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Markushevich Bases and Auerbach Bases in Banach Spaces

by
Apala Mandal

A Major Research Paper
Submitted to the Faculty of Graduate Studies
through the Department of Mathematics and Statistics
in Partial Fulfillment of the Requirements for
the Degree of Master of Science at the
University of Windsor

Windsor, Ontario, Canada
2018

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MARKUSHEVICH BASES AND AUERBACH BASES IN BANACH SPACES

by

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January 11, 2018

AUTHOR'S DECLARATION OF ORIGINALITY

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ABSTRACT

This paper studies Markushevich bases and Auerbach bases in Banach spaces. Firstly, a countable 1-norming Markushevich basis is constructed for any infinite-dimensional separable Banach space. Secondly, an Auerbach basis is constructed for any finite-dimensional Banach space. Thirdly, a Markushevich basis is constructed for a class of non-separable Banach spaces by applying projectional generators and projectional resolution identities, and the transfinite induction on the density character of the space.

To my father for keeping me company,
and Comet for keeping me happy.

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CHAPTER I

Introduction

In this major paper, we construct a Markushevich basis for any infinite-dimensional separable Banach space, an Auerbach basis for any finite-dimensional Banach space, and a Markushevich basis for a class of non-separable Banach spaces. A short answer to why we study Markushevich bases on separable Banach spaces would be to find a system of coordinates to represent any vector in such a space. A natural approach for a separable Banach space would be to consider the concept of a Schauder basis. Unfortunately, not every separable Banach space has such a basis as proved by Enflo in [4] (see also [20]). All such spaces do have, however, a Markushevich basis. We know this due to Markushevich's result which extends the classical Gram-Schmidt orthogonal process. In the final section of the paper, we prove that many non-separable Banach spaces also have Markushevich bases.

CHAPTER II

Markushevich Bases in Infinite-Dimensional Separable Banach Spaces

The main references for this chapter are [1], [11], [12], and [16].

2.1 Preliminaries

In this section, we look at some basic definitions concerning biorthogonal systems in Banach spaces, and discuss several results related to this structure and some results on 1-norming subspaces of Banach spaces. We also look at some auxiliary propositions, with the purpose of better understanding norming subspaces and Markushevich bases, leading up to the main Theorem 2.8 (see [1, Theorem A]).

Specifically, we define 1-norming and norming subspaces of a Banach space (see [12] for further details), and we define M-bases (see [11], [12], and [23]) and Σ -subspaces (see [1]). We also define Plichko spaces (see [11]), and look at some examples concerning 1-Plichko spaces (which are worked out in details in [12], [15], and [17]).

We begin with the following definition. For more details, see [12, page 41].

Definition 2.1. Let X be a Banach space, and D be a linear subspace of X^* .

- (i) We say that D is *1-norming* if $\|x\| = \sup\{|y^*(x)| : y^* \in D, \|y^*\| \leq 1\}$ for any $x \in X$.
- (ii) We say that D is *norming* if $\|x\|_D = \sup\{|y^*(x)| : y^* \in D, \|y^*\| \leq 1\}$ ($x \in X$) defines an equivalent norm on X .

The following proposition gives a useful property of norming subspaces of a Banach space.

Proposition 2.2. *Let X be a Banach space, and D be a norming linear subspace of X^* . Then D separates points of X .*

Proof. Suppose $x_1, x_2 \in X$ with $x_1 \neq x_2$. Then $\|x_1 - x_2\| > 0$, which means that $\|x_1 - x_2\|_D > 0$. This implies that there exists some $y^* \in D$ such that $y^*(x_1) \neq y^*(x_2)$. Therefore, D separates points of X . \square

The following useful characterization for 1-norming subspaces is summarily stated in [16, Proposition 2.1]. We prove it in details below.

Proposition 2.3. *Let D be a linear subspace of X^* . Then D is 1-norming if and only if $D \cap B_{X^*}$ is weak*-dense in B_{X^*} , where B_{X^*} is the closed unit ball of X^* .*

Proof. Suppose that $D \cap B_{X^*}$ is weak*-dense in B_{X^*} (i.e., $B_{X^*} \subseteq \overline{D \cap B_{X^*}}^{w^*}$). Note that $\|x\|_D = \|x\|$ if $x = 0$. Let $x \in X \setminus \{0\}$. Clearly, $\|x\|_D \leq \|x\|$. And, as a consequence of the Hahn-Banach Theorem (see [8, Corollary 2.5]), there exists some $f \in B_{X^*}$ such that $\|x\| = f(x)$. By the assumption, there

exists a net $\{f_i\}$ in $D \cap B_{X^*}$ such that $f_i \rightarrow f$ in the weak* topology. So, $\|x\| = |f(x)| = \lim_i |f_i(x)| \leq \|x\|_D$. Therefore, $\|x\| = \|x\|_D$ for all $x \in X$, and hence D is 1-norming.

Conversely, suppose that D is 1-norming. Assume that $D \cap B_{X^*}$ is not *weak**-dense in B_{X^*} . Then there exists $f \in B_{X^*}$ such that $f \notin \overline{D \cap B_{X^*}}^{w^*}$. Here we take X^* to be a complex Banach space. The proof for the real space case is similar. Then, since the singleton $\{f\}$ is w^* -compact and convex and $\overline{D \cap B_{X^*}}^{w^*}$ is w^* -closed and convex, by [2, Corollary IV.3.13], there exist $x \in X$, $\alpha \in \mathbb{R}$ and $\epsilon > 0$ such that for all $g \in \overline{D \cap B_{X^*}}^{w^*}$, $\operatorname{Re}(g(x)) \leq \alpha < \alpha + \epsilon \leq \operatorname{Re}(f(x))$.

