Optimal control of step frequency transitions in a phaselock loop.

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OPTIMAL CONTROL OF STEP FREQUENCY TRANSITIONS
IN A PHASELOCK LOOP

by

GEORGE J. TOMKO

A Thesis
Submitted to the Faculty of Graduate Studies through the Department of Electrical Engineering in Partial Fulfillment of the Requirements for the Degree of Master of Applied Science at the University of Windsor

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ABSTRACT

Pontryagin's maximum principle has been used to derive the conditions of optimality for minimum time frequency transitions in a 2nd order phaselock loop with a loop filter transfer function of the form

\[ \frac{1 + ts}{1 + ats} \quad (a > 1). \]

The equations were solved on a digital computer using an analogue simulation program. The results were implemented in a preliminary design for controlling a microwave phaselocking oscillator. Tests have highlighted the need for specific refinements and means of providing them have been suggested.
ACKNOWLEDGEMENTS

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GLOSSARY OF SYMBOLS

\( \tau = \text{RC time constant of loop filter} \)

\( \alpha = \frac{C_1 + C_2}{C_1} \quad \text{a capacitance ratio of the loop filter} \)

VCO = Voltage - controlled oscillator

\( \phi_s = \text{Input phase (radians)} \)

\( \phi_o = \text{VCO output phase (radians)} \)

\( V_s = \text{Amplitude of the input (volts)} \)

\( V_o = \text{Amplitude of the VCO output (volts)} \)

\( \varepsilon = \text{Error output of the phase comparator (volts)} \)

\( K_m = \text{Phase comparator constant of proportionality} \)

\( K_o = \text{Phase comparator gain (volts/rad.)} \)

\( K_1 = \text{DC gain of loop filter} \)

\( K_2 = \text{VCO gain (rad/sec/volt)} \)

\( K = \text{Overall loop gain, } K_o K_1 K_2 \), (rad/sec)

\( \omega_c = \text{VCO center frequency (rad/sec)} \)

\( . = \frac{d}{dt} \)

\( s' = \frac{d}{dt} \)

\( s = \text{Laplace operator} \)

\( v_2 = \text{Output of loop filter (volts)} \)

\( \phi_{ss} = \text{Steady state value of phase error due to step change in frequency (radians)} \)

\( \nu (o) = \text{Initial frequency of input signal (rad/sec)} \)
$U_F$ = Magnitude of frequency control

$X_1$ = Phase error (radians)

$X_2$ = Frequency error (rad/sec)

$X_3$ = Performance criterion to be minimize (time) (sec)

$Y$ = Phase of phase error at $t = 0$ (radians)

$\psi$ = Input signal phase state variable (radians)

$\nu$ = Filter output frequency state variable (rad/sec)

$\phi_0$ = VCO output phase state variable (radians)

$A$ = State variable matrix

$B$ = State control matrix

$L^{-1}$ = Inverse Laplace transformation

$I$ = Identity matrix

$\phi$ = Error phase (radians)

$\zeta$ = Damping factor

$\omega_n$ = Undamped natural frequency of loop (rad./sec.)

$\Delta \omega$ = Step change in frequency (rad./sec.)

$u(t)$ = Heaviside step function

$u_i$ = Control vectors

$u_f$ = Frequency control vector (volts)

$u_p$ = Phase control vector (volts)

$P$ = Pontryagin's function

$R_K$ = Constraint placed on system

$\mu$ = Lagrange multiplier
\( t_f \) = Final time
\( t' \) = Simulated time
\( t_s \) = Time of first switch for control vector
\( H \) = Hamiltonian
\( \mathbf{p} \) = Costate vector
\( U \) = Amplitude of control signal
\( \theta \) = State transition matrix
\( \tau' \) = Dummy time variable
\( \alpha' = \frac{\alpha - 1}{\alpha} \)
CHAPTER I

INTRODUCTION

1.1 The General Problem

During the past ten years a considerable amount of work has been carried out on the application of methods of optimization* to a wide range of physical and engineering problems. One field of research which has shown promise in this regard is the area of communications. By applying the methods of optimal control to communication systems a wide variety of problems are being solved which in the past were too difficult to solve by normal methods.

This paper studies the methods of optimal control - specifically the application of the classical calculus of variations applied to a phaselock receiver with the object of minimizing the criterion of phase and frequency error settling time when a step change in frequency is introduced into the input.

The mathematical theory of optimal control presupposes that the equations which describe the behaviour of the dynamic system to be controlled are in the form of either vector differential equations or vector difference equations. This necessitated the return to the time domain and the development of the state space description of a dynamical system. The two main approaches to the control optimization problem have been Bellman's dynamic programming method which is based on the principle of optimality and Pontryagin's maximum principle (1) which is an extension and application of the classical calculus of variations to the optimal control problem.

*A simple explanation of the meaning of optimization is that a system is designated to operate in such a fashion that some criterion is a minimum or maximum. The criterion used depends on the application. The most commonly used are minimum time, minimum fuel, maximum range and minimum energy.
The dynamic programming method was originally developed for discrete time problems, and later it was applied to continuous time problems, being referred to as the Hamilton-Jacobi-Bellman theory. Its major disadvantage lies in the large computer memory requirements.

The maximum principle was originally developed for continuous time problems. Its major disadvantage is that it provides, in general, only local necessary conditions for optimality. Its computational requirements are not as severe as those associated with dynamic programming.

It must be noted that the maximum principle gives only a set of necessary conditions on the existence of optimal controls. If a control satisfies the maximum principle it is classified as an extremal control. Only in special cases does the maximum principle provide a sufficiency condition. Extremal controls are usually locally, but not globally*, optimal. To find the globally optimal control, the value of the performance criterion for each extremal control is calculated. The extremal control which gives the optimum value for the performance criterion is the optimal control, for if an optimal control exists, then it is, of course, an extremal control.

Under certain conditions, however, the maximum principle is both a necessary and sufficient condition for optimality. It is for these types of systems that the maximum principle offers a decisive advantage over other methods.

*A control is globally optimal if it satisfies the condition of optimality over the entire state space.
1.2 Application to a Phaselock Loop

A common method used in tracking the frequency of an external signal is the phaselock loop. Criteria of interest in the performance of the loop are the lock-in range, and the time required for a loop to lock to a change in frequency of the input signal.

Both transients in phase and frequency take place in the loop as the frequency of the loop voltage controlled oscillator makes the frequency transition corresponding to the change in the frequency of the input signal. The times for these transients to decay determine the time to achieve lock. Therefore it would be desirable to control the phase or frequency of the input signal during the frequency transition in an optimum manner, so that the transients settle more rapidly.

Jaffe and Rechtin (2) performed the optimization of a second order loop in response to a white noise input plus a step change in the frequency of the input signal using Wiener filtering theory. The performance criterion minimized was the variance of the noise error with the transient integral squared error constrained. This procedure resulted in the type II second order loop having a damping ratio of 0.707.

Gupte and Sanneman (3) used this type II second order loop with a step change of frequency in the input signal in an analysis which utilized the maximum principle in decreasing the phase and frequency transients to zero in minimum time.

In the present work the lock-in time of a second order loop with a phase lag filter of the form \( \frac{1 + ts}{1 + a ts} \) is minimized for a step change in frequency of the input signal. It is assumed that the phaselock loop
initially tracks the input signal with zero phase error and that the input signal is noise free. The frequency of the signal source will be controlled such that transients in the loop decay to zero in minimum time as the loop VCO changes frequency by an amount equal to the signal source frequency change. Phase control of the input signal is analyzed theoretically, but only frequency control is experimentally implemented.

