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The Virasoro Algebra: Signatures of Highest Weight Modules and the Modular Group Action

Byung Chun

University of Windsor

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by
Byung Kyu Chun

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by:
Byung Kyu Chun

Approved by:

. . . .

Dr. Ilya Shapiro
Department of Mathematics and Statistics

. . . .

Dr. Eugene Kim
Department of Physics

. . . .

Dr. Wai Ling Yee, Supervisor
Department of Mathematics and Statistics

. . . .

Dr. Rick Caron, Chair of Defense
Department of Mathematics and Statistics

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Abstract

The Virasoro algebra (*Vir*) has important applications to the study of infinite dimensional Lie algebras, and specifically to areas of theoretical physics modeled by conformal field theories. The positive-energy representations of *Vir* play a key role in string theory. A vital piece of information is the signature of the positive-energy representations. The physicist Adrian Kent calculated the characters of the signatures of these positive-energy representations in his paper [Ken91].

In this thesis, we will provide mathematical proofs for Kent's signature formulas for all possible values of the central charge and lowest weight. Furthermore, we simplify Kent's formulas by adopting a different approach to viewing the formulas. An important consequence is a clean reformulation in the minimal model case, which is of tremendous interest to theoretical physicists who wish to understand the modular group action in order to apply Kent's formulas to the study of non-unitary conformal field theories. We discuss how to compute the modular group action.

Thesis Supervisor: Wai Ling Yee
Title: Assistant Professor of Mathematics and Statistics

Department Reader: Dr. Ilya Shapiro
Title: Assistant Professor of Mathematics and Statistics

External Reader: Dr. Eugene Kim
Title: Associate Professor of Physics

Dedication

This thesis is dedicated in memory to my father who passed away from cancer before he could see this finished.

Acknowledgements

I would like to acknowledge the incredible effort and support provided by my supervisor, Dr. Yee. Without her constant encouragement, unfailing support and late night work, this thesis simply could not have come together. This work is a testament to the understanding and dedication and words cannot truly thank her enough for the monumental task of completing this thesis.

I further want to recognize my wonderful wife, Pamela, who wasted weeks helping to typeset and provided relief and comfort when the stress seemed unmanageable. She was without a doubt my walking stick, and I could not have completed this journey without her.

Moreover, my unfailing friends: Dawn and Laura; who encouraged, supported, and even provided a place to stay throughout the writing process. Friends of this caliber are few and far between, and I appreciate all you have done.

Lastly, but of course not least, I want to recognize my family. Their continual and ongoing companionship lifted my spirits when the going was difficult.

Thank you all for your help. This thesis really should belong to all of you.

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Chapter 1

Introduction

1.1 Thesis Problem

In this thesis, we compute signatures of invariant Hermitian forms on irreducible highest weight modules over the Virasoro algebra. Such formulas were stated in [Ken91] without rigorous proof. We provide rigorous proofs and significantly simplify some of Kent's formulas, the most important simplification being in what is called the minimal model case. Mathematical physicists would like to use Kent's formulas in order to study non-unitary conformal field theories; however, in order to do so, they require the understanding of the modular group action on the signature formulas. The simplification should lead to a complete understanding of the modular group action.

1.2 Motivation

The Virasoro algebra is an infinite dimensional Lie algebra that naturally arises in numerous problems in theoretical physics. The Witt algebra models the differential operators on the unit circle and the Virasoro algebra is its (unique) central extension. It plays an important role in the study of Lie algebras with applications to theoretical physics. Of particular interest are the unitary representations of the Virasoro algebra. The unitary representations were classified by Friedan-Qui-Shenker, Kac- Wakimoto, and Goddard-Kent-Olive in the 1980s ([FQS84], [KW86], [GKO86]). In the 1991 paper [Ken91], the physicist Adrian Kent expanded upon the results by computing signatures of all representations admitting invariant Hermitian forms (the unitary representations are those for which the form is positive definite). It is desirable to physicists to use Kent's formulas to study non-unitary conformal field theories–however, in order to do so, the action of the modular group on signature characters must be understood.

1.3 Methodology

In order to compute our signature characters, we will be computing the change to the signature as we cross a single reducibility curve. We start from a unitary region where the signatures are known (possibly as a limiting case) and compute the signature of irreducible Verma modules as a sum of all of the changes over all of the reducibility curves crossed in other regions. We will then use these signatures to compute the signatures of irreducible highest weight modules corresponding to reducible Verma modules.

1.4 Outline

- In chapters 2-5, we discuss the background mathematics required for this thesis.
- In chapter 6, we discuss the invariant Hermitian form attached to a Verma module and define the reducibility curves that bound the irreducible regions of the c, h plane.
- In chapter 7, we discuss the submodule structure of Verma modules and also discuss the Jantzen filtration. We note how the Janzten filtration can be used to determine how the signature changes as we cross a reducibility curve.
- In chapter 8, we break down the c, h plane into specific regions and give signature formulas for irreducible Verma modules for each region. However, in order to fully define the formulas, we will need to compute some unknowns, denoted by ε, which will be computed in chapter 9.
- In chapter 9, we explicitly compute ε for $c > 1$ and conjecture ε for $c < 1$. We will detail how to compute ε in our final chapter.
- In chapter 10, we prove the equivalency of our formulas (for irreducible Verma modules) with those of Kent ([Ken91]).
- In chapter 11, we compute the signature character for the irreducible quotient for the reducible Verma modules by proving Kent's averaging formula.

- In chapter 12, we discuss partial results for the modular group action on signature characters.
- In chapter 13, we conclude by outlining the methodology to prove our conjectured formula for ε for $c < 1$. We discuss future work arising from this thesis.

Chapter 2

Introduction to Representation Theory

In this chapter, we review concepts in algebra and representation theory required for this thesis.

2.1 Groups

Groups are one of the most basic and fundamental algebraic structures.

Definition 2.1.1. *([DF04], p.16) A* ***group*** *is defined as a set $\mathcal{X}$ with a binary operation $\cdot$ (**multiplication**) satisfying $\forall a, b, c \in \mathcal{X}$:*

1. ***Closure:*** $a \cdot b \in \mathcal{X}$
2. ***Identity:*** $\exists\ 1 \in \mathcal{X}$ *such that* $a \cdot 1 = 1 \cdot a = a$
3. ***Inverse:*** $\exists\ a^{-1} \in \mathcal{X}$ *such that* $a^{-1} \cdot a = a \cdot a^{-1} = 1$
4. ***Associativity:*** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Example 2.1.2. *The unit circle $\mathbb{S}^1$ (in the complex plane) can be viewed as a group with the normal multiplication of complex numbers.*

Definition 2.1.3. *([DF04], p.16) Group multiplication is* ***commutative*** *if $a \cdot b = b \cdot a$. The elements are said to commute. The group may also be called an* ***abelian*** *group.*

In the previous example $\mathbb{S}^1$ was commutative. However, it should be noted that groups are not necessarily abelian.

Example 2.1.4. *An example of a non-abelian group which is frequently studied is $GL_n(\mathbb{C})$: the set of invertible $n \times n$ matrices with the usual matrix multiplication operation (or more generally composition of linear maps). The identity element of the group is the identity matrix.*

Groups are fundamental in the study of physics as interactions between particles can be modelled by the binary operation $\cdot$.

2.2 Topologies

Definition 2.2.1. *([Run05], p. 61) A **topology** on a set $\mathcal{X}$ consists of a set $\tau \subseteq \mathcal{P}(\mathcal{X})$ of subsets of $\mathcal{X}$ which satisfies the following three conditions:*

1. *$\mathcal{X}$ and $\emptyset$ are in τ.*
2. *$\forall A, B \in \tau, A \cap B \in \tau$.*
3. *$\forall \mathcal{A} = \{A_i : i \in I\} \subseteq \tau$, $\bigcup_{i \in I} A_i \in \tau$.*

*The elements of τ are the **open sets** of the topology.*

Example 2.2.2. *In the usual topology on the real line, the open sets are all possible unions of open intervals.*

Topologies act as generalized algebraic constructs that allow definitions of continuity for different structures. The elements of τ are the open sets allowing us to extend the definitions of continuity to maps between abstract sets; a map is continuous if the pre-image of any open set is also open. Generalizing the notion of continuity allows us to examine how continuity can be extended to different objects (in this case, extended to groups).

2.3 Lie Groups

Definition 2.3.1. *([War71], p. 81) A **Lie group** G is a group and a differentiable manifold that satisfies:*

1. *The binary operation* $\cdot : G \times G \to G$ *is smooth.*

2. *The inverse operation* $i : g \mapsto g^{-1}$ *is smooth.*

Because a Lie group is equipped with group operations and a topology arising from the manifold structure, it is ideal for describing interactions between particles (group operations). Lie groups, however, do not take advantage of much of the mathematics of algebras and so we look towards representations to study Lie groups.

2.4 Group Actions and Representations

Definition 2.4.1. *([DF04], p.112) A* ***(left) group action*** *(.) of a group G acting on a set X consists of maps for each $g \in G$:*

$$\begin{aligned} g : X &\to X \\ x &\mapsto g.x \end{aligned}$$

satisfying: $\forall\, g, h \in G, x \in X$

1. $g.(h.x) = (gh).x$

2. $1.x = x$.

Example 2.4.2. *The element g of the group $\mathbb{S}^1$ acts on $z \in \mathbb{C}$ by $g.z = gz$. In other words, the unit circle acts on the complex plane by rotation ($e^{i\theta}$ rotates by θ counterclockwise).*

Definition 2.4.3. *([DF04], p.840,843) A* ***representation of a group*** *is a group action on a vector space V (over a field $\mathbb{F}$) preserving linearity: i.e.*

$$g.(av + bw) = a(g.v) + b(g.w) \qquad \forall g \in G; v, w \in V; a, b \in \mathbb{F}.$$

In other words, one has a map from G to $GL(V) = \{$invertible linear maps from $V \to V\}$ preserving group structure. A representation can therefore be defined as a group homomorphism π from G to $GL(V)$. That is, $\exists\, \pi(g)$ in $GL(V)$ such that: $g.v = \pi(g)v \,\forall\, v \in V$ satisfying for all $g, h \in G$:

1. $\pi(gh) = \pi(g)\pi(h)$

2. $\pi(1) = Id$

3. $\pi(g^{-1}) = \pi(g)^{-1}$.

For a Lie group representation, the group homomorphism π must also be continuous.

2.5 Algebras

Definition 2.5.1. *([DF04], p.342) An **algebra** is defined to be a pair $(A, *)$, where A is a vector space over $\mathbb{F}$ and $* : A \times A \to A$ is bilinear:*

$$\begin{aligned} a * (b + c) &= a * b + a * c \\ (a + b) * c &= a * c + b * c \\ (fa) * b &= f(a * b) = a * (fb) \end{aligned}$$

*for all $a, b, c \in A$ and $f \in \mathbb{F}$. If $a * (b * c) = (a * b) * c$ for all $a, b, c \in A$, then the algebra and its multiplication are said to be **associative**.*

Example 2.5.2. *([EW06], p.6) $End(V)$, which is defined to be the set of linear maps $V \to V$ along with the composition operation as multiplication, is an associative algebra.*

Definition 2.5.3. *([EW06], p.173) Given an associative algebra $(A, *)$, a **representation on vector space** V is a homomorphism of associative algebras. That is,*

$$\begin{aligned} \pi : A &\to End(V) \text{ is a linear map satisfying} \\ \pi(a * b) &= \pi(a) \circ \pi(b). \end{aligned}$$

Definition 2.5.4. *Given an algebra $(A, *)$, a subspace $I \subset A$ is said to be a **left (respectively right) ideal** if $A * I \subset I$ (respectively $I * A \subset I$). If I is both a left and a right ideal, then it is said to be a **two-sided ideal**.*

2.6 Lie Algebras

Definition 2.6.1. *([Hum78], p.1) A **Lie algebra** is an algebra $(L, [\cdot, \cdot])$ that in addition to the usual conditions also satisfies:*

1. ***Anticommutativity:*** $[a, b] = -[b, a]$

2. ***Jacobi Identity:*** $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$

*for all $a, b, c \in L$. The operation $[\cdot, \cdot]$ is called the **Lie bracket**.*

Definition 2.6.2. *([Hum78], p.1) A **Lie subalgebra** of a Lie algebra $(L, [\cdot, \cdot])$ is a subspace $K \subset L$ such that $[K, K] \subset K$.*

Example 2.6.3. *An important example of a Lie algebra, particularly for finite dimensional Lie algebras, is $sl_2(\mathbb{C})$: the trace zero 2×2 matrices. The Lie bracket for the algebra is:*

$$[A, B] = AB - BA.$$

Example 2.6.4. *([Ken91]) The **Virasoro algebra** Vir is an infinite dimensional Lie algebra over $\mathbb{C}$ with basis $\{L_m\}_{m \in \mathbb{Z}} \cup \{z\}$ with the following defining relations:*

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \delta_{n,-m}\frac{n(n^2-1)}{12}z \\ [L_n, z] &= 0 \end{aligned}$$

for n,m$\in \mathbb{Z}$ and where δ is the Kronecker delta function.

Example 2.6.5. *([Jac79], p.6) One can construct a Lie algebra form any associative algebra has an associated Lie algebra by simply replacing the multiplication operation $*$ with a bracket operation: $[A, B] = A * B - B * A$. A commonly used Lie algebra is $\mathfrak{gl}(V)$ which arises from the associative algebra $End(V)$ so that $[A, B] = AB - BA$ for all $A, B \in \mathfrak{gl}(V)$. ([Hum78], p.2)*

Definition 2.6.6. *([Hum78], p.6) Given a Lie algebra $(L, [\cdot, \cdot])$, the Lie algebra (or the Lie bracket) is said to be **commutative** or **abelian** if $[X, Y] = 0$ for all $X, Y \in L$.*

Example 2.6.7. *By anticommutativity, any one-dimensional Lie algebra is abelian.*

Definition 2.6.8. *([Hum78], p.6) An **ideal** I of a Lie algebra L is a subspace of L that satisfies: $[x, y] \in I\ \forall x \in I, y \in L$.*

Definition 2.6.9. *([Hum78], p.6) A non-abelian Lie algebra whose only ideals are 0 and itself is a **simple** Lie algebra.*

Definition 2.6.10. *([Hum78], p.10) The **derived series** of L is the series of ideals $L^{(n)}$ where $L^{(0)} = L$ and $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$ for $i \in \mathbb{Z}^+$. L is **solvable** if for some n, $L^{(n)} = 0$.*

Definition 2.6.11. *([Hum78], p.11) The **radical** of L is the maximal solvable ideal of L. L is **semisimple** if its radical is 0.*

Lie algebras and Lie groups are related–for example the tangent space at the identity of any Lie group is a Lie algebra and may be thought of as a vector space approximation of the Lie group at the identity ([War71], p.86)–but there are many advantages of using Lie algebras. Algebras are vector spaces, and so the many tools of linear algebra are available to us.

2.7 Lie Algebra Representations

Definition 2.7.1. *([Hum78], p.8) Similar to a Lie group, a Lie algebra can also act on a vector space V. A **representation of a Lie algebra** L is a Lie algebra homomorphism from L to $\mathfrak{gl}(V)$.*

A Lie algebra homomorphism is a linear map $\pi\colon L \to \mathfrak{gl}(V)$ that satisfies:

$$\pi([x, y]) = [\pi(x), \pi(y)].$$

We may use action notation and write $X.v = \pi(X)(v)$ for $X \in L, v \in V$.

*A representation of L is also called an L-**module**.*

Remark 2.7.2. *Although $[x, y]$ may not be as simple as $xy - yx$, working within a representation allows us to use a simple formula for the Lie bracket: $[\pi(x), \pi(y)] = \pi(x)\pi(y) - \pi(y)\pi(x) \in \mathfrak{gl}(V)$.*

Example 2.7.3. *([Hum78], p.8) It can be shown using the Jacobi identity that given a Lie algebra L, the map $ad : L \to \mathfrak{gl}(L)$ where*

$$ad_X(Y) = [X, Y] \qquad \text{for every } X, Y \in L$$

*is a representation of L on the vector space $V = L$. It is called the **adjoint** representation of L.*

Example 2.7.4. *Given a Lie algebra L over $\mathbb{C}$, the trivial representation of L on $V = \mathbb{C}$ sends every element of L to $0 \in \mathfrak{gl}(\mathbb{C})$. Observe that the Lie bracket of any two elements of $\mathfrak{gl}(\mathbb{C})$ is zero by anticommutativity.*

Example 2.7.5. *The natural representation π of $\mathfrak{sl}_2(\mathbb{C})$ on the vector space $\mathbb{C}^2$ is matrix multiplication:*

$$\pi(A)v = Av \qquad \text{for all } A \in \mathfrak{sl}_2(\mathbb{C}), v \in \mathbb{C}^2.$$

Definition 2.7.6. *([Jac79], p.19) A **submodule** W is a subspace of an L-module (π, V) that is closed under π; that is: $\pi(x)(W) \subset W$ for all $x \in L$. (Note: submodules are also called **subrepresentations**.) Another way to state this is to say that W is closed under the action of L, i.e. $x.W \subset W \, \forall x \in L$.*

Example 2.7.7. *Given a representation V of the Lie algebra L, $\{0\} \subset V$ is the trivial submodule. V is also a submodule of itself.*

Definition 2.7.8. *([Hum78], p.25) A module is **irreducible** (or **simple**) if its only two submodules are the trivial submodule and itself.*

2.8 Constructing Representations

Definition 2.8.1. *([Jac79], p.19) Given two L-representations (π_1, V_1) and (π_2, V_2), the **direct sum** is also a representation:*

$$\pi := \pi_1 \oplus \pi_2 : L \to \mathfrak{gl}(V_1 \oplus V_2)$$

where $\pi(X)(v_1, v_2) = (\pi_1(X)v_1, \pi_2(X)v_2)$.

Definition 2.8.2. *([Hum78], p.26) Given two vector spaces U and V over the field $\mathbb{F}$, the **tensor product** $U \otimes V$ is the vector space generated by terms of the form $u \otimes v$ for every $u \in U, v \in V$. The tensor product $\otimes : U \times V \to U \otimes V$ is bilinear:*

1. *$(au) \otimes v = u \otimes (av) = a(u \otimes v)$ for all $a \in \mathbb{F}$, $u \in U$, $v \in V$;*
2. *$(u_1 + u_2) \otimes v = u_1 \otimes v + u_2 \otimes v$ for all $u_1, u_2 \in U$, $v \in V$;*
3. *$u \otimes (v_1 + v_2) = u \otimes v_1 + u \otimes v_2$ for all $u \in U$, $v_1, v_2 \in V$.*

Definition 2.8.3. *([Hum78], p.26) Given two representations* (π_1, V_1) *and* (π_2, V_2) *of the Lie algebra* L*, the **tensor product*** $V_1 \otimes V_2$ *is also a representation of* L*:*

$$\begin{aligned} \pi_1 \otimes \pi_2(X)(v_1 \otimes v_2) &= (\pi_1(X)(v_1)) \otimes v_2 + v_1 \otimes (\pi_2(X)(v_2)) \qquad \textit{for all } X \in L, v_i \in V_i \\ \textit{i.e. } X.(v_1 \otimes v_2) &= (X.v_1) \otimes v_2 + v_1 \otimes (X.v_2) \end{aligned}$$

Definition 2.8.4. *([EW06], p.57) Given an* L*-representation* π *on the vector space* V *containing a subrepresentation* U*, the **quotient*** $V/U := \{v + U : v \in V\}$ *has the natural structure of a representation* $\bar{\pi}$*:*

$$\bar{\pi}(X)(v + U) = \pi(X)(v) + U.$$

This is well-defined since U *is a submodule.*

Remark 2.8.5. *Note that there may not be a submodule* W *so that* $V = U \oplus W$ *as* L*-representations. We will see examples of this when we study Verma modules.*

2.9 Jordan-Hölder Series

Because representations may not be expressed necessarily as direct sums of irreducible subrepresentations, decomposing a representation into irreducible constituents requires the consideration of quotients.

Definition 2.9.1. *([Jac79], p.48) Given a Lie algebra* L *and an* L*-representation* V*, a **Jordan-Hölder series** (or **composition series**) for the representation* V *is a series of submodules:*

$$V = V_1 \supset V_2 \supset \cdots \supset V_n = \{0\}$$

such that V_i/V_{i+1} *is irreducible for* $i = 1, 2, \ldots, n-1$*. The* V_i/V_{i+1} *are called the **composition factors** of* V*.*

Our first method of understanding the structure of a representation is then to understand:

1. What irreducible representations appear as composition factors?

2. How many times does each irreducible representation occur?

We will study this for Verma modules over the Virasoro algebra in Chapter 5.

Chapter 3

Weight Theory

Weight theory is another means of understanding the structure of a Lie algebra representation. In linear algebra, for a square matrix representing a linear transformation from a vector space to itself, we decompose the vector space into eigenspaces where the behaviour of the linear transformation is easy to understand: it is simply scalar multiplication by the corresponding eigenvalue. Weight theory is the application and generalization of this theory to Lie algebras and their representations.