Let $g \in D \cap B_{X^*}$. Then there exists $\theta \in \mathbb{R}$ such that $|g(x)| = e^{i\theta}g(x)$. Note that $e^{i\theta}g \in D \cap B_{X^*}$. Thus we have

$$|g(x)| = (e^{i\theta}g)(x) = \operatorname{Re}((e^{i\theta}g)(x)) \leq \alpha.$$

It follows that

$$\|x\| = \|x\|_D = \sup_{g \in D \cap B_{X^*}} \{|g(x)|\} \leq \alpha < \alpha + \epsilon.$$

On the other hand, we have

$$\alpha + \epsilon \leq \operatorname{Re}(f(x)) \leq |f(x)| \leq \|f\| \|x\| \leq \|x\|.$$

So, we get $\|x\| < \alpha + \epsilon \leq \|x\|$, a contradiction. Therefore, $D \cap B_{X^*}$ is *weak**-dense in B_{X^*} . \square

The following definition of a Markushevich basis (one of some equivalent ones, whose equivalence will be proved later) can be found in [12, Definition 4.13].

Definition 2.4. Let X be a Banach space.

(i) A *Markushevich basis* (M-basis) of X is a family $\{x_\alpha, x_\alpha^*\}_{\alpha \in \Lambda}$ of elements of $X \times X^*$ such that

$$(P_M(i)) \quad \overline{\text{span}\{x_\alpha\}_{\alpha \in \Lambda}} = X;$$

$$(P_M(ii)) \quad x_\alpha^*(x_\alpha) = 1, \text{ and } x_\alpha^*(x_\beta) = 0 \text{ if } \alpha \neq \beta;$$

$$(P_M(iii)) \text{ for any } x \in X \setminus \{0\}, \text{ there is } \alpha \in \Lambda \text{ such that } x_\alpha^*(x) \neq 0.$$

(ii) An M-basis $\{x_\alpha, x_\alpha^*\}_{\alpha \in \Lambda}$ of X is called *norming* (resp. *1-norming*) if $\overline{\text{span}\{x_\alpha^*\}_{\alpha \in \Lambda}}$ is a norming (resp. 1-norming) subspace of X^* .

(iii) An M-basis $\{x_\alpha, x_\alpha^*\}_{\alpha \in \Lambda}$ is called *countably norming* (resp. *countably 1-norming*) if the space $\{x^* \in X^* : \{\alpha \in \Lambda : x^*(x_\alpha) \neq 0\} \text{ is countable}\}$ is a norming (resp. 1-norming) subspace of X^* .

Remark 2.5. We note that the above definitions indicate that every (1-)norming M-basis is a countably (1-)norming M-basis. However, the converse is not necessarily true. For example, the space $C[0, \omega_1]$ has a countably norming M-basis, but it has no norming M-basis (see [11, Theorem 5.25]). Here ω_1 is the first uncountable ordinal (see Chapter IV for the definition of an ordinal).

There is an alternate definition for Markushevich bases (see, for example, [11, page 4] and [23, page 266]), where $P_M(iii)$ is replaced by

$$P_M(iii)' \quad \overline{\text{span}\{x_\alpha^*\}^{w^*}} = X^*.$$

We prove the equivalence of these two properties in Proposition 2.7. Therefore, an alternative definition of an M-basis is as follows.

Definition 2.6. A *Markushevich basis* in a Banach space X is a *biorthogonal system* $\{x_i, u_i\}_{i \in I}$ such that $X = \overline{\text{span}\{x_i\}}$ and $X^* = \overline{\text{span}\{u_i\}}^{w^*}$.

Here, we call $\{x_i, u_i\}_{i \in I}$ a biorthogonal system if $\{x_i\} \subseteq X$, $\{u_i\} \subseteq X^*$, and $u_j(x_i) = \delta_{ij}$ ($i, j \in I$).

Proposition 2.7. Let X be a Banach space, and $\{x_\alpha^*\}_{\alpha \in \Lambda}$ be a family of functionals in X^* . Then the following statements are equivalent.

- (i) $\overline{\text{span}_{\alpha \in \Lambda} \{x_\alpha^*\}}$ is w^* -dense in X^* ;
- (ii) for any $x \in X \setminus \{0\}$, there is $\alpha \in \Lambda$ such that $x_\alpha^*(x) \neq 0$.

Proof. Assume that (ii) holds, but $\overline{\text{span}_{\alpha \in \Lambda} \{x_\alpha^*\}}^{w^*} \neq X^*$. Then there exists $x_0^* \in X^*$ such that $x_0^* \notin \overline{\text{span}_{\alpha \in \Lambda} \{x_\alpha^*\}}^{w^*}$. By Hahn-Banach Separation Theorem (see [2, Corollary IV.3.15]), there exists $x_0 \in X$ such that $x_0^*(x_0) \neq 0$ but $x_\alpha^*(x_0) = 0$ for all $\alpha \in \Lambda$. Now, since $x_0 \neq 0$, by the assumption, there exists α_0 such that $x_{\alpha_0}^*(x_0) \neq 0$, which is a contradiction. Hence $\overline{\text{span}_{\alpha \in \Lambda} \{x_\alpha^*\}}^{w^*} = X^*$.

Conversely, suppose $\overline{\text{span}_{\alpha \in \Lambda} \{x_\alpha^*\}}^{w^*} = X^*$. Let $x \in X \setminus \{0\}$. Then, by a corollary to the Hahn-Banach Theorem [8, Corollary 2.5], there exists f in X^* such that $f(x) = \|x\| \neq 0$. Now, we can find $g \in \text{span}\{x_\alpha^*\}$ such that $g(x) \neq 0$, which implies that there exists $\alpha \in \Lambda$ such that $x_\alpha^*(x) \neq 0$.

□

We now present the statement of [1, **Theorem A**], and we will prove in Section 4.2 the main part of the non-trivial implication (1) \Rightarrow (2). Specifically, we prove (1) \Rightarrow (2'), where (2') is the following statement:

(2') There is a Markushevich basis $\{x_\alpha, x_\alpha^*\}_{\alpha \in \Gamma}$ of X such that $\{x_\alpha\}_{\alpha \in \Gamma} \subseteq \text{span } M$ and $D \subseteq \{x^* \in X^* : \{\alpha \in \Gamma : x^*(x_\alpha) \neq 0\} \text{ is countable}\}$.