The phaselock loop incorporated in the experimental portion operates at microwave frequencies, with a klystron oscillator acting as the VCO.
2.1 A Review of the Phaselock Loop

A phaselock loop (4) contains three basic components (Figure 2.1); a phase comparator, a low pass filter and a voltage-controlled oscillator (VCO).

The phase comparator compares the phase of a periodic input signal against the phase of the VCO. When the phase differs from 90° (a condition which will exist when the input signal tries to change frequency) the phase comparator will immediately sense the phase difference and provide a dc error output voltage proportional to the cosine of the phase difference. The phase comparator does not permit a steady state frequency error to be developed between the VCO and the input signal. This is due to the integration of the frequency changes by the phase comparator which produces the error voltage proportional to phase difference.
This error voltage is applied to a low pass filter where high frequency components are suppressed. This filter also determines the dynamic performance of the loop.

The output of the loop filter is applied to the VCO to control its frequency about the center frequency. The center frequency of the klystron oscillator is determined by a mechanical tuner and by the reflector voltage. The error voltage output from the filter is summed with the reflector voltage from the klystron power supply and fed to the reflector of the klystron oscillator to control the output frequency of the klystron.

An analytic block diagram showing the operations of a phaselock loop is illustrated in Figure 2.2 (5).

\[ V_o \cos \left[ \omega_c t + K K_1 K_2 \frac{F(s')}{s'} \sin \phi \right] \]

Fig. 2.2 Analytic Block Diagram of Phaselock Loop
2.2 Analysis of Loop Equations

Consider the loop shown in Figure 2.2. The input signal has a phase of \( \phi_s(t) \), and the VCO output has a phase \( \phi_o(t) \). It is assumed that the loop is locked and the phase comparator is a perfect multiplier.

Suppose that the input signal is
\[
v_i(t) = V_s \sin(\phi_s)
\]
and the VCO waveform** is
\[
v_o(t) = V_v \cos(\phi_v)
\]
Output of the phase comparator (neglecting double frequency terms that will be removed by the loop filter) is:
\[
e = \frac{K m V_s V_v}{2} \sin \left[ 90^\circ - (\phi_s - \phi_v) \right]
\]
or
\[
e = \frac{K m V_s V_v}{2} \sin (\phi_s - \phi_v)
\]
where \( K_m \) is a constant of proportionality of phase comparator output, i.e., the output of the phase comparator is \( K_m V_v V_o \).

Using the notation of Figure 2.2
\[
K_o = \frac{K m V_s V_v}{2}
\]
Therefore,
\[
e = K_o \sin (\phi_s - \phi_v)
\]
\[
e = K_o \sin \phi
\]
**Note that \( v_i \) and \( v_o \) are really \( 90^\circ \) out of phase with one another.

The input has been written as sine and the VCO output is a cosine. The two phases \( \phi_s \) and \( \phi_o \) are referred to these quadrature references.
or \( \epsilon = K_o \sin (\phi_s(t) - \omega_c t) - K K_1 K_2 \frac{F(s)}{s'} \sin \phi \)  \hspace{1cm} (2-3)

Comparison of equations (2-2) and (2-3) yields

\[ \phi = \phi_s(t) - \omega_c t - K K_1 K_2 \frac{F(s)}{s'} \sin \phi \]

differentiating with respect to time and defining \( K = K_o K_1 K_2 \), the overall loop gain, gives

\[ \dot{\phi} + K F(s) \sin \phi = \dot{\phi}_s - \omega_c \]  \hspace{1cm} (2-4)

The transfer function for the loop filter in the present system is

\[ F(s) = \frac{1 + Ts}{1 + \alpha Ts} \]

where \( \tau \) is a time constant and \( \alpha \) is a factor of proportionality greater than one.

Equation (2-4) becomes

\[ \ddot{\phi} (1 + \alpha \tau s) + K (1 + \tau s) \sin \phi = (\dot{\phi}_s - \omega_c) (1 + \alpha \tau s) \]

or

\[ \ddot{\phi} + \frac{\dot{\phi}}{\alpha} + \frac{K \alpha}{\alpha} \sin \phi + \frac{K}{\alpha} \sin \phi = \ddot{\phi}_s + (\dot{\phi}_s - \omega_c) \frac{1}{\alpha} \]  \hspace{1cm} (2-5)

Since it was assumed that the loop was locked the linear approximation \( \sin \phi + \phi \) can be made. This gives

\[ \ddot{\phi} + \left( \frac{K + \frac{1}{\alpha}}{\alpha} \right) \dot{\phi} + \frac{K}{\alpha} \dot{\phi} = \ddot{\phi}_s + (\dot{\phi}_s - \omega_c) \frac{1}{\alpha} \]  \hspace{1cm} (2-6)

The left hand side of equation (2-6) can be compared to the second order equation in servo terminology.
\[ \ddot{\phi} + 2 \zeta \omega_n \dot{\phi} + \omega_n^2 \phi \]

Equating appropriate terms gives

\[ \omega_n = \sqrt{\frac{K}{\alpha \tau}} \]  

(2-7)

and

\[ \zeta = \frac{\omega_n}{2} \left( \frac{Kr + 1}{K} \right) \]  

(2-8)

substituting into equation (2-5) gives

\[ \ddot{\phi} + \omega_n^2 \left( \frac{1}{K} + \tau \cos \phi \right) \dot{\phi} + \omega_n^2 \sin \phi = \ddot{\phi}_s + (\dot{\phi}_s - \omega_c) \frac{\omega_n^2}{K} \]  

(2-9)

The input frequency \( \dot{\phi}_s \) is composed of the initial frequency plus the step change of frequency

\[ \dot{\phi}_s = \omega_c + \Delta \omega . u(t) \]  

(2-10a)

where

\[ u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \]

differentiating (2-10a) with respect to time gives

\[ \ddot{\phi}_s = \Delta \omega \frac{d}{dt} u(t) \]  

(2-10b)

On substitution of equations (2-10a and b) into equation (2-9) one obtains

\[ \ddot{\phi} + \omega_n^2 \left( \frac{1}{K} + \tau \cos \phi \right) \dot{\phi} + \omega_n^2 \sin \phi = \ddot{\phi}_s + \Delta \omega \frac{d}{dt} u(t) + \omega_n^2 \frac{u(t)}{K} \]

or upon integrating with respect to time,
\[ \dot{\phi} = \Delta \omega \left[ u(t) + \frac{\omega_n^2}{K} t \right] - \frac{\omega_n^2}{K} \phi - \omega_n^2 \tau \int_0^t \cos \phi \, dt - \omega_n^2 \int_0^t \sin \phi \, dt. \]

Integrating the third term on the R.H.S. we obtain

\[ \dot{\phi} = \Delta \omega \left[ u(t) + \frac{\omega_n^2}{K} t \right] - \frac{\omega_n^2}{K} \phi - \omega_n^2 \tau \sin \phi - \omega_n^2 \int_0^t \sin \phi \, dt \quad (2-11) \]

where integration has eliminated the "infinite discontinuity" \( \frac{du}{dt} \).

Equation (2-11) is programmed on a digital computer using analogue simulation and the resulting phase plane and time trajectories for an uncontrolled input step change in frequency are obtained. (See Chapter III).
2.2.1 Linear Analysis of Transient Phase Error

The output of the phase comparator is given as (equation 2-2)

\[ \epsilon = K_o \sin (\phi_s - \phi_o) \]  

(2-12)

now

\[ \phi_o = K_2 v_2 \]  

(2-13)

where \( v_2 \) is the output of the filter.