Definition 3.0.2. *([Hum78], p.11,80) A* ***Cartan subalgebra*** *H of a Lie algebra L is a Lie subalgebra such that:*

1. *the* ***normalizer*** *of H is H: $\{X : [X, Y] \in H \ \forall \ Y \in H\} = H$*

2. *H is* ***nilpotent****: defining $H^1 = [H, H]$ and $H^i = [H, H^{i-1}]$ for $i \geq 2$, H^n must be zero for some $n \in \mathbb{Z}^+$.*

In many situations, Cartan subalgebras are maximal commutative subalgebras (eg. for simple Lie algebras over fields of characteristic 0 [Hum78] p. 80, the Virasoro algebra).

Example 3.0.3. *Consider the Virasoro algebra. Let $H = span\{L_0, z\}$. Then H is a Cartan subalgebra of Vir.*

Proof. We note that H is nilpotent since $[L_0, z] = 0$ whence $[H, H] = 0$.

Recall that we had the basis $\{L_n\}_{n\in\mathbb{Z}} \cup \{z\}$ of Vir. From the commutation relations

$$\begin{aligned} [L_0, L_n] &= -nL_n \\ [z, L_n] &= 0 \end{aligned}$$

we note that the basis vectors are eigenvectors for ad_z and ad_{L_0} where L_n has eigenvalue $-n$ for ad_{L_0} and eigenvalue 0 for ad_z.

Assume Z is in the normalizer of H and let $Z = \sum_{n\in\mathbb{Z}} a_n L_n + bz$ for some scalars $a_n, b \in \mathbb{C}$, finitely many of which are non-zero. Then $[L_0, Z] = \sum_{n\in\mathbb{Z}} -na_n L_n \in H$. Therefore, $na_n = 0$ for all $n \neq 0$. Thus, $Z = a_0 L_0 + bz$. Therefore $Z \in H$ so the normalizer of H is H.

Observe that H is commutative since z is central, i.e. commutes with all of Vir. $\square$

Example 3.0.4. *Let* $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, *and let* $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *The set* $\{x, y, h\}$ *is a basis of* $\mathfrak{sl}_2(\mathbb{C})$ *and is called the* ***standard triple****.([CM93], p.35) Observe that:*

$$\begin{aligned} [h, x] &= 2x \\ [h, h] &= 0 \\ [h, y] &= -2y. \end{aligned}$$

Therefore, viewing x, y, h *as eigenvectors for* ad_h *as in the previous example,* $H := \mathbb{C}h$ *is self-normalizing and nilpotent and thus is a Cartan subalgebra of* $\mathfrak{sl}_2(\mathbb{C})$.

Observe in our previous two examples that the Cartan subalgebras are commutative and the bases that we have selected are eigenvectors for ad_h for all $h \in H$. Given any square matrices A and B, if A and B commute then A and B preserve each other's eigenspaces. This can be seen by looking any eigenvector x (of A). Let $Ax = ax$ for some scalar a. Then $A(Bx) = (AB)x = (BA)x = B(Ax) = B(ax) = a(Bx)$. Therefore, Bx is also in the a-eigenspace of A. Similarly A also preserves B's eigenspaces.

Definition 3.0.5. *([Hum78], p.35) A subalgebra* H *of a Lie algebra* L *is said to be a* ***maximal toral subalgebra*** *if it is maximal with respect to the property that the elements act diagonalizably under the adjoint action. Maximal toral subalgebras are abelian ([Hum78], p. 35).*

Since Cartan subalgebras are often maximal toral subalgebras and since commuting operators preserve each other's eigenspaces, we like to study joint eigenspace decompositions of the Cartan subalgebra. Given a representation (π, V) of the Lie algebra L and a Cartan subalgebra H of L, consider

$$V_\mu := \{v \in V \mid \exists\, \mu(h) : H \to \mathbb{F} \text{ such that } \pi(h)(v) = \mu(h)v \ \ \forall\, h \in H\}$$

where $\mu : H \to \mathbb{F}$ is the function which, at h, is equal to the eigenvalue of $\pi(h)$ for the eigenvector v. For this joint eigenspace, the linear map $\pi(h) : V \to V$ has eigenvalue $\mu(h)$ on V_μ. Since $\pi(h_1 + h_2) = \pi(h_1) + \pi(h_2)$, therefore $\mu(h_1 + h_2) = \mu(h_1) + \mu(h_2)$ for all $h_1, h_2 \in H$. Similarly, $\mu(ah) = a\mu(h)$ for all $a \in \mathbb{F}$, $h \in H$. Therefore $\mu : H \to \mathbb{F}$ is a linear map. In other words, $\mu \in H^*$, the dual space of H. This motivates the definition:

Definition 3.0.6. *([Hum78], p.107) Let L be a Lie algebra and H a maximal toral subalgebra of H. For $\mu \in H^*$, the vector space*

$$V_\mu := \{v \in V : h \cdot v = \mu(h)v \ \forall h \in H\}$$

is called the μ ***weight space of*** *V. If $V_\mu \neq \{0\}$, μ is called a* ***weight*** *of V. When equality holds,*

$$V = \bigoplus_{\mu \in H^*} V_\mu$$

is the ***weight space decomposition*** *of V.*

A particular example of immense importance is the case when the representation is the adjoint representation of L.

Definition 3.0.7. *([Hum78], p.35) Let L be a Lie algebra and H a maximal toral subalgebra. A* ***root*** *is a non-zero linear map $\alpha : H \to \mathbb{F}$ such that $L_\alpha \neq 0$. (Note that $L_0 = H$ since H is maximal abelian.) The set of roots with respect to H is denoted by $\Delta(L, H)$. The* ***root space decomposition*** *of L is*

$$L = H \oplus \bigoplus_{\alpha \in \Delta(L,H)} \mathbb{F}L_\alpha.$$

The elements of each L_α are called ***root vectors****.*

Note that weight space decompositions and root space decompositions are analogous to eigenspace decompositions.

Proposition 3.0.8. *([Hum78], p.107) Let L be a Lie algebra, (π, V) a representation of L, and μ be a weight of V for the Cartan subalgebra H. Let $\alpha \in \Delta(L, H)$. Observe that $\mu + \alpha \in H^*$. For every $X_\alpha \in L_\alpha$ and $v \in V_\mu$, $X_\alpha.v \in V_{\mu+\alpha}$.*

Proof. Let $h \in H$. Since $[\pi(h), \pi(X_\alpha)] = \pi(h)\pi(X_\alpha) - \pi(X_\alpha)\pi(h)$, we have $\pi(h)\pi(X_\alpha) = [\pi(h), \pi(X_\alpha)] + \pi(X_\alpha)\pi(h)$. Therefore,

$$\begin{aligned}
\pi(h)(\pi(X_\alpha)v) &= [\pi(h), \pi(X_\alpha)]v + \pi(X_\alpha)(\pi(h)v) \\
&= \pi([h, X_\alpha])v + \pi(X_\alpha)(\mu(h)v) \\
&= \pi(\alpha(h)X_\alpha)v + \mu(h)\pi(X_\alpha)v \\
&= \alpha(h)\pi(X_\alpha)v + \mu(h)\pi(X_\alpha)v \\
&= (\alpha(h) + \mu(h))\pi(X_\alpha)v \\
&= (\alpha + \mu)(h)\pi(X_\alpha)v
\end{aligned}$$

Therefore, $h.(X_\alpha.v) = (\alpha + \mu)(h)(X_\alpha.v)$ for all $h \in H$, so $X_\alpha.v$ has weight $\alpha + \mu$. □

Thus, we see as mentioned, weight theory provides us with our second means of understanding the structure of a module. The proposition above shows us how we may apply root vectors to move between weight spaces. To better understand the structure, we ask: given a representation, what is the dimension of each weight space? We can record the information as a formal sum called the character.

Definition 3.0.9. *([Hum78], p.124) Let V be a representation of the Lie algebra L admitting a weight space decomposition with respect to the Cartan subalgebra H. If each weight space is finite-dimensional, then the* ***(formal) character*** *of V is the formal sum*

$$charV = \sum_{\mu \in H^*} \dim(V_\mu)e^\mu.$$

Chapter 4

The Universal Enveloping Algebra

In this chapter, we discuss the universal enveloping algebra. It is useful for discussing repeated Lie algebra actions and it will be used later in the definition of Verma modules. The universal enveloping algebra $\mathcal{U}(L)$ of a Lie algebra L has the useful property that representations of the associative algebra $\mathcal{U}(L)$ are in one to one correspondence with Lie algebra representations of L. Elements of the universal enveloping algebra are in bijection with polynomials in L; however, multiplication in $\mathcal{U}(L)$ and in the polynomial algebra may be different.

4.1 Building a Universal Enveloping Algebra

Definition 4.1.1. *([Hum78], p.89) Given a vector space V over $\mathbb{C}$ (could be over any field) we can build the **tensor algebra** by constructing "polynomials":*

$$\begin{aligned} T^0V &= \mathbb{C} \\ T^1V &= V \\ T^2V &= V \otimes V \\ &\vdots \\ T^nV &= \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}} \\ &\vdots \end{aligned}$$

*We define the **tensor algebra** to be the direct sum of all T^nV as a set:*

$$T^{\cdot}V = \bigoplus_{n=0} T^nV.$$

Now $T^{\cdot}V$ has the standard addition operation as well as $\otimes$ for multiplication which is easily seen to be associative so that, for example, $(a \otimes b) \otimes (c \otimes d \otimes e) = a \otimes b \otimes c \otimes d \otimes e$. Thus $(T^{\cdot}V, \otimes)$ is an algebra. We can see that this multiplication even increases the degree as expected: $T^iV \otimes T^jV = T^{i+j}V$.

Definition 4.1.2. *([Hum78], p.91) Construct the tensor algebra $T^{\cdot}V$ in the case where V is already an algebra–specifically, $V = L$ is a Lie algebra (and therefore has its own multiplication $[\cdot, \cdot]$). In this case, generate a two sided ideal I by the elements of the form $\{x \otimes y - y \otimes x - [x, y]\}$ for all $x, y \in L$.*

*We then define the **universal enveloping algebra** of L as $\mathcal{U}(L) := T^{\cdot}L/I$. Multiplication in $\mathcal{U}(L)$ descends from $\otimes$ on $T^{\cdot}(L)$ to $\mathcal{U}(L)$.*

We use the notation $x \otimes y$, $x \otimes y + I$, xy interchangeably.

It should be noted that whether or not L was associative, $\mathcal{U}(L)$ is associative since the tensor product is associative.

Proposition 4.1.3. *([EW06],p.173) If V is an L-representation, then V is also a $\mathcal{U}(L)$-representation and vice versa.*

Proof. Suppose V is an L-representation.

Let $\pi : L \to \mathfrak{gl}(V)$. Since π is a Lie algebra homomorphism, we have $\pi([x, y]) = [\pi(x), \pi(y)]$ for all $x, y \in L$.

We wish to show that V is a $\mathcal{U}(L)$-representation. Let us take the map:

$$\begin{aligned} \Pi : \mathcal{U}(L) &\to \mathrm{End}(V) \\ x_1 \otimes x_2 \otimes \cdots \otimes x_n &\mapsto \pi(x_1)\pi(x_2)\cdots\pi(x_n) \end{aligned}$$

since for Π to be a homomorphism of associative algebras, it has to respect multiplication.

In order for this map to be well-defined, $\Pi(x \otimes y) - \Pi(y \otimes x) = \Pi([x, y])$ since $\mathcal{U}(L) =$

$T^{\cdot}(L)/I$. However,

$$\begin{aligned}\Pi(x \otimes y) - \Pi(y \otimes x) &= \Pi([x, y]) \\ \Leftrightarrow \pi(x)\pi(y) - \pi(y)\pi(x) &= \pi([x, y]) \\ \Leftrightarrow [\pi(x), \pi(y)] &= \pi([x, y])\end{aligned}$$

and the last statement is true since π is a Lie algebra homomorphism. Therefore, (Π, V) is a $\mathcal{U}(L)$-representation.

Conversely, let (Π, V) be a $\mathcal{U}(L)$-representation. Then $\Pi : \mathcal{U}(L) \to \text{End}(V)$ is an algebra homomorphism, and thus

$$\Pi(x \otimes y) = \Pi(x) \circ \Pi(y). \tag{4.1}$$

Since Π is well-defined with respect to the quotient by I, we have:

$$\Pi(x \otimes y - y \otimes x) = \Pi([x, y]). \tag{4.2}$$

To define an L-representation (π, V) from (Π, V), let us restrict Π from $\mathcal{U}(L)$ to $T^1(L) = L)$

$$\begin{aligned}\pi : L &\to \mathfrak{gl}(V) \\ x &\mapsto \Pi(x).\end{aligned}$$

In order for (π, V) to be an L-representation, it must satisfy $[\pi(x), \pi(y)] = \pi([x, y])$ for all $x, y \in L$. Now,

$$\begin{aligned}\Pi(x \otimes y) - \Pi(y \otimes x) &= \Pi([x, y]) && \text{by (4.2)} \\ \Leftrightarrow \Pi(x)\Pi(y) - \Pi(y)\Pi(x) &= \pi([x, y]) && \text{by (4.1)} \\ \Leftrightarrow \pi(x)\pi(y) - \pi(y)\pi(x) &= \pi([x, y]) \\ \Leftrightarrow [\pi(x), \pi(y)] &= \pi([x, y])\end{aligned}$$

Therefore, (π, V) is an L-representation. $\square$

4.2 The Poincaré - Birkhoff - Witt Theorem

If L is a Lie algebra over the field $\mathbb{C}$, there is a canonical linear map $h : L \to \mathcal{U}(L)$. We can use this map to extend a (totally ordered) basis of L to a (totally ordered) basis of $\mathcal{U}(L)$.

Theorem 4.2.1. *The Poincaré - Birkhoff - Witt Theorem ([Hum78] p. 92).*

If L has basis $\{x_i : i \in I\}$ then $\mathcal{U}(L)$ has a basis

$$\{x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_n}^{a_n} \,|\, a_1, \ldots, a_n, n \in \mathbb{Z}^+, i_1 < i_2 < \cdots < i_n\} \cup \{1\}.$$

The PBW Theorem allows us to view elements of the universal enveloping algebra as polynomials in L. The ordering that the PBW-Theorem provides is natural: first by degree then by ordering in the basis.

Example 4.2.2. *If L is n dimensional and abelian, $L = Span\{x_1, x_2, \ldots, x_n\}$. Since L is abelian, $[x, y] = 0$ so taking the quotient by the ideal generated by $x \otimes y - y \otimes x - [x, y]$ is equivalent to taking the quotient by the ideal generated by $x \otimes y - y \otimes x$. Therefore, $x \otimes y = y \otimes x$ and $\otimes$ is commutative in $\mathcal{U}(L)$ and so $\mathcal{U}(L) \cong \mathbb{C}[x_1, x_2, ..., x_n]$ the algebra of polynomials in the variables $x_1, \ldots, x_n$.*

Example 4.2.3. *If L is Vir then $\mathcal{U}(L) = Span\{L_{i_1} L_{i_2} ... L_{i_n} | i_j \leq i_k \text{ } if \text{ } j < k\}$.*

Chapter 5

Verma Modules

In this chapter, we introduce Verma modules which are the main focus of this thesis. They arise naturally from studying weight theory and are the basic building blocks for important categories of modules.

5.1 Positive and Negative Roots

Recall that

$$L = H \oplus \bigoplus_{\alpha \in \Delta(L,H)} L_\alpha.$$

A well known result for semisimple Lie algebras is that if α is a root of L, $-\alpha$ is also a root ([Hum78] p.37). Now, Vir is not semisimple, however, the Witt algebra is simple (and thus semisimple). We conclude that the result holds for the Witt algebra. Since Vir is the unique central extension of the Witt algebra by z and $z \in H$, our choice of Cartan subalgebra for Vir, $\alpha(z) = 0$ for every root $\alpha \in \Delta(Vir, H)$. Therefore, just as for the Witt algebra, if α is a root, $-\alpha$ is also a root.

Proposition 5.1.1. *([Hum78], p.48) If for the Lie algebra L and the Cartan subalgebra H,*

1. *if α is a root, then $-\alpha$ is a root;*

2. *there exists a fixed $h_0 \in H$ so that for every $\alpha \in \Delta(L, H)$, $\alpha(h_0) \neq 0$;*

then we can divide $\Delta(L, H)$ into positive and negative roots as follows:

$$\begin{aligned}\Delta^+(L,H) &:= \{\alpha : \alpha(h_0) > 0\} \\ \Delta^-(L,H) &:= \{\alpha : \alpha(h_0) < 0\}.\end{aligned}$$

Observe that $\Delta^-(L, H) = -\Delta^+(L, H)$.

Example 5.1.2. *Consider the Virasoro algebra where we selected $H = span\{L_0, z\}$. The roots are $\alpha_n(aL_0 + bz) := -an$ for $n \in \mathbb{Z}$. Fix $h_0 = L_0$. Then the corresponding positive and negative roots are:*

$$\Delta^+(Vir, H) = \{\alpha_n : n \in \mathbb{Z}^-\} \qquad \Delta^-(Vir, H) = \{\alpha_n : n \in \mathbb{Z}^+\}.$$

Definition 5.1.3. *([Hum78] p. 47) Given two elements $\lambda, \mu \in H^*$, we say that μ is less than λ, written $\mu \prec \lambda$, if and only if $\lambda - \mu$ is a sum of positive roots.*

This assigns a partial ordering to H^*.

5.2 Verma Modules M(λ)

Definition 5.2.1. *Define*

- $\boldsymbol{N} = span\{x_i | x_i is\ a\ root\ vector\ for\ a\ positive\ root\}$
- $\mathbf{N}^- = span\{x_i | x_i is\ a\ root\ vector\ for\ a\ negative\ root\}$
- $\boldsymbol{B} = H \oplus N$.

Proposition 5.2.2. *([Hum78], p.110) Given $\lambda \in H^*$, we can define a one-dimensional representation $(\pi_\lambda, \mathbb{C}_\lambda)$ of B with basis vector v_λ satisfying:*

1. $h.v_\lambda = \lambda(h)v_\lambda$
2. $n.v_\lambda = 0$

for all $h \in H$ and $n \in N$.

Proof. This is a B-representation as can be seen by observing that for all $h_1, h_2 \in H$ and $n_1, n_2 \in N$:

1. It is linear:

$$((h_1+n_1)+(h_2+n_2)).v_\lambda = \lambda(h_1+h_2)v_\lambda = \lambda(h_1)v_\lambda+\lambda(h_2)v_\lambda = (h_1+n_1).v_\lambda+(h_2+n_2).v_\lambda$$

2. Since

   (a) $[h_1, h_2] = 0$

   (b) $[h_1, x_\alpha] = \alpha(h_1)x_\alpha$ for all $x_\alpha \in L_\alpha$ so that $[h_1, n_1] \in N$ and

   (c) $[n_1, n_2] \in N$

   therefore $\pi_\lambda([h_1 + n_1, h_2 + n_2]) = \pi_\lambda(n) = 0$ for $n = [h_1 + n_1, h_2 + n_2] \in N$. Since $[\pi_\lambda(h_1 + n_1), \pi_\lambda(h_2 + n_2)] = 0$ since $\mathbb{C}_\lambda$ is one-dimensional, we see that π_λ is indeed a Lie algebra homomorphism from B to $\mathfrak{gl}(\mathbb{C}_\lambda)$.

□

We can take a B-representation and create an L-representation as follows.

Definition 5.2.3. *Given a B-module X,*

$$\mathcal{U}(L)\bigotimes_{\mathcal{U}(B)}X$$

has the structure of an L-representation.

In more detail, our previous tensor products allowed for scalars to be transferred over the tensor product. Implicitly, $U \otimes V$ meant $U \otimes_{\mathbb{C}} V$. However, in this case, we would like the B-action to commute across the tensor product.

$\mathcal{U}(L)$ may be viewed as a right B-module as follows (our previous definitions were left modules). By the PBW theorem, all of $\mathcal{U}(L), \mathcal{U}(B)$ and $\mathcal{U}(N^-)$ can be viewed as polynomials in L, B and N^-, respectively. We can choose the ordering of the basis of L appropriately so that N^- variables appear on the left and the B variables appear on the right. Then we can express

$$\mathcal{U}(L) = \mathcal{U}(N^-) \otimes \mathcal{U}(B)$$

which is a right B-module: $x.b = x \otimes b$ for $x \in \mathcal{U}(L), b \in B$. This is a Lie algebra action as $x.[b_1, b_2] = x.b_1.b_2 - x.b_2.b_1$ due to the quotient by the two-sided ideal I (of $\mathcal{U}(L)$).

