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Approximating Average Bounded-Angle Minimum Spanning Trees

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Approximating Average Bounded-Angle Minimum Spanning Trees

By

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Approximating Average Bounded-Angle Minimum Spanning Trees

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DECLARATION OF CO-AUTHORSHIP AND PREVIOUS PUBLICATION

I. Co-Authorship

I hereby declare that this thesis incorporates material that is the result of joint research, as follows:

Chapter 2 of the thesis include the outcome of publications which have the following other co-authors: Ahmad Biniaz, Prosenjit Bose. In all cases only my primary contributions towards these publications are included in this thesis, and the contribution of co-authors Dr. Biniaz and Dr. Bose was primarily through supervision. Dr. Biniaz had the initial idea that an improved algorithm could be designed. I proposed ideas for the algorithm and correctness proofs, and Dr. Biniaz and Dr. Bose provided consultation and discussion. I wrote the initial draft of the paper, and all authors contributed to editing and proofreading prior to submission for publication.

Chapter 3 of the thesis incorporates unpublished material co-authored with Ahmad Biniaz and Prosenjit Bose. In all cases only my primary contributions towards this material is included in this thesis, and the contribution of co-authors Dr. Biniaz and Dr. Bose was primarily through supervision. I proposed the idea that the approximation ratio lower bound proof from previous works could be generalized. I proposed ideas for new algorithms, and Dr. Biniaz and Dr. Bose provided consultation and discussion. I wrote up the results, and all authors contributed to proofreading prior to inclusion in this thesis.

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This thesis includes 1 original papers that have been previously published/submitted to journals for publication, as follows:

Thesis Chapter	Publication title/full citation	Publication Status
Chapter 2	*A $\frac{13}{9}$-approximation of the average $\frac{2\pi}{3}$-MST* A. Biniaz, P. Bose, P. Devaney In proceedings of the 34th Canadian Conference on Computational Geometry (CCCG), pp. 55–59, 2022.	Published

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ABSTRACT

Motivated by the problem of orienting directional antennas in wireless communication networks, we study average bounded-angle minimum spanning trees. Let P be a set of points in the plane and let α be an angle. An α-spanning tree (α-ST) of P is a spanning tree of the complete Euclidean graph induced by P with the restriction that all edges incident to each point $p \in P$ lie in a wedge of angle α with apex p. An α-minimum spanning tree (α-MST) of P is an α-ST with minimum total edge length.

An average-α-spanning tree (denoted by $\overline{\alpha}$-ST) is a spanning tree with the relaxed condition that incident edges to all points lie in wedges with average angle α. An average-α-minimum spanning tree ($\overline{\alpha}$-MST) is an $\overline{\alpha}$-ST with minimum total edge length. We first focus on $\alpha = \frac{2\pi}{3}$. Let $A\left(\alpha\right)$ be the smallest ratio of the length of the $\overline{\alpha}$-MST to the length of the standard MST, over all sets of points in the plane. Biniaz, Bose, Lubiw, and Maheshwari (Algorithmica 2022) showed that $\frac{4}{3} \leq A\left(\frac{2\pi}{3}\right) \leq \frac{3}{2}$. We improve the upper bound and show that $A\left(\frac{2\pi}{3}\right) \leq \frac{13}{9}$.

We then generalize the lower bound argument of Biniaz et al. (Algorithmica 2022) for $A\left(\frac{2\pi}{3}\right)$ to a formula giving a lower bound on $A\left(\alpha\right)$ for any $\alpha \leq \pi$. We further show how to modify the algorithm of Biniaz et al. (Algorithmica 2022) for the $\overline{\frac{2\pi}{3}}$-MST to compute the $\overline{\pi}$-MST, and show that $A\left(\pi\right) = 1$. Finally, we present an algorithm to compute the $\overline{\frac{\pi}{2}}$-MST, and show that $\frac{3}{2} \leq A\left(\frac{\pi}{2}\right) \leq 4$.

DEDICATION

To everyone who still wears a mask on campus.

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CHAPTER 1

Introduction

1.1 Basic Definitions

A *graph* $G = (V, E)$ is a pair consisting of a set V of *vertices* and a collection E of *edges* between pairs of vertices.

Edges can be assigned a numeric value called a *weight*.

In a *geometric graph*, the weight is the distance between the corresponding vertices.

A *tree* is a connected, acyclic graph.

A *minimum spanning tree* (MST) of a graph $G = (V, E)$ is a tree $T = (V, E' \subseteq E)$ with minimum total edge weight such that all vertices in V are connected. (e.g. Figure 1.1.1)

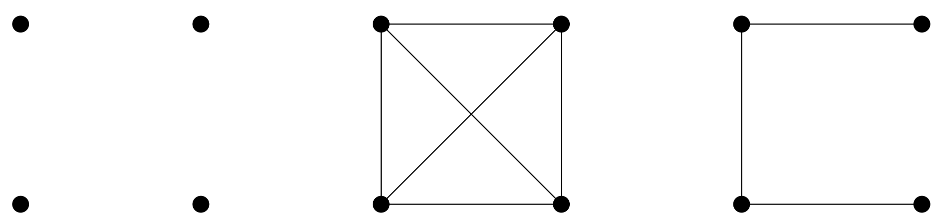

Fig. 1.1.1: A set of points (left), the complete geometric graph induced by the points (center), and an MST of this graph (right)

1.2 Background

A wireless communication network can be represented as a geometric graph in the plane. Each antenna is represented by a point p, its transmission range is represented by a disk with radius r centered at p, and there is an edge between two points if they are within each other's transmission ranges (see Figure 1.2.1). The problems of assigning transmission ranges to antennas to achieve networks possessing certain properties has been widely studied [3, 6, 10, 13, 15, 16, 17, 18].

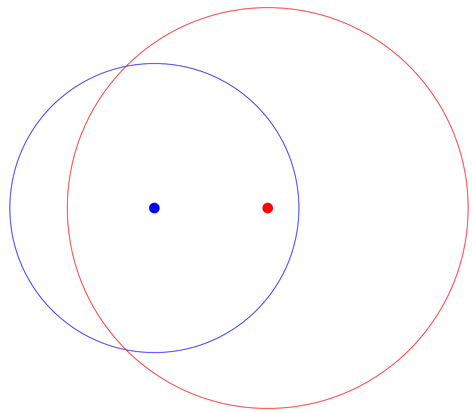

Fig. 1.2.1: Two points representing antennas, and disks representing their respective transmission ranges.

In recent years, there has been considerable research on the problem of replacing omni-directional antennas with directional antennas [1, 2, 5, 7, 9, 11, 12, 14, 15, 19]. Here, the transmission range of each point p is an oriented wedge with apex p and angle α (see Figure 1.2.2). Directional antennas provide several advantages over omni-directional antennas, including less potential for interference, lower power consumption, and reduced area where communications could be maliciously intercepted [3, 19].

