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The Price of Stocks, Geometric Brownian Motion, and Black Scholes Formula

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Abstract

In this paper, we discuss the stock price model as Geometric Brownian motion. After that, we obtain a closed form solution to the model using Itô's Lemma. Moreover, we use this solution to derive the Black Scholes formula.

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Contents

CHAPTER 1

Introduction

This major paper concerns a study of geometric Brownian motion that was assumed by Black and Scholes to be a model of a stock price and obtain a solution to this model which will be used in the derivation of Black Scholes formula.

We begin this paper with a brief overview of some mathematical foundations that are essential to understanding the model of a stock price and comprehend Black Scholes world. In the third chapter, we introduce Plain Vanilla European call and put options which is priced using the formula. After that, we study the put-call parity that is used in the derivation of the put option. The fourth chapter demonstrates the risk-neutral derivation of Black Scholes formula. After that, we explain some of the properties of the Black-Scholes formula to get a better understanding of it.

The fifth chapter aims to explain the model of a stock price and consider some of the justifications of this model. We start with Efficient Market Hypotheses. Then, we try to introduce the model in a simple way. After that, we construct the Brownian motion from a discrete random walk example, and we explain some of its properties. Ultimately, we discuss a Geometric Brownian motion.

The last chapter aims to understand Itô's Lemma and its applications. We start that chapter with a justification of Itô's process multiplication rules. Moreover, we study Itô's Lemma and mention some of its importance in stochastic calculus and financial world. Then, we prove Itô's Lemma. Furthermore, we consider some of the application of Itô's Lemma including solving our Geometric Brownian motion to obtain a lognormal model which used in the derivation of Black Scholes formula.

CHAPTER 2

Preliminaries

This chapter contains some basics of math that is essential to understand this paper.
We review some definitions and theorems of probability theory concerning normal and lognormal distribution and some of their properties.
The primary references for this section are [**1**] and [**11**].

2.1. Properties of the normal and lognormal distribution

The normal distribution is essential in this paper because it is used to model the future returns of the stock price. Therefore, we will start reviewing this distribution and some of its properties.

Definition 2.1.1. For reals $-\infty < \mu < \infty$ and $\sigma > 0$, the normal distribution (or Gaussian distribution) denoted $N(\mu, \sigma^2)$ with mean μ and variance σ^2 is a continuous random variable with probability density function defined by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Definition 2.1.2. The standard normal distribution is $N(0, 1)$.

Theorem 2.1.3. *The random variable X is a normal random variable if and only if $X = \mu + \sigma Z$ where Z is the standard normal variable and* $\mathrm{E}[X] = \mu$, $Var[X] = \sigma^2$.

Definition 2.1.4. If X is a continuous random variable having probability density function $f(x)$, then the expected value of X is given as:

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f(x) dx$$

and is called the mean of X.

DEFINITION 2.1.5. Let $f(x)$ be a probability density function with $\int_{-\infty}^{\infty} xf(x)dx < \infty$. Let $m = E[X]$ denote the expected value of X. Define the variance $Var[X]$ as:

$$Var[X] = E[(X - \mathrm{E}[X])^2] = \int_{-\infty}^{\infty} (x-m)^2 f(x)dx$$

LEMMA 2.1.6.

$$Var[X] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2$$

DEFINITION 2.1.7. The standard deviation of a random variable X is given by

$$\sigma[X] = \sqrt{Var[X]}$$

DEFINITION 2.1.8. Denote by $N(t)$ the cumulative distribution of the standard normal variable:

$$N(t) = P(Z \leq t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2} dx$$

LEMMA 2.1.9. *Let Z be the standard normal variable. Then $P(Z \geq a) = P(Z \leq -a)$ for all $a \in \mathbb{R}$, and*

$$1 - N(a) = N(-a), \quad \forall\, a \in \mathbb{R}$$

In the fourth chapter, we will see that the cumulative normal distribution function is a central part of the Black Scholes formula.

DEFINITION 2.1.10. A random variable Y is called lognormal if $Y > 0$ and

$$\ln(Y) = \mu + \sigma Z,$$

where Z is standard normal variable. We say Y is lognormal with parameters μ and σ.

THEOREM 2.1.11. *Let $X_1, ..., X_k$ be independent normal distributions with each*

$$X_j \sim N(\mu_j, \sigma_j^2).$$

Then

$$X_1 + ... + X_k \sim N(\mu_1 + ... + \mu_k, \sigma_1^2 + ... + \sigma_k^2)$$

THEOREM 2.1.12. *A product of independent lognormal variables is also lognormal with parameters*

$$\mu = \sum_j \mu_j$$

$$\sigma^2 = \sum_j \sigma_j^2$$

THEOREM 2.1.13 (Central Limit Theorem, (CLT),[**14**]). *If* $\{X_i\}$ *are independent identically distributed (IID) random variables with finite mean* $\mathrm{E}[X] = \mu$ *and* $Var[X] = \sigma^2$*, then*

$$Z_n := \frac{1}{\sigma\sqrt{n}}(\sum_{i=1}^{n} X_i - n\mu) \to N(0,1), n \to \infty$$

in distribution:

$$\lim_{n\to\infty} P(Z_n \leq x) = N(x),\ x \in \mathbb{R}$$

When $\mu = 0$*,* $\sigma = 1$*, then the CLT becomes*

$$Z_n := \frac{1}{\sqrt{n}}(\sum_{i=1}^{n} X_i) \to N(0,1), n \to \infty$$

Moreover, for any constant $c \neq 0$*,* $cN(0,1) \sim N(0,c^2)$*, and therefore with* $\mu = 0$*,* $\sigma = 1$*,* $c \neq 0$ *the CLT becomes*

$$c\frac{1}{\sqrt{n}}(\sum_{i=1}^{n} X_i) \to N(0,c^2), n \to \infty$$

We will see one of the applications of this theorem in the demonstration of Brownian Motion in chapter 5.