We have a right B module structure on $\mathcal{U}(L)$ and a left B-module structure on X. Therefore, we also have $\mathcal{U}(B)$-module structures on both $\mathcal{U}$ and X. Now for a left $\mathcal{U}(B)$-module X, if we define $\mathcal{U}(L) \bigotimes\limits_{\mathcal{U}(B)} X$ by

$$(u.b) \otimes_{\mathcal{U}(B)} x = u \otimes_{\mathcal{U}(B)} (b.x) \qquad \text{for } u \in \mathcal{U}(L), x \in X, b \in \mathcal{U}(B),$$

then we have formed a $\mathcal{U}(L)$-module. For $u \in \mathcal{U}(L)$, $v \otimes x \in \mathcal{U}(L) \otimes_{\mathcal{U}B} X$, $u.(v \otimes x) = (uv) \otimes x$.

Definition 5.2.4. *([Dix96], p.232) The **Verma module** of **highest weight** λ is defined as*

$$M(\lambda) := \mathcal{U}(L) \bigotimes_{\mathcal{U}(B)} \mathbb{C}_\lambda.$$

What this gives us is a $\mathcal{U}(L)$ module that is constructed on the highest weight vector v_λ. Note that λ is the highest weight of $M(\lambda)$ since by the PBW Theorem and the tensor product over $\mathcal{U}(B)$, we may view $M(\lambda) = \mathcal{U}(N^-) \otimes \mathbb{C}_\lambda$ as vector spaces. Viewing elements of $\mathcal{U}(N^-)$ as polynomials in a basis of root vectors for N^- (the negative roots), we note that all weights of $M(\lambda)$ are of the form $\lambda - \mu$ where μ is a (possibly empty) sum of positive roots, whence we call λ the highest weight. Applying a positive root vector $X_\alpha \in L_\alpha$ to v_λ gives us a weight vector of weight $\lambda + \alpha$. This can only happen if $X_\alpha.v_\lambda = 0$. Thus applying a positive root vector will kill v_λ (gives 0). All elements of the Verma module belong to $\mathcal{U}(N^-).v_\lambda$.([Hum78], p.110)

Example 5.2.5. *Recall that for $\mathfrak{sl}_2$ we had the standard triple $\{x, y, h\}$ and we chose the Cartan subalgebra $\mathbb{C}h$. Recall that x and y were eigenvectors for the action of ad_h. Let α be the root corresponding to x. Since $[h, x] = \alpha(h)x = 2x$, we must have $\alpha(h) = 2$. Observe that y is a root vector for the root $-\alpha$. Choosing positive and negative roots relative to h, we see that $\Delta^+(\mathfrak{sl}_2(\mathbb{C}), H) = \{\alpha\}$ and $\Delta^-(\mathfrak{sl}_2(\mathbb{C}), H) = \{-\alpha\}$. We set $B = span\{h, x\}$, $N^- = span\{y\}$. Then a basis for $M(\lambda) = \mathcal{U}(\mathfrak{sl}_2(\mathbb{C})) \otimes_{\mathcal{U}(B)} \mathbb{C}_\lambda$ is $\{y^k.v_\lambda : k \in \mathbb{Z}^{\geq 0}\}$.*

To better understand how $\mathcal{U}(L)$ acts on $M(\lambda)$, let's compute $x.(y^2.v_\lambda)$. We have [x,y]=h=xy-

yx, so xy=h+yx. Likewise, hy=yh-2y. Using these relations:

$$\begin{aligned} xyy.v_\lambda &= hy.v_\lambda + yxy.v_\lambda \\ &= yh.v_\lambda - 2y.v_\lambda + yh.v_\lambda + yyx.v_\lambda \\ &= \lambda(h)y.v_\lambda - 2y.v_\lambda + \lambda(h)y.v_\lambda + 0 \\ &= -2y.v_\lambda + 2\lambda(h)y.v_\lambda. \end{aligned}$$

Example 5.2.6. *For Vir, given a highest weight λ, we have the basis vectors $\{L_{i_1}L_{i_2}...L_{i_n}.v_\lambda | i_j \leq i_k < 0 \text{ if } j < k\}$ for $M(\lambda)$.*

5.3 Kostant's Partition Function

As we observed before, the PBW-theorem gives us a basis for a Verma module $M(\lambda) = \mathcal{U}(N^-).v_\lambda$:

Let $\{x_i\}_{i\in\mathcal{I}}$, where $\mathcal{I}$ indexes the negative roots, be a basis of root vectors for N^- where x_i is a root vector for the negative root α_i. Then

$$\{x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k} v_\lambda | \, a_1, \ldots, a_k, k \in \mathbb{Z}^+, i_1 < i_2 < \cdots < i_k \in \mathcal{I}\} \cup \{v_\lambda\}$$

is a basis of $M(\lambda)$.([Hum78], p.110)

We also know that the weight of the basis vector on the left side is simply $\lambda + \sum_{j=1}^{k} a_j \alpha_{i_j}$ since each time you act by x_{i_j}, the weight changes by α_{i_j}. Thus simple combinatorial counting determines the dimension of a given weight space: i.e. given a weight $\lambda+\mu$ of $M(\lambda)$, how many ways can $\mu = \sum_{j=1}^{k} a_j \alpha_{i_j}$ where the coefficients a_j are positive integers? This number is referred to as Kostant's partition function. Multiplying μ by -1 and, equivalently, counting the number of ways of expressing the result as a sum of positive roots, Kostant's partition function is defined by:

Definition 5.3.1. *([Hum78], p.136)* ***Kostant's partition function*** *is a map $P : \Lambda_r^+ \to \mathbb{Z}$ satisfying:*

$$P(\mu) := \#\{\{a_{i_1}, \ldots, a_{i_k}\} : a_{i_j} \in \mathbb{Z}^+, i_1 < i_2 < \cdots < i_k \in \mathcal{I}, a_{i_1}\alpha_{i_1} + \cdots + a_{i_n}\alpha_{i_n} = \mu\}.$$

Proposition 5.3.2. *([Hum78], p.136) Let* $\Lambda_r^+ := \mathbb{Z}^{\geq 0} - span\Delta^+(L,H)$. *Then*

$$chM(\lambda) = \sum_{\mu \in \Lambda_r^+} P(\mu) e^{\lambda - \mu}.$$

Example 5.3.3. *For* Vir, *recall from Example 5.1.2 that* $\Delta^-(Vir, H) = \{\alpha_n : n \in \mathbb{Z}^+\}$ *where* $\alpha_n(aL_0 + bz) = -an$. *Therefore* $\alpha_n = n\alpha_1$. *We specify weights* λ *using pairs* c, h: $\lambda(L_0) = h$, *the highest weight (abuse of notation since we only consider the eigenvalue of* L_0*), and* $\lambda(z) = c$ *the central charge. We write* $M(c,h)$ *in place of* $M(\lambda)$. *Then a basis of* $M(c,h)$ *is:*

$$\{L_{i_1}^{a_1} \cdots L_{i_k}^{a_k} v_{c,h} : 1 \leq i_1 < \cdots < i_k, k \in \mathbb{Z}^+, a_i \in \mathbb{Z}^+\} \cup \{v_{c,h}\}.$$

Observe that $L_{i_1}^{a_1} \cdots L_{i_k}^{a_k} v_{c,h}$ *has weight* $\lambda + a_1\alpha_{i_1} + \cdots + a_k\alpha_{i_k} = \lambda + (\sum_{j=1}^k a_j \cdot i_j)\alpha_1$. *We define the* n^{th} *level of the Verma module to be the* $\lambda + n\alpha_1$ *weight space (often we decompose by the eigenvalues of* L_0 *and ignore the action of* z *since it is central and abbreviate this to the* $h+n$ *weight space or simply consider the* n^{th} *weight space of* $\mathcal{U}(N^-)$*). From our formula, we observe that in fact* $P(n)$ *in this case is the (positive integer) partition function: ([DFMS97], p.204)*

$$\begin{aligned} P(n) &= \#\text{ ways of expressing } n \text{ as a sum of positive integers.} \\ &= P(n) = [t^n] \prod_{j=1}^{\infty} \frac{1}{(1-t^j)} \end{aligned}$$

The character of $M(c,h)$ *is thus:*

$$chM(c,h) = \sum_{n \in \mathbb{Z}^{\geq 0}} P(n) e^{\lambda + n\alpha_1} = t^h \sum_{n \in \mathbb{Z}^{\geq 0}} P(n) t^n = t^h \prod_{j=1}^{\infty} \frac{1}{(1-t^j)}$$

where $t = e^{\alpha_1}$.

Remark 5.3.4. *Although technically our choice of positive and negative roots make our* $M(c,h)$ *a highest weight module, it is more natural to make the opposite choice; i.e.* $\Delta^-(L,H) = \{\alpha_n : n \in \mathbb{Z}^-\}$. *Then our Verma modules are lowest weight modules.*

5.4 Unique Irreducible Quotient

In the study of modules, we would like to classify irreducible modules. Verma modules are not necessarily irreducible, but they have a unique irreducible quotient (by taking the quotient by the unique maximal proper submodule). ([Hum78], p.108)

5.4.1 Unique Maximal Proper Submodule

If A and B are both submodules of V then $A+B$ is also a submodule of V. (A and B are both invariant under L-actions.)

Let $M(\lambda)$ be a Verma module with highest weight vector v_λ. Consider the set defined by $\mathcal{W} = \{W | W \text{ is a proper submodule of } M(\lambda)\}$. Any submodule can be expressed as the direct sum of its weight spaces ([Hum78] p 108). Let $J(\lambda) = \langle W | W \in \mathcal{W} \rangle$, the subspace generated by all proper submodules. Note that v_λ cannot be in any W (since v_λ generates $M(\lambda)$ and the λ weight space is one-dimensional, any non-zero vector in the λ weight space can generate $M(\lambda)$, and W is proper). Therefore, λ is not a weight of W, and hence, is not a weight of $J(\lambda)$. $J(\lambda)$ is therefore proper and so is the unique maximal proper submodule of $M(\lambda)$.

Definition 5.4.1. *([Hum78], p.109) We define the* ***unique irreducible quotient*** *to be: $L(\lambda) := M(\lambda)/J(\lambda)$ by $L(\lambda)$. It is called the irreducible highest weight module of highest weight λ.*

Example 5.4.2. *We can now see that if $J(\lambda)$ is not trivial, then $M(\lambda)$ is not $J(\lambda) \oplus W$ for some submodule W. Suppose there exists some W such that $M(\lambda) = J(\lambda) \oplus W$. As we saw in the previous paragraph, $J(\lambda)$ cannot have weight λ and since submodules of Verma modules have weight space decompositions, v_λ must be in W. But then $\mathcal{U}(L).W = M(\lambda)$ which means $J(\lambda)$ must have been trivial–a contradiction.*

This is an example of a module that cannot be expressed as a direct sum of irreducible submodules. See 2.8.5.

Chapter 6

Invariant Hermitian Forms

The thesis problem is to compute the signature characters of irreducible highest weight modules of the Virasoro algebra. In physics, the settings for problems are often Hilbert spaces, which carry an inner product. However, problems can arise on vector spaces with non-definite Hermitian forms. Often, the representations which arise respect the Hermitian form. Thus, we will study, abstractly, invariant Hermitian forms on representations and associated data which is of interest to physicists and mathematicians: specifically, signatures of these forms.

6.1 Real Forms

Definition 6.1.1. *([Kna02], p.34) Given a real Lie algebra $L_{\mathbb{R}}$, the* ***complexification of*** *$L_{\mathbb{R}}$ is*

$$L := L_{\mathbb{R}} \oplus iL_{\mathbb{R}}.$$

Definition 6.1.2. *([Kna02], p.35) A* ***real form*** *of a complex Lie algebra L is a real subalgebra $L_{\mathbb{R}}$ for which $L = L_{\mathbb{R}} \oplus iL_{\mathbb{R}}$. In this situation, complex conjugation of elements of L is with respect to $L_{\mathbb{R}}$:*

$$\overline{X + iY} = X - iY \qquad \textit{where } X, Y \in L_{\mathbb{R}}.$$

Note that $L_{\mathbb{R}} = \{X \in L : \overline{X} = X\}$.

Example 6.1.3. *It is easy to show that $\mathfrak{sl}_2(\mathbb{R})$ is the set of fixed points of the usual complex* *conjugation on $\mathfrak{sl}_2(\mathbb{C})$. It is a real form of $\mathfrak{sl}_2(\mathbb{C})$.*

The previous example was very natural. There is a obvious choice real form for Vir: $span_{\mathbb{R}}\{L_n, z\}$ but we will instead use the (still natural) definition that arises from the real form for the Witt algebra. However, since it is a more complicated Lie algebra we will find that describing complex conjugation is somewhat more difficult.

Example 6.1.4. *([KR87],p.1) Since the Virasoro algebra is the central extension of the Witt algebra $\mathcal{D} = span_{\mathbb{C}}\{L_n : n \in \mathbb{Z}\}$, it makes sense to examine Vir by considering the real form of the Witt algebra. The Witt algebra can be viewed as the complexification of the real Lie algebra of vector fields on $\mathbb{S}^1$. The space $\mathcal{D}_{\mathbb{R}}$ of vector fields has a basis: $\frac{d}{d\theta}, \cos(n\theta)\frac{d}{d\theta}$, and $\sin(n\theta)\frac{d}{d\theta}$ for all $n \in \mathbb{Z}$. They form a basis (over $\mathbb{C}$) for the Witt algebra that is fixed under complex conjugation with respect to the real form $\mathcal{D}_{\mathbb{R}}$. We can relate this basis to the basis $\{L_n : n \in \mathbb{Z}\}$ for the Witt algebra as follows:*

$$L_n = ie^{in\theta}\frac{d}{d\theta} = -z^{n+1}\frac{d}{dz}$$

where $z = e^{i\theta}$ and $n \in \mathbb{Z}$. Now, since $e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$,

$$\begin{aligned} L_n - L_{-n} &= -2\sin(n\theta)\frac{d}{d\theta} \\ \Rightarrow \overline{L_n - L_{-n}} &= L_n - L_{-n} \qquad \textit{since } \sin(n\theta)\frac{d}{d_\theta} \in \mathcal{D}_{\mathbb{R}} \\ \\ L_n + L_{-n} &= 2i\cos(n\theta)\frac{d}{d\theta} \\ \Rightarrow \overline{L_n + L_{-n}} &= -(L_n + L_{-n}) \qquad \textit{since } \cos(n\theta)\frac{d}{d_\theta} \in \mathcal{D}_{\mathbb{R}} \\ \\ \Rightarrow \overline{L_n} &= -L_{-n} \end{aligned}$$

Therefore the natural real form on the Virasoro algebra arising in physics is the real form for which:

1. $\overline{L_n} = -L_{-n}$ *for* $n \in \mathbb{Z}$
2. $\overline{z} = -z$.

6.2 Hermitian Forms

Definition 6.2.1. *([Lan02], p.579) A* ***Hermitian form*** *is a sesquilinear pairing* $\langle \cdot, \cdot \rangle$*:* $V \times V \to \mathbb{C}$ *which satisfies* $\langle x, y \rangle = \overline{\langle y, x \rangle}$ *for* $x, y \in V$*. By sesquilinear, we mean it is linear in the first argument and complex conjugate linear in the second.*

Definition 6.2.2. *([Lan02], p.581) Given a linear map* $T : V \to V$*, if* $\langle \cdot, \cdot \rangle$ *is nondegenerate (see definition 6.3.4), there is a unique complex conjugate linear map* $T^* : V \to V$ *satisfying*

$$\langle Tx, y \rangle = \langle y, T^* x \rangle .$$

The map T^* *is called the* ***(Hermitian) adjoint*** *of* T*.*

Definition 6.2.3. *([Vog84], p.148) Given a real Lie group* G *and a Hermitian form* $\langle \cdot, \cdot \rangle$ *on the* G*-representation* V*, the form is said to be* ***invariant under the action of*** G *if*

$$\langle g.v, g.w \rangle = \langle v, w \rangle \qquad \forall\, v, w \in V.$$

Descending to the Lie algebra, this gives us:

Definition 6.2.4. *([Vog84], p.148) Let* $L_{\mathbb{R}}$ *be a real Lie algebra and* V *a representation of* $L_{\mathbb{R}}$ *on a complex vector space. A Hermitian form* $\langle \cdot, \cdot \rangle$ *on* V *is said to be* ***invariant*** *if*

$$\langle X.v, w \rangle + \langle v, X.w \rangle = 0 \qquad \forall\, X \in L_{\mathbb{R}}, v, w \in V.$$

Since Hermitian forms are conjugate linear in the second variable and satisfy $\overline{\langle v, w \rangle} = \langle w, v \rangle$, we have:

Definition 6.2.5. *([Vog84], p.148) Let* L *be a complex Lie algebra and* (π, V) *a representation of* L*. A Hermitian form* $\langle \cdot, \cdot \rangle$ *on* V *is said to be* ***invariant*** *if*

$$\langle X.v, w \rangle + \left\langle v, \overline{X}.w \right\rangle = 0 \qquad \forall\, X \in L, v, w \in V.$$

That is, $\langle X.v, w \rangle = - \left\langle v, \overline{X}.w \right\rangle$*.*

Note that what this says is that for every $X \in L$*, the adjoint operator to the linear map* $V \to V$ *defined by the action of* X *is the linear map defined by the action of* $-\bar{X}$*. I.e.* $\pi(X)^* = -\pi(\bar{X})$*.*

Proposition 6.2.6. *([KR87], Proposition 3.4) A non-zero invariant Hermitian form exists for $M(c,h)$ if and only if c,h are both real. Since $M(c,h)$ is generated from a single vector, the invariance condition implies that invariant Hermitian forms on $M(c,h)$ are unique up to a real scalar.*

Henceforth we only work with c,h real.

Definition 6.2.7. *([Yee05], Def. 2.5) Let $\langle\cdot,\cdot\rangle_{c,h}$ denote the invariant Hermitian form on $M(c,h)$ and assume that the form has been normalized so that the inner product of the choice $v_{c,h}$ of generating vector with itself is 1. This is called the* ***Shapovalov form****. Note that invariance and our computations in example 6.1.4 show that the form satisfies:*

$$\begin{aligned}\langle L_n.v,w\rangle_{c,h} &= \langle v,L_{-n}.w\rangle_{c,h} \qquad \forall n\in\mathbb{Z}, v,w\in M(c,h)\\ \langle z.v,w\rangle_{c,h} &= \langle v,z.w\rangle_{c,h}\,.\end{aligned}$$

In the literature (eg. [KR87]), you will find a complex conjugate linear map $\omega : Vir \to Vir$ defined by $\omega(L_n) = L_{-n}$ and $\omega(z) = z$ for which $\omega(aX) = \bar{a}\omega(X)$ for $a\in\mathbb{C}$, $X\in Vir$. The Shapovalov form is then described to be the unique form for which $\langle v_{c,h},v_{c,h}\rangle_{c,h} = 1$ and $\langle X.v,w\rangle_{c,h} = \langle v,\omega(X).w\rangle_{c,h}$ for $X\in Vir$, $v,w\in M(c,h)$. We can note that $\omega(X) = -\bar{X}$, making our discussion consistent with the literature.

6.3 Orthogonality of Weight Spaces for Verma Modules of Vir

Given an invariant Hermitian form, weight vectors may be seen to be orthogonal by comparing weights. This can easily be seen in an example.

Proposition 6.3.1. *([KR87] p 25.) In the case of Vir, let v_n and v_m lie in the n^{th} and m^{th} levels of $M(c,h)$ respectively (i.e. the $h+n$ and $h+m$ weight spaces of $M(c,h)$ respectively). Then we have:*

$$\begin{aligned}(h+n)\langle v_n,v_m\rangle &= \langle L_0v_n,v_m\rangle\\ &= \langle v_n,L_0v_m\rangle\\ &= (h+m)\langle v_n,v_m\rangle\end{aligned}$$

Therefore, if $n \neq m$, then $\langle v_n, v_m \rangle = 0$.

Orthogonality of the weight spaces allows us to study the invariant Hermitian form on one (finite dimensional) weight space at a time.

Definition 6.3.2. *([Vog84], p.148) The* ***radical*** *of a form $\langle \cdot, \cdot \rangle$ on V is the set of vectors $\{x \in V | \langle x, y \rangle = 0 \text{ for all } y \in V\}$.*

Definition 6.3.3. *([Itō87], p.1294) The* ***signature of the Hermitian form*** *$\langle \cdot, \cdot \rangle$ on a finite dimensional weight space X is the triple (p, q, r) where p is the dimension of the subspace where the form is positive definite; q is the dimension of the subspace where the form is negative definite; and r is the dimension of the radical of the form. Equivalently, p, q, r are the number of positive, negative and zero eigenvalues respectively for a matrix A describing the form where $\langle u, v \rangle = u^t A \bar{v}$ for vectors u, v in X expressed in coordinates for the same basis used to determine the entries of A.*

Definition 6.3.4. *([Vog84], p.148) A Hermitian form is* ***nondegenerate*** *if and only if its radical is 0.*

Although Verma modules are infinite dimensional, orthogonality of weight spaces allows us to make the following definition:

Definition 6.3.5. *([Wal84], p.132) Let $M(c, h)$ be a Verma module with a non-degenerate invariant Hermitian form. Let $(p(n), q(n), 0)$ be the signature of the form on the n^{th} level. Then the* ***signature character*** *for $M(c, h)$ is*

$$ch_s M(c, h) = \sum_{n \in \mathbb{Z}^{\geq 0}} (p(n) - q(n)) t^{h+n}.$$

It is convenient to associate all Verma modules with the same vector space $\mathcal{U}(N^-)$ so we often consider the normalized signature character:

$$\sigma(c, h)(t) = \sum_{n \in \mathbb{Z}^{\geq 0}} (p(n) - q(n)) t^{n}.$$

We will also abbreviate the normalized signature to $\sigma(c, h)$.

