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Extension of First Passage Probability

Yiping Zhang
zhang1u1@uwindsor.ca

EXTENSION OF FIRST PASSAGE PROBABILITY

by

Yiping Zhang

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FIRST PASSAGE PROBABILITY

by

Yiping Zhang

APPROVED BY:

M. Belalia

Department of Mathematics and Statistics

M. Hlynka, Co-Advisor

Department of Mathematics and Statistics

P. Brill, Co-Advisor

Department of Mathematics and Statistics

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Declaration of Co-authorship / Previous Publication

I hereby declare that this major paper incorporates material that is result of joint research, as follows:

Chapter 6 includes joint research with my co-supervisors Dr. Hlynka and Dr. Brill. In all cases, the key ideas, primary contributions, interpretation, and writing were performed by the author, and the contribution of co-authors was primarily through the provision of a method of analysis.

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Abstract

In this paper, we consider the extension of first passage probability. First, we present the first, second, third, and generally k-th passage probability of a Markov Chain moving from one state to another state through step-by-step calculation and two other matrix-version methods. Similarly, we compute the first passage probability of a Markov Chain moving from one state to multiple states. In all discussions, we take into account the situations that one state moves to a different state and returns to itself. Also, we find the mean number of steps needed from one state to another state in a Markov Chain for the first, second, third, and generally k-th passage. Besides, we find the probability generating function for the number of steps. This makes the calculation of passage probabilities, mean and variance of passage steps, easier. Additionally, if we extend a discrete Markov Chain to its corresponding continuous Markov Process with the same transition probabilities and transition time in the form of an exponential distribution with parameter 1 between two states, we can obtain the mean time needed from one state to another state by Laplace Transforms, which is the same as with the discrete situation. Subsequently, we can calculate the variance of the time needed from one state to another state in the same way.

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Contents

CHAPTER 1

Introduction

First passage time describes the amount of time (number of steps) required for a stochastic process starting from one initial state to another state for the first time. More mathematically, let T_{ij} denote the number of transitions the process takes for its first entrance into state j given that $X_0 = i$ where X_n is a Markov Chain with state space S and $i, j \in S$. The random variable T_{ij} is known as the first passage time from i to j ([10]).

Similarly, first passage probability considers the probability from one state to another for the first time in given time (number of steps) for a stochastic process. For any states i and j, define $f_{ij}^{(n)}$ to be the probability that starting in i, the first transition into j occurs at step n. That is to say,

$$f_{ij}^{(n)} = P(T_{ij} = n) = P(X_n = j, X_{n-1} \neq j, ..., X_1 \neq j | X_0 = i)$$

With the well-defined first passage probability, we can do some further research.

In chapter 2, we consider the first passage probability between two states. We use three methods, including a recursive method and two matrix-version methods, to calculate the first passage probability for two cases. In the first case a process moves from state i to state j for a given Markov Chain where i and j are different states. The other case is that a process moves from state i to state i, i.e., returns to state i.

In chapter 3, we consider the second, third, and generally k-th passage probability between two states for the same two cases. For each passage probability, we develop three methods to calculate it.

In chapter 4, we consider the first passage probability between multiple states. In the first part, we discuss the first passage probability from state i to states j or k, where i, j and k are different states, using three methods. This means when reaching one of the two destination states, the process stops. In the second part, we consider the first passage probability from state i to state i or to a different state j using three methods. In the third part, we talk about the first passage probability from state i to state j, k, where they are all different states, meaning the process can stop only when it reaches all states, using three methods. In the fourth part, we compute the first passage probability from state i to state i and a different state j.

In chapter 5, we focus on mean passage times. We study the mean number of steps between two and more states and show the relationship between the mean first passage time and the mean second passage time and the mean k-th passage time.

In chapter 6, we find the probability generating function, by using the method of collective marks, for first and second passage, with which we can obtain the passage probabilities using Taylor expansion, and the mean and variance of passage times.

In chapter 7, we extend the discrete Markov Chain to a continuous Markov process, the transition probabilities of which are the same as those of discrete Markov Chains where the time needed from one state to another has an exponential distribution with $\lambda = 1$. We apply Laplace Transforms to obtain the mean and variance of the first, second, third, and generalized k-th passage time.

CHAPTER 2

First Passage Probability Between Two States

In order to illustrate the first passage probability from one state to another, we present a typical Markov Chain with states 1, 2, and 3 and transition matrix

$$P = \begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.4 \\ 0.5 & 0.4 & 0.1 \end{bmatrix}$$

as an example.