Theorem 2.8 (Main Theorem). *Let X be a Banach space and let D be a norming linear subspace of X^* . Then the following statements are equivalent.*

(1) *There exists a linearly dense subset M of X such that*

$$D = \{x^* \in X^* : \{m \in M : x^*(m) \neq 0\} \text{ is countable}\}.$$

(2) *There is a Markushevich basis $\{x_\alpha, x_\alpha^*\}_{\alpha \in \Gamma}$ of X such that*

$$D = \{x^* \in X^* : \{\alpha \in \Gamma : x^*(x_\alpha) \neq 0\} \text{ is countable}\}.$$

(3) *There exists a net $(P_\lambda)_{\lambda \in \Lambda}$ of projections on X such that*

(i) $P_\lambda X$ is separable for each λ and $X = \bigcup_{\lambda \in \Lambda} P_\lambda X$;

(ii) $P_\lambda P_\mu = P_\mu P_\lambda = P_\lambda$ whenever $\lambda \leq \mu$;

(iii) if (λ_n) is an increasing sequence in Λ , it has a supremum $\lambda \in \Lambda$

$$\text{and } P_\lambda X = \overline{\bigcup_n P_{\lambda_n} X};$$

(iv) $P_\lambda P_\mu = P_\mu P_\lambda$ for all $\lambda, \mu \in \Lambda$;

(v) $D = \bigcup_{\lambda \in \Lambda} P_\lambda^* X^*$.

Definition 2.9. Let X be a Banach space.

- (i) A norming linear subspace D of X^* is called a Σ -subspace of X^* if D satisfies one of the equivalent conditions in Theorem 2.8.
- (ii) A Banach space admitting a (1-)norming Σ -subspace is said to be (1-)Plichko. So, Theorem 2.8 gives some equivalent definitions for Plichko spaces.

Definition 2.10. A family of projections on X satisfying conditions (i)-(iii) from Theorem 2.8 (3) is called a *projectional skeleton* in X .

Example 2.11.

- (1) For a compact abelian group K , $C(K)$ is 1-Plichko, which is an immediate consequence of the fact that $C(K)$ has a (norming) M-basis. See [15, Theorem 4.6] and [17, Theorem 17.14] for details.
- (2) Any abstract L^1 space is 1-Plichko (see [12, Example 6.10]).
- (3) Let T be any locally compact Hausdorff space. Then $C_0(T)^*$ is 1-Plichko. See [15, Theorem 5.5] for details. Therefore, the spaces $C[0, 1]^*$ and $C_0(\mathbb{R})^*$ are 1-Plichko (see [15, Example 5.6]).

Remark 2.12. The following containment is clear:

1-Plichko spaces \subseteq Plichko spaces \subseteq spaces with projectional skeletons.

2.2 M-Bases in Separable Banach Spaces

In this section, we present the classical Markushevich construction of a countable M-basis in any separable Banach space. The following theorem is summarily proved in [11, Lemma 1.21]. We prove it in depth below.

Theorem 2.13 (Markushevich). *Every infinite-dimensional separable Banach space has a countable M-basis which is 1-norming.*

Proof. Suppose X is an infinite-dimensional separable Banach space. Let $M = \{z_n\}_{n=1}^{\infty}$ be a dense subset of X . Since X is separable, the unit ball (B_{X^*}, w^*) of X^* with the w^* -topology is compact (by the Banach-Alaoglu Theorem) and metrizable. Thus (B_{X^*}, w^*) is separable, since every compact metric space is separable. Therefore, we can take a dense subset $N = \{z_n^*\}_{n=1}^{\infty}$ of (B_{X^*}, w^*) .

Since X separates points of X^* and M is dense in X , it is clear that

$$(M1) \quad M = \{z_n\}_{n=1}^{\infty} \text{ separates points of } X^*.$$

We show below that

$$(M2) \quad N = \{z_n^*\}_{n=1}^{\infty} \text{ separates points of } X.$$

Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. By the Hahn-Banach Theorem (see, for example, [2, Corollary 6.7]), there exists $x^* \in X^*$ such that $x^*(x_1 - x_2) = \|x_1 - x_2\| = \epsilon > 0$. Since N is w^* -dense in B_{X^*} , there exists n such that

$$|(z_n^* - x^*)(x_1 - x_2)| < \frac{\epsilon}{2}.$$

It follows that

$$|z_n^*(x_1 - x_2)| \geq |x^*(x_1 - x_2)| - |(z_n^* - x^*)(x_1 - x_2)| > \epsilon - \frac{\epsilon}{2}.$$

Therefore, $N = \{z_n^*\}_{n=1}^\infty$ separates points of X .

Now, we use the sequences $\{z_n\}$ and $\{z_n^*\}$ to construct an M-basis for X .

Step 1. Define $x_1 = z_{n_1}$, where n_1 is the first index such that $z_{n_1} \neq 0$. By (M2), there exists m_1 such that $z_{m_1}^*(x_1) \neq 0$. Let

$$x_1^* = \frac{1}{z_{m_1}^*(x_1)} z_{m_1}^*.$$

Then $x_1^*(x_1) = 1$.

Step 2. Let m_2 be the first index such that $z_{m_2}^* \notin \text{span}\{x_1^*\}$. Note that such m_2 exists since X is of infinite dimension. Let

$$x_2^* = z_{m_2}^* - z_{m_2}^*(x_1) x_1^*.$$

Then $x_2^*(x_1) = z_{m_2}^*(x_1) - z_{m_2}^*(x_1) x_1^*(x_1) = 0$ since $x_1^*(x_1) = 1$. Also $x_2^* \neq 0$, since $z_{m_2}^* \notin \text{span}\{x_1^*\}$. Hence, by (M1), we can find n_2 such that $x_2^*(z_{n_2}) \neq 0$. Let

$$x_2 = \frac{z_{n_2} - x_1^*(z_{n_2}) x_1}{x_2^*(z_{n_2})}.$$

Then

$$x_1^*(x_2) = \frac{x_1^*(z_{n_2}) - x_1^*(z_{n_2}) x_1^*(x_1)}{x_2^*(z_{n_2})} = 0,$$

since $x_1^*(x_1) = 1$. And, since $x_2^*(x_1) = 0$, we have

$$x_2^*(x_2) = \frac{x_2^*(z_{n_2}) - x_1^*(z_{n_2}) x_2^*(x_1)}{x_2^*(z_{n_2})} = \frac{x_2^*(z_{n_2})}{x_2^*(z_{n_2})} = 1.$$

