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Buckling loads of stayed columns using the finite element method.

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Buckling Loads of Stayed Columns Using The Finite Element Method

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ABSTRACT

The elastic buckling load of a concentrically loaded, pin-ended, slender metal column may be increased many times by reinforcing it with an assemblage of pre-tensioned stays and rigidly connected crossarm members. In this dissertation, the buckling solutions for different types of stayed columns are presented. The purpose of these solutions is to predict the buckling load and the corresponding modes of instability. The finite element method has been used to obtain such a buckling solution. The solution is applied with success to single, double and triple crossarm stayed columns. An attempt has been made to study the effect of stay eccentricity on the column strength for a triple crossarm stayed column. This was important since the length of the crossarms has a significant effect on the buckling load of a triple crossarm stayed column. Any other effort to optimize the column properties to give the maximum strength was not made in this report. Numerical results are presented and compared with the existing solutions.

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NOTATION

English Alphabet

A	= cross-sectional area
E	= modulus of elasticity
I	= moment of inertia of a cross-section
$[K]$	= master stiffness matrix
$[K_E]$	= master elastic stiffness matrix
$[K_G]$	= master geometrical stiffness matrix
$[K_G^*]$	= master geometrical stiffness matrix for unit values of the applied loading
$[\tilde{k}^{(i)}]$	= stiffness matrix for the i^{th} element in the local co-ordinate system
$[\tilde{k}_E^{(i)}]$	= elastic stiffness matrix for the i^{th} element in the local co-ordinate system
$[\tilde{k}_G^{(i)}]$	= geometrical stiffness matrix for the i^{th} element in the local co-ordinate system
L	= length
ℓ	= half column length
ℓ_{Ca}	= length of crossarm
ℓ'_{Ca}	= length of middle crossarm (see Fig. 4-3)
ℓ_1, m_1	= direction cosines of X-axis
ℓ_2, m_2	= direction cosines of Y-axis
$\{P\}$	= load vector
P_i	= initial loading
P_a	= applied load on the column
P_{cr}	= critical load
P_r	= residual pretension force left in the stays at the instant of buckling

$[T^{(i)}]$ = transformation matrix for the i^{th} element

$U^{(i)}$ = strain energy of the i^{th} element

U_0, U_1 & U_2 = see Eq. 3.15

$\{\tilde{u}^{(i)}\}$ = displacement vector for the i^{th} element in the local co-ordinate system

$\{u^{(i)}\}$ = displacement vector for the i^{th} element in the global co-ordinate system

$u(x), v(x)$ = the displacement functions

u = horizontal displacement at the nodes

v = vertical displacement at the nodes

x, y = local co-ordinate axes

$\bar{x}, \bar{y}$ = global co-ordinate axes

Greek Letters

$\{\Delta\}$ = vector of nodal displacements

ε = strain

ε_i = initial strain

ε_f = additional strain

θ = rotational displacement at the nodes

λ = eigenvalue

μ_s = strain energy per unit volume

σ_i = initial stress

Special Symbols

[ ] = square matrix

{ } = column matrix

$\int_V$ = integration over the volume

CHAPTER 1

INTRODUCTION

1.1 General

A stayed column can be defined as a simple column reinforced with an assemblage of pretensioned stays and crossarm members distributed along its length. A stayed column is shown in Fig. 1-1. These columns can be used as 1) supports to hold plates in place during the erection of large plate structures, 2) side booms for the mast of a derrick, and 3) masts for ships. Ties are pretensioned in order to prevent them from becoming slack before buckling starts. The purpose of pretensioned stays and crossarm members is to introduce, at several points along the length of the column, restraint against translation and rotation. This effectively reduces the maximum unsupported length of the column and helps in increasing the buckling strength of the column.

Figure 1-1 shows a stayed column which is a three dimensional symmetrical column. Because of symmetry, it can be considered as a two dimensional problem and buckling is assumed to occur in one plane.

Depending upon the number of crossarm members distributed along the length of the column, a stayed column is classified into different types i.e., single, double and triple crossarm stayed columns (Fig. 1-2).

1.2 Motivation and Choice of Method

The recent paper by Smith, McCaffrey and Ellis(1)* presented the formulation of the solutions governing the buckling behavior of the single crossarm stayed column and a procedure to determine the critical buckling load. The solution involved numerous equations and would involve many more equations for a double and a triple crossarm stayed column. This motivated the author to devise a different method to solve any stayed column for its buckling load. The method should take full advantage of the computer's capability and should involve few equations. The automatic choice was the finite element method which is a very versatile and widely used method for the analysis of Civil Engineering Structures.

1.3 The Finite Element Method

The finite element method is, essentially, a method by which the whole structure is divided into a number of "finite elements". These elements are assumed

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*Numbers in parenthesis refer to cited references in the Bibliography.

to be interconnected at a discrete number of nodal points. Since the finite element displacement method has been used, the displacements of these nodal points are the basic parameters. The minimization of the total potential energy, which is the sum total of the potential energy of the individual elements, will always result in a stiffness relationship given by:

$$[K]\{\Delta\} = \{P\} \tag{1.1}$$

where [K] is the stiffness matrix of the complete structure

{Δ} is the vector of nodal displacements

and {P} loads at the nodes

1.4 Scope of the Work

First of all, work done by different research workers has been discussed with special emphasis on the conclusions drawn by each worker.

Then the finite element method as used to solve stability problems is formulated. For nonlinear problems, the stiffness matrix is comprised of two components. One component of stiffness matrix $[K_E]$, which is termed the elastic stiffness matrix, is independent of load level while the other component $[K_G]$, the geometrical stiffness matrix, is dependent on the load level. The problem of finding the buckling load reduces to an eigenvalue problem of the form $|K_E - \lambda\ K_G^*| = 0$, which is comparatively easy to solve using subroutine EIGEN from the IBM System/360 Scientific Subroutine Package[2]. Alternatively, the problem can be reduced to the form $|K_E^{-1} K_G^* - \frac{1}{\lambda} I| = 0$, which is solved using Subroutine NROOT from the IBM System/360 Scientific Subroutine Package.

Different types of stayed columns i.e. single, double and triple crossarm stayed columns were then solved using a Fortran IV computer program. The buckling loads calculated were tabulated and compared with the existing results. The computer program calculates the buckling modes as well. The buckling configurations were sketched and discussed in each case.

1.5 Basic Assumptions

The following assumptions are made in this study of stayed columns:

1. The axial deformation of the crossarms has been neglected since it is so small to have a significant effect on the buckling of the column.

2. The connections between the crossarm members and column are assumed perfectly rigid. The connections between the stays and the column, and between the stays

and the crossarm members are assumed to be ideal hinges.

3. The stayed column is completely symmetrical and ideally centrally loaded. This means that there is no initial eccentricity and crookedness. There will be no lateral deflection of the column prior to buckling.

4. It is assumed that there is some residual pretension left in the stays at the instant of buckling. This means that all the stays remain effective when buckling occurs.

5. The geometrical stiffness matrix $[K_G]$ is neglected for the stays as well as the crossarm members since the initial stresses in the stays and crossarm members are negligible as compared to the column.

6. It is assumed that buckling occurs in the plane of a crossarm member.

CHAPTER 2

LITERATURE SURVEY

In 1963, Chu and Berge[3] analyzed a slender pin-ended column with tension ties arranged in equilateral rosettes around the column and bearing on several intermediate points along the column through hogging frames. The connections between the hogging frame and the column, and connections between the ties and the frame were assumed to be ideal hinges. The solution indicated that the maximum buckling load would be a four-fold strength increase over the Euler Column. Any increase in the number of symmetrically placed intermediate frames did not effect the strength increase. Models of stayed columns were also tested and found to agree satisfactorily with the analytical results.

To continue the work of Chu and Berge, Mauch and Felton[4] developed an analytical foundation for the rational design of the columns used by Chu and Berge[3], such as exists for simple columns. The "Structural Index" (P/L^2), which may be considered as a measure of the loading intensity, was used. Their analysis indicated that at low values of Structural Index, columns supported by tension ties offer potential savings of up to 50% of the weight of optimum simple tubular columns.

In 1970, as a design-build-test project, a single crossarm stayed column was studied by the Civil Engineering Undergraduates at The Royal Military College of Canada as their fourth year projects[5]. The crossarm members were welded to the column to provide restraint against rotation of the column. The work included the design and construction of the column as well. The results indicated a seven-fold strength increase over the Euler column. The seven-fold increase, as compared to Chu and Berge's[3] maximum of four-fold increase, was due to the fact that crossarms were firmly fixed to the columns. Thus the column had a rotational restraint in addition to the translational restraint it had in Chu and Berge's column.

In 1971, Pearson[6] examined the behaviour of a single crossarm stayed column with a high slenderness ratio when loaded to its buckling point. The effects on column strength of stay eccentricity and pretension force were examined experimentally but not theoretically. The results indicated that buckling strength is directly proportional to stay eccentricity and pretension.

Another experimental study on a single crossarm stayed column was carried out by Clarke[7] in 1972. The results verified the conclusions made by Pearson.

