delta_{i\rho} X_\ell + X_i \delta_{\ell\rho}) + \frac{h_2^{1'}}{r} \varepsilon_{\ell jk} a_{kj} X_\rho +$$
$$+ \frac{h_1^{2'}}{r} A_{imm} X_i X_\ell X_\rho + h_1^2 A_{imm} (\delta_{i\rho} X_\ell + X_i \delta_{\ell\rho}) +$$
$$+ \frac{h_2^{2'}}{r} A_{\ell mm} X_\rho + \frac{h_1^{3'}}{r} A_{mim} X_i X_\ell X_\rho + h_1^3 A_{mim} (\delta_{i\rho} X_\ell + X_i \delta_{\ell\rho}) +$$
$$+ \frac{h_2^{3'}}{r} A_{m\ell m} X_\rho + \frac{h_1^{4'}}{r} A_{mmi} X_i X_\ell X_\rho + h_1^4 A_{mmi} (\delta_{i\rho} X_\ell + X_i \delta_{\ell\rho}) +$$
$$+ \frac{h_2^{4'}}{r} A_{mm\ell} X_\rho + \frac{h_1^{5'}}{r} a_{ij} X_i X_j X_\ell X_\rho + h_1^5 a_{ij} (\delta_{i\rho} X_j X_\ell +$$
$$+ X_i \delta_{j\rho} X_\ell + X_i X_j \delta_{\rho\ell}) + h_2^5 a_{\ell j} \delta_{j\rho} + \frac{h_3^{5'}}{r} a_{j\ell} X_j X_\rho + \qquad \text{A.6}$$

$$+ h_3^{5} a_{j\ell}\delta_{j\rho} + \frac{h_1^{6\prime}}{r} A_{ijk}X_iX_jX_kX_\ell X_\rho + h_1^{6} A_{ijk}(\delta_{i\rho}X_jX_kX_\ell +$$

$$+ X_i\delta_{j\rho}X_kX_\ell + X_iX_j\delta_{\rho k}X_\ell + X_iX_jX_k\delta_{\rho\ell}) + \frac{h_2^{6\prime}}{r} A_{\ell jk}X_jX_kX_\rho +$$

$$+ h_2^{6} A_{\ell jk}(\delta_{j\rho}X_k + X_j\delta_{\rho k}) + \frac{h_3^{6\prime}}{r} A_{j\ell k}X_jX_kX_\rho + h_3^{6} A_{j\ell k}(\delta_{j\rho}X_k +$$

$$+ X_j\delta_{k\rho}) + \frac{h_4^{6\prime}}{r} A_{jk\ell}X_jX_kX_\rho + h_4^{6} A_{jk\ell}(\delta_{j\rho}X_k + X_j\delta_{k\rho}) + \frac{h_2^{5\prime}}{r} a_{\ell j}X_jX_\rho$$

$$h_{\ell,\rho q} = \left(\frac{h_1^{0\prime\prime}}{r^2} - \frac{h_1^{0\prime}}{r^3}\right) a_{ii}X_\ell X_\rho X_q + \frac{h_1^{0\prime}}{r} a_{ii}(\delta_{\ell q}X_\rho + \delta_{\ell\rho}X_q +$$

$$+ X_\ell\delta_{\rho q}) + \left(\frac{h_1^{1\prime\prime}}{r^2} - \frac{h_1^{1\prime}}{r^3}\right) \varepsilon_{ijk}a_{kj}X_iX_\ell X_\rho X_q +$$

$$+ \frac{h_1^{1\prime}}{r} \varepsilon_{ijk}a_{kj}(\delta_{iq}X_\ell X_\rho + X_i\delta_{\ell q}X_\rho + \delta_{i\rho}X_\ell X_q + X_i\delta_{\ell\rho}X_q +$$

$$+ X_iX_\ell\delta_{\rho q}) + h_1^{1} \varepsilon_{ijk}a_{kj}(\delta_{i\rho}\delta_{\ell q} + \delta_{i\rho}\delta_{\ell q}) + \left(\frac{h_2^{1\prime\prime}}{r^2} -\right.$$

$$\left. - \frac{h_2^{1\prime}}{r^3}\right) \varepsilon_{\ell jk}a_{kj}X_\rho X_q + \frac{h_2^{1\prime}}{r} \varepsilon_{\ell jk}a_{kj}\delta_{\rho q} + \left(\frac{h_1^{2\prime\prime}}{r^2} -\right.$$

$$\left. - \frac{h_1^{2\prime}}{r^3}\right) A_{imm}X_iX_\ell X_\rho X_q + \frac{h_1^{2\prime}}{r} A_{imm}(\delta_{iq}X_\ell X_\rho + X_i\delta_{\ell q}X_\rho +$$

$$+ \delta_{i\rho}X_\ell X_q + X_i\delta_{\ell\rho}X_q + X_iX_\ell\delta_{\rho q}) + h_1^{2} A_{imm}(\delta_{i\rho}\delta_{\ell q} +$$