Transposing to the frequency domain gives

\[ \phi_o (s) = \frac{K_2 v_2 (s)}{s} \]  

(2-14)

Referring to Figure 2.2 the output of the filter is given as

\[ V_2 (s) = F(s) \epsilon (s) \]

\[ = K_o \left[ \phi_s (s) - \phi_o (s) \right] F(s) \]  

(2-15)

where the linear approximation \( \sin \phi \approx \phi \) for \( \phi << \frac{\pi}{2} \) is included.

Substituting equation (2-15) into (2-14) and performing the required algebra yields

\[ \frac{\phi_o (s)}{\phi_s (s)} = \frac{K_o K_2 F(s)}{s + K_o K_2 F(s)} \]

and

\[ \frac{\phi_s (s) - \phi_o (s)}{\phi_s (s)} = \frac{s}{\phi_s (s)} = \frac{s}{s + K_o K_2 F(s)} \]  

(2-16)

where \( \phi \) is the phase error.

For a step change of frequency of magnitude \( \Delta \omega \)

\[ \phi_s (s) = \frac{\Delta \omega}{s^2} \]  

(2-17)
Substituting equation (2-17) into (2-16) and applying equations (2-7) and (2-8) gives

\[ \phi(s) = \frac{\Delta \omega (s + \frac{\omega_n^2}{K})}{s(s^2 + 2\zeta \omega_n s + \omega_n^2)} \]

Transforming back into the time domain yields

\[ \phi(t) = \Delta \omega \left[ \frac{1}{K} + \frac{1}{\omega_n^2 \sqrt{1 - \zeta^2} \left( \frac{\omega_n^2}{K^2} - \frac{2\zeta \omega_n}{K} + 1 \right)^{1/2}} \right] e^{-\zeta \omega_n t}

\[ \sin \left( \sqrt{1 - \zeta^2} \omega_n t + \gamma \right) \]

(2-18)

where

\[ \gamma = \left( \tan^{-1} \left( \frac{\sqrt{1 - \zeta^2}}{\zeta} \right) - \tan^{-1} \left( \frac{1 - \zeta^2}{\omega_n K} \right) \right) \]

The steady state value of phase error is

\[ \phi_{ss} = \frac{\Delta \omega}{K} \]

The time for the transient phase error to decay to zero varies with the values of \( \zeta \) and \( \omega_n \), the loop parameters. Once the loop is chosen the decay time depends only on the magnitude of the frequency step \( \Delta \omega \). Therefore to obtain small lock in times a compromise has to be made in the size of the frequency step. This would handicap most communication systems dependent on step frequency transitions.

2.3 State Variable Analysis of Loop

Two methods of controlling the signal source are considered. The first method is termed phase control and consists of controlling the phase
of the output of the signal source following a step change in frequency.
The second method, frequency control, consists of controlling only the
frequency of the signal source. The emphasis is centered on the latter
method of control in this paper. The two methods of control are
illustrated in Figures (2.3a and b)\textsuperscript{++}.

\textbf{Fig. 2.3 Signal Source} \\
(a) Physical System
(b) Voltage Analogue

\textsuperscript{++} $u_p(t)$, $\Delta \omega$, $u_p(t)$ and $v(o)$ may be considered as voltages which are
summed and fed into a VCO. The dimension of the gain constant for the
VCO is $\frac{mc}{\text{volt}}$.
During phase control, the oscillator frequency is stepped by an amount $\Delta \omega$ at $t = o$, the oscillator output is phase shifted by an amount $u_p(t)$ for $t > o$, and $u_p(t) = o$. During frequency control, the oscillator frequency is stepped by an amount $\Delta \omega$ at $t = o$, and is additionally changed by $u_p(t)$ for $t > o$. In this case $u_p(t) = o$.

For frequency control, the frequency of the oscillator is

$$\Delta \omega + u_p(t), \ t > o$$

For phase control, the phase of the phase shifter output is

$$\phi_s = u_p(t) + \int_{0}^{t} (\Delta \omega + \nu(o)) \ dt$$

$$= u_p(t) + (\Delta \omega + \nu(o)) t, \ t > o.$$  

The initial frequency of the input signal $\nu(o)$ is made equal to the VCO center frequency $\omega_c$ in order to eliminate tuning error from the analysis.

A block diagram showing the input phase $\phi_s$ and the phase comparator non-linearity is illustrated in Figure 2.4.

![Phase Lock Loop Diagram](image)

**Fig. 2.4 Phaselock Loop**

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The loop shown in Figure 2.4 is reconstructed with the signal source included as shown in Figure 2.5. This representation is known as the state variable diagram (6), and the output of each integrator is called a state variable. The state variables chosen are $\psi$, $v$, and $\phi_0$ as shown in Figure 2.5. The loop filter has been represented in terms of summers, amplifiers and integrators using the method of direct programming (6).

In the theoretical analysis the initial frequency of the input signal, $v(0)$, and the center frequency of the VCO, $w_c$, can be omitted since adjustments are made experimentally so that $v(0) = w_c$.

Fig. 2.5  State Variable Diagram of Loop
2.3.1. Optimum Frequency Control

Writing the differential equations of the loop by inspection of
Figure 2.5 gives

\[
\dot{\psi} = \Delta \omega + u_p \tag{2-19a}
\]
\[
\dot{\phi}_o = K_2 \left[ \frac{\nu}{\tau} + \frac{K}{\alpha} \sin (\psi - \phi_o) - \frac{\nu}{\alpha t} \right] \tag{2-19b}
\]
\[
\dot{\nu} = \frac{K}{\alpha} \sin (\psi - \phi_o) - \frac{\nu}{\alpha t} \tag{2-19c}
\]

These equations are the state differential equations for the loop.

Initially \( \phi (0) = 0 \).

The problem is to find \( u_p \) such that \( \phi (t_f) = 0 \) and the frequency
error at the terminal time, \( t_f \), is zero, \( t_f \) being a minimum.

In order to simplify equation (2-12) the definition

\[
x_1 = \psi - \phi_o \tag{2-20a}
\]

and

\[
x_2 = \Delta \omega - \frac{K_2}{\tau} \left[ 1 - \frac{1}{\alpha} \right] \nu \tag{2-20b}
\]

is made, where \( x_1 \) and \( x_2 \) are phase and frequency error respectively.

Substituting equations (2-20) into (2-19) and utilizing the servo
terminology of equations (2-7) and (2-8) yields

\[
\dot{x}_1 = -\omega_n^2 \tau \sin x_1 + x_2 + u_p \tag{2-21a}
\]

\[
\dot{x}_2 = -\omega_n^2 \frac{\alpha - 1}{\alpha} \sin x_1 - \omega_n^2 \frac{x_1}{K} x_2 + \omega_n^2 \Delta \omega \tag{2-21b}
\]
with initial and final conditions

\[ x_1 (o) = o, \quad x_2 (o) = \Delta \omega \quad \text{and} \quad x_1 (t_f) = x_2 (t_f) = o \]

Pontryagin's maximum principle, as outlined in Appendix I, is now applied to equations (2-21) in order to find the control, \( u_p^* \), which minimizes \( t_f \).

A new state variable

\[ x_3 = \int_0^t dt = t \quad (2-22) \]

is defined such that \( \dot{x}_3 = 1 \), and \( x_3 (o) = o \).

The Pontryagin function \( P \) is given by

\[ P = u_1 x_1 (t_f) + u_2 x_2 (t_f) + x_3 \quad (2-23) \]

The optimum design problem now reduces to the determination of the admissible control, \( u_p \leq |U_p| \) so that the Pontryagin function is minimized.

