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INFINITE SUMS, PRODUCTS, AND URN MODELS

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by

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Abstract

This paper considers an urn and its evolution in discrete time steps. The urn initially has two different colored balls(blue and red). We discuss different cases where k blue balls ($k = 1, 2, 3, \dots$) will be added (or removed) at every step if a blue ball is withdrawn, based on the goal of eventually withdrawing a red ball $P(\text{R eventually})$. We compute the probability of eventually withdrawing a red ball with two different methods–one using infinite sums and other using infinite products. One advantage of this is that we can obtain $P(\text{R eventually})$ in a complex but nicely patterned form using one method, and a simple form using the other method. Since the results must be equal, we obtain some interesting identities. We also present a general result and invent new identities, illustrated by an example using the Fibonacci numbers. Additionally, we transform Wallis Product(a) and Wallis Product(b) into urn models. Finally, we illustrate some results by simulating the urn processes in R.

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CHAPTER 1

Introduction

1.1. Scope and Motivation of Research

Urn models have been intensely studied since the eighteenth century. Early notable contributions were made by mathematicians such as Jacob Bernoulli and Laplace[1]. In Ars Conjectandi (1713), Bernoulli used the Latin word urna(a clay vessel) and considered a problem of determining the proportions of different colored pebbles inside an urn after a number of pebbles are drawn from the urn. This topic of research also attracted the attention of Abraham de Moivre and Thomas Bayes. During the nineteenth and twentieth century, a number of classical urn problems attributed to some eminent mathematicians researching in probability theory, were proposed and studied. For instance, the ballot problems concerning with the progress of a vote were first put forward the solved in 1878 by W.Whitworth. He viewed the progress of votes as white and blue balls being deposited in an urn. Other urn problems such as occupancy problems, coupon collector's problems, the gambler's ruin and banach's Matchbox problem were also raised and discussed before the early twentieth century. Then in 1977, the book by Johnson and Kotz[2] did a remarkable job of presenting a historical perspective on the simplest urn models and some of their applications in statistics, engineering, and genetics. They traced urns back to the post-Renaissance era and mentioned urns in works of Huygens, de Moivre, Laplace, Bernoulli and other distinguished mathematicians and scientists. Nowadays, urn models can be successfully used to model real world processes in fields as diverse as genetics, ecology, physics, and economics. An extensive overview of the literature on urn model can be found in Rafik(2018)[4], Eskenazis(2018)[5], Stenlund(2018)[6], Aguech(2018)[7], Barbier(2017)[8], etc.

In particular, Pólya urn models, as a popular topic of urn problems, seem to be first mentioned in 1906 by Markov in his seminal paper[9] and in [10]. A special form appeared in the work of Paul Ehrenfest and Tatyana Ehrenfest in 1907, and

was proposed as a model for the mixing of gases. However, it was named after the work of Pólya and Eggenberger(1923)[11]. They generalized Markov's urn and used it as a model for contagion processes.The interest in Pólya urn models mainly concentrated on the urn population and the sequence of colors of the balls drawn out. Furthermore, in Pólya urn models, objects of real interest (e.g. atoms, genes, social network users) are represented as colored balls, which can apply to a broad variety of areas. A recent work about Pólya urns is the book by Mahmoud(2008)[3] where a lot of applications, especially in computer science and bioscience, are discussed. And more recent literature includes Saraiva(2017)[12], Kuntschik(2017)[13], Hayhoe(2017)[14], Dosi(2017)[15], etc.

However, this paper focused on an urn with two different colored balls(blue and red) under different circumstances. We discussed various cases of withdrawing a ball from the urn with and without replacement on top of the goal of eventually withdrawing a red ball. More importantly, we computed the probability of eventually obtaining a red ball in two different forms—infinite sums and infinite products. Moreover, we generalized the results and invented new identities based on the infinite sums and infinite products obtained from the evolution of the urn.

CHAPTER 2

An Urn With 1 red ball and 1 blue ball

An urn contains one red ball and one blue ball. The urn evolves in discrete time steps. At each step, a ball is sampled at random. If a blue ball is withdrawn, then the ball will be returned to the urn(with replacement) with k blue balls added to the urn. This process stops when the red ball is withdrawn. We are interested in the probability of drawing the red ball at each step, which will be discussed in Section 7.1. In addition, we will especially research the probability of eventually drawing the red ball.

It needs to be clarified that k is the number of blue balls added to the urn if a blue ball is withdrawn, so $k = 1, 2, 3, 4, ...;$ n is the number of steps, so $n = 1, 2, 3, 4, ...$

And we define:

$$P(R_i) = P(B_1 B_2 B_3 \dots B_{i-1} R_i), \qquad i = 1, 2, 3, \dots, n$$

to be the probability of withdrawing the red ball at the ith step. We will simply use $P(R_i)$ instead of the form of $P(B_1 B_2 B_3 \dots B_{i-1} R_i)$ in some cases.

For example,

$$P(R_2) = P(B_1 R_2) = P(B_1) P(R_2|B_1)$$

meaning the probability of obtaining a blue ball at the first step and the red ball at the second step;

$$P(R_3) = P(B_1 B_2 R_3) = P(B_1) P(B_2|B_1) P(R_3|B_1, B_2)$$

meaning the probability of obtaining a blue ball at the first step, another blue ball at the second step and the red ball at the third step. So,

$$\begin{aligned} P(R_n) &= P(B_1 B_2 B_3 \dots B_{n-1} R_n) \\ &= P(B_1) P(B_2|B_1) P(B_3|B_1, B_2) \dots P(R_n|B_1, B_2, B_3, \dots, B_{n-1}) \end{aligned}$$

meaning the probability of withdrawing a blue ball for the first $(n-1)$ steps and the red ball at the nth step.

Then, as $n \to \infty$, the probability of eventually obtaining the red ball can be computed in two different ways.

$$\begin{aligned} &P(\text{R eventually}) \\ &= P(R_1) + P(B_1R_2) + P(B_1B_2R_3) + \ldots + P(B_1B_2B_3 \ldots B_{n-1}R_n) + \ldots \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &P(\text{R eventually}) \\ &= 1 - P(\text{obtaining a blue ball at every step}) \\ &= 1 - \lim_{n\to\infty} P(B_1B_2B_3 \ldots B_n) \end{aligned} \tag{2.2}$$

To be specific, we will use both (2.1) and (2.2) to calculate P(R eventually) for all the related problems in this paper. The advantages of doing this it that the infinite summation part in (2.1) is often difficult to calculate while the infinite product part in (2.2) may be easy to obtain. Thus, we can always try to obtain P(R eventually) using (2.2) whenever we can not solve (2.1) since (2.1) $\iff$ (2.2). Furthermore, calculating P(R eventually) in both ways can help to avoid computational mistakes and can result in some beautiful equalities.

The following sections illustrate the probability of eventually obtaining the red ball when $k = 1, 2, 3$ respectively using formulas (2.1) and (2.2). Adding 4 or more blue balls ($k \geqslant 4$) if a blue ball is withdrawn will not be discussed here since it is the same calculating process as adding 3 or fewer balls.Therefore, we will discuss the circumstances where

$k = 1$: adding 1 blue ball every step if a blue ball is withdrawn in Section 2.1;

$k = 2$: adding 2 blue balls every step if a blue ball is withdrawn in Section 2.2;
$k = 3$: adding 3 blue balls every step if a blue ball is withdrawn in Section 2.3.

In Section 2.4, we will discuss a special case where adding k blue balls if a blue ball is withdrawn at the kth step where $k = 1, 2, \ldots$.

2.1. $k = 1$:Adding One Blue Ball

In the case where $k = 1$,

$$\begin{aligned}(2.1) &= P(R_1) + P(B_1R_2) + P(B_1B_2R_3) + \ldots + P(B_1B_2B_3 \ldots B_{n-1}R_n) + \ldots \\ &= \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} \times \frac{1}{4} + \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{1}{5} \\ &+ \ldots + \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \ldots \times \frac{n-1}{n} \times \frac{1}{n+1} \\ &= \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{(n-1) \times n} + \frac{1}{n \times (n+1)} \\ &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)}\end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ requires partial fractions and telescoping series, we will calculate it again using (2.2).

$$\begin{aligned}(2.2) &= 1 - \lim_{n \to \infty} P(B_1B_2B_3 \ldots B_n) \\ &= 1 - \lim_{n \to \infty} \left(\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \ldots \times \frac{n-1}{n} \times \frac{n}{n+1} \right) \\ &= 1 - \lim_{n \to \infty} \frac{1}{n+1} \\ &= 1 - 0 = 1\end{aligned}$$

Therefore, the probability of eventually obtaining the red ball is 1 in the case where 1 blue ball is added if a blue ball is withdrawn, and we have found a new method to evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

2.2. $k = 2$:Adding Two Blue Balls

In the case where $k = 2$,

$$
\begin{aligned}
(2.1) &= P(R_1) + P(B_1R_2) + P(B_1B_2R_3) + \ldots + P(B_1B_2B_3 \ldots B_{n-1}R_n) + \ldots \\
&= \frac{1}{2} + \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{3}{4} \times \frac{1}{6} + \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{1}{8} + \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8} \times \frac{1}{10} + \ldots \\
&= \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1 \times 3 \times 5 \times \ldots \times (2n-3) \times 1}{2 \times 4 \times 6 \times \ldots \times (2n-2) \times 2n} \\
&= \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1 \times 2 \times 3 \times 4 \times 5 \times \ldots \times (2n-3) \times (2n-2)}{2 \times 4 \times 6 \times \ldots \times (2n-2) \times 2n \times 2 \times 4 \times 6 \times \ldots \times (2n-2)} \\
&= \sum_{n=1}^{\infty} \frac{(2n-2)!}{2^n \left[1 \times 2 \times 3 \times \ldots \times (n-1) \times n\right] \times 2^{n-1} \left[1 \times 2 \times 3 \times \ldots \times (n-1)\right]} \\
&= \sum_{n=1}^{\infty} \frac{(2n-2)!}{2^n \cdot 2^{n-1} \cdot n! \cdot (n-1)!} \\
&= \sum_{n=1}^{\infty} \frac{(2n-1)!}{n!(n-1)!} \cdot \frac{1}{2^{n-1}} \cdot \frac{1}{2^n} \cdot \frac{1}{2n-1} \\
&= \sum_{n=1}^{\infty} \binom{2n-1}{n} \left(\frac{1}{2}\right)^{n-1} \left(\frac{1}{2}\right)^{n} \cdot \frac{1}{2n-1}
\end{aligned}
$$