Proposition 6.3.6. *([KR87] p 25.) The radical of the Shapovalov form on $M(c, h)$ is the unique maximal proper submodule $J(c, h)$. Therefore the form descends to a non-degenerate form on $L(c, h) = M(c, h)/J(c, h)$.*

Corollary 6.3.7. *A Verma module is irreducible if and only if the radical of the Shapovalov form is* 0*.*

Definition 6.3.8. *If the Shapovalov form of the Verma module $M(c,h)$ is degenerate, then $J(c,h)$ is non-zero and the Verma module is reducible. In this case, we define the signature character for the form on $L(c,h)$. Let $(p(n), q(n), 0)$ be the signature of the form on the n^{th} level of $L(c,h)$. Then the* ***signature character for*** *$L(c,h)$ is*

$$ch_s L(c,h) = \sum_{n \in \mathbb{Z}^{\geq 0}} (p(n) - q(n)) t^{h+n}.$$

We likewise use $\sigma(c,h)$ to denote the normalized signature character of $L(c,h)$.

6.4 Kac's Determinant Formula

We can enumerate the basis vectors of a finite dimensional subspace of the Verma module $M(\lambda)$ in our orthogonal decomposition and thus express the Shapovalov form on that subspace as a square matrix M where the $(i,j)^{th}$ term is $\langle v_i, v_j \rangle$ and v_i, v_j are basis vectors. Since determinants are products of eigenvalues (and $\det M$ is independent of the choice of basis), the Shapovalov form is degenerate if and only if M has a zero eigenvalue, and thus a zero determinant. Therefore, the Verma module $M(\lambda)$ is irreducible if and only if $det(M) \neq 0$ for every choice of M.

Recall for Vir that the decomposition by level is orthogonal. Thus, $M(c,h)$ is reducible if and only if the Shapovalov form is degenerate for some level n.

Theorem 6.4.1. *([KR87] p 87) Kac's Determinant Formula: Let M be a matrix representing the Shapovalov form on the n^{th} level of the Verma module $M(c,h)$. Then up to a non-zero scalar k:*

$$det(M) = k \prod_{\substack{p,q \in \mathbb{N} \\ 1 \leq pq \leq n}} (h - h_{p,q}(c))^{P(n-pq)}$$

where $P(n)$ is the partition function and

$$h_{p,q}(c) = \frac{1}{48}[(13-c)(p^2+q^2) + \sqrt{(c-1)(c-25)}(p^2-q^2) - 24pq - 2 + 2c].$$

We can let $c = 1 - \frac{6}{m(m+1)}$ which simplifies $h_{p,q}(c)$ to

$$h_{p,q}(m) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \qquad ([DFMS97], p.210).$$

We will use $h_{p,q}(c)$ and $h_{p,q}(m)$ interchangeably with the understanding that $c = 1 - \frac{6}{m(m+1)}$. Note that $det(M) = 0 \Leftrightarrow h = h_{p,q}(c)$ for some $p, q \in \mathbb{Z}^+$, $1 \leq pq \leq n$, and $c \in \mathbb{R}$.

Definition 6.4.2. *([DFMS97], p.209) The determinant is zero only at $h = h_{p,q}(c)$, thus we define the curve $h_{p,q} : (c, h_{p,q}(c)) \subset \mathbb{R}^2$ on a plane (with c and h as axes). These curves are called **reducibility curves** (often called vanishing curves in the literature).*

6.5 Reducibility Curves for Vir

When studying signatures, it is important to understand where the reducibility curves lie, and their intersections. For Vir, the c, h plane can be broken down into five areas as shown: ([FF90])

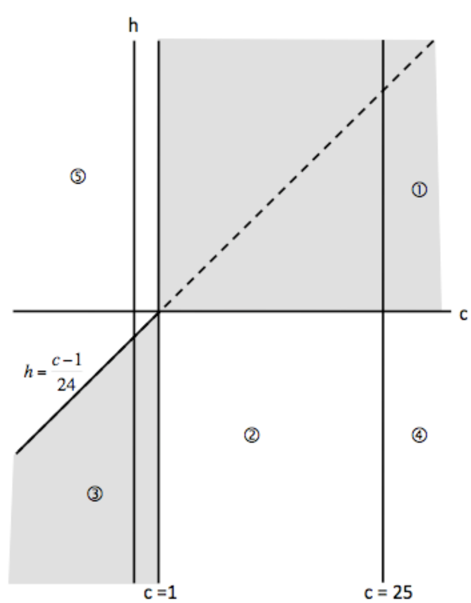


Figure 6-1: Decomposition of the c, h plane

These areas satisfy (discussed in further detail later):

1. $c > 1$ and $h > 0$

   This region is unitary; that is, the invariant Hermitian form on the Verma module

$M(c,h)$ is positive definite. There are no reducibility curves in this region. ([DFMS97], p. 209)

2. $1 < c < 25$ and $h < 0$

   $1 < c < 25$ which implies that m is complex in this region. Thus, $h_{p,q}(m)$ is real if and only if $p = q$. Therefore, the only reducibility curves in this region are $h_{p,p}$. These reducibility curves do not intersect each other.

3. $c < 1$ and $h < \frac{c-1}{24}$

   In this region, there are no reducibility curves, yet the invariant Hermitian form is not positive definite. We will examine the signature for this region later. ([KR87], p. 92)

4. $c > 25$ and $h < 0$

   $h_{p,q}$ is well-defined for all $p, q \in \mathbb{Z}^+$. Furthermore, for rational values of c, intersections of reducibility curves can occur. The structure behind the intersections is fascinating and again we will examine it in greater detail later. It should be noted, however, that in this region, there is only a finite number of curves intersecting at any given point.

5. $c < 1$ and $h > \frac{c-1}{24}$

   Similar to the previous region, $h_{p,q}$ is well-defined for all $p, q \in \mathbb{Z}^+$. However, while the previous region had only a finite number of curves intersecting at a given point, every intersection point in this region has an infinite number of curves intersecting it. Furthermore, for irrational c values, $\{(\text{c},h_{p,q}(c))|p, q \in \mathbb{Z}^+\}$ is dense on the vertical line $\{(c,h)|h > \frac{c-1}{24}\}$. However, we will see that this is not a problem when we examine the signature by looking at a weight space decomposition of the Verma module $M(c,h)$.

6.6 Signatures in Regions Bounded by Reducibility Curves

Here we review philosophies introduced in [Vog84]. Consider points (c_1, h_1) and (c_2, h_2) such that $\sigma(c_1, h_1) \neq \sigma(c_2, h_2)$ and their respective Verma modules are irreducible. The signatures must differ for some level. Consider a path from (c_1, h_1) to (c_2, h_2). Since the number of positive and negative eigenvalues for matrices representing the form on that level differ, there must be some point (c, h) on the path where the number changes. Then by continuity, there must be a zero eigenvalue at (c, h).

We can therefore conclude that signatures can only change when crossing a reducibility curve. It is not, however, sufficient to conclude that the signature must change at reducibility curves. For example, the changing eigenvalue could go from $+ \to 0 \to +$. Therefore, we need additional information in order to compute the signatures. Some of that information is given in the Jantzen filtration. Our philosophy to compute some signatures will be to track how signatures change as we cross reducibility curves.

Chapter 7

The Jantzen Filtration And Submodule Structure of Verma Modules over Vir

In this chapter, we discuss two key pieces of information associated to each Verma module: its Jantzen filtration and its submodule structure. The Jantzen filtration is a filtration by order of vanishing which, in conjunction with additional information, indicates how signatures change as you cross reducibility curves. The Janzen filtration is closely related to submodule structure, which we need to understand later to compute $ch_s L(c,h)$.

7.1 Module Structure and Jantzen Filtration For Verma Modules over Vir

In this section, we will use r, s to index reducibility curves instead of p, q in deference to the notation used in [DFMS97].

Fix (c,h). Let $\Phi_{r,s}(c,h) = (h - h_{r,s}(c))(h - h_{s,r}(c))$. In order to study module structure, we wish to understand to which reducibility curves (c,h) belongs. We seek integer solutions $(\alpha, \beta) = (r,s)$ to $\Phi_{r,s}(c,h) = 0$.

Proposition 7.1.1. *([FF90] p. 479, [DFMS97] p.240) The equation $\Phi_{r,s}(c,h) = 0$ has integer solutions $(r,s) = (\alpha, \beta)$ when $p\alpha + p'\beta + k = 0$ for some integers α, β where*

1. $c = 1 - 6\frac{(p-p')^2}{pp'}$ *and*

2. $h_{r,s}(c) = \frac{k^2-(p-p')^2}{4pp'}$

for some numbers p, p', k.

Furthermore, for some constant C,

$$\Phi_{r,s}(c,h) = C(p\alpha + p'\beta + k)(p\alpha + p'\beta - k)(p'\alpha + p\beta + k)(p'\alpha + p\beta - k).$$

Feigin and Fuchs note in [FF90] that you only need to consider one of the four lines. For example, if $pr + p's + k = 0$, then $p(-r) + p'(-s) - k = 0$. I.e. if (r, s) gives a solution for one line, then $(-r, -s)$ gives a solution for another line. Note that $M(c, h + rs) = M(c, h + (-r)(-s))$, etc.

Notation 7.1.2. *([FF90]) Consider the line* $p\alpha + p'\beta + k = 0$*. It can have 0, 1, or infinitely many lattice points lying on it. We consider the cases and subcases:*

I: If $p\alpha + p'\beta + k = 0$ *contains 0 lattice points.*

II: If $p\alpha + p'\beta + k = 0$ *contains 1 lattice point* (α_1, β_1)*.*

- II_+: $\alpha_1\beta_1 > 0$
- II_-: $\alpha_1\beta_1 < 0$
- II_o: $\alpha_1\beta_1 = 0$

III: If $p\alpha + p'\beta + k = 0$ *contains infinitely many lattice points.*

- III_+^o: $p\alpha + p'\beta + k = 0$ *has negative slope and intersects one axis at a lattice point.*
- III_+^{oo}: $p\alpha + p'\beta + k = 0$ *has negative slope and intersects both axes at lattice points.*
- III_+: $p\alpha + p'\beta + k = 0$ *has negative slope and does not intersect either axis at a lattice point.*
- III_-^o: $p\alpha + p'\beta + k = 0$ *has positive slope and intersects one axis at a lattice point.*
- III_-^{oo}: $p\alpha + p'\beta + k = 0$ *has positive slope and intersects both axes at lattice points.*
- III_-: $p\alpha + p'\beta + k = 0$ *has positive slope and does not intersect either axis at a lattice point.*

We then have the following module structure for cases:

Theorem 7.1.3. *([FF90])*

$I:$ *$M(c,h)$ is irreducible.*

$II_+:$ $\alpha_1\beta_1 > 0$*:* $M(c,h) \supset M(c,h+\alpha_1\beta_1) \supset 0$.

II_-*:* $\alpha_1\beta_1 < 0$*:* $M(c,h+\alpha_1\beta_1) \supset M(c,h) \supset 0$. *Note that $M(c,h)$ is irreducible.*

II_o*: If $\alpha\beta = 0$ then $M(c,h)$ is irreducible.*

III*: Consider the lattice points satisfying $p\alpha + p'\beta + k = 0$.*

$III_\pm^{oo}$*: The lattice points on the line can be paired into pairs giving the same product. We therefore only consider half the points: let P be the midpoint of the intersection points with the axes and we will take only points lattice in the upper half of the line above P, including P if it is a lattice point. We can order the points by:*

$$\cdots < \alpha_{-2}\beta_{-2} < \alpha_{-1}\beta_{-1} < 0 < \alpha_1\beta_1 < \alpha_2\beta_2 < \cdots .$$

We then have

$$\cdots \supset M(c,h+\alpha_{-1}\beta_{-1}) \supset M(c,h) \supset M(c,h+\alpha_1\beta_1) \supset M(c,h+\alpha_2\beta_2) \supset \cdots .$$

$III_\pm^{o}$*: Again order (α_i,β_i) as in the previous case, but this time all the points (since we do not have the same product appearing) and we have the module structure:*

$$\cdots \supset M(c,h+\alpha_{-1}\beta_{-1}) \supset M(c,h) \supset M(c,h+\alpha_1\beta_1) \supset M(c,h+\alpha_2\beta_2) \supset \cdots .$$

$III_\pm$*: Again, order (α_i,β_i) and draw a line parallel to $p\alpha + p'\beta + k = 0$ but passing through $(\alpha_i,-\beta_i)$. Order the lattice points on the new line whose coordinates have a positive product, i.e. $0 < \alpha'_1\beta'_1 < \alpha'_2\beta'_2 < \cdots$. Then, we have the following:*

$$0 < \alpha_1\beta_1 < \alpha_2\beta_2 < \alpha_1\beta_1 + \alpha'_1\beta'_1 < \alpha_1\beta_1 + \alpha'_2\beta'_2 < \alpha_3\beta_3 < \alpha_4\beta_4 < \alpha_1\beta_1 + \alpha'_3\beta'_3 < \cdots$$

and

$$\cdots < \alpha'_{-3}\beta'_{-3} + \alpha_1\beta_1 < \alpha_{-2}\beta_{-2} < \alpha_{-1}\beta_{-1} < \alpha'_{-2}\beta'_{-2} + \alpha_1\beta_1 < \alpha'_{-1}\beta'_{-1} + \alpha_1\beta_1 = 0$$

Giving us the module structure:

$$
\begin{tikzcd}
 & M(c,h+\alpha_2\beta_2) \arrow[r, dash] \arrow[ddr, dash] & M(c,h+\alpha_1\beta_1+\alpha_2'\beta_2') \arrow[r, dash] \arrow[ddr, dash] & M(c,h+\alpha_4\beta_4) \arrow[r, dotted, dash] \arrow[ddr, dotted, dash] & \cdots \\
M(c,h) \arrow[ur, dash] \arrow[dr, dash] & & & & \\
 & M(c,h+\alpha_1\beta_1) \arrow[r, dash] \arrow[uur, dash] & M(c,h+\alpha_1\beta_1+\alpha_1'\beta_1') \arrow[r, dash] \arrow[uur, dash] & M(c,h+\alpha_3\beta_3) \arrow[r, dotted, dash] \arrow[uur, dotted, dash] & \cdots
\end{tikzcd}
$$

where each module contains the modules to its right connected to it by lines. There is a similar diagram for modules which contain $M(c,h)$.

Remark 7.1.4. *We note that if the slope of the line is positive, then the number of lattice points on the line in quadrants one and three is infinite. If the slope of the line is negative, then the number of lattice points on the line in quadrants one and three is finite. Therefore there are infinitely many $\alpha_i\beta_i$ and finitely many $\alpha_{-i}\beta_{-i}$ for $i \in \mathbb{Z}^+$ when the slope is positive, while there are finitely many $\alpha_i\beta_i$ and infinitely many $\alpha_{-i}\beta_{-i}$ for $i \in \mathbb{Z}^+$ when the slope is negative. Thus our sequences of submodules terminate on one side: "above" in the - case and "below" in the + case, as shown in the following diagrams.*

The figures below provide complete module structure. Dots represent Verma modules with the solid dot indicating the Verma module $M(c,h)$. Lines indicate containment: Verma modules contain the modules connected to them by a downward path.

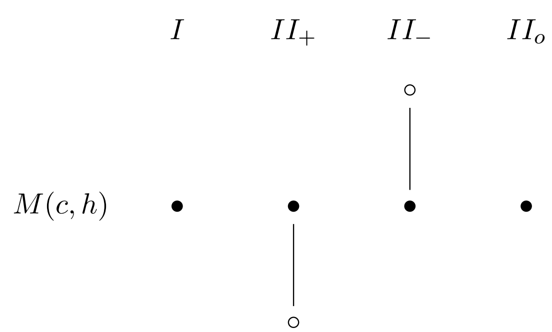


Figure 7-1: The Submodule Structure for Cases I and II

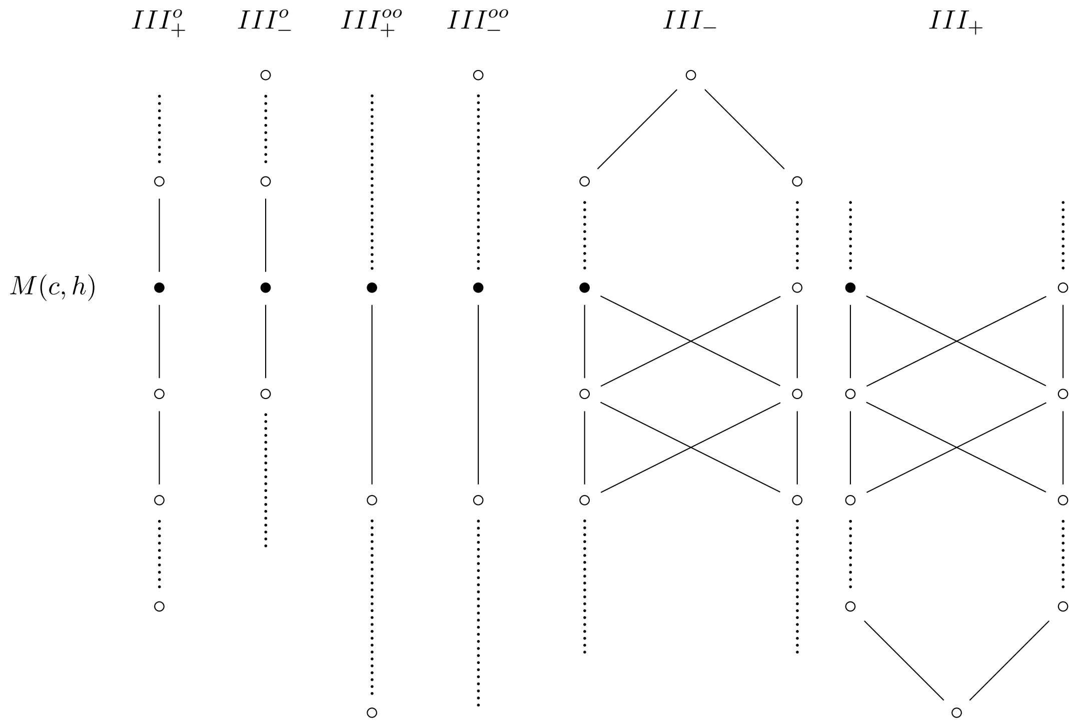


Figure 7-2: The Submodule Structure for Case III

Remark 7.1.5. *A few things of note:*

Case I occurs when (c,h) does not lie on a reducibility curve.

In Case II_+, if $\alpha_1\beta_1 > 0$ then (c,h) lies on h_{α_1,β_1} and only on this reducibility curve.

In Case II_-, if $\alpha_1\beta_1 < 0$ then $(c, h+\alpha_1\beta_1)$ lies on $h_{|\alpha_1|,|\beta_1|}$.

Case III only occurs at the intersection of reducibility curves. If $c < 1$ then there are an infinite number of reducibility curves intersecting and if $c > 25$ then the number of reducibility curves is finite. $c < 1$ and $c > 25$ also corresponds to positive/negative slope ([FF90]).

Case	*Regions*
I	*All*
II_+	*2, 4, 5*
II_-, II_o	*1, 4, 5*
III^*_+	*4*
III^*_-	*5*

Remark 7.1.6. *([FF90]) It is readily apparent from the figures that the cases X_- and X_+ are related. The antiautomorphism between Verma modules $M(c,h)$ and $M(26-c,1-h)$ that Feigin and Fuchs note gives rise to this phenomenon.*

Remark 7.1.7. *([DFMS97], p.216) One should note that the* ***minimal model*** *case corresponds to case III_- where $M(c,h)$ is not a submodule of any other Verma module (it is the top module in the III_- structure).*

7.2 The Jantzen Filtration

The Jantzen filtration of a Verma module is a decomposition that can be used in conjunction with additional information to calculate how the signature of a Verma module changes as it crosses a reducibility curve. Consider a vertical path $g(t)$:

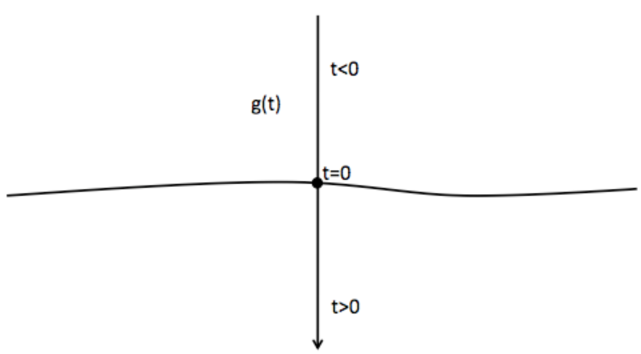


Figure 7-3: Jantzen Filtration and Paths

We define Verma modules $M(g(t))$ along path $g(t)$ with highest weight vectors $v_{g(t)}$ with invariant Hermitian forms $\langle\cdot,\cdot\rangle_t$.