$$+ \delta_{iq}\delta_{\ell\rho}) + \left(\frac{h_2^{2\prime\prime}}{r^2} - \frac{h_2^{2\prime}}{r^3}\right) A_{\ell mm}X_\rho X_q + \frac{h_2^{2\prime}}{r} A_{\ell mm}\delta_{\rho q} +$$

$$+ \left(\frac{h_1^{3\prime\prime}}{r^2} - \frac{h_1^{3\prime}}{r^3}\right) A_{mim}X_iX_\ell X_\rho X_q + \frac{h_1^{3\prime}}{r} A_{mim}(\delta_{iq}X_\ell X_\rho +$$

$$+ X_i\delta_{\ell q}X_\rho + \delta_{i\rho}X_\ell X_q + X_i\delta_{\ell\rho}X_q + X_iX_\ell\delta_{\rho q}) +$$

$$+ h_1^{3} A_{mim}(\delta_{i\rho}\delta_{\ell q} + \delta_{iq}\delta_{\ell\rho}) + \left(\frac{h_2^{3\prime\prime}}{r^2} - \frac{h_2^{3\prime}}{r^3}\right) A_{m\ell m}X_\rho X_q +$$

$$+ \frac{h_2^{3\prime}}{r} A_{m\ell m}\delta_{\rho q} + \left(\frac{h_1^{4\prime\prime}}{r^2} - \frac{h_1^{4\prime}}{r^3}\right) A_{mmi}X_iX_\ell X_\rho X_q +$$

$$+ \frac{h_1^{4\prime}}{r} A_{mmi}(\delta_{iq}X_\ell X_\rho + X_i\delta_{\ell q}X_\rho + \delta_{i\rho}X_\ell X_q + X_i\delta_{\ell\rho}X_q +$$

$$+ X_i X_\ell \delta_{\rho q}) + h_1^{4} A_{mmi}(\delta_{i\rho}\delta_{\ell q} + \delta_{iq}\delta_{\ell\rho}) + (\frac{h_2^{4''}}{r^2} -$$

$$- \frac{h_2^{4'}}{r^3}) A_{mm\ell} X_\rho X_q + \frac{h_2^{4'}}{r} A_{mm\ell}\delta_{\rho q} + (\frac{h_1^{5''}}{r^2} - \frac{h_1^{5'}}{r^3}) a_{ij} X_i X_j X_\ell X_\rho X_q +$$

$$+ \frac{h_1^{5'}}{r} a_{ij} (\delta_{iq} X_j X_\ell X_\rho + X_i \delta_{jq} X_\ell X_\rho + X_i X_j \delta_{\ell q} X_\rho + \delta_{i\rho} X_j X_\ell X_q +$$

$$+ X_i \delta_{j\rho} X_\ell X_q + X_i X_j \delta_{\ell\rho} X_q + X_i X_j X_\ell \delta_{\rho q}) + h_1^{5} a_{ij} (\delta_{i\rho}\delta_{jq} X_\ell +$$

$$+ \delta_{i\rho} X_j \delta_{\ell q} + \delta_{iq}\delta_{j\rho} X_\ell + X_i \delta_{j\rho}\delta_\ell + \delta_{iq} X_j \delta_{\ell\rho} + X_i \delta_{jq}\delta_{\ell\rho}) +$$

$$+ (\frac{h_2^{5''}}{r^2} - \frac{h_2^{5'}}{r^3}) a_{\ell j} X_j X_\rho X_q + \frac{h_2^{5'}}{r} a_{\ell j} (\delta_{jq} X_\rho + \delta_{j\rho} X_q + X_j \delta_{\rho q}) +$$

$$+ (\frac{h_3^{5''}}{r^2} - \frac{h_3^{5'}}{r^3}) a_{j\ell} X_j X_\rho X_q + \frac{h_3^{5'}}{r} a_{j\ell} (\delta_{jq} X_\rho + \delta_{j\rho} X_q + X_j \delta_{\rho q}) + \qquad \text{A.7}$$

$$+ (\frac{h_1^{6''}}{r^2} - \frac{h_1^{6'}}{r^3}) A_{ijk} X_i X_j X_k X_\ell X_\rho X_q + \frac{h_1^{6'}}{r} A_{ijk} (\delta_{iq} X_j X_k X_\ell X_\rho +$$

$$+ X_i \delta_{jq} X_k X_\ell X_\rho + X_i X_j \delta_{kq} X_\ell X_\rho + X_i X_j X_k \delta_{\ell q} X_\rho + \delta_{i\rho} X_j X_k X_\ell X_q +$$

$$+ X_i \delta_{j\rho} X_k X_\ell X_q + X_i X_j \delta_{k\rho} X_\ell X_q + X_i X_j X_k \delta_{\ell\rho} X_q + X_i X_j X_k X_\ell \delta_{\rho q}) +$$

$$+ h_1^{6} A_{ijk} (\delta_{i\rho}\delta_{jq} X_k X_\ell + \delta_{i\rho} X_j \delta_{kq} X_\ell + \delta_{i\rho} X_j X_k \delta_{q\ell} +$$