Step 3. Let n_3 be the first index such that $z_{n_3} \notin \text{span}\{x_1, x_2\}$. Let

$$x_3 = z_{n_3} - x_1^*(z_{n_3}) x_1 - x_2^*(z_{n_3}) x_2.$$

Then

$$x_1^*(x_3) = x_1^*(z_{n_3}) - x_1^*(z_{n_3})x_1^*(x_1) - x_2^*(z_{n_3})x_1^*(x_2) = 0,$$

since $x_1^*(x_1) = 1$ and $x_1^*(x_2) = 0$. Also, we have

$$x_2^*(x_3) = x_2^*(z_{n_3}) - x_2^*(z_{n_3})x_2^*(x_1) - x_2^*(z_{n_3})x_2^*(x_2) = 0,$$

since $x_2^*(x_1) = 0$ and $x_2^*(x_2) = 1$. By the choice of z_{n_3} , $x_3 \neq 0$. By (M2),

there exists m_3 such that $z_{m_3}^*(x_3) \neq 0$. Let

$$x_3^* = \frac{z_{m_3}^* - z_{m_3}^*(x_1)x_1^* - z_{m_3}^*(x_2)x_2^*}{z_{m_3}^*(x_3)}.$$

Then $x_3^*(x_1) = x_3^*(x_2) = 0$ and $x_3^*(x_3) = 1$.

Inductively, at Step $2n$, we can construct x_{2n}^* and then x_{2n} , and at Step $(2n + 1)$, we can construct x_{2n+1} and then x_{2n+1}^* . This gives us the biorthogonal system $\{x_n, x_n^*\}_{n=1}^\infty$. We can see that $\text{span}\{x_n\}_{n=1}^\infty \subseteq \text{span}\{z_n\}_{n=1}^\infty$. To see $\text{span}\{z_n\}_{n=1}^\infty = \text{span}\{x_n\}_{n=1}^\infty$, let $k \in \mathbb{N}$. Then we can choose l such that $n_{2l+1} > k$. Since n_{2l+1} is the first index such that $z_{n_{2l+1}} \notin \text{span}\{x_1, \dots, x_{2l}\}$, we have $z_k \in \text{span}\{x_1, \dots, x_{2l}\}$. Therefore, $\text{span}\{x_n\}_{n=1}^\infty = \text{span}\{z_n\}_{n=1}^\infty$.

Similarly, we have $\text{span}\{x_n^*\}_{n=1}^\infty = \text{span}\{z_n^*\}_{n=1}^\infty$ from the construction.

Therefore, we have

$$X = \overline{\text{span}\{z_n\}_{n=1}^\infty} = \overline{\text{span}\{x_n\}_{n=1}^\infty}$$

and

$$X^* = \overline{\text{span}\{z_n^*\}_{n=1}^\infty}^{w^*} = \overline{\text{span}\{x_n^*\}_{n=1}^\infty}^{w^*}.$$

It follows from Definition 2.6 that $\{x_n, x_n^*\}_{n=1}^\infty$ is an M-basis for X . Finally, since

$$B_{X^*} = \overline{\{z_n^*\}}^{w^*} = \overline{\{z_n^*\} \cap B_{X^*}}^{w^*} \subseteq \overline{\text{span}\{z_n^*\}_{n=1}^\infty \cap B_{X^*}}^{w^*} = \overline{\text{span}\{x_n^*\}_{n=1}^\infty \cap B_{X^*}}^{w^*},$$

we have

$$\overline{\text{span}\{x_n^*\}_{n=1}^\infty \cap B_{X^*}}^{w^*} \supseteq B_{X^*}.$$

By Proposition 2.3, $\text{span}\{x_n^*\}_{n=1}^\infty$ is a 1-norming linear subspace of X^* . Therefore,

$\{x_n, x_n^*\}_{n=1}^\infty$ is a countable 1-norming Markushevich basis for X . \square

CHAPTER III

Auerbach Bases in Finite-Dimensional Banach Spaces

It is known that every finite-dimensional Banach space X has a Hamel basis, which obviously can be taken to consist of vectors of norm 1. However, in this case, it is difficult in general to control the norm of the coefficient functionals. In the proposition of this chapter, we are showing that it is possible to choose a Hamel basis so that both the vectors and the coefficient functionals have norm 1 (cf. [11, Theorem 1.16]). This construction geometrically represents a parallelepiped of maximal volume inscribed in the closed unit ball of X . The following definition can be found in [7, page 181]. The main references for this chapter are [7] and [11].

Definition 3.1. For a Banach space X , a biorthogonal system $\{x_\gamma, f_\gamma\}_{\gamma \in \Gamma}$ in X is called an *Auerbach system* if $\|x_\gamma\| = \|f_\gamma\| = 1$ for all $\gamma \in \Gamma$. An Auerbach system is called an *Auerbach basis* if it is a Markushevich basis.

As shown in Theorem 2.13, every infinite-dimensional separable Banach space has a Markushevich basis which is 1-norming. Now, we show a stronger

property for finite-dimensional Banach spaces. In fact, we will show below that every finite-dimensional Banach space has an Auerbach basis (which is obviously 1-norming).

The following proposition can be found, for example, in [11, Theorem 1.16].

Proposition 3.2 (Auerbach). *Let X be a finite-dimensional Banach space. Then X contains an Auerbach basis.*

Proof. Let $\{v_i\}_{i=1}^n$ be an algebraic basis of X . For any finite sequence $\{u_i\}_{i=1}^n$ of vectors in X , let $\det(u_1, u_2, \dots, u_n)$ be the determinant of the $n \times n$ matrix whose j -th column consists of the coordinates of u_j with respect to the basis $\{v_i\}_{i=1}^n$.

Since the function $|\det(\cdot)|$ on the compact set $B_X \times \dots \times B_X$ is continuous, it attains its supremum at some $(x_1, x_2, \dots, x_n) \in B_X \times \dots \times B_X$. Then we get that $\|x_i\| = 1$ for $i = 1, \dots, n$, since $\det(\cdot)$ is a multilinear mapping on its columns. Note that the vectors u_1, \dots, u_n are linearly independent if and only if $\det(u_1, u_2, \dots, u_n) \neq 0$. So, x_1, \dots, x_n are linearly independent and hence $\{x_i\}_{i=1}^n$ is also an algebraic basis of X .

