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**Recommended Citation**
Xu, Yejun; Chen, Lei; Li, Kevin; and Wang, Huimin. (2015). A chi-square method for priority derivation in group decision making with incomplete reciprocal preference relations. *Information Sciences*, 306, 166-179.
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A chi-square method for priority derivation in group decision making with incomplete reciprocal preference relations

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Abstract: This paper proposes a chi-square method (CSM) to obtain a priority vector for group decision making (GDM) problems where decision-makers’ (DMs’) assessment on alternative is furnished as incomplete reciprocal preference relations with missing values. Relevant theorems and iterative algorithm about CSM are proposed. Satty’s consistency ratio concept is adapted to judge whether an incomplete reciprocal preference relation provided by a DM is of acceptable consistency. If its consistency is unacceptable, an algorithm is proposed to repair it until its consistency ratio reaches a satisfactory threshold. The repairing algorithm aims to rectify an inconsistent incomplete reciprocal preference relation to one with acceptable consistency in addition to preserving the initial preference information as much as possible. Finally, four examples are examined to illustrate the applicability and validity of the proposed method, and comparative analyses are provided to show its advantages over existing approaches.

Keywords: Group decision making; chi-square method; incomplete reciprocal preference relation; priority; consistency.

1. Introduction

Group decision making (GDM) [9, 13, 14, 18, 21, 24, 35] is a procedure of drawing on the combined wisdom and experience of experts from different domains
to rank a finite number of alternatives. Reciprocal preference relations [21, 23, 27, 34,
39] are commonly used to represent decision-makers (DMs)’ preferences over a set of
possible alternative solutions, and have received considerable research attention in the
past decades. However, owing to time pressure, lack of knowledge, and the DM’s
limited expertise in the specific problem domain [1, 4, 5, 33, 36, 37, 41-44],
sometimes a DM can at best furnish his/her judgment on alternatives as a reciprocal
preference relation with missing or incomplete entries. Therefore, the method to
derive priorities from incomplete reciprocal preference relations [3, 10, 11, 15, 41]
has presented itself as an important and promising research topic, and attracted
considerable research interest.

For example, Xu and Da [32] put forward a normalizing ranking aggregation
method (NRAM) to derive priorities from an incomplete reciprocal preference
relation. Xu and Wang [40] extended the well-known eigenvector method (EM) for
priority derivation for an incomplete reciprocal preference relation, and the
improvement method therein not only increases the consistency level but also
preserves the initial preference information as much as possible. It is worth noting that
the aforementioned NRAM and EM can only be applied to a single incomplete
reciprocal preference relation. Xu [41] proposed two goal programming models
(GPM) to obtain a collective priority vector from several incomplete reciprocal
preference relations. Gong [17] put forward a least-square method (LSM) to generate
a collective priority vector from incomplete reciprocal preference relations furnished
by multiple DMs. Gong’s approach results in a simple equation. But it cannot be
applied to obtain a priority vector when the matrix $Q$ is singular or $Q^{-1}$ does not exist.
In contrast to LSM, which is only applicable to the case with at least one
multiplicative inconsistent incomplete reciprocal preference relation, the logarithmic
least squares method (LLSM) put forward by Xu et al. [38] can be used for all
incomplete reciprocal preference relations regardless of their multiplicative
consistency property. In real-world decision processes, different DMs often carry
heterogeneous power in reaching the final recommendation. It is noted that the
aforementioned methods did not take into account DMs’ weights in the decision
This paper extends a chi-square method (CSM) to prioritize alternatives in a GDM context when DMs furnish their judgment as incomplete reciprocal preference relations. The CSM was initially developed for priorities by Jensen [19], and was later cited by Blankmeyer [7]. The original approach is complicated and has rarely been used. Wang and Fu [28] developed a convergent and simple iterative algorithm to facilitate its application in practice. Due to its nonlinear property, this improved algorithm has many advantages such as ease in computer implementation. As such, the extended CSM has arisen as a simple but efficient approach to deal with incomplete reciprocal preference relations.

The key motivations to adopt the CSM can be summarized as follows: (1) The CSM can be used to obtain a collective priority vector from several incomplete reciprocal preference relations, while other methods such as EM and NRAM can only be applied to a single incomplete reciprocal preference relation. This advantage makes it a natural choice for handling GDM. (2) The CSM is convenient in considering different DMs’ weights in the decision process while this issue has been largely omitted by other methods. (3) By properly setting model parameters, the CSM can be flexibly employed to handle both complete and incomplete reciprocal preference relations. (4) Compared with other methods, the CSM is known for its better fitting performance, rank preservation capability and discrimination power. After Wang and Fu [28]’s extension, the improved CSM has become an efficient and convenient tool to handle incomplete reciprocal preference relations. By exploiting CSM to derive priority weights from incomplete reciprocal preference relations in a GDM context, this article further enhances its applications and enriches the theory and methodology of priority derivation.

An important issue in GDM with incomplete reciprocal preference relations is consistency test and inconsistency repairing because consistency of the judgment given by DMs has a direct impact on the final decision result [22]. Xu and Wang [40] adapted Saaty [26]’s consistency ratio (CR) to a fuzzy context and introduced a so-called fuzzy consistency ratio (FCR), which can be applied to incomplete
reciprocal preference relations. By adopting Saaty’s suggested threshold, an incomplete reciprocal preference relation is deemed to be acceptably consistent if $FCR<0.1$ [24]. If an incomplete reciprocal preference relation given by the DM does not possess acceptable consistency, it has to be repaired so that its consistency reaches the acceptable threshold. This paper will put forward a CSM-based algorithm to accomplish this task.

The remainder of the paper is organized as follows. Section 2 provides a review on basic concepts of reciprocal preference relations, incomplete reciprocal preference relations and an acceptable FCR. An associated theorem is also presented. In Section 3, the CSM is extended to obtain a priority vector from incomplete reciprocal preference relations based on the multiplicative transitivity property, resulting in an iterative algorithm. Section 4 puts forward an approach to repair an unacceptably inconsistent incomplete reciprocal preference relation to derive one with acceptable consistency. In Section 5, four examples are examined to show how to apply the proposed CSM and its effectiveness in handling GDM problems. Comparative analyses with existing methods demonstrate its validity and advantages. Concluding remarks are furnished in Section 6.

2. Preliminaries

In this section, we will give the definitions of reciprocal preference relations, incomplete reciprocal preference relations and a FCR.

Denote $N = \{1, 2, \ldots, n\}$, $M = \{1, 2, \ldots, m\}$. Let $X = \{x_1, x_2, \ldots, x_n\}$ ($n \geq 2$) be a finite set of alternatives, where $x_i$ denotes the $i$th alternative. $E = \{e_1, \ldots, e_m\}$ be a finite set of experts, where $e_k$ stands for the $k$th expert. $H = (h_1, \ldots, h_m)^T$ be the weight vector of experts, where $\sum_{k=1}^{m} h_k = 1$, $h_k \geq 0$ and $h_k$ demonstrates the importance degree of expert $e_k$ in the decision process.

A fuzzy preference relation is defined as follows [9, 16, 44]. The preference information on $X$ is described by a fuzzy preference relation $R \subseteq X \times X$, 

\[ R(x, y) = \begin{cases} f(x, y) & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases} \] 

where $f(x, y)$ is a fuzzy number.
\[ R = (r_{ij})_{n \times n} \], with membership function \( \mu_R : X \times X \rightarrow [0,1] \), where \( \mu_R(x_i, x_j) = r_{ij} \) denotes the preference degree of alternative \( x_i \) over \( x_j \). \( r_{ij} = 0.5 \) indicates the DM’s indifference between \( x_i \) and \( x_j \). \( r_{ij} = 1 \) signifies that \( x_i \) is definitely preferred to \( x_j \). \( 0 \leq r_{ij} < 0.5 \) implies that \( x_j \) is preferred to \( x_i \) and the smaller \( r_{ij} \) the stronger the preference of alternative \( x_j \) over \( x_i \). \( 0.5 < r_{ij} < 1 \) means that \( x_i \) is preferred to \( x_j \) and the greater \( r_{ij} \) the stronger the preference of alternative \( x_i \) over \( x_j \).

**Definition 1** [21]. Let \( R = (r_{ij})_{n \times n} \) be a fuzzy preference relation, then \( R \) is called a reciprocal preference relation if
\[
r_{ij} \in [0,1], \quad r_{ij} + r_{ji} = 1, \quad r_{ii} = 0.5, \quad \text{for all } i, j \in N.
\]

**Definition 2** [12, 27]. Let \( R = (r_{ij})_{n \times n} \) be a reciprocal preference relation, then \( R \) has multiplicative transitivity property if
\[
\left( \frac{1}{r_{ij}} - 1 \right) \left( \frac{1}{r_{jk}} - 1 \right) = \frac{1}{r_{ik}} - 1, \quad i, j, k \in N.
\]

It has been found [17] that a perfectly multiplicative transitivity reciprocal preference relation \( R = (r_{ij})_{n \times n} \) can be precisely characterized by a priority vector \( W = (w_1, w_2, \ldots, w_n)^T \), where \( r_{ij} = w_i / (w_i + w_j) \), \( \sum_{i=1}^{n} w_i = 1 \) and \( w_i > 0 \) for \( i = 1, 2, \ldots, n \).