$$+ \delta_{iq}\delta_{j\rho} X_k X_\ell + X_i \delta_{j\rho}\delta_{kq} X_\ell + X_i \delta_{j\rho} X_k \delta_{q\ell} + \delta_{iq} X_j \delta_{k\rho} X_\ell +$$

$$+ X_i \delta_{jq}\delta_{k\rho} X_\ell + X_i X_j \delta_{k\rho}\delta_{\ell q} + \delta_{iq} X_j X_k \delta_{\ell\rho} + X_i \delta_{jq} X_k \delta_{\ell\rho} +$$

$$+ X_i X_j \delta_{kq}\delta_{\ell\rho}) + (\frac{h_2^{6''}}{r^2} - \frac{h_2^{6'}}{r^3}) A_{\ell jk} X_j X_k X_\rho X_q + \frac{h_2^{6'}}{r} A_{\ell jk} (\delta_{jq} X_k X_\rho +$$

$$+ X_j \delta_{kq} X_\rho + \delta_{j\rho} X_k X_q + X_j \delta_{k\rho} X_q + X_j X_k \delta_{\rho q}) + h_2^{6} A_{\ell jk} (\delta_{j\rho}\delta_{kq} +$$

$$+ \delta_{jq}\delta_{k\rho}) + (\frac{h_3^{6''}}{r^2} - \frac{h_3^{6'}}{r^3}) A_{j\ell k} X_j X_k X_\rho X_q + \frac{h_3^{6'}}{r} A_{j\ell k} (\delta_{jq} X_k X_\rho +$$

$$+ X_j \delta_{kq} X_\rho + \delta_{j\rho} X_k X_q + X_j \delta_{k\rho} X_q + X_j X_k \delta_{\rho q}) + h_3^{6} A_{j\ell k} (\delta_{j\rho}\delta_{kq} +$$

$$+ \delta_{jq}\delta_{kj}) + (\frac{h_4^{6''}}{r^2} - \frac{h_4^{6'}}{r^3}) A_{jk\ell} X_j X_k X_\rho X_q + \frac{h_4^{6'}}{r} A_{jk\ell} (\delta_{jq} X_k X_\rho +$$

$$+ X_j \delta_{kq} X_\rho + \delta_{j\rho} X_k X_q + X_j \delta_{k\rho} X_q + X_j X_k \delta_{\rho q}) + \overset{6}{h}_4 A_{jk\ell} (\delta_{j\rho} \delta_{kq} + \delta_{jq} \delta_{k\rho}).$$

In equation A.6 if N is the dimension of the space and if $\rho = \ell$, then

$$\begin{aligned} \nabla \cdot \underset{\sim}{h} = h_{\ell,\ell} = {} & (r\overset{0}{h}{}_1' + N\overset{0}{h}_1 + \overset{5}{h}_2 + \overset{5}{h}_3)\, a_{ii} + (r\overset{1}{h}{}_1' + (N+1)\overset{1}{h}_1 + \\ & + \frac{\overset{1}{h}{}_2'}{r})\, \varepsilon_{ijk} a_{kj} X_i + (r\overset{2}{h}{}_1' + (N+1)\overset{2}{h}_1 + \overset{6}{h}_3 + \frac{\overset{2}{h}{}_2'}{r} + \\ & + \overset{6}{h}_4)\, A_{imm} X_i + (r\overset{3}{h}{}_1' + (N+1)\overset{3}{h}_1 + \frac{\overset{3}{h}{}_2'}{r} + \\ & + \overset{6}{h}_2 + \overset{6}{h}_4)\, A_{mim} X_i + (r\overset{4}{h}{}_1' + (N+1)\overset{4}{h}_1 + \frac{\overset{4}{h}{}_2'}{r} + \\ & + \overset{6}{h}_2 + \overset{6}{h}_3)\, A_{mmi} X_i + (r\overset{5}{h}{}_1' + (N+2)\overset{5}{h}_1 + \frac{\overset{5}{h}{}_2'}{r} + \\ & + \frac{\overset{5}{h}{}_3'}{r})\, a_{ij} X_i X_j + (r\overset{6}{h}{}_1' + (N+3)\overset{6}{h}_1 + \frac{\overset{6}{h}{}_2'}{r} + \frac{\overset{6}{h}{}_3'}{r} + \\ & + \frac{\overset{6}{h}{}_4'}{r})\, A_{ijk} X_i X_j X_k \end{aligned} \qquad \text{A.8}$$