Similarly, we will obtain the probability using (2.2) as well.

$$
\begin{aligned}
(2.2) &= 1 - \lim_{n\to\infty} P(B_1B_2B_3 \ldots B_n) \\
&= 1 - \lim_{n\to\infty} \left(\frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8} \times \frac{9}{10} \times \ldots \times \frac{2n-1}{2n}\right) \\
&= 1 - \lim_{n\to\infty} \frac{1 \times 2 \times 3 \times 4 \times 5 \times \ldots \times (2n-1) \times 2n}{2 \times 4 \times 6 \times \ldots \times (2n-2) \times 2n \times 2 \times 4 \times 6 \times \ldots \times (2n-2) \times 2n} \\
&= 1 - \lim_{n\to\infty} \frac{(2n)!}{2^n \cdot 2^n \cdot n! \cdot n!} \\
&= 1 - \lim_{n\to\infty} \binom{2n}{n} \left(\frac{1}{2}\right)^{n} \left(\frac{1}{2}\right)^{n} \qquad (*) \\
&= 1 - \lim_{n\to\infty} \frac{(2n)^{2n} \cdot e^{-2n} \cdot \sqrt{4\pi n}}{n^n \cdot e^{-n} \cdot \sqrt{2\pi n} \cdot n^n \cdot e^{-n} \cdot \sqrt{2\pi n} \cdot 2^{2n}} \\
&= 1 - \lim_{n\to\infty} \frac{1}{\sqrt{\pi n}} \\
&= 1 - 0 = 1
\end{aligned}
$$

where $n! \approx n^n \cdot e^{-n}\sqrt{2\pi n}$ by Stirling's approximation

Note that (*) is intuitively equal to zero as it represents the limit as $n \to \infty$ of the probability of getting exactly n heads in $2n$ flips of a fair coin.

Therefore, the probability of eventually obtaining the red ball is 1 in the case where 2 blue balls are added if a blue ball is withdrawn. And we have obtained the sum of a series which would otherwise be difficult to obtain.

2.3. $k = 3$:Adding Three Blue Balls

In the case where $k = 3$,

$$
\begin{aligned}
(2.1) &= P(R_1) + P(B_1R_2) + P(B_1B_2R_3) + \ldots + P(B_1B_2B_3 \ldots B_{n-1}R_n) + \ldots \\
&= \frac{1}{2} + \frac{1}{2} \times \frac{1}{5} + \frac{1}{2} \times \frac{4}{5} \times \frac{1}{8} + \frac{1}{2} \times \frac{4}{5} \times \frac{7}{8} \times \frac{1}{11} + \frac{1}{2} \times \frac{4}{5} \times \frac{7}{8} \times \frac{10}{11} \times \frac{1}{14} + \ldots \\
&= \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1 \times 4 \times 7 \times \ldots \times (3n-5) \times 1}{2 \times 5 \times 8 \times \ldots \times (3n-4) \times (3n-1)} \\
&= \frac{1}{2} + \sum_{n=2}^{\infty} \left(\frac{\prod_{i=2}^{n} (3i-5)}{\prod_{i=1}^{n} (3i-1)} \right) \\
&= \sum_{n=1}^{\infty} \left(\frac{\frac{3^{n-1}\Gamma(n-\frac{2}{3})}{\Gamma(\frac{1}{3})}}{\frac{3^n\Gamma(n+\frac{2}{3})}{\Gamma(\frac{2}{3})}} \right) \qquad (*) \\
&= \frac{\Gamma(\frac{2}{3})}{3\Gamma(\frac{1}{3})} \sum_{n=1}^{\infty} \frac{\Gamma(n-\frac{2}{3})}{\Gamma(n+\frac{2}{3})}
\end{aligned}
$$

We prove $(*)$ as follows.

Recall $\Gamma(n+1) = n\Gamma(n)$,

$$\begin{aligned}
\prod_{i=2}^{n}(3i-5) &= \prod_{i=2}^{n} 3^{i-1}(i-\frac{5}{3}) \\
&= \prod_{i=2}^{n} 3^{i-1}\frac{\Gamma(i+1-\frac{5}{3})}{\Gamma(i-\frac{5}{3})} \\
&= 3^{n-1}\frac{\Gamma(3-\frac{5}{3})}{\Gamma(2-\frac{5}{3})}\cdot\frac{\Gamma(4-\frac{5}{3})}{\Gamma(3-\frac{5}{3})}\cdots\frac{\Gamma(n+1-\frac{5}{3})}{\Gamma(n-\frac{5}{3})} \\
&= 3^{n-1}\frac{\Gamma(n+1-\frac{5}{3})}{\Gamma(2-\frac{5}{3})} \\
&= 3^{n-1}\frac{\Gamma(n-\frac{2}{3})}{\Gamma(\frac{1}{3})}
\end{aligned}$$

Similarly,

$$\prod_{i=1}^{n}(3i-1) = 3^{n}\frac{\Gamma(n+\frac{2}{3})}{\Gamma(\frac{2}{3})}$$

This explains $(*)$.

We will obtain the probability using (2.2) as well.

$$\begin{aligned}
(2.2) &= 1-\lim_{n\to\infty} P(B_1B_2B_3\ldots B_n) \\
&= 1-\lim_{n\to\infty}\left(\frac{1}{2}\times\frac{4}{5}\times\frac{7}{8}\times\frac{10}{11}\times\frac{13}{14}\times\ldots\times\frac{3n-2}{3n-1}\right) \\
&= 1-\lim_{n\to\infty}\frac{\prod_{i=1}^{n}(3i-2)}{\prod_{i=1}^{n}(3i-1)} \\
&= 1-\lim_{n\to\infty}\prod_{i=1}^{n}(1-\frac{1}{3i-1}) \\
&= 1-0=1 \qquad \text{by using later result theorem 2.4.1}
\end{aligned}$$

By equating both sides, we have

$$1 = \frac{\Gamma(\frac{2}{3})}{3\Gamma(\frac{1}{3})} \sum_{n=1}^{\infty} \frac{\Gamma(n-\frac{2}{3})}{\Gamma(n+\frac{2}{3})}$$

So

$$\sum_{n=1}^{\infty} \frac{\Gamma(n-\frac{2}{3})}{\Gamma(n+\frac{2}{3})} = \frac{3\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})}$$

This is a very interesting summation. We obtained this result using urn models and very limited knowledge of $\Gamma(x)$.

We note that the command

$$\text{sum Gamma}(n-\frac{2}{3})/\text{Gamma}(n+\frac{2}{3}), \quad n = 1 \text{ to infinity}$$

in WolframAlpha gives the equivalent sum equal to $\dfrac{2\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{3})}$.

In summary, the probability of eventually obtaining the red ball is 1 when $k = 1, k = 2$ and $k = 3$.

2.4. Adding k Blue Balls

In this section, 1 blue ball will be added at the first step, 2 blue balls will be added at the second step, 3 blue balls will be added at the third step,..., k blue balls will be added at the kth step, if a blue ball is withdrawn. Also, we will try to calculate the probability of eventually obtaining the red ball using formulas (2.1) and (2.2).

$$\begin{aligned}(2.1) &= P(R_1) + P(B_1R_2) + P(B_1B_2R_3) + \ldots + P(B_1B_2B_3\ldots B_{n-1}R_n) + \ldots \\ &= \frac{1}{2} + \frac{1}{2}\times\frac{1}{3} + \frac{1}{2}\times\frac{2}{3}\times\frac{1}{5} + \frac{1}{2}\times\frac{2}{3}\times\frac{4}{5}\times\frac{1}{8} \\ &+ \frac{1}{2}\times\frac{2}{3}\times\frac{4}{5}\times\frac{7}{8}\times\frac{1}{12} + \frac{1}{2}\times\frac{2}{3}\times\frac{4}{5}\times\frac{7}{8}\times\frac{11}{12}\times\frac{1}{17} + \ldots\end{aligned}$$

It is complicated to find the pattern of this summation, so we try to obtain the probability using (2.2).

$$
\begin{aligned}
(2.2) &= 1 - \lim_{n\to\infty} P(B_1B_2B_3\ldots B_n) \\
&= 1 - \lim_{n\to\infty} \left(\frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} \times \frac{7}{8} \times \frac{11}{12} \times \frac{16}{17} \times \ldots \times \frac{1 + \dfrac{n(n-1)}{2}}{2 + \dfrac{n(n-1)}{2}} \right) \\
&= 1 - \prod_{n=1}^{\infty} \left(\frac{1 + \dfrac{n(n-1)}{2}}{2 + \dfrac{n(n-1)}{2}} \right)
\end{aligned}
$$

In order to prove $\displaystyle\prod_{n=1}^{\infty} \left(\frac{1 + \dfrac{n(n-1)}{2}}{2 + \dfrac{n(n-1)}{2}} \right)$ converges, we next introduce theorem 2.4.1.

THEOREM 2.4.1. *Let a_n be a sequence of real numbers with $0 \leqslant a_n < 1$. Then the infinite product*

$$P = \prod_{n=1}^{\infty} (1 - a_n)$$

converges to a nonzero real number if and only if the series

$$\sum_{n=1}^{\infty} a_n$$

converges.

PROOF. Taking the logarithm of the product gives the series

$$\ln P = \sum_{n=1}^{\infty} \ln(1 - a_n)$$

whose convergence is equivalent to the convergence of the product. But using L'Hópital's Rule, observe that

$$\lim_{x\to 0}\frac{\ln(1-x)}{x}=\lim_{x\to 0}\frac{\dfrac{\mathrm{d}}{\mathrm{d}x}\ln(1-x)}{\dfrac{\mathrm{d}}{\mathrm{d}x}(x)}=\lim_{x\to 0}\frac{\left(\dfrac{-1}{1-x}\right)}{1}=-1$$

If we assume that $a_n \to 0$, this gives us that

$$\lim_{n\to\infty}\frac{\ln(1-a_n)}{a_n}=-1$$

and the theorem follows by the limit comparison test. Q.E.D.[16] Using this theorem, everything about infinite series translates directly to the world of infinite products. For example, the product

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{n^p}\right)$$

converges iff $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^p}$ converages and $p>1$. □

Using theorem 2.4.1, we can prove $\prod\limits_{n=1}^{\infty}\left(\dfrac{1+\dfrac{n(n-1)}{2}}{2+\dfrac{n(n-1)}{2}}\right)$ converges to a real number $r, r\in(0,1]$.

