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CERTAIN FLOW PROBLEMS IN POLAR FLUIDS.

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CERTAIN FLOW PROBLEMS IN POLAR FLUIDS

by

Merzik Tawfik Kamel

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submitted to the Faculty of Graduate Studies Through
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ABSTRACT

This dissertation deals with the solution of the following flow problems in polar fluids:

1. The motion of a polar fluid flow through a curved pipe is studied. A perturbation method is used to solve the system of differential equations which governs the motion. The stream-lines in the central plane of the pipe and in the cross-section are sketched for particular values of the coupling number, the length ratio and Reynolds number and a comparison is made with the case of Newtonian fluids. The rate of flow through the pipe is calculated up to the second approximation and it is observed that it decreases due to the curvature of the pipe.

2. Two-dimensional internal and external flows of polar fluids for a circular cylinder are investigated. The motion within a circular cylinder generated by:

a) fluid entering and leaving through slots in the cylinder wall and

b) the rotation of part of the wall

are considered. In both cases an analytical solution is obtained, the stream-lines are sketched for special cases and the deviation from the Newtonian fluid is observed.

An investigation of creeping flow of a polar fluid past a circular cylinder is carried out and it is noticed that this motion is not possible as in the case of a Newtonian fluid.

3. The solution of the Hamel flow problem is discussed and it is shown that there does not exist a purely radial flow of a polar fluid which satisfies both the equations of motion and the boundary conditions when the micro-inertia of the particles are negligible.

If the micro-inertia terms are retained it is observed that the solution is possible, only for a particular relation between the material coefficients, and it has the same form as for the Newtonian fluid.

Also it is observed that the rotation of the particles of the fluid does not induce a secondary flow between two non-parallel infinite plates in the case of creeping flow of a polar fluid.

Dedicated to my beloved parents

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CHAPTER I

BASIC EQUATIONS AND OBJECTIVES

1.1 Introduction

The theory of polar fluids and related theories are models for fluids in which the microstructure of the material plays a significant role. The applications of these theories have been to physical and biological sciences, for example, to blood flow and suspensions. The theory can also be applied to fluids carrying charged particles and subjected to an external electromagnetic field that causes the particles to rotate relative to their neighbors. To treat the mechanics of fluids with microstructure, new continuum models - different from the classical continuum model for viscous fluids - have been suggested from different points of view by many authors.

The theory of polar fluids was originally developed by assuming non-central forces of interaction between the particles. If the interparticle forces are not central forces in a particle-particle interaction there is an interparticle couple as well as an interparticle force. Under the action of this couple the fluid particle will have a tendency to rotate relative to its neighbors. The essential idea of a polar fluid is obtained by introducing a kinematic variable to model the rotation of the particle relative to its neighbors and a skew-symmetric stress tensor to model the forces that balance the action of the couple. The presence of body couples and also of the micro-inertia of the particles are allowed in the theory of polar fluids.

In classical fluid mechanics the only kinematical vector field is the velocity field and a fluid element rotates with the local vorticity. In the case of polar fluids the rotation of the particles cannot be represented by the vorticity alone. Hence the theory employs two independent vector fields, the usual velocity field and an angular velocity field. Mechanically, polar fluid theory differs from the classical one in that in the former case angular momentum effects, such as couple stresses and asymmetry of the usually symmetric stress tensor, are considered.

Grad [14] introduced the linear constitutive equations for a polar fluid subsequent to a statistical mechanics study. These constitutive equations relate the skew-symmetric part of the usual stress tensor to the difference between the particle angular velocity and the vorticity field as well as the couple stresses to the gradient of the particle angular velocity.

The same constitutive equations have been advanced from different view points by Aero et al.[1], Cowin [4], Condiff and Dahler [3], Dahler [8] and Eringen [13]. Aero et al. [1] simply introduced the velocity and total angular velocity fields and postulated a disspation function that leads to Grad's constitutive equations. Cowin development was based on a Cosserat continuum. He assumed that each particle is outfitted with a rigid triad of vectors. The total angular velocity of the particle is then represented by the angular velocity of the rigid triad. Eringen [13] obtained the constitutive equations for a polar fluid by specializing his more general theory of simple microfluids [12]. Each of these developments has been from a continuum mechanics viewpoint. Dahler [8] developed the same constitutive equations from statistical mechanics considerations.

The theory of polar fluids has appeared in recent literature under a variety of different names. It is also called Cosserat fluids, asymmetric hydrodynamics, micropolar fluids, fluids with antisymmetric stress, oriented fluids, ... etc. A comprehensive account for the theory of polar fluids has been given by Cowin [6].

In Cowin's article, references to the solution of several boundary value problems in polar fluids have been listed. More recently, solutions to another set of flow problems in polar fluids have been discussed by Guram and Smith ([16]-[17]). Guram [15] also gave references to other works not mentioned in Cowin [6].

Cowin's terminology will be employed throughout the present work.

1.2 Summary of Linear Polar-Fluid Theory

(i) Kinematics

The two independent kinematical vector fields for the motion of polar fluids are: the usual velocity field $\underset{\sim}{v}(\underset{\sim}{x},t)$ and an axial vector field $\hat{\underset{\sim}{G}}(x,t)$ which represents the total angular velocity of the polar fluid particle. The usual or average angular velocity field $\hat{\underset{\sim}{W}}(\underset{\sim}{x},t)$ is given by

$$\hat{\underset{\sim}{W}} \equiv \frac{1}{2} \underset{\sim}{\nabla} \times \underset{\sim}{v} \quad . \tag{1.1}$$

The relative angular velocity $\hat{\underset{\sim}{H}}(\underset{\sim}{x},t)$, defined by

$$\hat{\underset{\sim}{H}} \equiv \hat{\underset{\sim}{G}} - \hat{\underset{\sim}{W}} \tag{1.2}$$

is the difference between the total angular velocity of a particle and the average angular velocity of the region in which the particle is embedded. The spatial gradient of the total angular velocity $\hat{\underset{\sim}{G}}$ will be denoted by

$$\hat{\Psi} \equiv \underset{\sim}{\nabla}\hat{\underset{\sim}{G}} \tag{1.3}$$

and the symmetric part of the velocity gradient is given by

$$\mathbf{D} \equiv \frac{1}{2}[\underset{\sim}{\nabla}\underset{\sim}{v} + (\underset{\sim}{\nabla}\underset{\sim}{v})^{*}] \tag{1.4}$$

(ii) Balance Equations

The usual form of the mass conservation applies for polar fluids which can be written as

$$\dot{\rho} + \rho \nabla \cdot \underset{\sim}{v} = 0 \qquad (1.5)$$

where ρ is the density and the superimposed dot indicates the material time derivative.

The well-known differential form of Cauchy's linear momentum principle is

$$\underset{\sim}{\nabla} \cdot \mathbf{T} + \rho \underset{\sim}{b} = \rho \dot{\underset{\sim}{v}} \qquad (1.6)$$

where $\mathbf{T}$ is the usual stress tensor and $\underset{\sim}{b}$ is the body force per unit mass. Equation (1.6) has the same form as in ordinary continuum except for the fact that the stress tensor $\mathbf{T}$ need not be symmetric.

The balance of angular momentum for polar fluids is

$$\underset{\sim}{\nabla} \cdot \hat{\mathbf{A}} + 2\hat{\underset{\sim}{T}} + \rho \hat{\underset{\sim}{c}} = \rho \, \mathbf{I} \cdot \dot{\hat{\underset{\sim}{G}}} \qquad (1.7)$$

where $\hat{\mathbf{A}}$ is the couple stress axial tensor, $\hat{\underset{\sim}{T}}$ is an axial vector associated with the skew-symmetric part of the usual stress tensor, $\hat{\underset{\sim}{c}}$ is a body couple and $\mathbf{I}$ is the inertia tensor of the polar fluid particles. If we assume that the particles are spherical, $\mathbf{I}$ can be written as

$$\mathbf{I} = k^2 \mathbf{1}$$

where k being the radius of gyration of the particles and $\mathbf{1}$ is the unit tensor. In this case equation (1.7) takes the form

$$\underset{\sim}{\nabla} \cdot \hat{\mathbf{A}} + 2\hat{\underset{\sim}{T}} + \rho\hat{\underset{\sim}{c}} = \rho k^2 \dot{\hat{\underset{\sim}{G}}} \ . \tag{1.8}$$

(iii) Constitutive Equations

The linear constitutive equations defining polar fluids are:

$$\mathbf{T} = -p\,\mathbf{1} + \lambda\,\mathbf{1}\,\mathrm{Tr}\,\mathbf{D} + 2\mu\mathbf{D} - 2\tau\,\mathbf{H} \tag{1.9}$$

$$\hat{\mathbf{A}} = \alpha\,\mathbf{1}\;\mathrm{Tr}\,\hat{\mathbf{\Psi}} + (\beta+\gamma)\,\hat{\mathbf{\Psi}} + (\beta-\gamma)\,\hat{\mathbf{\Psi}}^{*} \tag{1.10}$$

Here, p is the thermodynamic pressure, λ and μ are the usual viscosity coefficients and τ is a relative rotational viscosity called, simply, the rotational viscosity. The coefficients α, β and γ are viscosities associated with the total rotational gradient and are called, for simplicity, gradiant viscosities. The constitutive equation (1.9) reduces to the Newtonian law of viscosity when $2\tau\,\mathbf{H}$ vanishes. The viscosity coefficients λ, μ, τ, α, β and γ are restricted by the inequalities

$$\mu \geq 0 \ , \quad 3\lambda + 2\mu \geq 0 \ , \quad \tau \geq 0 \ ,$$

$$3\alpha + 2\beta \geq 0 \ , \quad \beta \geq 0 \ , \quad \gamma \geq 0 \ .$$

(iv) Equations of motion

When the material coefficients are assumed to be constants and the constitutive equations (1.9) and (1.10) are substituted into the expressions for balance of linear momentum and angular momentum (1.6) and (1.8), one obtains

$$(\lambda+\mu)\underset{\sim}{\nabla}\underset{\sim}{\nabla} \cdot \underset{\sim}{v} + \mu\nabla^2\underset{\sim}{v} + 2\tau\underset{\sim}{\nabla} \times \underset{\sim}{\hat{H}} + \rho\underset{\sim}{b} - \underset{\sim}{\nabla}p = \rho\dot{\underset{\sim}{v}} \qquad (1.11)$$

$$(\alpha+\beta-\gamma)\underset{\sim}{\nabla}\underset{\sim}{\nabla} \cdot \underset{\sim}{\hat{G}} + (\beta+\gamma)\nabla^2\underset{\sim}{\hat{G}} - 4\tau\underset{\sim}{\hat{H}} + \rho\underset{\sim}{\hat{c}} = \rho k^2\dot{\underset{\sim}{\hat{G}}} \qquad (1.12)$$

It is noticed that equation (1.11) reduces to Navier-Stokes equations when $2\tau\underset{\sim}{\nabla} \times \underset{\sim}{\hat{H}}$ vanishes. Equation (1.12) has no counterpart in ordinary continuum mechanics. If the polar fluid is assumed to be incompressible then (1.5) reduces to

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0 \qquad (1.13)$$

and (1.11) and (1.12) take the form

$$(\mu+\tau)\nabla^2\underset{\sim}{v} + 2\tau\ \underset{\sim}{\nabla} \times \underset{\sim}{\hat{G}} + \rho\underset{\sim}{b} - \underset{\sim}{\nabla}p = \rho\dot{\underset{\sim}{v}} \qquad (1.14)$$

$$(\alpha+2\beta)\underset{\sim}{\nabla}\underset{\sim}{\nabla} \cdot \underset{\sim}{\hat{G}} - (\beta+\gamma)\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla}\times\underset{\sim}{\hat{G}}) - 4\tau\underset{\sim}{\hat{G}} + 2\tau\underset{\sim}{\nabla} \times \underset{\sim}{v} + \rho\underset{\sim}{\hat{c}} = \rho k^2\dot{\underset{\sim}{\hat{G}}} \qquad (1.15)$$

The system of equations (1.13) - (1.15) is a system of seven differential equations in the seven scalar unknowns $\underset{\sim}{v}$, $\underset{\sim}{\hat{G}}$ and p .

(v) Dimensionless Parameters

In this subsection we introduce two dimensionless numbers. The first one is called the length ratio L and it is a ratio of a geometric characteristic length L_0 to the material characteristic length ℓ which is defined by

$$\ell = \left(\frac{\beta+\gamma}{4\mu}\right)^{\frac{1}{2}} . \tag{1.16}$$

We notice that ℓ is a real positive number which follows from the restrictions on the material coefficients. The second number to be introduced is called the coupling number N and is given by

$$N = \left(\frac{\tau}{\mu+\tau}\right)^{\frac{1}{2}} , \quad 0 \leq N \leq 1 . \tag{1.17}$$

This number N characterizes the coupling of the conservation of linear and angular momentum as may be seen from (1.11) and (1.12). The coupling number N is a property of the fluid and is a measure of the degree to which the particle is constrained to rotate with the average angular velocity in which it is embedded. The case $N \to 0$ corresponds to unconstrained rotations. For $N = 1$, the total angular velocity $\hat{G}$ coincides with the regional angular velocity $\hat{W}$ and the theory of polar fluids reduces to Stokes [28] couple stress theory. A complete discussion for the behaviour of polar fluids for the limiting cases of N and L is given by Cowin ([5], [7]).

(vi) Boundary Conditions

In classical fluid mechanics, one employs the no-slip condition for velocity on the solid boundary, namely,

$$\underset{\sim}{v} = \underset{\sim}{v}_0 \quad \text{at all solid boundaries} \tag{1.18}$$

where $\underset{\sim}{v}_0$ represents the velocity of the boundary. In polar fluids theory, the same boundary condition is used. For spin, stress and couple stress there have been several boundary conditions suggested in the literature. The no-spin condition on the boundary is the most frequently used one for the total angular velocity that is

$$\hat{\underset{\sim}{G}} = \hat{\underset{\sim}{G}}_0 \quad \text{at all solid boundaries} \tag{1.19}$$

where $\hat{\underset{\sim}{G}}_0$ is the angular velocity of the boundary.

Aero et al. [1] suggested that the couple stress on the solid boundary might be related to the difference between the angular velocity of the fluid at the boundary and the angular velocity of the boundary by a friction factor, that is,

$$\hat{\mathbf{\Lambda}}\, n = \mathbf{A}\cdot(\hat{\underset{\sim}{G}} - \hat{\underset{\sim}{G}}_0) \quad \text{at all solid boundaries} \tag{1.20}$$

where $\underset{\sim}{n}$ is the unit normal to the boundary and $\mathbf{A}$ is a second-rank tensor representing the fluid-wall friction. No couple stress condition on the boundary is fulfilled if $\mathbf{A}$ vanishes. If $\mathbf{A}^{-1}$ equal to zero then the condition (1.20) reduces to the no-spin condition (1.19).

Condiff and Dahler [3] discussed the no-spin condition and the vanishing of the skew-symmetric part of the stress tensor on the solid boundary. They noticed that these two conditions represent opposite extremes. These authors suggested the formulation of a compromise boundary condition as a homogeneous linear combination of the two extremes by introducing a new parameter s . This condition can be written as

$$\hat{\underset{\sim}{G}} - \hat{\underset{\sim}{G}}_0 = s\hat{\underset{\sim}{W}} , \quad 0 \leq s \leq 1 \quad \text{on the solid boundary.} \tag{1.21}$$

1.3 Scope of The Present Work

The main objective of this work is to study the solution of certain boundary value problems in the theory of polar fluids.

Chapter II deals with the study of the motion of a polar fluid through a curved pipe. A perturbation method is used to solve the system of differential equations which governs the motion. The stream-lines are sketched and compared with those for a Newtonian fluid. The rate of flow through the pipe is calculated up to the second approximation and the effect of curvature of the pipe is discussed.

The solutions for creeping two-dimensional internal and external flows of polar fluids for a circular cylinder are presented in chapter III. In the first part of the chapter the motion of a polar fluid within a circular cylinder generated by

(a) fluid entering and leaving through slots in the cylinder wall and

(b) the rotation of part of the wall

is considered. In both cases analytical solution are obtained and stream-lines are sketched.

The last part of the chapter is devoted to an investigation of a creeping polar fluid flow problem past a circular cylinder.

In chapter IV, the solution of the Hamel flow problem is studied in the case of polar fluids. The possibility of exhibiting a secondary flow for a polar fluid between two non-parallel plates is investigated.

CHAPTER II

POLAR FLUID FLOW THROUGH A CURVED PIPE

2.1 Introduction

During the past fifty years there has been an increasing interest in the study of fluid flow behaviour through curved pipes. There are many physical and industrial situations in which such work is of interest as in the flow of blood in human arteries. Dean ([9], [10]) first considered the problem of steady viscous fluid in a tube, which is coiled in an arc of a circle and which has a constant cross-section, under a constant pressure gradient. Dean [9] obtained the solution to the first approximation but failed to show that the relation between the pressure gradient and the rate of flow through the pipe is dependent on the curvature. In the subsequent paper, Dean [10] used a perturbation series to get higher order approximations and showed that the rate of flow through the pipe is reduced due to the curvature. Topakoglu [31] employed a somewhat different approach to the solution of such problems and also obtained results for the flow between two concentric torus shaped pipes. Topakoglu [31] observed that in order to show the effect of curvature on the volume flow rate it was necessary to consider solutions of higher order approximations than the first. The literature on viscous fluid flow in curved pipes is very extensive and a detailed account is given by Smith [27].

For non-Newtonian fluids in a curved pipe, Jones [19] obtained the solution to the first approximation only. He pointed out that the second approximation is necessary to show the effect of curvature on the volume flow rate but it is too difficult to calculate. Thomas and Walters [29] studied the problem for the case of visco-elastic liquids and showed the effect of elasticity on the fluid behaviour in the pipe and the flux through it.

In this chapter an investigation of a steady flow of a polar fluid through a curved pipe under a constant pressure gradient is carried out.

2.2 Formulation Of The Problem

Figure 1 shows the coordinate system used for the discussion of the motion of a polar fluid through a pipe of circular cross-section, coiled in the form of a circle. The axis of the circle in which the pipe is coiled is OZ and C is the centre of the section of the pipe cut by a plane that makes an angle θ with the fixed axial plane. OC is of length R which is the radius of curvature of the coiled tube. The plane passing through O and perpendicular to OZ will be called the "central plane" of the pipe and the circle traced out by C, its "central line". Any point M of the section θ = constant is referred to by the orthogonal curvilinear coordinates (r, ψ, θ) where r is the distance CM and ψ is the angle that CM makes with the line through

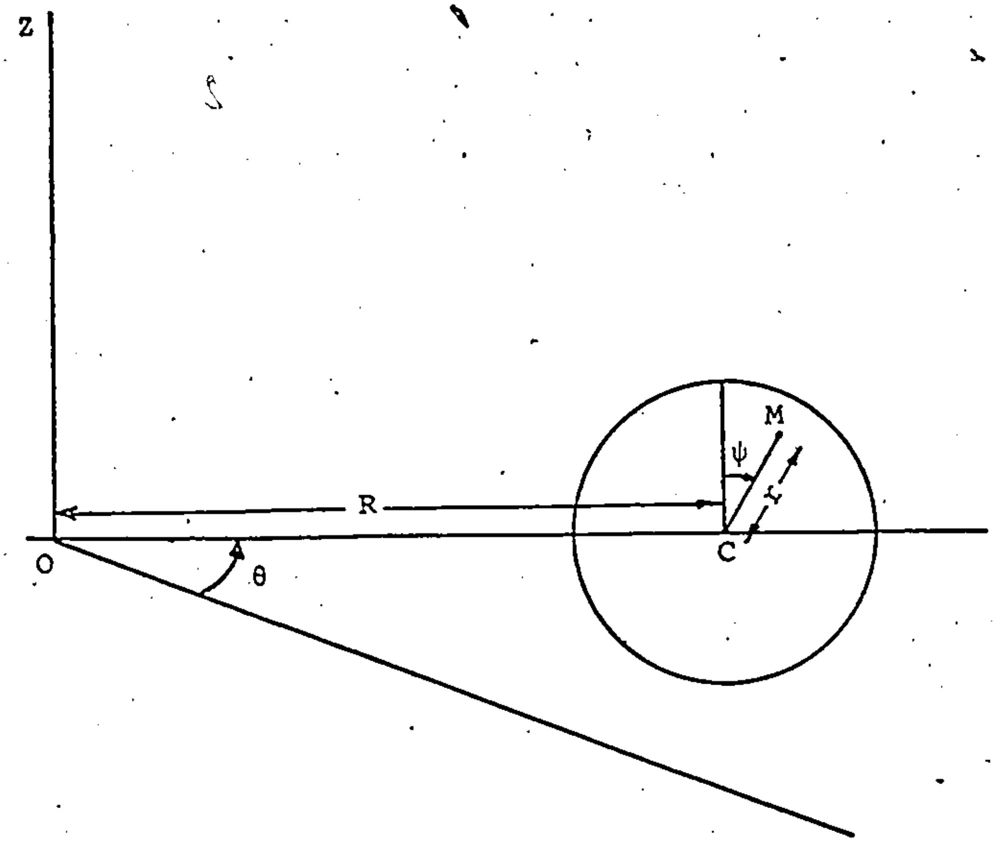

Figure 1. The coordinate system (r, ψ, θ) chosen to describe the motion in a curved pipe of circular cross-section.