**Definition 3** [2]. A membership function \( f : X \rightarrow Y \) is called partial if at least one element in the set \( X \) is not mapped to an element in the set \( Y \). If every element from the set \( X \) is mapped to an element in \( Y \), then we have a total function.

**Definition 4** [2]. A reciprocal preference relation \( P \) on a set of alternatives \( X \) with a partial membership function is an incomplete reciprocal preference relation.

For any \( i, j \in N \), let \( c_{ij} \) be the \( ij \)th entry of an incomplete preference relation.
\[ C = (c_{ij})_{n \times n}, \quad \delta_{ij} = \begin{cases} 1 & \text{if } c_{ij} \neq - \infty, \\ 0 & \text{if } c_{ij} = - \infty \end{cases}, \text{ and } c_{ij} = - \infty \text{ indicates a missing element } c_{ij}. \]

According to Definition 3, \( \delta_{ij} = 1 \) if and only if there exists \( c_{ij} = \mu_C(x_i, x_j) \), \( \delta_{ij} = 0 \) denotes that the preference value \( c_{ij} = \mu_C(x_i, x_j) \) is not furnished or missing.

**Theorem 1** [20]. Let \( C = (c_{ij})_{n \times n} \) be an incomplete reciprocal preference relation, then \( C \) can be completed by the known elements if there exists at least one known non-diagonal element in each row or column of \( C \). This implies that an incomplete reciprocal preference relation \( C \) which can be completed has at least \((n-1)\) non-diagonal judgments.

Let \( C = (c_{ij})_{n \times n} \) be an incomplete reciprocal preference relation, its fuzzy consistency index and fuzzy consistency ratio are denoted by \( FCI \) and \( FCR \) for short, and their formulas are presented as follows [38].

\[
\begin{align*}
\text{FCI} &= \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} \sigma_{ij} \delta_{ij} \left( \frac{c_{ij} w_j}{c_{ij} w_i} + \frac{c_{ji} w_i}{c_{ij} w_j} - 2 \right), \\
\text{FCR} &= \frac{\text{FCI}}{RI}
\end{align*}
\]

(1)

where \( \sigma_{ij} \) and \( \delta_{ij} \) are binary variables defined below and \( RI \) is the mean consistency index of randomly generated multiplicative preference matrices as given in Table 1.

\[
\sigma_{ij} = \begin{cases} 0, & \text{if } c_{ij} = 0 \text{ or } 1, \\ 1, & \text{otherwise}. \\ i, j \in N. \end{cases}
\]

(2)

\[
\delta_{ij} = \begin{cases} 0, & \text{if } c_{ij} = - \infty, \\ 1, & \text{if } c_{ij} \neq - \infty, \\ i, j \in N. \end{cases}
\]

(3)

By adapting the acceptable consistency threshold 0.1 proposed by Saaty, we have

**Definition 5** [38]. Let \( C = (c_{ij})_{n \times n} \) be an incomplete reciprocal preference relation, if \( FCR < 0.1 \), then \( C \) is of acceptable consistency, otherwise \( C \)'s consistency level is unacceptable.

**Table 1.** The mean consistency index of randomly generated matrix [26]

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( RI )</td>
<td>0</td>
<td>0</td>
<td>0.52</td>
<td>0.89</td>
<td>1.12</td>
<td>1.26</td>
<td>1.36</td>
<td>1.41</td>
<td>1.46</td>
<td>1.49</td>
<td>1.52</td>
<td>1.54</td>
<td>1.56</td>
<td>1.58</td>
<td>1.59</td>
</tr>
</tbody>
</table>
3. A chi-square method for priority derivation from group incomplete reciprocal preference relations

Consider a GDM problem, where \( m \) DMs give their preferences in the form of reciprocal preference relations, i.e. expert \( e_k \) describes his/her preference information as \( R^{(k)} = (r_{ij}^{(k)})_{n \times n} \).

Let \( W = (w_1, w_2, \ldots, w_n)^T \) be the priority weight vector for the reciprocal preference relations \( R^{(k)} = (r_{ij}^{(k)})_{n \times n} \), where \( \sum_{i=1}^n w_i = 1, \ w_i > 0, \ i \in N \). If \( R^{(k)} = (r_{ij}^{(k)})_{n \times n} \) is a complete reciprocal preference relation with multiplicative transitivity then it can be expressed as \[17\]

\[ r_{ij}^{(k)} = \frac{w_i}{w_i + w_j}, \ i, j \in N. \tag{4} \]

If some elements of \( R^{(k)} \) are missing or not given by the DM, then \( R^{(k)} \) is an incomplete reciprocal preference relation. We shall extend Eq. (4) to the case of incomplete reciprocal preference relations. For computational convenience, an indicator matrix \( \Delta = (\delta_{ij}^{(k)})_{n \times n} \) is constructed for incomplete reciprocal preference relation \( C^{(k)} = (c_{ij}^{(k)})_{n \times n} \) as follows

\[ \delta_{ij}^{(k)} c_{ij}^{(k)} = \delta_{ij}^{(k)} \frac{w_i}{w_i + w_j}, \ i, j \in N. \tag{5} \]

where \( \delta_{ij}^{(k)} \) is a binary variable defined as \[41\]:

\[ \delta_{ij}^{(k)} = \begin{cases} 0, & c_{ij}^{(k)} = -; \\ 1, & c_{ij}^{(k)} \neq -. \end{cases} \tag{6} \]

Due to additive reciprocity, it is easy to find that \( \delta_{ij}^{(k)} = \delta_{ji}^{(k)} \).

Next, we turn to find a priority vector \( W = (w_1, w_2, \ldots, w_n)^T \) to satisfy Eq.(5), where \( \sum_{i=1}^n w_i = 1, \ w_i \geq 0 \). To accomplish this, the following chi-square optimization model is constructed:
Min $F(W) = \sum_{k=1}^{m} \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij} \delta^{(k)}_{ij} \left[ \frac{(c^{(k)}_{ij} - w_i / (w_i + w_j))^2}{w_i / (w_i + w_j)} \right]$ \hspace{1cm} (7)

s. t. \begin{align*}
\sum_{i=1}^{n} w_i &= 1, \\
w_i &> 0, \quad i \in N.
\end{align*} \hspace{1cm} (8)

$D_w = \left\{ W = (w_1, w_2, \ldots, w_n)^T \mid \sum_{i=1}^{n} w_i = 1, w_i > 0, \quad i \in N \right\}$. \hspace{1cm} (9)

The idea is to minimize the overall deviation from Eq. (5). To solve this chi-square model, the following theorem is established.

**Theorem 2.** $F(W)$ has a unique minimum point $W^* = (w_1, w_2, \ldots, w_n)^T \in D_w$, which is also the unique solution of the following system of equations in $D_w$:

$$
\sum_{j=1}^{n} \sum_{k=1}^{m} h_{ik} \delta^{(k)}_{ij} \frac{w_i}{w_j} = \sum_{j=1}^{n} \sum_{k=1}^{m} h_{ik} \delta^{(k)}_{ij} \frac{w_j}{w_i}.
$$

\hspace{1cm} (10)

**Proof.** As $D_w$ is a bounded vector space and $F(W)$ is continuous function in $D_w$, for $F(W) \geq 0$, $w \in D_w$, $F(W)$ therefore has an infimum, namely there exists $w \in D_w$ such that function $F(W)$ reaches its minimum value.

In order to obtain the optimal priority vector $W^* = (w_1, w_2, \ldots, w_n)^T \in D_w$, the following Lagrangian function is constructed:

$$
L(W, \lambda) = F(W) + \lambda \left( \sum_{i=1}^{n} w_i - 1 \right). \hspace{1cm} (11)
$$

where $\lambda$ is the Lagrange multiplier. By setting the partial derivatives with respect to $w_i$ to be zero, we obtain the following set of equations:

$$
\frac{1}{w_i} \sum_{j=1}^{n} \sum_{k=1}^{m} h_{kj} \left( \delta^{(k)}_{ij} \frac{w_i}{w_j} - \delta^{(k)}_{ij} \frac{w_j}{w_i} \right) + \sum_{j=1}^{n} \sum_{k=1}^{m} h_{kj} \left( \frac{w_j}{w_i + w_j} \right)^2 \left( \delta^{(k)}_{ij} - \delta^{(k)}_{ij} \right) + \lambda = 0, \\
i \in N
$$

Given that $\delta^{(k)}_{ij} = \delta^{(k)}_{ji}$, Eq. (12) can be further simplified as follows:
\[
\frac{1}{W_i} \sum_{j=1}^{M} \sum_{k=1}^{N} h_k \left( \delta_{ji}^{(k)} c_{ji}^{(2k)} \frac{W_j}{W_i} - \delta_{ij}^{(k)} c_{ij}^{(2k)} \frac{W_j}{W_i} \right) + \lambda = 0, \quad i \in N
\]  
(13)

which is equivalent to

\[
\sum_{j=1}^{M} \sum_{k=1}^{N} h_k \left( \delta_{ji}^{(k)} c_{ji}^{(2k)} \frac{W_j}{W_i} - \delta_{ij}^{(k)} c_{ij}^{(2k)} \frac{W_j}{W_i} \right) + \lambda W_i = 0, \quad i \in N
\]  
(14)