The Janzen filtration of a Verma module $M(c,h)$ (at $g(0)=(c,h)$) with highest weight vector $v_{c,h}$ can be defined as:

Definition 7.2.1. *([Vog84], p.151) Let*

$$J_k = \{l.v_\lambda \mid l \in \mathcal{U}(L),\ \textit{and}\ \forall x \in \mathcal{U}(L), \lim_{t\to 0} t^{-k} \left\langle l.v_{g(t)}, x.v_{g(t)} \right\rangle_t \neq \pm\infty\}.$$

Clearly $M(\lambda) = J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$. This sequence of submodules is the ***Jantzen filtration*** *of a Verma module. The Jantzen filtration is independent of path and J_k consists* *of vectors which vanish to at least order k.*

Example 7.2.2. *By definition of the radical,* $J(\lambda) = J_1$*. If* $M(\lambda)$ *is irreducible, then* $J_1 = \{0\}$ *and the Jantzen filtration is thus:*

$$J_0 = M(\lambda) \supseteq \{0\} = J_1.$$

We will be focusing our attention on case II_+ as we will cross the reducibility curves one at a time. We will also look closely into the Janzten filtration of case III_- as it is of particular interest to physicists.

Theorem 7.2.3. *[FF90], p. 491 The Jantzen filtration for a Verma module over the Virasoro algebra is given by the following diagram.*

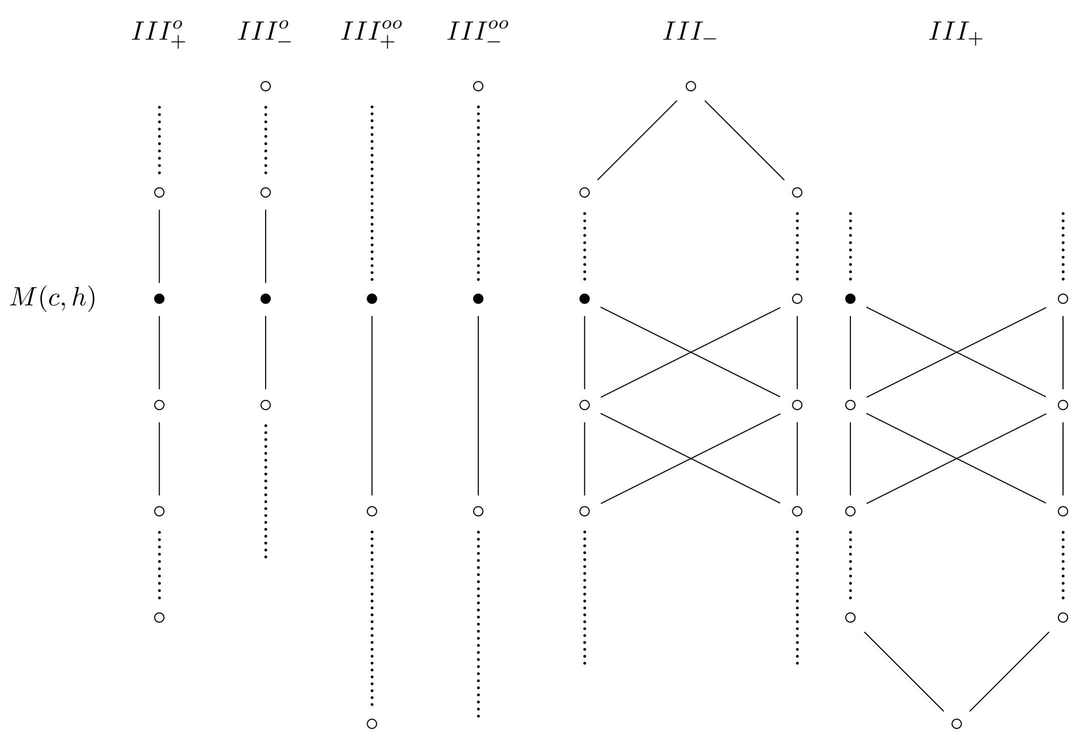


Figure 7-4: The Jantzen Filtration for Reducible Verma Modules

Note that in cases $III_\pm^{oo}$ $J_1 = J_2$ because any vector which vanishes at least to order one must vanish to order two. Therefore the quotient J_k/J_{k+1} is only non-trivial for even k.

7.2.1 Invariant Hermitian Form on J_k/J_{k+1}

Definition 7.2.4. *([Vog84], p.151) If we consider the module J_k/J_{k+1}, then we can define the* ***invariant Hermitian form on*** *J_k/J_{k+1} as*

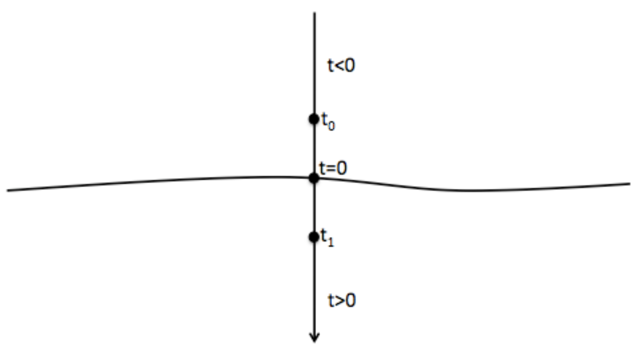
$$\left\langle l_1.v_{g(0)} + J_{k+1}, l_2.v_{g(0)} + J_{k+1} \right\rangle^k = \lim_{t \to 0} t^{-k} \left\langle l_1.v_{g(t)}, l_2.v_{g(t)} \right\rangle_t$$

for all $l_1, l_2 \in \mathcal{U}(L)$ such that $l_1.v_{g(0)}, l_2.v_{g(0)} \in J_k$.

7.3 Changes to Signature Characters When Crossing Reducibility Curves

Here, we see how the Jantzen filtration is related to understanding changes to signature characters as we cross reducibility curves.

Theorem 7.3.1. *([Vog84] Proposition 3.3) Consider the path $g(t) : (-\delta, \delta) \to \mathbb{R}^2$ crossing a reducibility curve only at t=0. Let $M(c, h)$ be the Verma module at t=0 and let $t_0 \in (-\delta, 0)$ and $t_1 \in (0, \delta)$.*



Let us again apply the Jantzen filtration to $M(g(0))$, i.e. $M(g(0)) = J_0 \supseteq J_1 \supseteq ...$ as in the prior subsection. Now consider the signature at t_0. Let the signature of the k^{th} level of the Jantzen filtration be (p_k, q_k); that is, the $h + n$ weight space in the k^{th} level of the Jantzen filtration has signature $(p_k(n), q_k(n), 0)$.

1. *The signature for the n^{th} level of $\langle \cdot, \cdot \rangle_{t_1}$ is*

$$\left(\sum_k p_k(n), \sum_k q_k(n) \right).$$

2. The signature for the n^{th} level of $\langle\cdot,\cdot\rangle_{t_0}$ is

$$\left(\sum_{k:even} p_k(n) + \sum_{k:odd} q_k(n)\ ,\ \sum_{k:odd} p_k(n) + \sum_{k:even} q_k(n)\right).$$

Note that only finitely many reducibility curves can affect any particular level.

Corollary 7.3.2. *Consider a path $g(t)$ from $(-\delta,\delta)$ to the c,h plane such that $g(0)$ lies on a single reducibility curve. Then, in the notation of the previous theorem,*

1. $\lim_{t\to 0^+}$ signature for the n^{th} level of $\langle\cdot,\cdot\rangle_t$ is

$$\left(\sum_{k} p_k(n), \sum_{k} q_k(n)\right).$$

2. $\lim_{t\to 0^+}$ signature for the n^{th} level of $\langle\cdot,\cdot\rangle_{t_0}$ is

$$\left(\sum_{k:even} p_k(n) + \sum_{k:odd} q_k(n)\ ,\ \sum_{k:odd} p_k(n) + \sum_{k:even} q_k(n)\right).$$

Chapter 8

Signatures for Irreducible Verma Modules for Vir

The philosophies which we will use to compute signatures for irreducible Verma modules are those from [Vog84]. First, we saw that signatures can only change as we cross reducibility curves. Then we saw that according to a result of Vogan (Theorem 7.3.1), as you cross a reducibility curve, the signature will change by the signature of the odd levels of the Jantzen filtration. When the crossing point only lies on a single reducibility curve, the Jantzen filtration is particularly simple–it has two levels. In this situation, the signature changes by the signature of the radical, an irreducible Verma module.

Beginning in a region where signatures are known and then crossing reducibility curves one at a time, we arrive at formulas for other regions up to some unknown variables, denoted by ε, which take values ± 1. ε will be computed in a subsequent chapter.

8.1 The Difference Equation For Crossing a Single Reducibility Curve $h_{p,q}$

Theorem 8.1.1. *Consider the Verma module $M(c, h)$ in the situation where (c, h) lies on only one reducibility curve and there are no reducibility curves between (c, h) and $(c, h \pm \delta)$ for small $\delta > 0$. Let's suppose $h = h_{p,q}(c)$. Then:*

1. *The Jantzen filtration is $M(c, h) = J_0 \supseteq J_1 = M(c, h + pq) \supseteq 0$.*

2. *Recall that* $\langle \cdot, \cdot \rangle^k$ *was the induced invariant Hermitian form on* J_k/J_{k+1}*. Then:*

$$ch_s M(c, h+\delta) = t^{2\delta} ch_s M(c, h-\delta) - 2\varepsilon t^{\delta} ch_s M(c, h+pq)$$

where ε *is one of* ± 1*.*

3. *Recall that* σ *is used to denote the normalized signature character. Then:*

$$\sigma(c, h-\delta) = \sigma(c, h+\delta) + 2\varepsilon t^{pq} \sigma(c, h+pq)$$

where ε *is one of* ± 1*.*

Proof. Now, by Kac's Determinant Formula (6.4.1) and Theorem 7.6.6 of [Dix96], we know that there is a unique singular vector w (up to scalar multiples) of weight $h+pq$. Since w is a singular vector, there exists a submodule of $M(c, h)$ with highest weight vector w which is isomorphic to the Verma module $M(c, h+pq)$. Thus $M(c, h) \supset M(c, h+pq)$.

Since we are in case 1 of section 7.1, the Jantzen filtration for $M(c, h)$ stabilizes quickly. Specifically, $M(c, h) = J_0 \supseteq J_1 = M(c, h+pq) \supseteq 0$.

To prove the final formula, let $t = h \pm \delta$ in Theorem 7.3.1. Then:

$$\begin{aligned} ch_s M(c, h+\delta) &= t^{\delta} ch_s \langle \cdot, \cdot \rangle^0 + t^{\delta} ch_s \langle \cdot, \cdot \rangle^1 = t^{\delta} ch_s \langle \cdot, \cdot \rangle^0 - t^{\delta} \varepsilon \ ch_s M(c, h+pq) \qquad (8.1) \\ ch_s M(c, h-\delta) &= t^{-\delta} ch_s \langle \cdot, \cdot \rangle^0 - t^{-\delta} ch_s \langle \cdot, \cdot \rangle^1 = t^{-\delta} ch_s \langle \cdot, \cdot \rangle^0 + t^{-\delta} \varepsilon \ ch_s M(c, h+pq) \end{aligned}$$

where $\varepsilon = \pm 1$. Subtracting $t^{2\delta}$ times the second formula from the first equation and then rearranging, we obtain our desired formula. ☐

An additional consideration which separates the Virasoro algebra case from the settings considered in [Vog84] and [Yee05] is the density of points around which reducibility curves are dense. However, we may apply the curve crossing philosophy nonetheless because of our orthogonal decomposition by level. As we saw in the previous theorem, there are only finitely many curves which may alter the signature at level n: the $h_{p,q}$ for which $pq \leq n$. It follows that:

Corollary 8.1.2. *For* m*,* p, q *such that* $(c, h_{p,q}(m))$ *lies on only one reduciblity curve, there* *exists* $\varepsilon(m,p,q) = \pm 1$ *such that*

$$\lim_{\delta \to 0^+} (\sigma(c, h_{p,q} - \delta) - \sigma(c, h_{p,q} + \delta)) = 2\varepsilon(m,p,q) t^{pq} \sigma(c, h + pq)$$

Remark 8.1.3. *Since* ε *is associated with our single curve crossing formula, we only define* ε *for points lying on a single reducibility curve.*

Note the t^{pq} in the curve crossing formula. This occurs in the formula due to σ being a normalized signature.

8.2 Region 1: $\sigma(c,h)$ for Irreducible $M(c,h)$ when $c > 1, h > 0$

Theorem 8.2.1. *([DFMS97], p. 360) In this region, every Verma module is irreducible and every matrix for the Shapovalov form is positive definite. Thus signature is equivalent to the character, and so*

$$\sigma(c,h) = \sum_{n=0}^{\infty} P(n) t^n = \prod_{n=1}^{\infty} (1 - t^n)^{-1}$$

We will denote this product as $\varphi(t)$.

Remark 8.2.2. *It should also be noted that if* $c < 1$ *then* $\lim_{h \to \infty} \sigma(c,h) = \varphi(t)$. *([Ken91])*

8.3 Region 2: $\sigma(c,h)$ for Irreducible $M(c,h)$ when $1 < c < 25, h < 0$

Lemma 8.3.1. *If* $1 < c < 25$*, then* m *is complex.*

Proof. Recall that $c = 1 - \frac{6}{m(m+1)}$.

Now if m is real,

$$\begin{aligned} m(m+1) &= (m + \frac{1}{2})^2 - \frac{1}{4} \\ \Rightarrow m(m+1) &> -\frac{1}{4} \end{aligned}$$

But if $-\frac{1}{4} < m(m+1) < 0$ then $c > 25$ and if $0 < m(m+1)$ then $c < 1$. Therefore, if $1 < c < 25$ then m must be a complex number (and not real). $\square$

Proposition 8.3.2. *The only possible p, q values for which $h_{p,q}(c)$ is real in region 2 is p, p.*

Proof. Recall that $h_{p,q}(m) = \frac{[(m+1)p-mq]^2-1}{4m(m+1)}$. Let $k = \frac{(m+1)}{m}$ (note that k is also not real) and we can simplify $h_{p,q}(m)$ as: ([DFMS97], p.208)

$$\begin{aligned} h_{p,q}(m) &= \frac{1}{4}(p^2-1)k + \frac{1}{4k}(q^2-1) - \frac{1}{2}(pq-1) \\ &= \frac{1}{4}(p^2-1)(k+\frac{1}{k}) - \frac{1}{2}(pq-1) + \frac{1}{4k}(q^2-p^2). \end{aligned}$$

Now it should be noted that $k + \frac{1}{k} = \frac{(m+1)}{m} + \frac{(m)}{m+1} = 2 + \frac{1}{m(m+1)}$ which is a real number.

Since p and q are both positive integers, $h_{p,q}(m)$ is real if only if $\frac{1}{4k}(q^2-p^2)$ is also a real number. But since k is not real, this is only possible if $p = q$. Hence the only reducibility curves in the range $1 < c < 25$ are $h_{p,p}$ for positive integers p. □

Lemma 8.3.3. *In region 2,*

1. $0 > h_{1,1}(c) > h_{2,2}(c) > \dots$
2. $h_{p,p}(c) + p^2 > 0$.

Proof. We first note that

$$\begin{aligned} h_{p,p}(c) &= \frac{[(m+1)p-mp]^2-1}{4m(m+1)} \\ &= \frac{p^2-1}{4m(m+1)} \\ &= \frac{(p^2-1)(1-c)}{24}. \end{aligned}$$

Since $1 < c < 25$, therefore $0 > h_{1,1}(c) > h_{2,2}(c) > \dots$. Now since $-24 < 1-c < 0$, $|\frac{(p^2-1)(1-c)}{24}| < |p^2-1| < p^2$. Therefore, $h_{p,p}(c) + p^2 > 0$. □

Theorem 8.3.4. *Given (c, h) in region 2 with $h_{p+1,p+1}(c) < h < h_{p,p}(c)$,*

$$\sigma(c,h) = [1 + \sum_{k=1}^{p} 2\varepsilon(m,k,k)t^{k^2}]\varphi(t).$$

Proof. Consider a vertical path from the unitary region to (c, h). Since the $h_{k,k}(c)$ decrease as k increases, we will intersect $h_{1,1}, h_{2,2}, \dots, h_{p,p}$ in order. Since $\sigma(c, h_{k,k}(c) + k^2) = \varphi(t)$, by Lemma 8.3.3 and Theorem 8.2.1, applying the difference equation Theorem 8.1.1 p times, we obtain the desired result. □

8.4 Region 3: $\sigma(c,h)$ for Irreducible $M(c,h)$ when $c<1$ and $h<\frac{c-1}{24}$

Lemma 8.4.1. *([KR87], p. 91) The region $c<1, h>\frac{c-1}{24}$ contains no reducibility curves.*

Fix c_0, h satisfying $c_0<1$ and $h<\frac{c_0-1}{24}$. Since there are no reducibility curves in this region, the signature here remains constant if we alter h, as long as $h<\frac{c_0-1}{24}$.

Since we have the formulas for $1<c<25$ we will use these formulas to calculate the signature for region 3. We will compare the signatures of the Verma modules at (c,h) in region 2 and at (c_0,h).

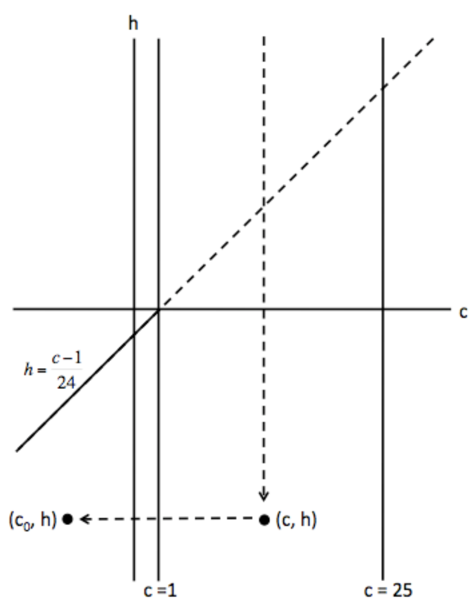


Figure 8-1: Path From the Unitary Region to $c<1$ and $h<\frac{c-1}{24}$

Lemma 8.4.2. *Let (c,h) be in region 2 and p be such that $h_{p+1,p+1}(c)<h<h_{p,p}(c)$. Then the reducibility curves that intersect the line segment from (c,h) to (c_0,h) are $h_{p+1,p+1}$, $h_{p+2,p+2}$, $h_{p+3,p+3}\ldots$.*

Proof. The only reducibility curves in regions 2 are of the form $h_{k,k}$ and region 3 has no reducibility curves. Now each curve partitions the plane into two regions: (c, h) for which $h > h_{k,k}(c)$ and (c, h) for which $h < h_{k,k}(c)$. Recall that the $h_{k,k}$ are monotonically increasing in region 2 as c decreases to 1, so any curves crossing our line segment cross exactly once. For $1 \leq k \leq p$, $h < h_{k,k}$. For every k, $h_{k,k}(c) \to 0 > h$ as $c \to 1$. Since $h_{k,k}(c) < h < 0$ for $k \geq p+1$, therefore as we travel horizontally from (c, h) to (c_0, h), we cross only the reducibility curves $h_{k,k}$ for each $k \geq p+1$. □

Theorem 8.4.3. *Given (c_0, h) in region 3,*

$$\sigma(c_0, h) = [1 + \sum_{k=1}^{\infty} 2\varepsilon(m, k, k)t^{k^2}]\varphi(t)$$

where $(c(m), h)$ lies in region 2.

Proof. Recall we chose $c \in (1, 25)$ and p so that $h_{p+1,p+1}(c) < h < h_{p,p}(c)$. The path tracing the line segment from (c_0, h) to (c, h) cross $h_{k,k}$ for $k \geq p+1$. That means that all of the reducibility curves crossed can only alter signatures at levels $(p+1)^2$ and higher. So $\sigma(c_0, h)$ and $\sigma(c_0, h)$ have the same signature for the eigenvectors of any level less than $(p+1)^2$.