Proof.

$$\prod_{n=1}^{\infty}\left(\frac{1+\dfrac{n(n-1)}{2}}{2+\dfrac{n(n-1)}{2}}\right)=\prod_{n=1}^{\infty}\left(1-\frac{1}{2+\dfrac{n(n-1)}{2}}\right)$$

,

which converges to a non-zero number bewteen 0 and 1 iff

$$\sum_{n=2}^{\infty}\left(\frac{1}{2+\frac{n(n-1)}{2}}\right)$$

converges.

And

$$\sum_{n=2}^{\infty}\left(\frac{1}{2+\frac{n(n-1)}{2}}\right)\leqslant\sum_{n=2}^{\infty}\frac{2}{(n-1)^2}$$

which converges.

Therefore,

$$\prod_{n=1}^{\infty}\left(\frac{1+\frac{n(n-1)}{2}}{2+\frac{n(n-1)}{2}}\right)$$

converges to a real number $r, r\in(0,1]$. □

Then,

$$(2.2)=1-\prod_{n=1}^{\infty}\left(\frac{1+\frac{n(n-1)}{2}}{2+\frac{n(n-1)}{2}}\right)=1-r$$

is a real number whose range is $[0,1)$, meaning that the probability of eventually obtaining the red ball is a number strictly between 0 and 1.

This is very interesting. Even though the partial products get smaller as n increases, the limit of the partial products does not tend to zero. Thus our method does not result in a closed form for the infinite sum.

Using WolframAlpha, the command

$$\text{product}(1 + n * (n-1)/2)/(2 + n * (n-1)/2), \quad n = 1 \text{ to } 10000$$

gives $0.1455419\ldots$ for an approximation to r. Or

$$1 - 0.1455419 = 0.8544581$$

as an approximation to the probability of eventually choosing a red ball.

Interestingly, we can change the condition that k blue balls will be added at the kth step if a blue ball is withdrawn to one red ball and k blue balls will be added at the kth step if a blue ball is withdrawn. In this way, we will obtain $P(\text{R eventually}) = 1$. We compute this result using (2.1) and (2.2) as follows.

$$\begin{aligned}
(2.1) &= P(R_1) + P(B_1R_2) + P(B_1B_2R_3) + \ldots + P(B_1B_2B_3\ldots B_{n-1}R_n) + \ldots \\
&= \frac{1}{2} + \frac{1}{2}\times\frac{2}{4} + \frac{1}{2}\times\frac{2}{4}\times\frac{3}{7} + \frac{1}{2}\times\frac{2}{4}\times\frac{4}{7}\times\frac{4}{11} \\
&+ \frac{1}{2}\times\frac{2}{4}\times\frac{4}{7}\times\frac{7}{11}\times\frac{5}{16} + \frac{1}{2}\times\frac{2}{4}\times\frac{4}{7}\times\frac{7}{11}\times\frac{11}{16}\times\frac{6}{22} + \ldots \\
&= \frac{1}{1\times 2} + \frac{2}{2\times 4} + \frac{3}{4\times 7} + \frac{4}{7\times 11} + \frac{5}{11\times 16} + \ldots \\
&= \sum_{i=1}^{\infty} \frac{i}{\left(1 + \dfrac{(i-1)i}{2}\right)\left(1 + \dfrac{(i+1)i}{2}\right)}
\end{aligned}$$

Using (2.2), we get

$$\begin{aligned}
(2.2) &= 1 - \lim_{n\to\infty} P(B_1B_2B_3\ldots B_n) \\
&= 1 - \left(\frac{1}{2}\times\frac{2}{4}\times\frac{4}{7}\times\frac{7}{11}\times\frac{11}{16}\ldots\right) \\
&= 1 - 0 = 1
\end{aligned}$$

Thus

$$\sum_{i=1}^{\infty} \frac{2i}{(2+(i-1)i)(2+(i+1)i)} = 1$$

so

$$\sum_{i=1}^{\infty} \frac{i}{(2+(i-1)i)(2+(i+1)i)} = \frac{1}{2}$$

This sum can also be obtained by a telescoping series.

CHAPTER 3

Urns Without Replacement

In this chapter we work with 50 red and 50 blue balls as a particular case of a general setting. If a blue ball is withdrawn, the ball will be returned to the urn, but k blue balls($k \geqslant 1$) are removed from the urn. The process stops when the red ball is withdrawn.

It needs to be clarified that k is the number of blue balls removed from the urn if a blue ball is withdrawn, so $k = 1, 2, 3, 4, ..., 50$; n is the number of steps, $n = 1, 2, 3, 4, ..., 51$.

Clearly, we can obtain the red ball within finite steps since there are finite balls in the urn at the beginning and we remove blue balls from the urn if a blue ball is withdrawn. Thus, a red ball can be withdrawn as early as the first step or as late as the 51th step. We obtain a red ball at the 51th step in the case where we remove one blue ball at each step if a blue ball is withdrawn. However, we are still interested in the probability of drawing a red ball at each step. Also, we would like to know the total probability of drawing a red ball after n steps even though we know P(R eventually) is 1. For example, we would like to know the total probability of obtaining a red ball within the first 10 steps $\sum_{i=1}^{10} P(R_i)$.

The total probability of obtaining a red ball within the first n continuous attempts is $\sum_{i=1}^{n} P(R_i)$, which is the summation of the probability of obtaining the red ball at the first step, the second step,..., until the nth step. Thus, it is still given by (2.1) from Chapter 2. That is,

$$\sum_{i=1}^{n} P(R_i) = P(R_1) + P(B_1R_2) + P(B_1B_2R_3) + \ldots + P(B_1B_2B_3 \ldots B_{n-1}R_n)$$

However, calculating the probability of obtaining the red ball using (2.2) will not be discussed in this chapter as the red ball will be obtained anyway within finitely many steps. Also, removing 4 or more blue balls ($k \geqslant 4$) if a blue ball is

withdrawn will not be discussed here since it is the same calculating process as dropping 3 or fewer balls.

Therefore, we will attain $\sum_{i=1}^{n} P(R_i)$ in the following sections and discuss the circumstances where

$k = 1$: removing 1 blue ball every step if a blue ball is withdrawn in Section 3.1;

$k = 2$: removing 2 blue balls every step if a blue ball is withdrawn in Section 3.2;

$k = 3$: removing 3 blue balls every step if a blue ball is withdrawn in Section 3.3;

3.1. $k = 1$:Removing One Blue Ball

In the case where $k = 1$,

$$\begin{aligned}
\sum_{i=1}^{n} P(R_i) &= P(R_1) + P(B_1R_2) + P(B_1B_2R_3) + P(B_1B_2B_3R_4) \\
&\quad + \ldots + P(B_1B_2 \ldots B_{n-1}R_n) \\
&= \frac{50}{100} + \frac{50}{100} \times \frac{50}{99} + \frac{50}{100} \times \frac{49}{99} \times \frac{50}{98} + \frac{50}{100} \times \frac{49}{99} \times \frac{48}{98} \times \frac{50}{97} + \ldots \\
&\quad + \frac{50}{100} \times \frac{49}{99} \times \frac{48}{98} \times \ldots \times \frac{50-n+2}{100-n+2} \times \frac{50}{100-n+1} \\
&= \sum_{i=1}^{n} \frac{\dfrac{50!}{(50-i+1)!}}{\dfrac{100!}{(100-i+1)!}} \times \frac{50}{100-i+1} \\
&= \sum_{i=1}^{n} \frac{\dfrac{(101-i)!}{50!(51-i)!}}{\dfrac{100!}{50!50!}} \times \frac{50}{101-i} \\
&= \sum_{i=1}^{n} \frac{\binom{101-i}{50}}{\binom{100}{50}} \times \frac{50}{101-i}
\end{aligned}$$

where $n = 2, 3, \ldots, 51$. In particular, if $n = 1$, $(1) = \dfrac{50}{100}$.

When $n = 51$,

$$\sum_{i=1}^{51} P(R_i) = \frac{50}{100} + \frac{50}{100} \times \frac{50}{99} + \frac{50}{100} \times \frac{49}{99} \times \frac{50}{98} + \frac{50}{100} \times \frac{49}{99} \times \frac{48}{98} \times \frac{50}{97}$$

$$
\begin{aligned}
&+ \ldots + \frac{50}{100} \times \frac{49}{99} \times \frac{48}{98} \times \frac{47}{97} \times \frac{46}{96} \times \ldots \times \frac{1}{51} \times \frac{50}{50} \\
&= \sum_{i=1}^{51} \frac{\binom{101-i}{50}}{\binom{100}{50}} \times \frac{50}{101-i} \\
&= 1 \quad \text{(since we must eventually choose a red ball)}
\end{aligned}
$$

This can be rewritten as

$$
\begin{aligned}
\binom{100}{50} &= \sum_{i=1}^{51} \binom{101-i}{50} \frac{50}{101-i} \\
&= \sum_{i=1}^{51} \binom{100-i}{49} \\
&= \sum_{i=1}^{51} \binom{100-i}{51-i} \\
&= \binom{99}{50} + \binom{98}{49} + \binom{97}{48} + \ldots + \binom{49}{0}
\end{aligned}
$$

This can be interpreted combinatorially as follows.