CHAPTER 3

European Options

We begin with an introduction of European put and call options. Then, we explain the relationship between them with the put-call parity, and we prove it. The main references for this chapter are [**1**] and [**2**].

3.1. Plain Vanilla European Call and Put Options

In this section, we need to know some basic definitions that will help us to understand the Black Scholes formula.

DEFINITION 3.1.1. [**3**] *A derivative security* or derivative is a financial instrument whose value depends on an underlying asset.

The term a derivative security is broad, but we will concentrate here on one of the commonest types of derivatives which is options. There are two main types of options which are European and American. The main difference between them is the time of exercise. European options have a predetermined date of exercise, whereas American option can be exercised in any date during the life of the option contract. The term Plain Vanilla options mean the options are ubiquitous. That is normal or has standard features such as the size. This term distinguish these types from other types that are complicated called Exotic.
Now, we will define our central concepts which are European call and put.

DEFINITION 3.1.2. *A Call Option* is a contract between two parties that gives the right, but not the obligation, to *buy* from the seller of the option an asset, *i.e.*, one share of the stock, at a certain date in the future that is called *maturity* T, or *expiration date*, for a specific price which is called *strike price* K or *exercise price*. For this right, the buyer of the option pays $C(t)$ at time $t < T$ to the seller of the option.

DEFINITION 3.1.3. *A Put Option* is a contract between two parties that gives the right, but not the obligation, to *sell* to the seller of the option an asset, *i.e.*, one share of the stock, at a certain date in the future that is called maturity T, or expiration date, for a specific price which is called strike price K or exercise price. For this right, the purchaser of the option pays $P(t)$ at time $t < T$ to the seller of the option.

A call and put option on an underlying asset that we assumed to be the stock price that has today price $S(t)$ and future price $S(T)$. We say call option expires in the money (ITM) when $S(T) > K$, at the money (ATM) when $S(T) = K$, or out of the money (OTM) if $S(T) < K$.
However, a put option expires in the money if $S(T) < K$, at the money $S(T) = K$, or out of the money if $S(T) > K$.

At maturity T, the payoff of European call option is

$$C(T) = \max(S(T) - K, 0) = \begin{cases} S(T) - K, & \text{if } S(T) > K \\ 0, & \text{if } S(T) \leq K \end{cases}$$

At maturity T, the payoff of European put option is

$$P(T) = \max(K - S(T), 0) = \begin{cases} 0, & \text{if } S(T) \geq K \\ K - S(T), & \text{if } S(T) < K. \end{cases}$$

The main goal of this paper is to know the fair value of the call and put option at the present time t which we denoted as $C(t)$ and $P(t)$, respectively, and that what Black Scholes formulas will tell us in the next chapter. However, we went to know the relationship between European call and put option, and that will be studied in the next section.

3.2. The Put-Call Parity for European Options

DEFINITION 3.2.1. [**10**] A portfolio is a collection of investments held by an individual or institution.

DEFINITION 3.2.2. Arbitrage is a trading strategy that hunts advantage of two or more securities being mispriced relative to each other.

That means, the arbitrage opportunity is a risk-less opportunity to earn money. It should be mentioned that if the Put-Call Parity does not hold, then there are arbitrage opportunities. However, In the Black Scholes world, it is assumed that there are no risk-less arbitrage opportunities.

Now, we want to know the Put-Call Parity which will show the relationship between European put and call Options, where they have the same expiration and strike price.

THEOREM 3.2.3. *(The Put-Call Parity). Let $C(t)$ and $P(t)$ be the values of European call and put options, respectively, with the same exercise price K and expiration date T, on the same non-dividend paying stock with spot price $S(t)$ and its future value at the maturity is $S(T)$.*
Therefore, The put-call parity states that

$$P(t) + S(t) = C(t) + Ke^{-r(T-t)}.$$

PROOF. To prove this parity we need the *The Law of One Price* which states that [**10**]: If, in an arbitrage-free world, portfolios A and B are such that at time T, A is worth at least as much as B, then at any time $t < T$, A will be worth at least as much as B.

Therefore, Suppose we have two portfolios A and B such that:

Portfolio A	made of one European put option and one share of the stock .
Portfolio B	made of one European call option and a risk-free bond that has a payoff of K at the expiration, so its present value is $K\,\mathrm{e}^{-r(T-t)}$.

Now, we want to examine the values of these portfolios at the maturity T:

$\boxed{\text{If } S(T) > K}$, then $C(T) = S(T) - K$ and the free-risk bond $= K$. Therefore, the value of the portfolio $B = S(T) - K + K = S(T)$. Moreover, the $P(T) = 0$, so the put option will not be exercised when $K < S(T)$ because it is worthless. Thus, the value of portfolio A is $S(T)$.

$\boxed{\text{If } S(T) < K}$, then $C(T) = 0$, which means the call option will be out of the money (OTM), so it will not be exercised. Therefore, the value of the portfolio $B = K$. Now, the put option will be in the money, and it should be exercised and its value as

$$P(T) = K - S(T)$$

and the share value $= S(T)$. Thus, the value of the portfolio $A = K - S(T) + S(T) = K$.
We realize the value of these portfolios A and B are equal at the maturity T regardless of whether $S(T) > K$ or $S(T) < K$, their values always worth

$$\max(S(T), K)$$

Applying the Law of One Price, these portfolios A and B must have identical values today at $(t < T)$.

Therefore, the value of portfolio $A(t) = P(t) + S(t)$, and the value of the portfolio B today $= C(t) + Ke^{-r(T-t)}$.

Thus,

$$P(t) + S(t) = C(t) + Ke^{-r(T-t)}.$$

$\square$

We can use this relation between European Put and call options to determine the value of one of them when we know the other. Therefore, we will see one of its application in the derivation of the Black Scholes formula for put option where we have derived the value of the European call option.