In January 1975, Smith, McCaffrey and Ellis[1] published a paper in which they developed an analytical method to predict the buckling load associated with each of two modes of failures for a single crossarm stayed column. Also, they demonstrated the influence of various stayed column parameters on its buckling behaviour and strength. The differential calculus approach was adopted to derive the theoretical solutions. Buckled shapes were assumed to derive the buckling solutions. There is a chance of missing the first mode when the buckled shape is assumed.

Work on stayed columns using a stiffness matrix approach has simultaneously been carried out by Temple[8] at the University of Windsor.

Experimental studies on stayed columns are in progress at The Royal Military College of Canada.

CHAPTER 3

METHOD OF ANALYSIS

3.1 General

Many problems of practical significance exist in which linearity is not preserved. Two types of nonlinearities occur in structural problems. They are

1) nonlinearity through material properties, and
2) nonlinearity through large deformations and geometrical changes in structure, so that the equations of equilibrium must be formulated for the deformed configuration.

The nonlinearity through large deformations is of great importance for stability problems. The matrix displacement method has been employed to include such a nonlinearity.

The linear relationship $\{P\} = [K]\{\Delta\}$ between the applied forces $\{P\}$ and the displacements $\{\Delta\}$ can no longer be used in the nonlinear regime. However, because of the presence of large deflections, as are encountered in most of the buckling problems, strain-displacement equations contain nonlinear terms, which must be included in calculating the stiffness matrix.

[K]. Including the appropriate nonlinear terms in the strain displacement relations, the stiffness matrix [K] can be modified as

$$[K] = [K_E] + [K_G] \tag{3.1}$$

in which $[K_E]$ = the conventional elastic stiffness matrix, and $[K_G]$ = the geometrical stiffness matrix. The elastic stiffness matrix is independent of load level while the geometrical stiffness matrix depends not only on the geometry but also on the initial internal forces existing at the start of the loading.

To solve any structure for its buckling load, the $[K_E]$ and $[K_G]$ matrices are calculated for each element of the structure and the total stiffness matrix $[K_E + K_G]$ is assembled for the possible nodal displacements.

The general stiffness relationship between the applied forces P and the displacements Δ can be expressed in a matrix form as

$$\{P\} = [K]\{\Delta\} \tag{3.2}$$

in which [K] is the stiffness matrix. For nonlinear problems [K] will be shown to comprise of $[K_E]$ and $[K_G]$ as explained in Eq. 3.1.

The external loading P can be expressed

$$P = \lambda P^* \tag{3.3}$$

in which λ is a constant and P^* represents the relative magnitudes of the applied forces. Also, since the geometrical stiffness matrix $[K_G]$ is proportional to the internal forces at the start of the loading step, then

$$[K_G] = \lambda [K_G^*] \tag{3.4}$$

where $[K_G^*]$ is the geometrical stiffness matrix for unit values of the applied loading ($\lambda = 1$). Hence Eq. 1.1 can be written as,

$$[K_E + \lambda \; K_G^*]\{\Delta\} = \lambda\{P^*\} \tag{3.5}$$

At the buckling load, stiffness of the structure becomes zero and no external disturbances are required to displace the structure. $\{\Delta\}$ cannot be zero, hence

$$|K_E + \lambda \; K_G^*| = 0 \tag{3.6}$$

The lowest value of λ multiplied by P^* gives the buckling load for the idealized structure which can be calculated using Subroutine EIGEN or Subroutine NROOT from IBM/360 Scientific Subroutine Package[2].

3.2 Generation of Elastic and Geometrical Stiffness Matrices

The first step in the finite element method is to divide the structure into a number of substructures (or elements). Having divided the structure into finite elements, a function (or functions) is chosen to define uniquely the state of strain within an element in terms of the nodal displacements. The appropriate terms for nonlinear problems must be included in the strain displacement relationship. Strain energy for the i^{th} element is then written in terms of these nodal displacements which results in a stiffness matrix for the element. The stiffness matrix for the complete structure is then assembled by summing up the contributions of the individual elements.

A stayed column consists of the main column, pretensioned stays and the crossarm members. The column and crossarm members can be idealized as beam-column members. The stiffness matrices for a beam-column member will be derived only. Essentially, a beam-column member is a modification of the axial force member.

It will be shown that the strain energy for the i^{th} element can always be expressed in quadratic form as follows:

$$U^{(i)} = \tfrac{1}{2}\{\tilde{u}^{(i)}\}^T[\tilde{k}^{(i)}]\{\tilde{u}^{(i)}\} \tag{3.7}$$

in which $\{\tilde{u}^{(i)}\}$ = the displacement vector of the i^{th} element in the local co-ordinate system, and $[\tilde{k}^{(i)}]$ = the stiffness matrix of the i^{th} element in the local co-ordinate system.

Let $\{u^{(i)}\}$ represent the displacement vector of the i^{th} element in the global or datum co-ordinate system. The relationship between $\{u^{(i)}\}$ and $\{\tilde{u}^{(i)}\}$ can always be expressed by

$$\{\tilde{u}^{(i)}\} = [T^{(i)}]\{u^{(i)}\} \tag{3.8}$$

where $[T^{(i)}]$ is a transformation matrix for the i^{th} element.

Then, Eq. 3.7 can be modified to read

$$U^{(i)} = \tfrac{1}{2}\{u^{(i)}\}^T[k^{(i)}]\{u^{(i)}\} \tag{3.9}$$

where

$$[k^{(i)}] = [T^{(i)}]^T[\tilde{k}^{(i)}][T^{(i)}] \tag{3.10}$$

3.2.1 Stiffness Matrices for a Beam-Column Member

Consider a straight, constant cross-section beam element for two-dimensional structures with nodal displacements as shown in Fig. 3-1. Let A be the cross-sectional area, EI the flexural rigidity, and L the length of the beam element shown.

The displacement functions u(x) and v(x) can be selected very conveniently for a simple beam element. For bending v(x) must be a cubic expression. This is necessary since the third derivative of v(x) (the shear) is then a constant, which is consistent with the nodal force pattern assumed for beam elements. Therefore, a cubic function in x for v(x) would fulfill the usual requirements of beam theory for the case of uniform shear along the member.

Since the total number of nodal degrees of freedom (u, v and θ at each node) is six, there should be six constants in the assumed displacement functions. Hence, having justified the choice of v(x), u(x) is accepted as a linear function. Consequently, the displacement functions are chosen as:

$$u(x) = C_1 + C_2x$$

$$v(x) = C_3 + C_4x + C_5x^2 + C_6x^3 \qquad (3.11)$$

Let ε_i be the initial strain. As actual deformation starts, additional strain ε_f develops. The total strain ε is then

$$\varepsilon = \varepsilon_i + \varepsilon_f \tag{3.12}$$

The total strain energy U may be expressed as:

$$U = \int_V \mu_s \, dV \tag{3.13}$$

where μ_s = strain energy per unit volume. μ_s can also be expressed as:

$$\mu_s = \frac{E}{2} \varepsilon^2 \tag{3.14}$$

Substituting for μ_s from Eq. 3.14 into Eq. 3.13 yields

$$U = \tfrac{1}{2} \int_V E \, \varepsilon^2 \, dV$$

Further, substituting from Eq. 3.12 the following is obtained

$$U = \tfrac{1}{2} AEL(\varepsilon_i)^2 + E\varepsilon_i \int_V \varepsilon_f \, dV + \tfrac{1}{2} E \int_V (\varepsilon_f)^2 \, dV$$

$$= U_0 + U_1 + U_2 \tag{3.15}$$

The first term on the right hand side of Eq. 3.15 is the strain energy before any additional disturbance is applied. The second term, U_1, depends on the initial stress and must yield $[K_G]$. The third term, U_2, depends on the additional strain and will yield $[K_E]$. Since U_0 does not contribute to [K] we can drop it out. Eq. 3.15 then becomes,

$$U = U_1 + U_2 \tag{3.16}$$

The nonlinear strain-displacement equation is represented by

$$\varepsilon_f = \frac{du}{dx} + \tfrac{1}{2}\left(\frac{dv}{dx}\right)^2 - Y \frac{d^2v}{dx^2} \tag{3.17}$$

where Y is the distance from the centroidal surface to any point on the section.