In equation A.7 if N is the dimension of the space and if $q = \rho$, then

$$\begin{aligned} \nabla^2 h_\ell = h_{\ell,\rho\rho} = {} & (\overset{0}{h}{}_1'' + \frac{N+1}{r}\overset{0}{h}{}_1' + 2\overset{5}{h}_1)\, a_{ii} X_\ell + (\overset{1}{h}{}_1'' + \\ & + \frac{N+3}{r}\overset{1}{h}{}_1')\, \varepsilon_{ijk} a_{kj} X_i + (\overset{1}{h}{}_2'' + \frac{N-1}{r}\overset{1}{h}{}_2' + 2\overset{1}{h}_1)\, \varepsilon_{\ell jk} a_{kj} \\ & + (\overset{2}{h}{}_1'' + \frac{N+3}{r}\overset{2}{h}{}_1' + 2\overset{6}{h}_1)\, A_{imm} X_i X_\ell + (\overset{2}{h}{}_2'' + \frac{N-1}{r}\overset{2}{h}{}_2' + \\ & + 2\overset{2}{h}_1 + 2\overset{6}{h}_2)\, A_{\ell mm} + (\overset{3}{h}{}_1'' + \frac{N+3}{r}\overset{3}{h}{}_1' + \end{aligned}$$

$$+ 2h_1^6)\, A_{mim} X_i X_\ell + (h_2^{3\prime\prime} + \frac{N-1}{r} h_2^{3\prime} + 2h_1^3 + 2h_3^6)\, A_{m\ell m} +$$

$$+ (h_1^{4\prime\prime} + \frac{N+3}{r} h_1^{4\prime} + 2h_1^6)\, A_{mmi} X_i X_\ell + (h_2^{4\prime\prime} + \frac{N-1}{r} h_2^{4\prime} + \qquad \text{A.9}$$

$$+ 2h_1^4 + 2h_4^6)\, A_{mm\ell} + (h_1^{5\prime\prime} + \frac{N+5}{r} h_1^{5\prime})\, a_{ij} X_i X_j X_\ell + (h_2^{5\prime\prime} +$$

$$+ \frac{N+1}{r} h_2^{5\prime} + 2h_1^5)\, a_{\ell j} X_j + (h_3^{5\prime\prime} + \frac{N+1}{r} h_3^{5\prime} + 2h_1^5)\, a_{j\ell} X_j +$$

$$+ (h_1^{6\prime\prime} + \frac{N+7}{r} h_1^{6\prime})\, A_{ijk} X_i X_j X_k X_\ell + (h_2^{6\prime\prime} + \frac{N+3}{r} h_2^{6\prime} +$$

$$+ 2h_1^6)\, A_{\ell jk} X_i X_k + (h_3^{6\prime\prime} + \frac{N+3}{r} h_3^{6\prime} + 2h_1^6)\, A_{j\ell k} X_j X_k + (h_4^{6\prime\prime} +$$

$$+ \frac{N+3}{r} h_4^{6\prime} + 2h_1^6)\, A_{jk\ell} X_j X_k$$

To solve a vector equation of the form $\nabla^2 \underset{\sim}{h} (\underset{\sim}{r}) = A \nabla H (\underset{\sim}{r})$ it is necessary, using equations A.2, A.9, to solve equations of the form A.10

$$h_1^{0\prime\prime} + \frac{N+1}{r} h_1^{0\prime} + 2h_1^5 = A \frac{H^{0\prime}}{r}$$

$$h_1^{5\prime\prime} + \frac{N+5}{r} h_1^{5\prime} = A \frac{H^{5\prime}}{r}$$

$$h_2^{5\prime\prime} + \frac{N+1}{r} h_2^{5\prime} + 2h_1^5 = A H^5 \qquad \text{A.11}$$

where $H^0 (r), H^5 (r)$ are known functions of r.

Homogeneous and inhomogeneous Helmholtz type equations are also solvable using this technique. For example, if a solution is needed to a problem of the form

$$(\nabla^2 - C^2)\, H (r) = 0 \qquad \text{A.12}$$

then with $H^{\infty}(r)$ as given by A.1 it is necessary to solve equations of the form

$$H^{0''} + \frac{N-1}{r} H^{0'} - C^2 H^0 = 2H^5 \qquad \text{A.13}$$
$$H^{5''} + \frac{N+3}{r} H^{5'} - C^2 H^5 = 0$$

The solution to the second of these equations with $\rho = Cr$ is

$$H^5(r) = \rho^{-\left(\frac{N+2}{2}\right)} \left[A_1^5 I_{\frac{N+2}{2}}(\rho) + A_2^5 K_{\frac{N+2}{2}}(\rho) \right] \qquad \text{A.14}$$

For the first of these equations if it is assumed that the particular solution is

$$H^0_\rho(r) = \rho^{\frac{2-N}{2}} \left[D\, I_{\frac{N+2}{2}}(\rho) + E\, K_{\frac{N+2}{2}}(\rho) \right]$$

then the complete solution is

$$H^0(r) = \rho^{\frac{2-N}{2}} \left[A_1^0 I_{\frac{N-2}{2}}(\rho) + A_2^0 K_{\frac{N+2}{2}}(\rho) - \frac{A_1^5}{NK^2} I_{\frac{N+2}{2}}(\rho) - \frac{A_2^5}{NK^2} K_{\frac{N+2}{2}}(\rho) \right] \qquad \text{A.15}$$