Motivated by this problem, Aschner and Katz [2] introduced the α-*Spanning Tree* (α-ST): a spanning tree of the complete Euclidean graph in the plane where all incident edges of each point p lie in a wedge of angle α with apex p (e.g. Figure 1.2.4). They also presented approximation algorithms for the cases where $\alpha = \frac{\pi}{2}, \frac{2\pi}{3}$, and π, with approximation factors of 16, 6, and 2, respectively, with respect to the MST. They further observed lower bounds on the approximation ratio of 2 when $\alpha \in \left[\frac{\pi}{3}, \pi\right)$ (e.g. points on a line) and $\frac{2+\sqrt{3}}{3}$ when $\alpha \in \left[\pi, \frac{4\pi}{3}\right)$ (e.g. 3 points forming an equilateral triangle with a fourth point in the center).

For the case where $\alpha = \frac{\pi}{2}$, the original algorithm of [2] approximates the TSP

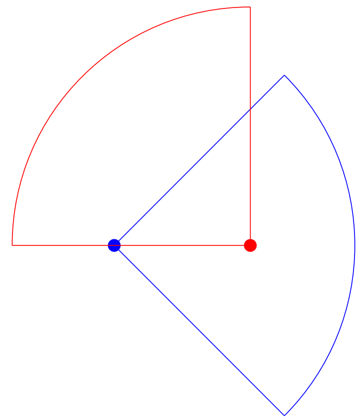

Fig. 1.2.2: Two points representing directional antennas, and wedges representing their respective transmission ranges.

tour from the underlying MST, and then arranges the points into consecutive groups of 8. Using a gadget described by Aschner et al. [4], it then arranges the leftmost and rightmost 4 points such that they are connected, and the combination of the two arrangements covers the entire plane. This gives a 16-approximation of the $\frac{\pi}{2}$-MST. Biniaz et al. [8] improved this to a 10-approximation by showing that an arrangement of groups of 5 points on the TSP tour could be oriented to cover the plane.

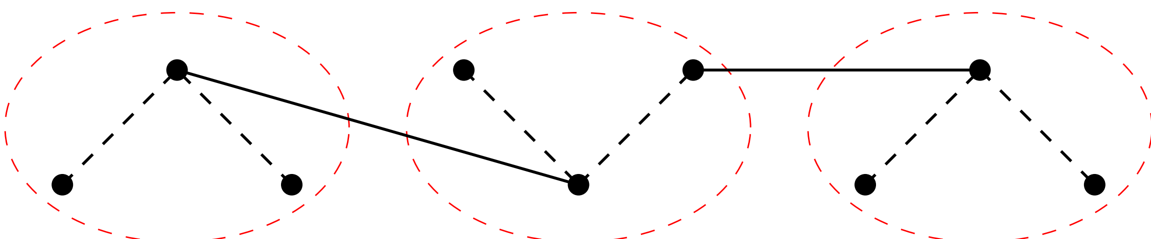

Fig. 1.2.3: A set of points in groups of 3 along the TSP tour. All points within a group are connected, and each group is connected. [2]

For the $\alpha = \frac{2\pi}{3}$ case, the algorithm of [2] again begins by approximating the TSP tour, then arranges the points into consecutive groups of 3 and orients them in a way that covers the entire plane (see Figure 1.2.3), giving a 6-approximation of the $\frac{2\pi}{3}$-MST. Biniaz et al. [7] improved this approximation ratio to $\frac{16}{3}$ using a similar configuration that takes advantage of the fact that a Hamiltonian path can be made to be non-crossing. Ashur and Katz [5] further improved this to a 4-approximation by showing that any path can be rearranged to a $\frac{2\pi}{3}$-ST with at most twice the weight of the original path. They also showed that 2 is a lower bound for their rearrangement procedure (and thus this approach cannot be improved any further), but note that other approaches might still yield a lower approximation factor.

For the $\alpha = \pi$ case, Aschner and Katz [2] showed that the vertices of the TSP tour can be oriented to cover both their incident edges, giving a simple 2-approximation.

Aschner and Katz [2] further proved the NP-hardness of the problem of computing the α-MST for the $\alpha = \frac{2\pi}{3}$ and $\alpha = \pi$ cases by showing reductions to the problem of finding Hamiltonian paths in hexagonal graphs and square grid graphs of degree at most 3, respectively.

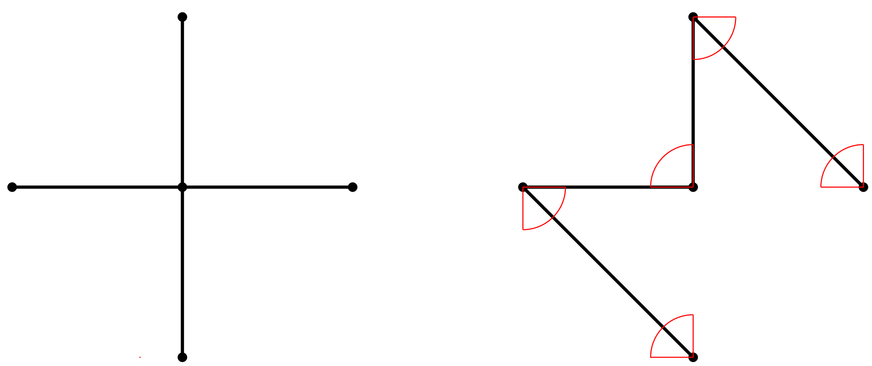

Fig. 1.2.4: Left: Euclidean MST, Right: Corresponding $\frac{\pi}{2}$-MST

Most previous research in this context has been done on the case where α is one fixed value for all antennas [7]. Biniaz et al. [7] extended this concept to an average-α-minimum spanning tree ($\overline{\alpha}$-MST): an α-MST with the relaxed restriction that the average angle of all the wedges is at most α (e.g. Figure 1.2.5). More formally, a total angle of αn must be allocated among n points p so that each point has a sufficient allowed angle to cover all incident edges. In the case where $\overline{\alpha} = \frac{2\pi}{3}$, they presented an algorithm that achieves an $\overline{\alpha}$-ST of length at most $\frac{3}{2}$ times the length of the MST. They also proved a lower bound of $\frac{4}{3}$ on the approximation factor with respect to the MST.