Summing up Eq. (14) with respect to \( w_i, \ i \in N \), we have

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{n} h_k \left( \delta_{ji}^{(k)} c_{ji}^{(2k)} \frac{W_j}{W_i} - \delta_{ij}^{(k)} c_{ij}^{(2k)} \frac{W_j}{W_i} \right) + \lambda \sum_{i=1}^{n} W_i = 0
\]  
(15)

Since \( \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{n} \delta_{ji}^{(k)} c_{ji}^{(2k)} \frac{W_j}{W_i} - \delta_{ij}^{(k)} c_{ij}^{(2k)} \frac{W_j}{W_i} = 0 \) and \( \sum_{i=1}^{n} W_i = n \), we have

\[ \lambda = 0. \]  

Plugging \( \lambda = 0 \) into Eq. (14), one has

\[
\sum_{j=1}^{m} \sum_{k=1}^{n} h_k \left( \delta_{ji}^{(k)} c_{ji}^{(2k)} \frac{W_j}{W_i} - \delta_{ij}^{(k)} c_{ij}^{(2k)} \frac{W_j}{W_i} \right) = 0, \quad i \in N
\]  
(16)

That is

\[
\sum_{j=1}^{m} \sum_{k=1}^{n} h_k \delta_{ji}^{(k)} c_{ji}^{(2k)} \frac{W_j}{W_i} = \sum_{j=1}^{m} \sum_{k=1}^{n} h_k \delta_{ij}^{(k)} c_{ij}^{(2k)} \frac{W_j}{W_i}, \quad i \in N
\]  
(17)

It is clear that the minimum point \( W^* \) is a solution to Eq. (10), if the solution is unique in \( D_w \), \( W^* \) can be uniquely determined. The uniqueness is proved by contradiction as follows.

Assume that \( V = (v_1, v_2, \ldots, v_n)^T \in D_w \) and \( W = (w_1, w_2, \ldots, w_n)^T \in D_w \) are two solutions to Eq. (10). Let \( u_i = w_i / v_i, \ i \in N \) and \( u_i = \max_{i \in N} \{ u_i \} \). If there exists \( j \in N \) such that \( u_j < u_i \), then we have

\[
\sum_{j=1}^{m} \sum_{k=1}^{n} h_k \delta_{ji}^{(k)} c_{ji}^{(2k)} \frac{V_j}{V_i} > \sum_{j=1}^{m} \sum_{k=1}^{n} h_k \delta_{ij}^{(k)} c_{ij}^{(2k)} \frac{V_j}{V_i} \cdot \frac{U_j}{U_i} = \sum_{j=1}^{m} \sum_{k=1}^{n} h_k \delta_{ji}^{(k)} c_{ji}^{(2k)} \frac{W_j}{W_i}
\]  
(18)

\[
\sum_{j=1}^{m} \sum_{k=1}^{n} h_k \delta_{ji}^{(k)} c_{ji}^{(2k)} \frac{V_j}{V_i} < \sum_{j=1}^{m} \sum_{k=1}^{n} h_k \delta_{ij}^{(k)} c_{ij}^{(2k)} \frac{V_j}{V_i} \cdot \frac{U_j}{U_i} = \sum_{j=1}^{m} \sum_{k=1}^{n} h_k \delta_{ji}^{(k)} c_{ji}^{(2k)} \frac{W_j}{W_i}
\]  
(19)

According to Eqs. (10), (18), (19), it can be deduced that
which contradicts Eq. (10), Thus, \( u_j < u_1 \) cannot hold. Therefore, for all \( j \in N \),

\[ u_j = u_1 \], namely, \( w_1 / v_1 = w_2 / v_2 = \ldots = w_n / v_n \). Due to the fact that \( \sum_{i=1}^{n} v_i = 1 \),

\[ \sum_{i=1}^{n} w_i = 1 \], we have \( w_i \equiv v_i \), \( \forall i \in N \). This proves the uniqueness of the solution to Eq. (10).

To solve Eq. (10), we put forward a simple convergent iterative algorithm as follows.

**Algorithm 1.**

Let \( C_k = (c_{ij}^{(k)})_{n \times n} \ (k \in M) \) be the initial incomplete reciprocal preference relations provided by the DMs.

**Step 1.** Using Theorem 1 to judge whether an incomplete reciprocal preference relation \( C_k (k \in M) \) given by the DM \( e_k \) can be completed. If not, it is returned to expert \( e_k \) for an updated reciprocal preference relation, otherwise, go to Step 2.

**Step 2.** Initiating the iteration by giving an initial priority vector \( W(0) = (w_1(0), w_2(0), \ldots, w_n(0))^T \) and specifying an error parameter \( \varepsilon \ (0 < \varepsilon < 1) \), for example, \( \varepsilon = 0.0001 \), and setting \( L=0 \).

**Step 3.** Calculating

\[
\eta_i (W(L)) = \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta_{ji}^{(k)} c_{ij}^{2(k)} \frac{W_j}{w_j} - \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta_{ij}^{(k)} c_{ij}^{2(k)} \frac{W_j}{w_j}, \ i \in N
\]  

(21)

If \( |\eta_i (W(L))| \leq \varepsilon \) holds for all \( i \in N \), then \( W^* = W(L) \) and stop, otherwise, continue to Step 4.

**Step 4.** Determining \( p \) such that \( \left| \eta_p (W(L)) \right| = \max_{i \in N} \left\{ \left| \eta_i (W(L)) \right| \right\} \) and computing
\[ T(L) = \frac{\sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta_{lj}^{(k)} c_{jp}^{(k)} w_j(L)}{w_p(L)} \]

\[ f_i(L) = \begin{cases} T(L)w_p(L), & i = p, \\ w_i(L), & i \neq p, \end{cases} \]

\[ w_i(L+1) = f_i(L) / \left( \sum_{i=1}^{n} f_i(L) \right), \quad i \in N. \]

**Step 5.** Let \( L = L + 1 \) and go to Step 3.

For Algorithm 1, we can establish the following theorem.

**Theorem 3.** Algorithm 1 is convergent for any \( \varepsilon > 0 \).

**Proof.** We shall examine how \( F(W) \) changes, when \( W(L) \) progresses to \( W(L+1) \).

Suppose that \( t > 0 \) and \( S(t) = F(f(L)) = F(w_i(L), \ldots, w_{p-1}(L), tw_p(L), w_{p+1}(L), \ldots, w_p(L)). \)

Then we have

\[ S(t) = \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta_{lj}^{(k)} \left[ \left( c_{jp}^{(k)} - \frac{tw_p(L)}{tw_p(L) + w_j(L)} \right)^2 \right] \frac{tw_p(L) + w_j(L)}{tw_p(L)} \]

\[ + \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta_{lj}^{(k)} \left[ \left( c_{ip}^{(k)} - \frac{w_i(L)}{w_i(L) + tw_p(L)} \right)^2 \right] \frac{w_i(L) + tw_p(L)}{w_i(L)} \]

\[ + \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta_{lj}^{(k)} \left[ \left( c_{jp}^{(k)} - \frac{w_j(L)}{w_j(L) + w_p(L)} \right)^2 \right] \frac{w_p(L) + w_j(L)}{w_j(L)} \]

(25)

which is equivalent to

\[ S(t) = \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta_{lj}^{(k)} \left[ c_{jp}^{(k)} - \frac{w_j(L)}{w_p(L) + 1} \right] + \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta_{lj}^{(k)} \left[ c_{ip}^{(k)} - \frac{w_i(L)}{w_p(L) + 1} \right] \]

\[ + \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \left[ \delta_{jp}^{(k)} c_{jp}^{(k)} + \delta_{jp}^{(k)} c_{jp}^{(k)} - 2 \left( \delta_{jp}^{(k)} c_{jp}^{(k)} + \delta_{jp}^{(k)} c_{jp}^{(k)} \right) \right] \]

(26)
\[ + \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \left( \delta^{(k)}_{jp} \frac{t w_p(L)}{t w_p(L) + w_j(L)} + \delta^{(k)}_{pj} \frac{w_j(L)}{t w_p(L) + w_j(L)} \right) \]

\[ + \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta^{(k)}_{ij} \left[ \left( c_{ij}^{(k)} - \frac{w_j(L)}{w_i(L) + w_j(L)} \right)^2 \right] \frac{w_i(L) + w_j(L)}{w_j(L)} \]  

\[ \text{(26)} \]

\[ q_1 = \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta^{(k)}_{pj} c_{pj}^{(2k)} \frac{w_j(L)}{w_p(L)}, \]

\[ q_2 = \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta^{(k)}_{jp} c_{jp}^{(2k)} \frac{w_p(L)}{w_j(L)}, \]

\[ q_3 = \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \left[ \delta^{(k)}_{jp} c_{jp}^{(2k)} + \delta^{(k)}_{pj} c_{pj}^{(2k)} - 2 \left( \delta^{(k)}_{jp} c_{jp}^{(k)} + \delta^{(k)}_{pj} c_{pj}^{(k)} \right) \right] \]

\[ + \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \left( \delta^{(k)}_{jp} \frac{t w_p(L)}{t w_p(L) + w_j(L)} + \delta^{(k)}_{pj} \frac{w_j(L)}{t w_p(L) + w_j(L)} \right) \]

\[ + \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta^{(k)}_{ij} \left[ \left( c_{ij}^{(k)} - \frac{w_j(L)}{w_i(L) + w_j(L)} \right)^2 \right] \frac{w_i(L) + w_j(L)}{w_j(L)} \]  

\[ \text{(29)} \]