Substituting Eq. 3.17 into Eq. 3.16

$$U_1 = E\varepsilon_i \int_V \left(\frac{du}{dx} - Y \frac{d^2v}{dx^2} \right) dV$$

$$+ \tfrac{1}{2} E\varepsilon_i \int_V \left(\frac{dv}{dx}\right)^2 dV \qquad (3.18)$$

In the same manner, U_2 can be written as

$$U_2 = \tfrac{1}{2} E \int_V \left[\left(\frac{du}{dx}\right)^2 - 2Y \frac{du}{dx} \frac{d^2v}{dx^2} + Y^2 \left(\frac{d^2v}{dx^2}\right)^2 \right] dV$$

$$+ \tfrac{1}{2} E \int_V \left[\tfrac{1}{4} \left(\frac{dv}{dx}\right)^4 + \frac{du}{dx} \left(\frac{dv}{dx}\right)^2 - Y \left(\frac{dv}{dx}\right)^2 \left(\frac{d^2v}{dx^2}\right) \right] dV$$

$$(3.19)$$

From Eq. 3.11

$$\frac{du}{dx} = C_2$$

$$\frac{dv}{dx} = C_4 + 2C_5x + 3C_6x^2 \qquad (3.20)$$

$$\frac{d^2v}{dx^2} = 2C_5 + 6C_6x$$

With these substitutions U_1 and U_2 may be expressed as follows:

$$U_1 = E\varepsilon_i \int_{x=0}^{L} \int_A \left[C_2 - Y(2C_5 + 6C_6 x) \right] dA\, dx$$

$$+ \tfrac{1}{2} E\varepsilon_i \int_{x=0}^{L} \int_A \left[C_4 \; C_5 \; C_6 \right] \begin{bmatrix} 1 & & \underline{SYM} \\ 2x & 4x^2 & \\ 3x^2 & 6x^3 & 9x^4 \end{bmatrix}$$

$$\begin{bmatrix} C_4 \\ C_5 \\ C_6 \end{bmatrix} dA\, dx \qquad (3.21)$$

$$U_2 = \tfrac{1}{2} E \int_{x=0}^{L} \int_A \left[C_2 \; C_5 \; C_6 \right] \begin{bmatrix} 1 & & \underline{SYM} \\ -2y & 4y^2 & \\ -6xy & 12xy^2 & 36x^2y^2 \end{bmatrix}$$

$$\begin{bmatrix} C_2 \\ C_5 \\ C_6 \end{bmatrix} dA\, dx + \tfrac{1}{2} E \int_{x=0}^{L} \int_A \Big[\tfrac{1}{4}(C_4 + 2C_5 x + 3C_6 x^2)^4$$

$$+ C_2 (C_4 + 2C_5 x + 3C_6 x^2)^2 - y(C_4 + 2C_5 x + 3C_6 x^2)^2$$

$$(2C_5 + 6C_6 x) \Big] \; dA\, dx \qquad (3.22)$$

Only the quadratic terms in the displacements will contribute to stiffness coefficients. Therefore, lower order terms and higher terms are dropped, as is also the case in the classical nonlinear theory. The first integral in Eq. 3.21 and the second integral in Eq. 3.22 are dropped for the reason given above.

Solving for constants in Eq. 3.11 by writing u, v and θ (dv/dx) at each node, then

$$C_1 = u_1$$

$$C_2 = \frac{u_2 - u_1}{L}$$

$$C_3 = v_1$$

$$C_4 = \frac{v_2 - v_1}{L}$$

$$C_5 = \frac{3}{L^2}(v_2 - v_1) - \frac{1}{L}(2\theta_1 + \theta_2)$$

$$C_6 = -\frac{2}{L^3}(v_2 - v_1) + \frac{1}{L^2}(\theta_1 + \theta_2)$$

(3.23)

On substituting from Eq. 3.23 and using the fact that $AE\varepsilon_i = A\sigma_i = P_i$, the initial loading, the quadratic part of U_1 can be written as:

$$U_1 = \tfrac{1}{2} \{\tilde{u}^{(i)}\}^T [\tilde{k}_1^{(i)}] \{\tilde{u}^{(i)}\} \qquad (3.24)$$

where

$$\{\tilde{u}^{(i)}\}^T = [v_1 \; \theta_1 \; v_2 \; \theta_2]$$

and

$$[\tilde{k}_1^{(i)}] = P_i \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 6/5L & & & \underline{\text{SYM}} \\ 1/10 & 2L/15 & & \\ -6/5L & -1/10 & 6/5L & \\ 1/10 & -L/30 & -1/10 & 2L/15 \end{bmatrix} \end{array}$$

$[\tilde{k}_1^{(i)}]$ is termed as $[\tilde{k}_G^{(i)}]$, the geometrical stiffness matrix of the i^{th} element in the local co-ordinate system.

It is now a simple matter to expand the stiffness matrix to order (6 x 6). This can be done by adding columns and corresponding rows of zeros for the added displacements (u_1 and u_2)

$$
[\tilde{k}_G^{(i)}] = P_i \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 0 & & & & & \underline{\text{SYM}} \\ 0 & 6/5L & & & & \\ 0 & 1/10 & 2L/15 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -6/5L & -1/10 & 0 & 6/5L & \\ 0 & 1/10 & -L/30 & 0 & -1/10 & 2L/15 \end{bmatrix} \end{array} \tag{3.25}
$$

Similarly writing U_2 in quadratic form as

$$
U_2 = \tfrac{1}{2}\{\tilde{u}^{(i)}\}^T[\tilde{k}_2^{(i)}]\{\tilde{u}^{(i)}\} \tag{3.26}
$$

where

$$
\{\tilde{u}^{(i)}\}^T = [u_1 \; v_1 \; \theta_1 \; u_2 \; v_2 \; \theta_2]
$$

$[\tilde{k}_2^{(i)}] = [\tilde{k}_E^{(i)}]$, is the elastic stiffness matrix of the i^{th} element in the local co-ordinate system and is given as follows:

$$
[\tilde{k}_E^{(i)}] = \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} EA/L & & & & & \underline{SYM} \\ 0 & 12EI/L^3 & & & & \\ 0 & 6EI/L^2 & 4EI/L & & & \\ -EA/L & 0 & 0 & EA/L & & \\ 0 & -12EI/L^3 & -6EI/L^2 & & 12EI/L^2 & \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \end{array} \tag{3.26}
$$

The stiffness matrices can be written in the global co-ordinate system using Eq. 3.10.

The transformation matrix, $[T^{(i)}]$, for a two-dimensional beam element can be written as:

$$
[T^{(i)}] = \begin{bmatrix} \ell_1^{(i)} & m_1^{(i)} & 0 & 0 & 0 & 0 \\ \ell_2^{(i)} & m_2^{(i)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ell_1^{(i)} & m_1^{(i)} & 0 \\ 0 & 0 & 0 & \ell_2^{(i)} & m_2^{(i)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.27}
$$

where ℓ_1, m_1 are the direction cosines of X-axis

ℓ_2, m_2 are the direction cosines of Y-axis.

Performing the matrix operations in Eq. 3.10, the elastic and geometrical stiffness matrices in global co-ordinate system are given as:

$$[K_E^{(i)}] = \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} k_{11} & & & & & \underline{\text{SYM}} \\ k_{21} & k_{22} & & & & \\ k_{31} & k_{32} & k_{33} & & & \\ -k_{11} & -k_{21} & -k_{31} & k_{11} & & \\ -k_{21} & -k_{22} & -k_{32} & k_{21} & k_{22} & \\ k_{31} & k_{32} & k_{33}/2 & -k_{31} & -k_{32} & k_{33} \end{bmatrix} \end{array} \qquad (3.28)$$

where,

$$k_{11} = \ell_1^2\left(\frac{AE}{L}\right) + \ell_2^2\left(\frac{12EI}{L^3}\right)$$

$$k_{21} = \ell_1 m_1\left(\frac{AE}{L}\right) + \ell_2 m_2\left(\frac{12EI}{L^3}\right)$$

$$k_{22} = m_1^2\left(\frac{AE}{L}\right) + m_2^2\left(\frac{12EI}{L^3}\right)$$

$$k_{31} = \ell_2\left(\frac{6EI}{L^2}\right)$$

$$k_{32} = m_2\left(\frac{6EI}{L^2}\right)$$

$$k_{33} = \frac{4EI}{L}$$

ℓ_1, m_1 are the directions, cosines of X-axis

ℓ_2, m_2 are the directions, cosines of Y-axis

Note: $\ell_2 = -m_1$ and $m_2 = \ell_1$.

Similar type of matrix is generated for $[K_G^{(i)}]$

$$\begin{array}{cc} & \begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\ [K_G^{(i)}] = P_i & \begin{bmatrix} k_{11} & & & & \text{SYM} & \\ k_{21} & k_{22} & & & & \\ k_{31} & k_{32} & k_{33} & & & \\ -k_{11} & -k_{21} & -k_{31} & k_{11} & & \\ -k_{21} & -k_{22} & -k_{32} & k_{21} & k_{22} & \\ k_{31} & k_{32} & k_{33}/4 & -k_{31} & -k_{32} & k_{33} \end{bmatrix} \end{array} \qquad (3.29)$$

where,

$$k_{11} = \ell_2^2\left(\frac{6}{5L}\right)$$

$$k_{21} = \ell_2 m_2\left(\frac{6}{5L}\right)$$

$$k_{22} = m_2^2\left(\frac{6}{5L}\right)$$

$$k_{31} = \ell_2(\frac{1}{10})$$

$$k_{32} = m_2(\frac{1}{10})$$

$$k_{33} = \frac{2}{15} L$$

3.2.2 Modifications for Stays and Crossarm Members

To modify these stiffness matrices for any tensile axial force member, the moment of inertia for these members are taken as zero since they do not have bending stiffness and hence all the terms containing I are reduced to zero. The geometrical stiffness matrices for stays (axial force tensile member) and crossarm members are dropped out since the initial stresses due to applied loading in these members are assumed to be negligible as compared to those in the column.

To assemble the stiffness matrices for the complete structure, the "variable correlation scheme" has been used. The procedure is simple and Reference 9 explains it thoroughly.