The hamiltonian for the system is

\[
H = p_1 (-\omega_n^2 \sin x_1 + x_2 + u_p) + p_2 (-\omega_n^2 \frac{a-1}{a} \sin x_1 \\
+ \frac{\omega_n^2}{K} (\Delta \omega - x_2)) + p_3 \quad (2-24)
\]

where the costate (or adjoint) vectors \( p_1 \), \( p_2 \) and \( p_3 \) are related by the canonical equations

\[
p_1 = -\frac{\partial H}{\partial x_1} = \omega_n^2 \cos x_1 (p_2 \frac{a-1}{a} + p_1 \tau) \quad (2-25a)
\]
\[ \dot{p}_2 = -\frac{\partial H}{\partial x_2} = -p_1 + \frac{\omega_n^2}{2K} p_2 \]  

(2-25b) 

\[ \dot{p}_3 = -\frac{\partial H}{\partial x_3} = 0 \]  

(2-25c)

which describe the adjoint system, subject to the boundary conditions

\[ p_1(t_f) = u_1 \text{ and } p_2(t_f) = u_2. \]

Since \( \dot{p}_3(t) = 0 \) and \( p_3(t_f) = -1 \),

\[ p_3(t) = -1. \]

Examination of the hamiltonian \( H \) reveals that it is a maximum with respect to \( u_p \) if

\[ u_p^* = |u_p| \text{ sgn } p_1 \]  

(2-26)

In order to obtain \( u_p^* \) with respect to time, equations (2-21) and (2-25) are solved in conjunction with (2-26).

The above equations are readily adaptable to an analogue computer.

A trial and error method of finding the initial conditions of the costate vectors \( p_1 \) and \( p_2 \) must be used. The initial conditions \( p_1(o) \) and \( p_2(o) \) must be chosen so that the final conditions \( x_1(t_f) = x_2(t_f) = 0 \) are satisfied. The final conditions on the costate vectors \( p_1(t_f) \) and \( p_2(t_f) \) are open since in Pontryagin's function the final conditions on \( x_1 \) and \( x_2 \) are zero.

2.3.2 Linear Analysis of Frequency Control

Allowing \( \Delta \omega = 0.5 \omega_n \) and \( |U| = \omega_n \) it may be assumed with small error that

\[ \sin x_1 = x_1 \]

By letting \( t' = \omega_n t \) where \( t' \) is simulated time and \( t \) is real time.
equations (2-21) are normalized. With these assumptions equations (2-21) become

\[ \dot{x}_1 = -\omega_n^2 x_1 + x_2 + \frac{u_p}{\omega_n} \tag{2-27a} \]

\[ \dot{x}_2 = -\frac{a-1}{a} x_1 - \frac{\omega_n}{K} x_2 + \frac{\Delta \omega}{K} \tag{2-27b} \]

with \( x_1 (0) = 0 \), and \( x_2 (0) = \frac{\Delta \omega}{\omega_n} \).

In matrix notation this is

\[ \dot{X} = AX + BU \]

where

\[ \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}; \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad U = \begin{bmatrix} u_p \\ \Delta \omega \end{bmatrix} \]

\[ A = \begin{bmatrix} -\omega_n^2 & 1 \\ -\frac{a-1}{a} \omega_n & -\frac{1}{K} \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{\omega_n} \\ 0 \end{bmatrix} \]

(2-28)

and

\[ X_0 = \begin{bmatrix} 0 \\ \frac{\Delta \omega}{\omega_n} \end{bmatrix} \]

Using the state transition method (7) of solution and solving by means of Laplace transforms yields

\[ X(t) = L^{-1}\left[ X(s)e^{-t(s)}\right] = L^{-1}\left[ \theta(s)X(0) + \theta(s)B U (s) \right] \]

where

\[ \theta(s) = (sI - A)^{-1} \]

and I is the identity matrix. The solutions to equations (2-27) in terms

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of normalized time, \( \omega_n t \) are

\[
x_1(t) = \Delta \omega \left[ \frac{1}{K} + \frac{1}{\omega_n (1 - \zeta^2)} \left( \frac{\omega_n^2}{K^2} - \frac{2 \zeta \omega_n}{K} + 1 \right)^{1/2} \right] e^{-\zeta \omega_n t} 
\cdot \sin \left( \sqrt{1 - \zeta^2} \omega_n t + \gamma_1 \right) + L^{-1} \left[ \frac{1}{\omega_n (s + \frac{\omega_n}{K})} u_p(s) \right]
\]

(2-29a)

\[
x_2(t) = \frac{\Delta \omega}{\omega_n} \left[ \frac{1 - \frac{1}{\sqrt{1 - \zeta^2}}}{\sqrt{1 - \zeta^2}} \right] e^{-\zeta \omega_n t} \sin \left( \sqrt{1 - \zeta^2} \omega_n t + \gamma_2 \right)
\]

\[
+ L^{-1} \left[ \frac{\omega_n - 1}{\omega_n} \frac{u_p(s)}{s^2 + 2 \zeta s + 1} \right]
\]

(2-29b)

where

\[
\gamma_1 = \left( \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta} - \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta - \frac{\omega_n}{K}} \right)
\]

and

\[
\gamma_2 = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta}
\]

These equations for phase and frequency error illustrate that when the loop parameters have been fixed, and a specified step in frequency introduced into the input that the decay times of \( x_1(t) \) and \( x_2(t) \) can still be controlled by applying a control \( u_p \). Since \( |u_p| = \pm U \), it is observed from equations (2-29) that \( u_p \) can be switched in a manner, so as to have a direct effect on the values of \( x_1 \) and \( x_2 \).

Comparison of equation (2-29a) with (2-18) demonstrates that applying an external control to the input signal has an effect on the times for the transients to decay to zero.
2.3.3 Optimum Phase Control

Setting \( u_p = 0 \) and writing the differential equations of the state variable diagram illustrated in Figure 2.5 yields

\[
\dot{\psi} = \Delta \omega \quad (2-30a)
\]

\[
\dot{\phi}_o = K_2 \left[ \frac{V}{T} (1 - \frac{1}{\alpha}) + \frac{K K_i}{\alpha} \sin \left( \psi - \phi_o + u_p \right) \right] \quad (2-30b)
\]

\[
\dot{\nu} = \frac{K K_i}{\alpha} \sin \left( \psi - \phi_o + u_p \right) - \frac{\nu}{\alpha T}
\]

Substituting State Variables \( x_1 \) and \( x_2 \) from equation (2-20) gives

\[
\dot{x}_1 = -\omega n^2 x_1 \sin (x_1 + u_p) + x_2 \quad (2-31a)
\]

\[
\dot{x}_2 = -\omega n^2 \frac{\alpha - 1}{\alpha} \sin (x_1 + u_p) - \frac{\omega n^2}{K} x_2 + \frac{\omega n^2}{K} \Delta \omega \quad (2-31b)
\]

The initial and final conditions are the same as before:

\[
x_1 (0) = 0 \quad , \quad x_2 (0) = \Delta \omega \quad , \quad x_1 (t_f) = x_2 (t_f) = 0
\]

The problem is to determine \( u_p \) such that \( t_f \) is minimized. Again the Pontryagin function is

\[
P = u_1 x_1 (t_f) + u_2 x_2 (t_f) + x_3 (t_f) \quad (2-32)
\]

with

\[
x_3 = \int_0^t \dot{t} = t \quad . \quad (2-33)
\]

The hamiltonian, as shown in Appendix I is
\[ H = - \left[ p_1 \omega_n^2 \tau + p_2 \omega_n^2 \left( 1 - \frac{1}{\alpha} \right) \right] \sin (x_1 + u_p) + \left[ p_1 \frac{-\omega_n^2}{K} + p_2 \right] x_2 \]
\[ + p_2 \frac{\omega_n^2}{K} \Delta \omega + p_3 \quad (2-34) \]