Now consider what happens as $h \to -\infty$. Choose $\{h_i | i \in \mathbb{Z}^+, h_{i+1,i+1}(c) < h_i < h_{i,i}(c)\}$. As $i \to \infty$:

$$\lim_{i \to \infty} \sigma(c, h_i) = [1 + \sum_{k=1}^{\infty} 2\varepsilon(m, k, k)t^{k^2}]\varphi(t).$$

We know that $\sigma(c_0, h_i)$ has the same first i^2 coefficients of $\sigma(c, h_i)$ for large enough i. But we also know that $\sigma(c_0, h_i) = \sigma(c_0, h_j)$ for all i, j as there are no intervening reducibility curves (provided the points are in region 3). This is only possible if for large enough i $\sigma(c_0, h_i) = \lim_{j \to \infty} \sigma(c, h_j)$. Therefore,

$$\sigma(c_0, h) = [1 + \sum_{k=1}^{\infty} 2\varepsilon(m, k, k)t^{k^2}]\varphi(t).$$

□

8.5 No Intersections of Reducibility Curves at Irrational c

Note that if c is irrational, then m must also be irrational.

Proposition 8.5.1. *Reducibility curves cannot intersect at any irrational c value.*

Proof. Suppose there exist two distinct pairs of integers, p_1, q_1 and p_2, q_2 such that $h = h_{p_1,q_1}(m) = h_{p_2,q_2}(m)$. But then

$$\begin{aligned} [(m+1)p_1 - mq_1]^2 &= [(m+1)p_2 - mq_2]^2 \\ \implies (m+1)p_1 - mq_1 &= \mp[(m+1)p_2 - mq_2] \\ \implies \frac{(m+1)}{m} &= \frac{q_1 \pm q_2}{p_1 \pm p_2} \end{aligned}$$

Which implies that m (and therefore c) is rational. □

Thus if we take paths following irrational c values, we will only intersect one reducibility curve at a time which will permit us to use the difference equation. We will use this in regions 4 and 5 where our formulas become more complicated.

8.6 Region 4: $\sigma(c, h)$ for Irreducible $M(c, h)$ when $c > 25, h < 0$

Lemma 8.6.1. *Let (c, h) lie in region 4. There are finitely many reducibility curves between (c, h) and $(c, 0)$ and only finitely many reducibility curves may intersect at any given point in the region.*

Proof. Since $-\frac{1}{2} < m < 0$, we have $(m+1)p > 0$ and $-mq > 0$. Thus there are only finitely many $h_{p,q}$ curves such that $h_{p,q} > h$ for any fixed h. □

The reducibility curves divide this region into open regions. Therefore for $\delta > 0$ small enough, $\sigma(c, h) = \sigma(c + \delta, h)$.

When c is irrational, reducibility curves may be crossed one at a time if one takes a vertical path to the unitary region. Since $\sigma(c, h) = \sigma(c + \delta, h)$ for δ small enough, applying the curve crossing difference equation to each of those curves:

Proposition 8.6.2. *For (c, h) in region 4,*

$$\sigma(c, h) = \varphi(t) + \sum_{\substack{p,q \in \mathbb{Z}^+ \\ 0 > h_{p,q}(c) > h}} 2\varepsilon(m, p, q) t^{pq} \sigma(c, h + pq)(t).$$

Unfortunately, in this case, $\sigma(c, h+pq)$ may not equal $\varphi(t)$. If $h+pq > 0$, then it does. Otherwise, it may not. However, each $(c, h+pq)$ is closer to the unitary region and we find by recursion:

Theorem 8.6.3. *For (c, h) in region 4 lying on no reducibility curves,*

$$\sigma(c,h) = \sum_{\substack{(p_1,q_1),\ldots,(p_r,q_r) \\ h+\sum_{i=1}^{j-1} p_i q_i < h_{p_j,q_j}(c) < 0 \text{ for } j=1,\ldots,r}} 2^r \prod_{i=1}^{r} \varepsilon(m, p_i, q_i) t^{p_i q_i} \varphi(t).$$

Note: include $r = 0$ for the empty list which satisfies the second condition vacuously.

Corollary 8.6.4. *For (c, h) in region 4,*

$$\sigma(c,h) = Q_{c,h}(t)\varphi(t)$$

for some polynomial $Q_{c,h}(t)$.

8.7 Region 5: $\sigma(c, h)$ for Irreducible $M(c, h)$ when $c < 1, h > \frac{c-1}{24}$

Lemma 8.7.1. *Let (c, h) lie in region 5. There are infinitely many curves on the line between (c, h) and (c, ∞). Where any two reducibility curves intersect in this region, there are in fact infinitely many reducibility curves intersecting at that point.*

The same reasoning for region 4 may be applied to region 5, except the unitary region we move towards can only be reached in a limit. This gives us:

Theorem 8.7.2. *For (c, h) in region 5,*

$$\sigma(c,h) = \sum_{\substack{(p_1,q_1),\ldots,(p_r,q_r) \\ h+\sum_{i=1}^{j-1} p_i q_i < h_{p_j,q_j}(c) \text{ for } j=1,\ldots,r}} 2^r \prod_{i=1}^{r} \varepsilon(m, p_i, q_i) t^{p_i q_i} \varphi(t).$$

Like region 4, we follow a vertical path along $c + \delta$ irrational; however, since infinitely many reducibility curves intersect the line segment between (c, h) and $(c + \delta, h)$, we need to take appropriate limits along irrational $c + \delta \to c$.

Remark 8.7.3. *Comparing these equations to those in [Ken91], one notices that our formulas have 2^r while Kent has $(-2)^r$. This difference is due to the choice of difference* *equation; that is, if we had instead chosen*

$$\lim_{\delta \to 0^+} (\sigma(c, h_{p,q} + \delta) - \sigma(c, h_{p,q} - \delta)) = 2\varepsilon(m, p, q) t^{pq} \sigma(c, h + pq)$$

then our equations would match those Kent have already calculated. This change will also propagate in the ε values. I have made this change to provide a more intuitive understanding on the ε values which will be calculated later.

Chapter 9

Computing ε

In this chapter, we assume $c > 1$. Our proofs apply for $c > 1$ and we conjecture the formulas hold for $c < 1$.

A necessary step in completing the formulas in the previous chapter is understanding the values of $\varepsilon(m, p, q)$.

Since the potential values of ε are limited to ± 1, we seek means of eliminating one of the options. The asymptotic behaviour of $P(n)$, the n^{th} coefficient of $\varphi(t)$, in many cases is enough to force ε to take certain values. Those values are related to the number of reducibiility curves separating two particular points. Counting the number of such curves may be formulated as counting the number of lattice points within a certain parallelogram.

9.1 Asymptotic Approximation of $P(n)$

Recall that $\varphi(t) = \displaystyle\sum_{n=0}^{\infty} P(n)t^n$.

Theorem 9.1.1. *([Apo76], p.316) $P(n)$ is asymptotically* $\frac{e^{\left(\pi\sqrt{\frac{2n}{3}}\right)}}{4n\sqrt{3}}$.

9.2 $Q(t)\varphi(t)$

Proposition 9.2.1. *Due to our curve crossing algorithm, we can express each signature in the form*

$$\sigma(c,h) = Q_{c,h}(t)\varphi(t)$$

where $Q_{c,h}(t)$ is a power series in t with integer coefficients.

Remark 9.2.2. *Note that for $c > 1$ (i.e. all regions except 3 and 5), $Q_{c,h}(t)$ is in fact a polynomial as we only cross a finite number of reducibility curves to reach (c,h) from the unitary region. We will see that in such a situation, we may compute ε. Since region 3 does not contain any reducibility curves, we are able to compute ε in all regions except region 5. We will conjecture a formula for region 5.*

Lemma 9.2.3. *Let (c,h) lie on only the reducibility curve $h_{p,q}$. Then*

$$\lim_{\delta\to 0^+} Q_{c,h-\delta}(t) - Q_{c,h+\delta}(t) = 2\varepsilon(m,p,q)t^{pq}Q_{c,h+pq}(t).$$

Proof. Recall (formula 8.1.2) that when a path crosses a reducibility curve:

$$\lim_{\delta\to 0^+} \sigma(c,h_{p,q}(m)-\delta) = \lim_{\delta\to 0^+} \sigma(c,h_{p,q}(m)+\delta) + 2\varepsilon(m,p,q)t^{pq}\sigma(c,h_{p,q}(m)+pq).$$

Substituting in the appropriate $Q(t)$ for each signature and dividing by $\varphi(t)$ gives us the desired result. □

We have the following important theorem when $Q_{c,h}(t)$ is in fact a polynomial:

Theorem 9.2.4. *If $c > 1$, then for all h, $Q_{c,h}(1) = \pm 1$.*

Proof. Consider a vertical path π from $(c,0)$ to (c,h). Since $M(c,0)$ is unitary, $Q_{c,0}(t) = 1$. Now when π crosses the $h_{p,q}$ reducibility curve, $Q_{c,h}(t)$ changes by $2t^{pq}Q_{c,h+pq}(t)$. So $Q_{c,h}(1)$ will change by $2Q_{c,h+pq}(1)$ which is an even integer. Since $Q_{c,0}(1) = 1$ which is odd, $Q_{c,h}(1)$ is always an odd integer.

Assume that for some (c,h), $|Q_{c,h}(1)| \geq 3$.

Let $Q_{c,h}(t) = 1 + a_1t^1 + \ldots + a_mt^m$ for some integers k and a_i where $i = 1,2,\ldots,m$.

We can expand out the terms of $\sigma(c,h) = Q_{c,h}(t)\varphi(t)$. For large n, the coefficient of t^n is $P(n) + a_1P(n-1) + a_2P(n-2) + ... + a_mP(n-m)$.

However,

$$\begin{aligned}
\lim_{n\to\infty} \frac{P(n)}{P(n-k)} &= \lim_{n\to\infty} \frac{\frac{e^{\left(\pi\sqrt{\frac{2n}{3}}\right)}}{4n\sqrt{3}}}{\frac{e^{\left(\pi\sqrt{\frac{2(n-k)}{3}}\right)}}{4(n-k)\sqrt{3}}} \\
&= \lim_{n\to\infty} \frac{e^{\left(\pi\left(\sqrt{\frac{2n}{3}}-\sqrt{\frac{2(n-k)}{3}}\right)\right)}}{\frac{4n\sqrt{3}}{4(n-k)\sqrt{3}}} \\
&= \lim_{n\to\infty} \frac{e^{\left(\frac{\pi}{\sqrt{3}}\left(\frac{2k}{\sqrt{2n}+\sqrt{2n-k}}\right)\right)}}{\frac{n}{n-k}} \\
&= 1
\end{aligned}$$

as both the numerator and the denominator tend to 1.

So for large n, the the coefficient of t^n in $|Q_{c,h}(t)\varphi(t)|$ is

$$|P(n) + a_1P(n-1) + \cdots + a_mP(n-m)| \approx |Q(1)P(n)| \geq 3P(n)$$

or more importantly, larger than $P(n)$. This is impossible however, since $P(n)$ represents the the dimension of the entire $(n+h)^{th}$ eigenspace.

Therefore $|Q(1)| = 1$. □

Corollary 9.2.5. *Let π be a vertical path from $(c,0)$ to (c,h) for irrational $c > 1$. Then $Q_{c,h}(1) = (-1)^k$ where k is the number of reducibility curves k crosses.*

Proof. This follows directly from the fact that $|2\varepsilon t^{pq}Q_{c,h+pq}(t)|_{t=1}$=2 and from Proposition 8.5.1. □

Corollary 9.2.6. *If $c > 1$ is irrational, $\varepsilon(m,p,q) = (-1)^k$ where k is the number of reducibility curves on the path between $(c, h_{p,q} - \delta)$ and $(c, h_{p,q} + pq)$ for very small $\delta > 0$.*

Proof. Let i be the number of reducibility curves crossing the vertical path from $(c,0)$ to $(c, h_{p,q} - \delta)$. Then $Q_{c,h_{p,q}-\delta}(1) = (-1)^i$, $Q_{c,h_{p,q}+\delta}(1) = (-1)^{i-1}$ and $Q_{c,h_{p,q}+pq}(1) =$

$(-1)^{i-k}$. Using Lemma 9.2.3, evaluating at $t = 1$ we have:

$$\begin{aligned} Q_{c,h_{p,q}-\delta}(1) - Q_{c,h_{p,q}+\delta}(1) &= 2\varepsilon(m,p,q)1^{pq}Q_{c,h_{p,q}+pq}(1) \\ (-1)^i - (-1)^{i-1} &= 2\varepsilon(m,p,q)(-1)^{i-k} \\ 2(-1)^i &= 2\varepsilon(m,p,q)(-1)^{i-k} \\ (-1)^k &= \varepsilon(m,p,q). \end{aligned}$$

□

9.3 Computing ε for irrational $c > 1$

Recall that $h_{p,q}(m) = \frac{[(m+1)p-mq]^2-1}{4m(m+1)}$. We then note the following lemma:

Lemma 9.3.1. *For positive integers i, k such that $h_{p,q}(m) < h_{i,k}(m) < h_{p,q}(m) + pq$,*

$$(m+1)p - mq < (m+1)i - mk < |(m+1)p + mq|.$$

That is, the number of reducibility curves between $h_{p,q}(m)$ and $h_{p,q}(m) + pq$ is the number of i, k pairs satisfying the above inequality.

Proof. This follows immediately by noting that for $c > 1$, $-\frac{1}{2} < m < 0$ and thus $(m+1)p - mq$ is positive and that $h_{p,q}(m) + pq = h_{p,-q}(m)$. □

This leads us to the following theorem for $\varepsilon(m,p,q)$ for $c > 1$.

Theorem 9.3.2. *Given $c > 1$ and irrational c and positive integers p, q we have the following formulas:*

1. *If $c > 1$ and $(m+1)p + mq > 0$ then $\varepsilon(m,p,q) = (-1)^{\lfloor \frac{q}{m+1} \rfloor + 1}$.*
2. *If $c > 1$ and $(m+1)p + mq < 0$ then $\varepsilon(m,p,q) = (-1)^{\lfloor \frac{-p}{m} \rfloor}$.*

Proof. **Case 1: $c > 1$ and $(m+1)p + mq > 0$:**
By Lemma 9.2.6 we only have to calculate the parity of the number of reducibility curves between $(c, h_{p,q}(c) - \delta)$ and $(c, h_{p,q}(c) + pq)$. Consider the following diagram with lines of slope $\frac{m+1}{m}$ passing through all possible lattice points.

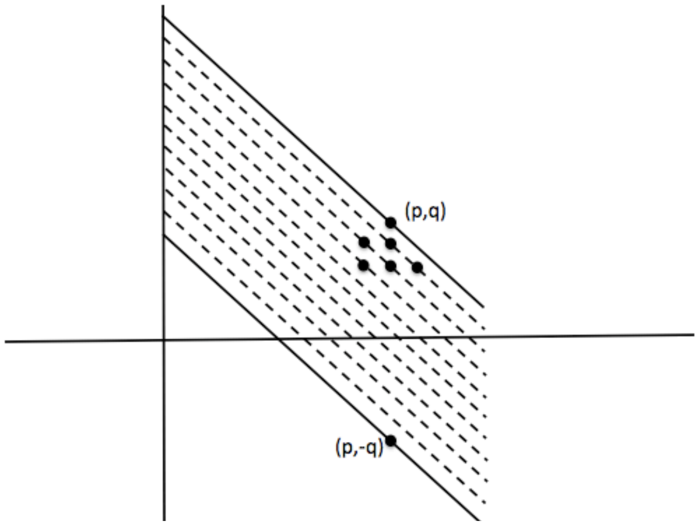


Figure 9-1: Lattice Points Between (p, q) and $(p, -q)$

Now for any lattice point (a, b) we can express the equation of the line as: $y = \frac{m+1}{m}x + \frac{k}{m}$ giving $k^2 = [(m+1)a - bm]^2$. We can see immediately that:

$$h_{p,q}(c) - \delta < h_{a,b}(c) < h_{p,q}(c) + pq$$

if and only if

1. $(a, b) = (p, q)$

2. $[(m+1)p + mq]^2 < k^2 < [(m+1)p - mq]^2$.

The second condition holds when the line passing through (a, b) lies between the line passing through (p, q) and the line passing through $(p, -q)$.

Thus we only need to determine the parity of the number of first quadrant lattice points lying between the two lines (including (p,q)). We divide up the areas as shown in the following diagram:

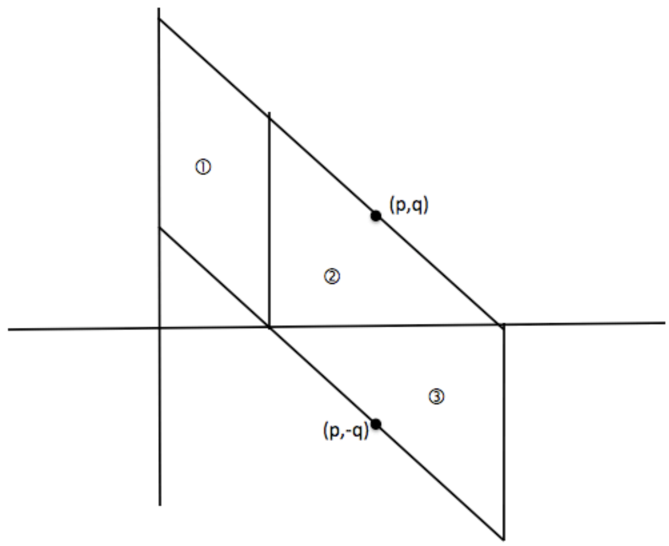


Figure 9-2: Computing the Number of Reducibility Curves Between $h_{p,q}$ and $h_{p,-q}$, Case 1

Area 1: Note that each vertical lattice line segment has length $2q$. Since the slope is irrational, neither end point can lie on a lattice point. Therefore we must have an even number of lattice points ($2q$ lattice points per line segment), so the number of lattice points in the entire area has an even parity.

Areas 2 and 3 are symmetric by a 180° rotation around the point $(p, 0)$. Now since the lines have slope $\frac{m+1}{m}$ and pass through points (p, q) and $(p, -q)$, the x-intercepts are $p \pm \frac{qm}{m+1}$ so there are $2\lfloor \frac{qm}{m+1} \rfloor + 1$ vertical lattice lines passing through the parallelogram. There are, therefore, $2q[2\lfloor | \frac{qm}{m+1} | \rfloor + 1] + 1 = 2q[2\lfloor \frac{-qm}{m+1} \rfloor + 1] + 1$ lattice points in the parallelogram. (The $+1$ is due to the fact that there are $2q + 1$ lattice points on the $x = p$ vertical line).

We can now calculate the parity of the number of lattice points in area 2. We will combine areas 2 and 3 and divide by 2.

Since each vertical lattice line passes through the x-axis, we subtract $2\lfloor \frac{qm}{m+1} \rfloor + 1$ from the total number of lattice points in the parallelogram. Of these, half will lie in area 2. So the number of lattice points in area 2 is:

$$\begin{aligned}
\frac{2q[2\lfloor\frac{-qm}{m+1}\rfloor+1]+1-[2\lfloor\frac{-qm}{m+1}\rfloor+1]}{2} &= q[2\lfloor\frac{-qm}{m+1}\rfloor+1]-\lfloor\frac{-qm}{m+1}\rfloor \\
&= (2q-1)\lfloor\frac{-qm}{m+1}\rfloor+q \\
&\equiv \lfloor\frac{-qm}{m+1}\rfloor+q \\
&\equiv \lfloor\frac{q}{m+1}-q\rfloor+q \\
&\equiv -\lfloor\frac{q}{m+1}\rfloor \\
&\equiv \lfloor\frac{q}{m+1}\rfloor \pmod 2.
\end{aligned}$$

Therefore $\varepsilon(m,p,q)=(-1)^{\lfloor\frac{q}{m+1}\rfloor}$.

Case 2: $c>1$ **and** $(m+1)p+mq<0$:

We have a similar argument here, this time noting that $h_{p,-q}=h_{-p,q}$. We instead count the lattice points between the lines passing through (p,q) and $(-p,q)$. We now have the following diagram:

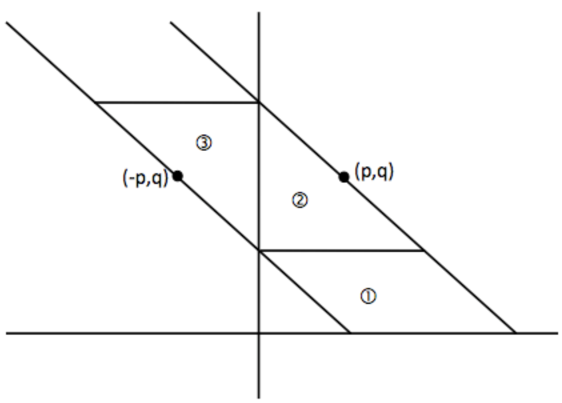


Figure 9-3: Computing the Number of Reducibility Curves Between $h_{p,q}$ and $h_{-p,q}$, Case 2

Following the same argument, but this time with horizontal lattice lines (each contain

$2q$ lattice points) and y-intercepts, we get that the the number of lattice points is:

$$\begin{aligned}\frac{2p[2\lfloor\frac{-p(m+1)}{m}\rfloor+1]+1-[2\lfloor\frac{-p(m+1)}{m}\rfloor+1]}{2} &\equiv (2p-1)\lfloor\frac{-p(m+1)}{m}\rfloor+p\\ &\equiv \lfloor\frac{-p(m+1)}{m}\rfloor+p\\ &\equiv \lfloor\frac{-p}{m}\rfloor \pmod 2.\end{aligned}$$

Therefore $\varepsilon(m,p,q)=(-1)^{\lfloor\frac{-p}{m}\rfloor}$. □

9.4 Formulas for ε and σ for Irreducible $M(c,h)$ by Region

9.4.1 ε and σ for Region 1: $c>1, h>0$

Theorem 9.4.1. *1. There are no reducibility curves in this region, hence ε is not defined.*

2. $\sigma(c,h)=\varphi(t)=\prod_{n=1}^{\infty}(1-t^n)^{-1}$.