In this thesis, we improve the upper bound on $A\left(\frac{2\pi}{3}\right)$ from $\frac{3}{2}$ to $\frac{13}{9}$. In fact we modify the algorithm of [7] and obtain an $\overline{\alpha}$-ST of length at most $\frac{13}{9}$ times the length of the MST. Our algorithm involves a stronger exploitation of the Euclidean metric than the previous work. Our improved upper bound immediately gives an approximation

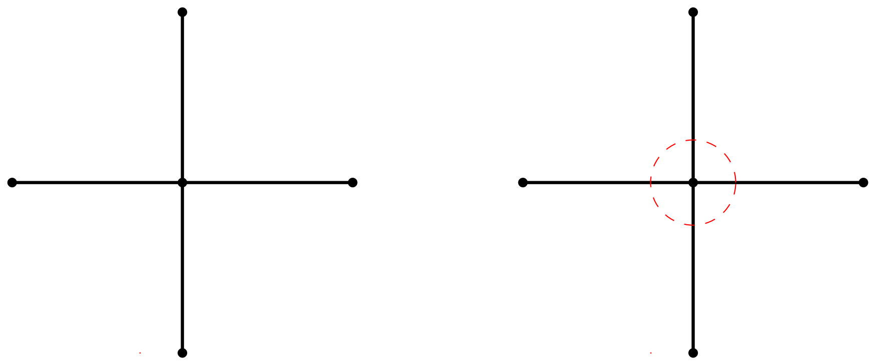

Fig. 1.2.5: Left: Euclidean MST, Right: Corresponding $\frac{\overline{\pi}}{2}$-MST

Angle	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	avg-$\frac{2\pi}{3}$
Approx. Ratio	10 [8]	4 [5]	2 [2]	1.5 [7]
Lower Bound	2 [2]	2 [2]	$\frac{2+\sqrt{3}}{3}$ [2]	$\frac{4}{3}$ [7]
NP-Hard	?	✓[2]	✓[2]	?

Table 1.2.1: Summary of previous bounded-angle MST results. Approximation ratios and lower bounds are w.r.t the underlying MST

algorithm with ratio $\frac{13}{9}$ (with respect to the MST) for the $\overline{\alpha}$-MST problem for any $\alpha \geq \frac{2\pi}{3}$.

We further show bounds on $A(\alpha)$ for the other two values of α studied in [2], $\alpha = \frac{\pi}{2}$ and $\alpha = \pi$. We begin by generalizing the lower bound argument of [7] to get a lower bound formula for any $A(\alpha)$ with $\alpha \leq \pi$. We then show that the approach of [7] can easily be modified to exactly compute the exact $\overline{\pi}$-MST, and present a new approximation algorithm for the $\frac{\overline{\pi}}{2}$-MST. Combining these results shows that $\frac{3}{2} \leq A\left(\frac{\pi}{2}\right) \leq 4$ and $A(\pi) = 1$.

Similar to that of [7], our algorithms run in linear time after computing the MST.

1.3 Notation

We use the terms point and vertex interchangeably depending on the context.

To facilitate comparison, we borrow the following notation from [7]. A *maximal path* in a tree is a path with at least two edges where all internal vertex degrees are

2, and the end vertex degrees are not 2. To *contract* a maximal path is to remove all vertices of degree 2 on the path and the edges between them, and add an edge connecting the end vertices. The angle that the incident edges of a vertex in an $\overline{\alpha}$-MST are allowed to fall within is called its *charge*. Charges can be redistributed between vertices. We denote the total length of edges of a geometric graph G by $w(G)$.

As the length of the optimal solution is often not known, we use the underlying MST of the points as a lower bound in our analysis. We denote the smallest ratio of the length of the $\overline{\alpha}$-MST to the length of the standard MST over all points in the plane as $A(\alpha)$. In [7], it was shown that $\frac{4}{3} \leq A\left(\frac{2\pi}{3}\right) \leq \frac{3}{2}$.

1.4 Outline

The approximation algorithm of [7] for the $\overline{\frac{2\pi}{3}}$-MST starts with a standard MST that has maximum degree 5 (which always exists). Then it re-assigns angle charges from leaves to inner vertices. Their approach first considers the MST with all maximal paths contracted, and then introduces edge shortcuts in each contracted path.

In Chapter 2, we improve upon this algorithm. By exploiting additional geometric properties we ensure the connectivity of path vertices with less total charge. This enables us to save some charges. The saved charges allow us to introduce fewer shortcuts than the original algorithm, resulting in a shorter $\overline{\frac{2\pi}{3}}$-ST.

In Chapter 3, we first observe that the lower bound argument used for $A\left(\frac{2\pi}{3}\right)$ in [7] can easily be generalized to give a lower bound formula for all $A(\alpha)$ with $\alpha \leq \pi$. We then apply this formula to obtain lower bounds on $A(\pi)$ and $A\left(\frac{\pi}{2}\right)$. We then show how the algorithm of [7] for the $\overline{\frac{2\pi}{3}}$-MST can easily be modified to exactly compute the $\overline{\pi}$-MST. Finally, we present a new algorithm to approximate the $\overline{\frac{\pi}{2}}$-MST.

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CHAPTER 2

A $\frac{13}{9}$-approximation of the average-$\frac{2\pi}{3}$-MST

Ahmad Biniaz, Prosenjit Bose, Patrick Devaney

In Proceedings of the 34th Canadian Conference on Computational Geometry (CCCG), pages 55–59, 2022

In this chapter we focus on the case where $\alpha = \frac{2\pi}{3}$. We first briefly describe the algorithm of Biniaz et al. [1], which operates in two phases by first contracting all maximal paths and then reintroducing them using new "shortcut" edges. By more carefully considering the geometry of the contracted paths, we then show how to improve the algorithm by reversing some shortcuts in the second phase to obtain a better approximation ratio.

2.1 The Algorithm of Biniaz et al.

We begin by briefly describing the algorithm of Biniaz et al. [1], which we refer to by "Algorithm 1".