The adjoint equations are then

\[ p_1 = -\frac{\partial H}{\partial x_1} = \left[ p_1 \omega_n^2 \tau + p_2 \omega_n^2 \left( 1 - \frac{1}{\alpha} \right) \right] \cos (x_1 + u_p) \quad (2-35a) \]
\[ p_2 = -\frac{\partial H}{\partial x_2} = -p_1 + \frac{\omega_n^2}{K} p_2 \quad (2-35b) \]
\[ p_3 = 0 \quad (2-35c) \]

with the final conditions on the costate vectors \( p_1 \) and \( p_2 \) open as in frequency control. The hamiltonian, \( H \), is maximized with respect to \( u_p \) if

\[ \sin (x_1 + u_p) = -\text{sgn} \left( p_1 \omega_n^2 \tau + p_2 \omega_n^2 \left( 1 - \frac{1}{\alpha} \right) \right) \]

or

\[ u_p^* = -x_1 - \frac{\pi}{2} \text{sgn} \left( p_1 \omega_n^2 \tau + p_2 \omega_n^2 \left( 1 - \frac{1}{\alpha} \right) \right) \quad (2-36) \]

With this value of \( u_p^* \), the system state variable differential equations (2-31) and the adjoint equations (2-35) reduce to

\[ \dot{x}_1 = +\omega_n^2 \tau \text{sgn} \left( p_1 \omega_n^2 \tau + p_2 \omega_n^2 \left( 1 - \frac{1}{\alpha} \right) \right) + x_2 \quad (2-37a) \]
\[ \dot{x}_2 = \omega_n^2 \frac{\alpha-1}{\alpha} \text{sgn} \left( p_1 \omega_n^2 \tau + p_2 \omega_n^2 \left( 1 - \frac{1}{\alpha} \right) \right) - \frac{\omega_n^2}{K} x_2 \]
\[ + \frac{\omega_n^2}{K} \Delta \omega \quad (2-37b) \]
\[ \dot{p}_1 = 0 \quad \text{or} \quad p_1 = p_1(o) \quad (2-37c) \]

\[ \dot{p}_2 = -p_1(o) + \frac{\omega_n^2}{K} p_2 \quad (2-37d) \]

Solving equation (2-37d) gives

\[ p_2(t) = \frac{Kp_1(o)}{\omega_n^2} - p_2(o) - \frac{Kp_1(o)}{\omega_n^2} \frac{\omega_n^2 t}{e^K} \quad (2-38) \]

where \( p_1(o) \) and \( p_2(o) \), the costate vector initial conditions, are chosen by a trial and error method to give the optimum control \( u^* \).

An interesting feature to note in the above analysis is that the introduction of phase control has eliminated the sine non-linearity in the State Variables \( x_1 \) and \( x_2 \). The above system may now be analyzed by ordinary linear methods without incurring the restrictions due to the use of a linear approximation.

2.3.4 Analysis of Optimum Phase Control

Rewriting equations (2-37a) and (2-37b) in terms of normalized time, \( \omega_n t \), yields

\[ \dot{x}_1 = x_2 + \omega_n t \text{sgn} \, q \quad (2-39a) \]

\[ \dot{x}_2 = -\frac{\omega_n}{K} x_2 + \frac{a-1}{a} \text{sgn} \, q + \frac{\Delta \omega}{K} \quad (2-39b) \]

where

\[ q = \omega_n t \, p_1 + p_2 \frac{a-1}{a} \omega_n \]

which in matrix notation becomes

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\[
\dot{x} = \begin{bmatrix}
0 & 1 \\
-\frac{\alpha}{K} & 0 \\
\frac{\omega_n^2}{K} & 0 \\
\end{bmatrix}
\begin{bmatrix}
x \\
u \\
\end{bmatrix}
+ \begin{bmatrix}
\omega_n^\top & 0 \\
\frac{\alpha-1}{\alpha} & \frac{1}{K} \\
\end{bmatrix}
\begin{bmatrix}
x \\
u \\
\end{bmatrix}
\]
(2-40)

or

\[
\dot{x} = Ax + Bu
\]

Solution by state transition method (8) gives

\[
x = \theta(t) x_0 + \int_0^t \theta(t-t') Bu \, dt
\]
(2-41)

where

\[
x_0 = \begin{bmatrix}
0 \\
\frac{\Delta\omega}{\omega_n} \\
\end{bmatrix}; \quad u = \begin{bmatrix}
\text{sgn} \, q \\
\Delta\omega \\
\end{bmatrix}
\]

\[
\theta(t) = L^{-1} (sI-A)^{-1}, \text{ the transition matrix}
\]

Expansion of equation (2-41) yields

\[
x_1(t) = \theta_2 \frac{\Delta\omega}{\omega_n} + d_1 \text{sgn} \, q + d_2 \Delta\omega
\]
(2-42a)

\[
x_2(t) = \theta_4 \frac{\Delta\omega}{\omega_n} + d_2 \text{sgn} \, q + d_4 \Delta\omega
\]
(2-42b)

where

\[
\theta_2 = \sqrt{\frac{K}{\omega_n}} \sin \sqrt{\frac{\omega_n}{K}} \, t, \quad \theta_4 = \cos \sqrt{\frac{\omega_n}{K}} \, t
\]

\[
d_1 = \omega_n \left[\frac{\alpha-1}{\alpha^2} + \left(\frac{K}{\alpha} \omega_n^2 + \frac{K}{\alpha-1} \frac{\omega_n}{K}\right)^{1/2}\left(\frac{\omega_n^2}{K} t - \psi_1\right)\right]
\]
As in frequency control, $\text{sgn } q$, has an effect on the phase and frequency transients.

Phase trajectories of equation (2-42) can be plotted for $\text{sgn } q = 1$ and $\text{sgn } q = -1$. The optimum trajectory may be determined from these trajectories.
CHAPTER III

ANALOGUE COMPUTER SIMULATION

3.1 Procedure

The differential equations for phase and frequency error derived in Chapter II, autonomous for $t > 0$ allow an accurate solution to be obtained by analogue methods. The state variable system is simulated in conjunction with the adjoint system which supplies the control vector. The trajectories of the system are obtained by Pactolus Digital Analogue Simulator Program (9). The program is incorporated into an IBM 1620 MK II digital computer and the trajectories are plotted by a CALCOMP on-line plotter. Pactolus allows direct control of the costate initial conditions via an on-line typewriter.

The equations are simulated in terms of normalized time, $\omega_n t$, in order to avoid exceeding the numerical bounds on the computer.

3.2 Locking Without External Control

Rewriting equation (2-11) in terms of normalized time yields

$$\dot{\phi} = \frac{\Delta \omega}{\omega_n} \left[ u(t) + \frac{\omega_n}{K} t \right] - \frac{\omega_n}{K} \phi - \frac{1}{\omega_n} \sin \phi - \int_0^t \sin \phi \, dt \quad (3-1)$$

An analogue block diagram of equation (3-1) which consists of integrators, summers, amplifiers, inverters and other blocks which are elements of the Pactolus program is shown in Figure 3.1. The parameter values for the loop are given in Appendix II. The related Pactolus program for Figure 3.1 is given in Appendix III.
Figure 3.2a shows the phase plane trajectories\textsuperscript{0} for step frequency inputs of $\Delta \omega = 0.5 \omega_n$. Figure 3.2b and c illustrate a plot of phase and frequency error versus time, respectively. Figures 3.3 show the above curves for $\Delta \omega = \omega_n$.

The phase plots of the uncontrolled systems (Figures 3.2a and 3.3a) start at a frequency error of $\Delta \omega$ at time $t = 0$ and spiral to a point of zero frequency error and positive phase error. This point is the steady state phase error of the system which is 0.032 (radians) for $\Delta \omega = 0.5 \omega_n$ and 0.064 (radians) for $\Delta \omega = \omega_n$.