C parallel to OZ; the line element is given by

$$ds^2 = dr^2 + r^2 d\psi^2 + (R + r\sin\psi)^2 d\theta^2 \tag{2.1}$$

The motion is expected to be a steady one in which the velocity $\underset{\sim}{V}$ and the total angular velocity $\underset{\sim}{\hat{G}}$ (but not the pressure p) are independent of θ. It is assumed that the motion of the fluid is due to a fall of pressure along the pipe and that the general direction of flow is the direction in which θ increases.

The components of $\underset{\sim}{V}$, $\underset{\sim}{\hat{G}}$ respectively are U, $\hat{G}_1$ in the direction of CM, V, $\hat{G}_2$ perpendicular to U, $\hat{G}_1$ and in the plane of the cross-section and W, $\hat{G}_3$ perpendicular to this plane.

Using equations (1.13) - (1.15) the motion of such a fluid is described by

$$\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{U\sin\psi}{R + r\sin\psi} + \frac{1}{r}\frac{\partial V}{\partial\psi} + \frac{V\cos\psi}{R + r\sin\psi} = 0 \tag{2.2}$$

$$\rho\left[U\frac{\partial U}{\partial r} + \frac{V}{r}\frac{\partial U}{\partial\psi} - \frac{V^2}{r} - \frac{W^2\sin\psi}{R + r\sin\psi}\right] =$$

$$-\frac{\partial p}{\partial r} + \rho b_1 - (\mu+\tau)\left(\frac{1}{r}\frac{\partial}{\partial\psi} + \frac{\cos\psi}{R + r\sin\psi}\right)\left(\frac{\partial V}{\partial r} + \frac{V}{r} - \frac{1}{r}\frac{\partial U}{\partial\psi}\right)$$

$$+ 2\tau\left(\frac{1}{r}\frac{\partial\hat{G}_3}{\partial\psi} + \frac{\hat{G}_3\cos\psi}{R + r\sin\psi}\right) \tag{2.3}$$

$$\rho\left[U\frac{\partial V}{\partial r} + \frac{V}{r}\frac{\partial V}{\partial \psi} + \frac{UV}{r} - \frac{W^2 \cos\psi}{R + r\sin\psi}\right] =$$

$$-\frac{1}{r}\frac{\partial p}{\partial \psi} + \rho b_2 + (\mu+\tau)\left(\frac{\partial}{\partial r} + \frac{\sin\psi}{R + r\sin\psi}\right)\left(\frac{\partial V}{\partial r} + \frac{V}{r} - \frac{1}{r}\frac{\partial U}{\partial \psi}\right)$$

$$-2\tau\left(\frac{\partial \hat{G}_3}{\partial r} + \frac{\hat{G}_3 \sin\psi}{R + r\sin\psi}\right) \qquad (2.4)$$

$$\rho\left[U\frac{\partial W}{\partial r} + \frac{V}{r}\frac{\partial W}{\partial \psi} + \frac{UW\sin\psi}{R + r\sin\psi} + \frac{VW\cos\psi}{R + r\sin\psi}\right] =$$

$$-\frac{1}{R + r\sin\psi}\frac{\partial p}{\partial \theta} + \rho b_3 + (\mu+\tau)\left[\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\left(\frac{\partial W}{\partial r} + \frac{W\sin\psi}{R + r\sin\psi}\right)\right.$$

$$\left. + \frac{1}{r}\frac{\partial}{\partial \psi}\left(\frac{1}{r}\frac{\partial W}{\partial \psi} + \frac{W\cos\psi}{R + r\sin\psi}\right)\right] + 2\tau\left(\frac{\partial \hat{G}_2}{\partial r} + \frac{\hat{G}_2}{r} - \frac{1}{r}\frac{\partial \hat{G}_1}{\partial \psi}\right) \qquad (2.5)$$

$$(\alpha+2\beta)\frac{\partial}{\partial r}\left\{\frac{\partial \hat{G}_1}{\partial r} + \frac{\hat{G}_1}{r} + \frac{\hat{G}_1\sin\psi}{R + r\sin\psi} + \frac{1}{r}\frac{\partial \hat{G}_2}{\partial \psi} + \frac{\hat{G}_2\cos\psi}{R + r\sin\psi}\right\}$$

$$-(\beta+\gamma)\left(\frac{1}{r}\frac{\partial}{\partial \psi} + \frac{\cos\psi}{R + r\sin\psi}\right)\left(\frac{\partial \hat{G}_2}{\partial r} + \frac{\hat{G}_2}{r} - \frac{1}{r}\frac{\partial \hat{G}_1}{\partial \psi}\right)$$

$$-4\tau\hat{G}_1 + 2\tau\left(\frac{1}{r}\frac{\partial W}{\partial \psi} + \frac{W\cos\psi}{R + r\sin\psi}\right) + \rho c_1 = \rho k^2\dot{\hat{G}}_1 \qquad (2.6)$$

$$(\alpha+2\beta)\frac{1}{r}\frac{\partial}{\partial \psi}\left\{\frac{\partial \hat{G}_1}{\partial r} + \frac{\hat{G}_1}{r} + \frac{\hat{G}_1\sin\psi}{R + r\sin\psi} + \frac{1}{r}\frac{\partial \hat{G}_2}{\partial \psi} + \frac{\hat{G}_2\cos\psi}{R + r\sin\psi}\right\}$$

$$+(\beta+\gamma)\left(\frac{\partial}{\partial r} + \frac{\sin\psi}{R + r\sin\psi}\right)\left(\frac{\partial \hat{G}_2}{\partial r} + \frac{\hat{G}_2}{r} - \frac{1}{r}\frac{\partial \hat{G}_1}{\partial \psi}\right) - 4\tau\hat{G}_2$$

$$-2\tau\left(\frac{\partial W}{\partial r} + \frac{W\sin\psi}{R + r\sin\psi}\right) + \rho c_2 = \rho k^2\dot{\hat{G}}_2 \qquad (2.7)$$

$$(\beta+\gamma)\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\left(\frac{\partial \hat{G}_3}{\partial r}+\frac{\hat{G}_3 \sin\psi}{R+r\sin\psi}\right)+\frac{1}{r}\frac{\partial}{\partial\psi}\left(\frac{1}{r}\frac{\partial \hat{G}_3}{\partial\psi}+\frac{\hat{G}_3\cos\psi}{R+r\sin\psi}\right)$$

$$-4\tau\hat{G}_3+2\tau\left(\frac{\partial V}{\partial r}+\frac{V}{r}-\frac{1}{r}\frac{\partial U}{\partial\psi}\right)+\rho c_3=\rho k^2\dot{\hat{G}}_3 \tag{2.8}$$

The system of equations (2.2) - (2.8) is somewhat complicated and it is difficult to get its general solution. Following Dean [10] one can assume that the curvature of the pipe is small: that is a/R is small where a is the radius of the pipe. As a result it is possible to replace

$$\frac{1}{R+r\sin\psi},\quad \frac{\partial}{\partial r}\pm\frac{\sin\psi}{R+r\sin\psi}\quad\text{and}\quad \frac{1}{r}\frac{\partial}{\partial\psi}\pm\frac{\cos\psi}{R+r\sin\psi}$$

by

$$\frac{1}{R},\quad \frac{\partial}{\partial r}\quad\text{and}\quad \frac{1}{r}\frac{\partial}{\partial\psi}$$

respectively.

Moreover, if one assumes that the body force $\underset{\sim}{b}$, the body couple $\underset{\sim}{\hat{C}}$ and the micro-inertia of the particles all vanish, the system (2.2) - (2.8) reduce to

$$\frac{\partial U}{\partial r}+\frac{U}{r}+\frac{1}{r}\frac{\partial V}{\partial\psi}=0 \tag{2.9}$$

$$U\frac{\partial U}{\partial r}+\frac{V}{r}\frac{\partial U}{\partial\psi}-\frac{V^2}{r}-\frac{W^2\sin\psi}{R}=-\frac{\partial}{\partial r}\left(\frac{p}{\rho}\right)$$

$$-\left(\frac{\mu+\tau}{\rho}\right)\frac{1}{r}\frac{\partial}{\partial\psi}\left(\frac{\partial V}{\partial r}+\frac{V}{r}-\frac{1}{r}\frac{\partial U}{\partial\psi}\right)+\frac{2\tau}{\rho}\frac{1}{r}\frac{\partial \hat{G}_3}{\partial\psi} \tag{2.10}$$

$$U\frac{\partial V}{\partial r} + \frac{V}{r}\frac{\partial V}{\partial \psi} + \frac{UV}{r} - \frac{W^2 \cos\psi}{R} = -\frac{1}{r}\frac{\partial}{\partial \psi}\left(\frac{p}{\rho}\right)$$

$$+\left(\frac{\mu+\tau}{\rho}\right)\frac{\partial}{\partial r}\left(\frac{\partial V}{\partial r} + \frac{V}{r} - \frac{1}{r}\frac{\partial U}{\partial \psi}\right) - \frac{2\tau}{\rho}\frac{\partial \hat{G}_3}{\partial r} \tag{2.11}$$

$$U\frac{\partial W}{\partial r} + \frac{V}{r}\frac{\partial W}{\partial \psi} = -\frac{1}{R}\frac{\partial}{\partial \theta}\left(\frac{p}{\rho}\right) + \left(\frac{\mu+\tau}{\rho}\right)\left(\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \psi^2}\right)$$

$$+\frac{2\tau}{\rho}\left(\frac{\partial \hat{G}_2}{\partial r} + \frac{\hat{G}_2}{r} - \frac{1}{r}\frac{\partial \hat{G}_1}{\partial \psi}\right) \tag{2.12}$$

$$(\alpha+2\beta)\frac{\partial}{\partial r}\left(\frac{\partial \hat{G}_1}{\partial r} + \frac{\hat{G}_1}{r} + \frac{1}{r}\frac{\partial \hat{G}_2}{\partial \psi}\right) - (\beta+\gamma)\frac{1}{r}\frac{\partial}{\partial \psi}\left(\frac{\partial \hat{G}_2}{\partial r} + \frac{\hat{G}_2}{r} - \frac{1}{r}\frac{\partial \hat{G}_1}{\partial \psi}\right)$$

$$+ 2\tau\frac{1}{r}\frac{\partial W}{\partial \psi} - 4\tau\hat{G}_1 = 0 \tag{2.13}$$

$$(\alpha+2\beta)\frac{1}{r}\frac{\partial}{\partial \psi}\left(\frac{\partial \hat{G}_1}{\partial r} + \frac{\hat{G}_1}{r} + \frac{1}{r}\frac{\partial \hat{G}_2}{\partial \psi}\right) + (\beta+\gamma)\frac{\partial}{\partial r}\left(\frac{\partial \hat{G}_2}{\partial r} + \frac{\hat{G}_2}{r} - \frac{1}{r}\frac{\partial \hat{G}_1}{\partial \psi}\right)$$

$$- 2\tau\frac{\partial W}{\partial r} - 4\tau\hat{G}_2 = 0 \tag{2.14}$$

$$(\beta+\gamma)\left(\frac{\partial^2 \hat{G}_3}{\partial r^2} + \frac{1}{r}\frac{\partial \hat{G}_3}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \hat{G}_3}{\partial \psi^2}\right) + 2\tau\left(\frac{\partial V}{\partial r} + \frac{V}{r} - \frac{1}{r}\frac{\partial U}{\partial \psi}\right)$$

$$- 4\tau\hat{G}_3 = 0 \tag{2.15}$$

As U, V, W, $\hat{G}_1$, $\hat{G}_2$ and $\hat{G}_3$ are independent of θ it follows from (2.12) that $\frac{p}{\rho}$ must be of the form

$\theta f_1(r,\psi) + f_2(r,\psi)$, and then from (2.10) and (2.11) that $f_1(r,\psi)$ must be a constant. We can, therefore, write

$$-\frac{1}{R}\frac{\partial}{\partial\theta}\left(\frac{p}{\rho}\right) = \frac{A}{\rho} \tag{2.16}$$

where A is a constant which may be termed the mean pressure gradient; it is equal to the space-rate of decrease in the pressure along the central line.

From equations (2.13) and (2.14) one gets

$$(\nabla^2-\lambda_1^2)\left(\frac{\partial\hat{G}_1}{\partial r} + \frac{\hat{G}_1}{r} + \frac{1}{r}\frac{\partial\hat{G}_2}{\partial\psi}\right) = 0 \tag{2.17}$$

and

$$(\nabla^2-\lambda_2^2)\left(\frac{\partial\hat{G}_2}{\partial r} + \frac{\hat{G}_2}{r} - \frac{1}{r}\frac{\partial\hat{G}_1}{\partial\psi}\right) - \frac{\lambda_2^2}{2}\nabla^2 W = 0 \tag{2.18}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\psi^2}, \quad \lambda_1^2 = \frac{4\tau}{\alpha + 2\beta}$$

and

$$\lambda_2^2 = \frac{4\tau}{\beta + \gamma}.$$

The continuity equation (2.9) suggests that one can write

$$rU = -\frac{\partial f}{\partial\psi}, \quad V = \frac{\partial f}{\partial r} \tag{2.19}$$

where f, the stream function of the secondary flow, is a function of r and ψ only. Inserting these expressions of U and V in (2.10) and (2.11) and eliminating p yields

$$\left(\frac{\partial f}{\partial \psi}\frac{\partial}{\partial r} - \frac{\partial f}{\partial r}\frac{\partial}{\partial \psi}\right)\nabla^2 f + \frac{2W}{R}\left(r\cos\psi\frac{\partial W}{\partial r} - \sin\psi\frac{\partial W}{\partial \psi}\right) =$$

$$- \left(\frac{\mu+\tau}{\rho}\right) r\nabla^4 f + \frac{2\tau}{\rho} r\nabla^2 \hat{G}_3 \ . \qquad (2.20)$$

Also, using (2.19) in (2.12) and (2.15), one gets

$$\frac{1}{r}\left(\frac{\partial f}{\partial r}\frac{\partial W}{\partial \psi} - \frac{\partial f}{\partial \psi}\frac{\partial W}{\partial r}\right) = \frac{A}{\rho} + \left(\frac{\mu+\tau}{\rho}\right)\nabla^2 W + \frac{2\tau}{\rho}\left(\frac{\partial \hat{G}_2}{\partial r} + \frac{\hat{G}_2}{r} - \frac{1}{r}\frac{\partial \hat{G}_1}{\partial \psi}\right) \qquad (2.21)$$

and

$$(\nabla^2 - \lambda_2^2)\hat{G}_3 + \frac{\lambda_2^2}{2}\nabla^2 f = 0 \qquad (2.22)$$

respectively.

Equations (2.17), (2.18), (2.20), (2.21) and (2.22) can be put in non-dimensional form by using the following substitutions:

$$f = \frac{\mu+\tau}{\rho}\phi \ , \quad W = W_0 w \ , \quad \hat{G}_1 = G g_1 \ ,$$
$$\hat{G}_2 = G g_2 \ , \quad \hat{G}_3 = G g_3 \ , \quad r = a r' \qquad (2.23)$$

where W_0 is assumed to have the dimensions of velocity and G has the dimensions of angular velocity. Thus, on using (2.23) in (2.17), (2.18) and (2.20) - (2.22) one obtains:

$$(\nabla_1^2 - k_1^2)\left(\frac{\partial g_1}{\partial r'} + \frac{g_1}{r'} + \frac{1}{r'}\frac{\partial g_2}{\partial \psi}\right) = 0 \tag{2.24}$$

$$(\nabla_1^2 - k_2^2)\left(\frac{\partial g_2}{\partial r'} + \frac{g_2}{r'} - \frac{1}{r'}\frac{\partial g_1}{\partial \psi}\right) - n_1\nabla_1^2 w = 0 \tag{2.25}$$

$$\left(\frac{\partial \phi}{\partial \psi}\frac{\partial}{\partial r'} - \frac{\partial \phi}{\partial r'}\frac{\partial}{\partial \psi}\right)\nabla_1^2\phi + \overline{K}w\left(r' \cos \psi \frac{\partial w}{\partial r'} - \sin \psi \frac{\partial w}{\partial \psi}\right) =$$

$$- r'\nabla_1^4\phi + n_3 r'\nabla_1^2 g_3 \tag{2.26}$$

$$\frac{1}{r'}\left(\frac{\partial \phi}{\partial r'}\frac{\partial w}{\partial \psi} - \frac{\partial \phi}{\partial \psi}\frac{\partial w}{\partial r'}\right) = \overline{C} + \nabla_1^2 w + n_4\left(\frac{\partial g_2}{\partial r'} + \frac{g_2}{r'} - \frac{1}{r'}\frac{\partial g_1}{\partial \psi}\right) \tag{2.27}$$

$$(\nabla_1^2 - k_2^2)g_3 + n_2\nabla_1^2\phi = 0 \ . \tag{2.28}$$

In the above equations:

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial r'^2} + \frac{1}{r'}\frac{\partial}{\partial r'} + \frac{1}{r'^2}\frac{\partial^2}{\partial \psi'^2}$$

$$k_1^2 = \frac{4\tau a^2}{\alpha + 2\beta} \ , \qquad k_2^2 = \frac{4\tau a^2}{\beta + \gamma}$$

$$n_1 = \frac{k_2^2 W_0}{2aG} \ , \qquad n_2 = \frac{2\tau(\mu+\tau)}{\rho(\beta+\gamma)G} \tag{2.29}$$

$$n_3 = \frac{2\tau G a^2 \rho}{(\mu+\tau)^2} \ , \qquad n_4 = \frac{2\tau G a}{(\mu+\tau)W_0}$$

and

$$\overline{K} = \frac{2W_0^2 a^3}{R}\left(\frac{\rho}{\mu+\tau}\right)^2 , \quad \overline{C} = \frac{Aa^2}{W_0(\mu+\tau)} . \tag{2.30}$$

The constants $\overline{K}$ and $\overline{C}$ are related respectively to K and C, as defined by Dean [10], by the relations of the form:

$$\overline{K} = (1-N^2)^2 K \quad \text{and} \quad \overline{C} = C(1-N^2) \tag{2.31}$$

where N is the coupling number defined by (1.17).

Following Dean [10], if the motion is slow, W_0 can be taken to be the θ-component of the velocity of the fluid at any point of the central line; and in this case, the distribution of θ-component of velocity approximates to that occurring in the straight-pipe problem. Consequently, $aW_0\rho/\mu$ is then approximately equal to the Reynolds number n, defined as $2\overline{V}a\rho/\mu$ where $\overline{V}$ is the mean velocity over the cross-section. Thus for slow motion it follows that

$$\overline{K} = 2n^2(1-N^2)^2 \frac{a}{R} \tag{2.32}$$

2.3 Solution

In order to solve the system (2.24) - (2.28), one can use the method of successive approximation in which it is assumed that g_1, g_2, g_3, w and ϕ can be expanded in ascending powers of $\overline{K}$ (cf. Dean [10]).