We have 100 objects and we want to choose 50 of them(left hand side). If we forbid the first object then there are $\binom{99}{50}$ choices. If we choose the first object but forbid the second object, there are $\binom{98}{49}$. If we choose the first two objects but forbid the third object, there are $\binom{97}{48}$ choices. Continuing in this manner we achieve our result. The result above is a classical result from combinatorics and results from repeated application of the identity

$$
\binom{n}{n} = \binom{n-1}{n-1} + \binom{n-1}{n}
$$

3.2. $k = 2$:Removing Two Blue Balls

In the case where $k = 2$,

$$\begin{aligned}
\sum_{i=1}^{n} P(R_i) &= P(R_1) + P(B_1R_2) + P(B_1B_2R_3) + P(B_1B_2B_3R_4) \\
&\quad + \ldots + P(B_1B_2 \ldots B_{n-1}R_n) \\
&= \frac{50}{100} + \frac{50}{100} \times \frac{50}{98} + \frac{50}{100} \times \frac{48}{98} \times \frac{50}{96} + \frac{50}{100} \times \frac{48}{98} \times \frac{46}{96} \times \frac{50}{94} \\
&+ \frac{50}{100} \times \frac{48}{98} \times \frac{50}{96} + \frac{50}{100} \times \frac{48}{98} \times \frac{46}{96} \times \frac{44}{94} \times \frac{50}{92} + \ldots \\
&= \frac{50}{100} + \sum_{i=2}^{n} \frac{2^{i-1}[(25 - (i-2)) \times \ldots \times 23 \times 24 \times 25]}{2^{i-1}[(50 - (i-2)) \times \ldots \times 48 \times 49 \times 50]} \times \frac{50}{100 - 2(i-1)} \\
&= \sum_{i=1}^{n} \frac{\dfrac{25!}{(27-i-1)!}}{\dfrac{50!}{(52-i-1)!}} \times \frac{25}{51-i} \\
&= \sum_{i=1}^{n} \frac{\dfrac{(51-i)!}{25!(26-i)!}}{\dfrac{50!}{25!25!}} \times \frac{25}{51-i} \\
&= \sum_{i=1}^{n} \frac{\binom{51-i}{25}}{\binom{50}{25}} \times \frac{25}{51-i}
\end{aligned}$$

where $n = 1, 2, \ldots, 26$.

When $n = 26$,

$$\begin{aligned}
\sum_{i=1}^{26} P(R_i) &= \frac{50}{100} + \frac{50}{100} \times \frac{50}{98} + \frac{50}{100} \times \frac{48}{98} \times \frac{50}{96} + \frac{50}{100} \times \frac{48}{98} \times \frac{46}{96} \times \frac{50}{94} \\
&\quad + \ldots + \frac{50}{100} \times \frac{48}{98} \times \frac{46}{96} \times \frac{2}{52} \times \frac{50}{50} \\
&= \sum_{i=1}^{26} \frac{\binom{51-i}{25}}{\binom{50}{25}} \frac{25}{51-i} = 1
\end{aligned}$$

As in the previous case, this is equivalent to

$$\sum_{i=1}^{26} \binom{51-i}{26-i} = \binom{50}{25}$$

which has a combinatorial interpretation similar to the previous. Note that we obtained our result probabilistically rather than combinatorially.

3.3. $k = 3$:Removing Three Blue Balls

In the case where $k = 3$, we do not get factorial or binomial coefficient expressions.

$$\begin{aligned}
\sum_{i=1}^{n} P(R_i) &= P(R_1) + P(B_1R_2) + P(B_1B_2R_3) + P(B_1B_2B_3R_4) \\
&\quad + \ldots + P(B_1B_2 \ldots B_{n-1}R_n) \\
&= \frac{50}{100} + \frac{50}{100} \times \frac{50}{97} + \frac{50}{100} \times \frac{47}{97} \times \frac{50}{94} + \frac{50}{100} \times \frac{47}{97} \times \frac{44}{94} \times \frac{50}{91} + \ldots \\
&= \frac{50}{100} + \sum_{i=2}^{n} \frac{[(50-3(i-2)] \times \ldots \times 44 \times 47 \times 50}{[(100-3(i-2)] \times \ldots \times 94 \times 97 \times 100} \times \frac{50}{100-3(i-1)} \\
&= \frac{50}{100} + \sum_{i=2}^{n} \left(\frac{50 \prod_{j=2}^{i} (56-3j)}{\prod_{j=1}^{i} (103-3j)} \right) \\
&= \frac{50}{100} + \sum_{i=2}^{n} \frac{50(3^{i-1})\Gamma(\frac{103-3i}{3})\Gamma(\frac{53}{3})}{3^i \Gamma(\frac{56-3i}{3})\Gamma(\frac{103}{3})}
\end{aligned}$$

where $n = 2, 3, \ldots, 17$. In particular, if $n = 1$, $(1) = \dfrac{50}{100}$.

Specifically, after we draw a blue ball every step for 16 steps, there will be 2 blue balls and 50 red balls left since 3 blue balls are dropped out of the urn every step. Then, at the 17th step if we draw a blue ball again, 2 blue balls and 1 red ball will be dropped out of the urn. At last, at the 18th step, there will be 49 red balls left and we will definitely obtain a red ball.

So, the probability of obtaining a red ball at the 18th step is

$$\frac{\prod_{i=2}^{18}(56-3i)}{\prod_{i=1}^{17}(103-3i)} = \frac{23}{118885713076}$$

And when $n = 18$,

$$\begin{aligned}
\sum_{i=1}^{18} P(R_i) &= \frac{50}{100} + \frac{50}{100}\times\frac{50}{97} + \frac{50}{100}\times\frac{47}{97}\times\frac{50}{94} + \frac{50}{100}\times\frac{47}{97}\times\frac{44}{94}\times\frac{50}{91} + \dots \\
&\quad + \frac{50}{100}\times\frac{47}{97}\times\dots\times\frac{5}{55}\times\frac{50}{52} + \frac{50}{100}\times\frac{47}{97}\times\frac{50}{94}\times\dots\times\frac{5}{55}\times\frac{2}{52}\times\frac{49}{49} \\
&= \frac{50}{100} + \sum_{n=2}^{17}\frac{50\prod_{i=2}^{n}(56-3i)}{\prod_{i=1}^{n}(103-3i)} + \frac{\prod_{i=2}^{18}(56-3i)}{\prod_{i=1}^{17}(103-3i)} \\
&= \frac{50}{100} + \sum_{n=2}^{17}\frac{50\Gamma(-\frac{100}{3})\Gamma(n-\frac{53}{3})}{53\Gamma(-\frac{53}{3})\Gamma(n-\frac{100}{3})} + \frac{23}{118885713076} \\
&= \frac{50}{100} + \frac{59442856515}{118885713076} + \frac{23}{118885713076} = 1
\end{aligned}$$

CHAPTER 4

An Urn With m red balls and 1 blue ball

This chapter considers an urn containing m red balls and one blue ball at the beginning. The urn evolves in discrete time steps. At each step, a ball is sampled at random. If a blue ball is withdrawn at a step, then the ball will be returned to the urn(with replacement) with 1 blue ball added to the urn. This process stops when a red ball is withdrawn. We will still study the probability of eventually drawing a red ball $P(\text{R eventually})$ using (2.1) as well as (2.2).

It needs to be clarified that m is the number of red balls in the urn. Specially, when $m = 1$, it means that there are 1 red ball and 1 blue ball in the urn at the beginning and one blue ball is added if a blue ball is withdrawn. So, it becomes the same case as that in Section 2.1 and will not be discussed again in this chapter. Here, n is the number of steps, so $n = 1, 2, 3, 4, ...$

We will discuss the circumstances where $m = 2$, an urn starting with 2 red balls and 1 blue ball in Section 4.1. In Section 4.2, we will discuss and summarize the results where $m = M$, an urn starting with M red balls and 1 blue ball.

4.1. An urn starting with 2 red balls and 1 blue ball

In the case where $m = 2$,

$$\begin{aligned}(2.1) &= P(R_1) + P(B_1R_2) + P(B_1B_2R_3) + P(B_1B_2B_3R_4) \\ &+ \ldots + P(B_1B_2 \ldots B_{n-1}R_n) + \ldots \\ &= \frac{2}{3} + \frac{1}{3} \times \frac{2}{4} + \frac{1}{3} \times \frac{2}{4} \times \frac{2}{5} + \frac{1}{3} \times \frac{2}{4} \times \frac{3}{5} \times \frac{2}{6} \\ &+ \frac{1}{3} \times \frac{2}{4} \times \frac{3}{5} \times \frac{4}{6} \times \frac{2}{7} + \ldots + \frac{2^2}{n(n+1)(n+2)} + \ldots \\ &= \sum_{n=1}^{\infty} \frac{2^2}{n(n+1)(n+2)}\end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{2^2}{n(n+1)(n+2)}$ is uneasy to obtain, we will calculate it using (2.2).

$$\begin{aligned}(2.2) &= 1 - \lim_{n\to\infty} P(B_1B_2B_3\ldots B_n) \\ &= 1 - \lim_{n\to\infty} \left(\frac{1}{3} \times \frac{2}{4} \times \frac{3}{5} \times \frac{4}{6} \times \ldots \times \frac{n-2}{n} \times \frac{n-1}{n+1} \times \frac{n}{n+2}\right) \\ &= 1 - \lim_{n\to\infty} \frac{2}{(n+1)(n+2)} = 1 - 0 = 1\end{aligned}$$

Therefore, the probability of eventually obtaining a red ball $P(\text{R eventually})$ is 1 in the case where an urn starts with 2 red balls and 1 blue ball and 1 blue ball will be added if a blue ball is withdrawn.

By equating (2.1) and (2.2) results, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$$

.

This result is of interest in itself and a discussion of it appears in Mathematics Stack Exchange at questions 560816, 1108626, 721749, 308555, 1686516[17].

4.2. An urn starting with M red balls and 1 blue ball

This section summarizes the formulas obtained from previous sections under the circumstances where $m = 1, m = 2$ and gives the general formulas for both (2.1) and (2.2) when $m = M$.