When the size of the step frequency input was doubled from $0.5 \omega_n$ to $\omega_n$ the time for the frequency transients to decay from their maximum values to within 5% of their steady state values increased from 8.15 to 8.50 units of normalized time.

3.3 Locking with Frequency Control

For convenience, equations (2-21) along with the adjoint equations (2-25) are rewritten in terms of normalized time:

\begin{align*}
\dot{x}_1 &= - \tau \omega_n \sin x_1 + x_2 + \frac{u_p}{\omega_n} \\
\dot{x}_2 &= -\frac{a-1}{a} \sin x_1 - \frac{\omega_n}{K} x_2 + \frac{\Delta \omega}{K} 
\end{align*}

with $x_1(0) = 0$, and $x_2(0) = \frac{\Delta \omega}{\omega_n}$

\textsuperscript{0}Phase plane trajectories are a plot of frequency error versus phase error with time a parameter along the path length.
\[ \dot{p}_1 = \cos x_1 \left[ p_2 \frac{\alpha - 1}{\alpha} + p_2 \right] \]  
(3-2c)

\[ \dot{p}_2 = -p_1 + \frac{\omega_n}{K} p_2 \]  
(3-2d)

These equations are simulated for Pactolus programming by the analogue block diagram of Figure 3.4. The corresponding program is given in Appendix III.

Figure 3.2a shows the phase trajectories for a step frequency input of \( \Delta \omega = 0.5 \omega_n \) and control vector amplitudes of \( U_F = \omega_n, 2\omega_n \) and \( 3\omega_n \).

Figure 3.2b and c show the corresponding plots for phase and frequency error respectively, versus time.

Figure 3.3a shows the phase trajectories for a step frequency input of \( \Delta \omega = \omega_n \) with control vector amplitudes of \( U_F = 0.5 \omega_n, \omega_n \) and \( 1.5 \omega_n \).

Figure 3.3b and c show the corresponding time domain plots for phase and frequency error respectively. Superimposed on the above curves are plots of uncontrolled trajectories for comparison.

Figure 3.5 illustrates the shape of the control \( u_F^* \). The numerical values for the parameters in Figure 3.5 are given in Table 3.1. Table 3.1 gives the resulting switching times \( t_s \) and \( t_f \). It is observed that as the magnitude of the control is increased, the decay times of the transients decrease.
It was found that for $\Delta \omega = \omega_n$ the loop could not achieve lock for $U_p > 2\omega_n$. This occurs because the phase error has surpassed the lock range of the loop.

The portion of the phase plane curve prior to the first switch occurring at time $t_s$ is found to be identical for all combinations of costate initial conditions tried with $\Delta \omega$ and $U_p$ remaining constant. The costate initial conditions determine only the time of the first

<table>
<thead>
<tr>
<th>$\frac{U_p}{\omega_n}$</th>
<th>$\omega_n t_s$</th>
<th>$\omega_n t_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>---</td>
<td>8.150</td>
</tr>
<tr>
<td>1.0</td>
<td>0.650</td>
<td>1.330</td>
</tr>
<tr>
<td>2.0</td>
<td>0.510</td>
<td>0.995</td>
</tr>
<tr>
<td>3.0</td>
<td>0.450</td>
<td>0.850</td>
</tr>
</tbody>
</table>

Table 3.1 Transient Decay Times. (a) $\frac{\Delta \omega}{\omega_n} = 0.5$ (b) $\frac{\Delta \omega}{\omega_n} = 1.0$
switch $t_s$. In addition the portion of the curves following the switching time $t_s$ are nearly linear translations of the same curve.

This allows the optimal initial conditions to be found with relative ease; in fact five trials of different initial conditions prove to be sufficient to find the optimal control.

Table 3.1 illustrates the great improvements possible in minimizing the settling times when control is applied.
Fig. 3.1 Analogue Block Diagram of Uncontrolled Loop

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Fig. 3.2a Phase Plane Trajectories for $\Delta \omega = 0.5 \omega_n$
Fig. 3.2b Phase Error Trajectories for $\Delta \omega = 0.5 \omega_n$

Fig. 3.2c Frequency Error Trajectories for $\Delta \omega = 0.5 \omega_n$
Fig. 3.3a Phase Plane Trajectories for $\Delta \omega = \omega_n$
Fig. 3.3b  Phase Error Trajectories for $\Delta \omega = \omega_n$

Fig. 3.3c  Frequency Error Trajectories for $\Delta \omega = \omega_n$
Fig. 3.5 Control Vector Waveform
4.1 Experimental Setup

The phaselock loop utilized in the experiment is incorporated in a Dymec Oscillator Synchronizer Model DY-2650A. The DY-2650A is used to stabilize a voltage controlled oscillator in the frequency range of 1 to 12.4 Gc. The VCO used is a Varian X-13 reflex klystron which operates in the x band (8.2 - 12.4 Gc). A small sample of the klystron's signal is mixed with a harmonic of the signal from a temperature-stabilized crystal r-f reference oscillator, either internal or external, to produce a 30 mc i-f signal. (See Figure 4.1). This i-f signal is then phase compared to an i-f reference signal. When lock is obtained any attempts by the klystron to shift frequency will produce a phase error and a phase comparator output voltage which is applied in series with the reflector supply to stop the frequency shift.

The approach taken to obtain experimental results is to first lock the klystron to a known frequency, ensuring that the phase error is zero. The output of the phase comparator is monitored on an oscilloscope. The step change in frequency plus the control are fed to the reflector via the external modulation terminals on the klystron power supply. These voltages are fed repetitively so that a visible trace can be observed on the oscilloscope. The repetition rate is set to give the loop time to relock to its initial state.

The oscilloscope gives a trace of the phase error versus time.
Figure 4.1: DY-2650A Oscillator Synchronizer Block Diagram
Fig. 4.2 Experimental Setup Block Diagram

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4.2 Method of Introducing Frequency Step and Control Function Input

In the theoretical analysis the frequency control is summed with the step frequency change and applied to the phase comparator. It was assumed that there is zero tuning error, i.e. that a condition is set up so that the input initial frequency is equal to the VCO centre frequency. In a practical system this requires a very stable rf generator which could be frequency modulated with a pulse and have an output which could furnish the required amplitude of at least one volt to the VHF terminal of the harmonic mixer.

The rf reference circuit which consists of a crystal oscillator, provides the stability necessary to give zero tuning error, but it is not possible to frequency modulate the rf reference circuit. This difficulty is avoided by applying the step change in frequency at a portion of the loop other than the input.

Figure 4.3a illustrates a simple negative feedback control loop

![Negative Feedback Control Loop Diagram]

Fig. 4.3 Negative Feedback Loop

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The output of the summer is given as

$$\delta (s) = r (s) - h (s) = - (-r (s) + h (s))$$  \hspace{1cm} (4-1)$$

now \(h (s)\) is given as \(c (s) G (s)\). Therefore equation (4-1) becomes

$$\delta (s) = - (-r (s) \frac{G (s)}{G (s)} + c (s) G (s))$$

or

$$\delta (s) = - \left( c (s) - \frac{r (s)}{G (s)} \right) G (s)$$  \hspace{1cm} (4-2)$$

In other words, the output of the summer will not change if the input, \(r\), is summed with the output, \(c\), in the manner shown in Figure 4.3b. Applying this technique to Figure 2.5 results in Figure 4.4, where the filter is lumped into one block.

Fig. 4.4 Revised State Variable Diagram

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\( \Delta \omega \) and \( u^* \) will have their amplitudes changed by a factor of \( \frac{1}{K^2} \).