When the pipe is straight, a/R and $\overline{K}$ are zero, and also $g_1 = g_3 = \phi = 0$. Equations (2.25) and (2.27) then reduce to

$$\left[\left(\frac{d^2}{dr'^2} + \frac{1}{r'}\frac{d}{dr'}\right) - k_2^2\right]\left(\frac{dg_2}{dr'} + \frac{g_2}{r'}\right) - n_1\left(\frac{d^2w}{dr'^2} + \frac{1}{r'}\frac{dw}{dr'}\right) = 0 \qquad (2.33)$$

and

$$\overline{C} + \left(\frac{d^2w}{dr'^2} + \frac{1}{r'}\frac{dw}{dr'}\right) + n_4\left(\frac{dg_2}{dr'} + \frac{g_2}{r'}\right) = 0 \qquad (2.34)$$

respectively. Equations (2.33) and (2.34), subject to:

$$w(1) = g_2(1) = 0\ ,\quad w(0) \text{ and } g_2(0) \text{ are finite} \qquad (2.35)$$

are satisfied by

$$w = \frac{Aa^2}{4\mu W_0}\left[1 - r'^2 - \frac{2N}{L}\left(\frac{I_0(NL) - I_0(NLr')}{I_1(NL)}\right)\right] \qquad (2.36)$$

$$g_2 = \frac{Aa}{4\mu G}\left[r' - \frac{I_1(NLr')}{I_1(NL)}\right] \qquad (2.37)$$

where I_0 and I_1 are the modified Bessel functions of the first kind and order 0 and 1 respectively, L is the length ratio and N is the coupling number.

When the pipe is curved, and a/R and $\overline{K}$ are sufficiently small, following Dean [10] it is assumed that

$$\left.\begin{aligned}
g_1 &= \overline{K}g_1^{(1)} + \overline{K}^2 g_1^{(2)} + \ldots\ldots \\
g_2 &= g_2^{(0)} + \overline{K}g_2^{(1)} + \overline{K}^2 g_2^{(2)} + \ldots\ldots \\
g_3 &= \overline{K}g_3^{(1)} + \overline{K}^2 g_3^{(2)} + \ldots\ldots \\
w &= w_0 + \overline{K}w_1 + \overline{K}^2 w_2 + \ldots\ldots \\
\phi &= \overline{K}\phi_1 + \overline{K}^2 \phi_2 + \ldots\ldots
\end{aligned}\right\} \tag{2.38}$$

where g's, w's and ϕ's are functions of r' and ψ only.

On substituting (2.38) in (2.24) - (2.28) and equating the coefficients of similar powers of $\overline{K}$, g's, w's and ϕ's can successively be found. The coefficients of zero order in $\overline{K}$ are coincident with (2.33) and (2.34) and, therefore, the solution for w_0 and $g_2^{(0)}$ is given by (2.36) and (2.37) respectively.

Equating the coefficients of $\overline{K}$ one obtains:

$$(\nabla_1^2 - k_1^2)\left(\frac{\partial g_1^{(1)}}{\partial r'} + \frac{g_1^{(1)}}{r'} + \frac{1}{r'}\frac{\partial g_2^{(1)}}{\partial \psi}\right) = 0 \quad , \tag{2.39}$$

$$(\nabla_1^2 - k_2^2)\left(\frac{\partial g_2^{(1)}}{\partial r'} + \frac{g_2^{(1)}}{r'} - \frac{1}{r'}\frac{\partial g_1^{(1)}}{\partial \psi}\right) - \eta_1 \nabla_1^2 w_1 = 0 \quad , \tag{2.40}$$

$$-r'\nabla_1^4\phi_1 + \eta_3 r'\nabla_1^2 g_3^{(1)} = r' w_0 \frac{\partial w_0}{\partial r'} \cos\psi \quad , \tag{2.41}$$

$$\nabla_1^2 w_1 + n_4\left(\frac{\partial g_2^{(1)}}{\partial r'} + \frac{g_2^{(1)}}{r'} - \frac{1}{r'}\frac{\partial g_1^{(1)}}{\partial \psi}\right) = -\frac{1}{r'}\frac{\partial \phi_1}{\partial \psi}\frac{\partial w_0}{\partial r'}\,, \tag{2.42}$$

$$(\nabla_1^2 - k_2^2)g_3^{(1)} + n_2\nabla_1^2\phi_1 = 0\,. \tag{2.43}$$

Elimination of ϕ_1 between (2.41) and (2.43) gives

$$\nabla_1^2(\nabla_1^2 - N^2L^2)g_3^{(1)} = n_2 w_0 \frac{\partial w_0}{\partial r'} \cos\psi \tag{2.44}$$

which suggests that $g_3^{(1)}$ can be written as

$$g_3^{(1)} = \bar{g}_3^{(1)}(r')\cos\psi\,. \tag{2.45}$$

Using (2.45) in (2.44) we get the differential equation

$$\left(\frac{d^2}{dr'^2} + \frac{1}{r'}\frac{d}{dr'} - \frac{1}{r'^2}\right)\left\{\frac{d^2}{dr'^2} + \frac{1}{r'}\frac{d}{dr'} - \left(\frac{1}{r'^2} + N^2L^2\right)\right\}\bar{g}_3^{(1)} = n_2 w_0 \frac{dw_0}{dr'} \tag{2.46}$$

At this point it is convenient to write w_0 as

$$w_0 = A_0\,[A_1 + A_2 r'^2 + A_3 I_0(NLr')] \tag{2.47}$$

where

$$A_0 = \frac{Aa^2}{4\mu W_0}\,,\qquad A_1 = 1 - \frac{2N}{L}\frac{I_0(NL)}{I_1(NL)}\,,$$

$$A_2 = -1\,,\qquad A_3 = \frac{2N}{L}\frac{1}{I_1(NL)}\,. \tag{2.48}$$

In order to evaluate the particular integral of (2.46) it is convenient to approximate the right hand side of this equation. It is known that the modified Bessel function $I_n(z)$ can be expressed as an infinite series:

$$I_n(z) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}z)^{n+2m}}{\Gamma(m+1)\Gamma(n+m+1)} \quad . \tag{2.49}$$

This is a uniformly convergent series for all finite z and n. Hence, if a finite number of terms from the infinite series is used, $w_0 \frac{dw_0}{dr'}$ can be approximated by

$$w_0 \frac{dw_0}{dr'} = A_0^2 \sum_{m=0}^{M} \gamma_m r'^{\,2m+1} \tag{2.50}$$

where M is finite but arbitrary such that the remainder of the series tends to zero. The constants γ's are given by

$$\gamma_0 = 2A_1A_2 + E_0 \quad , \quad \gamma_1 = 2A_2^2 + E_1 + A_2A_3NLC_0$$

$$\gamma_m = E_m + A_2A_3NLC_{m-1} \quad , \quad m \geq 2 \tag{2.51}$$

where

$$E_m = 2A_2A_3B_m + A_1A_3NLC_m + A_3^2NLD_m$$

and

$$\left.\begin{aligned} B_m &= \frac{\left(\frac{NL}{2}\right)^{2m}}{(m!)^2}\ , \qquad C_m = \frac{\left(\frac{NL}{2}\right)^{2m+1}}{m!(m+1)!} \\ D_m &= \frac{\left(\frac{NL}{2}\right)^{2m+1}(2m+1)!}{\{m!(m+1)!\}^2}\ . \end{aligned}\right\} \tag{2.52}$$

Using (2.50) in (2.46) and employing the properties of modified Bessel function (cf. Magnus et al. [21]), the general solution of (2.46) is given by

$$\bar{g}_3^{(1)} = F_3 I_1(NLr') + F_4 K_1(NLr') - \frac{1}{N^2L^2}\left(F_1 r' + \frac{F_2}{r'}\right) - \frac{A_0^2 n_2}{N^2L^2}\sum_{m=0}^{M+1} \kappa_m r'^{\,2m+1}\ . \tag{2.53}$$

where

$$\kappa_m = \sum_{j=0}^{M} \frac{\gamma_j}{4(j+1)(j+2)}\,\sigma_{j-m+1,j}\ ,$$

$$\sigma_{k,m} = \begin{cases} \left(\frac{2}{NL}\right)^{2k} \dfrac{\Gamma(m+2)\Gamma(m+3)}{\Gamma(m+2-k)\Gamma(m+3-k)} & k \geq 0 \\ 0 & k < 0 \end{cases} \tag{2.54}$$

If one approximates $I_1(NLr')$ by a polynomial of degree $2M+1$, and employs the boundary conditions that $\bar{g}_3^{(1)}(0)$ is finite and $\bar{g}_3^{(1)}(1) = 0$, then

$$\overline{g}_3^{(1)} = F_3 \sum_{m=0}^{M} C_m r'^{\,2m+1}$$

$$- \frac{1}{N^2L^2} \left\{ F_1 r' + A_0^2 \eta_2 \sum_{m=0}^{M+1} \kappa_m r'^{\,2m+1} \right\} \tag{2.55}$$

subject to the relation

$$F_3 \sum_{m=0}^{M} C_m - \frac{1}{N^2L^2} \left\{ F_1 + A_0^2 \eta_2 \sum_{m=0}^{M+1} \kappa_m \right\} = 0 \ . \tag{2.56}$$

Equation (2.56) contains two constants F_1 and F_3, and since there are no more boundary conditions on $\overline{g}_3^{(1)}$ one proceeds to determine ϕ_1.

If one uses (2.55) in (2.43) and sets

$$\phi_1 = \overline{\phi}_1(r')\cos\psi \tag{2.57}$$

then

$$r'^2 \frac{d^2\overline{\phi}_1}{dr'^2} + r' \frac{d\overline{\phi}_1}{dr'} - \overline{\phi}_1 = - \frac{k_2^2}{\eta_2} \frac{1}{N^2L^2} F_1 r'^3$$

$$- \frac{1}{\eta_2} F_3 \left\{ \sum_{m=0}^{M} 4m(m+1) C_m r'^{\,2m+1} - k_2^2 \sum_{m=0}^{M} C_m r'^{\,2m+3} \right\}$$

$$+ \frac{A_0^2}{N^2L^2} \left\{ \sum_{m=0}^{M+1} 4m(m+1)\kappa_m r'^{\,2m+1} - k_2^2 \sum_{m=0}^{M+1} \kappa_m r'^{\,2m+3} \right\} \tag{2.58}$$

The general solution of (2.58) is given as

$$\overline{\phi}_1 = F_5 r' + \frac{F_6}{r'} - \frac{k_2^2}{8\eta_2}\frac{1}{N^2L^2}F_1 r'^3$$

$$- \frac{F_3}{\eta_2}\left\{\sum_{m=0}^{M} C_m r'^{2m+1} - k_2^2 \sum_{m=0}^{M} \frac{C_m}{4(m+1)(m+2)} r'^{2m+3}\right\}$$

$$+\frac{A_0^2}{N^2L^2}\left\{\sum_{m=0}^{M+1} \kappa_m r'^{2m+1} - k_2^2 \sum_{m=0}^{M+1} \frac{\kappa_m}{4(m+1)(m+2)} r'^{2m+3}\right\}. \qquad (2.59)$$

Employing the boundary conditions:

$\frac{\partial\phi_1}{\partial r'}$ is finite at $r' = 0$ and $\frac{\partial\phi_1}{\partial r'} = \frac{\partial\phi_1}{\partial\psi} = 0$ at $r' = 1$

one gets

$$F_6 = 0 \qquad (2.60)$$

$$F_5 - \frac{3k_2^2}{8\eta_2}\frac{1}{N^2L^2}F_1 - \frac{1}{\eta_2}F_3\left\{\sum_{m=0}^{M} C_m(2m+1) - k_2^2 \sum_{m=0}^{M} \frac{C_m(2m+3)}{4(m+1)(m+2)}\right\}$$

$$+\frac{A_0^2}{N^2L^2}\left\{\sum_{m=0}^{M+1} \kappa_m(2m+1) - k_2^2 \sum_{m=0}^{M+1} \frac{\kappa_m(2m+3)}{4(m+1)(m+2)}\right\} = 0 \qquad (2.61)$$

$$F_5 - \frac{k_2^2}{8\eta_2}\frac{1}{N^2L^2}F_1 - \frac{1}{\eta_2}F_3\left\{\sum_{m=0}^{M} C_m - k_2^2 \sum_{m=0}^{M} \frac{C_m}{4(m+1)(m+2)}\right\}$$

$$+\frac{A_0^2}{N^2L^2}\left\{\sum_{m=0}^{M+1} \kappa_m - k_2^2 \sum_{m=0}^{M+1} \frac{\kappa_m}{4(m+1)(m+2)}\right\} = 0 . \qquad (2.62)$$

Now (2.56), (2.61) and (2.62) are three equations in three unknowns F_1, F_3 and F_5 . Hence $g_3^{(1)}$ and ϕ_1 are completely determined.

The function ϕ_1 can be written as

$$\phi_1 = \sum_{m=0}^{M+2} \overline{\nu}_m \, r'^{\,2m+1} \cos\psi \tag{2.63}$$

where

$$\overline{\nu}_0 = F_5 - \frac{1}{\eta_2} F_3 C_0 + \frac{A_0^2}{N^2 L^2} \kappa_0$$

$$\overline{\nu}_1 = -\frac{1}{\eta_2} F_3 C_1 + \frac{k_2^2}{8\eta_2}\left(C_0 F_3 - \frac{1}{N^2 L^2} F_1\right) + \frac{A_0^2}{N^2 L^2}\left(\kappa_1 - \frac{k_2^2 \kappa_0}{8}\right)$$

$$\begin{aligned}\overline{\nu}_m = &-\frac{1}{\eta_2} F_3 \left(C_m - k_2^2 \frac{C_{m-1}}{4m(m+1)}\right) \\ &+ \frac{A_0^2}{N^2 L^2}\left(\kappa_m - k_2^2 \frac{\kappa_{m-1}}{4m(m+1)}\right), \quad 2 \le m \le M\end{aligned} \tag{2.64}$$

$$\overline{\nu}_{M+1} = \frac{k_2^2}{4(M+1)(M+2)}\left\{\frac{F_3}{\eta_2} C_M - \frac{A_0^2}{N^2 L^2} \kappa_M\right\},$$

$$\overline{\nu}_{M+2} = -\frac{k_2^2}{4(M+2)(M+3)} \frac{A_0^2}{N^2 L^2} \kappa_{M+1} .$$

To obtain $g_1^{(1)}$, $g_2^{(1)}$ and w_1 one uses equations (2.39), (2.40) and (2.42). On eliminating w_1 between (2.40) and (2.42) one gets

$$(\nabla_1^2 - N^2 L^2)\left(\frac{\partial g_2^{(1)}}{\partial r'} + \frac{g_2^{(1)}}{r'} - \frac{1}{r'} \frac{\partial g_1^{(1)}}{\partial \psi}\right) = -\eta_1 \frac{1}{r'} \frac{\partial \phi_1}{\partial \psi} \frac{\partial w_0}{\partial r'} . \tag{2.65}$$

Using (2.47) and (2.63) the right hand side of (2.65) will take the form

$$-n_1 \frac{1}{r'} \frac{\partial \phi_1}{\partial \psi} \frac{\partial w_0}{\partial r'} = n_1 A_0^2 \sum_{m=0}^{2M+2} \delta_m r'^{\,2m+1} \sin \psi \tag{2.66}$$

where

$$\left.\begin{aligned}
\delta_m &= \sum_{k=0}^{m} \nu_k \mu_{m-k} \\
\nu_i &= \begin{cases} \dfrac{\bar{\nu}_i}{A_0^2}, & 0 \le i \le M+2 \\ 0, & i > M+2 \end{cases} \\
\mu_i &= \begin{cases} A_0(2A_2 + A_3 NLC_0), & i = 0 \\ A_0 A_3 NLC_i, & 1 \le i \le M \\ 0, & i > M . \end{cases}
\end{aligned}\right\} \tag{2.67}$$

On substituting (2.66) in (2.65) and setting

$$g_1^{(1)} = \bar{g}_1^{(1)}(r')\cos\psi , \quad g_2^{(1)} = \bar{g}_2^{(1)}(r')\sin\psi \tag{2.68}$$

one obtains

$$\left\{r'^2 \frac{d^2}{dr'^2} + r'\frac{d}{dr'} - (1 + N^2L^2r'^2)\right\} \cdot \left(\frac{d\bar{g}_2^{(1)}}{dr'} + \frac{\bar{g}_2^{(1)}}{r'} + \frac{\bar{g}_1^{(1)}}{r'}\right)$$

$$= A_0^2 n_1 \sum_{m=0}^{2M+2} \delta_m r'^{\,2m+3} . \tag{2.69}$$

The solution of (2.69) is given as

$$\frac{d\bar{g}_2^{(1)}}{dr'} + \frac{\bar{g}_2^{(1)}}{r'} + \frac{\bar{g}_1^{(1)}}{r'} = H_1 I_1(NLr') + H_2 K_1(NLr')$$

$$- \frac{A_0^2 n_1}{N^2 L^2} \sum_{m=0}^{2M+2} \bar{\lambda}_m r'^{\,2m+1} \tag{2.70}$$

where

$$\bar{\lambda}_m = \sum_{j=0}^{2M+2} \delta_j \beta_{j-m,j}$$

$$\beta_{k,m} = \left(\frac{2}{NL}\right)^{2k} \frac{\Gamma(m+1)\Gamma(m+2)}{\Gamma(m+1-k)\Gamma(m+2-k)} ; \tag{2.71}$$

$$\beta_{k,m} = 0 \quad \text{if} \quad k < 0 .$$

Using (2.68) in (2.39) one gets

$$\left\{r'^2 \frac{d^2}{dr'^2} + r' \frac{d}{dr'} - (1 + k_1^2 r'^2)\right\}\left(\frac{d\bar{g}_1^{(1)}}{dr'} + \frac{\bar{g}_1^{(1)}}{r'} + \frac{\bar{g}_2^{(1)}}{r'}\right) = 0 \tag{2.72}$$

which possesses the solution

$$\frac{d\bar{g}_1^{(1)}}{dr'} + \frac{\bar{g}_1^{(1)}}{r'} + \frac{\bar{g}_2^{(1)}}{r'} = H_3 I_1(k_1 r') + H_4 K_1(k_1 r'). \tag{2.73}$$

The solution of (2.70) and (2.73) with the boundary conditions: $\bar{g}_1^{(1)}(0)$ and $\bar{g}_2^{(1)}(0)$ are finite is given as

$$\bar{g}_1^{(1)} = \frac{1}{2}\left[\frac{H_3}{k_1}\{I_2(k_1r') + I_0(k_1r')\} + \frac{H_1}{NL}\{I_2(NLr') - I_0(NLr')\} + H_5 + \frac{A_0^2 n_1}{N^2L^2}\sum_{m=0}^{2M+2}\frac{\bar{\lambda}_m}{2(m+1)(m+2)}r'^{\,2m+2}\right] \quad (2.74)$$

$$\bar{g}_2^{(1)} = \frac{1}{2}\left[\frac{H_3}{k_1}\{I_2(k_1r') - I_0(k_1r')\} + \frac{H_1}{NL}\{I_2(NLr') + I_0(NLr')\} - H_5 - \frac{A_0^2 n_1}{N^2L^2}\sum_{m=0}^{2M+2}\frac{\bar{\lambda}_m(2m+3)}{2(m+1)(m+2)}r'^{\,2m+2}\right]. \quad (2.75)$$

On employing the boundary conditions

$$\bar{g}_1^{(1)}(1) = \bar{g}_2^{(1)}(1) = 0$$

one gets

$$\frac{H_3}{k_1}\{I_2(k_1) + I_0(k_1)\} + \frac{H_1}{NL}\{I_2(NL) - I_0(NL)\} + H_5 = -\frac{A_0^2 n_1}{N^2L^2}\sum_{m=0}^{2M+2}\frac{\bar{\lambda}_m}{2(m+1)(m+2)} \quad (2.76)$$

$$\frac{H_3}{k_1}\{I_2(k_1) - I_0(k_1)\} + \frac{H_1}{NL}\{I_2(NL) + I_0(NL)\} - H_5 = \frac{A_0^2 n_1}{N^2L^2}\sum_{m=0}^{2M+2}\frac{\bar{\lambda}_m(2m+3)}{2(m+1)(m+2)}. \quad (2.77)$$