The algorithm starts by computing a minimum spanning tree T of the point set with maximum degree 5, where each vertex holds a charge of $\frac{2\pi}{3}$. Then the algorithm goes through two phases that redistribute the charges and also modify the tree. In the first phase, all maximal paths of T are contracted (to edges), resulting in a tree with no vertices of degree 2, and all other vertices having the same degree as in T (see Figure 2.1.2). The charge from the leaves are then redistributed among the internal vertices so that each vertex of degree 3, 4, and 5 has a charge of $\frac{4\pi}{3}$, 2π, and $\frac{8\pi}{3}$,

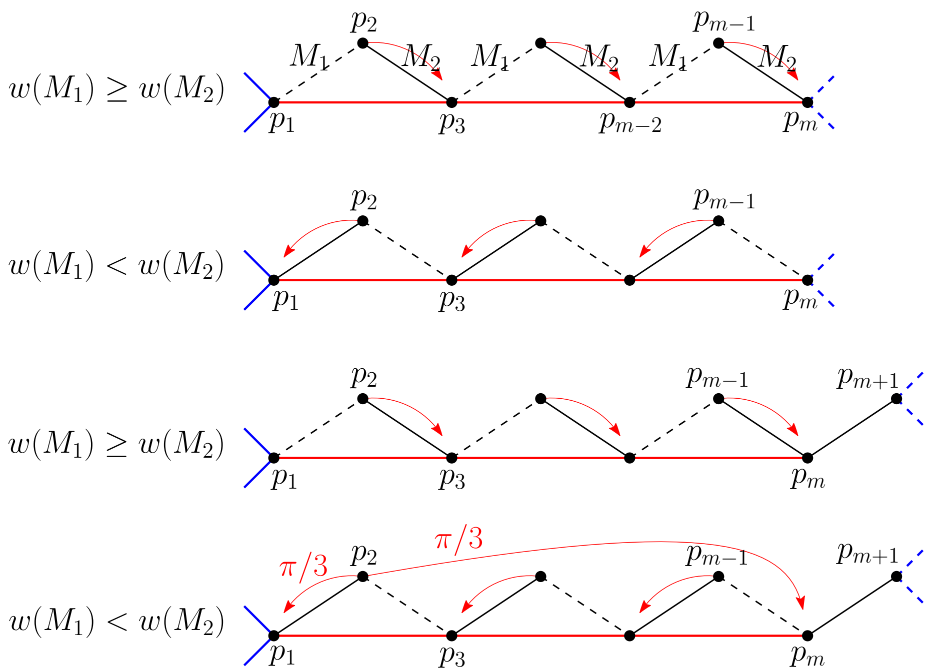

Fig. 2.1.1: (borrowed from [1]) The contracted path is shown by black segments. The dashed-black edges belong to M_2 and the red edges belong to S.

respectively (see Figure 2.1.3). Since the charge of each internal vertex with degree n is at least $\left(1 - \frac{1}{n}\right) 2\pi$, which covers any set of n edges, all vertices can cover their incident edges. After redistribution, degree-1 vertices have 0 charge and each degree-2 vertex holds its original $\frac{2\pi}{3}$ charge. This redistribution retains a pool of $\frac{4\pi}{3}$ charge that can be split among all leaves in the tree at the end of the algorithm.

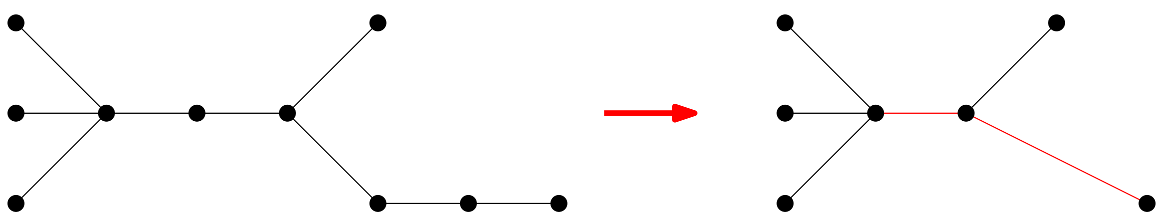

Fig. 2.1.2: The original MST (left) and MST with contracted maximal paths shown as red lines (right)

In the second phase, the edges of each path $p_1, p_2, \ldots, p_m$ that was contracted in phase 1 are split into two matchings, M_1 and M_2 with equal number of edges. If the path has odd number of edges then the last edge is not in either matching (see Figure 2.1.4). The edges of the matching with the larger weight are removed, and a set $S = \{(p_1, p_3), (p_3, p_5), ...\}$ of new edges called *shortcuts* are introduced (see Figure 15

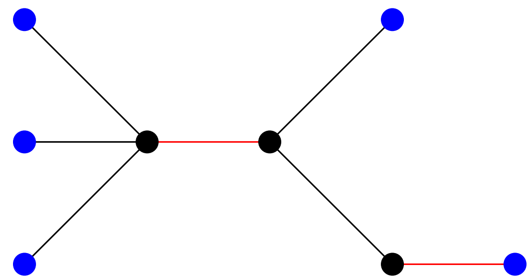

Fig. 2.1.3: The MST with contracted maximal paths (red edges). Leaves are blue, and internal vertices are black.

of [1], which we include here as Figure 2.1.1). By this process, the charge of every new degree-1 vertex is redistributed among other vertices so that each new degree-2 and degree-3 vertex along the path has a charge of π and $\frac{4\pi}{3}$, respectively; this is handled in four cases based on which matching is heavier and whether the path length is even or odd, as shown in Figure 2.1.1. Note that the charge given to vertices assigned degree 2 and 3 allows them to cover any set of 2 and 3 edges, respectively.

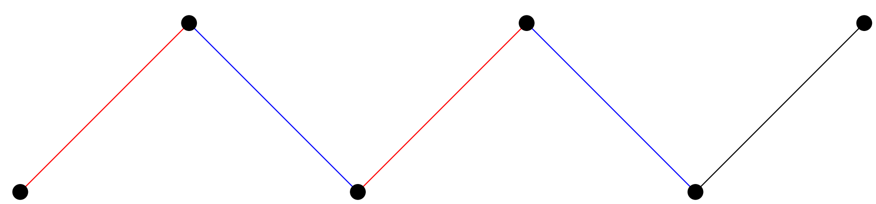

Fig. 2.1.4: A path is split into two matchings (shown as red and blue edges). The last edge in a path of odd length (shown in black) is not part of either matching.

We also note that both phases can be performed without using the assumption that the underlying MST has maximum degree 5, as each vertex of degree 4 or greater receives at least 2π charge from the leaves it introduces, which covers any number of incident edges.

Let M_1' and M_2' be the union of the edges in the smaller and larger-weight matchings of all contracted paths, respectively. Let T' be the final tree obtained by the above algorithm, and let E be the set of edges of T not in $M_1' \cup M_2'$. Then $w(T) = w(E) + w(M_1') + w(M_2')$. By the triangle equality we have $w(S) \leq w(M_1') + w(M_2')$. Since $w(M_2') \geq w(M_1')$ we get

$$\begin{aligned} w(T') &= w(E) + w(M_1') + w(S) \\ &\leq w(E) + w(M_1') + w(M_1') + w(M_2') \\ &= w(T) + w(M_1') \leq \frac{3}{2} w(T). \end{aligned}$$

2.2 The Improved Algorithm

We begin by modifying the charge-redistribution of phase 2 of Algorithm 1 with a more careful charge redistribution. In particular we show that the 3 edges, that are incident to new degree-3 vertices, can be covered by $\frac{4\pi}{3} - \frac{\pi}{12}$ charge (meaning that we can *save* the $\frac{\pi}{12}$ charge). We then use the saved charge of $\frac{\pi}{12}$ to achieve a better approximation with respect to the original MST. The following lemma, although very simple, plays an important role in the design of the modified algorithm.