It is now possible to add \( \frac{1}{K^2} (\Delta \omega + u^*) \) in series with the reflector supply. This gives the required conditions of experimental analysis.

4.3 Design of Frequency Control Function

The control function of Figure 3.5 can be broken down into the addition of two pulses, one positive and one negative, of equal amplitude. The negative pulse is delayed by \( t_s \) seconds before being summed with the positive pulse. The positive pulse is obtained from an emitter-coupled monostable multivibrator whose pulse duration is made equal to \( \frac{t_s}{\omega_n} \). A second multivibrator furnishes a pulse of duration \( t_f - t_s \) to an inverter circuit. This second multivibrator is triggered by a delayed trigger. This delay is equal to \( t_s \). The step change of frequency \( \Delta \omega \) is obtained from a third monostable multivibrator with a long duration pulse and a magnitude of \( \Delta \omega \). All three pulse are algebraically summed by a high gain dc amplifier. The output of the high gain amplifier is fed in series with the reflector power supply. The delay of \( t_s \) seconds is obtained by passing a pulse through the required number of feet of transmission line. The above circuit implementation is summarized in the block diagram of Figure 4.5.
Fig. 4.5 Generation of Frequency Control Function
CHAPTER V

EXPERIMENTAL RESULTS

5.1 Response of Loop to Step Frequency Changes

A step change in frequency was obtained by turning off the first and second multivibrators, MV1 and MV2 in Figure 4.5. An exact replica of a step change can not be obtained since there is capacitance in the leads leading from the summer to the reflector, and approximately 20 pf of capacitance shunting the reflector to ground. This causes a large increase in rise time of the desired step change.

Table 5.1 gives the values of phase error settling times obtained by analogue and experimental methods for \( \Delta \omega = 0.5 \omega_n \) and \( \omega_n \). The experimental results are accurate to within 20%. The reason for this is poor definition of waveforms available on the oscilloscope (see Figures 5.1). The thickness of the waveform is due to 30 Me leaking through the phase comparator.

Although there is no direct method of measuring the decay time of the frequency error, or, the exact amount of frequency change by the klystron oscillator, an indication of frequency change is noticed when the repetitive pulse returns to its initial condition by a disturbance of the phase error. This disturbance in phase error approaches that due to a quasi-static change because of the long decay time of the switch to the stable state.

5.2 Controlled Step Frequency Changes

When the step change plus control (Figure 5.2a) is applied to the
reflector of the klystron, the waveform visible at the reflector (Figure 5.2b) is distorted and attenuated. This distortion due to capacitance of leads and reflector to ground capacitance can be removed by using a compensated probe to apply the signal from the summer amplifier to reflector. Since the compensated probes have an attenuation factor of 10:1, the amplitude of the pulses at the summer have to be magnified by a factor of 10 which causes saturation of the summer amplifier.

Another method of applying the control signals to the reflector is by an active probe using an emitter or cathode follower configuration. The emitter follower is not applicable since it is required to be able to follow a signal which switches between ± 20 volts.

A cathode follower circuit was constructed using a pentode tube. This circuit cannot follow the speed of switching of the control signal which has a rise time of 20 nanoseconds.

When the signal is applied using the cathode follower circuit the waveform which appears at the reflector is shown in Figure 5.3a. The output of the phase comparator is not an optimum response. This is expected since the control signal is not the optimum control designed.

Application of the distorted control signal does indicate that the phase error response is changed by a significant amount (Figure 5.3b) from a simple frequency step. The phase error response does not approach the response calculated by analogue methods since the control follows a ramp more closely than an ideal bang-bang relay.
### Table 5.1 Experimental Decay Time for Step Change in Frequency

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<tr>
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<th>$\frac{\Delta \omega}{\omega_n} = 0.5$</th>
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<td><strong>Experimental</strong></td>
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<td><strong>Computer</strong></td>
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<td>$22.4 \mu s$</td>
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<td><strong>Results</strong></td>
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Fig. 5.1 Phase Transient Trajectories

(a) $\Delta \omega = 0.5 \omega_n$

(b) $\Delta \omega = \omega_n$
Fig. 5.2 Control Function Input (a) Control Generated (b) Control After Applying to Reflector

(Time scale of (b) is tripled)
Fig. 5.3 Application of Distorted Control Function
(a) Pulse Applied to Reflector
(b) Phase Transient

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CHAPTER VI
DISCUSSION OF RESULTS

6.1 Theoretical

Application of the maximum principle to a second order phase-lock loop has produced an extremal frequency control which is locally optimal for a specified magnitude of control vector $U_\omega$. The theoretical derivation was relatively straightforward.

Optimum phase control has removed the sine non-linearity in the state variable equations allowing a linear analysis approach to the problem. Although phase control is less difficult to handle theoretically, it would be an experimental problem of considerable magnitude. The construction of a phase control vector being the main problem, for although it would switch in the same manner as $u_\omega^*$, its amplitude between switching depends on $x_1$.

The results from analogue simulation illustrate that a considerable improvement in minimizing transient decay times is obtained. (See Table 3.1).

The optimum control vectors are $u_\omega^* = 3 \omega_0 \text{ sgn } p_1$ for $\Delta \omega = 0.5 \omega_0$ and $u_\omega^* = 1.5 \omega_0 \text{ sgn } p_1$ for $\Delta \omega = \omega_0$. This leads one to the obvious conclusion that the optimal control is obtained by increasing the magnitude $U_\omega$ to maximum limits without the loop going out of lock.

Solution of the boundary value problem to obtain $p_1(t)$ is accomplished by a trial and error method of searching for the costate initial conditions which satisfy the final conditions on $x_1$ and $x_2$. The trial and error method of search is made easier if an on-line plot of phase trajectories is available to allow visualization of the effect of changing $p_1(0)$ and $p_2(0)$. 

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has on the switching times of the control vector.

6.2 Experimental

The difficulty encountered in the acquisition of experimental results is the "state of the art" problem of transferring a fast rise pulse from point a to b with minimum distortion at b. The most logical approach to this problem is construction of an active probe which presents a very high impedance to the output of the summer amplifier and a low output impedance which is capable of driving the klystron reflector.

The results that were obtained demonstrate the feasibility of this type of control if experimental problems can be overcome. These problems are construction of the control cycle waveform, introducing the control, and an accurate method of measuring the results of application of a control signal.

By applying the control of the input of the VCO as shown in Figure 4.3b a degree of adaptability is obtained. Since the adjoint system which generates the control vector depends only on the cosine of the phase error and costate initial conditions, a black box could be constructed with the input being phase error and the output being the control signal which has a predetermined maximum amplitude. The input signal could then change frequency and the control would change its switching time appropriately to minimize the settling time of the transients.

6.3 Conclusions

This paper has investigated the method of optimal control using the maximum principle. Although only token experimental results are obtained
it has been shown that a control vector can be constructed and that it does affect the performance criteria.

The theoretical derivation of optimality does not present any major difficulties aside from handling cumbersome equations. The control vector obtained is optimal. This is shown by the analogue computational results.

An application of this work would be to minimize the time required to change the frequency of the transmitter by an amount greater than the lock range of the loop. This could be done by changing the frequency of the transmitter in discrete steps, each step being the maximum lock-in range for a step change in frequency. This would produce a staircase transition from one frequency level to the next.

A particular extension of this work would be to find the relation of the magnitude of the step change in frequency and optimal control vector to the value of the costate initial conditions. Such information would lay the path for a completely adaptable receiver for frequency transitions of the input. Such a receiver would be applicable to following a frequency pulse modulated signal.