CHAPTER 4

Black Scholes Formula

The goal of this chapter is to derive the famous option pricing formula using the lognormal model of the stock price.

We derive the Black Scholes formula for a call option using risk-neutral concept and the lognormal model of stock. Also, we obtain the Black Scholes formula for a put option using put-call parity and the value of the call. Finally, we conclude this chapter with some of the interpretation of the formula.
The main references for this chapter are [**1**] and [**2**].

4.1. Risk-neutral derivation of the Black Scholes formula

European put and call options can be priced using the risk-neutral assuption, which defined as discounted expected values of their payoffs at maturity T. Let r be the risk-free interest rate, and σ be the volatility. Therefore, at the present time $t = 0$ risk-neutral valuations of European call and put option, respectively, are as follow:

$$C(0) = e^{-rT}\,\mathrm{E}[C(T)]$$

$$P(0) = e^{-rT}\,\mathrm{E}[P(T)].$$

The expected value of put and call is computed with respect to the future stock price $S(T)$ which is given by the following expression:

$$S(T) = S(0)\; e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}\mathbb{Z}}. \tag{4.1.1}$$

This expression will be clarified later on in this paper.

Theorem 4.1.1. *Assuming risk-neutrality and the price of a stock has a log-normal distribution, and we have non-dividend paying stock where $S(T)$ and $S(0)$ represent the future stock price and its initial value, respectively. Let r be the* *risk-free interest rate, σ is the constant volatility of the stock. Let $C(T)$ and $C(0)$ represent the value of European call option at the maturity T and today $t = 0$, respectively. Similarly, $P(T)$ and $P(0)$ represent the value of European put option at maturity and in the present, respectively. Also, K is the strike price, and $\mathcal{N}(x)$ denotes the cumulative distribution function of a standard normal variable. Therefore, the Black-Scholes formulas are*

$$C(0) = S(0)\mathcal{N}(d_1) - K\,\mathrm{e}^{-rT}\,\mathcal{N}(d_2)$$

whereas,

$$P(0) = K\,\mathrm{e}^{-rT}\,\mathcal{N}(-d_2) - S(0)\mathcal{N}(-d_1)$$

where,

$$d_1 = \frac{\ln(S(0)/K) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln(S(0)/K) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

PROOF. We know that the value of the call option at the maturity T is

$$C(T) = \max[S(T) - K, 0].$$

We use risk-neutral valuation of call option

$$\begin{aligned} C(0) &= e^{-rT}\ E[C(T)] \\ &= e^{-rT}\ E[\max(S(T) - K, 0)]. \end{aligned} \tag{4.1.2}$$

Then, we can see here the price of a call option depends on the value of stock price. Therefore, One of the advantages of using the lognormal model of a stock is that helps us to compute the probability of call option expires in the money (ITM); that is equivalent to computing the probability that $S(T) \geq K$.

Therefore:

$$
\begin{aligned}
P(S(T) \geq K) &= P\left(S(0)\; e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}Z} \geq K\right) \\
&= P\left(e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\;\sqrt{T}Z} \geq \frac{K}{S(0)}\right) \\
&= P\left(\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}Z \geq \ln\left(\frac{K}{S(0)}\right)\right) \\
&= P\left(Z \geq \frac{\ln\left(\frac{K}{S(0)}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\;\sqrt{T}}\right) \\
&= P\left(Z \geq -d_2\right)
\end{aligned}
$$

Then,

$$
\max[S(T)-K,0] = \begin{cases} S(0)\; e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}Z} - K & \text{, if } Z \geq -d_2 \\ 0 & \text{, if } Z < -d_2 \end{cases}
$$

back to equation (4.1.2)

$$
\begin{aligned}
C(0) &= e^{-rT}E[\max(S(T)-K,0)] \\
&= e^{-rT}\int_{-d_2}^{\infty}\left[S(0)\; e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}\boldsymbol{x}} - K\right]\frac{1}{\sqrt{2\;\pi}}\; e^{-\frac{\boldsymbol{x}^2}{2}}\; d\boldsymbol{x} \\
&= \frac{e^{-rT}}{\sqrt{2\;\pi}}\int_{-d_2}^{\infty} S(0)\; e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\;\sqrt{T}\boldsymbol{x}} e^{-\frac{\boldsymbol{x}^2}{2}}\; d\boldsymbol{x} \\
&- \frac{Ke^{-rT}}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{-\frac{\boldsymbol{x}^2}{2}}\; d\boldsymbol{x}
\end{aligned} \tag{4.1.3}
$$

Let's focus on the first integral in the equation (4.1.3) for now,

$$\frac{1}{\sqrt{2\pi}}\ S(0)\ e^{-rT}\int_{-d_2}^{\infty}\ e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}\boldsymbol{x}-\frac{\boldsymbol{x}^2}{2}}\ d\boldsymbol{x}$$

$$\Longrightarrow \frac{1}{\sqrt{2\pi}}\ S(0)\int_{-d_2}^{\infty}\ e^{-\frac{\sigma^2}{2}T+\sigma\sqrt{T}\boldsymbol{x}-\frac{\boldsymbol{x}^2}{2}}\ d\boldsymbol{x}$$

$$\Longrightarrow \frac{1}{\sqrt{2\pi}}\ S(0)\int_{-d_2}^{\infty}\ e^{-\frac{1}{2}\left(\boldsymbol{x}^2-2\sigma\sqrt{T}\boldsymbol{x}+\sigma^2\ T\right)}\ d\boldsymbol{x}$$

$$\Longrightarrow \frac{1}{\sqrt{2\pi}}\ S(0)\int_{-d_2}^{\infty}\ e^{-\frac{1}{2}\left(\boldsymbol{x}-\sigma\sqrt{T}\right)^2}\ d\boldsymbol{x} \tag{4.1.4}$$