For $m = 1$,

$$(2.1) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$(2.2) = 1 - \lim_{n\to\infty} \frac{1}{n+1}$$

For $m = 2$,

$$(2.1) = \sum_{n=1}^{\infty} \frac{2^2}{n(n+1)(n+2)}$$

$$(2.2) = 1 - \lim_{n\to\infty} \frac{2}{(n+1)(n+2)}$$

For $m = 3$,

$$(2.1) = \sum_{n=1}^{\infty} \frac{2 \times 3^2}{n(n+1)(n+2)(n+3)}$$

$$(2.2) = 1 - \lim_{n\to\infty} \frac{2 \times 3}{(n+1)(n+2)(n+3)}$$

For $m = 4$,

$$(2.1) = \sum_{n=1}^{\infty} \frac{2 \times 3 \times 4^2}{n(n+1)(n+2)(n+3)(n+4)}$$

$$(2.2) = 1 - \lim_{n\to\infty} \frac{4!}{(n+1)(n+2)(n+3)(n+4)}$$

$$\vdots$$

For $m = M$,

$$(2.1) = \sum_{n=1}^{\infty} \frac{(M-1)! \cdot M^2}{n(n+1)(n+2)(n+3)(n+4)\dots(n+M)}$$

$$= \sum_{n=1}^{\infty} \frac{(M-1)! \cdot M^2}{\frac{(n+M)!}{(n-1)!}}$$

$$= \sum_{n=1}^{\infty} \frac{M \cdot M! \cdot n!}{n(n+M)!}$$

$$= \sum_{n=1}^{\infty} \frac{M}{n} \frac{1}{\binom{n+M}{n}}$$

$$(2.2) = 1 - \lim_{n\to\infty} \frac{M!}{(n+1)(n+2)(n+3)(n+4)\dots(n+M)}$$

$$= 1 - \lim_{n\to\infty} \frac{M!}{\frac{(n+M)!}{n!}}$$

$$= 1 - \lim_{n\to\infty} \frac{1}{\binom{n+M}{n}} = 1 - 0 = 1$$

Therefore, when $m = M$, the probability of eventually obtaining a red ball is

$$P(\text{R eventually}) = \sum_{n=1}^{\infty} \frac{M}{n} \cdot \frac{1}{\binom{n+M}{n}} = 1 - \lim_{n\to\infty} \frac{1}{\binom{n+M}{n}} = 1$$

.

CHAPTER 5

General Theorem

This chapter presents a general result based on the previous chapters. The result presented can be useful in inventing new identities. We illustrate with one example after stating the main theorem.

THEOREM 5.0.1. *Let* $0 < a_i \leqslant 1$, *then*

$$1 - \prod_{i=1}^{n} a_i = \sum_{i=1}^{n} (1 - a_i) \prod_{j=1}^{i-1} a_j \tag{5.1}$$

If convergence holds, then

$$1 - \prod_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} (1 - a_i) \prod_{j=1}^{i-1} a_j \tag{5.2}$$

or

$$1 - a_1 a_2 a_3 \ldots = (1 - a_1) + a_1(1 - a_2) + a_1 a_2 (1 - a_3) + \ldots \tag{5.3}$$

PROOF. Assume we have an urn with red and blue balls. We draw balls repeatedly and independently until we obtain a red ball. Assume that the probability of drawing a blue ball on the ith draw is $a_i, i = 1, 2, \ldots, n$. We compute $P(\text{R eventually}) = P$ in two different ways.

$$\begin{aligned} P &= P(R) + P(BR) + P(BBR) + \ldots + P(BBB \ldots BR) \\ &= (1 - a_1) + a_1(1 - a_2) + a_1 a_2 (1 - a_3) + \ldots + a_1 \ldots a_{n-1}(1 - a_n) \\ &= \sum_{i=1}^{n} (1 - a_i) \prod_{j=1}^{i-1} a_j \end{aligned}$$

Also,

$$P = 1 - P(\text{all B}) = 1 - P(BB\ldots B)$$
$$= 1 - a_1 a_2 \ldots a_n = 1 - \prod_{i=1}^{n} a_i$$

From this, we get (5.1), (5.2), (5.3). □

Example: Let $f(n)$ be the nth Fibonacci number. So $f(1) = 1$, $f(2) = 1$, $f(3) = 2$, $f(4) = 3$, $f(5) = 5$, and

$$f(n) = f(n-1) + f(n-2) \qquad \text{for } n = 3, 4, \ldots$$

.
Choose

$$a_n = \frac{2f(n+1)}{f(n+3)} \qquad \text{in theorem 5.0.1}$$

Note that

$$f(n) \uparrow \text{ and } f(n+3) = f(n+2) + f(n+1)$$

So

$$2f(n+1) < f(n+3) \qquad \text{for } n = 1, 2, \ldots$$

Hence

$$0 < a_i < 1 \qquad \text{for } n = 1, 2, \ldots$$

It is well known that

$$\lim_{n\to\infty} \frac{f(n+2)}{f(n+1)} = \frac{1+\sqrt{5}}{2} \text{ (golden ratio)[18]}$$

Hence

$$a_n = \frac{2f(n+1)}{f(n+3)} = \frac{2f(n+1)}{f(n+2)}\frac{f(n+2)}{f(n+3)} \to 2(\frac{2}{1+\sqrt{5}})(\frac{2}{1+\sqrt{5}})$$

So

$$a_n < \frac{8}{9} \quad , \quad \text{for large n}$$

Since

$$\left(\frac{8}{9}\right)^n \to 0,$$

it follows that

$$\prod_{i=1}^{\infty} a_i = 0 \quad \text{so } 1 - \prod_{i=1}^{\infty} a_i = 1 - 0 = 1$$

We observe that

$$a_1 = \frac{2f(2)}{f(4)} = \frac{2(1)}{3} = \frac{2}{3}, \quad a_2 = \frac{2f(3)}{f(5)} = \frac{2(2)}{5} = \frac{4}{5}$$

$$a_3 = \frac{2f(4)}{f(6)} = \frac{2(3)}{8} = \frac{6}{8}, \quad a_4 = \frac{2f(5)}{f(7)} = \frac{2(5)}{13} = \frac{10}{13},$$

We transform our situation to an urn model. The urn initially has 2 blue balls and 1 red ball. The process is with replacement. If a blue ball is withdrawn at the 1st step, then $2f(1)$ blue balls will be added to the urn; If a blue ball is withdrawn at the 2nd step, then $2f(2)$ blue balls and $f(1)$ red ball will be added to the urn; If a blue ball is withdrawn at the 3rd step, then $2f(3)$ blue balls and $f(2)$ red ball will be added to the urn; If a blue ball is withdrawn at the 4th step, then $2f(4)$ blue balls and $f(3)$ red balls will be added to the urn; If a blue ball is withdrawn

at the ith step, then $2f(i)$ blue balls and $f(i-1)$ red ball will be added to the urn and so forth.

We know

$$1 - \prod_{i=1}^{\infty} a_i = 1 - \frac{2f(2)}{f(4)}\frac{2f(3)}{f(5)}\frac{2f(4)}{f(6)}\ldots = 1 - 0 = 1$$

Using (5.3), we have

$$\begin{aligned}
1 &= (1-a_1) + a_1(1-a_2) + a_1a_2(1-a_3) + a_1a_2a_3(1-a_4) + \ldots \\
&= \left(1 - \frac{2f(2)}{f(4)}\right) + \frac{2f(2)}{f(4)}\left(1 - \frac{2f(3)}{f(5)}\right) \\
&+ \frac{2f(2)}{f(4)}\frac{2f(3)}{f(5)}\left(1 - \frac{2f(4)}{f(6)}\right) + \frac{2f(2)}{f(4)}\frac{2f(3)}{f(5)}\frac{2f(4)}{f(6)}\left(1 - \frac{2f(5)}{f(7)}\right) + \ldots \\
&= \left(1 - \frac{2f(2)}{f(4)}\right) + \frac{2f(2)}{f(4)}\left(1 - \frac{2f(3)}{f(5)}\right) + \sum_{i=2}^{\infty} \frac{2^{i+1}}{f(2+i)f(3+i)}\left(1 - \frac{2f(2+i)}{f(4+i)}\right) \\
&= 1 - \frac{2}{3} + \frac{2}{3}\left(1 - \frac{4}{5}\right) + \sum_{i=2}^{\infty} \frac{2^{i+1}}{f(2+i)f(3+i)}\frac{f(4+i) - 2f(2+i)}{f(4+i)} \\
&= 1 - \frac{8}{15} + \sum_{i=2}^{\infty} \frac{2^{i+1}}{f(2+i)f(3+i)}\frac{f(3+i) - f(2+i)}{f(4+i)} \\
&= 1 - \frac{8}{15} + \sum_{i=2}^{\infty} \frac{2^{i+1}}{f(2+i)f(3+i)}\frac{f(1+i)}{f(4+i)}
\end{aligned}$$

Hence

$$\sum_{i=2}^{\infty} \frac{2^{i+1}f(1+i)}{f(2+i)f(3+i)f(4+i)} = \frac{8}{15} = 0.5333\ldots$$

Using Wolfram Alpha command

$$\text{sum } 2^{(n+1)} * F_{(n+1)}/(F_{(n+2)} * F_{(n+3)} * F_{(n+4)}), n = 2 \text{ to } 1000$$

we obtain $0.5333\ldots$, which confirms our result.

There are many possible applications of theorem 5.0.1, as illustrated by our one example.

CHAPTER 6

Infinite Products and Urn Models

This chapter examines two well-known infinite products for π—Wallis Product(a) and Wallis Product(b) and how they are transformed into urn models.

DEFINITION 6.1 (Infinite Product). For a sequence of complex numbers $a_1, a_2, a_3,$ ... , the infinite product

$$\prod_{n=1}^{\infty} a_n = a_1 a_2 a_3 \dots$$

is defined to be the limit of the partial products $a_1 a_2 ... a_n$ as n increases without bound. The product is said to converge when the limit exists and is not zero. Otherwise the product is said to diverge.

Wallis Product(a) is given by

$$\begin{aligned} \frac{\pi}{2} &= \prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1}\right) \\ &= \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \frac{6}{7} \times \frac{8}{7} \times \frac{8}{9} \times \frac{10}{9} \times \dots \end{aligned} \tag{6.1}$$

Wallis Product(b) is given by

$$\begin{aligned} \frac{2}{\pi} &= \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right) \\ &= \prod_{n=1}^{\infty} \left(\frac{4n^2 - 1}{4n^2}\right) \\ &= \frac{3}{4} \times \frac{15}{16} \times \frac{35}{36} \times \frac{63}{64} \times \frac{99}{100} \times \dots \end{aligned} \tag{6.2}$$

6.1. Wallis Product(a) Transformed to an Urn Problem

This section demonstrates how Wallis' Product(a) is turned into the form of (2.2) in terms of a certain type of urn model with blue and red balls. Specifically, we indicate the initial urn and how it works with the purpose of eventually drawing the red ball. In addition, we will write formulas —(2.1) and (2.2) for this urn model to obtain the probability of eventually drawing the red ball.