Another inhomogeneous Helmboltz equation which may occur quite often is

$$g'' + \frac{N+2\alpha+1}{r} g' - C^2 g = \ell\, r^{\beta} \qquad \text{A.16}$$

The Solution of this Bessel type differential equation is (G. N. Watson 1944)

$$g(r) = \rho^{-\frac{(N+2\alpha)}{2}}\left[A I_{\frac{N+2\alpha}{2}}(\rho) + B K_{\frac{N+2\alpha}{2}}(\rho) + \frac{\ell}{C^{\beta+2}} S_{\beta+1,\frac{N+2\alpha}{2}}(i\rho)\right] \qquad \text{A.17}$$

where $I_k(\rho)$, $K_k(\rho)$ are modified Bessel functions of the first and second kind respectively and $S_{a,b}(q)$ is a Lommel function.

Throughout the main body of the esis the only Bessel functions encountered were of half integer order. These Bessel functions are expressable in terms of more elementary functions.

$$\sqrt{\frac{\pi}{2}}\rho^{-\frac{1}{2}} I_{-\frac{1}{2}}(\rho) = \frac{1}{\rho}\cosh\rho = \tfrac{1}{2}\left[\frac{1}{\rho}e^{\rho} + \frac{1}{\rho}e^{-\rho}\right] = \tfrac{1}{2}\left(d_1(\rho)\, e^{\rho} + d_2(\rho)\, e^{-\rho}\right)$$

$$\sqrt{\frac{\pi}{2}}\rho^{-\frac{1}{2}} I_{\frac{1}{2}}(\rho) = \frac{1}{\rho}\sinh\rho =$$

$$\tfrac{1}{2}\left[\frac{1}{\rho}e^{\rho} - \frac{1}{\rho}e^{-\rho}\right] = \tfrac{1}{2}\left(d_1(\rho)\, e^{\rho} - d_2(\rho)\, e^{-\rho}\right)$$

$$\sqrt{\frac{\pi}{2}}\rho^{-3/2} I_{-3/2}(\rho) = \frac{1}{\rho^2}\sinh\rho - \frac{1}{\rho^3}\cosh\rho =$$

$$\tfrac{1}{2}\left[\left(\frac{1}{\rho^2} - \frac{1}{\rho^3}\right)e^{\rho} - \left(\frac{1}{\rho^2} + \frac{1}{\rho^3}\right)e^{-\rho}\right] = \tfrac{1}{2}\left(d_3(\rho)\, e^{\rho} - d_4(\rho)\, e^{-\rho}\right)$$

$$\sqrt{\frac{\pi}{2}}\rho^{-3/2} I_{3/2}(\rho) = \frac{1}{\rho^2}\cosh\rho - \frac{1}{\rho^3}\sinh\rho = \qquad \text{A.18}$$

$$\tfrac{1}{2}\left[\left(\frac{1}{\rho^2} - \frac{1}{\rho^3}\right)e^{\rho} + \left(\frac{1}{\rho^2} + \frac{1}{\rho^3}\right)e^{-\rho}\right] = \tfrac{1}{2}\left(d_3(\rho)\, e^{\rho} + d_4(\rho)\, e^{-\rho}\right)$$

$$\sqrt{\frac{\pi}{2}}\,\rho^{-\frac{5}{2}}\, I_{-\frac{5}{2}}(\rho) = -\frac{3}{\rho^4}\sinh\rho + \left(\frac{1}{\rho^3}+\frac{3}{\rho^5}\right)\cosh\rho = \frac{1}{2}\left(\left(\frac{1}{\rho^3}-\frac{3}{\rho^4}+\frac{3}{\rho^5}\right)e^{\rho} + \left(\frac{1}{\rho^3}+\frac{3}{\rho^4}+\frac{3}{\rho^5}\right)e^{-\rho}\right) - \frac{1}{2}\left(d_5(\rho)\,e^{\rho} + d_6(\rho)\,e^{-\rho}\right)$$

$$\sqrt{\frac{\pi}{2}}\,\rho^{-5/2}\, I_{5/2}(\rho) = -\frac{3}{\rho^4}\cosh\rho + \left(\frac{1}{\rho^3}+\frac{3}{\rho^5}\right)\sinh\rho = \frac{1}{2}\left(\left(\frac{1}{\rho^3}-\frac{3}{\rho^4}+\frac{3}{\rho^5}\right)e^{\rho} - \left(\frac{1}{\rho^3}+\frac{3}{\rho^4}+\frac{3}{\rho^5}\right)e^{-\rho}\right) - \frac{1}{2}\left(d_5(\rho)\,e^{\rho} - d_6(\rho)\,e^{-\rho}\right)$$