Since the first passage probability depicts the probability from one state to another state for the first time in some specific number of steps, we can easily obtain the probability of one step from the transition matrix. But for more steps, we need further calculation.

2.1. From State i to State j (i≠j)

In this section, we propose three methods to calculate the first passage probability from state i to state j. For each method, we present the thought behind it and use the situation from state 1 to state 3 to demonstrate.

Method 1

Define $p_{ij}^{(n)}$ to be n-step transition probability from state i to state j for a Markov Chain. When $n = 1$, we denote $p_{ij}^{(1)}$ as p_{ij}, which means one step transition probability. We can calculate $f_{ij}^{(k)}$ from

$$p_{ij}^{(n)} = \sum_{k=1}^{n} p_{jj}^{(n-k)} f_{ij}^{(k)}$$

where $f_{ij}^{(0)} = 0$ for all i, j, and $p_{ij}^{(0)} = 0$ for $i \neq j$; $p_{jj}^{(0)} = 1$.

Proof. Let T denote the number of steps of the first transition into state j, if any. By conditioning on T, we obtain

$$\begin{aligned} p_{ij}^{(n)} &= \sum_{k=1}^{n} P(X_n = j | T = k, X_0 = i) P(T = k | X_0 = i) \\ &= \sum_{k=1}^{n} p_{jj}^{(n-k)} f_{ij}^{(k)} \end{aligned}$$

Applying this method to our example, we first compute P^2, P^3, P^4. Then
$p_{13}^{(1)} = f_{13}^{(1)} = 0.4$
Thus $f_{13}^{(1)} = 0.4$.
$p_{13}^{(2)} = f_{13}^{(1)} p_{33}^{(1)} + f_{13}^{(2)} = 0.04 + f_{13}^{(2)} = 0.28$
Thus $f_{13}^{(2)} = 0.24$.
$p_{13}^{(3)} = f_{13}^{(1)} p_{33}^{(2)} + f_{13}^{(2)} p_{33}^{(1)} + f_{13}^{(3)} = 0.148 + 0.024 + f_{13}^{(3)} = 0.316$
Thus $f_{13}^{(3)} = 0.144$.
$p_{13}^{(4)} = f_{13}^{(1)} p_{33}^{(3)} + f_{13}^{(2)} p_{33}^{(2)} + f_{13}^{(3)} p_{33}^{(1)} + f_{13}^{(4)} = 0.1166 + 0.0888 + 0.0144 + f_{13}^{(4)} = 0.3052$
Thus $f_{13}^{(4)} = 0.0864$.

Method 2

If we think about this question in another way, the first passage from state i to state j at step k means during the first $k-1$ steps, the process never enters j and it could only reach states other than j. Therefore, the probabilities of all the states other than j to j during the first $k-1$ steps are 0s and the transition probabilities between other states are the same as the original transition probabilities. Under such circumstances, we can construct a new matrix P_0 with the j-th column in P replaced by 0s. So we can use P_0 as the transition matrix for the first $k-1$ steps and for the last step, we still use P since the process can reach every state at that stage. Finally we take the (i, j) entry of the matrix which is the product of the k matrices as the first passage probability from state i to j at step k.

When we consider state 3 as the destination, the new matrix P_0 is P with the third column replaced by 0s.

$$P_0 = \begin{bmatrix} 0.2 & 0.4 & 0 \\ 0.3 & 0.3 & 0 \\ 0.5 & 0.4 & 0 \end{bmatrix}$$

To get $f_{13}^{(1)}$, we take the $(1,3)$ entry of

$$P_0^0 P = \begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.4 \\ 0.5 & 0.4 & 0.1 \end{bmatrix}$$

So $f_{13}^{(1)} = 0.4$.

To get $f_{13}^{(2)}$, we take the $(1,3)$ entry of

$$P_0^1 P = \begin{bmatrix} 0.16 & 0.2 & 0.24 \\ 0.15 & 0.21 & 0.24 \\ 0.22 & 0.32 & 0.36 \end{bmatrix}$$

So $f_{13}^{(2)} = 0.24$.

To get $f_{13}^{(3)}$, we take the $(1,3)$ entry of

$$P_0^2 P = \begin{bmatrix} 0.092 & 0.124 & 0.144 \\ 0.093 & 0.123 & 0.144 \\ 0.14 & 0.184 & 0.216 \end{bmatrix}$$

So $f_{13}^{(3)} = 0.144$.