First we transform Wallis Product(a) into the form of (2.2) for a certain urn model,

$$\begin{aligned}(6.1) &= \frac{2}{1}\times\frac{2}{3}\times\frac{4}{3}\times\frac{4}{5}\times\frac{6}{5}\times\frac{6}{7}\times\frac{8}{7}\times\frac{8}{9}\times\frac{10}{9}\times\ldots=\frac{\pi}{2}\\ &\Longrightarrow \frac{1}{1}\times\frac{2}{3}\times\frac{4}{3}\times\frac{4}{5}\times\frac{6}{5}\times\frac{6}{7}\times\frac{8}{7}\times\frac{8}{9}\times\frac{10}{9}\times\ldots=\frac{\pi}{4}\\ &\Longrightarrow 1-\frac{1}{1}\times\frac{2}{3}\times\frac{4}{3}\times\frac{4}{5}\times\frac{6}{5}\times\frac{6}{7}\times\frac{8}{7}\times\frac{8}{9}\times\frac{10}{9}\times\ldots=1-\frac{\pi}{4}\\ &\Longrightarrow 1-\frac{8}{9}\times\frac{24}{25}\times\frac{48}{49}\times\frac{80}{81}\times\ldots=1-\frac{\pi}{4}\end{aligned}$$

The formula above can be seen as the form of (2.2), so the corresponding urn model starts with 1 red ball and 8 blue balls. And 16 blue balls will be added if a blue ball is withdrawn at the first step; 24 blue balls will be added if a blue ball is withdrawn at the second step; ... ; $8(n+1)$ blue balls will be added if a blue ball is withdrawn at the nth step. The whole process is still with replacement and it stops when the red ball is withdrawn. So the probability of eventually drawing the red ball is already obtained above, which is 1 - $\frac{\pi}{4}$.

We next write the formula in the form of (2.1) as well as (2.2), then

$$\begin{aligned}(2.1) &= P(R_1)+P(B_1R_2)+P(B_1B_2R_3)+P(B_1B_2B_3R_4)\\ &\quad+\ldots+P(B_1B_2B_3\ldots B_{n-1}R_n)+\ldots\\ &=\frac{1}{9}+\frac{8}{9}\times\frac{1}{25}+\frac{8}{9}\times\frac{24}{25}\times\frac{1}{49}+\frac{8}{9}\times\frac{24}{25}\times\frac{48}{49}\times\frac{1}{81}+\ldots\end{aligned}$$

$$= \frac{1}{9} + \sum_{n=2}^{\infty} \frac{(3^2-1)\times(5^2-1)\times(7^2-1)\times\ldots\times[(2n-1)^2-1]\times 1}{3^2\times 5^2\times 7^2\times\ldots\times(2n-1)^2\times(2n+1)^2}$$

$$= \frac{1}{9} + \sum_{n=2}^{\infty} \frac{(2\times 4)\times(4\times 6)\times(6\times 8)\times\ldots\times[(2n-2)\times 2n]}{3^2\times 5^2\times 7^2\times\ldots\times(2n-1)^2\times(2n+1)^2}$$

$$= \frac{1}{9} + \sum_{n=2}^{\infty} \frac{\prod_{i=2}^{n}(2i-2)\cdot 2i}{\prod_{i=1}^{n}(2i+1)^2}$$

$$= \frac{1}{9} + \sum_{n=2}^{\infty} \frac{4^{2n-1}(n!)^3(n-1)!}{((2n+1)!)^2}$$

$$(2.2) = 1 - \lim_{n\to\infty} P(B_1B_2B_3\ldots B_n)$$

$$= 1 - \frac{8}{9}\times\frac{24}{25}\times\frac{48}{49}\times\frac{80}{81}\times\ldots$$

$$= 1 - \prod_{n=2}^{\infty}\frac{(2n-2)2n}{(2n-1)^2}$$

$$= 1 - \lim_{n\to\infty}\frac{4^{2n}(n!)^4(n+1)}{((2n+1)!)^2}$$

We know that both the sum and the product equal $1-\frac{\pi}{4}$ by Wallis' formula.

6.2. Wallis Product(b) Transformed to an Urn Problem

Similarly, this section illustrates how Wallis Product(b) is turned into the form of (2.2) for a certain type of urn model with blue and red balls. Specifically, the initial and addition of blue balls with the purpose of eventually drawing the red ball will be explained.

First we transform Wallis Product(b) into the form of (2.2) for a certain urn model,

$$(6.2) = \frac{3}{4}\times\frac{15}{16}\times\frac{35}{36}\times\frac{63}{64}\times\frac{99}{100}\times\ldots = \frac{2}{\pi}$$

$$\Longrightarrow 1 - \frac{3}{4}\times\frac{15}{16}\times\frac{35}{36}\times\frac{63}{64}\times\frac{99}{100}\times\ldots = 1-\frac{2}{\pi}$$

The formula above also can be seen as a form of (2.2), so the corresponding urn model is an urn starting with 1 red ball and 3 blue balls. 12 blue balls will be added if a blue ball is withdrawn at the first step; 20 blue balls will be added if a blue ball is withdrawn at the second step; ... ; $[12 + 8(n-1)]$ blue balls will be added if a blue ball is withdrawn at the nth step. The whole process is still with placement and it stops when the red ball is withdrawn. So the probability of eventually drawing the red ball is already obtained above, which is 1 - $\frac{2}{\pi}$.

We next write the formula in the form of (2.1) as well as (2.2), then

$$
\begin{aligned}
(2.1) &= P(R_1) + P(B_1R_2) + P(B_1B_2R_3) + P(B_1B_2B_3R_4) \\
&\quad + \ldots + P(B_1B_2B_3 \ldots B_{n-1}R_n) + \ldots \\
&= \frac{1}{4} + \frac{3}{4} \times \frac{1}{16} + \frac{3}{4} \times \frac{15}{16} \times \frac{1}{36} + \frac{3}{4} \times \frac{15}{16} \times \frac{35}{36} \times \frac{1}{64} + \ldots \\
&= \frac{1}{4} + \sum_{n=1}^{\infty} \frac{3 \times 15 \times 35 \times \ldots \times [(2n)^2 - 1] \times 1}{2^2 \times 4^2 \times 6^2 \times \ldots \times (2n)^2 \times (2n+2)^2} \\
&= \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(1 \times 3) \times (3 \times 5) \times (5 \times 7) \times \ldots \times [(2n-1) \times (2n+1)]}{2^2 \times 4^2 \times 6^2 \times \ldots \times (2n)^2 \times (2n+2)^2} \\
&= \frac{1}{4} + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{n}(2i-1)(2i+1)}{\prod_{i=0}^{n}(2i+2)^2} \\
&= \frac{1}{4} + \sum_{n=1}^{\infty} \frac{((2n)!)^2(2n+1)}{2^{4n+2}(n!)^4(n+1)^2} \\
(2.2) &= 1 - \lim_{n\to\infty} P(B_1B_2B_3 \ldots B_n) \\
&= 1 - \frac{3}{4} \times \frac{15}{16} \times \frac{35}{36} \times \frac{63}{64} \times \frac{99}{100} \times \ldots \\
&= 1 - \prod_{n=1}^{\infty} \frac{(2n-1)(2n+1)}{(2n)^2} \\
&= 1 - \lim_{n\to\infty} \frac{((2n)!)^2(2n+1)}{2^{4n}(n!)^4}
\end{aligned}
$$

By Wallis Product(b), both the sum and the product equal $1 - \frac{2}{\pi}$

CHAPTER 7

Urn Model Simulation in R

7.1. Simulation for Adding Blue Balls in Chapter 2

In this section, we will simulate the urn models discussed in Chapter 2 with the goal of withdrawing the red ball under the circumstances where $k = 1, 2, 3$ respectively in R. We also use the notation [increasing] to indicate that we add i extra blue balls to the urn if the ith ball is blue. In addition to the $k = 1, 2, 3$ cases, we also consider the [increasing] case. In particular, we will obtain and study the estimated probabilities as well as the real probability of withdrawing the red ball at each step using R.

For example, consider $k = 1$ (adding one extra blue ball after each draw) :

In R, firstly, we simulate the urn with one red ball and one blue ball initially. Then, we simulate the process so that if a blue ball is randomly withdrawn then the ball will be replaced with 1 blue ball added to the urn. If the red ball is randomly withdrawn then the process stops and we record the step at which the red ball is withdrawn. This procedure will be simulated N times. As observed in Chapter 2, this procedure will result in a red ball eventually with probability 1. Since extra blue balls are added each time a blue ball is selected, the probability of a blue ball on the subsequent step increases (and the probability of a red ball decreases). So there is a good possibility for the number of steps required to observe a red ball to be very large. For practical reasons, we will stop after 10000 steps if a red ball has not yet been drawn.

When $k = 1$ and $N = 10$ simulations, a typical set of results for the number of trials needed to see a red ball is $1, 1, 3, 1, 26, 1, 2, 5, 1, 1$. We denote the true probability that X trials are needed to see the first red ball by $f_1(x) = P(X = x)$ and observe that $f_1(x) = \dfrac{1}{x(x+1)}$, $x = 1, 2, 3, \ldots$. We can estimate this probability by

$$\hat{f}_1(x) = \frac{\text{Frequency(x)}}{N}, \qquad x = 1, 2, 3, \ldots$$

where N is the number of simulations, $\hat{f}_1(x)$ is the estimated probability of withdrawing the first red ball at the xth step; $Frequency(x)$ is the frequency (among N) of obtaining the red ball at the xth step;

For our simulation results with $N = 10$, we obtain
$\hat{f}_1(1) = 6/10$, $\hat{f}_1(2) = 1/10$, $\hat{f}_1(3) = 1/10$,
$\hat{f}_1(5) = 1/10$, $\hat{f}_1(26) = 1/10$,
and $\hat{f}_1(x) = 0$ for all other values of x. These estimates are plotted in Figure 7.1 using "+".

We repeat the procedure for $N = 1000$. The estimates for $N = 1000$ are also plotted in Figure 7.1 using "*". In addition, the true probabilities are plotted in Figure 7.1 using straight line segments.