For $i = 1, \dots, n$, define $f_i \in X^*$ by

$$f_i(x) = \frac{\det(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)}{\det(x_1, x_2, \dots, x_n)} \quad (x \in X).$$

We see that for each $1 \leq i \leq n$,

$$\|f_i\| = \sup_{x \in B_X} |f_i(x)| = \sup_{x \in B_X} \left| \frac{\det(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)}{\det(x_1, x_2, \dots, x_n)} \right|.$$

Since $|\det(\cdot)|$ attains its supremum at (x_1, \dots, x_n) , for all $x \in B_X$, we have

$$\begin{aligned} |\det(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)| &\leq |\det(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)| \\ &= |\det(x_1, \dots, x_n)|. \end{aligned}$$

This implies that

$$|f_i(x)| = \left| \frac{\det(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)}{\det(x_1, x_2, \dots, x_n)} \right| \leq \|x\| \text{ for all } x \in X.$$

Since $|f_i(x_i)| = 1$ and $x_i \in B_X$, $\|f_i\| = \sup_{x \in B_X} |f_i(x)| = 1$. By the definition, $f_i(x_k) = \delta_{ik}$ for all $1 \leq i, k \leq n$. Therefore, $\{x_i, f_i\}_{i=1}^n$ is a system in $B_X \times B_{X^*}$ satisfying $P_M(i)$ and $P_M(ii)$.

To see $P_M(iii)$, let $x = \sum_{i=1}^n \alpha_i x_i \in X \setminus \{0\}$. We look at the smallest k with $1 \leq k \leq n$ such that $\alpha_k \neq 0$. Since $\{x_1, \dots, x_{k-1}, x, x_{k+1}, \dots, x_n\}$ is a linearly independent set, we have $\det(x_1, \dots, x_{k-1}, x, x_{k+1}, \dots, x_n) \neq 0$. So, there exists k such that $f_k(x) \neq 0$.

Therefore, $\{x_i, f_i\}_{i=1}^n$ is an Auerbach basis for X . □

See [11, page 5] for more results on Auerbach bases, and [9, page 54] for a couple of open problems on Auerbach bases.

CHAPTER IV

Markushevich Bases in Non-Separable Banach Spaces

In this chapter, we look at Banach spaces which have projections onto smaller subspaces. A possible such class of Banach spaces is those with a projectional resolution of the identity (PRI), introduced by Lindenstrauss [18, 19]. PRIs are basically defined to be a well ordered continuous chain of projections onto smaller subspaces (see [13, 21, 22] for further details). We use PRIs as well as projectional generators and the transfinite induction to prove Theorem 4.8, which covers the main part of the proof of $(1) \Rightarrow (2)$ in Theorem 2.8.

PRIs together with transfinite induction are also useful in proving various other properties of non-separable Banach spaces. See [3] and [6] for more information concerning the PRI method. The main references for this chapter are [5], [12], [13] and [14].

4.1 Basic Definitions

In this section, we define (weak and strong) PRIs and projectional generators on Banach spaces. We begin with some basic definitions from set theory. See [5] for more detailed definitions and examples of partial and total orders, and [10, page 2] for notations.

Definition 4.1. Let S be a set.

(i) A *partial order* on S is a binary relation \preceq that possesses the following properties: for all $a, b, c \in S$,

$$(1) \text{ [Reflexivity]} \quad a \preceq a;$$

$$(2) \text{ [Anti-Symmetry]} \quad a \preceq b \wedge b \preceq a \Rightarrow a = b;$$

$$(3) \text{ [Transitivity]} \quad a \preceq b \wedge b \preceq c \Rightarrow a \preceq c.$$

(ii) A partial order \preceq on S is called a *total order* if for all $a, b \in S$, either $a \preceq b$ or $b \preceq a$.

(iii) A total order \preceq on S is called a *well-order* if every non-empty subset of S has a least element.

Definition 4.2.

- (i) (Ordinal) Every ordinal is the well-ordered set of all smaller ordinals.
- (ii) (Limit Ordinal) An ordinal λ is called a *limit ordinal* if whenever $\beta \prec \lambda$, there exists γ such that $\beta \prec \gamma \prec \lambda$.
- (iii) (Cardinal) A *cardinal number* (or *cardinal* for short) is an ordinal number α such that if it can be mapped bijectively with an ordinal number β , then $\alpha \preceq \beta$. If X is a set, then $\text{card}(X)$ (the cardinality of X) is the cardinal number that can be mapped bijectively to X .
- (iv) (Density Character) The *density character* of a topological space X , denoted by $\text{dens}(X)$, is the smallest cardinal Ω such that X has a dense subset with cardinality Ω .

Remark 4.3. The smallest infinite ordinal ω is a cardinal (also denoted as \aleph_0), and ω is a limit ordinal because for any smaller ordinal n (i.e., $n \in \mathbb{N}$), there exists $n+1$ with $n \prec n+1 \prec \omega$. The *transfinite induction* is an extension of the induction on ω to general ordinals.

The following definition of PRIs can be found in [13, Definition 8].

Definition 4.4. Let X be a non-separable Banach space with $\text{dens}(X) = \mu$. A *projectional resolution of the identity (shortly PRI)* on X is a family $\{P_\alpha : \omega \preceq \alpha \preceq \mu\}$ of projections on X such that

- (i) $P_\omega = 0, P_\mu = \text{Id}_X$;
- (ii) $\|P_\alpha\| = 1$ if $\omega \prec \alpha \preceq \mu$;
- (iii) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ if $\omega \preceq \alpha \preceq \beta \preceq \mu$;
- (iv) $\text{dens}(P_\alpha X) \preceq \text{card}(\alpha)$ if $\omega \prec \alpha \preceq \mu$;
- (v) $\bigcup_{\beta \prec \alpha} P_\beta X$ is dense in $P_\alpha X$ whenever $\alpha \preceq \mu$ is a limit ordinal.

We now define weak and strong PRIs. The following definitions can be found in [14, Definition 3].

Definition 4.5. Let X be a non-separable Banach space with $\text{dens}(X) = \mu$.