$$\sqrt{\frac{\pi}{2}}\,\rho^{-7/2}\, I_{-7/2}(\rho) = \left(\frac{1}{\rho^4}+\frac{15}{\rho^6}\right)\sinh\rho - \left(\frac{6}{\rho^5}+\frac{15}{\rho^7}\right)\cosh\rho = \frac{1}{2}\left(\left(\frac{1}{\rho^4}-\frac{6}{\rho^5}+\frac{15}{\rho^6}-\frac{15}{\rho^7}\right)e^{\rho} - \left(\frac{1}{\rho^4}+\frac{6}{\rho^5}+\frac{15}{\rho^6}+\frac{15}{\rho^7}\right)e^{-\rho}\right) - \frac{1}{2}\left(d_7(\rho)\,e^{\rho} - d_8(\rho)\,e^{-\rho}\right)$$

$$\sqrt{\frac{\pi}{2}}\,\rho^{-7/2}\, I_{7/2}(\rho) = \left(\frac{1}{\rho^4}+\frac{15}{\rho^6}\right)\cosh\rho - \left(\frac{6}{\rho^5}+\frac{15}{\rho^7}\right)\sinh\rho = \frac{1}{2}\left(\left(\frac{1}{\rho^4}-\frac{6}{\rho^5}+\frac{15}{\rho^6}-\frac{15}{\rho^7}\right)e^{\rho} + \left(\frac{1}{\rho^4}+\frac{6}{\rho^5}+\frac{15}{\rho^6}+\frac{15}{\rho^7}\right)e^{-\rho}\right) - \frac{1}{2}\left(d_7(\rho)\,e^{\rho} + d_8(\rho)\,e^{-\rho}\right)$$

B. Angular velocity and stress vectors

In chapter III and IV the vorticity vector, the microrotation vector, the shear stress vector, the normal stress and the couple stress vector are needed in the application of various boundary conditions. Each of these will be given first for the uniform flow case and then for the linear shear flow case.

For micropolar fluids subjected to uniform flow

$$V_\ell(\underset{\sim}{r}) = f_1(r)\ \varepsilon_{ijk}a_{kj}X_iX_\ell + f_2(r)\ \varepsilon_{\ell jk}a_{kj} \qquad \text{B.1}$$

$$U_\ell(\underset{\sim}{r}) = F_1(r)\ \varepsilon_{ijk}a_{kj}X_iX_\ell + F_2\ (r)\ \varepsilon_{\ell jk}a_{kj} \qquad \text{B.2}$$

$$\nu_p(\underset{\sim}{r}) = \frac{\mu+\kappa}{\kappa}\left[\ \varepsilon_{pq\ell}u_{\ell,q} + \frac{\gamma}{2\kappa}\ \varepsilon_{pq\ell}v_{\ell,q}\right] = \frac{\mu+\kappa}{\kappa}\left[\left[(F_1 - \frac{F_2'}{r}) + \right.\right.$$

$$\left.\left. + \frac{\gamma}{2\kappa}\left(\frac{f_1 - f_2'}{r}\right)\right](a_{pj} - a_{jp})\ X_j\right] \qquad \text{B.3}$$

where use has been made of the identity

$$\varepsilon_{pq\ell}\varepsilon_{pjk} = \delta_{qj}\delta_{\ell k} - \delta_{qk}\delta_{\ell j} \qquad \text{B.4}$$

Another relation which is to be used subsequently is

$$\varepsilon_{ijk}a_{kj}X_iX_\ell = \varepsilon_{\ell jk}\ (a_{jp} - a_{pj})\ X_pX_k + r^2\ \varepsilon_{\ell jk}a_{kj} \qquad \text{B.5}$$

From equation (1.1) the stress vector $t_\ell\ (\underset{\sim}{r})$ is deduced to

$$t_\ell(\underset{\sim}{r}) = -\rho X_\ell + \frac{2\mu+\kappa}{2}\ (u_{m,\ell} + u_{\ell,m})\ X_m + \kappa\ \varepsilon_{\ell m\rho}\ (\omega_\rho - \nu_\rho)\ X_m =$$

$$- \rho X_\ell + \frac{(2\mu+\kappa)}{2}\ (2rF_1' + 4F_1 + \frac{1}{L_2}\ (f_1 - \frac{f_2'}{r}))\varepsilon_{\ell jk}(a_{j\rho}-a_{\rho j})X_\rho X_k +$$

$$\frac{(2\mu+\kappa)}{2}\ r^2\ (2rF_1' + 4F_1 + \frac{2F_2'}{r})\ \varepsilon_{\ell jk}a_{kj} \qquad \text{B.6}$$

$$t(\underset{\sim}{r}) = t_\ell X_\ell = r^2\left[\ -\rho + \frac{2\mu+\kappa}{2}\ (2rF_1' + 4F_1 + \frac{2F_2'}{r})\right]\varepsilon_{\ell jk}a_{kj}X_\ell \qquad \text{B.7}$$

$$t_\ell(\underset{\sim}{r}) - \frac{t}{r^2}X_\ell = \frac{2\mu+\kappa}{2}\quad (-\frac{2F_2'}{r} + \frac{1}{L_2}\ \ (f_1 - \frac{f_2'}{r}))\ \varepsilon_{\ell jk}\ (a_{j\rho}-a_{\rho j}) \qquad \text{B.8}$$