Note that the probability mass function $f_1(x) = \dfrac{1}{x(x+1)}$, $x = 1, 2, 3, \ldots$ is a heavy tailed distribution since

$$\lim_{x\to\infty} e^{tx} P(X > x) = \lim_{x\to\infty} e^{tx} \sum_{i=x+1}^{\infty} \frac{1}{i(i+1)} = \lim_{x\to\infty} e^{tx} \frac{1}{x+1} = \infty \text{ for all } t > 0$$

Note as well, that if X has pmf $f_1(x)$, then

$$E(X) = \sum_{x=1}^{\infty} x f_1(x) = \sum_{x=1}^{\infty} x \frac{1}{x(x+1)} = \sum_{x=1}^{\infty} \frac{1}{(x+1)} = \infty.$$

The procedure that led to Figure 7.1 (using $k = 1$ additional ball) is repeated for $k = 2$, giving Figure 7.2, and for $k = 3$, giving Figure 7.3, and for the [increasing] case, giving Figure 7.4.

Finally, in Figure 7.5, the true probabilities for all four cases are plotted together. It is clear that since we have pmf's, and the sum of the probabilities is one for $f_k(x)$, $k = 1, 2, 3$ that the curves must cross each other. The [increasing] case is included in the Figure 7.5 as well.

model/urn plot in major paper/k=1.png

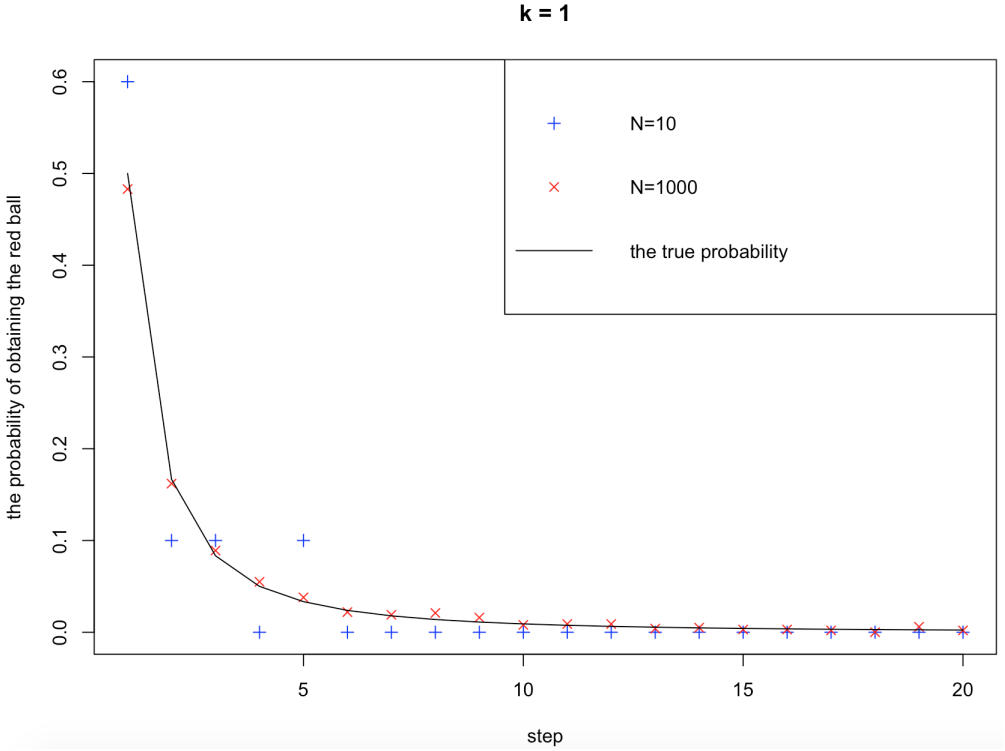

FIGURE 7.1. Comparing $N = 10$ and $N = 1000$ when $k = 1$

model/urn plot in major paper/k=2.png

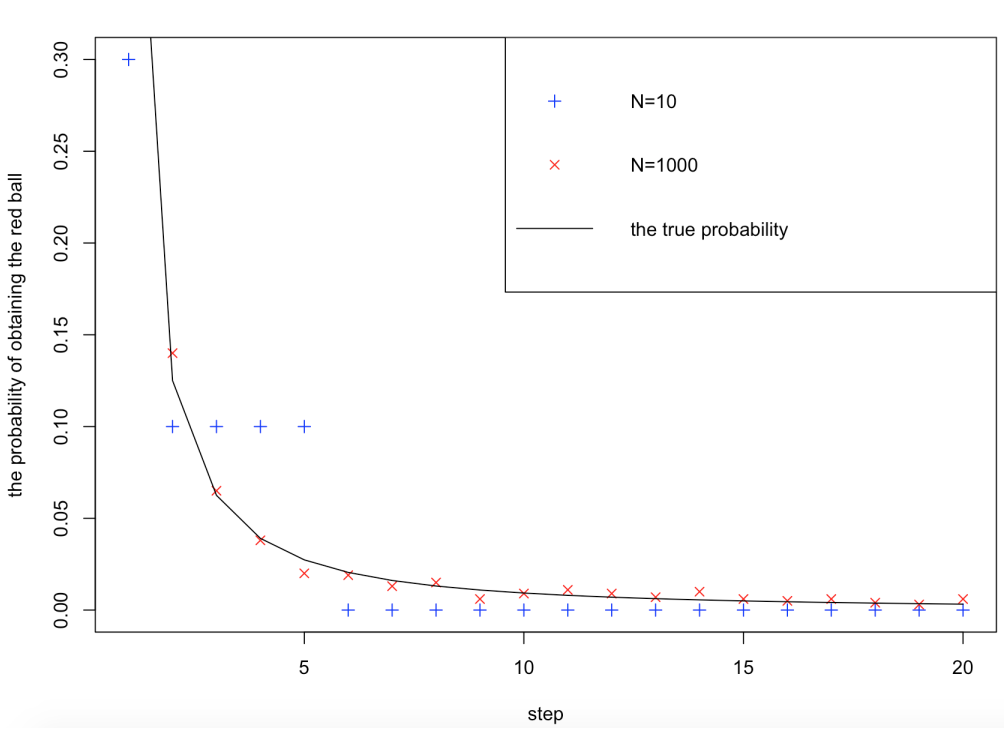

FIGURE 7.2. Comparing $N = 10$ and $N = 1000$ when $k = 2$

model/urn plot in major paper/k=3.png

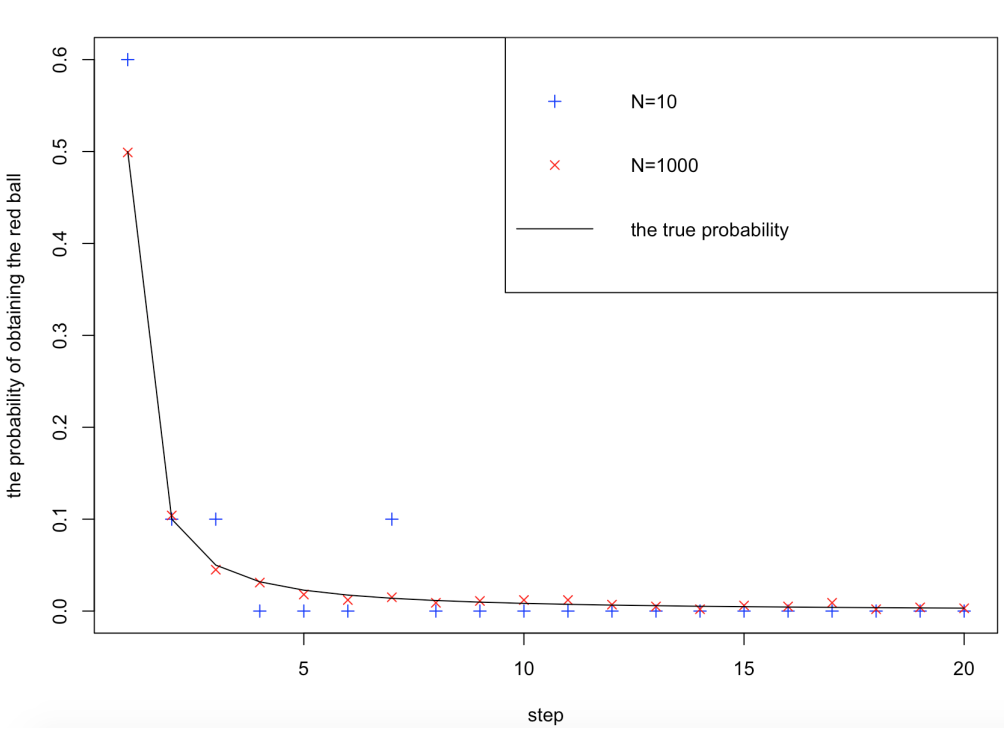

FIGURE 7.3. Comparing $N = 10$ and $N = 1000$ when $k = 3$

model/urn plot in major paper/[increasing].png

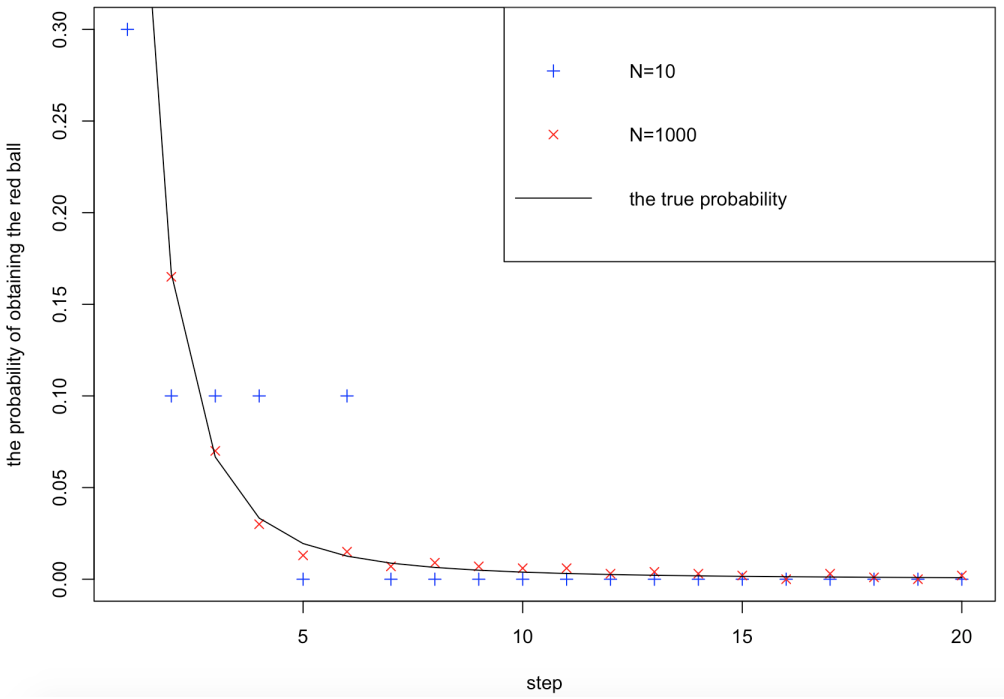

FIGURE 7.4. Comparing $N = 10$ and $N = 1000$ when [increasing]