- (1) A *weak PRI* on X is a family $\{P_\alpha : \omega \preceq \alpha \preceq \mu\}$ of projections on X which satisfies the same conditions as that of a PRI with the exception of condition (iv) replaced by

$$(iv') \quad \text{dens}(P_\alpha X) \prec \mu \text{ if } \omega \prec \alpha \prec \mu.$$

- (2) A *strong PRI* on X is a family $\{P_\alpha : \omega \preceq \alpha \preceq \mu\}$ of projections on X which satisfies the same conditions as that of a PRI with the exception of condition (iv) replaced by

$$(iv'') \quad \text{dens}(P_\alpha X) = \text{card}(\alpha) \text{ if } \omega \prec \alpha \prec \mu.$$

Next, we define projectional generators. Equivalent definitions can be found in multiple papers (see, for example, [13, Definition 7]).

Definition 4.6. Let X be a Banach space. A *projectional generator* on X is a pair (D, Φ) such that

- (a) D is a 1-norming \mathbb{Q} -linear subspace of X^* ;
- (b) $\Phi : D \rightarrow 2^X$ is a countably valued mapping, where countably valued means that any value of Φ is at most a countable subset of X ;
- (c) $(\Phi(B))^\perp \cap \overline{B}^{w^*} = \{0\}$ for any \mathbb{Q} -linear subspace B of D , where $\Phi(B) = \bigcup_{b \in B} \Phi(b)$.

4.2 M-Bases in Non-separable Banach Spaces

In this section, we prove the last theorem of this paper, Theorem 4.8, covering the main part of the proof (1) \Rightarrow (2) in Theorem 2.8. In the proof of Theorem 4.8, we use the auxiliary Theorem 4.7, a short proof of which can be found in [12, Proposition 4.9].

Theorem 4.7. *Let X be a non-separable Banach space which admits a projectional generator (D, Φ) . Let $M \subseteq X$ be such that for every $d \in D$, the set $\{x \in M : d(x) \neq 0\}$ is at most countable. Let μ denote the density character of*

X . Then X has a strong PRI $\{P_\alpha : \omega \preceq \alpha \preceq \mu\}$ such that

$$M \subseteq \bigcup_{\omega \preceq \alpha \prec \mu} (P_{\alpha+1} - P_\alpha)X.$$

The following important theorem constructs Markushevich bases for a class of non-separable Banach spaces. This theorem is summarily proved in [12, Lemma 4.19]. We prove it in details below.

Theorem 4.8. *Let X be a Banach space, and let $M \subseteq X$ be such that $\overline{\text{span } M} = X$ and*

$$D = \{x^* \in X^* : \{m \in M : x^*(m) \neq 0\} \text{ is countable}\}$$

is a 1-norming linear subspace of X^ . Then the following holds.*

(i) *If $\text{dens}(X) = \mu \succ \omega$, there is a strong PRI $\{P_\alpha : \omega \preceq \alpha \preceq \mu\}$ on X such that the following conditions are fulfilled.*

(a) $M \subseteq \bigcup_{\omega \preceq \alpha \prec \mu} (P_{\alpha+1} - P_\alpha)X;$

(b) $\overline{\text{span}(M \cap (P_{\alpha+1} - P_\alpha)X)} = (P_{\alpha+1} - P_\alpha)X$ for all $\alpha \in [\omega, \mu);$

(c) *the set $D_\alpha = \{d|_{(P_{\alpha+1}-P_\alpha)X} : d \in D\}$ is a 1-norming linear subspace of $((P_{\alpha+1} - P_\alpha)X)^*$ for every $\alpha \in [\omega, \mu)$.*

(ii) *There is an M -basis $\{x_\alpha, x_\alpha^*\}_{\alpha \in \Lambda}$ of X such that $\{x_\alpha\}_{\alpha \in \Lambda} \subseteq \text{span } M$ and for every $d \in D$, the set $\{\alpha \in \Lambda : d(x_\alpha) \neq 0\}$ is countable.*

Proof. (i) Suppose $\text{dens}(X) = \mu \succ \omega$. For every $d \in D$, let

$$\Phi(d) = \{m \in M : d(m) \neq 0\}.$$

Then, by the assumption, $\Phi : D \mapsto 2^X$ is a countably valued mapping. We show below that (D, Φ) is a projectional generator on X . Let B be a \mathbb{Q} -linear subspace of D . Then we have

$$\begin{aligned} (\Phi(B))^\perp &= \{z^* \in X^* : z^*(m) = 0 \text{ for all } m \in \Phi(B)\} \\ &= \{z^* \in X^* : z^*(m) = 0 \text{ if } m \in M \text{ and } b(m) \neq 0 \text{ for some } b \in B\}. \end{aligned}$$

Let $z^* \in (\Phi(B))^\perp \cap \overline{B}^{w^*}$. Suppose by contradiction that $z^* \neq 0$. Since z^* is linear and continuous and $\overline{\text{span } M} = X$, we have $z^*(m) \neq 0$ for some $m \in M$. Since $z^* \in (\Phi(B))^\perp$, we have that $m \notin \Phi(B)$. On the other hand, since $z^* \in \overline{B}^{w^*}$ and $z^*(m) \neq 0$, there exists $w^* \in B \subseteq D$ such that $w^*(m) \neq 0$. It follows from the definition of Φ that $m \in \Phi(w^*) \subseteq \Phi(B)$, contradicting that $m \notin \Phi(B)$. Therefore, $(\Phi(B))^\perp \cap \overline{B}^{w^*} = \{0\}$.

Hence, (D, Φ) is a projectional generator on X . By Theorem 4.7, there is a strong PRI $\{P_\alpha : \omega \preceq \alpha \preceq \mu\}$ on X such that

$$M \subseteq \bigcup_{\omega \preceq \alpha < \mu} (P_{\alpha+1} - P_\alpha)X.$$

Next, we claim that

$$\overline{\text{span}(M \cap (P_{\alpha+1} - P_\alpha)X)} = (P_{\alpha+1} - P_\alpha)X \quad \text{for every } \alpha \in [\omega, \mu).$$

To see this, we fix $\omega \preceq \alpha_0 \prec \mu$. Since $(P_{\alpha_0+1} - P_{\alpha_0})X$ is a closed linear subspace of X , we have $\overline{\text{span}(M \cap (P_{\alpha_0+1} - P_{\alpha_0})X)} \subseteq (P_{\alpha_0+1} - P_{\alpha_0})X$.