$$X_\rho X_k$$

From equation (1.2) the couple stress vector is calculated as

$$M_i\ \ (\underset{\sim}{r}) = \frac{\mu+\kappa}{\kappa}\left[\beta(rF_1' - F_2'' + \frac{F_2'}{r} + \frac{\gamma}{2\kappa}\ rf_1' + \frac{f_2'}{r} - f_2''\ ) + \right. \qquad \text{B.9}$$

$$\left. + (\beta-\gamma)\ (F_1 - \frac{F_2'}{r} + \frac{\gamma}{2\kappa}\ f_1 - \frac{f_2'}{r} - f_2'')\right](a_{ij} - a_{ji})\ X_j$$

For viscous Newtonian fluids subjected to uniform flow

$$u_\ell(\underset{\sim}{r}) = g_1(r)\ \varepsilon_{ijk}a_{kj}X_iX_\ell + g_2(r)\ \varepsilon_{\ell jk}a_{kj} \qquad \text{B.10}$$

$$\omega_\rho(\underset{\sim}{r}) = \tfrac{1}{2}\ (g_1 - \frac{g_2}{r})\ (a_{\rho j} - a_{j\rho})\ X_j \qquad \text{B.11}$$

$$t_\ell(\underset{\sim}{r}) = -\rho X_\ell + \mu(2rg_1' + 3g_1 + \frac{g_2'}{r})\ \varepsilon_{\ell jk}\ (a_{j\rho} - a_{\rho j})\ X_\rho X_k \qquad \text{B.12}$$

$$+ \mu\ r^2\ (2rg_1' + 4g_1 + \frac{2g_2'}{r})\ \varepsilon_{\ell jk}a_{kj}$$

$$t\ (\underset{\sim}{r}) = t_\ell X_\ell = r^2\Big[-\rho + \mu(2rg_1' + 4g_1 + \frac{2g_2'}{r})\Big]\varepsilon_{\ell jk}a_{kj}X_j \qquad \text{B.13}$$

$$t_\ell(\underset{\sim}{r}) - \frac{t}{r^2}\ X_\ell = -\mu(g_1' + \frac{g_2'}{r})\ \varepsilon_{\ell jk}\ (a_{j\rho} - a_{\rho j})\ X_\rho X_k \qquad \text{B.14}$$

If a viscous Newtonian fluid is under the influence of a linear shear flow, then

$$u_\ell(\underset{\sim}{r}) = g_3(r)\ a_{ij}X_iX_jX_\ell + g_4(r)\ a_{\ell j}X_j + g_5(r)\ a_{j\ell}X_j \qquad \text{B.15}$$

$$\omega_\rho(\underset{\sim}{r}) = \tfrac{1}{2}\ (g_3 - \frac{g_4'}{r})\ \varepsilon_{\rho mn}a_{mj}X_jX_n + \tfrac{1}{2}\ (g_3 - \frac{g_4'}{r})\ \varepsilon_{\rho mn}a_{jm}X_jX_n +$$

$$+ \tfrac{1}{2}\ (g_4 - g_5)\ \varepsilon_{\rho mn}a_{nm} \qquad \text{B.16}$$

$$t_\ell(\underset{\sim}{r}) = -\rho X_\ell + \mu\Big[2rg_3' + 4g_3 + \frac{(g_4 + g_5)}{r}\Big]a_{mn}X_mX_nX_\ell +$$

$$+ \mu\big[r^2\ g_3 + rg_4' + (g_4 + g_5)\big]\ a_{\ell m}X_m + \qquad \text{B.17}$$

$$+ \mu\big[r^2\ g_3 + rg_5' + (g_4 + g_5)\big]a_{m\ell}X_m$$

$$t = t_\ell X_\ell = r^2\Big[-\rho + \mu(2rg_3' + 6g_3 + \frac{2(g_4 + g_5)'}{r} +$$

$$+ (2\frac{(g_4 + g_5)}{r^2})\Big]a_{mn}X_mX_n \qquad \text{B.18}$$

$$t_\ell - \frac{t}{r^2}\ X_\ell = \mu\Big[-2g_3 - \frac{(g_4 + g_5)'}{r} - 2\frac{(g_4 + g_5)}{r^2}\Big]\ a_{mn}X_mX_nX_\ell +$$

$$+ \mu\Big[r^2\ g_3 + rg_4' + (g_4 + g_5)\Big]a_{\ell m}X_m + \qquad \text{B.19}$$

$$+\mu\left[r^2 g_3 + r g_4' + (g_4 + g_5)\right] a_{m\ell} X_m$$

For a micropolar fluid subjected to a linear shear flow

$$v_\ell(\underset{\sim}{r}) = f_3(r)\, a_{ij} X_i X_j X_\ell + f_4(r)\, a_{\ell j} X_j + f_5(r)\, a_{j\ell} X_j \qquad B.20$$