To get $f_{13}^{(4)}$, we take the $(1,3)$ entry of

$$P_0^3 P = \begin{bmatrix} 0.0556 & 0.074 & 0.0864 \\ 0.0555 & 0.0741 & 0.0864 \\ 0.0832 & 0.1112 & 0.1296 \end{bmatrix}$$

So $f_{13}^{(4)} = 0.0864$

Method 3

We can also compute the first passage probability from state i to state j by adding an absorbing state j^* to the original Markov Chain and thus getting a new transition matrix P^*. Whenever the process reaches state j, it gets into state j^* immediately and cannot get out. So the entries in j-th column of the P^* are 0s and the entries in j^* column are probabilities to state j. This is to say when we multiply this new transition matrix by itself k times, the (i, j^*) entry stands for the total probability from state i to state j after the k step. Similarly, if we multiply

P^* by itself $k-1$ times, the (i, j^*) entry stands for the probability from state i to state j after $k-1$ step. The difference between them is the probability from i to j for the first time at exactly k-th step.

In our example from state 1 to state 3, P^* is a 4×4 matrix with state space $\{1, 2, 3, 3^*\}$ where 3^* is the absorbing state.

$$P^* = \begin{bmatrix} 0.2 & 0.4 & 0 & 0.4 \\ 0.3 & 0.3 & 0 & 0.4 \\ 0.5 & 0.4 & 0 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To get $f_{13}^{(1)}$, we take the $(1, 4)$ entry of

$$P^* = \begin{bmatrix} 0.2 & 0.4 & 0 & 0.4 \\ 0.3 & 0.3 & 0 & 0.4 \\ 0.5 & 0.4 & 0 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore $f_{13}^{(1)} = 0.4$.

To get $f_{13}^{(2)}$, we take the $(1, 4)$ entry of

$$(P^*)^2 - P^* = \begin{bmatrix} -0.04 & -0.2 & 0 & 0.24 \\ -0.15 & -0.09 & 0 & 0.24 \\ -0.28 & -0.08 & 0 & 0.36 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore $f_{13}^{(2)} = 0.24$.

To get $f_{13}^{(3)}$, we take the $(1, 4)$ entry of

$$(P^*)^3 - (P^*)^2 = \begin{bmatrix} -0.068 & -0.076 & 0 & 0.144 \\ -0.057 & -0.087 & 0 & 0.144 \\ -0.08 & -0.136 & 0 & 0.266 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore $f_{13}^{(3)} = 0.144$.

To get $f_{13}^{(4)}$, we take the $(1,4)$ entry of

$$(P^*)^4 - (P^*)^3 = \begin{bmatrix} -0.0364 & -0.05 & 0 & 0.0864 \\ -0.0375 & -0.0481 & 0 & 0.0864 \\ -0.0568 & -0.0728 & 0 & 0.1296 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore $f_{13}^{(4)} = 0.0864$.

With three different methods, we obtain the same results, meaning that the methods we mentioned are reasonable and reliable.

2.2. From State i to State i

In this section, we discuss the situation that a process moves from state i to state i, in other words, returns to state i. The difference between this section and section 2.1 lies in the construction of matrices, since in section 2.1, we can easily distinguish start state and end state in transition matrix, so we are able to replace a destination state column by 0s. But in this case, we cannot just do the same thing because if the start state column was replaced by 0s, it cannot be reached any more. Therefore, we have to modify the methods. We use the process from state 1 to state 1 as an example.

Method 1

If we want to calculate the first passage probability step by step, we just put $j = i$ in the formula in section 2.1.1 because this is a special case when $j = i$, and we get

$$p_{ii}^{(n)} = \sum_{k=1}^{n} p_{ii}^{(n-k)} f_{ii}^{(k)}$$

Applying this method to our example where state 1 moves to state 1, we have

$p_{13}^{(1)} = f_{11}^{(1)} = 0.2$

Thus $f_{11}^{(1)} = 0.2$.

$p_{11}^{(2)} = f_{11}^{(1)} p_{11}^{(1)} + f_{11}^{(2)} = 0.04 + f_{11}^{(2)} = 0.36$

Thus $f_{11}^{(2)} = 0.32$.

$p_{11}^{(3)} = f_{11}^{(1)} p_{11}^{(2)} + f_{11}^{(2)} p_{11}^{(1)} + f_{11}^{(3)} = 0.072 + 0.064 + f_{11}^{(3)} = 0.32$

Thus $f_{11}^{(3)} = 0.184$.