Let

\[ q_1 = \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta^{(k)}_{pj} c_{pj}^{(2k)} \frac{w_j(L)}{w_p(L)}, \]

\[ q_2 = \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta^{(k)}_{jp} c_{jp}^{(2k)} \frac{w_p(L)}{w_j(L)}, \]

Since \( \delta^{(k)}_{ij} = \delta^{(k)}_{ji} \), the second double summation term in Eq. (29) can be rewritten

\[ \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \left( \delta^{(k)}_{jp} \frac{t w_p(L)}{t w_p(L) + w_j(L)} + \delta^{(k)}_{pj} \frac{w_j(L)}{t w_p(L) + w_j(L)} \right) \]

\[ = \begin{cases} (n-1) & \text{if } \delta^{(k)}_{jp} = \delta^{(k)}_{pj} = 1 \\ 0 & \text{if } \delta^{(k)}_{jp} = \delta^{(k)}_{pj} = 0 \end{cases} \]

\[ \text{(30)} \]

Therefore, \( q_3 \) can be further simplified as

\[ q_3 = \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \left[ \delta^{(k)}_{jp} c_{jp}^{(2k)} + \delta^{(k)}_{pj} c_{pj}^{(2k)} - 2 \left( \delta^{(k)}_{jp} c_{jp}^{(k)} + \delta^{(k)}_{pj} c_{pj}^{(k)} \right) \right] \]

\[ + \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta^{(k)}_{ij} \left[ \left( c_{ij}^{(k)} - \frac{w_j(L)}{w_i(L) + w_j(L)} \right)^2 \right] \frac{w_i(L) + w_j(L)}{w_j(L)} \]
\[+ \begin{cases} (n-1) \text{ if } \delta^{(k)}_{jn} = \delta^{(k)}_{pn} = 1, \\ 0 \text{ if } \delta^{(k)}_{jn} = \delta^{(k)}_{pn} = 0, \end{cases}\] (31)

275 This indicates that \( q_3 \) is independent of \( t \). Then Eq. (26) can be equivalently expressed as
\[S(t) = q_1 / t + q_2 \cdot t + q_3.\] (32)

278 By setting \( \frac{dS(t)}{dt} \) to be zero, we have
\[t^* = \sqrt{\frac{q_1}{q_2}} = \sqrt{\frac{\sum_{j=1, j \neq p}^{n} \sum_{k=1}^{m} h_k \delta^{(k)}_{jp} c_{2jk}^{(k)} w_j(L)}{\sum_{j=1, j \neq p}^{n} \sum_{k=1}^{m} h_k \delta^{(k)}_{jp} c_{2jk}^{(k)} w_j(L)}},\] (33)

280 where \( t^* \) stands for the minimum point, and \( S(t^*) \) gives the minimum value of \( S(t) \).

283 If \( t^* = 1 \), Eq. (33) is equivalent to
\[\sum_{j=1, j \neq p}^{n} \sum_{k=1}^{m} h_k \delta^{(k)}_{jp} c_{2jk}^{(k)} w_j(L) = \sum_{j=1, j \neq p}^{n} \sum_{k=1}^{m} h_k \delta^{(k)}_{jp} c_{2jk}^{(k)} w_j(L),\] (35)

285 which also holds for \( j = p \), therefore, we have
\[\sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta^{(k)}_{jp} c_{2jk}^{(k)} w_j(L) = \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta^{(k)}_{jp} c_{2jk}^{(k)} w_j(L).\] (36)

287 That is
\[\eta_p (W(L)) = \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta^{(k)}_{jp} c_{2jk}^{(k)} w_j(L) = 0.\]

289 By the definition of \( p \) in Step 3, we have \( |\eta_p (W(L))| = 0 \). Since \( p \) is the subscript such that \( |\eta_i (W(L))| \) is maximized, we thus have \( |\eta_i (W(L))| = 0 \) for all \( i \in N \).

291 Therefore, the algorithm terminates and \( W^* = W(L) \).

292 If \( t^* \neq 1 \), then
\[ F(W(L)) - F(f(L)) = S(I) - S(t^*) = q_1 + q_2 - 2\sqrt{q_1q_2} = \left(\sqrt{q_1} - \sqrt{q_2}\right)^2 > 0 \quad (37) \]

Since \( F(W) \) is a homogenous function, \( F(f(L)) = F(W(L+1)) \). Inequality (37) shows that \( F(W(L+1)) < F(W(L)) \), for any \( L \geq 0 \). Therefore, \( F(W(L)) \) is a monotonically decreasing sequence with an infimum in \( D_\omega \) and, hence, convergent.

4. A method for repairing inconsistency of incomplete reciprocal preference relations

If the consistency level of an incomplete reciprocal preference relation is too low and deemed unacceptable, it can be returned to the DM for a reassessment until the updated one reaches an acceptable consistency level. This approach is presumably more reliable and accurate, but it is often impracticable because the iteration process can be tedious and time-consuming. To facilitate the decision process, this section puts forwards an automated procedure to improve the consistency level of a given incomplete reciprocal preference relation with unacceptable consistency \( FCR \geq 0.1 \).

Whenever possible, the DM’s intervention should be called upon, but this procedure serves as a convenient tool and can be employed by the analyst to facilitate the DM in eliciting his/her preference expeditiously. We first introduce the consistency deviation variable for \( c_{ij} \) as follows.

\[ d_{ij} = |c_{ij} - \frac{w_i}{(w_i + w_j)}| \quad (38) \]

If \( d_{ij} = 0 \), for all \( i, j \in N \) and \( c_{ij} \neq - \), then \( C \) is a perfectly consistent incomplete reciprocal preference relation. The priority vector \( W \) is able to precisely represent \( C \). The higher the deviation \( d_{ij} \), the more likely \( c_{ij} \) should be updated.

Conceptually, a judgment \( c_{ij} \) should be as close to \( \frac{w_i}{(w_i + w_j)} \) as possible to make it more consistent. In addition, the known element \( c_{ij} \) given by the expert often falls in the set \( U = \{0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1\} \). To avoid
excessive distortion of the DM’s original judgment, the improved preference relation
should not only increase the consistency level but also try to preserve the initial
preference information. The procedure starts with identifying the unusual and false
elements (UFEs) that are the most inconsistent element with the biggest \(d_{ij}\). Once the
UFEs are identified, the initial UFE \((c_{ij})\) will be updated with \(c'_{ij}\), where
\[c'_{ij} = \text{round}(w_i / (w_i + w_j) \times 10) \times 10^{-1}\]
and “round” is the usual round operation. This function ensures the updated judgment values are between 0 and 1 and have one
decimal place. It is trivial to make adjustment to accommodate the case when the
analyst or DM prefers to express the judgment in more decimal places.

Given the aforesaid discussion, the following algorithm is devised to repair
inconsistency of an incomplete reciprocal preference relation.

**Algorithm 2**

Let \(C = (c_{ij})_{n \times n}\) be an incomplete reciprocal preference relation given by the DM.

**Step 1.** Using the CSM algorithm in Section 3 to obtain the priority vector
\[W = (w_1, w_2, ..., w_n)^T\].

**Step 2.** Determining the consistency ratio of the incomplete reciprocal preference
relation as per Eq. (1), if \(FCR < 0.1\), go to Step 5, otherwise, go to Step 3.

**Step 3.** Computing deviations \(d_{ij}\)'s by using Eq. (38), and identifying the
maximum deviation to find the corresponding UFEs.

**Step 4.** Updating the UFEs \((c_{ij})\) with \(c'_{ij}\), where \(c'_{ij} = \text{round}(w_i / (w_i + w_j) \times 10) \times 10^{-1}\)
and go to Step 1.

**Step 5.** Ranking the alternatives according the priority vector \(W^*\).

**Step 6.** End.

5. **Illustrative examples**

In this section, four numerical examples are examined to demonstrate the
applications and advantages of the proposed CSM framework. Example 1 is a GDM
problem with incomplete reciprocal preference relations and a comparative analysis is conducted between CSM and three existing methods. Example 2 is a single incomplete reciprocal preference relation with unacceptable consistency and Algorithm 2 is utilized to repair it until its consistency becomes acceptable. Example 3 considers a single incomplete reciprocal preference relation with acceptable consistency. The purpose is to compare the result derived from CSM with those from EM, NRAM, GPM, LSM and LLSM on three performance evaluation criteria: FCR, MAD and MD. Example 4 discusses a GDM problem with incomplete reciprocal preference relations with a purpose to show the advantages of CSM.