The remainder of this section is devoted to the calculation of w_1. Rewriting equation (2.42) as

$$\nabla_1^2 w_1 = -n_4\left(\frac{\partial g_2^{(1)}}{\partial r'} + \frac{g_2^{(1)}}{r'} - \frac{1}{r'}\frac{\partial g_1^{(1)}}{\partial \psi}\right) - \frac{1}{r'}\frac{\partial \phi_1}{\partial \psi}\frac{\partial w_0}{\partial r'} \quad \text{and}$$

substituting the expressions for $\frac{\partial g_2^{(1)}}{\partial r'} + \frac{g_2^{(1)}}{r'} - \frac{1}{r'}\frac{\partial g_1^{(1)}}{\partial \psi}$

and $\frac{1}{r'}\frac{\partial \phi_1}{\partial \psi}\frac{\partial w_0}{\partial r'}$, the last equation becomes

$$\nabla_1^2 w_1 = \left[A_0^2 \sum_{m=0}^{2M+2} \delta_m r'^{2m+1} - n_4\left\{H_1 I_1(NLr') - \frac{A_0^2 n_1}{N^2L^2}\sum_{m=0}^{2M+2} \bar{\lambda}_m r'^{2m+1}\right\}\right] \sin\psi \; . \tag{2.78}$$

Equation (2.78) suggests that w_1 can be written as

$$w_1 = \bar{w}_1(r') \sin\psi \; .$$

Hence the differential equation for w_1 takes the form

$$r'^2 \frac{d^2\bar{w}_1}{dr'^2} + r'\frac{d\bar{w}_1}{dr'} - \bar{w}_1 = -n_4 H_1 r'^2 I_1(NLr') + A_0^2 \sum_{m=0}^{2M+2}\left(\delta_m + \frac{n_1 n_4}{N^2L^2}\bar{\lambda}_m\right) r'^{2m+3} \; . \tag{2.79}$$

The solution of (2.79) subject to the boundary condition $\bar{w}_1(0)$ is finite turns out to be

$$\overline{w}_1 = H_7 r' - H_1 \frac{\eta_4}{N^2L^2} I_1(NLr')$$

$$+ \frac{A_0^2}{4} \sum_{m=0}^{2M+2} \frac{\left(\delta_m + \frac{\eta_1\eta_4}{N^2L^2} \overline{\lambda}_m\right)}{(m+1)(m+2)} r'^{2m+3} . \qquad (2.80)$$

Hence the condition that there is no-slip at the boundary requires

$$H_7 - \frac{\eta_4 I_1(NL)}{N^2L^2} H_1 = - \frac{A_0^2}{4} \sum_{m=0}^{2M+2} \frac{\left(\delta_m + \frac{\eta_1\eta_4}{N^2L^2} \overline{\lambda}_m\right)}{(m+1)(m+2)} . \qquad (2.81)$$

The equations (2.74), (2.75) and (2.80) give the solutions for $\overline{g}_1^{(1)}$, $\overline{g}_2^{(1)}$ and $\overline{w}_1$, which contain four arbitrary constants H_1, H_3, H_5 and H_7. It is observed that equations (2.76), (2.77) and (2.81) are three equations in these four unknowns. Hence one more relation between these constants is required. Since no more boundary conditions can be employed, the fourth relation is obtained in the following manner.

The dimensionless form of equation (2.13) is

$$\frac{\partial}{\partial r'}\left(\frac{\partial g_1}{\partial r'} + \frac{g_1}{r'} + \frac{1}{r'}\frac{\partial g_2}{\partial \psi}\right) - \frac{k_1^2}{k_2^2}\frac{1}{r'}\frac{\partial}{\partial \psi}\left(\frac{\partial g_2}{\partial r'} + \frac{g_2}{r'} - \frac{1}{r'}\frac{\partial g_1}{\partial \psi}\right)$$

$$+ \eta_1 \frac{k_1^2}{k_2^2}\frac{1}{r'}\frac{\partial w}{\partial \psi} - k_1^2 g_1 = 0 . \qquad (2.82)$$

Inserting the relations (2.38) in (2.82) and equating the coefficients of $\bar{K}$ one gets

$$\frac{\partial}{\partial r'}\left(\frac{\partial g_1^{(1)}}{\partial r'} + \frac{g_1^{(1)}}{r'} + \frac{1}{r'}\frac{\partial g_2^{(1)}}{\partial \psi}\right)$$
$$- \frac{k_1^2}{k_2^2}\frac{1}{r'}\frac{\partial}{\partial \psi}\left(\frac{\partial g_2^{(1)}}{\partial r'} + \frac{g_2^{(1)}}{r'} - \frac{1}{r'}\frac{\partial g_1^{(1)}}{\partial \psi}\right)$$
$$+ n_1 \frac{k_1^2}{k_2^2}\frac{1}{r'}\frac{\partial w_1}{\partial \psi} - k_1^2\, g_1^{(1)} = 0 . \qquad (2.83)$$

Moreover, substituting the expressions for $g_1^{(1)}$, $g_2^{(1)}$ and w_1, which have been obtained before, one finds that

$$-H_5 + \frac{2n_1}{k_2^2} H_7 = 0 . \qquad (2.84)$$

Equation (2.84) together with (2.76), (2.77) and (2.81) constitute a system of four linear algebraic equations in the four unknowns H_1, H_3, H_5 and H_7. Consequently $g_1^{(1)}$, $g_2^{(1)}$ and w_1 are now completely determined.

2.4 Stream-Line Projections

The differential equation of any stream-line is given by

$$\frac{dr}{U} = \frac{r d\psi}{V} = \frac{(R \pm r \sin \psi) d\theta}{W} . \qquad (2.85)$$

Dean [9] has pointed out that the relation between r, ψ and θ is of little interest, and has drawn attention to the useful projections of stream-lines represented by (r,θ) - and (r,ψ) -relations. The motion of the fluid is of special simplicity in the central plane of the pipe. At any point on OC, ψ is either $\pi/2$ or $3\pi/2$; in either case $\cos\psi$, and with it V, vanishes. At any such point the direction of the velocity of the fluid lies in the central plane; hence a particle of the fluid once in this plane does not leave it in the subsequent motion. The motion in one half of the pipe is, therefore, quite distinct from that in the other half and the central plane is clearly a plane of symmetry of the motion.

The differential equation of the stream-lines in the central plane is

$$\frac{dr}{U} = \frac{(R \pm r)d\theta}{W} \tag{2.86}$$

but with sufficient accuracy one can ignore r in comparison with R and write w_0 for W. Then (2.86) is reduced to

$$\frac{dr}{U} = \frac{R\,d\theta}{w_0} . \tag{2.87}$$

Writing $U = -\overline{K}(\frac{\mu+\tau}{\rho})\frac{1}{r}\frac{\partial\phi_1}{\partial\psi}$ and putting $\sin\psi = 1$ and

$w_0 = \frac{Aa^2}{4\mu W_0}[A_1 + A_2 r'^2 + A_3 I_0(NLr')]$ in equation (2.87), one gets

$$\frac{dr'}{2\bar{n}\frac{1}{r'}\bar{\phi}_1} = \frac{d\theta}{\left(\frac{A_1 + A_2 r'^2 + A_3 I_0(NLr')}{A_1 + A_3}\right)} \tag{2.88}$$

where

$$\bar{n} = n(1 - N^2) \tag{2.89}$$

and n is the classical Reynolds number.

Equation (2.88) will give a first approximation to the stream-lines, but only to those parts of them on the outside of the central line. It is noticed that (2.88) was derived by putting $\sin\psi = 1$, thus to obtain the other parts one writes $\sin\psi = -1$, and the sign of (2.88) must be reversed.

The relation between r' and θ can be written as

$$d\theta = \frac{1}{2\bar{n}} \frac{A_1 + A_2 r'^2 + A_3 I_0(NLr')}{(A_1+A_3)\frac{1}{r'}\bar{\phi}_1} dr' . \tag{2.90}$$

It is clear from (2.90) that it is not possible to get a closed expression for the relation between r' and θ. Hence numerical methods for integrating the right hand side of that equation can be used. The values of θ for corresponding values of r' and for different values of N

and L are compared with the values for Newtonian fluid in table 1. Numerical integration was carried out by using Simpson's rule, and Reynolds number was considered to be n = 63.3 as in Dean [9]. The values of θ are measured from the point where the stream-line corsses the central line $r' = 0$.

Table 1 shows that the values of θ increase steadily as N increases and therefore the curvature of the stream-lines in the central plane increases. It is observed that for small N the values of θ do not differ much from the ones for the Newtonian fluid. Moreover, for fixed N but L increasing, the values of θ approach closer to the values for Newtonian fluid. Figure 2 illustrates the dependence of the form of the stream-lines on N and L in the central plane, curves being plotted for the Newtonian fluids and for Polar fluids when N = 0.5, L = 2.0.

The other set of equations of interest are those giving the movement of fluid elements in relation to the central line. This can be visualized by constructing the projection of a stream-line on the section θ = constant, taking the projections as sufficiently represented by ϕ_1 = constant, where ϕ_1 is given by (2.63).

Figure 3 shows the paths of particles projected on the cross-section of the pipe, in the case of polar fluids for which N = 0.8 and L = 1.25, compared with the one for

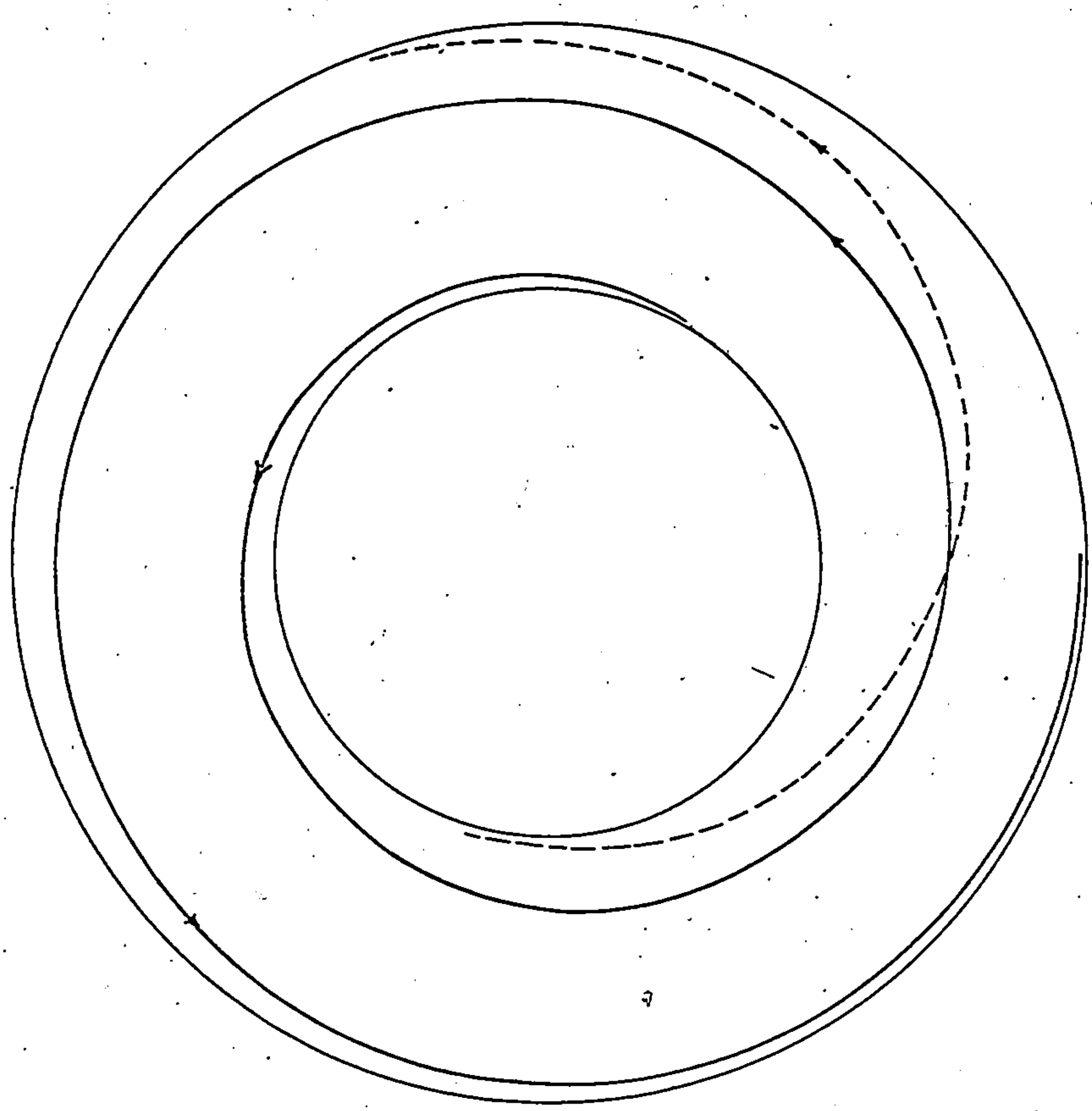

Figure 2. The path of a particle in the central plane of the pipe for a polar fluid when N = 0.5, L = 2 (full line) and for a Newtonian fluid(broken line).

TABLE 1: The values of θ corresponding to different r', N and L

Type of Fluid / r'	Newtonian	Polar N=0.1, L=5	Polar N=0.1, L=10	Polar N=0.2, L=5	Polar N=0.5, L=2
0.0	0.0	0.0	0.0	0.0	0.0
0.1	6.6	6.8	6.79	7.63	19.22
0.2	13.3	13.78	13.76	15.46	38.93
0.3	20.3	21.14	21.10	23.71	59.69
0.4	28.0	29.12	29.07	32.67	82.22
0.5	36.6	38.10	38.02	42.73	107.54
0.6	46.8	48.65	48.56	54.57	137.29
0.7	59.5	61.86	61.75	69.38	174.50
0.8	77.0	80.14	80.00	89.86	225.92
0.9	106.8	111.15	110.96	124.61	313.07

Newtonian fluid. It can be seen that the form of the projections of the stream-lines are not strongly dependent on the interaction between the particles; the position of the neutral point, where the velocity in the section vanishes, are slightly nearer the centre of the pipe in the case of polar fluids, being at $r' = 0.428$, $\psi = 0$ or π when $N = 0.8$, $L = 1.25$, compared with $r' = 0.429$, $\psi = 0$ or π for Newtonian fluid. The data for Newtonian fluid are taken from Dean [9].

In table 2a below, $\overline{\phi}_1$ and $d\overline{\phi}_1/dr'$ are tabulated corresponding to $r' = 0.0(0.1)1.0$ for various values of the parameters N and L . Using equations (2.19), (2.23), (2.38) and (2.57), the first approximation for the velocity components U and V can be written as

$$U_1 = \frac{\mu+\tau}{\rho} \frac{\overline{K}}{a} \overline{\phi}_1 \sin\psi ,$$

$$V_1 = \frac{\mu+\tau}{\rho} \frac{\overline{K}}{a} \frac{d\overline{\phi}_1}{dr'} \cos\psi . \qquad (2.91)$$

Expressions (2.91) and the values of $\overline{\phi}_1$ and $\frac{d\overline{\phi}_1}{dr'}$, given in Table 2a, show that as N increases the speed of the particles in the cross-section of the pipe decreases. We also notice that, for a fixed value of N , the speed increases as L increases and becomes closer to the Newtonian one. The data for Newtonian fluid is given in Table 2b. Hence it can be concluded that the motion in the cross-section of the

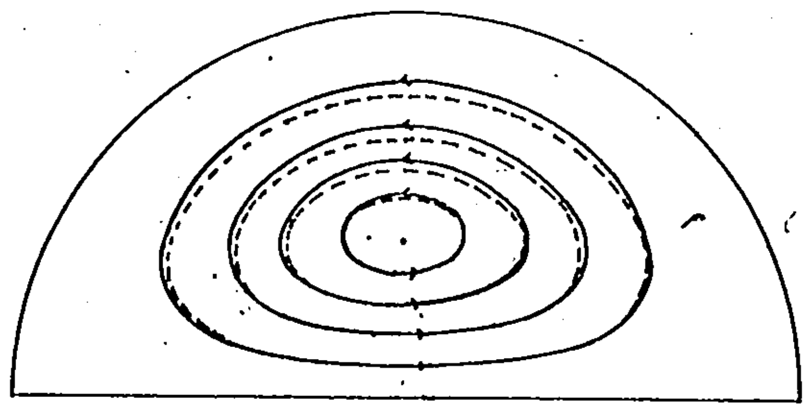

Figure 3. Paths of particles projected on the cross-section of the pipe for a polar fluid when $N = 0.8$, $L = 1.25$ (full line) and for a Newtonian fluid (broken line).

TABLE 2a

The values of $\bar{\phi}_1$ and $\frac{d\bar{\phi}_1}{dr'}$ for polar fluid corresponding to various r', N and L .

r'	N = 0.1, L = 5		N = 0.1, L = 10		N = 0.5, L = 2		N = 0.8, L = 1.25	
	$\bar{\phi}_1$	$d\bar{\phi}_1/dr'$	$\bar{\phi}_1$	$d\bar{\phi}_1/dr'$	$\bar{\phi}_1$	$d\bar{\phi}_1/dr'$	$\bar{\phi}_1$	$d\bar{\phi}_1/dr'$
0.0	0.000000	0.006809	0.000000	0.006818	0.000000	0.004125	0.000000	0.001198
0.1	0.000666	0.006355	0.000667	0.006363	0.000403	0.003848	0.000117	0.001116
0.2	0.001243	0.005052	0.001244	0.005058	0.000752	0.003054	0.000218	0.000882
0.3	0.001654	0.003078	0.001656	0.003081	0.001000	0.001853	0.000290	0.000529
0.4	0.001845	0.000714	0.001847	0.000714	0.001115	0.000419	0.000322	0.000110
0.5	0.001794	-0.001676	0.001798	-0.001678	0.001084	-0.001026	0.000312	-0.000307
0.6	0.001523	-0.003675	0.001525	-0.003680	0.000918	-0.002229	0.000263	-0.000649
0.7	0.001088	-0.004853	0.001089	-0.004859	0.000655	-0.002932	0.000187	-0.000844
0.8	0.000593	-0.004812	0.000594	-0.004818	0.000356	-0.002897	0.000101	-0.000827
0.9	0.000176	-0.003246	0.000177	-0.003250	0.000106	-0.001948	0.000030	-0.000553
1.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	-0.000001

TABLE 2b: The values of $\overline{\phi}_1$ and $\frac{d\overline{\phi}_1}{dr'}$ for Newtonian fluid

r'	$\overline{\phi}_1$	$d\overline{\phi}_1/dr'$
0.0	0.000000	0.006830
0.1	0.000669	0.006373
0.2	0.001245	0.005065
0.3	0.001659	0.003087
0.4	0.001850	0.000715
0.5	0.002804	-0.001682
0.6	0.001528	-0.003685
0.7	0.001092	-0.004866
0.8	0.000596	-0.004819
0.9	0.000179	-0.003255
1.0	0.000000	0.000000

pipe is slower in the theory of polar fluids as compared to the theory of Newtonian fluids.

2.5 The Flux Of Fluid Through The Pipe

The rate of flow of a fluid, Q_c, flowing in a curved pipe of circular cross-section can be obtained by the integration of the component of the velocity normal to the plane of the cross-section over the area of the cross-section of the pipe, i.e.,

$$Q_c = \int_0^a \int_0^{2\pi} rW d\psi \, dr . \qquad (2.92)$$

Since w_1 is proportional to $\sin\psi$, it makes no contribution to this integral. Consequently, to the first-order approximation, the flux through the pipe is independent of the curvature, a fact observed in previous studies Dean [10], Thomas and Walters [29] and Topakoglu [31]; in order to study the variation of the flux with curvature one must, therefore, introduce terms of higher order.

Although the relevant equations are complicated, it is possible to simplify the working when the variation of flux with $\bar{K}^2$ only is required; the method of simplification used is an extension of the method used already by Dean [10].