Once the elastic and geometrical stiffness matrices for the complete structure have been generated, Eq. 3.6 can be used to determine the buckling load.

$$|K_E + \lambda\ K_G^*| = 0 \tag{3.30}$$

The buckling load is then determined as the load at which the minimum eigenvalue goes to zero. It must be noted that it yields the critical load in the column, and not the cirtical applied load. The applied load can be given by

$$P_a = P_{cr} - P_r \tag{3.31}$$

in which P_a is the applied load on the column

P_{cr} is the critical load

and P_r is the residual pretension force left in the stays at the instant of buckling.

CHAPTER 4

APPLICATIONS

4.1 Introduction

A general computer program has been developed to solve any type of stayed column for its buckling load and the corresponding modes of instability.

4.2 Solution Procedure

Chapter 3 describes the detailed method but the computer algorithm can be summarized as follows:

1. Knowing the co-ordinates of the nodes and member properties, calculate the lengths and direction cosines of the members.

2. Calculate the element elastic and geometrical stiffness matrices in the global co-ordinate system using equations (3.28) and (3.29). Neglect the geometrical stiffness matrix for the stays and crossarm members for the reason mentioned in 3.2.3.

3. Calculate the master elastic and geometrical stiffness matrices for the whole structure using the "Variable Correlation Scheme".

4. Combine the two matrices to obtain Eq. 3.30.

5. Obtain the buckling load using subroutine EIGEN from IBM System/360 Scientific Subroutine Package[2]. This subroutine calculates the buckling load, the load at which the minimum eigenvalue goes to zero. This subroutine also gives the corresponding modes of instability for the structure which is the eigen vector associated with the minimum eigenvalue.

To obtain the buckling load directly, i.e. without using an iterative method, Subroutine NROOT from IBM System/360 Scientific Subroutine Package[2] can be used. Subroutine NROOT calculates the eigenvalues of $[K_E]^{-1}[K_G^*]$ which are the reciprocal $(1/\lambda)$ of the eigenvalues of $[K_G^*]^{-1}[K_E]$. The reciprocal of the maximum eigenvalue from Subroutine NROOT gives the lowest eigenvalue which corresponds to the buckling load of the structure.

4.3 Applications

The method has been applied to single, double and triple crossarm stayed columns to study the following aspects:

1. Increase in buckling strength of a column by using stays and crossarm members.

2. Significant change in buckling strength of a triple crossarm stayed column with the change in crossarms lengths.

3. To show that the accuracy increases with the number of elements used. To get the reliability of the results by comparing with the existing solutions.

4. To study the buckling behaviour of different types of stayed columns.

4.3.1 Example 1

In this problem the method of solution was applied to show how much the buckling strength increases over a simple column by using single, double and triple crossarm stayed columns. A Fortran IV computer program was used throughout.

The column and crossarm members were assumed to be circular steel tubes with an outside diameter of 2.25 in. and an inside diameter of 1.75 in. The length of the column, 2ℓ, was selected to be 16 ft. and the stays were assumed to be 7/16 inch steel wire ropes. The modulus of elasticity of the steel tubing (which implies, $E_C = E_{Ca}$) was taken to be 29.6×10^6 psi where E_C stands for modulus of elasticity of column and E_{Ca} stands for modulus of elasticity of crossarm members. The modulus of elasticity of the wire rope was taken to be 9.4×10^6 psi. The crossarm member length was assumed to be 1 ft. for both single and double crossarm stayed columns while for triple crossarm stayed columns, the lengths of crossarm members were assumed to be 1 ft., 2 ft. and 1 ft. (see Fig. 4-3). The length 1 ft. applies to upper and lower crossarm members while length of 2 ft. applies to the middle crossarm. A simple Euler column of same length (i.e. 16 feet) was also solved for its buckling load. The results are tabulated in Table I. The buckled configurations of all the columns studied in this section are shown in Figs. 4-1 to 4-3. The results were as expected because, by adding stays and crossarm members, restraint is introduced against translation and rotation and thereby decreasing the effective unsupported buckling length of the column. Hence an increase in strength was resulted. Also, the buckled configuration shows that the stays provide a horizontal force at the column-crossarm connection and thus providing extra strength to the column.

The results for the triple crossarm stayed column show that there is not much strength increase from that of double crossarm stayed column. This may be due to the fact that with the lengths of crossarm members used in Example 1, the stays remain straight and thus offers very little resistance at the column-crossarm connection. Thus the need arises to study the triple crossarm stayed column with varying lengths of crossarm members.

4.3.2. Example 2

Two triple crossarm columns were analyzed for their buckling load using the computer program developed and the results were compared with the triple crossarm column of Example 1.

The column, crossarm members and stays have the same member properties as described in Example 1 except that two sets of crossarm lengths were used. In one, lengths of 1.5 ft., 2 ft. and 1.5 ft. were used while in the other 0.5 ft., 2 ft. and 0.5 ft. were used. The results are tabulated in Table II. The buckled configurations are shown in Figs. 4-4 and 4-5. The results indicate that as the slopes of the stays are changed, there is a significant change in the buckling load of the column.

4.3.3 Example 3

The purpose of carrying out this example was twofold. Firstly, to show that the accuracy increases with the number of elements used and secondly to get the reliability of the results by comparing results with the existing solutions. Three stayed columns with one, two and three sets of crossarm members were analyzed for their buckling load. The columns were divided into twice as many elements as were used in Example 1 (see Fig. 4-6). Table III shows the comparison of the results with the results given by References 1 and 8. The results are in close agreement. The buckled configurations were not plotted since they are essentially of the same shapes as shown in Figs. 4-1 to 4-3.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

The finite element method as applied to stability problems was reviewed. The method of solution was applied to stayed columns. Single, double and triple crossarm stayed columns were solved for their buckling load. A completely general computer program, requiring a minimum of data preparation, was written. The first two buckling modes were studied since they are of greatest interest. An attempt was made to study the triple crossarm stayed column with varying stay slopes. The following conclusions may be stated:

1) It is possible to predict the buckling load and the corresponding buckled shape for a single, double and triple crossarm stayed columns. The solution can very well be used to solve a multiple crossarm stayed column.

2) The results have indicated that the buckling load of a column may be increased many times by

reinforcing it with a system of pretensioned stays and rigidly connected crossarm members.

3) The buckled shapes show the effect of restraint provided by the pretensioned stays at the column-crossarm connection.

4) The buckling load of a triple crossarm stayed column is greatly influenced by the slopes of the stays.

5) The solution procedure for a stayed column requires that all the pretension stays should remain effective at the instant of buckling.

6) The results indicate that with complete fixity of crossarms with column, the buckling load can be increased many times in comparison to Chu and Berge's[3] pin connected column where the maximum increase was four-fold.

7) The results for a single crossarm stayed column were in close agreement with Smith, McCaffrey and Ellis[1]. The results for all the three types, i.e. single, double and triple crossarm stayed columns, were in good agreement with Temple[8].

5.2 Future Research

Experimental studies should be undertaken to determine the residual pretension left in the stays at the instant of buckling. The buckling strength as obtained in this study should also be determined experimentally.

A theoretical study should be made to determine the amount of pretension needed in the stays to prevent the slackening of the stays, and to provide effective lateral support.

The influence of various stayed column parameters for double and triple crossarm stayed columns must also be studied.

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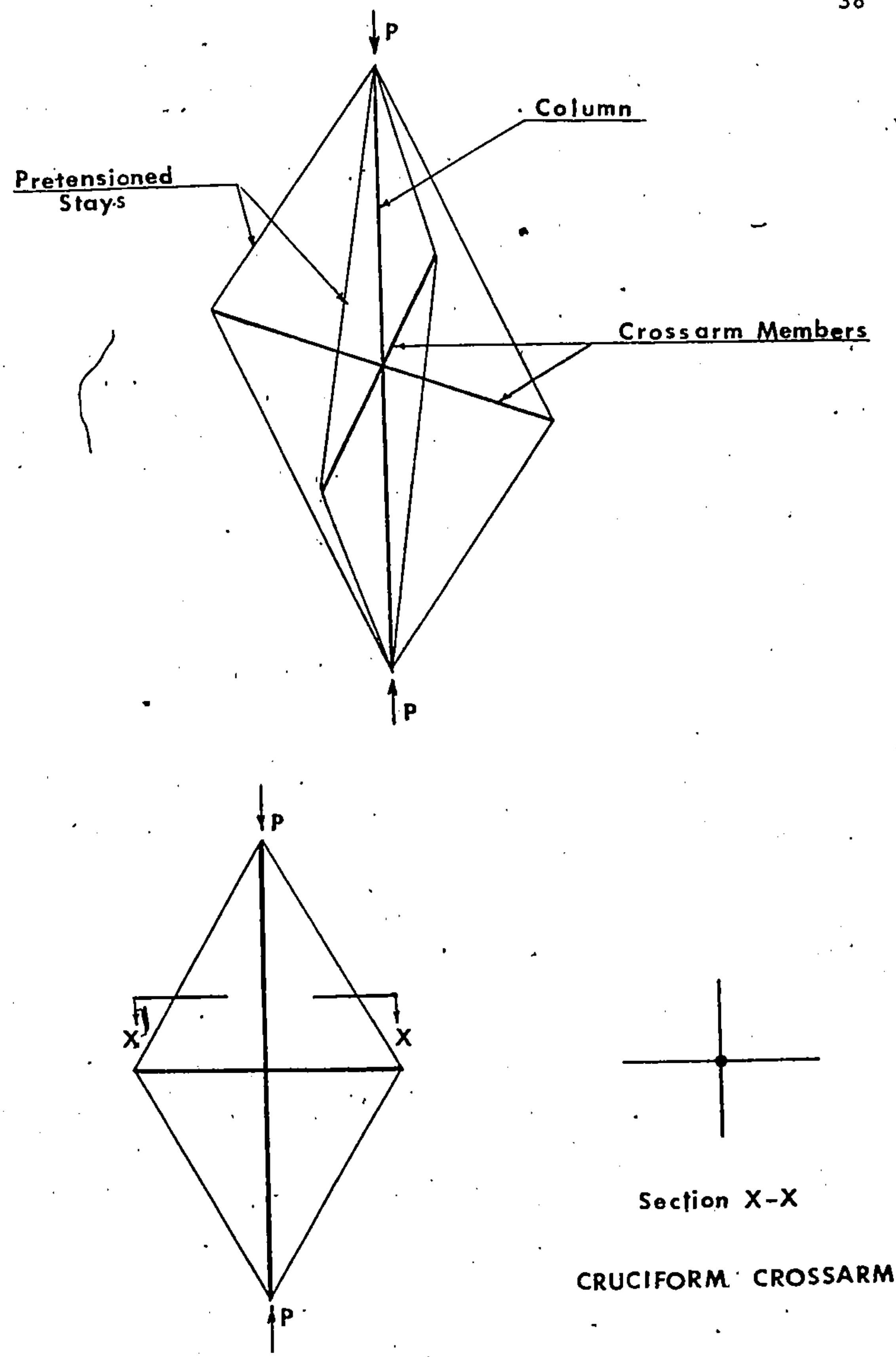