$$u_\ell\,(\underset{\sim}{r}) = F_3\,(r) a_{ij} X_i X_j X_\ell + F_4\,(r) a_{\ell j} X_j + F_5(r)\, a_{j\ell} X_j \qquad B.21$$

$$\psi_\nu(\underset{\sim}{r}) = H(r)\, \varepsilon_{ijk} a_{kj} X_i \qquad B.22$$

$$\frac{\kappa}{(\mu+\kappa)}\; \nu_\ell(\underset{\sim}{r}) = \left[(F_3 - \frac{F_4'}{r}) + \frac{\gamma}{2\kappa}\,(f_3 - \frac{f_4'}{r}) + \frac{\delta}{2(\mu+\kappa)}\,\frac{H'}{r}\right]$$

$$\varepsilon_{\ell mn} a_{m\rho} X_\rho X_n + \left[(F_3 - \frac{F_5'}{r}) + \frac{\gamma}{2\kappa}\,(f_3 - \frac{f_5'}{r}) - \frac{\delta}{2(\mu+\kappa)}\,\frac{H'}{r}\right] \qquad B.23$$

$$\varepsilon_{\ell mn} a_{\rho m} X_\rho X_n + \left[(F_4 - F_5) + \frac{\gamma}{-2\kappa}\,(f_4 - f_5) + \frac{\delta}{2(\mu+\kappa)}\,(rH' + H)\right]$$

$$\varepsilon_{\ell mn} a_{nm} = \frac{\kappa}{\mu+\kappa}\left[G_1(r)\, \varepsilon_{\ell mn} a_{m\rho} X_\rho X_n + G_2(r)\, \varepsilon_{\ell mn} a_{\rho m} X_\rho X_n + \right.$$

$$\left. G_3(r)\, \varepsilon_{\ell mn} a_{nm}\right] \qquad B.24$$

$$t_\ell(\underset{\sim}{r}) = \rho X_\ell + \frac{\eta}{2}\left[2rF_3' + 6F_3 + \frac{1}{L_2}\,(2f_3 - \frac{(f_4' + f_5')}{r})\right] a_{ij} X_i X_j X_\ell +$$

$$+ \frac{\eta}{2}\left[2rF_4' + 2F_4 - \frac{1}{L_2}\,(r^2 f_3 - r f_4' - (f_4 - f_5)) + \frac{\delta}{\eta} H\right] a_{\ell m} X_m + \qquad B.25$$

$$+ \frac{\eta}{2}\left[2rF_5' + 2F_5 + \frac{1}{L_2}\,(r^2 f_3 - r f_5' + (f_4 - f_5)) - \frac{\delta}{\eta} H\right] a_{m\ell} X_m$$

$$t\,(\underset{\sim}{r}) = t_\ell X_\ell = r^2\left[-\rho + \frac{\eta}{2}\,(2rF_3' + 6F_3 + 2\frac{(F_4' + F_5')}{r}) + \right. \qquad B.26$$

$$\left. + 2\frac{(F_4 + F_5)}{r^2}\right] a_{ij} X_i X_j$$

$$t_\ell - \frac{t}{r^2} X_\ell = \frac{\eta}{2}\left[-2\frac{(F_4 + F_5)'}{r} - 2\frac{(F_4 + F_5)}{r^2} + \frac{1}{L_2}\,(2f_3 - \frac{(f_4' + f_5')}{r})\right]$$

$$a_{mn} X_m X_n X_\ell + \frac{\eta}{2}\left[2rF_4' + 2F_4 - \frac{1}{L_2}\,(r^2 f_3 - r f_4' - (f_4 - f_5)) + \right. \qquad B.27$$

$$\left. + \frac{\delta}{\eta} H\right] a_{\ell m} X_m + \frac{\eta}{2}\left[2rF_5' + 2F_5 - \frac{1}{L_2}\,(r^2 f_3 - r f_5' + (f_4 - f_5)) - \right.$$

$$- \frac{\delta}{\eta} H \Big] a_{m\ell} X_m$$

$$M_i = \alpha\, \upsilon_{r,r} X_i + \beta \upsilon_{i,j} X_j + \gamma\, \upsilon_{j,i} X_j$$

$$- \Big[\beta\, r\, G_1' + (\alpha + 2\beta - \gamma)\, G_1 - \alpha G_2 + (\alpha + \gamma) \frac{G_3'}{r} \Big] \varepsilon_{imn} a_{m\rho} X_\rho X_n +$$

$$+ \Big[\beta\, r\, G_2' + (\alpha + 2\beta - \gamma)\, G_2 - \alpha\, G_1 - (\alpha + \gamma) \frac{G_3'}{r} \Big] \varepsilon_{imn} a_{\rho m} a_\rho X_n +$$

$$+ \Big[\alpha (G_1 - G_2) + (\alpha + \beta + \gamma) \frac{G_3'}{r} \Big] \varepsilon_{imn} a_{nm} X_\rho X_\rho \qquad \text{B.28}$$

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