For the reverse inclusion, let $x \in (P_{\alpha_0+1} - P_{\alpha_0})X$ and let $\epsilon > 0$. Then, $(P_{\alpha_0+1} - P_{\alpha_0})x = x$. Since $\overline{\text{span } M} = X$, there exists $y \in \text{span } M$ such

that $\|y - x\| < \epsilon$. Write $y = \sum_{i=1}^n \alpha_i x_i$, where $\alpha_1, \dots, \alpha_n$ are scalars and $x_1, \dots, x_n \in M$. Since $M \subseteq \bigcup_{\omega \leq \alpha < \mu} (P_{\alpha+1} - P_\alpha)X$, we can suppose $x_i \in (P_{\alpha_i+1} - P_{\alpha_i})X$ ($1 \leq i \leq n$). Then $x_i \in M \cap (P_{\alpha_i+1} - P_{\alpha_i})X$ and hence

$$(P_{\alpha_0+1} - P_{\alpha_0})x_i = \begin{cases} 0 & \text{if } \alpha_i \neq \alpha_0, \\ (P_{\alpha_i+1} - P_{\alpha_i})x_i = x_i & \text{if } \alpha_i = \alpha_0. \end{cases}$$

Let $z = (P_{\alpha_0+1} - P_{\alpha_0})y$. Then

$$z = \sum_{i=1}^n \alpha_i (P_{\alpha_0+1} - P_{\alpha_0})x_i = \sum_{\alpha_i = \alpha_0} \alpha_i (P_{\alpha_0+1} - P_{\alpha_0})x_i = \sum_{\alpha_i = \alpha_0} \alpha_i x_i.$$

So, $z \in \text{span}(M \cap (P_{\alpha_0+1} - P_{\alpha_0})X)$. Now

$$\|z - x\| = \|(P_{\alpha_0+1} - P_{\alpha_0})(y - x)\| \leq \|y - x\| < \epsilon.$$

Therefore, $x \in \overline{\text{span}(M \cap (P_{\alpha_0+1} - P_{\alpha_0})X)}$.

Finally, since D is a 1-norming subspace of X^* , for all $\alpha \in [\omega, \mu)$, $D_\alpha = \{d|_{(P_{\alpha+1}-P_\alpha)X} : d \in D\}$ is a 1-norming linear subspace of $((P_{\alpha+1}-P_\alpha)X)^*$.

To see this, we note that for all $x \in (P_{\alpha+1} - P_\alpha)X$,

$$\|x\| = \|x\|_D = \sup_{\substack{d \in D \\ \|d\| \leq 1}} \{|d(x)|\} = \sup_{\substack{d \in D_\alpha \\ \|d\| \leq 1}} \{|d(x)|\} = \|x\|_{D_\alpha}.$$

(ii) The assertion will be proved by the transfinite induction on $\text{dens}(X)$. The following is the induction hypothesis.

I.H. The assertion (ii) holds for Banach spaces X with (M, D) associated, where $M \subseteq X$ is such that $\overline{\text{span } M} = X$ and $\text{card}(M) = \text{dens}(X)$, and $D = \{x^* \in X^* : \{m \in M : x^*(m) \neq 0\} \text{ is countable}\}$ is a 1-norming linear subspace of X^* .

From the proof of Theorem 2.13, it is seen that the I.H. holds for separable Banach spaces. To see this, suppose $\text{dens}(X) = \omega$ with (M, D) associated as in I.H. Then $\text{card}(M) = \omega$. Thus there exists $M' \subseteq \text{span } M$ such that M' is countable and $\overline{M'} = \overline{\text{span } M} = X$. In fact, if we let M' be the rational linear span of M , then $\overline{M'} = \overline{\text{span } M}$, and M' is countable (since M is countable). By the proof of Theorem 2.13, we get a countable M-basis $\{x_n, x_n^*\}_{n=1}^\infty$ of X such that

$$\text{span } \{x_n\}_{n=1}^\infty = \text{span } M' \subseteq \text{span } M$$

Clearly, the set $\{n \in \mathbb{N} : d(x_n) \neq 0\}$ is now countable for every $d \in D$. Therefore, the I.H. holds for every separable Banach space.

In the following, assume that $\mu = \text{dens}(X) \succ \omega$. Let $\{P_\alpha : \omega \preceq \alpha \prec \mu\}$ be a strong PRI on X as obtained in (i). Let $\alpha \in [\omega, \mu)$. Define

$$M_\alpha = M \cap (P_{\alpha+1} - P_\alpha)X \quad \text{and} \quad D_\alpha = \{d|_{(P_{\alpha+1} - P_\alpha)X} : d \in D\}.$$

Then by (b) and (c) in (i), $\overline{\text{span } M_\alpha} = (P_{\alpha+1} - P_\alpha)X$, and D_α is a 1-norming linear subspace of $((P_{\alpha+1} - P_\alpha)X)^*$. Pick $M'_\alpha \subseteq M_\alpha$ such that $\text{card}(M'_\alpha) = \text{dens}((P_{\alpha+1} - P_\alpha)X)$ and $\overline{\text{span } M'_\alpha} = \overline{\text{span } M_\alpha} = (P_{\alpha+1} - P_\alpha)X$.

Let

$$D'_\alpha = \{x^* \in ((P_{\alpha+1} - P_\alpha)X)^* : \{m \in M'_\alpha : x^*(m) \neq 0\} \text{ is countable}\}.$$

Then $D_\alpha \subseteq D'_\alpha$, and hence D'_α is also a 1-norming linear subspace of $((P_{\alpha+1} - P_\alpha)X)^*$. By the Induction Hypothesis I.H., there exists an M-basis $\{x_\gamma, x_\gamma^*\}_{\gamma \in \Lambda_\alpha}$ of $(P_{\alpha+1} - P_\alpha)X$ such that

$$\{x_\gamma\}_{\gamma \in \Lambda_\alpha} \subseteq \text{span } M'_\alpha \subseteq \text{span } M_\alpha \subseteq \text{span } M,$$

and the set $\{\gamma \in \Lambda_\alpha : d(x_\gamma) \neq 0\}$ is countable for each $d \in D'_\alpha$. Let

$$\widetilde{x}_\gamma^* = x_\gamma^* \circ (P_{\alpha+1} - P_\alpha) \quad (\gamma \in \Lambda_\alpha).$$

Then each $\widetilde{x}_\gamma^* \in X^*$ and $\widetilde{x}_\gamma^* = 0$ on $P_\alpha X \oplus (\text{Id} - P_{\alpha+1})X$.