**Example 1.** For a GDM problem with four decision alternatives $x_i$ ($i=1,2,3,4$) and three DMs $e_k$ ($k=1,2,3$). The DMs provide their preferences over the four decision alternatives as three incomplete reciprocal preference relations [41].

$$C_1 = \begin{bmatrix} 0.5 & 0.6 & - & 0.7 \\ 0.4 & 0.5 & 0.2 & 0.8 \\ - & 0.8 & 0.5 & 0.4 \\ 0.3 & 0.2 & 0.6 & 0.5 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.5 & 0.8 & 0.4 & - \\ 0.2 & 0.5 & 0.3 & 0.6 \\ 0.6 & 0.7 & 0.5 & 0.3 \\ - & 0.4 & 0.7 & 0.5 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0.5 & 0.3 & 0.4 & 0.6 \\ 0.7 & 0.5 & - & 0.5 \\ 0.6 & - & 0.5 & 0.7 \\ 0.4 & 0.5 & 0.3 & 0.5 \end{bmatrix}.$$  

Xu [41] employed goal programming (GP) models to derive a priority vector $W^* = (0.265,0.236,0.276,0.223)^T$ from the aforesaid three incomplete reciprocal preference relations. The research leads to a final ranking: $x_3 \succ x_1 \succ x_2 \succ x_4$, which is the same as the ranking generated by the logarithmic least square method (LLSM) [38] but slightly differs from the one obtained by the least-square method (LSM) [17] with the order between $x_2$ and $x_4$ being reversed. We now examine the problem using the CSM. In order to offer a fair comparison with Xu [41]'s method, we also set $h_1 = h_2 = h_3 = 1/3$.

**Step 1.** According to Theorem 1, we know that $C_k$ ($k=1,2,3$) can all be completed by known elements.

**Step 2.** Given an initial priority vector $W(0) = (0.25,0.25,0.25,0.25)^T$, specify the
Step 3. Calculate $\eta_i(W(0))$, we have

$$|\eta_1(W(0))| = 0.2 > \varepsilon, \quad |\eta_2(W(0))| = 0.2 > \varepsilon,$$

$$|\eta_3(W(0))| = 0.4 > \varepsilon, \quad |\eta_4(W(0))| = 0.4 > \varepsilon.$$  

As $|\eta_i(W(0))| > \varepsilon$ holds for all $i = 1, 2, 3, 4$, we continue to Step 4.

Step 4. Determine $p$ such that $|\eta_p(W(L))| = \max_{i \in N}\{|\eta_i(W(L))|\}$, we can set $p = 3$, and compute $T(0)$, $f(0)$ and $W(1)$.

$$T(0) = 1.3650, \quad f(0) = (0.2500, 0.2500, 0.3413, 0.2500)^T,$$

$$W(1) = (0.2291, 0.2291, 0.3127, 0.2291)^T.$$  

Step 5. Let $L = L + 1 = 1$ and go to Step 3.

The computation processes are detailed in Table 2. It is clear that iterations terminates at $L = 3$, when $|\eta_1| = 0.0296 < 0.1$, $|\eta_2| = 0.0072 < 0.1$, $|\eta_3| = 0.0227 < 0.1$,

$$|\eta_4| = 2.6357 \times 10^{-4} < 0.1$$,
indicating that the derived priority vector has reached an acceptable level of $\varepsilon$. The optimal priority vector is thus found to be $W^* = (0.2797, 0.2197, 0.3, 0.2007)^T$, resulting in a ranking of the four alternatives $x_3 \succ x_1 \succ x_2 \succ x_4$.

Remark 1. Computation results in Table 2 demonstrate that $F(W(L))$ decreases in iteration step $L$. However, for $|\eta_i(W(L))|$, this monotonicity does not hold any more and there may have ups and downs when $L$ increases, but eventually $|\eta_i(W(L))|$ will decrease to a value below the threshold $\varepsilon$ as ascertained by Theorem 3. As three of the four aforesaid methods derive an identical ranking with the other one yielding a slightly different order, this result demonstrates the robustness and credibility of the proposed CSM framework. To further compare the performance with the other three methods in fitting the three incomplete reciprocal preference relations, the following evaluation criteria are introduced:
Maximum deviation (MD) for incomplete reciprocal preference relations

\[ MD = \max_{i,j,k} \left\{ \delta_{ij}^{(k)} \left( \frac{c_{ij} w_j}{c_{ji} w_i} + \frac{c_{ji} w_i}{c_{ij} w_j} - 2 \right) \right\}, \quad i, j \in N, k \in M \]  

(39)

Maximum absolute deviation (MAD) for incomplete reciprocal preference relations

\[ MAD = \max_{i,j,k} \left\{ \delta_{ij}^{(k)} \left| \frac{c_{ij}^{(k)}}{w_i w_j} - \frac{w_i}{w_i + w_j} \right| \right\}, \quad i, j \in N, k \in M \]  

(40)

where \( \delta_{ij}^{(k)} \) is defined by Eq. (6). \( d_{ij}^{(k)} = c_{ij}^{(k)} / (w_i + w_j) \) is the consistency deviation for \( c_{ij}^{(k)} \) in the incomplete reciprocal preference relation \( C^{(k)} = (c_{ij}^{(k)})_{n \times n} \). If the priority vector \( W = (w_1, \ldots, w_n)^T \) is able to precisely characterize the reciprocal preference relation \( C^{(k)} \), then \( |d_{ij}^{(k)}| = 0 \), otherwise, \( |d_{ij}^{(k)}| > 0 \).

Table 3 indicates that CSM results in an identical ranking as GPM and LLSM while the ranking derived by LSM is slightly different. CSM has a comparable MAD as GPM, which is smaller than both LSM and LLSM. In terms of MD, CSM outperforms all the other three methods as it yields the smallest deviation. This partly shows the advantage of the CSM.

**Remark 2.** To facilitate a comparative study with GPM, LSM and LLSM, the weights of three reciprocal preference relations were set to be equal \( h_1 = h_2 = h_3 = 1/3 \). However, CSM allows an analyst to set different weights as per the practical situation, to properly reflect different experts’ varying influences in the GDM problem at hand. It is worth noting that if \( \delta_{ij} = 1 \), for all \( i, j \in N \), then the proposed CSM can still be utilized to derive a priority vector from reciprocal preference relations. This means that CSM can be used for both complete and incomplete reciprocal preference relations. In addition, by setting \( h_1 = 1 \) and \( h_k = 0 \), for \( k = 2, \ldots, m \), the CSM can be conveniently applied to derive a priority vector from a single incomplete reciprocal preference relation. This allows CSM to be used for a single expert's decision making problems in Examples 2 and 3.
Furthermore, by using Algorithm 1, we can get the values of $L$, $W$, $F(W)$ and the ranking of alternatives for different $\varepsilon$'s as listed in Table 4.
Table 2
The iterative processes for Example 1.

| Iterative steps | $|\eta_i(W(L))|$ | $\eta_i(W(L))$ | $\eta_4(W(L))$ | $\eta_3(W(L))$ | $\eta_2(W(L))$ | $\eta_1(W(L))$ |
|-----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $L$             | $|\eta_1|$ | $|\eta_2|$ | $|\eta_3|$ | $|\eta_4|$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ |
| 0               | 0.2          | 0.2           | 0.4           | 0.4            | 0.25           | 0.25           | 0.25           | 0.25           | 0.7067          |
| 1               | 0.3031       | 0.0835        | 1.156 × 10^{-4} | 0.2197        | 0.2291         | 0.2291         | 0.3127         | 0.2291         | 0.6449          |
| 2               | 6.197 × 10^{-5} | 0.0571       | 0.0792        | 0.1362         | 0.2744         | 0.2256         | 0.2943         | 0.2156         | 0.6086          |
| 3               | 0.0296       | 0.0072        | 0.0227        | 2.6357 × 10^{-4} | 0.2797         | 0.2197         | 0.3000         | 0.2007         | 0.6024          |

Table 3
Performance comparisons for Example 1.

<table>
<thead>
<tr>
<th>Methods</th>
<th>$W^*$</th>
<th>Ranking</th>
<th>MD</th>
<th>MAD</th>
</tr>
</thead>
<tbody>
<tr>
<td>CSM (this article)</td>
<td>$(0.2797, 0.2197, 0.3000, 0.2007)^T$</td>
<td>$x_3 \succ x_1 \succ x_2 \succ x_4$</td>
<td>1.9277</td>
<td>0.2992</td>
</tr>
<tr>
<td>LSM [17]</td>
<td>$(0.2822, 0.1968, 0.3202, 0.2009)^T$</td>
<td>$x_3 \succ x_1 \succ x_4 \succ x_2$</td>
<td>2.3282</td>
<td>0.3145</td>
</tr>
<tr>
<td>GPM [41]</td>
<td>$(0.265, 0.236, 0.276, 0.223)^T$</td>
<td>$x_3 \succ x_1 \succ x_2 \succ x_4$</td>
<td>2.0442</td>
<td>0.2858</td>
</tr>
<tr>
<td>LLSM [38]</td>
<td>$(0.2806, 0.2105, 0.3189, 0.1900)^T$</td>
<td>$x_3 \succ x_1 \succ x_2 \succ x_4$</td>
<td>2.1717</td>
<td>0.3266</td>
</tr>
</tbody>
</table>

Table 4
The values of $L$, $W$, $F(W)$ and ranking order for different $\varepsilon$ in Example 1.