$$\text{Let}\quad y=\boldsymbol{x}-\sigma\sqrt{T}$$

$$dy=d\boldsymbol{x}$$

For the lower limit of the integral

$$y=-d_2-\sigma\sqrt{T}$$

$$=\left[\left(\frac{\ln(K/S(0))-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)-\sigma\sqrt{T}\right]$$

$$=\frac{\ln(K/S(0))-rT+\left(\frac{\sigma^2}{2}\right)T-\sigma^2T}{\sigma\sqrt{T}}$$

$$=\left(\left[\ln(K/S(0))-rT+\frac{\sigma^2T-2\sigma^2T}{2}\right]\Big/\sigma\sqrt{T}\right)$$

$$=\left(\left[\ln(K/S(0))-\left(r+\frac{\sigma^2}{2}\right)T\right]\Big/\sigma\sqrt{T}\right)$$

$$=-d_1$$

$\therefore$ the lower limit of the integral will be $y = -d_1$. Then, back to equation (4.1.4) and substitute y and dy

$$\frac{1}{\sqrt{2\pi}}\, S(0) \int_{-d_1}^{\infty} e^{-\frac{1}{2}y^2}\, dy$$

$$\Longrightarrow S(0)\, [1 - \mathcal{N}(-d_1)]$$

$$\Longrightarrow S(0)\mathcal{N}(d_1) \tag{4.1.5}$$

Now, let's back to the second integral in equation (4.1.3)

$$K\ e^{-rT} \left[\frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} \mathrm{e}^{\frac{-\boldsymbol{x}^2}{2}}\ d\boldsymbol{x} \right]$$

$$\Longrightarrow K\, \mathrm{e}^{-rT}\, [1 - \mathcal{N}(-d_2)]$$

$$\Longrightarrow K\, \mathrm{e}^{-rT}\, \mathcal{N}(d_2) \tag{4.1.6}$$

substitution equations (4.1.5) and (4.1.6) in equation (4.1.3), we obtain

$$C(0) = S(0)\mathcal{N}(d_1) - K\ e^{-rT}\mathcal{N}(d_2) \tag{4.1.7}$$

This is the Black-Scholes formula for the European Call option.

To obtain the Black-Scholes formula for the European Put option, it can be done similarly.

However, we can calculate it easily using Put and Call parity as follow:

$$P(0) + S(0) = C(0) + Ke^{-rT}$$

We want to calculate $P(0)$, so

$$\begin{aligned} P(0) &= C(0) - S(0) + Ke^{-rT} \\ &= \left[S(0)\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2)\right] - S(0) + Ke^{-rT} \\ &= Ke^{-rT}[1 - \mathcal{N}(d_2)] - S(0)[1 - \mathcal{N}(d_1)] \\ &= Ke^{-rT}\mathcal{N}(-d_2) - S(0)\mathcal{N}(-d_1). \end{aligned}$$

Therefore, the Black-Scholes formula for European Put option is

$$P(0) = Ke^{-rT}\mathcal{N}(-d_2) - S(0)\mathcal{N}(-d_1). \tag{4.1.8}$$

□

4.2. Properties of the Black Scholes formula

We want to see if some parameters of the formula take an extreme value, what will happen? If $S(0)$ becomes large, both d_1 and d_2 will be very large; as a consequence, $\mathcal{N}(d_1)$ and $\mathcal{N}(d_2)$ will be close to 1. So, the Black Scholes call option equation almost will be close to

$$S(0) - K\,\mathrm{e}^{-rT}$$

However, European put option value approaches zero when $S(0)$ is large because $\mathcal{N}(-d_1)$ and $\mathcal{N}(-d_2)$ are both close to zero.
Moreover, let's consider the case that $S(0)$ becomes very small. Therefore, the call option will be close to zero since d_1 and d_2 become very large in negative, and then $\mathcal{N}(d_1)$ and $\mathcal{N}(d_2)$ are both close to zero. Regarding put option price that will become close to

$$Ke^{-rT} - S(0)$$

because $\mathcal{N}(-d_1)$ and $\mathcal{N}(-d_2)$ become close to 1.

Understanding $\mathcal{N}(d_1)$ and $\mathcal{N}(d_2)$.

There is a straightforward interpretation of $\mathcal{N}(d_2)$ in the Black Scholes formula for pricing European call option which is the probability of the exercise in the risk-neutral world. We have seen that in the derivation of this formula where we compute the probability of the stock price at maturity will be above or at the strike price.

However, the term $\mathcal{N}(d_1)$ is not easy to interpret. Let's first write the expression of the expected payoff of the call option in a risk-neutral world at maturity T as

$$C(T) = S(0)e^{rT}\mathcal{N}(d_1) - K\mathcal{N}(d_2) \tag{4.2.1}$$

where $S(0)e^{rT}\mathcal{N}(d_1)$ is the expected stock price in risk-neutral world at time T; when $S(T) < K$, $S(0)e^{rT}\mathcal{N}(d_1)$ counted as zero. Regarding K that is the strike price will be paid only if the $S(T) > K$, which has the probability $\mathcal{N}(d_2)$.

If we want to write the present value of the equation (4.2.1), we multiply both sides by the present value factors e^{-rT}. As a result, that will be the same as the Black Scholes formula for call option:

$$C(0) = S(0)\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2) \tag{4.2.2}$$

In other world, if we rewrite the last equation above as

$$C(0) = e^{-rT}\mathcal{N}(d_2)\left[S(0)\ e^{rT}\ \mathcal{N}(d_1)/\mathcal{N}(d_2) - K\right]$$

The term $e^{rT}\mathcal{N}(d_1)/\mathcal{N}(d_2)$ means the expected percentage increase in the stock if the option exercise[**2**].