Again on substituting (2.38) in (2.25) - (2.28) and equating the coefficients of $\overline{K}^2$ one obtains

$$(\nabla_1^2 - k_2^2)\left(\frac{\partial g_2^{(2)}}{\partial r'} + \frac{g_2^{(2)}}{r'} - \frac{1}{r'}\frac{\partial g_1^{(2)}}{\partial \psi}\right) - \eta_1 \nabla_1^2 w_2 = 0 \tag{2.93}$$

$$\left(\frac{\partial \phi_1}{\partial \psi}\frac{\partial}{\partial r'} - \frac{\partial \phi_1}{\partial r'}\frac{\partial}{\partial \psi}\right)\nabla_1^2 \phi_1 + w_0\left(r' \cos \psi \frac{\partial w_1}{\partial r'} - \sin \psi \frac{\partial w_1}{\partial \psi}\right)$$

$$+ r' w_1 \frac{\partial w_0}{\partial r'} \cos \psi = -r' \nabla_1^4 \phi_2 + \eta_3 r' \nabla_1^2 g_3^{(2)} \tag{2.94}$$

$$\frac{1}{r'}\left(\frac{\partial \phi_1}{\partial r'}\frac{\partial w_1}{\partial \psi} - \frac{\partial \phi_1}{\partial \psi}\frac{\partial w_1}{\partial r'}\right) - \frac{1}{r'}\frac{\partial \phi_2}{\partial \psi}\frac{\partial w_0}{\partial r'} =$$

$$\nabla_1^2 w_2 + \eta_4\left(\frac{\partial g_2^{(2)}}{\partial r'} + \frac{g_2^{(2)}}{r'} - \frac{1}{r'}\frac{\partial g_1^{(2)}}{\partial \psi}\right) \tag{2.95}$$

$$(\nabla_1^2 - k_2^2) g_3^{(2)} + \eta_2 \nabla_1^2 \phi_2 = 0\ . \tag{2.96}$$

On eliminating $g_3^{(2)}$ between (2.94) and (2.96), and w_2 between (2.93) and (2.95) one gets

$$-(\nabla_1^2 - N^2L^2)\nabla_1^4 \phi_2 = (\nabla_1^2 - k_2^2)\left[\frac{1}{r'}\left(\frac{\partial \phi_1}{\partial \psi}\frac{\partial}{\partial r'} - \frac{\partial \phi_1}{\partial r'}\frac{\partial}{\partial \psi}\right)\nabla_1^2 \phi_1\right.$$

$$\left. + w_0\left(r' \cos \psi \frac{\partial w_1}{\partial r'} - \sin \psi \frac{\partial w_1}{\partial \psi}\right) + w_1 \frac{\partial w_0}{\partial r'} \cos \psi\right] \tag{2.97}$$

$$(\nabla_1^2 - N^2L^2)\left(\frac{\partial g_2^{(2)}}{\partial r'} + \frac{g_2^{(2)}}{r'} - \frac{1}{r'}\frac{\partial g_1^{(2)}}{\partial \psi}\right) =$$

$$n_1\left[\frac{1}{r'}\left(\frac{\partial \phi_1}{\partial r'}\frac{\partial w_1}{\partial \psi} - \frac{\partial \phi_1}{\partial \psi}\frac{\partial w_1}{\partial r'}\right) - \frac{1}{r'}\frac{\partial \phi_2}{\partial \psi}\frac{\partial w_0}{\partial r'}\right] \tag{2.98}$$

respectively.

Since $\phi_1 = \overline{\phi}_1(r')\cos\psi$ and $w_1 = \overline{w}_1(r')\sin\psi$, equation (2.97) shows that ϕ_2 is proportional to $\sin 2\psi$ and consequently (2.98) implies that

$$\left(\frac{\partial g_2^{(2)}}{\partial r'} + \frac{g_2^{(2)}}{r'} - \frac{1}{r'}\frac{\partial g_1^{(2)}}{\partial \psi}\right)$$

takes the form $M_1(r') + M_2(r')\cos 2\psi$. Using this in (2.93) or (2.95) one concludes that w_2 can be written as $h_1(r') + h_2(r')\cos 2\psi$. The second of these terms need not be calculated, as it does not affect the flux to order $\overline{K}^2$. Hence, it follows that ϕ_2, M_2 and h_2 need not be evaluated.

In order to determine that part of w_2 which is a function of r' only, $M_1(r')$ must be calculated. Equation (2.98) together with

$$\frac{\partial g_2^{(2)}}{\partial r'} + \frac{g_2^{(2)}}{r'} - \frac{1}{r'}\frac{\partial g_1^{(2)}}{\partial \psi} = M_1(r') + M_2(r')\cos 2\psi \tag{2.99}$$

implies that the differential equation for $M_1(r')$ is

$$\left(\frac{d^2}{dr'^2} + \frac{1}{r'}\frac{d}{dr'} - N^2L^2\right) M_1(r') = n_1 \frac{1}{2r'}\frac{d}{dr'}\left(\overline{w}_1 \frac{d\overline{\phi}_1}{dr'}\right)$$

$$= A_0^2 n_1 \sum_{j=0}^{3M+5} \chi_j (j+1) r'^{2j} \qquad (2.100)$$

where

$$\chi_j = \sum_{i=0}^{M+2} \nu_i \alpha_{j-i} \qquad (2.101)$$

and

$$\alpha_j = \begin{cases} H_7 - \dfrac{n_4}{2NL} H_1 , & j = 0 \\[2ex] -H_1 \dfrac{n_4}{4}\left(\dfrac{NL}{2}\right)^{2j-1} \dfrac{1}{j!(j+1)!} + \dfrac{A_0^2}{4} \dfrac{\left(\delta_{j-1} + \dfrac{n_1 n_4}{N^2L^2}\overline{\lambda}_{j-1}\right)}{j(j+1)} , & j = 1,\ldots,M \\[2ex] \dfrac{A_0^2}{4} \cdot \dfrac{\left(\delta_{j-1} + \dfrac{n_1 n_4}{N^2L^2}\overline{\lambda}_{j-1}\right)}{j(j+1)} , & j = M+1,\ldots,2M+3 . \end{cases} \qquad (2.102)$$

The solution of (2.100) turns out to be

$$M_1(r') = T_1 I_0(NLr') + T_2 K_0(NLr')$$

$$- \frac{A_0^2 n_1}{N^2L^2} \sum_{j=0}^{3M+5} \omega_j r'^{2j} \qquad (2.103)$$

where

$$\omega_j = \sum_{m=j}^{3M+5} \chi_m (m+1)\beta''_{m-j,m} ,$$

$$\beta''_{k,m} = \left(\frac{2}{NL}\right)^{2k} \left\{\frac{\Gamma(m+1)}{\Gamma(m-k+1)}\right\}^2 . \quad (2.104)$$

In order to evaluate the constants in (2.103), we note that

$$\frac{\partial}{\partial r'}\left(r' g_2^{(2)}\right) - \frac{\partial g_1^{(2)}}{\partial \psi} = \overline{M}_1(r') + \overline{M}_2(r') \cos 2\psi \quad (2.105)$$

where

$$\overline{M}_i(r') = r' M_i(r') , \quad i = 1,2 . \quad (2.106)$$

The above equations suggest that one can write

$$r' g_2^{(2)} = N_1(r') + N_2(r') \cos 2\psi$$

$$g_1^{(2)} = N_3(r') - \frac{1}{2} N_4(r') \sin 2\psi . \quad (2.107)$$

Hence, equations (2.105) and (2.107) imply that

$$N_1(r') = \int \overline{M}_1(r') dr' \quad (2.108)$$

Using (2.103) and (2.106) in (2.108), yields

$$N_1(r') = T_1 r' \frac{I_1(NLr')}{NL} - T_2 r' \frac{K_1(NLr')}{NL}$$

$$- \frac{A_0^2 n_1}{N^2 L^2} \sum_{j=0}^{3M+5} \frac{\omega_j}{2(j+1)} r'^{2j+2} + T_3 . \tag{2.109}$$

When the boundary conditions $g_2^{(2)}(0)$ is finite and $g_2^{(2)}(1) = 0$ are employed in (2.109) and use is made of (2.107), one gets

$$T_2 = T_3 = 0 , \quad T_1 = \frac{A_0^2 n_1}{NLI_1(NL)} \sum_{j=0}^{3M+5} \frac{\omega_j}{2(j+1)} .$$

Hence, $M_1(r')$ becomes

$$M_1(r') = \frac{A_0^2 n_1}{NL} \left\{ \left(\sum_{j=0}^{3M+5} \frac{\omega_j}{2(j+1)} \right) \frac{I_0(NLr')}{I_1(NL)} \right.$$

$$\left. - \frac{1}{NL} \sum_{j=0}^{3M+5} \omega_j r'^{2j} \right\} . \tag{2.110}$$

Using the fact that $w_2(r')$ can be written as $h_1(r') + h_2(r') \cos 2\psi$ in equation (2.95) and substituting $M_1(r')$ from (2.110), the differential equation for $h_1(r')$ takes the form

$$\left(\frac{d^2}{dr'^2} + \frac{1}{r'}\frac{d}{dr'}\right) h_1 = -\frac{A_0^2 n_1 n_4}{NL}\left(\sum_{j=0}^{3M+5} \frac{\omega_j}{2(j+1)}\right)\frac{I_0(NLr')}{I_1(NL)}$$

$$+ A_0^2 \sum_{j=0}^{3M+5} \left\{\frac{n_1 n_4}{N^2L^2}\omega_j + \chi_j (j+1)\right\} r'^{2j+1} . \qquad (2.111)$$

The solution of (2.111) subject to the boundary conditions: $w_2(0)$ is finite and $w_2(1) = 0$, turns out to be

$$h_1(r') = \frac{A_0^2 n_1 n_4}{(NL)^3}\left(\sum_{j=0}^{3M+5} \frac{\omega_j}{2(j+1)}\right)\left(\frac{I_0(NL) - I_0(NLr')}{I_1(NL)}\right)$$

$$-\frac{A_0^2}{4}\sum_{j=0}^{3M+5}\left\{\frac{n_1 n_4}{N^2L^2}\omega_j + \chi_j(j+1)\right\}\frac{(1-r'^{2j+2})}{(j+1)^2} . \qquad (2.112)$$

If one substitutes

$$W = W_0[w_0 + \overline{K}w_1 + \overline{K}^2 w_2 + \ldots.]$$

in (2.92) the flux through the curved pipe is then given by:

$$Q_c = 2\pi a^2 W_0 \int_0^1 r'[w_0 + \overline{K}^2 h_1(r') + \ldots.]dr'$$

$$= \frac{\pi A a^4}{8\mu}\left[1 - 8\left(\frac{N}{2L}\frac{I_0(NL)}{I_1(NL)} - \frac{1}{L^2}\right)\right]$$

$$+ 2\pi a^2 W_0 \overline{K}^2 \left[\frac{A_0^2 n_1 n_4}{4(NL)^3}\frac{I_2(NL)}{I_1(NL)}\left(\sum_{j=0}^{3M+5}\frac{\omega_j}{j+1}\right)\right.$$

$$\left. - \frac{A_0^2}{8}\left\{\frac{n_1 n_4}{(NL)^2}\sum_{j=0}^{3M+5}\frac{\omega_j}{(j+1)(j+2)} + \sum_{j=0}^{3M+5}\frac{\chi_j}{j+2}\right\}\right] . \qquad (2.113)$$

The first term on the right-hand side of (2.113) is the value of the flux of a polar fluid through a straight pipe and it is clear that this amount of flux is less than the one in a Newtonian fluid, which equals $\frac{\pi A a^4}{8\mu}$. The second term represents the effect of the curvature of the pipe on the rate of flow through it. Since it is not easy to determine the nature of this quantity in the general case, this term can be computed for special values of the parameters N and L. If one takes $N = 0.5$, $L = 2$ and $G = \frac{W_0}{a}$, the rate of flow in the curved pipe is decreased by the value

$$2\pi a^2 W_0 A_0^2 \overline{K}^2 \frac{2.2633575\, I_2(k_1) + 1.0283436\, I_0(k_1)}{1.9666840\, I_2(k_1) + 0.2714940\, I_0(k_1)} \qquad (2.114)$$

where

$$k_1^2 = \frac{4\tau a^2}{\alpha+2\beta}.$$

On substituting

$$A_0 = \frac{Aa^2}{4\mu W_0}, \qquad \overline{K} = 2n^2(1-N^2)^2 \frac{a}{R}$$

in (2.114) and employing the resulting expression in (2.113), the rate of flow in this case turns out to be

$$Q_c = \frac{\pi A a^4}{8\mu} \left[(1 - 0.2401926) - A_0^4 n^4 \left(\frac{a}{R}\right)^2 (0.019775) \right.$$

$$\left. \cdot \frac{2.2633575\ I_2(k_1) + 1.0283436\ I_0(k_1)}{1.966684\ I_2(k_1) + 0.271494\ I_0(k_1)} \right] . \quad (2.115)$$

Hence, it can be concluded that the flux of a polar fluid through a curved pipe is decreased due to the curvature of the pipe.

2.6 Conclusion

Collecting the main results it is observed that, in comparison to the results of Dean ([9], [10]), in the case of a polar fluid the curvature of the stream-lines in the central plane increases, the motion of the particles in the cross-section is slower and there is a decrease in the volume flow rate of the fluid flowing through the pipe.

CHAPTER III

TWO DIMENSIONAL INTERNAL AND EXTERNAL FLOWS OF POLAR FLUIDS FOR A CIRCULAR CYLINDER

3.1 Introduction

Internal flows of viscous fluids within a circular cylinder have not received considerable attention in comparison to flows past cylindrical bodies. In the case of internal flows, the motion within the cylinder is assumed to be in a plane perpendicular to the generators of the cylinder and is generated by:

(a) injection of the fluid into the cylinder through its wall

(b) the rotation of part (or all) of the cylinder wall.

The applications of type (a) problems arise, for instance, in the ventilation of the confined spaces while type (b) problems arise in the recirculating motion in cavities in aerodynamic surfaces.

These fluid motions for viscous fluids were first considered by Rayleigh [25] for simplified type (a) and (b) problems. Dennis [11] gave a numerical solution for a problem of type (a) and a problem of type (b) have been studied by Burgraff [2]. Recently, Mills [24] gave analytical and numerical solutions for certain type (a) and type (b) problems.

In the present chapter solutions are given for the inflow-outflow problem and for a moving-wall problem in the case of polar fluids. Also, in the last section the problem of creeping polar fluid flow past a circular cylinder is investigated.

3.2 Governing Equations

Consider a two-dimensional motion and take plane polar coordinates (r, θ) with velocity components v_r in the radial direction of the motion and v_θ in the tangential direction. The components of the total angular velocity $\hat{G}$ are $\hat{G}_1$ in the radial direction, $\hat{G}_2$ in the tangential direction and $\hat{G}_3$ in a direction perpendicular to the plane of the motion. Moreover, if the fluid is incompressible, the equations (1.13) - (1.15) take the form:

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0 \tag{3.1}$$

$$\rho\left[v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r}\right] = -\frac{\partial p}{\partial r} + (\mu+\tau)\left[\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + 2\tau\frac{1}{r}\frac{\partial \hat{G}_3}{\partial \theta} + \rho b_1 \tag{3.2}$$

$$\rho\left[v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r}\right] = -\frac{1}{r}\frac{\partial p}{\partial \theta} + (\mu+\tau)\left[\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] - 2\tau\frac{\partial \hat{G}_3}{\partial r} + \rho b_2 \tag{3.3}$$

$$\rho k^2 \left[v_r \frac{\partial \hat{G}_1}{\partial r} + \frac{v_\theta}{r} \frac{\partial \hat{G}_1}{\partial \theta} - \frac{v_\theta \hat{G}_2}{r} \right] =$$

$$(\alpha+2\beta)\frac{\partial}{\partial r}\left(\frac{\partial \hat{G}_1}{\partial r} + \frac{\hat{G}_1}{r} + \frac{1}{r}\frac{\partial \hat{G}_2}{\partial \theta}\right) - (\beta+\gamma)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial \hat{G}_2}{\partial r} + \frac{\hat{G}_2}{r} - \frac{1}{r}\frac{\partial \hat{G}_1}{\partial \theta}\right)$$

$$- 4\tau\hat{G}_1 + \rho c_1 \qquad (3.4)$$

$$\rho k^2 \left[v_r \frac{\partial \hat{G}_2}{\partial r} + \frac{v_\theta}{r} \frac{\partial \hat{G}_2}{\partial \theta} + \frac{v_\theta \hat{G}_1}{r} \right] =$$

$$(\alpha+2\beta)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial \hat{G}_1}{\partial r} + \frac{\hat{G}_1}{r} + \frac{1}{r}\frac{\partial \hat{G}_2}{\partial \theta}\right) + (\beta+\gamma)\frac{\partial}{\partial r}\left(\frac{\partial \hat{G}_2}{\partial r} + \frac{\hat{G}_2}{r} - \frac{1}{r}\frac{\partial \hat{G}_1}{\partial \theta}\right)$$

$$- 4\tau\hat{G}_2 + \rho c_2 \qquad (3.5)$$

$$\rho k^2 \left[v_r \frac{\partial \hat{G}_3}{\partial r} + \frac{v_\theta}{r} \frac{\partial \hat{G}_3}{\partial \theta} \right] = (\beta+\gamma)\nabla^2\hat{G}_3 - 4\tau\hat{G}_3$$

$$+ 2\tau\left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) + \rho c_3 \qquad (3.6)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} .$$

For a class of flows in which the inertia terms, the micro-inertia, the body forces, the body couples and $\hat{G}_1$, $\hat{G}_2$ all vanish, the governing equations of the motion of the fluid reduce to

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0 \tag{3.7}$$

$$-\frac{\partial p}{\partial r} + (\mu+\tau)\left[\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + 2\tau\,\frac{1}{r}\frac{\partial \hat{G}_3}{\partial \theta} = 0 \tag{3.8}$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + (\mu+\tau)\left[\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] - 2\tau\,\frac{\partial \hat{G}_3}{\partial r} = 0 \tag{3.9}$$

$$(\beta+\gamma)\nabla^2\hat{G}_3 - 4\tau\hat{G}_3 + 2\tau\left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) = 0 \tag{3.10}$$

Equation (3.7) suggests that one can write

$$v_r = \frac{1}{r}\frac{\partial f}{\partial \theta}, \qquad v_\theta = -\frac{\partial f}{\partial r}. \tag{3.11}$$

Introducing the dimensionless variables

$$v_r = Uv_r', \qquad v_\theta = -Uv_\theta', \qquad r = ar'$$

$$\hat{G} = \frac{U}{a}G \qquad \text{and} \qquad f = Ua\psi$$

where U has the dimension of velocity and a being a characteristic length, in the equations of motion (3.8) - (3.10) one finds that these equations can be expressed in terms of the dimensionless stream function ψ and the dimensionless angular velocity G as

$$\nabla^4\psi + 2N^2\nabla^2 G = 0 \tag{3.12}$$

$$(\nabla^2-\lambda^2)G - \frac{\lambda^2}{2}\nabla^2\psi = 0 \tag{3.13}$$

where

$$\lambda^2 = \frac{N^2L^2}{1-N^2} \quad , \quad N \neq 1 \ . \tag{3.14}$$

On eliminating G between (3.12) and (3.13) one obtains

$$\nabla^4(\nabla^2 - N^2L^2)\psi = 0 \ . \tag{3.15}$$

The solution of (3.15) can be written as

$$\psi = \psi_0 + \psi_1 \tag{3.16}$$

where ψ_0 is the solution of the biharmonic equation

$$\nabla^4\psi_0 = 0 \tag{3.17}$$

and ψ_1 satisfies the equation

$$(\nabla^2 - N^2L^2)\psi_1 = 0 \ . \tag{3.18}$$

The most general solution of (3.17) was given by Michell [23] (see Timoshenko and Goodier [30]) as

$$\psi_0 = a_0 + a_0' \ln r + b_0 r^2 + c_0 r^2 \ln r + d_0 r^2 \theta + a_0'' \theta$$

$$+ \frac{a_1''}{2} r\,\theta \sin\theta + (a_1 r + b_1 r^3 + a_1' r^{-1} + b_1' r \ln r)\cos\theta$$

$$- \frac{c_1''}{2} r\,\theta \cos\theta + (c_1 r + d_1 r^3 + c_1' r^{-1} + d_1' r \ln r)\sin\theta$$

$$+ \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2})\cos n\theta$$

$$+ \sum_{n=2}^{\infty} (c_n r^n + d_n r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2})\sin n\theta \; . \qquad (3.19)$$

Similarly the solution of (3.18) can be written as

$$\psi_1 = f_0 I_0(NLr) + f_0' K_0(NLr)$$

$$+ \sum_{n=1}^{\infty} \{f_n I_n(NLr) + f_n' K_n(NLr)\}\cos n\theta$$

$$+ \sum_{n=1}^{\infty} \{h_n I_n(NLr) + h_n' K_n(NLr)\}\sin n\theta \; . \qquad (3.20)$$

In exceptional cases fractional values of n would be needed.