Lemma 1. *It is possible to save at least $\frac{\pi}{12}$ charge from every shortcut performed by phase 2 of Algorithm 1.*

Proof. Consider a shortcut ac between two consecutive edges ab and bc of a contracted path as depicted in Figure 2.2.1. Up to symmetry we may assume that ab is in M_2 and thus it has been removed in phase 2 of Algorithm 1. Denote the angle $\angle bca$ by β. Since the path (a, b, c) is part of the MST, ac is the largest edge of the triangle $\triangle abc$, and thus $\angle abc$ is its largest angle. Therefore $\beta \leq \frac{\pi}{2}$.

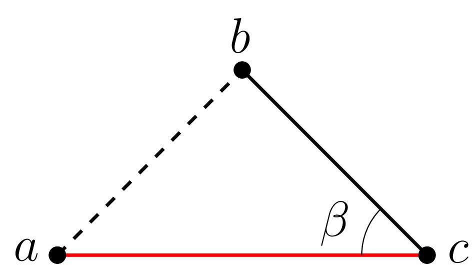

Fig. 2.2.1: Illustration of a shortcut between the points a and c.

The replacement of ab by the shortcut ac has not changed the degree of a, has decreased the degree of b by 1, and has increased the degree of c by 1. Thus the charge assigned to a by Algorithm 1 remains enough to cover its incident edges. Since b has degree 1, its $\frac{2\pi}{3}$ charge is free. Algorithm 1 transfers this free charge to c to cover its new edge (see Figure 2.2.2). We show how to cover all edges incident to c while saving $\frac{\pi}{12}$ charge. If c's original degree (i.e. after phase 1 and before phase 2) was 4 or 5 then it carries at least 2π charge which is sufficient to cover its edges. We may assume that the original degree of c is 1, 2, or 3, in which case it holds a charge of 0, $\frac{2\pi}{3}$, or $\frac{4\pi}{3}$, respectively. Thus the new degree of c (after phase 2) is 2, 3, or 4. Based on this we distinguish three cases.

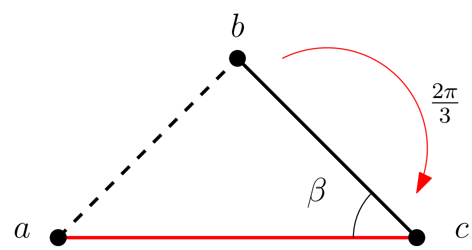

Fig. 2.2.2: Algorithm 1 transfers charge from b to c

- If $\deg(c) = 2$ then the two incident edges of c are ac and bc. We can cover these edges by a charge of β ($\leq \frac{\pi}{2}$). Thus we transfer $\frac{\pi}{2}$ charge from b to c (as shown in Figure 2.2.3) and we save $\frac{\pi}{6}$.

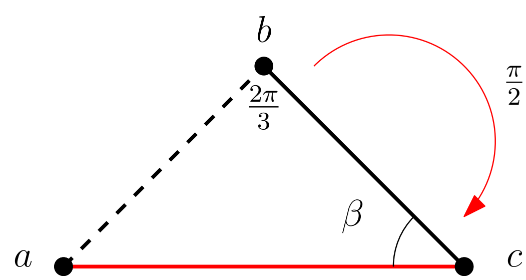

Fig. 2.2.3: Lemma 1: degree 2 case

- If $\deg(c) = 3$ then we cover β and the smaller of the other two angles at c. Thus the three incident edges to c can be covered by charge of

$$\beta + \left(\frac{2\pi - \beta}{2}\right) = \frac{2\pi + \beta}{2} \leq \frac{2\pi + \frac{\pi}{2}}{2} = \frac{5\pi}{4}.$$

Thus by transferring $\frac{7\pi}{12}$ from b to c it will have charge of $\frac{5\pi}{4}$ (including its original $\frac{2\pi}{3}$ charge – see Figure 2.2.4). Thus we save charge of $\frac{2\pi}{3} - \frac{7\pi}{12} = \frac{\pi}{12}$ from b.

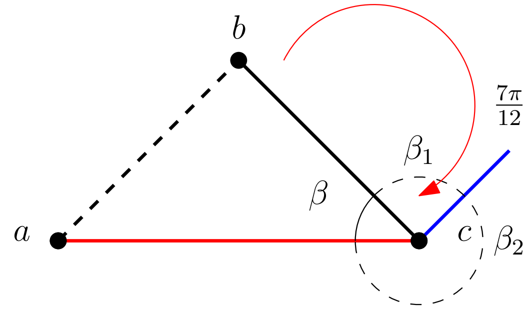

Fig. 2.2.4: Lemma 1: degree 3 case

- If $\deg(c) = 4$ then we transfer $\frac{\pi}{6}$ charge from b to c and save the remaining $\frac{\pi}{2}$ charge of b. The vertex c now holds $\frac{3\pi}{2}$ charge (including its charge $\frac{4\pi}{3}$ after phase 1) which covers its four incident edges (see Figure 2.2.5).

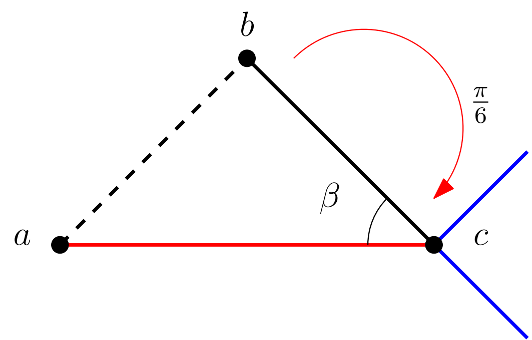

Fig. 2.2.5: Lemma 1: degree 4 case

□

The following is a direct implication of Lemma 1.

Corollary 2. *It is possible to save $\frac{\pi}{3}$ charge from every four shortcuts that are performed by Algorithm 1.*

2.3 Reversing Shortcuts

In this section, we present an approximation algorithm that uses fewer shortcuts than Algorithm 1. In fact the new algorithm reverses a constant fraction of the shortcuts performed by Algorithm 1.

Theorem 3. *Given a set of n points in the plane and an angle $\alpha \geqslant \frac{2\pi}{3}$, there is an $\overline{\alpha}$-spanning tree of length at most $\frac{13}{9}$ times the length of the MST. Furthermore, there is an algorithm to find such an $\overline{\alpha}$-ST that runs in linear time after computing the MST.*

Proof. Let T be a degree-5 minimum spanning tree of the point set, and T' be the $\overline{\frac{2\pi}{3}}$-spanning tree obtained from T by Algorithm 1.