CHAPTER 5

Geometric Brownian Motion

The goal of this chapter is to describe and justify the model of a stock price.

The first section explains an economic justification of the stock price model which is Efficient Market Hypothesis.

The second section introduces the main assumption of Black Scholes formula which is a return of the stock is normally distributed over independent intervals.

What we do in the third section is that to grasp the ideas of Brownian motion, we start by a construction of a standard Brownian motion using a simple example of a symmetric random walk. Then, we study some of the properties of a discrete and continuous random walk. Finally, we discuss Geometric Brownian motion. We introduce its definition and its properties.
The main references for this chapter is [**4**] in the first and second sections, and [**12**], [**8**], [**7**], [**13**], and[**1**] in the rest.

5.1. Efficient Market Hypothesis

Because one of our goals in this paper is to justify the model of stock price, one of the economic reasons is Efficient Market Hypothesis.

When you think of the stock price and what factors may affect its value, you may find many factors in the market. For example, the stock price can be influenced by company policy, competition with others, inventions, current price, its history, and some others depend on the type of stocks. Therefore, coming up with a model for the future stock price is not easy to include all these factors, so mathematical modelers have come up with some approximation assumptions to simplify that. Then, they could obtain an equation by assuming the market is efficient.

Therefore, the randomness that must be in the asset prices due to *Efficient Market Hypothesis*, which basically can be summarized in two points [**4**]:

- The past history of the asset price is fully reflected in the current price.
- Markets respond immediately to any new information about an asset.

As a result of the Efficient Market Hypothesis, the model has Markov property which merely means no memory, so the model is independent of the past, and it models the arrival of new information which affects the price. That gives economical reasons justifying the model. This concept will be discussed with further details later on in this chapter.

It should be mentioned that even though Efficient Market Hypothesis may be an unrealistic assumption, it helps the mathematical modelers to come up with a model to value future stock price and reduce some of the riskiness that associate with the financial world.

Professor Donald G. Saari mentioned that the Efficient Market Hypothesis is a nice, most convenient assumption for mathematical construction model; however, it is not valid in general (2014).

5.2. Assumptions about How Stock Prices Evolve

There is a simple assumption of the model of the future stock price as geometric Brownian motion. Considering a non-dividend paying stock, Black, Scholes and Merton assumed that return on the stock in a short period of time is normally distributed. Also, in two different non-overlapping intervals, their returns are supposed to be independent [**2**].

Let's first, discuss the meaning of the return and then explain its distribution. The return which is the main concern for anyone who invests their money is defined as the change in the stock price divided by its initial value.
Let S_i denote the stock price on the i_{th} day, the return from day i to day $i+1$ is

$$R_i = \frac{S_{i+1} - S_i}{S_i} \tag{5.2.1}$$

Investors concentrate on the percentage price change instead of the absolute change which is not a useful quantity by itself. Let S be the stock price and throughout this paper we assume non-dividend paying stock. Let dS the instantaneous change in the stock. As it is mentioned above the return on stock price $\left(\frac{dS}{S}\right)$ between now and a very short time (dt) – where dt represents small time interval – is normally distributed with mean rdt, and variance $\sigma^2 dt$, that is

$$\frac{dS}{S} \sim \mathcal{N}(rdt, \sigma^2 dt). \tag{5.2.2}$$

For interpretation of the above equation, we will write it in the form of a stochastic differential equation as follow:

$$\frac{dS}{S} = rdt + \sigma dB. \tag{5.2.3}$$

We will describe our model which is equation (5.2.3); however, it will be investigated in-depth later in this chapter, and then we obtained its solution in the following chapter.
The model contains two parts:

(1) One part is predictable which is rdt.

It represents a deterministic effect or change such as interest, where r is the continuously compounded expected return which is known as drift, and it is assumed to be constant in a simple model like the model that we are discussing here. However, r can be a function of S and t in more complicated models.

(2) The second part of the model σ represents the randomness of the stock price due to a variety of influences such as unexpected news which make the price vary up or down in short time.

Where σ is called *Volatility*, that is assumed to be constant and its measures the standard deviation of the return.

So far, we know what we meant by all symbols in equation(5.2.3) except dB will be explored in details in the following section.

If we ignore the randomness of the stock for a moment and assume $\sigma = 0$, we would have the ordinary differential equation:

$$\frac{dS}{S} = rdt \tag{5.2.3}$$

or

$$dS = rSdt.$$

Therefore, the return is like invested money in a risk-free bank, and that is like saving account which has some fixed rate of interest. Thus, we can predict with certainty the balance of the account at any future date by knowing the current balance and interest rate. We solve our ordinary differential equation(5.2.4), we get the following:

$$S = S_0 e^{r(t-t_0)}. \tag{5.2.4}$$

Unfortunately, the absence of randomness cannot happen in the stock price since it is a feature of it; hence, the random term must be part of the model of this risky investment.

Let's consider some of the properties of dB:

(1) dB is a random variable drawn from a normal distribution.

(2) the mean of dB is zero.

(3) the variance of dB is dt.

The term dB is called *Wiener process* and it is also known as *Brownian motion.*

We need the expected value of dB to be zero, and the variance to be dt because we want the random term dB to be just oscillation around the drift with standard deviation $\sigma\sqrt{dt}$.