Fig. 1-1. The Stayed Column.

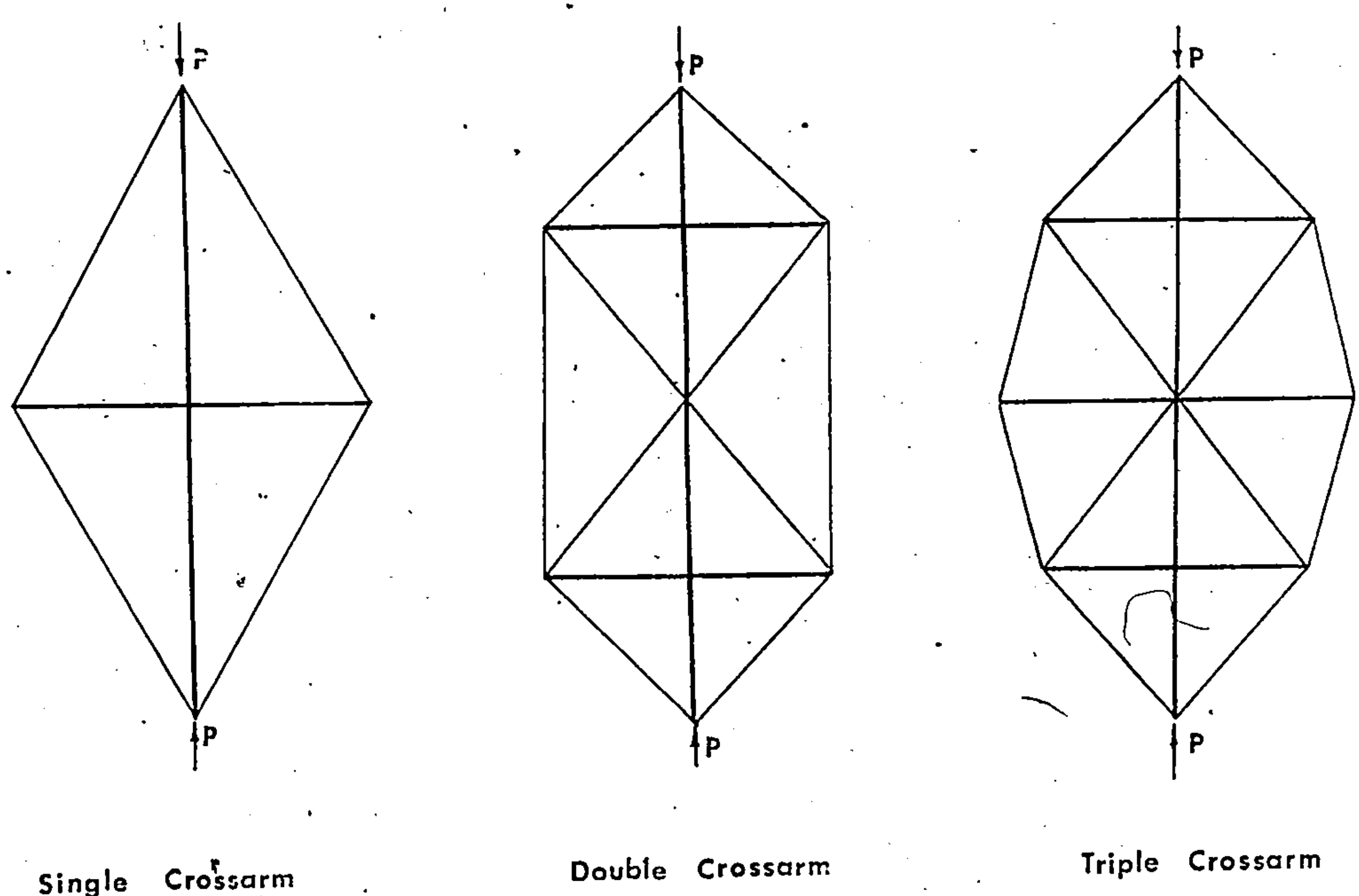


Fig. 1-2. Types of Stayed Columns.

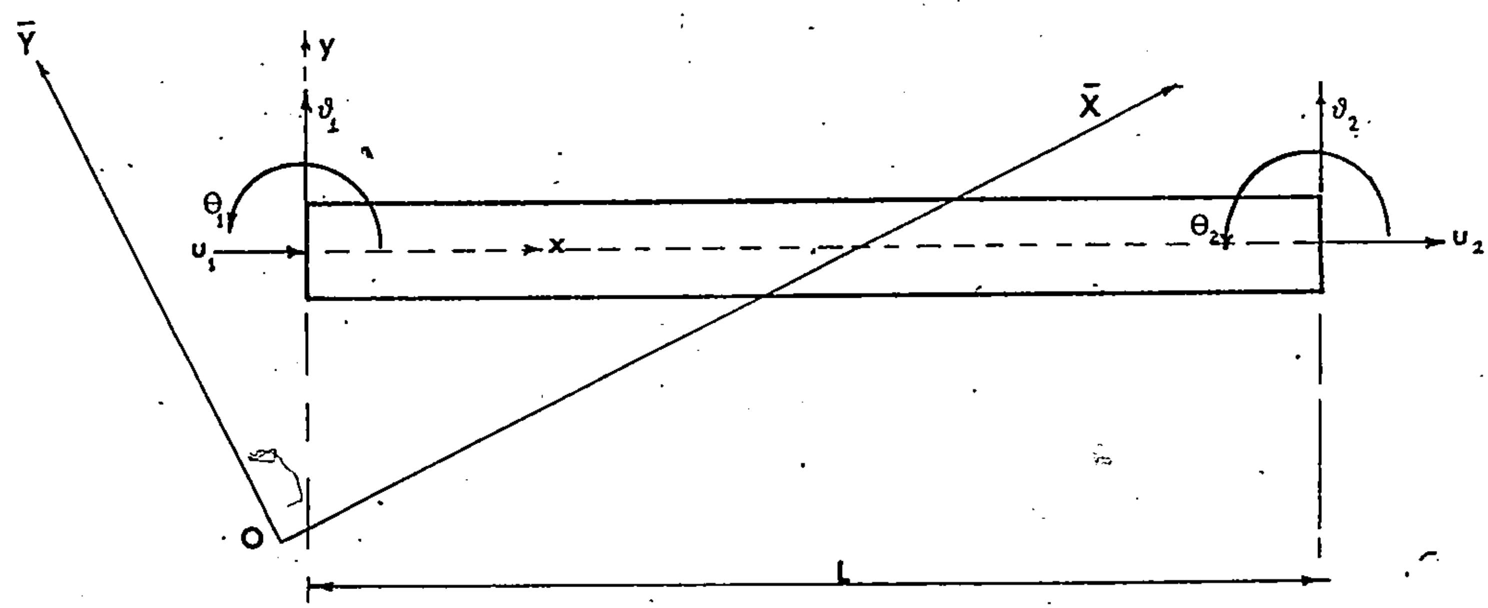
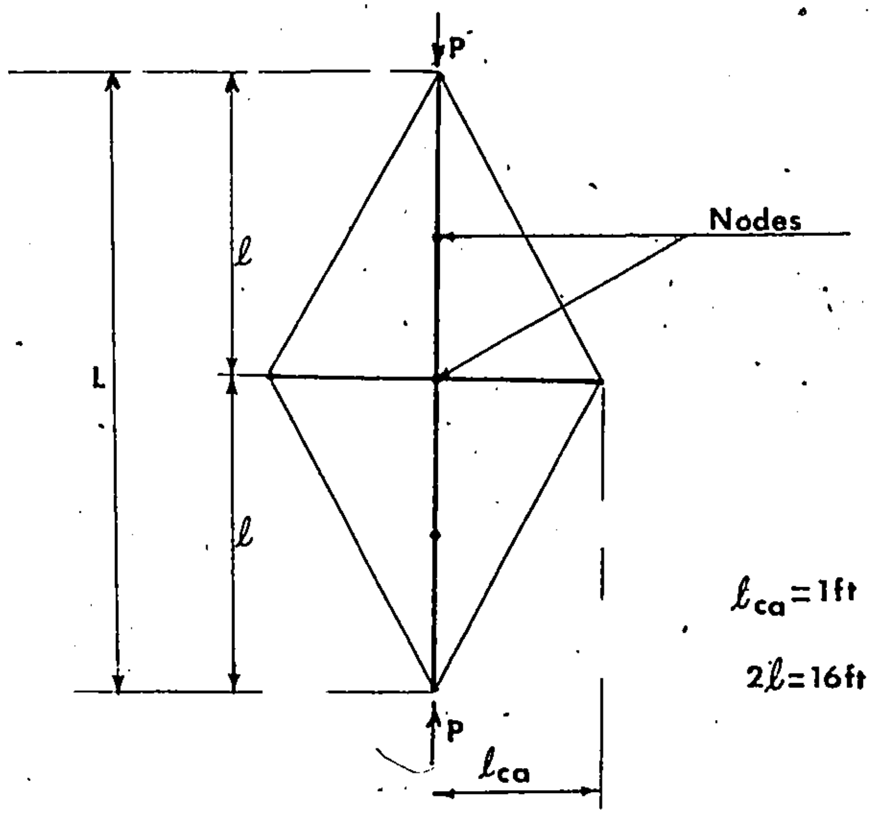