Consider the sequence of shortcuts introduced by Algorithm 1 along each contracted path. Let $s_1, s_2, ..., s_m$ be the concatenation of the sequences for all contracted paths. We split these shortcuts into nine sets $S_0, \dots, S_8$ such that $s_i \in S_{(i \mod 9)}$ for each $i \in \{1, \dots, m\}$. Note that no two adjacent shortcuts in the same contracted path will be in the same set S_i. Moreover the number of shortcuts in any two sets S_i and S_j differ by at most 1. Recall that the edges of each contracted path in Algorithm 1 are split into two matchings M_1 and M_2. Let M_1' be the set of edges that are kept in the tree (i.e. M_1' is the union of the smaller-weight matchings from each contracted path), and let the set of edges in the heavier matchings be M_2'. Among $S_0, \dots, S_8$, let S_8 be the one whose corresponding edges in M_1' have the largest total weight.

Our plan now is to reverse the shortcuts in S_8, i.e., to replace them by their corresponding edges in M_2'. Let S' be the union of $S_0, \dots, S_7$. Notice that $|S'| \geq 8 \cdot (|S_8| - 1)$. Let C denote the pool of charges that is obtained after phase 1 of Algorithm 1, and recall that it contains $\frac{4\pi}{3}$ charge. For each shortcut in S' we reassign the charges between its corresponding points to save at least $\frac{\pi}{12}$ charge (as shown in Lemma 1), and add this charge to C. Thus the total charge of C is at least

$$\frac{4\pi}{3} + 8 \cdot (|S_8| - 1) \cdot \frac{\pi}{12} = (|S_8| + 1) \cdot \frac{2\pi}{3}.$$

We will show that to reverse each shortcut from S_8 it suffices to take $\frac{2\pi}{3}$ charge from C.

Consider any shortcut ac from S_8 between two consecutive edges ab and bc of a contracted path as depicted in Figure 2.3.1. We *reverse* this shortcut by replacing ac with the removed edge ab. We also reclaim any portion of b's charge that was transferred to c. Thus the reverse operation brings the charges of b and c back to what it was after phase 1 and before phase 2; in particular it brings the charge of b back to $\frac{2\pi}{3}$. There is one exceptional case where $w(M_1) < w(M_2)$ and the path has odd number of edges (the last case in Figure 2.1.1 where p_3, p_2, p_1 play the roles of a, b, c, respectively). In this case the charge of b (i.e. p_2) would be $\frac{\pi}{3}$ as p_m holds the other $\frac{\pi}{3}$ portion. (Since no two shortcuts in S_8 are adjacent in the same contracted path, we can analyze a reverse operation independently of others. Notice, however, that it is possible that two or more shortcuts of S_8 are adjacent at a vertex that has degree at least 3 after phase 1. In this case, the charge of such a vertex suffices to cover its edges after reversing the shortcuts since it will have at least $\frac{\pi}{6}$ charge added for each new edge introduced by the process described in Lemma 1.) The reverse operation does not change the degree of a and thus its charge remains sufficient to cover its edges. The reverse operation makes b of degree 2 and decreases the degree of c by 1.

We take $\frac{\pi}{3}$ charge from C for b to bring it to a charge of π, which covers its two incident edges. If $\deg(c) = 1$ or $\deg(c) \geq 3$, its charge is sufficient to cover its edges. If $\deg(c) = 2$ then we take an additional charge of $\frac{\pi}{3}$ from C for c to cover its two incident edges. In the exceptional where $w(M_1) < w(M_2)$ and the path has odd number of edges (the last case in Figure 2.1.1), $p_2 = b$ holds $\frac{\pi}{3}$ charge, so we take $\frac{2\pi}{3}$ from C for p_2 to cover its two incident edges. Since $p_1 = c$ is of degree 1 or at least 3 (as the contracted path is maximal), its charge (acquired after phase 1) is sufficient to cover its edges. Thus, in the worst case we take $\frac{2\pi}{3}$ from C to reverse every shortcut.

After reversing all shortcuts in S_8, the pool C is left with at least $\frac{2\pi}{3}$ charge which can be distributed among the leaves of the resulting tree.

Let T'' be the $\overline{\frac{2\pi}{3}}$-ST tree obtained from T' after reversing all shortcuts in S_8. Let

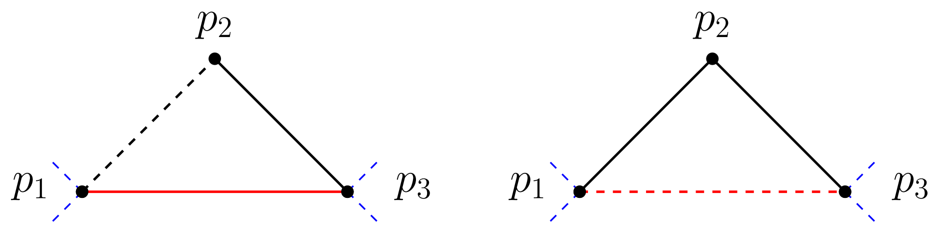

Fig. 2.3.1: Left: The tree T' before reversing shortcut ac. Right: The tree T'' after reversing ac.

E be the set of edges of T'' not in $M'_1 \cup M'_2$. Let E' be the set of all edges of $M'_1 \cup M'_2$ that correspond to the shortcuts in S_8. Let $M''_1 = M'_1 \setminus E'$ and $M'_2 = M'_2 \setminus E'$ (i.e. all edges in M'_1 and M'_2, respectively, with a shortcut between their endpoints in T''). Then,

$$\begin{aligned} w(T'') &= w(E) + w(E') + w(S') + w(M''_1) \\ &\leq w(E) + w(E') + w(M''_1) + w(M''_2) + w(M''_1) \\ &= w(T) + w(M''_1). \end{aligned}$$

Since S_8 has the largest corresponding M'_1 weight, $w(M''_1) \leq \frac{8}{9}w(M'_1) \leq \frac{8}{9} \cdot \frac{1}{2}w(T) = \frac{4}{9}w(T)$. Thus,

$$w(T'') \leq w(T) + \frac{4}{9}w(T) = \frac{13}{9}w(T).$$

□

With Theorem 3 in hand, we report the following bound for $A\left(\frac{2\pi}{3}\right)$.

Corollary 4. $\frac{4}{3} \leq A\left(\frac{2\pi}{3}\right) \leq \frac{13}{9}$.

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DOI: 10.1007/s00453-021-00889-6. URL: https://doi.org/10.1007/s00453-021-00889-6.