Once ψ is obtained, the total angular velocity G can be easily calculated by using (3.12) and (3.13).

3.3 Inflow-Outflow Problem

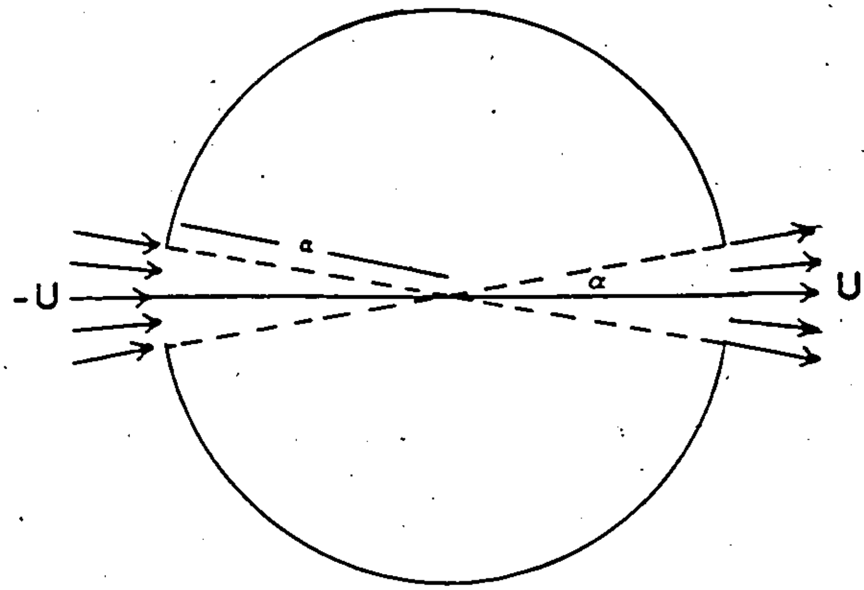

Figure 4. Inflow-outflow problem

Consider a steady two-dimensional motion of an incompressible polar fluid which flows radially into a circle with velocity $v_r = -U$ and flows outward radially with velocity $v_r = U$ as shown in figure 4. The inflow-outflow occurs over an arc subtending an angle 2α at the centre, the outflow being over the arc for which $-\alpha \leq \theta \leq \alpha$.

From figure 4 it can be noticed that

$$\psi(r', \theta) = -\psi(r', -\theta) \tag{3.21}$$

where θ is measured from the line bisecting the angle 2α . Thus $\psi = 0$ when $\theta = 0$, $\theta = \pi$ for all r' , and the domain of the problem may be restricted to $0 \leq r' \leq 1$, $0 \leq \theta \leq \pi$.

It is possible to take $\hat{G} = (0, 0, G)$. If the fluid is a creeping one and if the micro-inertia, body forces and body couples are negligible, then the motion is governed by (3.12) and (3.13). On the boundary it is assumed that v'_θ vanishes for $0 \leq \theta \leq \pi$ and v'_r also vanishes for $\alpha < \theta < \pi - \alpha$. These boundary conditions can be expressed in terms of the stream function ψ as

$$\psi = \begin{cases} \theta & \text{for} \quad 0 \leq \theta \leq \alpha \\ \alpha & \text{for} \quad \alpha \leq \theta \leq \pi - \alpha \\ \pi - \theta & \text{for} \quad \pi - \alpha \leq \theta \leq \pi \end{cases}$$

and

$$\frac{\partial \psi}{\partial r'} = 0 \quad \text{for} \quad 0 \leq \theta \leq \pi. \tag{3.22}$$

The total angular velocity G is assumed to vanish on $r' = 1$ for $0 \leq \theta \leq \pi$.

3.3.1 Solution

If one assumes that the solution of the inflow-outflow problem can be written as

$$\psi = \sum_{n=1}^{\infty} A_n(r') \sin n\theta \tag{3.23}$$

then from (3.19) and (3.20), after dropping the primes,

$$A_n(r) = c_n r^n + d_n r^{n+2} + h_n I_n(NLr) \tag{3.24}$$

where c_n, d_n and h_n are constants. On employing the boundary conditions (3.22), one obtains

$$c_n + d_n + h_n I_n(NL) = \frac{2}{\pi} \frac{\{1-(-1)^n\}}{n^2} \sin n\alpha \tag{3.25}$$

$$nc_n + (n+2)d_n + h_n \left.\frac{dI_n(NLr)}{dr}\right|_{r=1} = 0 . \tag{3.26}$$

Equations (3.25) and (3.26) are two equations in three unknowns c_n, d_n and h_n. One more equation is required for the complete determination of the constants. If one uses (3.24) and (3.23) in (3.13), then the differential equation for the total angular velocity G is

$$(\nabla^2-\lambda^2)G = \frac{\lambda^2}{2} \sum_{n=1}^{\infty} \{4(n+1)d_n r^n + h_n N^2 L^2 I_n(NLr)\} \sin n\theta \tag{3.27}$$

The last equation suggests that $G(r, \theta)$ can be written as

$$G = \sum_{n=1}^{\infty} g_n(r) \sin n\theta \tag{3.28}$$

Inserting (3.28) in (3.27), one finds that $g_n(r)$ satisfies the differential equation

$$\frac{d^2 g_n}{dr^2} + \frac{1}{r}\frac{dg_n}{dr} - \left(\frac{n^2}{r^2} + \lambda^2\right) g_n = \frac{\lambda^2}{2} [4(n+1)d_n r^n + h_n N^2 L^2 I_n(NLr)]$$

which has the solution

$$g_n(r) = D_n I_n(\lambda r) - 2(n+1)d_n r^n - \frac{1}{2} h_n L^2 I_n(NLr) .$$

Hence

$$G(r,\theta) = \sum_{n=1}^{\infty} \{D_n I_n(\lambda r) - 2(n+1)d_n r^n - \frac{1}{2} h_n L^2 I_n(NLr)\}\sin n\theta \quad (3.29)$$

This solution must now satisfy equation (3.12) together with the value of ψ already obtained. Substituting (3.29) and the value of ψ in (3.12) one gets

$$2N^2\lambda^2 \sum_{n=1}^{\infty} D_n I_n(\lambda r)\sin n\theta = 0 .$$

This equation implies that $G(r,\theta)$ is finally given by

$$G(r,\theta) = - \sum_{n=1}^{\infty} \{2(n+1)d_n r^n + \frac{1}{2} h_n L^2 I_n(NLr)\}\sin n\theta . \quad (3.30)$$

On employing the boundary condition

$$G = 0 , \quad 0 \leq \theta \leq \pi \quad \text{on} \quad r = 1$$

one finds that

$$2(n+1)d_n + \frac{1}{2} h_n L^2 I_n(NL) = 0 . \quad (3.31)$$

Hence (3.25), (3.26) and (3.31) are three linear algebraic equations in the three unknowns c_n, d_n and h_n. Their solution is given by

$$h_n = \frac{2}{\pi}\{1-(-1)^n\}\left(\frac{n+1}{n}\right)\frac{\sin n\alpha}{\frac{1}{2}L^2 I_n(NL) - (n+1)NLI_{n+1}(NL)}, \tag{3.32}$$

provided that

$$\frac{1}{2}L^2 I_n(NL) - (n+1)NLI_{n+1}(NL) \neq 0$$

$$d_n = -\frac{1}{4}\frac{L^2 I_n(NL)}{(n+1)} h_n \tag{3.33}$$

and

$$c_n = -d_n - h_n I_n(NL) + \frac{2}{\pi}\frac{\{1-(-1)^n\}}{n^2}\sin n\alpha . \tag{3.34}$$

At this point it is interesting to study the convergence of the infinite series

$$\sum_{n=1}^{\infty}\{c_n r^n + d_n r^{n+2} + h_n I_n(NLr)\}\sin n\theta \tag{3.35}$$

where h_n, d_n and c_n are given by (3.32) - (3.34).

When n is large enough, the modified Bessel function $I_n(z)$ is given by (cf. MacLachlan [22])

$$I_n(z) \simeq \frac{\left(\frac{1}{2}z\right)^n}{n!}\left\{1 + \frac{\left(\frac{1}{2}z\right)^2}{(n+1)} + \frac{\left(\frac{1}{2}z\right)^4}{2!(n+1)(n+2)}\right\} .$$

After simplification one finds that

$$h_n I_n(NLr) \simeq e_n$$

where

$$e_n = \frac{2}{\pi}\{1-(-1)^n\}\left(\frac{n+1}{n}\right) \sin n\alpha \, \frac{\gamma_1 n^5 + \ldots + \gamma_6}{\gamma_1' n^5 + \ldots + \gamma_6'} \, r^n \tag{3.36}$$

and $\gamma_1, \ldots, \gamma_6$; $\gamma_1', \ldots, \gamma_6'$ are quantities independent of n . As a result one can write

$$|e_n \sin n\theta| \le \frac{4}{\pi}\left(\frac{n+1}{n}\right) \frac{\gamma_1 n^5 + \ldots + \gamma_6}{\gamma_1' n^5 + \ldots + \gamma_6'} \, r^n . \tag{3.37}$$

The ratio test implies that the series

$$\sum_{n=1}^{\infty} \frac{4}{\pi}\left(\frac{n+1}{n}\right) \frac{\gamma_1 n^5 + \ldots + \gamma_6}{\gamma_1' n^5 + \ldots + \gamma_6'} \, r^n$$

converges for $|r| < 1$.

Hence, by the comparison test one concludes that the series $\sum_{n=1}^{\infty} e_n \sin n\theta$ converges for $|r| < 1$.

Consequently $\sum_{n=1}^{\infty} h_n I_n(NLr) \sin n\theta$ is convergent inside the unit circle.

Similarly one can prove that the series

$\sum_{n=1}^{\infty} d_n r^{n+2} \sin n\theta$ and $\sum_{n=1}^{\infty} c_n r^n \sin n\theta$ also converge in $|r| < 1$. Hence it can be concluded that the series (3.35) converges inside the circle $|r| < 1$. Also it can be shown that the series (3.30), which represents the solution for the total angular velocity G, is convergent if $|r| < 1$.

The regional angular velocity $\hat{\underset{\sim}{W}}$ can be expressed in terms of the stream function ψ as

$$\hat{\underset{\sim}{W}} = (0,\ 0,\ -\frac{1}{2}\nabla^2\psi)$$

$$= (0,\ 0,\ -\frac{1}{2}\sum_{n=1}^{\infty}\{4(n+1)d_n r^n + h_n N^2 L^2 I_n(NLr)\}\sin n\theta) . \qquad (3.38)$$

Hence the relative angular velocity $\hat{\underset{\sim}{H}}$ is given by

$$H_3 = -\frac{1}{2} L^2(1-N^2)\sum_{n=1}^{\infty} h_n I_n(NLr)\sin n\theta . \qquad (3.39)$$

If $N = 1$, (3.39) implies that H_3 vanishes and in this case the particles are constrained to rotate with the vorticity.

3.3.2 Numerical Results and Discussion

The stream function ψ for the inflow-outflow problem is calculated for different values of r and θ by using the series (3.35). Computations are carried out for $\alpha = 6^{\circ}$. and various values for the coupling number N and the length ratio L . The series is truncated after 35 terms which exhibits reasonable accuracy. The stream-lines corresponding to these values of the stream function are sketched for $N = 0.5$, $L = 6$ and $N = 0.9$, $L = 6$ and are compared with the corresponding ones for the Newtonian fluid in each case, as shown in figures 5 and 6. From these figures it is observed that, for fixed L , the deviation of the polar fluid from the Newtonian one increases as the coupling number N increases.

If the coupling number N is relatively small the values of the stream function do not vary much from the values for a Newtonian fluid as can be seen from tables 3 and 4.

Tables 5 and 6 show that, for fixed N but L is increasing, the values of ψ increase steadily with L at all points (r, θ) in the domain of the motion.

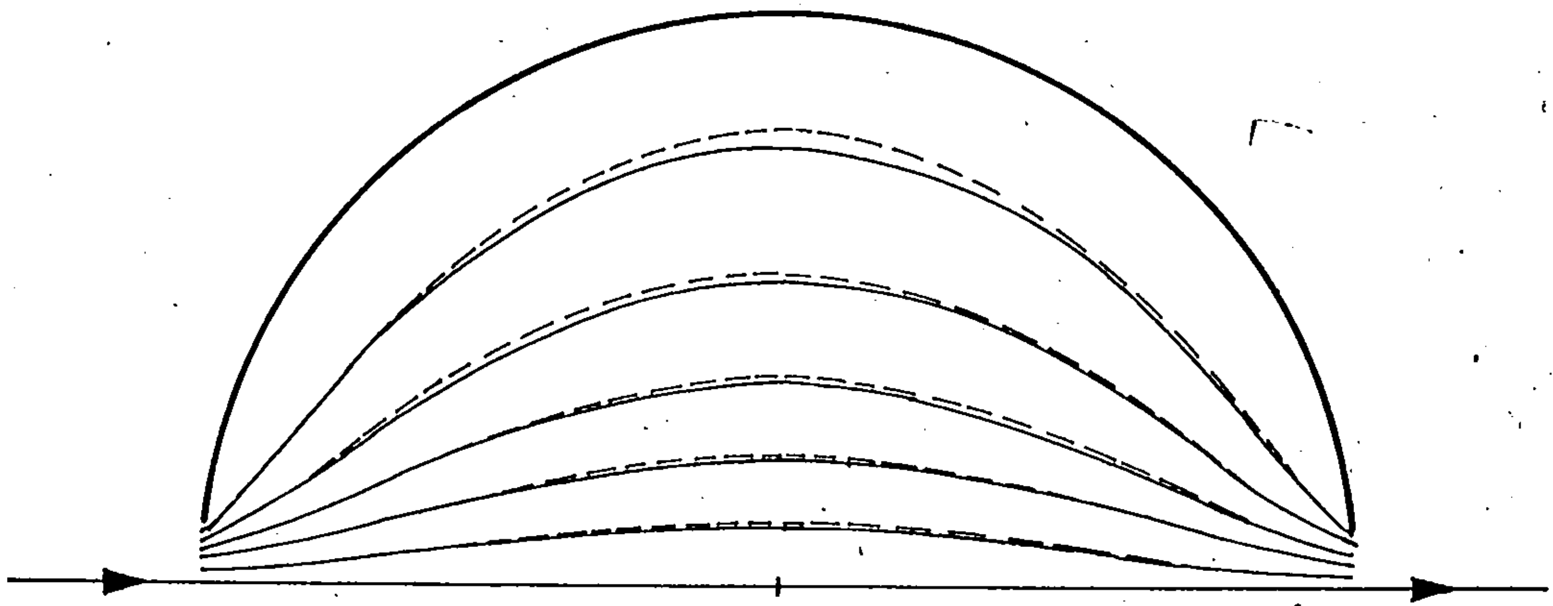

Figure 5. Stream-lines for a polar fluid when $N = 0.5$, $L = 6$ (full line) and for a Newtonian fluid (broken line).

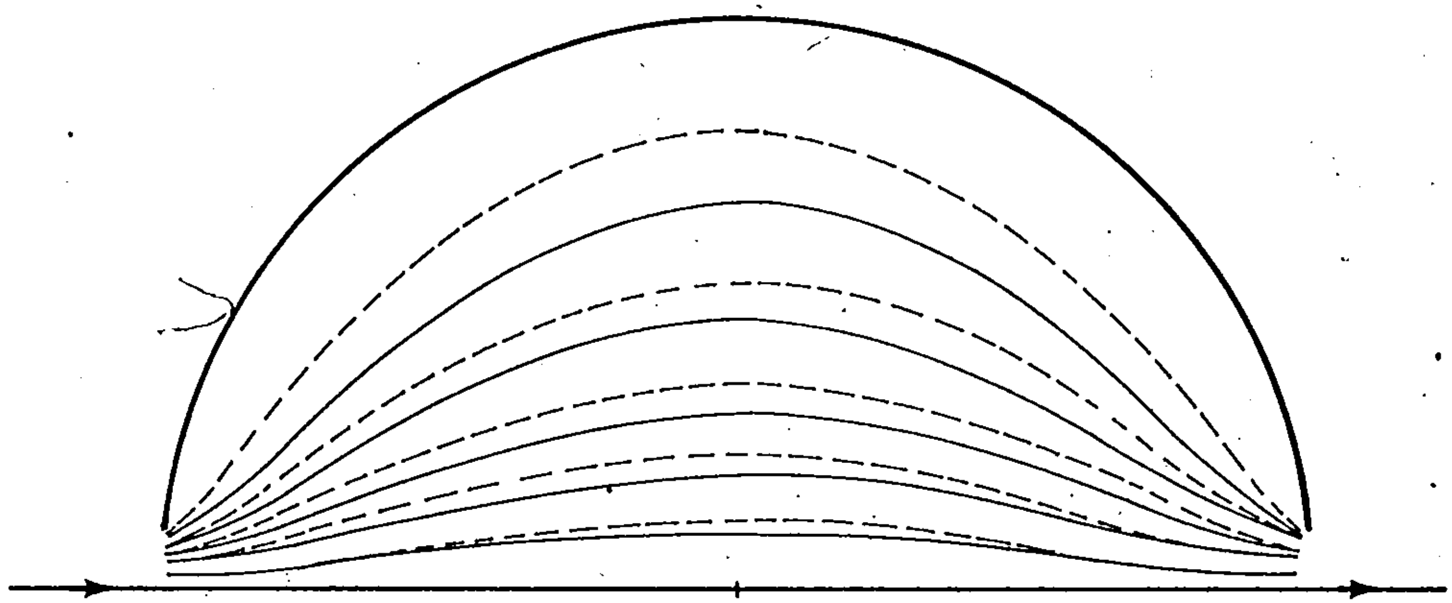

Figure 6. Stream-lines for a polar fluid when $N = 0.9$, $L = 6$ (full line) and for a Newtonian fluid (broken line).