Fig. 3-1. Beam Element for Two-Dimensional Structures.



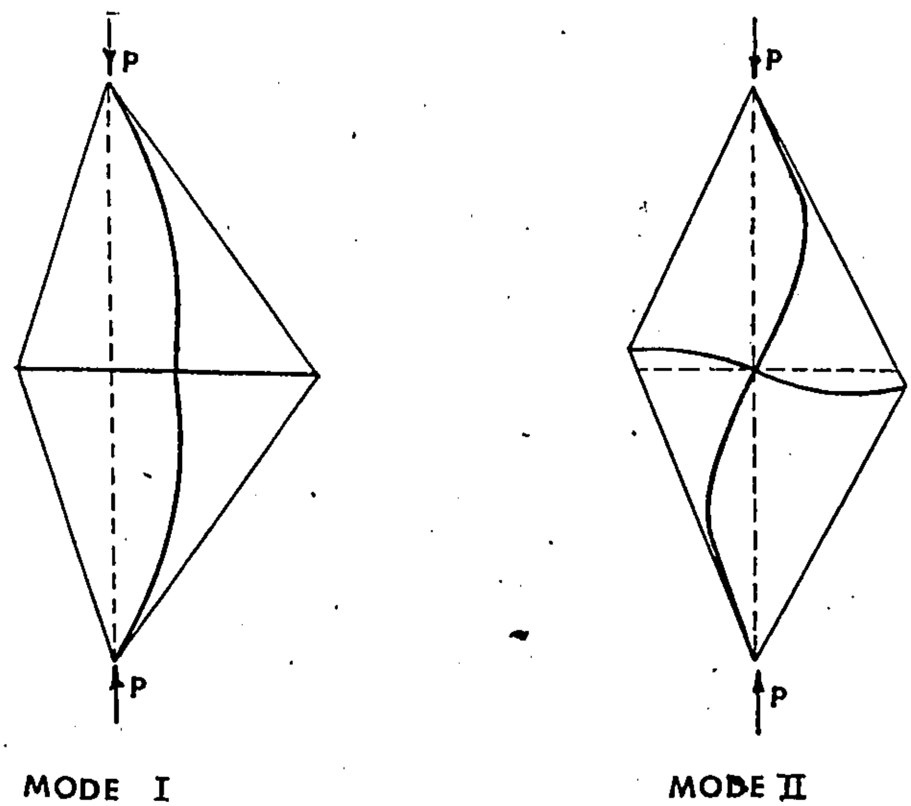
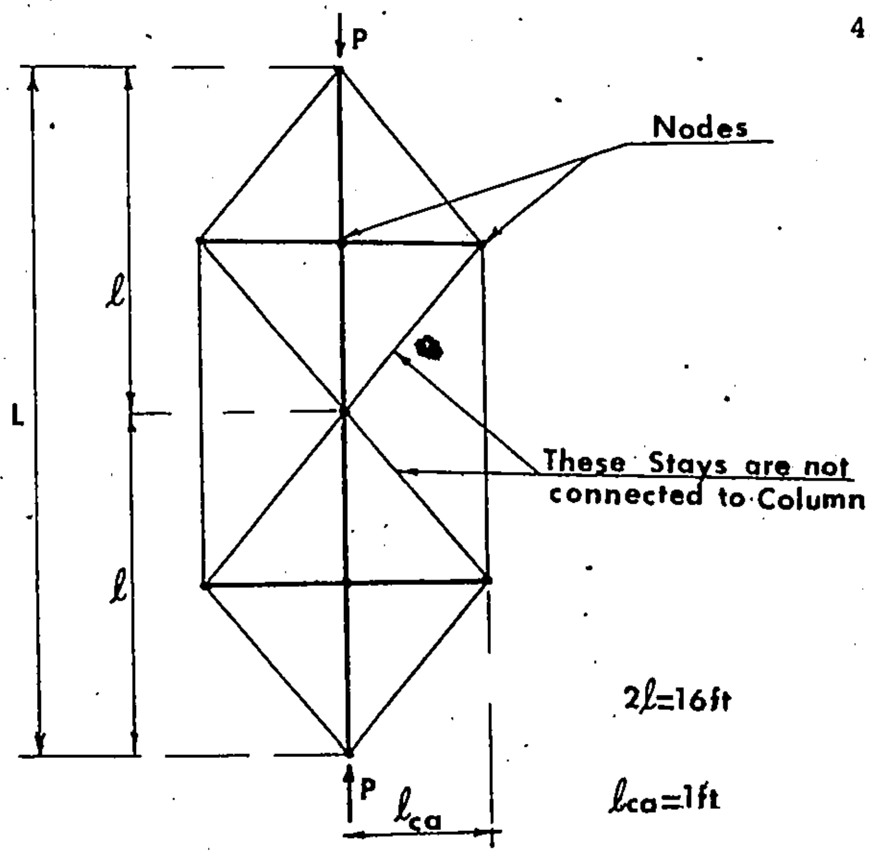


Fig. 4-1. Buckling Configuration of a Single Crossarm Stayed Column.



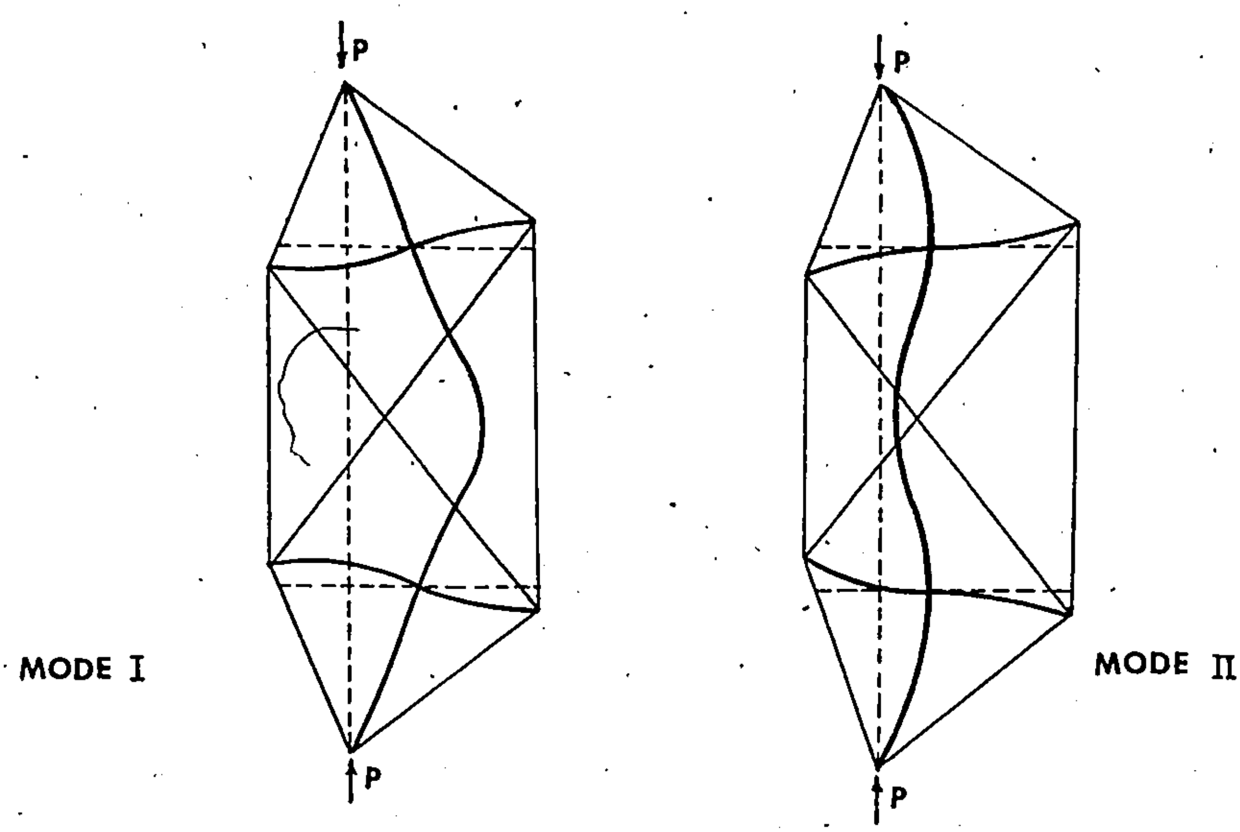


Fig. 4-2. Buckling Configuration of a Double Crossarm Stayed Column.