CHAPTER 3

Approximating the $\overline{\alpha}$-MST for Other Angles

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In this chapter, we first observe that an argument by Biniaz et al. [2] for the lower bound of $A\left(\frac{2\pi}{3}\right)$ can be generalized to obtain a formula that gives a lower bound on $A(\alpha)$ for any $\alpha \leq \pi$. We further use this formula to obtain lower bounds of 1 and $\frac{3}{2}$ for $A(\pi)$ and $A\left(\frac{\pi}{2}\right)$, respectively. We then show how Algorithm 1 of Biniaz et al. [2] (described in Section 2.1) can be modified to exactly compute the $\overline{\pi}$-MST, implying that $A(\pi) = 1$. Finally, we present an algorithm to compute a 4-approximation of the $\overline{\frac{\pi}{2}}$-MST, and conclude that $\frac{3}{2} \leq A\left(\frac{\pi}{2}\right) \leq 4$.

3.1 A General Lower Bound Formula

In [2], Biniaz et al. gave a proof that $A\left(\frac{2\pi}{3}\right) \geq \frac{4}{3}$, based on an argument that, for every point on the $\overline{\frac{2\pi}{3}}$-MST, there is a corresponding unique interval covered by one of the edges of the tree. We observe that this argument can easily be generalized to obtain a formula giving a lower bound on $A(\alpha)$, for any $\alpha \leq \pi$.

Theorem 5. *For any angle $\alpha \leq \pi$, there exists a set X of points in the plane such that the length of the corresponding $\overline{\alpha}$-MST is at least $\frac{(2\pi-\alpha)n}{\pi} - 3$ times the length of MST.*

Proof. Consider any $\overline{\alpha}$-minimum spanning tree $\overline{T}$ on X, where $\overline{\alpha} \leq \pi$. We show that $w(\overline{T}) \geq \frac{(2\pi-\alpha)n}{\pi} - 3$. Partition the vertices of $\overline{T}$ into X_1 and X_2, where X_1 is the set

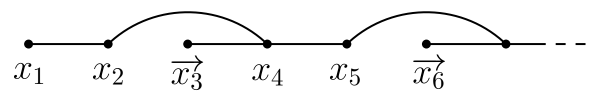

Fig. 3.1.1: (borrowed from [2]) $\overline{\alpha}$-ST of length $\frac{(2\pi-\alpha)n}{\pi} - 3$.

of vertices with wedges of angle strictly less than π and X_2 is the set of vertices with wedges of angle at least π. Since the total available charge is αn, $|X_2| \leq \frac{\alpha n}{\pi}$. Thus $|X_1| = n - |X_2| \geq \frac{(\pi-\alpha)n}{\pi}$. Observe that every interval (between consecutive vertices of X) is covered by an edge of $\overline{T}$. Every vertex $x_i \in X_1$ sees the vertices that are either to its left or to its right. We denote x_i by $\overleftarrow{x_i}$ if it sees the vertices to its left, and by $\overrightarrow{x_i}$ otherwise (see Figure 3.1.1(b)). For every $\overleftarrow{x_i}$ the interval $[i-1, i]$ is covered by at least two edges otherwise connectivity is lost: one edge is incident to $\overleftarrow{x_i}$ and another edge connects a point to the left of $\overleftarrow{x_i}$ with a point to the right (assuming $i \neq n$). Similarly for every $\overrightarrow{x_i}$ the interval $[i, i+1]$ is covered by an edge that is incident to $\overrightarrow{x_i}$ and by an edge that connects a point to the right of $\overrightarrow{x_i}$ with a point to the left (assuming $i \neq 1$), as in Figure 3.1.1(b). Thus, for every $\overleftarrow{x_i}$ (except possibly $\overleftarrow{x_n}$) there exists a unique interval that is covered by two edges of $\overline{T}$. Similarly, for every $\overrightarrow{x_i}$ (except possibly $\overrightarrow{x_1}$) there exists a unique interval that is covered by two edges of $\overline{T}$. (If x_i is oriented to the left and x_{i+1} is oriented to the right then—by the minimality of the tree—(x_i, x_{i+1}) is an edge of $\overline{T}$ and the interval $[i, i+1]$ is covered by three edges.) Therefore the length of $\overline{T}$ is at least $(n-1) + (|X_1| - 2) \geq n + \frac{(\pi-\alpha)n}{\pi} - 3 = \frac{(2\pi-\alpha)n}{\pi} - 3$. □

We now use this formula to compute lower bounds on the values of α that we present algorithms for in the following sections.

Corollary 6. $A(\pi) \geq \frac{2\pi-\pi}{\pi} = 1$.

Corollary 7. $A\left(\frac{\pi}{2}\right) \geq \frac{2\pi-\frac{\pi}{2}}{\pi} = \frac{3}{2}$.

3.2 Approximating the $\overline{\pi}$-MST

In this section, we show how to modify Algorithm 1 of Biniaz et al. [2] for the $\overline{\frac{2\pi}{3}}$-MST to compute the $\overline{\pi}$-MST. We further show that $A(\pi) = 1$.

Theorem 8. *Given a set of n points in the plane and an angle $\alpha \geqslant \pi$, the charge of the minimum spanning tree can be redistributed to make it an $\overline{\alpha}$-spanning tree. Furthermore, there is an algorithm to find such an $\overline{\alpha}$-ST that runs in linear time after computing the MST.*

Proof. Recall that Algorithm 1 works in two phases, as described in Section 2.1. Let X be a set of n points in the plane, and T be a degree-5 minimum spanning tree of the complete Euclidean graph induced by X. In the first phase, all maximal paths of T are contracted, giving a tree T' with no vertices of degree 2. Every tree has at least 2 leaves, and every vertex of degree 3, 4, or 5 introduces 1, 2, and 3 additional leaves, respectively. We put the charge from all leaves into a pool C, and then redistribute π, 2π, and 3π from this pool to each vertex of degree 3, 4, and 5, respectively. Note that C still contains 2π charge that can be distributed among the leaves if desired. Now every vertex with degree at least 3 has charge of at least 2π, which covers any set of incident edges.

Now the second phase consists only of reintroducing each contracted path, as each degree-2 vertex already has a charge of π, which covers its 2 incident edges. The result is the original tree T with the charge redistributed so each vertex can cover its incident edges. In particular, the weight of T has not changed from that of the original MST. □

Corollary 9. $A(\pi) = 1$

3.3 Approximating the $\overline{\frac{\pi}{2}}$-MST

In this section we present an algorithm to compute the $\overline{\frac{\pi}{2}}$-MST.