Table 3: Values of ψ/α for the inflow-outflow problem in the case of a Newtonian fluid

θ \ r	0.0	0.2	0.4	0.6	0.8	1.0
5	0.000000	0.0350172	0.0825511	0.1681147	0.3741202	0.8137647
10	0.000000	0.0696131	0.1626463	0.3215761	0.6402321	1.0022830
20	0.000000	0.1359984	0.3074649	0.5538513	0.8619916	1.0008257
30	0.000000	0.1963711	0.4244886	0.6919740	0.9243127	0.9976502
40	0.000000	0.2487763	0.5126371	0.7699215	0.9477870	0.9960517
50	0.000000	0.2920055	0.5758919	0.8147672	0.9589918	0.9969048
60	0.000000	0.3255727	0.6193923	0.8412508	0.9655082	0.9990913
70	0.000000	0.3493949	0.6475007	0.8657726	0.9685019	1.0006274
80	0.000000	0.3635801	0.6632215	0.8649633	0.9702845	1.0005715
90	0.000000	0.3683017	0.6682820	0.8675371	0.9708438	1.0004553

Table 4: Values of ψ/α for the inflow-outflow problem in polar fluids when $N = 0.2$, $L = 10$

θ \ r	0.0	0.2	0.4	0.6	0.8	1.0
5	0.0	0.0350707	0.0826637	0.1682721	0.3742771	0.8134657
10	0.0	0.0697243	0.1628640	0.3218783	0.6404852	1.0023060
20	0.0	0.1362093	0.3078844	0.5543736	0.8623121	1.0005000
30	0.0	0.1966895	0.4250587	0.6925998	0.9246307	0.9993493
40	0.0	0.2491694	0.5133162	0.7705962	0.9480953	0.9987471
50	0.0	0.2924722	0.5766416	0.8154503	0.9592867	0.9994117
60	0.0	0.3260902	0.6201890	0.8419316	0.9653530	1.0000900
70	0.0	0.3499468	0.6483187	0.8574394	0.9687764	1.0000267
80	0.0	0.3641634	0.6640539	0.8656286	0.9705351	1.0000830
90	0.0	0.3688820	0.6691145	0.8681934	0.9710874	0.9999528

Table 5: Values of ψ/α for the inflow-outflow problem in polar fluids when $N = 0.4$, $L = 5$

θ \ r	0.0	0.2	0.4	0.6	0.8	1.0
5	0.0	0.0355527	0.0837468	0.1700632	0.3762865	0.8134657
10	0.0	0.0706811	0.1649842	0.3252576	0.6438564	1.0023060
20	0.0	0.1380667	0.3117942	0.5598809	0.8663659	1.0005000
30	0.0	0.1883458	0.4302562	0.6989128	0.9282628	0.9993493
40	0.0	0.2524950	0.5193128	0.7769424	0.9511855	0.9987471
50	0.0	0.2963273	0.5830659	0.8215233	0.9619374	0.9994117
60	0.0	0.3303387	0.6268049	0.8476766	0.9676890	1.0000900
70	0.0	0.3544632	0.6549999	0.8629110	0.9708960	1.0000267
80	0.0	0.3688345	0.6707476	0.8709270	0.9725485	1.0000830
90	0.0	0.3736041	0.6758081	0.8734329	0.9730606	0.9999528

Table 6: Values of ψ/α for the inflow-outflow problem in polar fluids when $N = 0.4$, $L = 10$

θ \ r	0.0	0.2	0.4	0.6	0.8	1.0
5	0.0	0.0352695	0.0830629	0.1688366	0.3748087	0.8134657
10	0.0	0.0701195	0.1636490	0.3229556	0.6414003	1.0023060
20	0.0	0.1369796	0.3093564	0.5562022	0.8636076	1.0005200
30	0.0	0.1977979	0.4270642	0.6848141	0.9258070	0.9993493
40	0.0	0.2505676	0.5156970	0.7729555	0.9491864	0.9987471
50	0.0	0.2951059	0.5792663	0.8178334	0.9602936	0.9994117
60	0.0	0.3279038	0.6229618	0.8442902	0.9662928	1.0000900
70	0.0	0.3518863	0.6511751	0.8597632	0.9696591	1.0000267
80	0.0	0.3661770	0.6669519	0.8679262	0.9714002	1.0000830
90	0.0	0.3709206	0.6720250	0.8704815	0.9719407	0.9999528

3.4 Moving-wall Problem

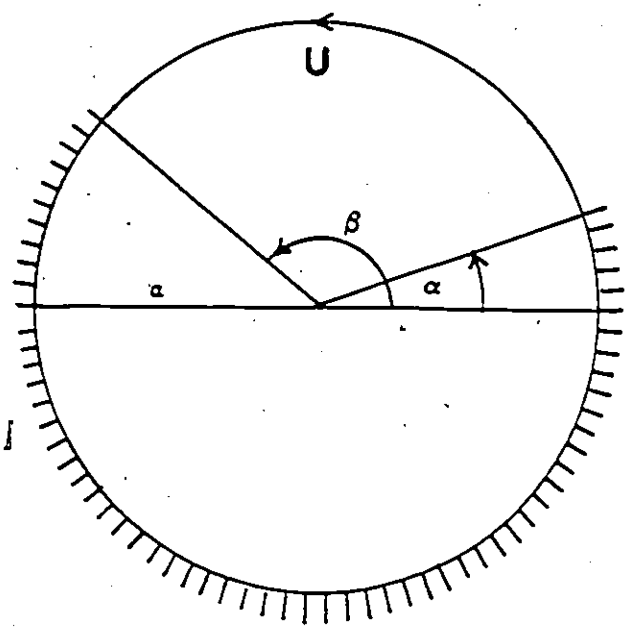

Figure 7: Moving-wall problem.

In this problem the motion within the circular cylinder is generated by rotating the part of the circumference which is bounded by $\alpha \leq \theta \leq \beta$ while the other part is fixed, as shown in figure 7. The moving part rotates with tangenial velocity U .

If the motion is assumed to be a steady one, the fluid is incompressible and the inertia, micro-inertia, body forces and body couples are negligible, then the polar fluid motion is governed again by (3.12) and (3.13).

The boundary conditions, for this problem are on $r = 1$:

$$\left.\begin{array}{ll}
\psi = 0 & \alpha \leq \theta \leq 2\pi + \alpha \\[2ex]
\dfrac{\partial\psi}{\partial r} = \begin{cases} -1 \\ 0 \end{cases} & \begin{array}{l} \alpha < \theta < \beta \\ \beta < \theta < 2\pi + \alpha \end{array} \\[2ex]
G = \begin{cases} 1 \\ 0 \end{cases} & \begin{array}{l} \alpha < \theta < \beta \\ \beta < \theta < 2\pi + \alpha \end{array}
\end{array}\right\} \qquad (3.40)$$

3.4.1 Solution

Assume that the stream function ψ for the moving-wall problem is

$$\psi = B_0(r) + \sum_{n=1}^{\infty} \{B_n(r)\cos n\theta + C_n(r)\sin n\theta\} . \qquad (3.41)$$

Hence, by choosing the appropriate terms from the expressions (3.19) and (3.20) one finds that ψ is given by

$$\begin{aligned}
\psi = {} & a_0 + b_0 r^2 + f_0 I_0(NLr) \\
& + \sum_{n=1}^{\infty} \{a_n r^n + b_n r^{n+2} + f_n I_n(NLr)\}\cos n\theta \\
& + \sum_{n=1}^{\infty} \{c_n r^n + d_n r^{n+2} + h_n I_n(NLr)\}\sin n\theta
\end{aligned} \qquad (3.42)$$

where the constants a_n, b_n, c_n, d_n, f_n and h_n are to be determined from the bondary conditions.

The total angular velocity G is obtained by a procedure similar to the one used in section 3.3.1. In the present case one finds that

$$G = -2b_0 - \frac{1}{2} L^2 f_0 I_0(NLr)$$

$$- \frac{1}{2} \sum_{n=1}^{\infty} \{L^2 f_n I_n(NLr) + 4(n+1) b_n r^n\} \cos n\theta$$

$$- \frac{1}{2} \sum_{n=1}^{\infty} \{L^2 h_n I_n(NLr) + 4(n+1) d_n r^n\} \sin n\theta \qquad (3.43)$$

On employing the boundary conditions (3.40) in (3.42) and (3.43) one obtains

$$\left.\begin{aligned} a_0 + b_0 + f_0 I_0(NL) &= 0 \\ a_n + b_n + f_n I_n(NL) &= 0 \\ c_n + d_n + h_n I_n(NL) &= 0 \end{aligned}\right\} \qquad (3.44)$$

$$\left.\begin{aligned} 2b_0 + f_0 \left.\frac{dI_0(NLr)}{dr}\right|_{r=1} &= \frac{1}{2}\,\frac{\alpha - \beta}{\pi} \\ n(a_n+b_n) + 2b_n + f_n \left.\frac{dI_n(NLr)}{dr}\right|_{r=1} &= \frac{1}{\pi}\,\frac{\sin n\alpha - \sin n\beta}{n} \\ n(c_n+d_n) + 2d_n + h_n \left.\frac{dI_n(NLr)}{dr}\right|_{r=1} &= \frac{1}{\pi}\,\frac{\cos n\beta - \cos n\alpha}{n} \end{aligned}\right\} \qquad (3.45)$$

$$\left.\begin{aligned} 2b_0 + \frac{1}{2} f_0 L^2 I_0(NL) &= \frac{1}{2}\frac{\alpha - \beta}{\pi} \\ 2(n+1)b_n + \frac{1}{2} L^2 f_n I_n(NL) &= \frac{1}{\pi}\frac{\sin n\alpha - \sin n\beta}{n} \\ 2(n+1)d_n + \frac{1}{2} L^2 h_n I_n(NL) &= \frac{1}{\pi}\frac{\cos n\beta - \cos n\alpha}{n} \end{aligned}\right\} . \tag{3.46}$$

The solution of the above equations for the constant coefficients turns out to be

$$f_0 = 0 \quad , \quad b_0 = -a_0 = \frac{\alpha - \beta}{4\pi} \tag{3.47}$$

$$\left.\begin{aligned} f_n &= \frac{1}{\pi}\frac{\sin n\beta - \sin n\alpha}{\frac{1}{2} L^2 I_n(NL) - (n+1)NLI_{n+1}(NL)} \\ b_n &= \frac{1}{2\pi}\frac{\sin n\alpha - \sin n\beta}{n} - \frac{1}{2} f_n NLI_{n+1}(NL) \\ a_n &= - b_n - f_n I_n(NL) \end{aligned}\right\} \tag{3.48}$$

$$\left.\begin{aligned} h_n &= \frac{1}{\pi}\frac{\cos n\alpha - \cos n\beta}{\frac{1}{2} L^2 I_n(NL) - (n+1)NLI_{n+1}(NL)} \\ d_n &= \frac{1}{2\pi}\frac{\cos n\beta - \cos n\alpha}{n} - \frac{1}{2} h_n NLI_{n+1}(NL) \\ c_n &= - d_n - h_n I_n(NL) \end{aligned}\right\} . \tag{3.49}$$

If one inserts the coefficients (3.47), (3.48) and (3.49) in the expression (3.42), then the stream function for the moving-wall problem takes the form

$$\psi = \left[\frac{(1-r^2)}{4\pi}\left\{(\beta-\alpha) + 2\sum_{n=1}^{\infty}\left(\frac{\sin n\beta - \sin n\alpha}{n}\right) r^n \cos n\theta + 2\sum_{n=1}^{\infty}\left(\frac{\cos n\alpha - \cos n\beta}{n}\right) r^n \sin n\theta\right\}\right]$$
$$+ \left[\frac{1}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(\sin n\beta - \sin n\alpha)\cos n\theta + (\cos n\alpha - \cos n\beta)\sin n\theta}{\frac{1}{2}L^2 I_n(NL) - (n+1)\, NL\, I_{n+1}(NL)}\right\} \cdot \left\{\frac{1}{2}NL\, I_{n+1}(NL)\cdot(1-r^2)r^n - I_n(NL)r^n + I_n(NLr)\right\}\right]. \tag{3.50}$$

The first term between the square brackets in (3.50) represents the solution for the problem in the case of a Newtonian fluid and the second term is the contribution due to the rotation of the fluid particles.

With regard to the convergence of the above series, one can prove in a similar manner to one used in section 3.3.1 that the series representing the stream function and the total angular velocity for the moving-wall problem are convergent in $|r| < 1$.

The regional angular velocity $\hat{W}$ is given by

$$\hat{W} = \left(0,\ 0,\ -\frac{1}{2}\left[4b_0 + f_0\ N^2L^2I_0(NLr) + \sum_{n=1}^{\infty}\{4(n+1)b_nr^n + f_n\ N^2L^2I_n(NLr)\}\cos n\theta + \sum_{n=1}^{\infty}\{4(n+1)d_nr^n + h_n\ N^2L^2I_n(NLr)\}\sin n\theta\right]\right). \quad (3.51)$$

From (3.43) and (3.51), the relative angular velocity turns out to be

$$\hat{H} = \left(0,\ 0,\ -\frac{1}{2}\ L^2(1-N^2)\sum_{n=1}^{\infty}(f_n \cos n\theta + h_n \sin n\theta)I_n(NLr)\right). \quad (3.52)$$

It is observed that when $N = 1$, the relative angular velocity vanishes and the rotation of the fluid particles is represented by the vorticity.

3.4.1 Numerical Results and Discussion

In this section computations are carried out in a special case when $\alpha = 0$ and $\beta = \pi$ which represents the rotation of the upper half of the cylinder wall while the lower half is fixed. The series (3.50) is used and truncated after 35 terms.

In figure 8 the stream-lines are sketched for a polar fluid when $N = 0.9$, $L = 6$ and also the corresponding ones for a Newtonian fluid. The values of ψ for a Newtonian fluid and for a polar fluid for various values of the parameters N and L are tabulated in tables 7 - 10. From these tables it can be noticed that, for a fixed value of r, ψ attains its maximum value at $\theta = \frac{\pi}{2}$ and its minimum value at $\theta = \frac{3\pi}{2}$. The values of ψ for a polar fluid for any values of N and L are the same as the values for a Newtonian fluid on the line $\theta = 0$ and $\theta = \pi$. It is also observed that the values of ψ in the case of a polar fluid are greater than the corresponding values of ψ for a Newtonian fluid when $0 < \theta < \pi$. The situation, however, is reversed when $\pi < \theta < 2\pi$.

From tables 8 and 9 one observes that, for fixed L but N increasing, ψ increases when $0 < \theta < \pi$ and decreases when $\pi < \theta < 2\pi$. Also it is noticed from tables 9 and 10 that, if N is fixed but L is allowed to increase, the values of ψ increase for $0 < \theta < \pi$ and decrease for $\pi < \theta < 2\pi$.

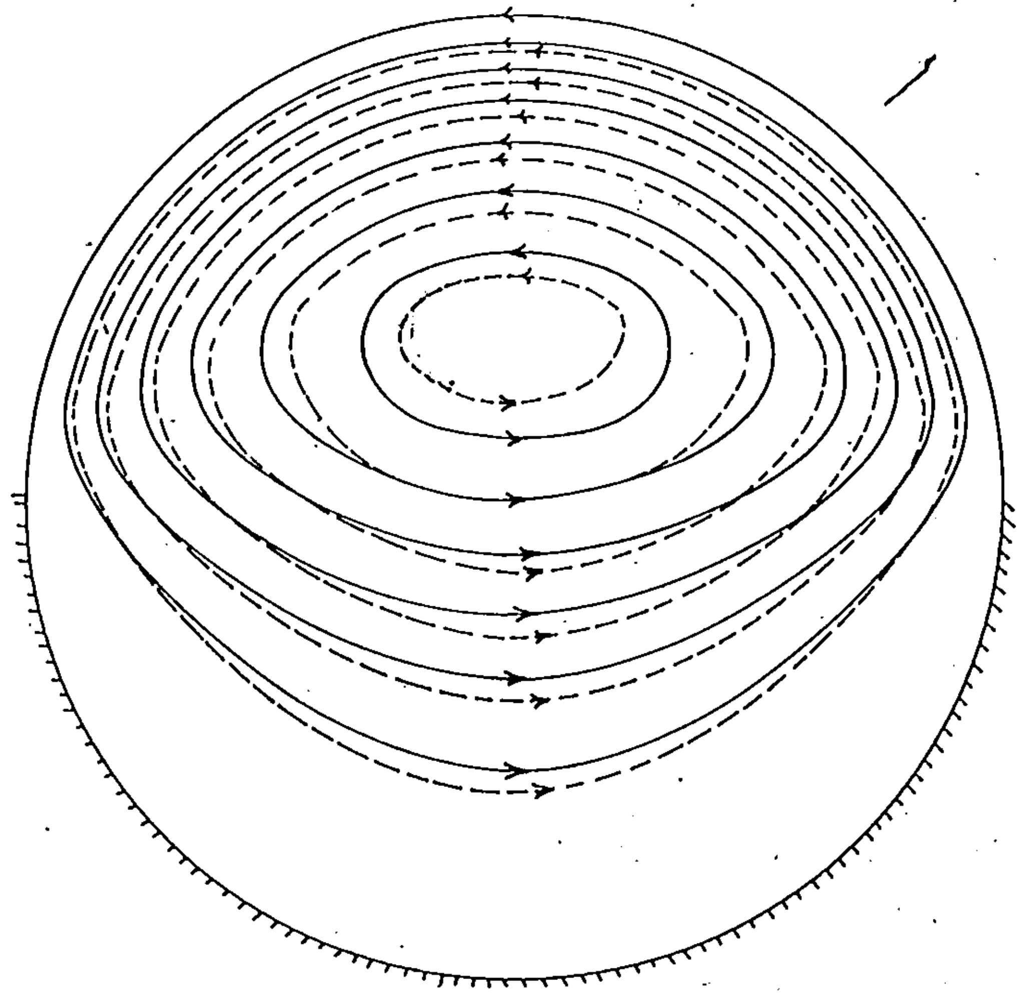

Figure 8. Stream-lines for a polar fluid when $N = 0.9$, $L = 6$ (full line) and for a Newtonian fluid (broken line).

Table 7: Values of ψ for the moving-wall problem in the case of a Newtonian fluid

r \ θ	0, 180, 360	30, 150	60, 120	90	210, 330	240, 300	270
0.1	0.247500	0.263362	0.274790	0.278908	0.231638	0.220209	0.216091
0.2	0.240000	0.271382	0.292911	0.300319	0.208618	0.187089	0.179680
0.3	0.227500	0.273622	0.302642	0.311924	0.181379	0.152358	0.143076
0.4	0.210000	0.269415	0.302202	0.311739	0.150586	0.117798	0.108260
0.5	0.187500	0.257688	0.289805	0.298188	0.117312	0.085195	0.076812
0.6	0.160000	0.236715	0.263775	0.270093	0.083285	0.056225	0.049907
0.7	0.127500	0.203892	0.222681	0.226644	0.051107	0.032319	0.028356
0.8	0.090000	0.155771	0.165435	0.167319	0.024229	0.014564	0.012681
0.9	0.047500	0.088673	0.091311	0.091801	0.006327	0.003689	0.003199
1.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

Table 8: Values of ψ for the moving-wall problem in polar fluids when $N = 0.9$, $L = 6$

r \ θ	0, 180, 360	30, 150	60, 120	90	210, 330	240, 300	270
0.1	0.247500	0.266762	0.280582	0.285542	0.228238	0.214418	0.209459
0.2	0.240000	0.278178	0.303925	0.312639	0.201822	0.176075	0.167361
0.3	0.227500	0.283725	0.317731	0.328212	0.171275	0.137269	0.126788
0.4	0.210000	0.282488	0.319673	0.329808	0.137512	0.100327	0.090192
0.5	0.187500	0.272899	0.307550	0.315734	0.102101	0. 67450	0.059266
0.6	0.160000	0.252475	0.279554	0.285052	0.067525	0.040446	0.034948
0.7	0.127500	0.217781	0.234553	0.237502	0.037219	0.020447	0.017498
0.8	0.090000	0.165083	0.172295	0.173422	0.014917	0.007705	0.006578
0.9	0.047500	0.092027	0.093493	0.093706	0.002973	0.001564	0.001293
1.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

Table 9: Values of ψ for the moving-wall problem in polar fluids when $N = 0.5$, $L = 6$

r \ θ	0, 180, 360	30, 150	60, 120	90	210, 330	240, 300	270
0.1	0.247500	0.263861	0.275648	0.279895	0.231139	0.219352	0.215105
0.2	0.240000	0.272351	0.294532	0.302160	0.207649	0.185468	0.177840
0.3	0.227500	0.274994	0.304842	0.314374	0.180006	0.150158	0.140626
0.4	0.210000	0.271084	0.304721	0.314479	0.148916	0.115279	0.105521
0.5	0.187500	0.259497	0.292335	0.300866	0.115503	0.082665	0.074134
0.6	0.160000	0.238457	0.266001	0.272386	0.081543	0.054000	0.047614
0.7	0.127500	0.205325	0.224338	0.228307	0.049675	0.030662	0.026693
0.8	0.090000	0.156678	0.166383	0.168247	0.023322	0.013617	0.011753
0.9	0.047500	0.088986	0.091608	0.092086	0.006014	0.003392	0.002913
1.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

Table 10: Values of ψ for the moving-wall problem in polar fluids when $N = 0.5$, $L = 10$

r \ θ	0, 180, 360	30, 150	60, 120	90	210, 330	240, 300	270
0.1	0.247500	0.264028	0.275932	0.280219	0.230972	0.219068	0.214781
0.2	0.240000	0.272694	0.295080	0.302769	0.207306	0.184920	0.177231
0.3	0.227500	0.275524	0.305613	0.315195	0.179476	0.149387	0.139805
0.4	0.210000	0.271801	0.305647	0.315416	0.148200	0.114353	0.104584
0.5	0.187500	0.260372	0.293318	0.301812	0.114628	0.081682	0.073188
0.6	0.160000	0.239408	0.266922	0.273233	0.080592	0.053077	0.046767
0.7	0.127500	0.206204	0.225075	0.228959	0.048796	0.029925	0.026042
0.8	0.090000	0.157297	0.166838	0.168639	0.022702	0.013162	0.011361
0.9	0.047500	0.089223	0.091764	0.092218	0.005777	0.003236	0.002781
1.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

3.5 Polar Fluid Flow Past a Cylinder

Consider the motion of an incompressible flow of a polar fluid past a circular cylinder. As in the preceding sections the motion is assumed to be a slow one, and the micro-inertia, body forces, body couples are negligible. Moreover, one can assume that $\hat{G}_1$ and $\hat{G}_2$ vanish. Hence the motion is governed by (3.12) and (3.13).