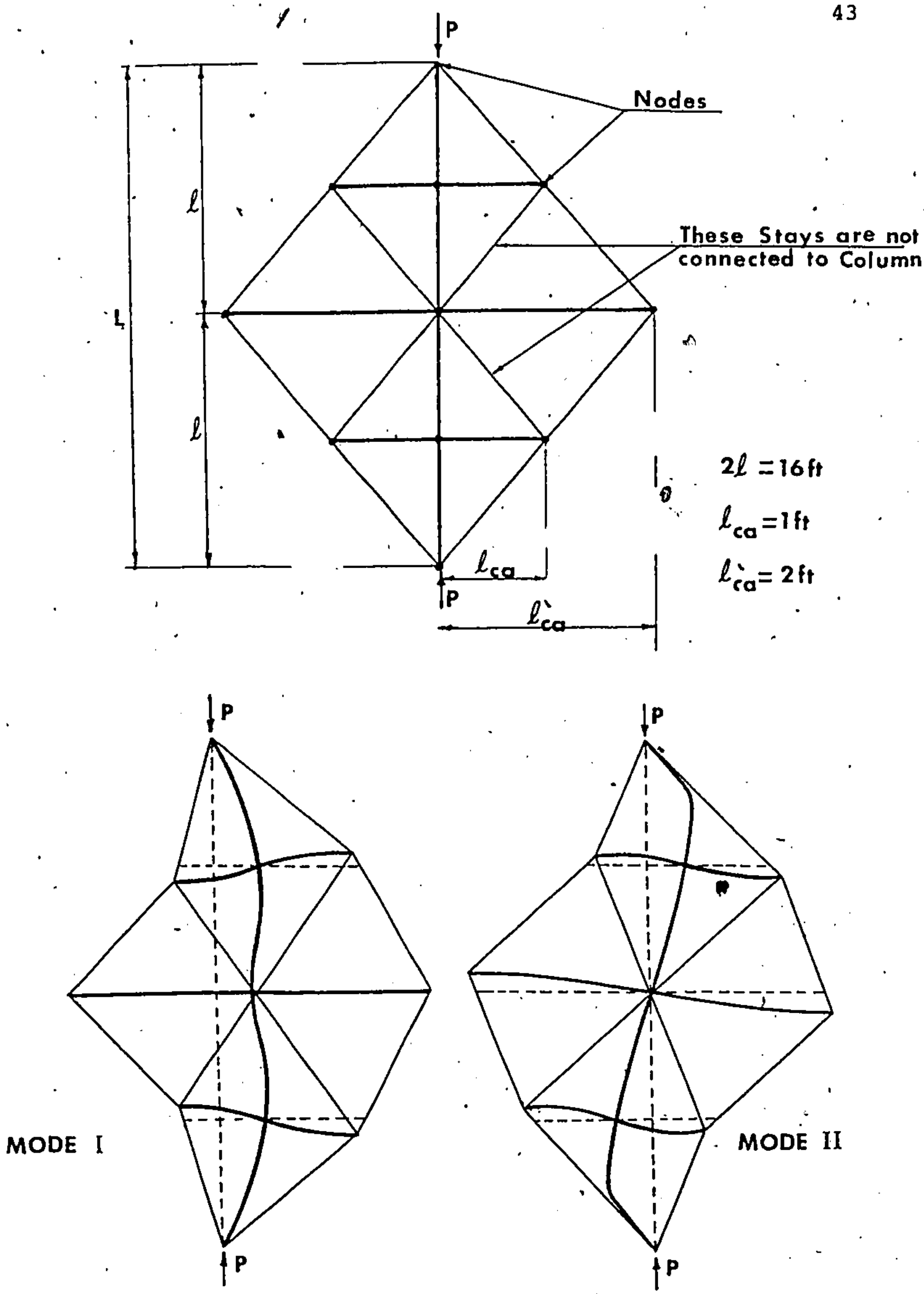


Fig. 4-3. Buckling Configuration of a Triple Crossarm Stayed Column (Example 1).

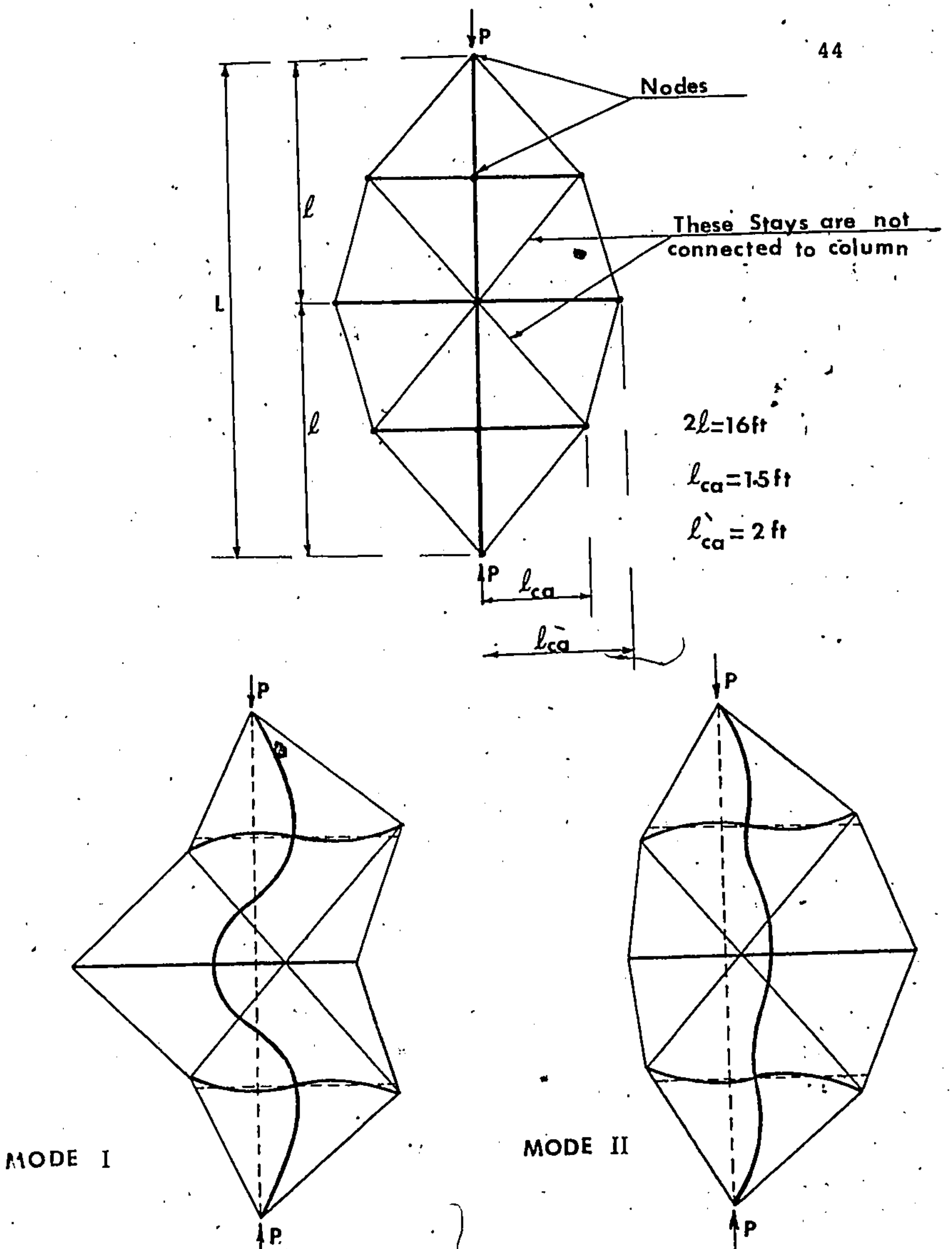


Fig. 4-4. Buckling Configuration of a Triple Crossarm Stayed Column (Example 2).