Theorem 10. *Given a set of n points in the plane and an angle $\alpha \geqslant \frac{\pi}{2}$, there is an $\overline{\alpha}$-spanning tree of length at most 4 times the length of the MST. Furthermore, there is an algorithm to find such an $\overline{\alpha}$-ST that runs in linear time after computing the MST.*

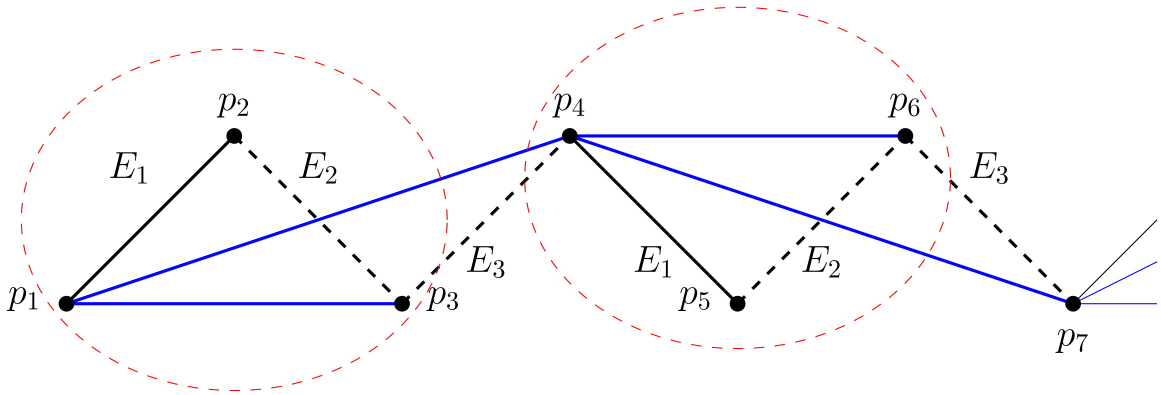

Fig. 3.3.1: Groups of 3 points on the TSP path circled – original edges in black, removed edges dashed, and shortcut edges in blue.

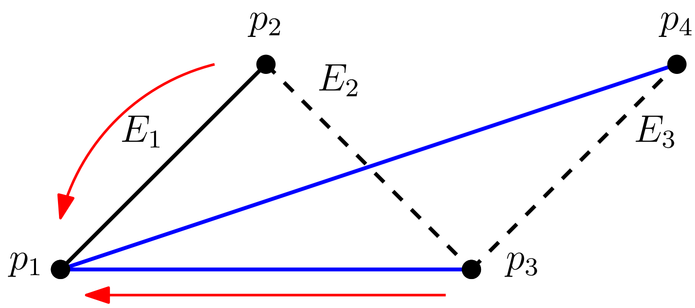

Fig. 3.3.2: A group of points p_1, p_2, p_3 with shortcuts (blue). Charge is redistributed to p_1.

Proof. Given a set of n points X in the plane, let T be the minimum spanning tree of the complete Euclidean graph induced by X. As with algorithms that approximate the $\frac{\pi}{2}$-MST [1, 3], we begin by computing the metric TSP approximation of T, giving a path P with $w(P) \leq 2w(T)$. We then split the vertices of this path into disjoint groups of three consecutive points. Let E_1, E_2, and E_3 be the sets of all first, second, and third (outgoing) edges of each group, respectively (see Figure 3.3.1). After suitable relabeling, let E_1 be the matching with minimum total edge weight and E_3 be the matching with maximum total edge weight. Among the points p_1, p_2, and p_3 of each group, redistribute all charge to the first point incident to E_1 (without loss of generality, say this is p_1). Then p_1 now has total charge $\frac{3\pi}{2}$, which allows it to cover any set of four edges. Let p_4 be the vertex incident to the outgoing edge of the group (if applicable). We remove the edges from E_2 and E_3, and introduce shortcut edges to connect each corresponding point directly to p_1 (see Figure 3.3.2). The shortcut corresponding to E_2 has weight at most $E_1 + E_2$, and the shortcut corresponding to E_3 has weight at most $E_1 + E_2 + E_3$. So the total weight of these edges (and thus the resulting tree T') is at most $3w(E_1) + 2w(E_2) + w(E_3)$. So

$$\begin{aligned} w(T') &\leq 3w(E_1) + 2w(E_2) + w(E_3) \\ &= (w(E_1) + w(E_2) + w(E_3)) + (w(E_1) + w(E_2)) + w(E_1) \end{aligned}$$

Since $w(E_1) + w(E_2) + w(E_3) = w(P)$,

$$w(T') \leq w(P) + (w(E_1) + w(E_2)) + w(E_1)$$

Since E_3 is the set of edges with largest weight, $w(E_3) \geq \frac{1}{3}w(P)$, so

$$w(E_1) + w(E_2) \leq \frac{2}{3}w(P)$$

Similarly, since E_1 has smallest total edge weight, $w(E_1) \leq \frac{1}{3}P$. Thus,

$$\begin{aligned} w(T') &\leq w(P) + \frac{2}{3}w(P) + \frac{1}{3}w(P) \\ &= 2w(P) \\ &\leq 4w(T) \end{aligned}$$

□

Corollary 11. *$\frac{3}{2} \leq A\left(\frac{\pi}{2}\right) \leq 4$.*

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[2] Ahmad Biniaz, Prosenjit Bose, Anna Lubiw, and Anil Maheshwari. "Bounded-Angle Minimum Spanning Trees". In: *Algorithmica* 84.1 (2022), pp. 150–175. DOI: 10.1007/s00453-021-00889-6. URL: https://doi.org/10.1007/s00453-021-00889-6.

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CHAPTER 4

Conclusion

In this thesis, we have presented approximation algorithms for the $\overline{\alpha}$-MST in the cases where $\alpha = \frac{\pi}{2}$, $\frac{2\pi}{3}$, and π, and shown a generalized lower bound formula for $A(\alpha)$ for any $\alpha \leq \pi$. We have further used these to show that $\frac{3}{2} \leq A\left(\frac{\pi}{2}\right) \leq 4$, $\frac{4}{3} \leq A\left(\frac{2\pi}{3}\right) \leq \frac{13}{9}$, and $A(\pi) = 1$.

4.1 Open Problems

The obvious open problems are that the bounds on $A\left(\frac{\pi}{2}\right)$ and $A\left(\frac{2\pi}{3}\right)$ are not tight. These could be improved by developing new algorithms with better approximation factors, or finding new sets of points whose $\overline{\alpha}$-MST must have a larger minimum weight with respect to the underlying MST. In particular, we believe that the upper bound on $A\left(\frac{2\pi}{3}\right)$ and the upper bound on $A\left(\frac{\pi}{2}\right)$ can be further improved.

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