Let $\Lambda = \bigcup_{\omega \preceq \alpha \prec \mu} \Lambda_\alpha$ be the disjoint union of $\{\Lambda_\alpha\}_{\alpha \in [\omega, \mu]}$. Then, by above, $\{x_\gamma\}_{\gamma \in \Lambda} \subseteq \text{span } M$.

Let $d \in D$. We show below that the set $\{\gamma \in \Lambda : d(x_\gamma) \neq 0\}$ is countable. By the definition of the set D , we can write $\{m \in M : d(m) \neq 0\} = \{y_1, y_2, \dots\}$.

Since $M = \bigcup_{\omega \preceq \alpha \prec \mu} M_\alpha$, we can suppose $y_n \in M_{\alpha_n}$ ($n \geq 1$). For any $\alpha \in [\omega, \mu)$,

since $(P_{\alpha+1} - P_\alpha)X = \overline{\text{span } M_\alpha}$, we have $d = 0$ on $(P_{\alpha+1} - P_\alpha)X$ if

$\alpha \notin \{\alpha_n\}_{n \geq 1}$. In particular, $d(x_\gamma) = 0$ if $\gamma \notin \bigcup_{n \geq 1} \Lambda_{\alpha_n}$. On the other hand, for

each $n \geq 1$, since $D_{\alpha_n} \subseteq D'_{\alpha_n}$, the set $\{\gamma \in \Lambda_{\alpha_n} : d(x_\gamma) \neq 0\}$ is countable.

It follows that the set

$$\{\gamma \in \Lambda : d(x_\gamma) \neq 0\} = \bigcup_{n \geq 1} \{\gamma \in \Lambda_{\alpha_n} : d(x_\gamma) \neq 0\}$$

is countable.

Now, we show that our newly constructed system $\{x_\gamma, \widetilde{x}_\gamma^*\}_{\gamma \in \Lambda}$ is indeed a Markushevich basis of X .

($P_M(i)$) For each $\alpha \in [\omega, \mu)$, since $\{x_\gamma, x_\gamma^*\}_{\gamma \in \Lambda_\alpha}$ is a Markushevich basis of $(P_{\alpha+1} - P_\alpha)X$, $\overline{\text{span}_{\gamma \in \Lambda_\alpha} \{x_\gamma\}} = (P_{\alpha+1} - P_\alpha)X$. Then

$$M \subseteq \bigcup_{\omega \preceq \alpha \prec \mu} (P_{\alpha+1} - P_\alpha)X = \bigcup_{\omega \preceq \alpha \prec \mu} \overline{\text{span}_{\gamma \in \Lambda_\alpha} \{x_\gamma\}} \subseteq \overline{\text{span}_{\gamma \in \Lambda} \{x_\gamma\}}.$$

Therefore, $X = \overline{\text{span } M} \subseteq \overline{\text{span}_{\gamma \in \Lambda} \{x_\gamma\}}$; that is, $X = \overline{\text{span}_{\gamma \in \Lambda} \{x_\gamma\}}$.

($P_M(ii)$) For each $\alpha \in [\omega, \mu)$, since $\{x_\gamma, x_\gamma^*\}_{\gamma \in \Lambda_\alpha}$ is a Markushevich basis of $(P_{\alpha+1} - P_\alpha)X$, we have $x_\gamma^*(x_\gamma) = 1$ and $x_\gamma^*(x_\zeta) = 0$ for all $\gamma, \zeta \in \Lambda_\alpha$ with $\gamma \neq \zeta$. By the construction of \widetilde{x}_γ^* from x_γ^* , we have $\widetilde{x}_\gamma^*(x_\delta) = 0$ and $\widetilde{x}_\gamma^*(x_\gamma) = 1$ for all $\gamma, \delta \in \Lambda$ with $\gamma \neq \delta$. Thus $\{x_\gamma, \widetilde{x}_\gamma^*\}_{\gamma \in \Lambda}$ is biorthogonal.

($P_M(iii)$) Let $x \in X \setminus \{0\}$. Using property (ii) from [7, Proposition 13.14], we know that $(P_{\alpha_0+1} - P_{\alpha_0})x \neq 0$ for some $\alpha_0 \in [\omega, \mu)$. Since $\{x_\gamma, x_\gamma^*\}_{\gamma \in \Lambda_{\alpha_0}}$ is an M-basis of $(P_{\alpha_0+1} - P_{\alpha_0})X$, there exists $\gamma \in \Lambda_{\alpha_0}$ such that $x_\gamma^*((P_{\alpha_0+1} - P_{\alpha_0})x) \neq 0$; that is, $\widetilde{x}_\gamma^*(x) = (x_\gamma^* \circ (P_{\alpha_0+1} - P_{\alpha_0}))(x) \neq 0$. Hence, for all $x \in X \setminus \{0\}$, there exists $\gamma \in \Lambda$ such that $\widetilde{x}_\gamma^*(x) \neq 0$.

This concludes the proof of Theorem 4.8. \square

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VITA AUCTORIS

Apala Mandal graduated with honours from the University of Waterloo with a Bachelor of Mathematics in Pure Mathematics, being the only female in her undergraduate program. She moved to her home country Canada when she was 10 years old from Calcutta, India, where she was born.

Her journey in mathematics started with top results in the CEMC contests such as Gauss, Fermat, Fryer, Hypatia and others, leading to her being welcomed by the University of Waterloo. After starting her Master's at the University of Windsor, the author was diagnosed with a severe vitamin B12 deficiency, having been a vegetarian for over a decade.

During the writing of this thesis, she has had the opportunity to rest with the support of the University of Windsor and get her regular injections and checkups due to apt healthcare. She is now ready to be back in action, with the immediate first step being to graduate with her MSc. in Pure Mathematics from the University of Windsor.