Technical Point: Parameter Estimation

Fortunately, σ is the only parameter that is needed to be estimated because it appears in the Black Scholes formula. A simple approach to the estimation of volatility from historical data is all about calculated the standard deviation of short-term returns as follow:

- We fix a standard time period dt (*i.e.* one day), and then we express it in a year unit (*i.e.* $\frac{1}{365}$ or mostly used $\frac{1}{252}$ since 252 is the number of trading days per annum).
- We collect the data regarding the price of the stock in the past.
- We compute the daily return for this data using the following:

$$r_i = \ln(S_{i+1}/S_i)$$

- we compute the average of the sample returns. Let's have a total of $n+1$ returns which are $r_0 + r_1 + \cdots + r_n$

$$\bar{r} = \frac{1}{n+1}(r_0 + r_1 + \cdots + r_n)$$

- we compute the standard deviation as

$$\sigma = \frac{1}{\sqrt{dt}}\sqrt{\frac{1}{n}((r_0 - \bar{r})^2 + (r_1 - \bar{r})^2 + \cdots + (r_n - \bar{r})^2)}$$

We divided by $\sqrt{dt}$ because in our model the volatility of a period of time dt is $\sigma\sqrt{dt}$. Also, inside the square root, we divided by n and not $n+1$ because we are estimating the volatility not computing the actual value of it.

5.3. Standard Brownian Motion

5.3.1. Construction of Brownian Motion. In this section, we will discuss an example of a simple symmetric random walk, and some of its properties. Then we build on that to construct a Brownian motion which is a version of a random walk.

A stochastic process is a random process that is a function of time.

A random walk is a stochastic process with independent increments.

We say a random walk is symmetric if the probability of an increase in the value of the random walk equals the probability of decrease in the random walk.

Example 5.3.1. As a motivation example of a discrete random walk is tossing a coin.
Let's consider this game of tossing a coin, every time you get heads, you win \$1. However, if you get tails, you lose \$1. We want to know how much money you have after n tosses. Consider a sequence of random payouts R_j of $\pm\$1$. Assume these IID, with $P(R=1) = P(R=-1) = 0.5$. Then,

$$\mathrm{E}[R] = 0,$$

and

$$Var[R] = 1.$$

As we know in this experiment, these expectations are conditional on the past or not; it does not really matter since the outcome on the n_{th} toss does not affected with the previous outcomes.
Suppose X_i is the total amount of money that you have won up to and include the payout at time $t = i$.
So,

$$X_i = \sum_{j=1}^{i} R_j$$

It will be useful later on when we reach the Brownian motion point if we start with no money as $X_i = 0$. Now, let's calculate the expectation of X_i. In this case, It does matter what information we have. Before the experiment has even started, the calculation of the future events will be

$\mathrm{E}[X_i] = 0$ that is because of $\mathrm{E}[R] = 0$.

Then,

$$Var[X_i] = i.$$

That follows from the below calculation

$$\begin{aligned}
Var[X_i] &= \mathrm{E}[X_i^2] - (\mathbf{E}[X_i])^2 \\
&= E[X_i X_i] - 0 \\
&= E[(R_1 + - - - + R_i)(R_1 + - - - + R_i)] \\
&= E[R_1^2 + R_1 R_2 + - - - + R_i R_{i-1} + R_i^2] \\
&= E[R_1^2 + R_2^2 + - - - + R_i^2] \\
&= i
\end{aligned}$$

That happens because the only terms that are matter are the squared terms since the rest will be zero due to independence of R_i and R_j where i is not equal to j.

That is $E[R_i R_j] = E[R_i]E[R_j] = 0$

Therefore, the main property of such a payout process which is a simple example of a random walk is the following:

- Quadratic Variation: the quadratic variation is property of a random walk which can be defined as

$$\sum_{j=1}^{i}(X_j - X_{j-1})^2$$

Therefore, in the experiment,

$$\sum_{j=1}^{i}(X_j - X_{j-1})^2 = \sum_{j=1}^{i}(R_j)^2 = \sum_{j=1}^{i}1 = i$$

The main objective of using a coin tossing experiment is to modify it to reach a continuous time random walk which is the Brownian motion that we aimed to demonstrate.

Changing the rules of the coin tossing experiment, set t to be the number of time periods, and n to be the number of steps in the random walk. Then, the length of one time step t/n, and we change the bet size (increment) instead of being \$1 to be $\sqrt{t/n}$. So

$$R_i^* = \pm\sqrt{t/n}$$

With $X_i = \sum_{j=1}^{i} R_j^*$, we see that

$$\mathrm{E}[X_i] = 0,$$

That is due to $\mathrm{E}[R_i^*] = 0$

$$\mathrm{E}[X_i^2] = i(\sqrt{t/n})^2 = i(t/n)$$

$$\mathrm{E}[X_n^2] = n(\sqrt{t/n})^2 = n(t/n) = t$$

That can be expressed in the form of increments as

$$\sum_{j=1}^{n}(X_j - X_{j-1})^2 = t$$

It should be noted that the final result is independent of n which is the number of the step in the random walk, so it does not matter how many time steps there, the result should be equal to t. Therefore, for this reason, the increment scales with the square root of the time step($\sqrt{t/n}$), so the variance remains finite.

Now, it is time to move from discrete random walk to continuous random walk by consider the limit as $n \to \infty$, and call the resulting process X_t, that is the value of the random variable after a time t (the money you have own). Then, for each $t > 0$,

$$\mathrm{E}[X(t)] = 0$$

and

$$Var[X(t)] = \mathrm{E}[X(t)^2] = t$$

As the time step goes to zero, the limiting process for this random walk is called *Brownian motion* that will be denoted in this paper by $B(t)$, and we will study it in further details in the following section.

From our experiment, we can write:

$$B(t) = \sqrt{t}[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i] \tag{5.3.1}$$

Considering the term in the square bracket, we apply the Central Limit Theorem that is stated in the second chapter. As a result, as n goes to infinity, we have

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} R(ih) \to N(0,1),$$

and, therefore, equation (5.3.1) implies that:

$$B(T) = \sqrt{T}Z; \;\; with \;\; Z \sim N(0,1).$$

Also, we can write above expression in the differential form of Brownian motion which is the following stochastic differential equation as

$$dB(t) = \sqrt{dt}Z. \tag{5.3.2}$$

Now let's study the definition of Brownian motion and some of its main properties.