9.4.2 ε and σ for Region 2: $1<c<25, h<0$

Theorem 9.4.2. *(cf. [Ken91] (11))*

1. *If (c,h) lies on a single reducibility curve $h_{p,p}$ in region 2, then*

$$\varepsilon(m,p,p)=(-1)^p.$$

2. *For (c,h) in the region corresponding to an irreducible Verma module with $h_{p+1,p+1}(c)<h<h_{p,p}(c)$,*

$$\sigma(c,h)=[1+\sum_{k=1}^{p}2(-1)^k t^{k^2}]\varphi(t).$$

Proof. The number of reducibility curves between (c,h) and $(c,0)$ is p by Lemma 8.3.3. Thus by Corollary 9.2.5, $\varepsilon(m,p,p)=(-1)^p$. Substituting into Theorem 8.3.4 gives the second formula. □

9.4.3 ε and σ for Region 3: $c < 1, h < \frac{c-1}{24}$

Theorem 9.4.3. *(cf. [KR87] p.91)*

1. *There are no reducibility curves in this region, hence ε is not defined.*
2. *For (c, h) in region 3,*

$$\sigma(c,h) = [1 + \sum_{k=1}^{\infty} 2(-1)^k t^{k^2}]\varphi(t) = \prod_{n=1}^{\infty}(1+t^n)^{-1}.$$

Proof. Substituting values for ε for region 2 into Theorem 8.4.3 gives us the first equality. □

9.4.4 ε and σ for Region 4: $c > 25, h < 0$

Theorem 9.4.4. *(cf. [Ken91] (10), (21))*

1. *For (c, h) in region 4, there are finitely many reducibilty curves crossing the vertical path from $(c, 0)$ to (c, h). Here,*

$$\varepsilon(m,p,q) = \begin{cases} (-1)^{\lfloor \frac{-p}{m} \rfloor} & \text{if } (m+1)p - mq > 0 \\ (-1)^{\lfloor \frac{q}{m+1} \rfloor + 1} & \text{if } (m+1)p - mq < 0. \end{cases}$$

2. *For (c, h) in region 4 lying on no reducibility curves,*

$$\sigma(c,h) = \sum_{\substack{(p_1,q_1),\ldots,(p_r,q_r) \\ h + \sum_{i=1}^{j-1} p_i q_i < h_{p_j,q_j}(c) < 0 \text{ for } j=1,\ldots,r}} 2^r \prod_{i=1}^{r} \varepsilon(m, p_i, q_i) t^{p_i q_i} \varphi(t)$$

where $\varepsilon(m, p_i, q_i)$ is given above.

9.5 Conjecture for ε in Region 5

We end this chapter with the conjecture for the cases where $c < 1$:

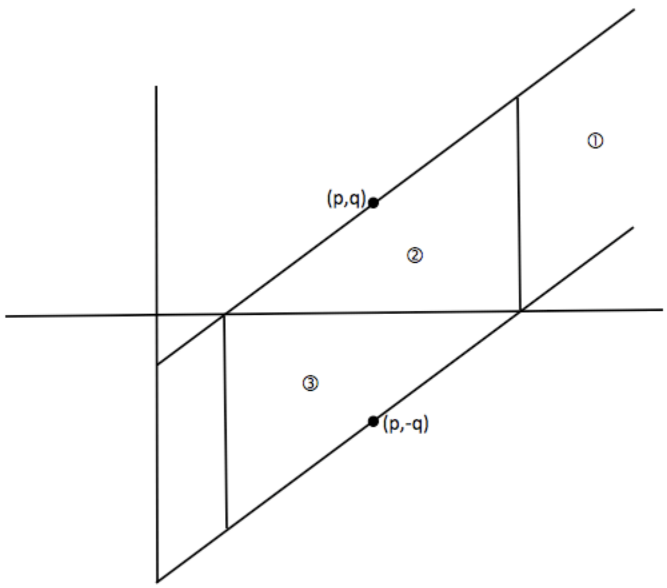
Conjecture 9.5.1. *([Ken91] (10))*
The formulas are equivalent for $c < 1$. That is:

1. *If $c < 1$ and $(m+1)p + mq > 0$ then $\varepsilon(m, p, q) = (-1)^{\lfloor \frac{q}{m+1} \rfloor + 1}$.*

2. *If $c < 1$ and $(m+1)p + mq < 0$ then $\varepsilon(m, p, q) = (-1)^{\lfloor \frac{p}{m} \rfloor}$.*

The diagrams for these cases (if we were to try to follow the same logic) are:

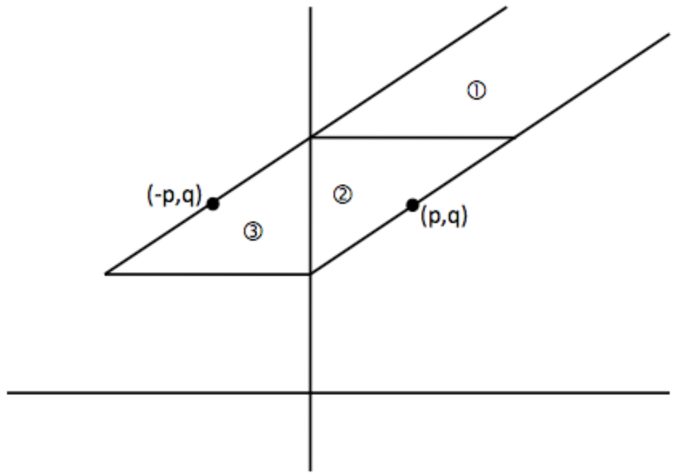
Case 1: $c < 1$ and $(m+1)p + mq > 0$



Giving $\varepsilon(m, p, q) = (-1)^{\lfloor \frac{q}{m+1} \rfloor + 1}$.

Figure 9-4: Computing the Number of Reducibility Curves Between $h_{p,q}$ and $h_{p,-q}$: $c < 1$

Case 2: $c < 1$ and $(m+1)p + mq < 0$



Giving $\varepsilon(m,p,q) = (-1)^{\lfloor \frac{-p}{m} \rfloor}$.

Figure 9-5: Computing the Number of Reducibility Curves Between $h_{p,q}$ and $h_{-p,q}$: $c < 1$

We arrive at the equivalent formula for $\sigma(c,h)$ except for the requirement that $h < 0$. That is:

$$\sigma(c,h) = \sum_{\substack{(p_1,q_1),\ldots,(p_r,q_r) \\ h+\sum_{i=1}^{j-1} p_i q_i < h_{p_j,q_j}(c) \text{ for } j=1,\ldots,r}} 2^r \prod_{i=1}^{r} \varepsilon(m,p_i,q_i) t^{p_i q_i} \varphi(t)$$

where $\varepsilon(m,p_i,q_i)$ is given above.

The issue with this method is area 1, while still having an even number of lattice points per line segment, the total number of lattice points is now infinite. However, we will outline a method for future study that should rectify this problem.

Remark 9.5.2. *At this point, we shall return to Remark 8.7.3 to note that the ε values calculated in this chapter all contain a factor of -1 as to those given by [Ken91]. These two changes combine to give the same signature formulas provided by Kent. We will thus appropriately modify Kent's formulas in future chapters.*

Comparing these signature to those of Kent's, we see that in region 2 we have a closed formula while Kent has a difference formula. Furthermore, in region 4, we have a closed formula while Kent provided a description (repeated use of the difference formula).

Chapter 10

Equivalence With Kent's Formula in Region 5

We would like to show that the previous section's formulas match Kent's formulas.

10.1 Kent's Formula

Theorem 10.1.1. *([Ken91] (21)) Kent's formula is for rational m. Let r, s be coprime integers satisfying $m = \frac{r}{s}$.*

Define $h(a, m) = \frac{(a^2-s^2)}{4r(r+s)}$

Then for $h(a-1, m) < h < h(a, m)$,

$$\sigma(h, c(m)) = \varphi(t)(1 + \sum_{(p_1,q_1),\ldots,(p_r,q_r)} 2^r \prod_{i=1}^{r} \varepsilon^{+}(m, p_i, q_i) t^{p_i q_i})$$

where:

$$\varepsilon^{\pm}(m, p, q) = \lim_{\delta \to 0^+} \varepsilon(m \pm \delta, p, q)$$

and the sum is over all finite sequences of pairs of positive integers satisfying:

1. *$|(r + s)p_1 - rq_1| \geq a$*
2. *For $i \geq 1, |(r + s)p_{i+1} - rq_{i+1}| > (r + s)p_i + rq_i$ or $(r + s)p_{i+1} - rq_{i+1} = -((r + s)p_i + rq_i)$.*

Now let us see how this is equivalent to our formula 5.

Since $h(a-1,m) < h < h(a,m)$, (c,h) does not lie on a reducibility curve. Therefore, we could follow a path from $\infty \to h$ along $m+\delta$, irrational. As $\delta \to 0^+$, this path will give us the same signature.

Recall for region 5 we had

$$\sigma(c,h) = \sum_{\substack{(p_1,q_1),\ldots,(p_r,q_r) \\ h+\sum_{i=1}^{j-1} p_i q_i < h_{p_j,q_j}(c) \text{ for } j=1,\ldots,r}} 2^r \prod_{i=1}^{r} \varepsilon^+(m,p_i,q_i) t^{p_i q_i} \varphi(t).$$

Note that we changed the formula slightly from before in that we now use $\varepsilon^+(m,p,q)$ due to the limiting argument. What now remains to be shown is that the polynomial in t is equivalent in both formulas. That is, we need to show that we sum over the same set. In order to do so, we require the following result:

10.2 The Ordering of Reducibility Curves Around an Intersection Point

Theorem 10.2.1. *The reducibility curves that intersect at point (c,h) have slopes that are ordered by (from largest to smallest):*
If $m < 0$, or $(c > 1)$ then the slopes are greatest with greatest q values.
If $m > 0$, or $(c < 1)$ then the slopes are greatest first by
All $(p_i(m+1) - q_i m) < 0$ greatest with greatest q-values followed by
All $(p_i(m+1) - q_i m) > 0$ greatest with smallest q values.

Proof. We follow a perturbed path from $(c+\delta,\infty)$ to $(c+\delta,h)$ along $c+\delta$, irrational. Taking $\delta \to 0^+$, in the limit, the path will cross reducibility curves in an order determined by the slopes of the tangents to each reducibility curve at (c,h).

Let us look at an intersection point (c,h) of the reducibility curves $\{h_{p_i,q_i}(c)\}$. We examine the slopes of the reducibility curves at (c,h) by considering the $h_{p,q}(m)$ values. Recall that $h_{p,q}(m) = \frac{[(m+1)p-mq]^2-1}{4m(m+1)}$. We reparamaterize with $x = \frac{m+1}{m}$ ([DFMS97],p.208) so

$$h_{p,q}(x) = \frac{1}{4}(p^2-1)x + \frac{1}{4x}(q^2-1) - \frac{1}{2}(pq-1)$$

Now since these reducibility curves intersect,

$$\begin{aligned} h = \frac{[(m+1)p_i - mq_i]^2 - 1}{4m(m+1)} &= \frac{[(m+1)p_j - mq_j]^2 - 1}{4m(m+1)} \\ \Longrightarrow (m+1)p_i - mq_i &= \pm[(m+1)p_j - mq_j] \end{aligned}$$

That is, $[(m+1)p_i - mq_i]^2$ is constant for all of the curves at the intersection point.

If we then take the derivative with respect to x, we see that

$$\begin{aligned} h'_{p,q}(x) &= \frac{1}{4}(p^2-1) - \frac{1}{4x^2}(q^2-1) \\ &= \frac{1}{4x^2}[(p^2x^2 - q^2) - (x^2-1)] \\ &= \frac{1}{4x^2}[(px-q)(px+q) - (x^2-1)] \\ &= \frac{1}{4x^2}\left[\frac{(p(m+1)-qm)(p(m+1)+qm)}{m^2} - (x^2-1)\right] \\ &= \frac{1}{4x^2}\left[\frac{(p(m+1)-qm)^2 + (p(m+1)-qm)2qm)}{m^2} - (x^2-1)\right] \end{aligned}$$

Notice that $\frac{1}{4x^2}\left[\frac{(p(m+1)-qm)^2 + (\mathbf{p(m+1)-qm)2qm)}}{m^2} - (x^2-1)\right]$ is fixed for all p,q values except for $(p(m+1)-qm)2qm)$. So $h'_{p,q}(x)$ is greatest when $(p(m+1)-qm)2qm$ is greatest.

Now since $\frac{\partial x}{\partial m} = -\frac{1}{m^2}$,

$$\frac{\partial(h_{p_i,q_i})}{\partial m} = -\frac{1}{m^2}\left(\frac{1}{4x^2}\left[\frac{(p(m+1)-qm)^2 + (p(m+1)-qm)2qm)}{m^2} - (x^2-1)\right]\right)$$

Therefore, $\frac{\partial(h_{p,q})}{\partial m}$ is greatest when $(p(m+1)-qm)2qm$ is smallest.

Also, $c = 1 - \frac{6}{m(m+1)} \Rightarrow \frac{\partial c}{\partial m} = \frac{6(2m+1)}{(m(m+1))^2} > 0 \Rightarrow \frac{\partial m}{\partial c} > 0$.

Therefore, $\frac{\partial(h_{p,q})}{\partial c}$ is greatest when $(p(m+1)-qm)2qm$ is smallest.

Region 4: $c > 25$, $-\frac{1}{2} < m < 0$

$(m+1)p_i - mq_i > 0$ so $p_i > p_j \Longrightarrow q_i > q_j$ as $[(m+1)p_i - mq_i]^2$ is constant. But since $(p_i(m+1) - q_im)$ and m are fixed, $\frac{\partial h_{p,q}}{\partial m}$ is completely dependent on q_i. Thus $\frac{\partial(h_{p_i,q_i})}{\partial m}$ is greater when q_i is greater and so the path from $(c+\delta, 0) \to (c+\delta, h)$ intersects the reducibility curve with the largest q_i first.

Region 5: $c < 1$, $m > 0$

We have a similar situation but there are now two subcases: $p_i(m+1) - q_i m > 0$ and $p_j(m+1) - q_j m < 0$. Note that in both cases, $|p_i(m+1) - q_i m| > 0$ is constant. Now it is clear that the entire first subcase has a smaller slope then the entire second subcase as $(p_i(m+1) - q_i m)2qm > 0 > (p_j(m+1) - q(j)m)2qm$.

Subcase A: $(p(m+1) - qm) > 0$

$(p_i(m+1) - q_i m) > 0$ and $m > 0$ so $\frac{\partial h_{p,q}}{\partial m}$ is greater when q is smaller

Subcase B: $(p(m+1) - qm) < 0$

$(p_j(m+1) - q_j m) < 0$ and $m > 0$ so $\frac{\partial h_{p,q}}{\partial m}$ will be greater when q is greater.

Thus the order of the p, q pairs by greatest to smallest slope around $m > 0$ intersection points is:

1. All $(p(m+1) - qm) < 0$ ordering by largest q values (only if $c < 1$)followed by
2. All $(p(m+1) - qm) > 0$ ordering by smallest q values.

$\square$

Let us now examine the restrictions on Kent's sequences of integers.

10.3 Kent's Sequences of Integers

We will show the equivalence by looking at the expansion of Region 5's formula and finding a 1-1 correspondence with the indexing set for Kent's formula.

First, let us look at all sequences in Kent's formula ([Ken91] (21)). The restrictions on the sequences are:

1. $|(r+s)p_1 - rq_1| \geq a$
2. For $i \geq 1, |(r+s)p_{i+1} - rq_{i+1}| > (r+s)p_i + rq_i$ or $(r+s)p_{i+1} - rq_{i+1} = -((r+s)p_i + rq_i)$.

Recall that $h_{p,q}(m) = \frac{((m+1)p-mq)^2-1}{4m(m+1)}$ and that $h_{p,q}(m) + pq = h_{p,-q}(m)$. Let's look closely at these restrictions.

Restriction 1: $|(r+s)p_1 - rq_1| \geq a$

Since both sides are positive,

$$\begin{aligned}
|(r+s)p_1 - rq_1| &\geq a \\
\Longleftrightarrow [(r+s)p_1 - rq_1]^2 &\geq a^2 \\
\Longleftrightarrow \left[\frac{(r+s)}{s}p_1 - \frac{r}{s}q_1\right]^2 &\geq \frac{a^2}{s^2} \\
\Longleftrightarrow [(m+1)p_1 - mq_1]^2 &\geq \frac{a^2}{s^2} \\
\Longleftrightarrow \frac{[(m+1)p_1 - mq_1]^2 - 1}{4m(m+1)} &\geq \frac{\frac{a^2}{s^2} - 1}{4m(m+1)}
\end{aligned}$$

But $\frac{\frac{a^2}{s^2}-1}{4m(m+1)} = \frac{a^2-s^2}{4m(m+1)s^2} = h(a,m)$

$$\therefore h_{p_1,q_1}(m) \geq h(a,m)$$

So restriction 1 can be understood as any reducibility curve above h since $h(a,m)$ is the first reducibility curve above h.

Restriction 2. For $i \geq 1, |(r+s)p_{i+1} - rq_{i+1}| > (r+s)p_i + rq_i$ OR $(r+s)p_{i+1} - rq_{i+1} = -((r+s)p_i + rq_i)$.

Conducting similar calculations to those above, we can reinterpret the restriction as:

$$h_{p_{i+1},q_{i+1}(m)} > h_{p_i,-q_i}(m) \text{ OR } h_{p_{i+1},q_{i+1}}(m) = -h_{p_i,-q_i}(m)$$

Now the first condition is expected, as these correspond to the curves that have h values strictly greater then $h_{p_i,q_i} + p_iq_i$. However, the second condition requires a bit more attention.

Recall that we evaluated the signatures by following a path along $m+\delta$, $\delta \to 0^+$. Since we have perturbed c, the curves no longer intersect. So we need to examine where $h_{p_i,q_i}(m+\delta) + p_iq_i$ lies in relation to the other reducibility curves around $h_{p_i,q_i}(m+\delta) + p_iq_i$. We know that $h_{p_i,q_i}(m) + p_iq_i = h_{p_i,-q_i}(m)$. Now $(p_i(m+1) - (-q_i)m) > 0$ and since $-q_i$ is negative, $h_{p_i,q_i}(c+\delta) > h_{r,s}$ where r, s falls into subcase

B. However, since $(p_i(m+1) - (-q_i)m) > 0$, $h_{p_i,q_i}(c+\delta) < h_{x,y}$ where x, y falls into subcase A. Therefore, any reducibility curve $h_{p,q}$ satisfying $(p(m+1) - (q)m) < 0$ is above $h_{p_i,q_i}(m) + p_i q_i$ and so must be included in the summation.

Thus we are summing over the same sets.

Chapter 11

Signatures for Irreducible Highest Weight Modules $L(c, h)$ over Vir

In this chapter, we see that $\sigma L(c, h)$ can be expressed as (limits of) averages of signature characters of irreducible Verma modules. (We abuse notation and insert L's and M's for clarity.)