| $\varepsilon$ | $L$ | $W^*$ | Ranking | $F(W)$ | $|\eta_1|$ | $|\eta_2|$ | $|\eta_3|$ | $|\eta_4|$ |
|----------------|-----|-------|---------|--------|--------------|--------------|--------------|--------------|
| $10^{-1}$      | 3   | $(0.2797, 0.2197, 0.3000, 0.2007)^T$ | $x_3 \succ x_1 \succ x_2 \succ x_4$ | 0.6024 | 0.0001       | 0.0728       | 0.0868       | 0.0141       |
| $10^{-2}$      | 15  | $(0.2753, 0.2204, 0.3032, 0.2011)^T$ | $x_3 \succ x_1 \succ x_2 \succ x_4$ | 0.6020 | 0.0031       | 0            | 0.0083       | 0.0052       |
| $10^{-3}$      | 24  | $(0.2746, 0.2201, 0.3041, 0.2012)^T$ | $x_3 \succ x_1 \succ x_2 \succ x_4$ | 0.6019 | 3.6858 × 10^{-4} | 0            | 9.8063 × 10^{-4} | 6.1205 × 10^{-4} |
| $10^{-4}$      | 34  | $(0.2745, 0.2201, 0.3043, 0.2012)^T$ | $x_3 \succ x_1 \succ x_2 \succ x_4$ | 0.6019 | 5.9215 × 10^{-5} | 2.6310 × 10^{-5} | 8.5525 × 10^{-5} | 0            |
| $10^{-5}$      | 44  | $(0.2746, 0.2201, 0.3040, 0.2012)^T$ | $x_3 \succ x_1 \succ x_2 \succ x_4$ | 0.6019 | 1.1102 × 10^{-16} | 6.4088 × 10^{-16} | 8.2368 × 10^{-16} | 1.8281 × 10^{-16} |
| $10^{-6}$      | 53  | $(0.2746, 0.2201, 0.3040, 0.2012)^T$ | $x_3 \succ x_1 \succ x_2 \succ x_4$ | 0.6019 | 1.1102 × 10^{-16} | 6.4088 × 10^{-16} | 8.2368 × 10^{-16} | 1.8281 × 10^{-16} |
Example 2. Consider a single DM’s decision problem with six alternatives \( x_i \) \((i=1,2,\ldots,6)\). The DM provides his/her preferences over the six decision alternatives, as an incomplete reciprocal preference relation which is shown below (adapted from [42]).

\[
C = \begin{bmatrix}
0.5 & - & - & 0.3 & 0.8 & 0.3 \\
- & 0.5 & - & - & - \\
- & - & 0.5 & - & - \\
0.7 & - & - & 0.5 & 0.4 & 0.8 \\
0.2 & - & - & 0.6 & 0.5 & 0.7 \\
0.7 & - & - & 0.2 & 0.3 & 0.5 \\
\end{bmatrix}
\]

Step 1. According to Theorem 1, it is easy to tell that \( C \) can be completed as no non-diagonal elements are furnished in the second or third row (column) of \( C \). Therefore, the initial judgment matrix has to be returned to the DM for an update, resulting in the following incomplete reciprocal preference relation:

\[
C = \begin{bmatrix}
0.5 & 0.3 & - & 0.3 & 0.8 & 0.3 \\
0.7 & 0.5 & 0.7 & - & 0.6 & - \\
- & 0.3 & 0.5 & 0.4 & - & - \\
0.7 & - & 0.6 & 0.5 & 0.4 & 0.8 \\
0.2 & 0.4 & - & 0.6 & 0.5 & 0.7 \\
0.7 & - & - & 0.2 & 0.3 & 0.5 \\
\end{bmatrix}
\]

Without loss of generality, let the original weight vector be \( W(0) = (1/6, 1/6, 1/6, 1/6, 1/6)^T \). Using Algorithm 1, one can get the values of \( L, W, F(W), FCR, |\eta(W(L))| \) and ranking results by setting different \( \varepsilon \) values as listed in Table 5. When \( \varepsilon \) is sufficiently small, the weight vector approaches

\[
W^* = (0.1301, 0.2714, 0.1281, 0.2090, 0.1509, 0.1106)^T,
\]

Step 2. Computing \( FCR \) by Eq. (1).

\[
FCI = 0.1870, \quad FCR = \frac{FCI}{RI} = \frac{0.1870}{1.26} = 0.1484 > 0.1.
\]

Since \( FCR > 0.1 \), the incomplete reciprocal preference relation \( C \) does not possess satisfactory consistency. We need to find its UFEs to repair this preference relation.
Step 3. Calculating the deviations between original judgment $c_{ij}$ and its corresponding consistent representation, we have

\[
D = \begin{bmatrix}
0 & 0.0241 & 0 & 0.0837 & 0.3369 & 0.2405 \\
0.0241 & 0 & 0.0206 & 0 & 0.0427 & 0 \\
0 & 0.0206 & 0 & 0.0201 & 0 & 0 \\
0.0837 & 0 & 0.0201 & 0 & 0.1808 & 0.1461 \\
0.3369 & 0.0427 & 0 & 0.1808 & 0 & 0.1231 \\
0.2405 & 0 & 0 & 0.1461 & 0.1231 & 0
\end{bmatrix}.
\]

Obviously, the maximum deviations are $d_{15}$ and $d_{31}$, so the UFEs are $c_{15}$ and $c_{31}$.

Step 4. Updating the UFEs $c_{ij}$ with $c'_{ij} = \text{round}(w_i/(w_i + w_j) \times 10) \times 10^{-1}$, one has $c'_{15} = 0.5$ and $c'_{31} = 0.5$.

Thus $C$ is updated as

\[
C' = \begin{bmatrix}
0.5 & 0.3 & - & 0.3 & 0.5 & 0.3 \\
0.7 & 0.5 & 0.7 & - & 0.6 & - \\
- & 0.3 & 0.5 & 0.4 & - & - \\
0.7 & - & 0.6 & 0.5 & 0.4 & 0.8 \\
0.5 & 0.4 & - & 0.6 & 0.5 & 0.7 \\
0.7 & - & - & 0.2 & 0.3 & 0.5
\end{bmatrix}
\]

Using Algorithm 1, one can obtain the values of $L$, $W$, $F(W)$, $FCR$, $|\eta(W(L))|$ and ranking of alternatives with different $\varepsilon$’s as listed in Table 6. When $\varepsilon$ is sufficiently small, the final priority vector is obtained as

\[
W^* = (0.0994, 0.2699, 0.1275, 0.2083, 0.1889, 0.1061)^T
\]

Computing $FCR$ by Eq. (1).

\[
FCI = 0.0661, \quad FCR = FCI / RI = 0.0661 / 1.26 = 0.0525 < 0.1.
\]

Thus, this updated $C$ is deemed to have acceptable consistency.

Step 5. Using the final priority vector $W^*$ to rank the alternatives as

\[
W^* = (0.0994, 0.2699, 0.1275, 0.2083, 0.1889, 0.1061)^T.
\]

$x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$.
By changing only $c_{15}$ and $c_{51}$, we were able to rectify an incomplete reciprocal preference relation to derive one with acceptable consistency. This allows the analyst to avoid the hassle of returning the inconsistent preference relation to the DM for reconsideration.

Remark 3. Numerical results in Tables 4, 5 and 6 demonstrate that iteration step $L$ increases when error parameter $\varepsilon$ decreases. In general, $F(W)$ and the consistency ratio of an incomplete reciprocal preference relation $C$ gets smaller when $\varepsilon$ decrease. When the error parameter $\varepsilon$ is sufficiently small, $W$, $F(W)$, $FCR$, and ranking results will converge to a set of values and remain unchanged.

In order to show the effectiveness of CSM, the other three methods EM [40], LSM [17], and LLSM [38] are also applied to the rectified C’ and assessed in terms of the criteria FCR, MD and MAD. Table 7 lists the ranking results by the four methods. It is clear that CSM and LLSM yield the same ranking $x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$, but EM and LSM generate slightly different rankings. Most notably, the EM and LSM reverse the order of $x_1$ and $x_6$, while the DM’s original judgment points to $x_6 \succ x_1$ because $c_{61} = c_{64} = 0.7$. It is apparent that this reverse is unwarranted and undesirable.

Moreover, CSM produces the smallest MD and MAD among the four methods, and the FCR from CSM is marginally larger than that from LLSM, but is smaller than those derived from EM and LSM. Across the three metrics, FCR, MD and MAD, Table 7 shows that CSM overall performs better than the other three methods EM, LSM and LLSM.
Table 5. The values of \( L, W, F(W), \) FCR and rankings for different \( \varepsilon \) of \( C \) in Example 2.