model/urn plot in major paper/trueprobs.png

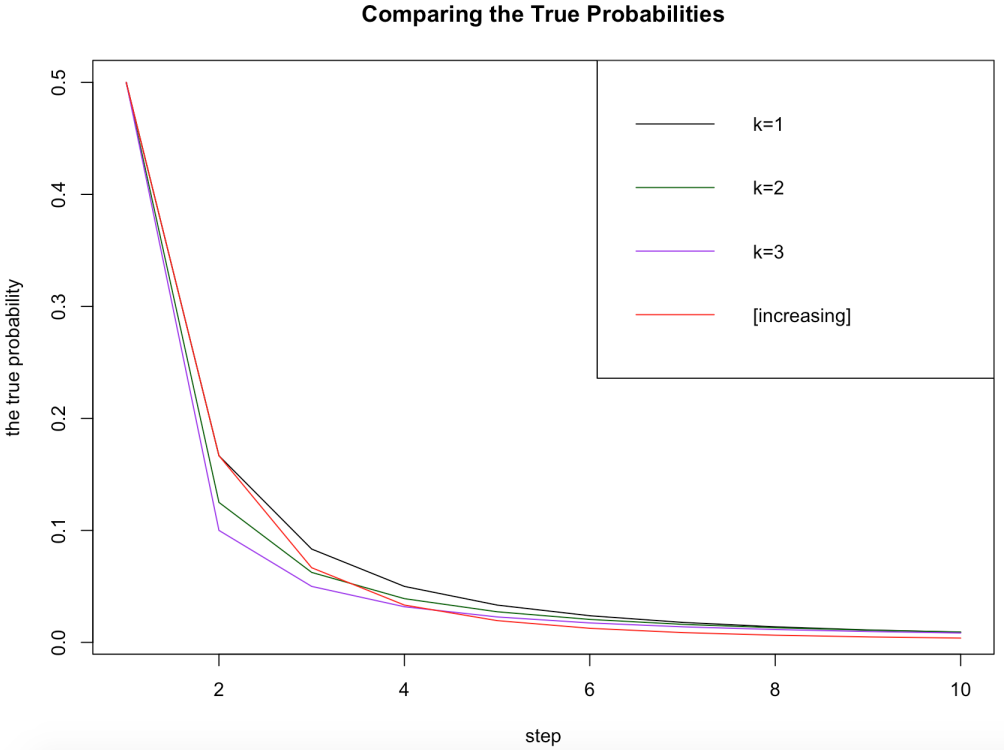

FIGURE 7.5. Comparing the True Probabilities When $k = 1, 2, 3$ and [increasing]

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Appendix A

Codes and Programs

This section shows the R programming code and commands used in the study:

```
###Simulation when k=1 and N=10###

red=1;blue=1;n=10000;N=10;  #simply change "N=10" to "N=1000" when doing 1000 trials
t <- rep(NA,N);
for(j in 1:N){
red=1;blue=1;
for(i in 1:n){
t[j]=i;
x=sample(red+blue,1,replace = TRUE);
if(x==1){
break;}
else{
blue=blue+1}
}
}
t.table <- table(factor(t,levels=1:n))
t.frequency <- as.matrix(t.table,header=FALSE)
p.simulation110 <- rep(NA,n)

#the estimated probability of obtaining the red ball at each step when k=1

for(k in 1:n){
p.simulation110[k]=t.frequency[k]/N
}
print(p.simulation110)

#the true probability of obtaining the red ball at each time when k=1

true.prob_add1=rep(NA,n)
for (y in 1:n) {
true.prob_add1[y]=1/(y*(y+1));
```

```
y=y+1;
}

##############################
###Simulation when k=2 and N=10###

red=1;blue=1;n=10000;N=10; #simply change "N=10" to "N=1000" when doing 1000 trials
t <- rep(NA,N);
for(j in 1:N){
red=1;blue=1;
for(i in 1:n){
t[j]=i;
x=sample(red+blue,1,replace = TRUE);
if(x==1){
break;}
else{
blue=blue+2}
}
}
t.table <- table(factor(t,levels=1:n))
t.frequency <- as.matrix(t.table,header=FALSE)
p.simulation210 <- rep(NA,n)

#the estimated probability of obtaining the red ball at each step when k=2

for(k in 1:n){
p.simulation210[k]=t.frequency[k]/N
}
print(p.simulation210)

#the true probability of obtaining the red ball at each time when k=2

true.prob_add2=rep(NA,n)
for (y in 1:n) {
true.prob_add2[y]=factorial(2*y-2)/(factorial(y)*factorial(y-1)*(2^y)*(2^(y-1)));
y=y+1;
}

##############################
###Simulation when k=3 and N=10###
```

```
red=1;blue=1;n=10000;N=10; #simply change "N=10" to "N=1000" when doing 1000 trials
t <- rep(NA,N);
for(j in 1:N){
red=1;blue=1;
for(i in 1:n){
t[j]=i;
x=sample(red+blue,1,replace = TRUE);
if(x==1){
break;}
else{
blue=blue+3}
}
}
t.table <- table(factor(t,levels=1:n))
t.frequency <- as.matrix(t.table,header=FALSE)
p.simulation310 <- rep(NA,n)

#the estimated probability of obtaining the red ball at each step when k=3

for(k in 1:n){
p.simulation310[k]=t.frequency[k]/N
}
print(p.simulation310)

#the true probability of obtaining the red ball at each time when k=3

true.prob_add3=rep(NA,n)
true.prob_add3[1]=0.5
for (y in 2:n) {
true.prob_add3[y]=factorial(3*y-3)/factorial(3*y-1)*(prod(3*(2:y)-2)/prod(3*(2:y)-4));
y=y+1;
}

##############################
###Simulation when [increasing] and N=10###

red=1;blue=1;n=10000;N=10; #simply change "N=10" to "N=1000" when doing 1000 trials
t <- rep(NA,N);
for(j in 1:N){
```

```
red=1;blue=1;
for(i in 1:n){
t[j]=i;
x=sample(red+blue,1,replace = TRUE);
if(x==1){
break;}
else{
blue=blue+i}
}
}
t.table <- table(factor(t,levels=1:n))
t.frequency <- as.matrix(t.table,header=FALSE)
p.simulationn10 <- rep(NA,n)

#the estimated probability of obtaining the red ball at each step when [increasing]

for(k in 1:n){
p.simulationn10[k]=t.frequency[k]/N
}
print(p.simulationn10)

#the true probability of obtaining the red ball at each time when [increasing]

a=b=c=rep(NA,n)
b[1]=1/2
c[2]=(1/2)*(2/3)
true.prob_addn=rep(NA,n)
true.prob_addn[1]=0.5;true.prob_addn[2]=(1/2)*(1/3)

for (y in 2:n) {
a[y]=y*(y-1)/2;
b[y]=(1+a[y])/(2+a[y])
y=y+1;
}

for (z in 2:n) {

c[z+1]=c[z]*b[z+1];
true.prob_addn[z+1]=c[z+1]*(1/(1+a[z+1]));
z=z+1;
```

```
}

##############################

#Figure 7.1: Comparing N=10 and N=1000 when k=1
plot(1:20, p.simulation110[1:20],col="blue",pch=3,main = "k = 1",
ylab = "the probability of obtaining the red ball", xlab = "step")
points(1:20, p.simulation11000[1:20],col="red",pch=4)
lines(1:20, true.prob_add1[1:20],col="black")
legend("topright", legend = c("N=10","N=1000","the true probability"),
col = c("blue","red","black"),pch = c(3,4,NA),lty = c(NA,NA,1))

#Figure 7.2: Comparing N=10 and N=1000 when k=2
plot(1:20, p.simulation210[1:20],col="blue",pch=3,main = "k = 2",
ylab = "the probability of obtaining the red ball", xlab = "step")
points(1:20, p.simulation21000[1:20],col="red",pch=4)
lines(1:20, true.prob_add2[1:20],col="black")
legend("topright", legend = c("N=10","N=1000","the true probability"),
col = c("blue","red","black"),pch = c(3,4,NA),lty = c(NA,NA,1))

#Figure 7.3: Comparing N=10 and N=1000 when k=3
plot(1:20, p.simulation310[1:20],col="blue",pch=3,main = "k = 3",
ylab = "the probability of obtaining the red ball", xlab = "step")
points(1:20, p.simulation31000[1:20],col="red",pch=4)
lines(1:20, true.prob_add3[1:20],col="black")
legend("topright", legend = c("N=10","N=1000","the true probability"),
col = c("blue","red","black"),pch = c(3,4,NA),lty = c(NA,NA,1))

#Figure 7.4: Comparing N=10 and N=1000 when [increasing]
plot(1:20, p.simulationn10[1:20],col="blue",pch=3,main = "[increasing]",
ylab = "the probability of obtaining the red ball", xlab = "step")
points(1:20, p.simulationn1000[1:20],col="red",pch=4)
lines(1:20, true.prob_addn[1:20],col="black")
legend("topright", legend = c("N=10","N=1000","the true probability"),
col = c("blue","red","black"),pch = c(3,4,NA),lty = c(NA,NA,1))

#Figure 7.5: Comparing the true probabilities when k=1,2,3 and [increasing]
plot(1:10, true.prob_add1[1:10],col="black","l",xlab = "step",ylab =
 "the true probability",main = "Comparing the True Probabilities")
lines(1:10, true.prob_add2[1:10],col="dark green")
```

```
lines(1:10, true.prob_add3[1:10],col="purple")
lines(1:10, true.prob_addn[1:10],col="red")
legend("topright",legend = c("k=1","k=2","k=3","[increasing]"),
col=c("black","dark green","purple","red"),lty=1)
```

Vita Auctoris

Yiyan Ni was born in China in 1990. She studied Statistics at Yanbian University from 2010 to 2014. She is currently a masters candidate in the Statistics program at the University of Windsor, and is expected to graduate in December 2018.