In general it is very difficult to design an optimal system in accordance with the mathematical theory. The reason for this is that the design of a system involves many engineering judgements regarding such facts as simplicity, reliability, cost, etc., which cannot be neatly expressed in a mathematical statement. Although the exact optimal system is seldom constructed, knowledge of the truly optimal system will provide the engineer with a well defined starting point and a yardstick for ultimate performance.

A particular advantage of the necessary conditions for optimality
provided by the maximum principle is that, with very little analytical effort, one can often deduce certain properties of the optimal control and of the structure of the optimal control system, without ever going through a complete solution to any given problem.
APPENDIX I

FUNDAMENTALS OF PONTRYAGIN'S MAXIMUM PRINCIPLE

Consider a nth-order control process characterized by

\[ \dot{x} = f(x, u, t) \]

where \( x \) is the \( n \times 1 \) state vector and \( u \) is the input \( n \times 1 \) control vector.

At each moment the control signals \( u_i \) must satisfy the inequality constraint \( g(u) \leq 0 \) which reflects the restriction imposed upon the control system. The control vectors which satisfy the constraint condition are referred to as admissible control vectors.

The above process is optimized by forming a criterion of performance for the above system and maximizing or minimizing this criterion by an external control in accordance with given constraints.

This criterion of performance is measured in terms of some functional

\[ P = \sum_{i=1}^{n} b_i x_i(t_f) \]

or \[ P = b' x(t_f) \]

where \( b \) is a \( n \times 1 \) column vector. \( P \) is called Pontryagin's function.

When the final state of the process is constrained by

\[ R_k \left[ x(t_f) \right] = 0 \quad k = 1, \ldots, v; v \leq n \]

Pontryagin's function takes the form

\[ P = b x(t_f) + \mu R \left[ x(t_f) \right] \]

where \( \mu \) is the vector Lagrange multiplier.
The maximum principle states that if the control vector \( \mathbf{u} \) is optimum, i.e., if it extremizes the Pontryagin function \( P \), then the hamiltonian for the system is extremized in the opposite direction with respect to \( \mathbf{u} \) over the control interval.

The hamiltonian is defined as

\[
H \left( \mathbf{x}, \mathbf{p}, \mathbf{u}, t \right) = \sum_{j=1}^{n} p_j f_j
\]  

(A 1-1)

where \( \mathbf{p} \) analogous to the canonical momenta in classical mechanics, will be defined later. The \( f_j \)'s are the system's state variable equations. The above statement implies that maximum \( H \) will give minimum \( P \) and vice-versa. Thus a necessary condition for the control vector \( \mathbf{u} \) to optimize the Pontryagin function is the fulfillment of the maximum condition for \( H \).

To apply the maximum principle in the solution of optimum-control problems, the design procedure is initiated with the maximization of hamiltonian \( H \) with respect to the control vector \( \mathbf{u} \).

This results in an optimum control vector \( \mathbf{u}^* \) as a function of the momentum vector \( \mathbf{p} \). In order to find \( \mathbf{u}^* \) as a function of \( \mathbf{x} \) or \( t \), equations providing the relationships between \( \mathbf{u} \) and \( \mathbf{p} \) must be established.

We first define the momentum vector \( \mathbf{p} \) as the solution to the differential equation

\[
p_i = - \sum_{j=1}^{n} p_j \frac{\partial f_j}{\partial x_i} \quad i = 1, 2, \ldots, n.
\]  

(A 1-2)

where \( p_i (t_f) = - b_i + \sum_{k=1}^{v} u_k \frac{\partial R_k x_i (t_f)}{\partial x_i} \)  

(A 1-3a)

if there are constraints, otherwise

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\[ p_i (t_f) = - b_i \]  \hspace{1cm} (A 1-3b)

where \( b_i \) is a known constant specified by the designer in the Pontryagin function \( P \) and

\[ \dot{x}_i = f_i (x, u, t) \]  \hspace{1cm} (A 1-4)

Differentiating equation (A 1-1) with respect to \( p_i \) yields

\[ \frac{\partial H}{\partial p_i} = f_i (x, u, t) \]  \hspace{1cm} (A 1-5)

Differentiating equation (A 1-1) with respect to \( x_i \) gives

\[ \frac{\partial H}{\partial x_i} = \sum_{j=1}^{n} p_j \frac{\partial f_i}{\partial x_i} \]  \hspace{1cm} (A 1-6)

Making use of these two equations reduces equations (A 1-2) and (A 1-4) to the Hamilton canonical form

\[ \dot{x}_i = \frac{\partial H}{\partial p_i} \]  \hspace{1cm} (A 1-7a)

\[ \dot{p}_i = - \frac{\partial H}{\partial x_i} \]  \hspace{1cm} (A 1-7b)

subject to the boundary conditions on \( x_i (t_0) \) and \( p_i (t_f) \).

The minimum time problem may be stated as the determination of an admissible control vector \( u \) so that the process is taken from a specified initial state \( x_0 \) to a desired final state \( x_f \) in the shortest possible time. The control signals being subject to the constraint

\[ |u| \leq U \]

where \( U \) is the largest magnitude available for \( u \).
By introducing a new co-ordinate

$$x_{n+1}(t) = \begin{cases} 
\dd t = t - t_0 & \\
0 & 
\end{cases}$$

the optimum control problem becomes a minimization of the new co-ordinate which is our criterion of performance. The differential equations of the system are now

$$\dot{x}_1 = f_1(x, u, t)$$

$$\dot{x}_{n+1} = 1$$

subject to the initial conditions $x(t_o) = x_0$

The Pontryagin function will be

$$P = x_{n+1}(t_f) + \sum_{j=1}^{v} \mu_j x_j(t_f)$$

where $b_i = 0, i = 1, \ldots, n,$

$$b_{n+1} = 1.$$ 

This gives (see equation A 1-3)

$$p_i(t_f) = 0,$$

$$p_j(t_f) = \mu_j,$$

$$p_{n+1}(t_f) = -1.$$ 

The hamiltonian for the system will be

$$H = \sum_{i=1}^{n} p_i \dot{x}_i + p_{n+1} \quad \text{(A 1-8)}$$
Maximizing the hamiltonian with respect to $u$ and applying equation (A 1-7b) to (A 1-8) furnishes the proper relations between $u$ and $t$. 

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APPENDIX II
VALUE OF LOOP PARAMETERS

The loop filter is shown in Figure (A.1)

![Fig. A.1 Loop Filter](image)

The transfer function can be written as

\[ F(s) = \frac{1 + \tau s}{1 + \alpha \tau s} \]

with \( \tau = RC_1 \) and \( \alpha = \frac{C_1 + C_2}{C_2} \)

\( R = 39k \), \( C_1 = 39 \text{ pf} \), \( C_2 = .001 \text{ uf} \)

which gives \( \alpha = 27.4 \)

\( \tau = 1.52 \times 10^{-6} \)

now \( K = K_0 K_1 K_2 = 20 \times 1 \times .3 = 6 \text{ mc} \). This yields (equations 2.7 and 2.8)

\[ \omega_n = .38 \text{ mc} \]
\[ \zeta = .32 \]
APPENDIX III

PACTOLUS DIGITAL ANALOG SIMULATOR PROGRAM

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REFERENCES


VITA AUCTORIS

1944  Born on August 2 in Amherstburg, Ontario.

1956  Completed elementary education at St. Anthony's School, Amherstburg, Ontario.

1961  Graduated from St. Rose Highschool, Amherstburg, Ontario.

1966  Bachelor of Applied Science Degree in Engineering Science from University of Windsor, Windsor, Ontario.

1967  Candidate for the degree of M.A.Sc. in Electrical Engineering at the University of Windsor.