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( L )</th>
<th>( W )</th>
<th>Ranking</th>
<th>( F(W) )</th>
<th>FCR</th>
<th>( \eta(W(L)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^{-1} )</td>
<td>26</td>
<td>(0.1319, 0.2565, 0.1302, 0.2147, 0.1540, 0.1128)( ^T )</td>
<td>( x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3 \succ x_6 )</td>
<td>1.0276</td>
<td>0.1490</td>
<td>( 4.4409 \times 10^{-16}, 0.098, 0.0256, 0.0093, 0.0291, 0.034 )</td>
</tr>
<tr>
<td>( 10^{-2} )</td>
<td>66</td>
<td>(0.1305, 0.2699, 0.1280, 0.2094, 0.1511, 0.1110)( ^T )</td>
<td>( x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3 \succ x_6 )</td>
<td>1.0239</td>
<td>0.1484</td>
<td>( 0.0041, 0.0095, 1.1102 \times 10^{-16}, 0.0045, 2.2204 \times 10^{-16}, 9.3303 \times 10^{-4} )</td>
</tr>
<tr>
<td>( 10^{-3} )</td>
<td>107</td>
<td>(0.1301, 0.2712, 0.1281, 0.2090, 0.1509, 0.1107)( ^T )</td>
<td>( x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3 \succ x_6 )</td>
<td>1.0239</td>
<td>0.1484</td>
<td>( 4.2799 \times 10^{-4}, 9.7803 \times 10^{-4}, 2.2542 \times 10^{-4}, 2.2980 \times 10^{-4}, 2.2204 \times 10^{-16}, 9.4821 \times 10^{-5} )</td>
</tr>
<tr>
<td>( 10^{-4} )</td>
<td>146</td>
<td>(0.1301, 0.2714, 0.1281, 0.2090, 0.1509, 0.1106)( ^T )</td>
<td>( x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3 \succ x_6 )</td>
<td>1.0239</td>
<td>0.1484</td>
<td>( 2.2204 \times 10^{-16}, 9.6242 \times 10^{-5}, 2.2204 \times 10^{-16}, 3.3698 \times 10^{-5}, 2.8917 \times 10^{-5}, 3.3627 \times 10^{-5} )</td>
</tr>
<tr>
<td>( 10^{-5} )</td>
<td>187</td>
<td>(0.1301, 0.2714, 0.1281, 0.2090, 0.1509, 0.1106)( ^T )</td>
<td>( x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3 \succ x_6 )</td>
<td>1.0239</td>
<td>0.1484</td>
<td>( 2.2204 \times 10^{-16}, 9.6242 \times 10^{-5}, 2.2204 \times 10^{-16}, 3.3698 \times 10^{-5}, 2.8917 \times 10^{-5}, 3.3627 \times 10^{-5} )</td>
</tr>
<tr>
<td>( 10^{-6} )</td>
<td>227</td>
<td>(0.1301, 0.2714, 0.1281, 0.2090, 0.1509, 0.1106)( ^T )</td>
<td>( x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3 \succ x_6 )</td>
<td>1.0239</td>
<td>0.1484</td>
<td>( 2.2204 \times 10^{-16}, 9.6242 \times 10^{-5}, 2.2204 \times 10^{-16}, 3.3698 \times 10^{-5}, 2.8917 \times 10^{-5}, 3.3627 \times 10^{-5} )</td>
</tr>
</tbody>
</table>
### Table 6. The values of $L$, $W$, $F(W)$, FCR and rankings for different $\varepsilon$ of $C^*$ in Example 2.

| $\varepsilon$ | $L$ | $W$ | Ranking | $F(W)$ | FCR | $|\eta(W(L))|$ |
|---------------|-----|-----|---------|--------|-----|----------------|
| $10^{-1}$     | 22  | $(0.1014, 0.2551, 0.1288, 0.2126, 0.1914, 0.1108)^T$ | $x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$ | 0.4193 | 0.0532 | 0.0143, 0.0943, 0.0127, 0.0406, 0.0267 |
| $10^{-2}$     | 61  | $(0.0994, 0.2685, 0.1278, 0.2086, 0.1892, 0.1064)^T$ | $x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$ | 0.4154 | 0.0525 | 2.2204×$10^{-16}$, 0.0092, 0.003, 0.003, 0.0022, 0.001 |
| $10^{-3}$     | 97  | $(0.0994, 0.2698, 0.1275, 0.2083, 0.1889, 0.1061)^T$ | $x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$ | 0.4154 | 0.0525 | 2.345×$10^{-4}$, 8.782×$10^{-4}$, 2.196×$10^{-4}$, 1.094×$10^{-4}$, 3.234×$10^{-4}$ |
| $10^{-4}$     | 133 | $(0.0994, 0.2699, 0.1275, 0.2083, 0.1889, 0.1061)^T$ | $x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$ | 0.4154 | 0.0525 | 2.192×$10^{-5}$, 9.265×$10^{-5}$, 3.112×$10^{-5}$, 3.822×$10^{-5}$, 2.220×$10^{-5}$, 3.224×$10^{-5}$ |
| $10^{-5}$     | 170 | $(0.0994, 0.2699, 0.1275, 0.2082, 0.1889, 0.1061)^T$ | $x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$ | 0.4154 | 0.0525 | 2.949×$10^{-6}$, 9.625×$10^{-6}$, 1.388×$10^{-6}$, 1.547×$10^{-6}$, 0, 3.739×$10^{-6}$ |
| $10^{-6}$     | 209 | $(0.0994, 0.2699, 0.1275, 0.2082, 0.1889, 0.1061)^T$ | $x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$ | 0.4154 | 0.0525 | 0, 9.251×$10^{-7}$, 2.497×$10^{-7}$, 1.957×$10^{-7}$, 1.116×$10^{-7}$, 3.680×$10^{-7}$ |

### Table 7. Performance comparisons for Example 2.

<table>
<thead>
<tr>
<th>Method</th>
<th>$W^*$</th>
<th>Ranking</th>
<th>FCR</th>
<th>MD</th>
<th>MAD</th>
</tr>
</thead>
<tbody>
<tr>
<td>CSM</td>
<td>$(0.0994, 0.2699, 0.1275, 0.2083, 0.1889, 0.1061)^T$</td>
<td>$x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$</td>
<td>0.0525</td>
<td>0.6434</td>
<td>0.1837</td>
</tr>
<tr>
<td>EM</td>
<td>$(0.1038, 0.2780, 0.1262, 0.2017, 0.1949, 0.0953)^T$</td>
<td>$x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_1 \succ x_6$</td>
<td>0.054</td>
<td>0.9349</td>
<td>0.2213</td>
</tr>
<tr>
<td>LSM</td>
<td>$(0.1017, 0.3036, 0.1354, 0.1849, 0.1937, 0.0808)^T$</td>
<td>$x_2 \succ x_5 \succ x_4 \succ x_3 \succ x_1 \succ x_6$</td>
<td>0.0607</td>
<td>1.2774</td>
<td>0.2573</td>
</tr>
<tr>
<td>LLSM</td>
<td>$(0.0965, 0.2682, 0.1288, 0.2166, 0.1901, 0.0998)^T$</td>
<td>$x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$</td>
<td>0.0519</td>
<td>0.6994</td>
<td>0.1916</td>
</tr>
</tbody>
</table>
Example 3. Given a decision problem with six alternatives $x_i$ ($i=1,2,...,6$), the DM provides his/her preferences over the six decision alternatives, as an incomplete reciprocal preference relation (adapted from [38])

$$C = \begin{bmatrix}
0.5 & 0.4 & - & 0.3 & 0.3 & 0.3 \\
0.6 & 0.5 & 0.6 & 0.5 & - & 0.4 \\
- & 0.4 & 0.5 & 0.3 & 0.6 & - \\
0.7 & 0.5 & 0.7 & 0.5 & 0.4 & 0.8 \\
0.7 & - & 0.4 & 0.6 & 0.5 & 0.7 \\
0.7 & 0.6 & - & 0.2 & 0.3 & 0.5
\end{bmatrix}$$

This incomplete reciprocal preference relation was investigated by Xu et al. [38], in which the optimal priority vector is derived by LLSM as $W^\star = (0.0878, 0.1599, 0.1551, 0.2464, 0.2208, 0.1301)^T$. This yields a ranking of the six alternatives $x_4 \succ x_5 \succ x_2 \succ x_3 \succ x_6 \succ x_1$. We now examine the problem using CSM as follows.

According to Theorem 1, we know that $C$ can be completed. Without loss of generality, we set the original weight vector as $W(0) = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6)^T$. When $\varepsilon$ is set to $10^{-3}$, the values of $W$, $F(W)$, $FCR$ and ranking of alternatives will stabilize and remain unchanged. At $L=55$, one has $F(W)=0.5860$, $|\eta_1|=1.1863\times10^{-5} < \varepsilon$, $|\eta_2|=0 < \varepsilon$, $|\eta_3|=1.2503\times10^{-5} < \varepsilon$, $|\eta_4|=9.6445\times10^{-5} < \varepsilon$, $|\eta_5|=2.9292\times10^{-5} < \varepsilon$, $|\eta_6|=4.2788\times10^{-5} < \varepsilon$, $FCR=0.0728$, $W^\star = (0.0884, 0.1615, 0.1581, 0.2365, 0.2185, 0.1370)^T$, implying a ranking of these six alternatives as: $x_4 \succ x_5 \succ x_2 \succ x_3 \succ x_6 \succ x_1$.

For this single incomplete reciprocal preference relation, it can also be solved by EM [40], NRAM[32], LSM[17], LLSM[38] and GPM [41]. The results are shown in Table 8, from which we can see that CSM achieves the same ranking as EM, NRAM, LSM and LLSM, $x_4 \succ x_5 \succ x_2 \succ x_3 \succ x_6 \succ x_1$, while GPM yields a slightly different ranking, $x_4 \sim x_5 \succ x_2 \succ x_3 \sim x_6 \sim x_1$, which fails to discriminate $x_4$ and $x_5$, as well as $x_1$, $x_3$ and $x_6$. Furthermore, both NRAM and GPM lead to unacceptable
consistency ratio $FCR > 0.1$, and have larger MD and MAD values than other methods. A further examination reveals that CSM results in the smallest MAD value among these six methods and outperforms NRAM, GPM and LSM in all the three criteria.