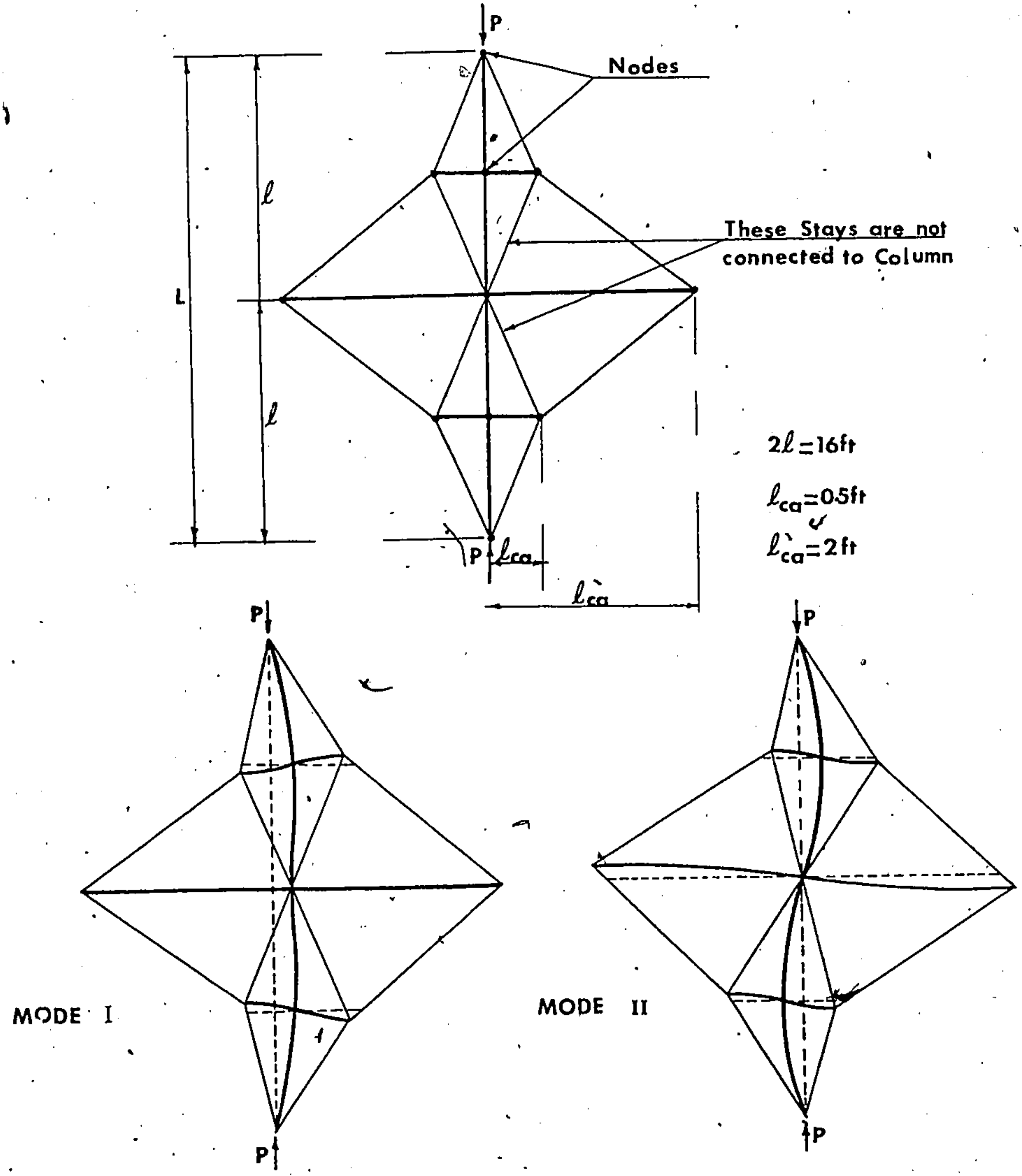


Fig. 4-5. Buckling Configuration of a Triple Crossarm Stayed Column (Example 2).

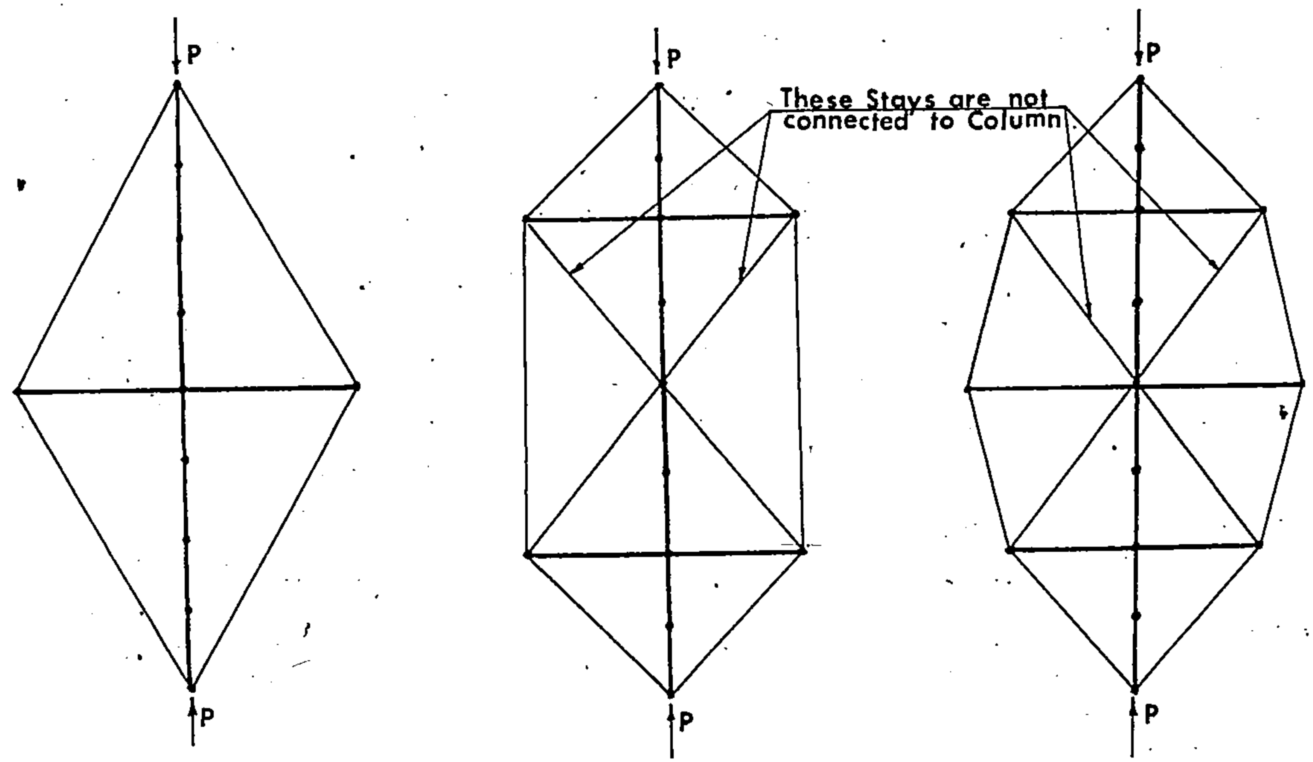


Fig. 4-6. Stayed Columns used for Example 3.

TABLE 4-1

Buckling Loads of Stayed Columns

S.No.	Description of the Column	Mode I Buckling Load (kips)	Mode II Buckling Load (kips)
1.	Simple Euler Column	6.32	25.47
2.	Single Crossarm Stayed Column (length of crossarm: 1 ft.) Fig. 4-1	35.64	44.33
3.	Double Crossarm Stayed Column (lengths of crossarms: 1 ft. and 1 ft.) Fig. 4-2	70.23	109.64
4.	Triple Crossarm Stayed Column (lengths of crossarms: 1 ft., 2 ft. and 1 ft.) Fig. 4-3	72.39	146.49

TABLE 4-2

Results of Triple Crossarm Stayed Columns

S.No.	Description of the Column	Mode I Buckling Load (kips)	Mode II Buckling Load (kips)
1.	Triple Crossarm Stayed Column (lengths of crossarms: 0.5 ft., 2 ft. and 0.5 ft.) Fig. 4-5	33.02	65.92
2.	Triple Crossarm Stayed Column (lengths of crossarms: 1.5 ft., 2 ft. and 1.5 ft.) Fig. 4-4	103.41	170.00
3.	Triple Crossarm Stayed Column of Example 1. (lengths of cross-arms: 1 ft., 2 ft. and 1 ft.) Fig. 4-3	72.39	146.49

TABLE 4-3

Comparison of Results

S. No.	Description of the Column	Smith[1]		Temple[8]		Author Example 3		Author Example 1	
		Mode I Buckling Load (kips)	Mode II Buckling Load (kips)	Mode I Buckling Load (kips)	Mode II Buckling Load (kips)	Mode I Buckling Load (kips)	Mode II Buckling Load (kips)	Mode I Buckling Load (kips)	Mode II Buckling Load (kips)
1.	Single Crossarm Stayed Column (length of crossarm: 1 ft.) Fig. 4-6	35.40	43.00	35.48	43.69	35.49	43.73	35.64	44.33
2.	Double Crossarm Stayed Column (lengths of crossarms: 1 ft. and 1 ft.) Fig. 4-6	Not Available	Not Available	69.86	106.28	69.94	106.78	70.23	109.64
3.	Triple Crossarm Stayed Column (lengths of crossarms: 1.5 ft., 2 ft. and 1.5 ft.) Fig. 4-6	Not Available	Not Available	102.33	167.58	102.96	169.28	103.41	170.00

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