5.3.2. Brownian Motion.

DEFINITION 5.3.2 (Brownian Motion). Brownian motion or as it is called standard Brownian motion is a continuous stochastic process $B(t)$ with the following characteristics:

- $B(0) = 0$
- For all $t \geq 0$ and all $h > 0$, the increment $B(t+h) - B(t)$ is a normal random variable with mean 0 and variance h.
- $B(t)$ has independent increments: Consider a sequence of non-overlapping time intervals $(t_j, t_j + h_j]$ (touching in the endpoints is OK) then the increments $B(t_j + h_j) - B(t_j)$ are independent.

Now it is time to review some essential *properties of Brownian motion*.

- *Finiteness*: It is crucial that scaling the increment with square root of the time step because any other scaling would have ended with random walk going to infinity in a finite time or a limit with no motion at all.

- *Continuity*: there are no discontinuities in the path of Brownian motion, and its path is fractal and not differentiable anywhere.

- *Markov:* the conditional distribution of $B(t)$ given information up to time $\tau < t$ only depends on $B(\tau)$.That means the future state given the present state is independent of the past [**14**].

- *Quadratic Variation*: when we divide up the time 0 to t in a partition with n+1 Partition points $t_j = jt/n$ then

$$\sum_{j=1}^{n} (B(t_j) - B(t_{j-1}))^2 \to t \ (almost\ surely)$$

since

$$\begin{aligned}\lim_{n\to\infty}\sum_{j=1}^{n}(B(t_j)-B(t_{j-1}))^2 &= \lim_{n\to\infty}\sum_{j=1}^{n}(R_{jh}\sqrt{h})^2\\ &= \lim_{n\to\infty}\sum_{j=1}^{n}h\\ &= \lim_{n\to\infty}\sum_{j=1}^{n}\frac{T}{n}\\ &= T\end{aligned}$$

Where we have used that $R_{jh} = \pm 1$ has a binomial distribution.

Thus, the quadratic variation of Brownian process $B(t)$ is defined as the sum of the squared increments to the process. We have shown that using the approximation of Brownian motion as the sum of the independent binomial random variable. Therefore, the quadratic variation of Brownian motion is finite and equal to T, which means is not a random variable. Hence, there is an important application of the finiteness of the quadratic variation which is that the higher order variations are zero. This result is important, and we will refer to it throughout this paper.

EXAMPLE 5.3.3. the sum of the cubed increments of the standard Brownian motion B is equal to zero. That is:

$$\lim_{n\to\infty}\sum_{i=1}^{n}[B(ih)-B((i-1)h)]^3 = 0$$

That can be shown as the same way where we did the quadratic variation.

Now we want to know the *total variation* of the standard Brownian motion.

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}|B(ih)-B((i-1)h)| &= \lim_{n\to\infty}\sum_{i=1}^{n}|R_{ih}|\sqrt{h} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{h} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{\sqrt{T}}{\sqrt{n}} \\
&= \sqrt{T}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{n}} \\
&= \sqrt{T}\lim_{n\to\infty}\sqrt{n} \\
&= \infty
\end{aligned}
$$

Therefore, over any finite interval $[0,T]$ the absolute length of the Brownian path is infinite. As a result, the Brownian motion path moves up and down rapidly, and that implies infinite crossing property, so in the interval, $[0,T]$, the path of Brownian motion will cross its starting point an infinite number of times.

5.4. Geometric Brownian Motion

The drift and volatility can be written as a function of the random variable $X(t)$ as follow:

$$dX(t)=\alpha(X(t))dt+\sigma(X(t))dB(t)$$

Solutions of this equation are called an *Ito process*, when the drift α and the volatility σ depend on $X(t)$. Here $dB(t)$ is the change in the Brownian motion over a short period of time. In particular if $\alpha(X(t))=\alpha X(t)$ and $\sigma(X(t))=\sigma X(t)$, so the previous equation becomes:

$$dX(t)=\alpha X(t)dt+\sigma X(t)dB(t)$$

so

$$\frac{dX(t)}{X(t)}=\alpha dt+\sigma dB(t) \tag{5.4.1}$$

This equation is known as *Geometric Brownian Motion.* It says the percentage change in the random variable $X(t)$ is normally distributed with instantaneous mean α and instantaneous variance σ^2.

NOTE 5.4.1. this equation is almost the same as equation (3.2.3),but the stock price is our random variable, and $\alpha = r$ which is the constant risk-free interest rate. Therefore, the price of stocks and other assets follows geometric Brownian motion as it is assumed by Black and Scholes.

To obtain a solution for this equation, we need a mathematical tool called *Itô's Lemma* and that will be discussed in details in the next chapter.

Now, let's consider some of Geometric Brownian Motion Properties.

Geometric Brownian Motion Properties

Suppose that $S(t)$ is Geometric Brownian Motion with drift parameter μ, volatility σ and initial value S_0 then

(1) S is a stochastic process with initial value S_0
(2) S has independent growth factors: for any sequence on non-overlapping intervals $(t_j, t_j + h_j]$ the growth factors $S(t_j + h_j)/S(t_j)$ are independent.
(3) For all $t \geq 0$ and $h > 0$ the growth factor $S(t + h)/S(t)$ is log-normal $e^{(N(\mu h, \sigma^2 h))}$ with mean $e^{(\mu+1/2\sigma^2)h}$ and variance $e^{(2\mu+\sigma^2)h}(e^{\sigma^2 h} - 1)$

Now, it will be illustrated that stock price has independent growth factors that are log-normally distributed. Suppose S_T denote stock price at future time T and let S_0 be the initial value of the stock at time 0. Suppose the ratios $S(T)/S(T-1)$, where $t \geq 1$ are independent and identically distributed, then we can write:

$$S_T = \frac{S_T}{S_{T-1}} S_{T-1}$$

which upon iteration gives

$$\begin{aligned} S_T &= \frac{S_T}{S_{T-1}} \frac{S_{T-1}}{S_{T-2}} S_{T-2} \\ &= \frac{S_T}{S_{T-1}} \frac{S_{T-1}}{S_{T-2}} ... S_0 \end{aligned}$$