**Example 4.** Consider a GDM problem with three DMs providing the following incomplete reciprocal preference relations $C_i$ ($i = 1, 2, 3$) for a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$:

\[
C_1 = \begin{bmatrix} 0.5 & 0.3 & - & 0.5 \\ 0.7 & 0.5 & 0.6 & 0.6 \\ - & 0.4 & 0.5 & 0.7 \\ 0.5 & 0.4 & 0.3 & 0.5 \end{bmatrix}, C_2 = \begin{bmatrix} 0.5 & 0.2 & 0.6 & 0.7 \\ 0.8 & 0.5 & 0.8 & - \\ 0.4 & 0.2 & 0.5 & 0.8 \\ 0.3 & - & 0.4 & 0.5 \end{bmatrix}, C_3 = \begin{bmatrix} 0.5 & 0.2 & 0.5 & 0.6 \\ 0.8 & 0.5 & - & 0.7 \\ 0.5 & 0.5 & 0.8 \\ 0.4 & 0.3 & 0.2 & 0.5 \end{bmatrix}.
\]

Let $h_1 = h_2 = h_3 = 1/3$ and $\varepsilon = 0.0001$. After several iterations, \(|\eta_1| = 5.7495 \times 10^{-5} < \varepsilon\), \(|\eta_2| = 8.8394 \times 10^{-5} < \varepsilon\), \(|\eta_3| = 3.0899 \times 10^{-5} < \varepsilon\), \(\eta_4| = 1.1102 \times 10^{-16} < \varepsilon\), indicating that the derived priority vector has reached an acceptable error level. Therefore, the optimal priority vector is found to be $W^* = (0.1954, 0.4386, 0.2316, 0.1344)^T$.

The comparative result is shown in Table 9. It is clear that CSM preforms the best in both MD and MAD. CSM obtains the same ranking as LLSM and LSM, $x_2 \succ x_3 \succ x_1 \succ x_4$, while GPM yields a slightly different ranking $x_2 \succ x_1 \sim x_3 \sim x_4$, as it fails to discriminate $x_1$ and $x_3$ and underperforms the proposed CSM and the other two methods in both MD and MAD.
### Table 8. Performance comparisons for Example 3

<table>
<thead>
<tr>
<th>Method</th>
<th>$W^*$</th>
<th>Ranking</th>
<th>$FCR$</th>
<th>MD</th>
<th>MAD</th>
</tr>
</thead>
<tbody>
<tr>
<td>CSM (This article)</td>
<td>(0.0884, 0.1615, 0.1581, 0.2365, 0.2185, 0.1370)$^T$</td>
<td>$x_4 \succ x_5 \succ x_2 \succ x_3 \succ x_6 \succ x_1$</td>
<td>0.0728</td>
<td>0.7487</td>
<td>0.1802</td>
</tr>
<tr>
<td>EM [40]</td>
<td>(0.0896, 0.1671, 0.1594, 0.2355, 0.2225, 0.1258)$^T$</td>
<td>$x_4 \succ x_5 \succ x_2 \succ x_3 \succ x_6 \succ x_1$</td>
<td>0.0729</td>
<td>0.6047</td>
<td>0.1826</td>
</tr>
<tr>
<td>NRAM [32]</td>
<td>(0.1204, 0.1681, 0.1648, 0.2000, 0.1931, 0.1537)$^T$</td>
<td>$x_4 \succ x_5 \succ x_2 \succ x_3 \succ x_6 \succ x_1$</td>
<td>0.1014</td>
<td>1.3993</td>
<td>0.2345</td>
</tr>
<tr>
<td>GPM [41]</td>
<td>(0.1091, 0.1636, 0.1091, 0.2545, 0.2545, 0.1091)$^T$</td>
<td>$x_4 \approx x_5 \succ x_2 \approx x_3 \approx x_6 \approx x_1$</td>
<td>0.1033</td>
<td>1.7849</td>
<td>0.2999</td>
</tr>
<tr>
<td>LSM [17]</td>
<td>(0.0978, 0.1765, 0.1591, 0.2263, 0.2220, 0.1183)$^T$</td>
<td>$x_4 \succ x_5 \succ x_2 \succ x_3 \succ x_6 \succ x_1$</td>
<td>0.0778</td>
<td>0.6848</td>
<td>0.1987</td>
</tr>
<tr>
<td>LLSM [38]</td>
<td>(0.0878, 0.1599, 0.1551, 0.2464, 0.2208, 0.1301)$^T$</td>
<td>$x_4 \succ x_5 \succ x_2 \succ x_3 \succ x_6 \succ x_1$</td>
<td>0.0719</td>
<td>0.6037</td>
<td>0.1874</td>
</tr>
</tbody>
</table>

### Table 9

Performance comparisons for Example 4

<table>
<thead>
<tr>
<th>Methods</th>
<th>$W^*$</th>
<th>Ranking</th>
<th>MD</th>
<th>MAD</th>
</tr>
</thead>
<tbody>
<tr>
<td>CSM(This article)</td>
<td>(0.1954, 0.4386, 0.2316, 0.1344)$^T$</td>
<td>$x_2 \approx x_3 \succ x_1 \succ x_4$</td>
<td>0.7520</td>
<td>0.1672</td>
</tr>
<tr>
<td>LSM[17]</td>
<td>(0.1822, 0.4611, 0.2160, 0.1408)$^T$</td>
<td>$x_2 \approx x_3 \succ x_1 \approx x_4$</td>
<td>0.9909</td>
<td>0.1946</td>
</tr>
<tr>
<td>GPM [41]</td>
<td>(0.2000, 0.4667, 0.2000, 0.1333)$^T$</td>
<td>$x_2 \approx x_3 \approx x_1 \approx x_4$</td>
<td>1.0411</td>
<td>0.1999</td>
</tr>
<tr>
<td>LLSM[38]</td>
<td>(0.1864, 0.4587, 0.2274, 0.1275)$^T$</td>
<td>$x_2 \approx x_3 \succ x_1 \approx x_4$</td>
<td>0.8154</td>
<td>0.1825</td>
</tr>
</tbody>
</table>
6. Concluding remarks

This paper proposes a chi-square method to handle decision problems with incomplete reciprocal preference relations and develops a convergent iterative algorithm to determine a priority vector. An adapted acceptable consistency ratio is employed to judge whether an incomplete reciprocal preference relation is acceptably consistent. If its consistency is not acceptable, an algorithm is put forward to repair it until its consistency reaches Saaty’s suggested threshold. This extended CSM not only improves the consistency level but also aims to preserve the initial preference information as much as possible.

Four numerical examples are examined to illustrate how to apply the proposed CSM and its effectiveness. Comparative studies with existing methods reveal the following features of the proposed CSM:

(1) In contrast to LSM, GPM and LLSM where DM’s weights are not considered, the proposed CSM allows the analyst to assign proper weights to different experts to reflect their varying influences in GDM problems.

(2) By setting $h_i = 1$ and $h_k = 0$ for $k = 2, \ldots, m$, CSM can be conveniently applied to derive a priority vector from a single incomplete reciprocal preference relation. This implies that the proposed CSM model can be employed to handle both group and individual decision problems.

(3) By setting $\delta_{ij} = 1$, for all $i, j \in N$, CSM can be utilized to derive a priority vector from complete reciprocal preference relations. This indicates that it can be flexibly used to handle decision problems with both complete and incomplete reciprocal preference relations.

(4) Numerical experiments demonstrate that CSM often outperforms the other methods such as EM, GPM, LSM, LLSM, and NRAM in terms of FCR, MD, and MAD when handling incomplete reciprocal preference relations.

(5) As illustrated in Example 2, CSM tends to have better rank preservation capability and discrimination power.

Current research establishes CSM as a viable and effective tool to handle decision
problems with incomplete reciprocal preference relations. In reality, DMs may provide their preference judgment in different formats of preference relations. As a worthy future research topic, it would be interesting to explore how the CSM framework can be extended to tackle other types of decision inputs such as incomplete intuitionistic fuzzy preference relations [29], incomplete linguistic preference relations [8, 25] and related consensus problems [6, 30, 31].

Acknowledgments

The authors are very grateful to the Associate Editor and the two anonymous reviewers for their constructive comments and suggestions that have helped to improve the quality and presentation of this paper. Yejun Xu would like to acknowledge the financial support of National Natural Science Foundation of China (NSFC) Grants (No. 71101043, 71471056 and 71433003), the Fundamental Research Funds for the Central Universities (No. 2014B09214), Program for Excellent Talents in Hohai University. Kevin W. Li would like to acknowledge the financial support of a Natural Sciences and Engineering Research Council of Canada (NSERC) Discovery Grant, and NSFC Grants (No. 71272129 and 71271188).

References

[25] C. Porcel, E. Herrera-Viedma, Dealing with incomplete information in a fuzzy linguistic recommender system to disseminate information in university digital libraries, Knowledge-Based