11.1 Case II_+:

Theorem 11.1.1. *Case: II_+ If (c, h) lies on only one reducibility curve $h_{p,q}$ then*

$$\sigma L(c, h) = \lim_{\delta \to 0^+ \,:\, M(c, h\pm\delta) irred} \frac{\sigma M(c, h + \delta) + \sigma M(c, h - \delta)}{2}.$$

Proof. Recall the equations (8.1) give when normalized and altered to accommodate the more general setting of Corollary 8.1.2:

$$\begin{aligned}
\lim_{\delta \to 0^+} \sigma M(c, h + \delta) &= \sigma \langle \cdot, \cdot \rangle^0 + t^{pq} \sigma \langle \cdot, \cdot \rangle^1 = \sigma \langle \cdot, \cdot \rangle^0 + t^{pq} \varepsilon \; \sigma M(c, h + pq) \\
\lim_{\delta \to 0^+} \sigma M(c, h - \delta) &= \sigma \langle \cdot, \cdot \rangle^0 - t^{pq} \sigma \langle \cdot, \cdot \rangle^1 = \sigma \langle \cdot, \cdot \rangle^0 - t^{pq} \varepsilon \; \sigma M(c, h + pq).
\end{aligned}$$

We take the limit over $M(c, h \pm \delta)$ irreducible–however, note that since only finitely many reducibility curves affect a given level, we can simply take the limit $\delta \to 0^+$ (except we have not defined σ for reducible $M(c, h)$). We see that from averaging the two equations,

$$\sigma L(c,h) = \langle \cdot, \cdot \rangle^0 = \lim_{\delta \to 0^+} \frac{\sigma M(c,h+\delta) + \sigma M(c,h-\delta)}{2}.$$

$\square$

Remark 11.1.2. *We can generalize the formulas for $\sigma(c,h)$ by looking at the full Jantzen filtration, i.e.*

$$\lim_{\delta \to 0^+} ch_s M(h+\delta) = ch_s \langle \cdot, \cdot \rangle^0 + ch_s \langle \cdot, \cdot \rangle^1 + ch_s \langle \cdot, \cdot \rangle^2 + \cdots$$

and

$$\lim_{\delta \to 0^+} ch_s M(h-\delta) = ch_s \langle \cdot, \cdot \rangle^0 - ch_s \langle \cdot, \cdot \rangle^1 + ch_s \langle \cdot, \cdot \rangle^2 - \cdots .$$

If, for example, our module structure is:

$$\begin{tikzcd}[column sep=small]
& M(c,h+\alpha_2\beta_2) \arrow[r, dash] \arrow[ddr, dash] & M(c,h+\alpha_1\beta_1+\alpha_2'\beta_2') \arrow[r, dash] \arrow[ddr, dash] & M(c,h+\alpha_4\beta_4) \arrow[r, dotted, dash] \arrow[ddr, dotted, dash] & {} \\
M(c,h) \arrow[ur, dash] \arrow[dr, dash] & & & & \\
& M(c,h+\alpha_1\beta_1) \arrow[r, dash] \arrow[uur, dash] & M(c,h+\alpha_1\beta_1+\alpha_1'\beta_1') \arrow[r, dash] \arrow[uur, dash] & M(c,h+\alpha_3\beta_3) \arrow[r, dotted, dash] \arrow[uur, dotted, dash] & {}
\end{tikzcd}$$

then we have

$$\lim_{\delta \to 0^+} ch_s M(c,h+\delta) = ch_s L(c,h) + \varepsilon_1 ch_s L(c,h+\alpha_1\beta_1) + \varepsilon_2 ch_s L(c,h+\alpha_2\beta_2) + \cdots$$

for some $\varepsilon_i = \pm 1$.

Now we can compute the signatures at intersections of reducibility curves.

11.2 Case $III_{\pm}$:

Theorem 11.2.1. *Case $III_{\pm}$: If (c, h) is such that $M(c, h)$ has the module structure:*

$$\begin{tikzcd}
 & M(c,h+p_2q_2) \arrow[r, dash] \arrow[ddr, dash] & M(c,h+p_1q_1+p'_2q'_2) \arrow[r, dash] \arrow[ddr, dash] & M(c,h+p_4q_4) \arrow[r, dotted, dash] \arrow[ddr, dotted, dash] & {} \\
M(c,h) \arrow[ur, dash] \arrow[dr, dash] & & & & \\
 & M(c,h+p_1q_1) \arrow[r, dash] \arrow[uur, dash] & M(c,h+p_1q_1+p'_1q'_1) \arrow[r, dash] \arrow[uur, dash] & M(c,h+p_3q_3) \arrow[r, dotted, dash] \arrow[uur, dotted, dash] & {}
\end{tikzcd}$$

then

$$\sigma(c,h) = \lim_{\delta \to 0 : c+\delta \ irrat} \frac{\sigma(c+\delta, h_{p_1,q_1}(c+\delta)) + \sigma(c-\delta, h_{p_1,q_1}(c-\delta))}{2}.$$

Proof. We will prove the formula for Case III_+ with the understanding that case III_- is the same but with a limiting argument.

For brevity, we will refer to the Verma modules as $M(i) = M(c, h+n_i)$ as shown in the following diagram.

$$\begin{tikzcd}
 & M(2) \arrow[r, dash] \arrow[ddr, dash] & M(4) \arrow[r, dash] \arrow[ddr, dash] & M(6) \arrow[r, dotted, dash] \arrow[ddr, dotted, dash] & {} & {} \arrow[dr, dash] & \\
M(0) \arrow[ur, dash] \arrow[dr, dash] & & & & & & M(m) \\
 & M(1) \arrow[r, dash] \arrow[uur, dash] & M(3) \arrow[r, dash] \arrow[uur, dash] & M(5) \arrow[r, dotted, dash] \arrow[uur, dotted, dash] & {} & {} \arrow[ur, dash] &
\end{tikzcd}$$

Consider the following diagram. The curves labelled by h_{p_1,q_1} and h_{p_2,q_2} correspond to $M(1)$ and $M(2)$, respectively. The dotted curves represent other reducibility curves. We focus on $M(1)$ and $M(2)$ in particular because they are maximal Verma submodules of $M(c, h)$. Note that there are no reducibility curves (other than those that pass through (c, h)) in a neighbourhood of (c, h).

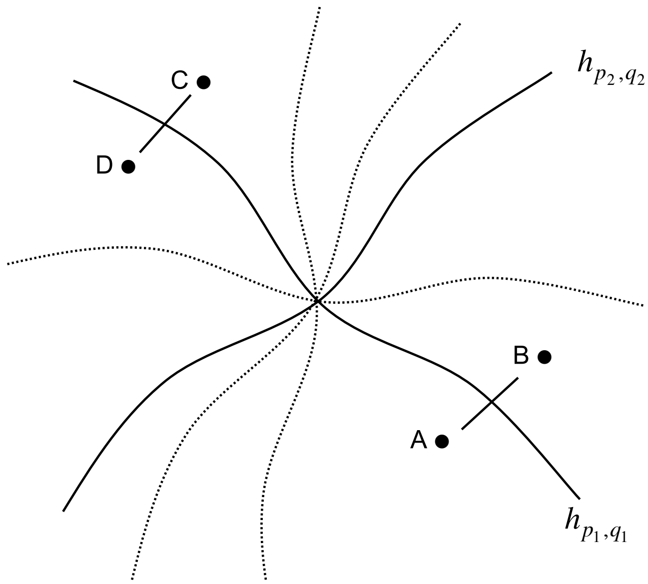


Figure 11-1: Reducibility Curves in the Neighbourhood of (c, h)

We consider points A,B and C,D which straddle the reducibility curve h_{p_1,q_1} on different sides of h_{p_2,q_2}. In any open region containing no reducibility curves, the signature must remain the same. Considering the Jantzen filtration, we therefore know that

$$\sigma(M(A)) = \sigma(L(0))+\varepsilon_1 t^{n_1}\sigma(L(1))+\varepsilon_2 t^{n_2}\sigma(L(2))+\varepsilon_3 t^{n_3}\sigma(L(3))+\cdots+\varepsilon_m t^{n_m}\sigma(L(m))$$

for some $\varepsilon_i = \pm 1$ by Theorem 7.3.1. There is an analogous formula for $\sigma(M(B))$.

Now we know that the path from A to B crosses h_{p_1,q_1} and the signature must therefore change by the signature of some $M(c_1, h_{p_1,q_1}(c_1)+p_1q_1)$ which may be taken so that there is a path from it to $M(1) = M(c, h + p_1q_1)$ crossing no reducibility curves. Considering the Jantzen filtration of $M(1)$, the signature character of $M(c_1, h_{p_1,q_1}(c_1)+p_1q_1)$ must be a ± 1-linear combination of the irreducible constituents of $M(1)$. We see therefore that the signature of $M(B)$ may be obtained from the signature of $M(A)$ by switching all of the coefficients in front of $L(1)$, $L(3)$, $L(4)$, $\ldots$, $L(m)$. Thus

$$\sigma(M(B)) = \sigma L(0)-\varepsilon_1 t^{n_1}\sigma(L(1))+\varepsilon_2 t^{n_2}\sigma(L(2))-\varepsilon_3 t^{n_3}\sigma(L(3))-\cdots-\varepsilon_m t^{n_m}\sigma(L(m)).$$

Note that ε_2 did not change signs, as $L(2)$ is not a composition factor of $M(1)$. Adding these two equations together, we have

$$\sigma(M(A)) + \sigma(M(B)) = 2\sigma(L(0)) + 2\varepsilon_2 t^{n_2}\sigma(L(2)).$$

Applying the same argument to $M(C)$ and $M(D)$, we have

$$\sigma(M(C)) + \sigma(M(D)) = 2\sigma(L(0)) + 2\varepsilon_2' t^{n_2}\sigma(L(2)).$$

However, $\varepsilon_2 = -\varepsilon_2'$ since:

1. A, B and C, D lie on opposite sides of h_{p_2,q_2}.
2. No reducibility curve other then h_{p_2,q_2} can change the coefficient of $\sigma(L(2))$ as $M(2)$ is a maximal Verma submodule.
3. Crossing h_{p_2,q_2} does change ε_2 since when you cross curves one at a time the signature changes by the signature of the maximal submodule.

Thus $\varepsilon_2 = -\varepsilon_2'$.

Now if we sum the two equations together, we arrive at

$$\frac{\sigma(M(A)) + \sigma(M(B)) + \sigma(M(C)) + \sigma(M(D))}{4} = \sigma(L(0)).$$

Applying the previous theorem, we are done. □

This theorem allows us, with the computations from chapter 9, to compute the signatures at most intersection points and in particular, those of minimal models.

11.3 Case $III_{\pm}^{o}$ and Case $III_{\pm}^{oo}$

These cases are even more straightforward:

Theorem 11.3.1. *Given $M(c,h)$ with the module structure of Case $III_{\pm}^{o}$ or Case $III_{\pm}^{oo}$, let $h_{p,q}$ be the reducibility curve of minimal level on which (c,h) lies. Then*

$$\sigma L(c,h) = lim_{\delta\to 0}\frac{\sigma M(c+\delta, h_{p_1,q_1}(c+\delta)) + \sigma M(c-\delta, h_{p,q}(c-\delta))}{2}.$$

Proof. Again, we will abbreviate our notation for the Verma submodules. For submodule structure, we have $M(0) \supset M(1) \supset M(2) \supset \cdots$. We apply a similar argument as in the previous case except this time note that $\sigma((A)) + \sigma((B)) = 2\sigma(L(0))$ and so we are done. $\square$

Remark 11.3.2. *Note that although the Jantzen filtration for Case $III_{\pm}^{oo}$ only has even levels, we perturb from (c, h) and cross the curves one at a time, and the Jantzen filtration after the perturbation has odd levels.*

Chapter 12

Modular Group Actions

In this chapter we briefly outline the motivation for studying modular group actions and discuss the action of $\mathcal{T}$ on signatures. The action of $\mathcal{S}$ is left for future work.

12.1 The Torus

One means of constructing a torus is by taking a unit square, gluing two opposite sides to make a cylinder, and then gluing the remaining two opposite sides (now circles) to form a torus. Described algebraically, a torus may be viewed as a quotient:

Definition 12.1.1. *A **torus** can be identified with the quotient of a plane by a lattice; that is if ω_1 and ω_2 are two vectors forming a basis for $\mathbb{R}^2$, define $\Lambda := \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$. Then a torus is in bijection with $\mathbb{R}^2/\Lambda$.*

Equivalently, choose $\omega_1, \omega_2 \in \mathbb{C}$ which are linearly independent over $\mathbb{R}$. Define the lattice Λ as above. $\mathbb{C}/\Lambda$ is in bijection with a torus.

Note that this makes sense since $k = k+\omega_1 = k+\omega_2$ for $k \in \mathbb{R}^2$ under the quotient while such an equality holds by considering our gluing.

Definition 12.1.2. $\tau = \frac{\omega_2}{\omega_1} \in \mathbb{C}$ *is called the **modular parameter**.*

12.2 Modular Invariance

Conformal field theories play an important role in theoretical physics. However, we would like the lattice to define the theory, not the choice of basis. This means that any formulas must remain invariant under a change of basis. Given this, consider two bases ω_1, ω_2 and ω_1', ω_2' which form the same lattice.

Consider the automorphisms of this lattice. Now since ω_1' and ω_2' are on the lattice, ω_1' and ω_2' are integral linear combinations of ω_1 and ω_2. So there exist $a, b, c, d \in \mathbb{Z}$ such that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Likewise, there must also be a set of integers such that

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}.$$

Substituting this into the previous equation we have

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

Since $\omega_1', ' \omega_2'$ are linearly independent in $\mathbb{R}$. But since $a, b, c, d, a', b', c', d'$ are all integers, $ad - bc$ and $a'd' - b'c'$ are both integers. Thus, $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \pm 1$ Now interchanging w_1' and w_2' will change the determinant by a factor of -1 (but leaves the lattice the same), thus we will study those matrices with determinant 1. Note that multiplying by negative the identity matrix does not change the determinant. The set of all integral matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant 1 is $SL(2, \mathbb{Z})$. The group we are interested in is $SL(2, \mathbb{Z})/\mathbb{Z}_2$, which is also called $PSL(2, \mathbb{Z})$.

12.3 Modular Group

Definition 12.3.1. *([DFMS97], p.339) The group $PSL(2,\mathbb{Z})$ is known as the* ***modular group*** Γ*, and can be generated by the two matrices*

$$\begin{aligned} T &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ S &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

Expressed in terms of generators and relations,

$$\Gamma = \{S, T : S^2 = I, (ST)^3 = I\}.$$

Definition 12.3.2. *Given* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$*, the corresponding* ***fractional linear transformation*** *of* $\mathbb{C}$ *is*

$$z \mapsto \frac{az+b}{cz+d}.$$

This defines a group action of Γ *on* $\mathbb{C}$*.*

Proposition 12.3.3. *([DFMS97] p. 338) If* $\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$*, then the modular parameters* τ' *and* τ *are related by*

$$\tau' = \frac{a\tau+b}{c\tau+d}.$$

Proposition 12.3.4. *([DFMS97] ,p.339)* T *and* S *correspond to the transformations* $\mathcal{T}$ *and* $\mathcal{S}$ *respectively which act on the modular parameter* τ *by:*

$$\begin{aligned} \mathcal{T} : \tau &\to \tau + 1 \\ \mathcal{S} : \tau &\to -\frac{1}{\tau} \end{aligned}$$

12.4 Modular Group Actions on Signature Characters

For applications to conformal field theories, we want to understand how the modular group acts on our signature formulas.

Notation 12.4.1. *Let the* ***upper half plane*** *be* $\mathbb{H} := \{z \in \mathbb{C} : Im\ z > 0\}$.

Proposition 12.4.2. *([DFMS97], p.340)* Γ *maps the upper half plane to itself.*

Since Γ acts on the upper half plane, we may view characters and signature characters as functions on the upper half plane, and then Γ acts on characters and signature characters. Our formulas involve polynomials in t; in conformal field theory, t is defined as

$$t = e^{2\pi i \tau}.$$

Given an element $g \in \Gamma$, how can we let g act on signature formulas, which are functions of t and hence functions of τ? A natural way would be to let g act on the argument τ of the function:

Definition 12.4.3. *Given* $g \in \Gamma$ *and* f *a function on* $\mathbb{H}$*, define the right action of* g *on* f *by*

$$(f.g)(\tau) = f(g.\tau).$$

Therefore, our previous definitions for $\mathcal{T}$ and $\mathcal{S}$ can be generalized to:

Definition 12.4.4. *Let* f *be a function on the upper half plane. Define*

$$\begin{aligned} \mathcal{T}(f) &= f.T \\ \mathcal{S}(f) &= f.S, \end{aligned}$$

which we observe to be consistent with Proposition 12.3.4.

Notation 12.4.5. *To simplify computations, there is a shift in the literature of* $t^{-\frac{c}{24}}$ *which we build into not just characters but also signature characters:*

1. $\chi_{c,h}(t) := t^{-\frac{c}{24}} chL(c,h)$
2. $\tilde{\chi}_{c,h}(t) := t^{-\frac{c}{24}} ch_s L(c,h)$.

We may act on $\chi_{c,h}$ and $\tilde{\chi}_{c,h}$ by the modular group Γ. It is known ([DFMS97]) that acting by $g \in \Gamma$ on $\chi_{c,h}$ returns a linear combination of $\chi_{c,h'}$'s. It suffices to understand the behaviour of $\mathcal{T}$ and $\mathcal{S}$ since they generate Γ. We wish to develop similar formulas for $\tilde{\chi}_{c,h}$'s.

We finish this chapter by computing how $\mathcal{T}$ acts on signature characters.

Lemma 12.4.6. *If k is an integer, then $\mathcal{T}(t^k) = t^k$.*

Proof. Recall that $\mathcal{T} : \tau \to \tau + 1$ and $t = e^{2\pi i \tau}$. $\square$

Corollary 12.4.7. *Polynomials in t and the formal power series $\varphi(t)$ are both unchanged under the action of $\mathcal{T}$.*

Theorem 12.4.8. $\mathcal{T}(\tilde{\chi}_{c,h}(t)) = e^{2\pi i(h-\frac{c}{24})}\tilde{\chi}_{c,h}(t)$

Proof. Recall that $\tilde{\chi}_{c,h}(t) = t^{h-\frac{c}{24}}Q_{c,h}(t)\varphi(t)$. Thus

$$\begin{aligned}
\mathcal{T}(\tilde{\chi}_{c,h}(t)) &= e^{2\pi i(\tau+1)(h-\frac{c}{24})}\mathcal{T}(Q_{c,h}(t))\mathcal{T}(\varphi(t)) \\
&= e^{2\pi i(\tau)(h-\frac{c}{24})}e^{2\pi i(h-\frac{c}{24})}Q_{c,h}(t)\varphi(t) \\
&= e^{2\pi i(h-\frac{c}{24})}e^{2\pi i\tau(h-\frac{c}{24})}Q_{c,h}(t)\varphi(t) \\
&= e^{2\pi i(h-\frac{c}{24})}\tilde{\chi}_{c,h}(t)
\end{aligned}$$

$\square$

Chapter 13

Future Work

13.1 Computing $\varepsilon(m, p, q)$ for $c < 1$

In Feigin and Fuch's paper ([FF90]), they prove that there is an antiautomorphism that takes $M(c, h) \to M(26 - c, 1 - h)$. In particular,

$$\begin{aligned} M(c,h) &\supset M(c, h + pq) \\ \Rightarrow M(26 - c, 1 - h) &\subset M(26 - c, 1 - h - pq). \\ (c, h) \text{ on } h = h_{p,q} &\iff (26 - c, 1 - h) \text{ on } h = h_{-p,q} = h_{p,-q}. \end{aligned}$$

Where $h = h_{p,q}$ detected if a Verma module has a non-trivial submodule, $h = h_{p,-q}$ detects if a Verma module has a non-trivial embedding into another Verma module. In region 5, every Verma module has finitely many (up to scaling) non-trivial embeddings into other Verma modules. Furthermore, formulas in [KR87] (eg. p. 27) indicate how invariant Hermitian forms on $M(c, h)$ and on $M(26 - c, 1 - h)$ are related.

If (c, h) lies on $h = h_{p,-q}$ so that $(c, h - pq)$ lies on $h = h_{p,q}$, the difference equation gives

$$\sigma(c, h - pq + \delta) = \sigma(c, h - pq - \delta) + 2t^{pq}\varepsilon(m, p, q)\sigma(c, h).$$

Rearranging, we find that:

$$\sigma(c, h) = \frac{\sigma(c, h - pq + \delta) - \sigma(c, h - pq - \delta)}{2t^{pq}\varepsilon(m, p, q)}.$$

13.2 Conformal Field Theory and Modular Transformations

To completely understand the action of the modular group, we must understand $\mathcal{S}$. We have the following philosophy for computing $\mathcal{S}(\tilde{\chi}_{c,h}(t))$. If $\mathcal{S}(\chi_{c,h}(t))$ is understood, we simply note that any $\tilde{\chi}_{c,h}(t)$ is a linear combination of χ's and therefore the problem reduces to understanding $\mathcal{S}(\chi_{c,h}(t))$. The solution to the latter problem is known for example in the minimal model case ([DFMS97], p. 363).

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Appendix - Listing of Virasoro Examples

This is a listing of the examples used throughout the thesis that highlight specific properties of the Virasoro algebra (and of related structures).

Definition: 2.6.4

Cartan Subalgebra: 3.0.3

Universal Enveloping Algebra: 4.2.3

Positive and Negative Roots: 5.1.2

Basis for Verma Modules: 5.2.6

Character Formula: 5.3.3

Real Form: 6.1.4

Invariant Hermitian Form: 6.2.7

Orthogonality of Weight Spaces: 6.3.1

Kac's Determinant Formula: 6.4.1

Reducibility Curves: 6.4.2

Submodule Structure and Jantzen Filtration: 7.1

Vita Auctoris

Author: Byung Kyu Chun

Birthplace: Vancouver, 1979

Educational Degrees:

- B.Math, University of Windsor, 2008
- B.Ed, Brock University, 2011
- M.Sc, University of Windsor, 2011