This equation represents stock price as a product of a large number of IID random variables, so we apply natural logarithm which will change the multiplication to addition:

$$\ln(S_T) = \ln(\frac{S_T}{S_{T-1}}) + \ln(\frac{S_{T-1}}{S_{T-2}}) + ... + \ln(S_0)$$

So:

$$\ln(S_T) - \ln(S_0) = \ln(\frac{S_T}{S_{T-1}}) + \ln(\frac{S_{T-1}}{S_{T-2}}) + ...$$

So:

$$\ln(S_T/S_0) = \sum_{i=1}^{T} \ln(S_i/S_{i-1})$$

The sum of independent random variables, each of small variance is approximately normal due to Central Limit Theorem. Therefore, $ln(S_T/S_0)$ will be suitable normalize, and S_T/S_0 will be approximately Geometric Brownian Motion[**9**].

CHAPTER 6

Itô's Lemma.

We need an important mathematical tool called Itô's Lemma to deal with the stochastic process that we have and obtain a solution for our model.

In this chapter, we start by explaining some of Itô's process multiplication rules that we need. Then, we discuss and prove Itô's Lemma. Moreover, we demonstrate some of Itô's Lemma application which is our main target in this chapter, where we obtain a solution for the geometric Brownian motion model.

In this chapter, the main references are [**4**], [**15**], [**16**].

6.1. Itô process multiplication rules

Let's recall some symbols from the previous chapter, dB_t and dt which represent the changes in Brownian motion and changes in times, respectively. Since we are working with Itô process and geometric Brownian motion, there is some situations where we need to multiply dB_t by itself, dt by itself, or dB_t by dt. These product rules are called *Itô process multiplication rules*, and will see their application in Itô's Lemma derivation.
First rule is

$$(dt)^n = 0, \qquad , \qquad n \geq 2$$

as we Know from regular Calculus.

The second rule will be used is

$$dtdB_t = 0,$$

where

$$dB_t = B_{t+dt} - B_t$$

$$= B_{dt} = \sqrt{dt}Z, \quad Z \sim \mathcal{N}(0,1).$$

In order to show this rule, we will write it in the integration form as:

$$\int_0^T dt dB_t = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{T}{n} \left(\sqrt{T/n} Z_i\right)$$

$$= T(\sqrt{T} \lim_{n\to\infty} \frac{1}{\sqrt{n}} (\frac{1}{n} \sum_{i=1}^{n} Z_i)$$

$$= T\sqrt{T} \cdot 0 \cdot (E(Z))$$

$$= 0$$

The third rule is that

$$dB_t^2 = dt$$

We can write it as follow:

$$\int_0^T (dB_t)^2 = T = \int_0^T dt$$

i.e.,

$$\int_0^T (dB)^2 = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\sqrt{T/n} Z_i\right)^2$$

$$= \lim_{n\to\infty} T \left[\frac{1}{n}\sum_{i=1}^{n} Z_i^2\right]$$

$$= TE(Z^2)$$

$$= TVar(Z)$$

$$= T$$

So, $(dB_t)^2 = dt$.

Also, $(dB_t)^{n\geq 3} = 0$ because $(dB_t)^{n\geq 3} = dt(dB_t)^{n\geq 1} = 0$

6.2. Itô's Lemma

The rules for differentiation and integration are different in stochastic calculus from those in classical calculus. Itô's Lemma is one of the main theorems in a stochastic environment. It plays a rule to stochastic variables as Taylor series do to deterministic variables. Itô's Lemma is a way of expanding functions or approximating in a series in dt. However, the difference between Itô's Lemma and Taylor series is using the multiplication rules explained in the previous section, in particular $(dB_t)^2$ is not equal to zero, but dt^2 and $dtdB_t$ does.

Now, we want to know what is an Itô's Lemma.

THEOREM 6.2.1. *(Itô's Lemma). Let X_t be a stochastic process, and suppose we have $f(t,x)$, $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. We have $X_t = f(t, B_t), where B_t$ is the standard Brownian motion.*

Then,

$$dX_t = df(t, B(t)) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right) dt + \frac{\partial f}{\partial x} dB(t)$$

PROOF. From Taylor series,

$$df(t,B(t)) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dB(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dB(t))^2$$

$$+ \text{ other terms that are 0 by Itô process multiplication rules.}$$

And so, we obtained

$$df(t,B(t)) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dB(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dt$$

$$= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \frac{\partial f}{\partial x}dB(t).$$

$\square$

6.3. Application of Itô's Lemma

The main target of this chapter is to obtain a closed form solution for the stock price as a Geometric Brownian Motion. We have referred to the lognormal model which is the solution of the stochastic deferential equation for the stock price, but here we will show how we get that by applying Itô's lemma. This solution is the one we use in the derivation of the Black Scholes formula.

Let $f(t,x) = e^{\mu t + \sigma x}$, let $S_t = f(t, B(t))$

Then, by applying Itôs lemma, we obtain

$$dS_t = df(t,B(t)) = \left[\mu e^{\mu t+\sigma B(t)} + \frac{1}{2}\sigma^2 e^{\mu t+\sigma B(t)}\right]dt + \sigma e^{\mu t+\sigma B(t)}dB(t)$$

$$= \left(\mu + \frac{1}{2}\sigma^2\right)S_t dt + \sigma S_t dB(t)$$

Therefore, if we set $r = \mu + \frac{1}{2}\sigma^2$, we get

$$dS_t = rS_t dt + \sigma S_t dB(t).$$

That is our Geometric Brownian motion which satisfies

$$S = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}Z}.$$

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