The boundary conditions of zero velocity at the surface can be written as

$$\psi(1, \theta) = \psi_r(1, \theta) = 0 \tag{3.53}$$

and the condition of uniform flow upstream is

$$\psi(r, \theta) \sim r \sin\theta \quad \text{as} \quad r \to \infty . \tag{3.54}$$

3.5.1 Solution

The boundary condition (3.54) suggests that the stream function ψ takes the form

$$\psi = f(r)\sin\theta . \tag{3.55}$$

Hence, by choosing the appropriate terms from (3.19) and (3.20) one obtains

$$\psi = \left[c_1 r + d_1 r^3 + \frac{c_1'}{r} + d_1' \, r \ln r + h_1 I_1(NLr) + h_1' K_1(NLr)\right] \sin\theta . \tag{3.56}$$

It can be shown that the total angular velocity G is given by

$$G = \left[-4d_1 r - \frac{d_1'}{r} - \frac{h_1}{2} L^2 I_1(NLr) - \frac{h_1'}{2} L^2 K_1(NLr) \right] \sin\theta \tag{3.57}$$

Employing the boundary condition (3.54) in (3.56) one finds that

$$c_1 = 1 \ , \quad d_1 = d_1' = h_1 = 0 \ .$$

Hence, the expressions (3.56) and (3.57) reduce to

$$\psi = \left\{ \frac{c_1'}{r} + r + h_1' K_1(NLr) \right\} \sin\theta \tag{3.58}$$

$$G = -\frac{h_1'}{2} L^2 K_1(NLr) \sin\theta \tag{3.59}$$

The expressions (3.58) and (3.59) contain two constants c_1' and h_1' which can be determined from the boundary conditions. However, there are two conditions on ψ and a third condition for the total angular velocity G which is more than the number of unknowns. To examine the consistency of the equations and the conditions, one can employ the conditions (3.53) in (3.58), leading to

$$c_1' + h_1' K_1(NL) = -1$$

$$-c_1' + h_1' NLK_1'(NL) = -1 \ .$$

The last two equations imply that

$$h_1' = \frac{2}{NLK_0(NL)} . \tag{3.60}$$

and

$$c_1' = -\left(1 + \frac{2K_1(NL)}{NLK_0(NL)}\right) \tag{3.61}$$

But if one uses (3.60) in (3.59), the no-spin condition on the boundary is not satisfied. Hence, one concludes that the equations of creeping polar fluid flow past a circular cylinder have no solution of the form (3.55) which satisfies both the far and near boundary conditions, simultaneously.

3.6 Conclusion

Summing up the results of this chapter, one observes that basic feature of polar and Newtonian fluids for internal and external creeping flows for a circular cylinder are similar. For both types of fluids, it is noticed that no external creeping flow past a circular cylinder exists while the solutions are possible for interior problems. For internal flows, the stream function for polar and Newtonian fluids are different due to the consideration of the rotation of the particles and hence the stream-lines are deviated from each other. This deviation is found to be strongly dependent on the coupling number as well as on the length ratio.

CHAPTER IV

HAMEL FLOW OF POLAR FLUIDS

4.1 Introduction

Hamel [18] considered a steady two-dimensional flow of an incompressible viscous fluid in a channel bounded by two infinite non-parallel plane walls due to a sink or a source at the intersection of the walls. He found that a purely radial flow can be obtained which satisfies exactly both equations of motion and boundary conditions. Hamel expressed his solution in terms of elliptic functions. Further contributions to this problem were made by various authors leading up to a very extensive treatment by Rosenhead [26].

Langlois [20] considered the analogous problem for a three-dimensional flow of a viscous fluid and showed that there can be no purely radial flow in a cone in the absence of body forces, and that there must be a component of velocity in the θ-direction.

In the present work the existence of a purely radial flow of a polar fluid in convergent and divergent channels is investigated. The possibility of exhibiting a secondary flow in the case of a creeping flow of a polar fluid between two non-parallel plates is also studied.

4.2 Radial flow in converging and diverging channels

Consider that an incompressible polar fluid is contained in the trough between two non-parallel walls. Consider further that a line source (or sink) of uniform output Q per unit length lies along the line of intersection of the walls as shown in Figure 9.

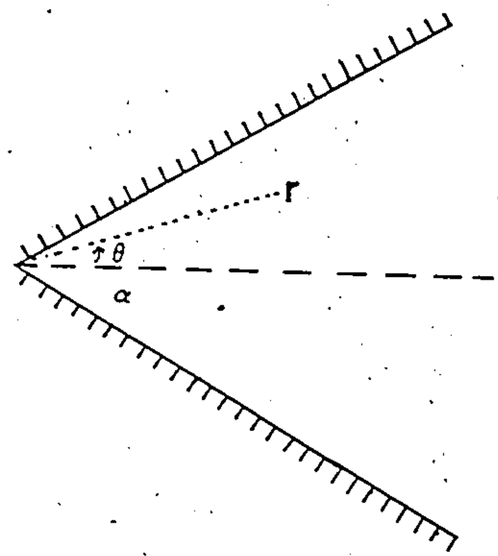

Figure 9. Hamel flow problem.

Let a plane polar coordinate system r, θ be defined so that the walls correspond to $\theta = \pm\alpha$. For a radial flow one seeks a solution of the equations of motion of a polar fluid with $v_r \neq 0$, $v_\theta = 0$ and the boundary conditions

$$v_r = 0 \quad \text{at} \quad \theta = \pm\alpha \tag{4.1}$$

along with the volume flow condition

$$\int_{-\alpha}^{\alpha} r v_r d\theta = \pm Q \tag{4.2}$$

If one assumes that the body forces and the body couples are negligible and $\hat{\underset{\sim}{G}} = (0, 0, G)$ then the governing equations of the radial flow are given by

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} = 0 , \tag{4.3}$$

$$\rho v_r \frac{\partial v_r}{\partial r} = - \frac{\partial p}{\partial r} + (\mu+\tau) \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} \right) + 2\tau \frac{1}{r} \frac{\partial G}{\partial \theta} \tag{4.4}$$

$$0 = - \frac{1}{r} \frac{\partial p}{\partial \theta} + (\mu+\tau) \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - 2\tau \frac{\partial G}{\partial r} , \tag{4.5}$$

$$\rho k^2 v_r \frac{\partial G}{\partial r} = (\beta+\gamma) \left(\frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} + \frac{1}{r^2} \frac{\partial^2 G}{\partial G^2} \right) - 4\tau G - 2\tau \frac{1}{r} \frac{\partial v_r}{\partial \theta} . \tag{4.6}$$

The continuity equation (4.3) requires that

$$v_r = \frac{f(\theta)}{r} . \tag{4.7}$$

Using (4.7) in (4.4) - (4.6) and eliminating the pressure one gets

$$(\mu+\tau)f''' + 4(\mu+\tau)f' + 2\rho ff' + 2\tau r^4\nabla^2 G = 0 \,, \tag{4.8}$$

$$\left(\nabla^2 - \frac{4\tau}{\beta+\gamma}\right)G - \frac{2\tau}{\beta+\gamma}\frac{f'}{r^2} = \frac{\rho k^2}{\beta+\gamma}\frac{f}{r}\frac{\partial G}{\partial r} \tag{4.9}$$

where prime denotes derivative with respect to θ .

The solution of (4.8) and (4.9) is investigated in the following two cases:

Case I: When micro-inertia is negligible

When the micro-inertia terms of the particles vanish equation (4.9) takes the form

$$\left(\nabla^2 - \frac{4\tau}{\beta+\gamma}\right)G - \frac{2\tau}{\beta+\gamma}\frac{f'}{r^2} = 0 \,. \tag{4.10}$$

On eliminating G between (4.8) and (4.10) one obtains, after some simplification,

$$f^{(V)} + [20 + r^2(4\lambda_1^2 - \lambda^2)]f''' + 4[16 + r^2(4\lambda_1^2 - \lambda^2)]f'$$

$$+ \frac{\rho}{\mu+\tau}(16 - \lambda^2 r^2)(f^2)' + \frac{2\rho}{\mu+\tau}(ff')'' = 0 \tag{4.11}$$

where

$$\lambda^2 = \frac{4\tau}{\beta+\gamma}$$

and

$$\lambda_1^2 = \frac{\tau^2}{(\beta+\gamma)(\mu+\tau)} .$$

Integration of equation (4.11) once with respect to θ yields

$$f^{(IV)} + [20 + r^2(4\lambda_1^2-\lambda^2)]f'' + 4[16 + r^2(4\lambda_1^2-\lambda^2)]f$$

$$+ \frac{\rho}{\mu+\tau}(16 - \lambda^2 r^2)f^2 + \frac{2\rho}{\mu+\tau}(ff')' = h(r) \qquad (4.12)$$

where h is an arbitrary function of r only.

Equation (4.12) can be rewritten as

$$\left\{f^{(IV)} + 4f'' + \frac{2\rho}{\mu+\tau}(ff')'\right\} + 16\left\{f'' + 4f + \frac{\rho}{\mu+\tau} f^2\right\}$$

$$+ r^2\left[(4\lambda_1^2-\lambda^2)f'' + 4(4\lambda_1^2-\lambda^2)f - \frac{\rho}{\mu+\tau}\lambda^2 f^2\right] = h(r) . \qquad (4.13)$$

Recalling that

$$4\lambda_1^2 - \lambda^2 = -\frac{\lambda^2\mu}{\mu+\tau} ,$$

one finds that (4.13) takes the form

$$\left(\frac{d^2}{d\theta^2} + 16\right)\left(f'' + 4f + \frac{\rho}{\mu+\tau} f^2\right) - \frac{\lambda^2\mu}{\mu+\tau} r^2 \left(f'' + 4f + \frac{\rho}{\mu} f^2\right) = h(r).$$

$$(4.14)$$

The first term on the left hand side of (4.14) and the expression $(f'' + 4f + \frac{\rho}{\mu} f^2)$ are functions of θ only. Hence one must have

$$\left(\frac{d^2}{d\theta^2} + 16\right)\left(f'' + 4f + \frac{\rho}{\mu+\tau} f^2\right) = c_1 \tag{4.15}$$

$$f'' + 4f + \frac{\rho}{\mu} f^2 = c_2 \; , \tag{4.16}$$

where c_1 and c_2 are arbitrary constants. From equation (4.15) one obtains

$$f'' + 4f + \frac{\rho}{\mu+\tau} f^2 = c_3 \cos 4\theta + c_4 \sin 4\theta + \frac{c_1}{16} \; . \tag{4.17}$$

Equations (4.16) and (4.17) imply that

$$\frac{\rho\tau}{\mu(\mu+\tau)} f^2 = c_2 - \frac{c_1}{16} + c_3 \cos 4\theta + c_4 \sin 4\theta \; . \tag{4.18}$$

The last equation determines f in terms of the unknown arbitrary constants. However, this value of f, in its general form, does not satisfy (4.16). Hence one can conclude that there does not exist a solution for f of the type given by (4.18), which satisfies simultaneously (4.16) and (4.17).

When the constants c_3 and c_4 are equal to zero, then (4.17) reduces to

$$f'' + 4f + \frac{\rho}{\mu+\tau} f^2 = \frac{c_1}{16} . \tag{4.19}$$

Equation (4.16) can be written as

$$f'' + 4f + \frac{\rho}{\mu+\tau} f^2 + \frac{\rho\tau}{\mu(\mu+\tau)} f^2 = c_2 . \tag{4.20}$$

From (4.19) and (4.20) one finds that

$$\frac{\rho\tau}{\mu(\mu+\tau)} f^2 = c_2 - \frac{c_1}{16} ,$$

which means that $f(\theta)$ is a constant. Hence, the boundary condition (4.1) implies that $f(\theta)$ is identically zero.

Only when $\tau = 0$, it is noticed that (4.16) and (4.19) are consistent if $c_1 = 16c_2$ and the velocity component in this case is the same as the velocity in the case of a Newtonian fluid.

The total angular velocity G is then determined independently by solving Laplace equation

$$\nabla^2 G = 0 \tag{4.21}$$

subject to the no-spin condition on the walls $\theta = \pm \alpha$ and G remaining finite as r increases indefinitely. The solution of (4.21) can be obtained by the method of separation of variables and in the present case the solution is not unique. One possible solution is of the form

$$G = \sum_{n=1}^{\infty} \frac{B_n}{r^{\frac{n\pi}{\alpha}}} \sin \frac{n\pi}{\alpha} \theta \tag{4.22}$$

where B_n are arbitrary constants.

Case II: <u>When micro-inertia terms do not vanish</u>

The presence of the micro-inertia terms makes it difficult to eliminate f between the equations

$$(\mu+\tau)f''' + 4(\mu+\tau)f' + 2\rho ff' + 2\tau r^4\nabla^2 G = 0 \tag{4.8}$$

$$\left(\nabla^2 - \frac{4\tau}{\beta+\gamma}\right) G - \frac{2\tau}{\beta+\gamma} \frac{f'}{r^2} = \frac{k^2\rho}{\beta+\gamma} \frac{f}{r} \frac{\partial G}{\partial r} \; . \tag{4.9}$$

Equation (4.8) suggests that G can be assumed to be of the form

$$G(r, \theta) = \frac{g(\theta)}{r^2} \; . \tag{4.23}$$

Using (4.23) in (4.8) and (4.9) one gets

$$(\mu+\tau)f''' + 4(\mu+\tau)f' + 2\rho ff' + 2\tau\{g'' + 4g\} = 0 \tag{4.24}$$

$$g'' + 4g + \frac{2k^2\rho}{\beta+\gamma} fg - \frac{2\tau}{\beta+\gamma} r^2\{2g + f'\} = 0 \; . \tag{4.25}$$

From (4.25) one deduces that

$$2g + f' = 0 \qquad (4.26)$$

and

$$g'' + 4g + \frac{2k^2\rho}{\beta+\gamma} fg = 0 \,. \qquad (4.27)$$

Using (4.26) in (4.24) and (4.27), one finds

$$f''' + 4f' + \frac{2k^2\rho}{\beta+\gamma} ff' = 0 \qquad (4.28)$$

and

$$f''' + 4f' + 2\frac{\rho}{\mu} ff' = 0 \qquad (4.29)$$

respectively.

When $k^2\mu = (\beta+\gamma)$, equations (4.28) and (4.29) are identical and coincide with the governing equation of the radial flow for a Newtonian fluid. Hence in this case the solution reduces to Hamel solution for viscous fluids.

One concludes therefore that if the micro-inertia terms are not negligible the solution to this problem is identically zero unless $k^2\mu = (\beta+\gamma)$, in which case the solution is the same as for a Newtonian fluid.

Some remarks as to why, in the case of a polar fluid, there does not exist a radial flow are in order. In the case of a viscous fluid one notes that the relevant physical parameters are the fluid density, its viscosity, the half-angle α and the source output. The dimensions of these quantities are

$$[\rho] = ML^{-3} \quad , \quad [\mu] = ML^{-1}T^{-1}$$

$$\alpha : \text{dimensionless} \qquad [Q] = L^2T^{-1} \quad .$$

No combination of these parameters yield a length. However, in the case of a polar fluid the combination $\frac{Q\rho}{\beta}$ is a length, where β is a gradient viscosity of dimension MLT^{-1}. Thus from the point of view of dimensional considerations, the differential equations for the polar fluids have the characteristic length inherent in them. It is because of this reason, that one can possibly assume the occurrence of a secondary type of flow, as opposed to a radial flow, in the case of polar fluids. This suggests that one can include the θ-component of the velocity into the analysis. Because of the complicated nature of the resulting equations, only solutions to the creeping flow are studied in the following section.

4.3 Creeping Flow in Converging and Diverging Channels

In this section it is assumed that there is a tangential component of the velocity in addition to the radial one. Hence the motion is a two-dimensional one in which $v_r \neq 0$, $v_\theta \neq 0$. One assumes that the flow is a creeping, body force, body couple, and micro-inertia are all negligible. It is also possible to assume that $\underset{\sim}{G} = (0, 0, G)$. Hence the equations governing the motion are given by

$$\nabla^4\psi + 2N^2\nabla^2 G = 0 \qquad (3.12)$$

$$(\nabla^2-\lambda^2)G - \frac{\lambda^2}{2}\nabla^2\psi = 0 \qquad (3.13)$$

where

$$\lambda^2 = \frac{N^2L^2}{1-N^2}, \quad N \neq 1 \ . \qquad (3.14)$$

The boundary conditions are

$$v_r(r, \pm\alpha) = 0 \quad , \quad v_\theta(r, \pm\alpha) = 0 \qquad (4.29)$$

along with the volume flow condition

$$\int_{-\alpha}^{\alpha} r v_r d\theta = \pm Q \ . \qquad (4.30)$$

The conditions (4.29) and (4.30) can be expressed in terms of the stream function ψ as

$$\left.\frac{\partial\psi}{\partial\theta}\right|_{\theta=\pm\alpha} = 0 \ , \quad \left.\frac{\partial\psi}{\partial r}\right|_{\theta=\pm\alpha} = 0 \qquad (4.31)$$

and

$$\psi(r, \alpha) - \psi(r, -\alpha) = \pm Q \ . \qquad (4.32)$$

If one eliminates G between (3.12) and (3.13) and puts in consideration that ψ is an odd function in θ, one finds

$$\psi_s = [A_s r^{-s} + B_s r^{-s+2} + c_s K_s(NLr)] \sin s\theta . \tag{4.33}$$

In (4.33) the positive powers of r, $\ln r$ and $I_s(NLr)$ were excluded because of their behaviour at infinity. On employing the conditions (4.31) and (4.32) one finds that

$$s[A_s r^{-s} + B_s r^{-s+2} + c_s K_s(NLr)] \cos s\alpha = 0 \tag{4.34}$$

$$[-sA_s r^{-s-1} + (-s+2)B_s r^{-s+1} + c_s NLK_s'(NLr)] \sin s\alpha = 0 \tag{4.35}$$

and

$$[A_s r^{-s} + B_s r^{-s+2} + c_s K_s(NLr)] \sin s\alpha = \pm \frac{Q}{2} . \tag{4.36}$$

Equation (4.34) implies that either

$$A_s r^{-s} + B_s r^{-s+2} + c_s K_s(NLr) = 0$$

or $\cos s\alpha = 0$.

The first of these possibilities shows that the solution is the trivial one while the second one implies that

$$s = \frac{n\pi}{2\alpha} , \quad n = 1, 3, 5, \ldots .$$

Using this in (4.36) one finds that

$$[A_s r^{-s} + B_s r^{-s+2} + c_s K_s(NLr)] \sin \frac{n\pi}{2} = \pm Q \ . \qquad (4.37)$$

But (4.37) cannot be true for all values of r. Hence the equations and the conditions in the present case are not compatible and there is no solution for such motion.

4.4 Conclusion

It can be concluded that there exists no solution for Hamel flow in the case of a polar fluid except in two trivial cases. The first of these is when the micro-inertia vanishes and the rotational viscosity is equal to zero. The second case is when the micro-inertia is not negligible and if the material coefficients are related by $k^2\mu = (\beta+\gamma)$. In both the cases the velocity component is the same one as for a Newtonian fluid.

Also, it has been shown that the coupling between the particles does not exhibit a secondary flow for creeping polar fluid flow between two non-parallel plates.

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