$p_{11}^{(4)} = f_{11}^{(1)} p_{11}^{(3)} + f_{11}^{(2)} p_{11}^{(2)} + f_{11}^{(3)} p_{11}^{(1)} + f_{11}^{(4)} = 0.064 + 0.1152 + 0.0368 + f_{11}^{(4)} = 0.3312$
Thus $f_{11}^{(4)} = 0.1152$.

Method 2

To obtain the first passage probability from state i to state i, we need to replicate the states, so now the state space is $\{1, 1', 2, 2', ..., i, i', ...\}$. For the original states other than destination state i, the transition probabilities between them remains the same. For all the replicated states, they could enter any other replicated states with the transition probabilities the same with the original chain. For state i, the process starting from it could only go to the replicated states with corresponding transition probabilities. And the matrix constructed by this procedure is the transition matrix P under this method. Then by replacing i' column in P by 0s to get P_0, we can take (i, i') entry of the product of $P_0^{n-1}P$ as the probability of first returning to state i at some step n.

When we considering state 1 as the destination, the new matrix P is a enlarged 6×6 matrix with state space $\{1, 1', 2, 2', 3, 3'\}$.

$$P = \begin{bmatrix} 0 & 0.2 & 0 & 0.4 & 0 & 0.4 \\ 0 & 0.2 & 0 & 0.4 & 0 & 0.4 \\ 0.3 & 0 & 0.3 & 0 & 0.4 & 0 \\ 0 & 0.3 & 0 & 0.3 & 0 & 0.4 \\ 0.5 & 0 & 0.4 & 0 & 0.1 & 0 \\ 0 & 0.5 & 0 & 0.4 & 0 & 0.1 \end{bmatrix}$$

The corresponding matrix P_0 is P with the second column replaced by 0s.

$$P_0 = \begin{bmatrix} 0 & 0 & 0 & 0.4 & 0 & 0.4 \\ 0 & 0 & 0 & 0.4 & 0 & 0.4 \\ 0.3 & 0 & 0.3 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0.4 \\ 0.5 & 0 & 0.4 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0.1 \end{bmatrix}$$

To get $f_{11}^{(1)}$, we take the $(1,2)$ entry of

$$P_0^0 P = \begin{bmatrix} 0 & 0.2 & 0 & 0.4 & 0 & 0.4 \\ 0 & 0.2 & 0 & 0.4 & 0 & 0.4 \\ 0.3 & 0 & 0.3 & 0 & 0.4 & 0 \\ 0 & 0.3 & 0 & 0.3 & 0 & 0.4 \\ 0.5 & 0 & 0.4 & 0 & 0.1 & 0 \\ 0 & 0.5 & 0 & 0.4 & 0 & 0.1 \end{bmatrix}$$

So $f_{11}^{(1)} = 0.2$.

To get $f_{11}^{(2)}$, we take the $(1,2)$ entry of

$$P_0^1 P = \begin{bmatrix} 0 & 0.32 & 0 & 0.28 & 0 & 0.2 \\ 0 & 0.32 & 0 & 0.28 & 0 & 0.2 \\ 0.29 & 0.06 & 0.25 & 0.12 & 0.16 & 0.12 \\ 0 & 0.29 & 0 & 0.25 & 0 & 0.16 \\ 0.17 & 0.1 & 0.16 & 0.2 & 0.17 & 0.2 \\ 0 & 0.17 & 0 & 0.16 & 0 & 0.17 \end{bmatrix}$$

So $f_{11}^{(2)} = 0.32$.

To get $f_{11}^{(3)}$, we take the $(1,2)$ entry of

$$P_0^2 P = \begin{bmatrix} 0 & 0.184 & 0 & 0.164 & 0 & 0.132 \\ 0 & 0.184 & 0 & 0.164 & 0 & 0.132 \\ 0.155 & 0.154 & 0.139 & 0.2 & 0.116 & 0.176 \\ 0 & 0.155 & 0 & 0.139 & 0 & 0.116 \\ 0.133 & 0.194 & 0.116 & 0.208 & 0.081 & 0.168 \\ 0 & 0.133 & 0 & 0.116 & 0 & 0.081 \end{bmatrix}$$

So $f_{11}^{(3)} = 0.184$.

To get $f_{11}^{(4)}$, we take the $(1,2)$ entry of

$$P_0^3 P = \begin{bmatrix} 0 & 0.1152 & 0 & 0.102 & 0 & 0.0788 \\ 0 & 0.1152 & 0 & 0.102 & 0 & 0.0788 \\ 0.0997 & 0.179 & 0.0881 & 0.1924 & 0.0672 & 0.1596 \\ 0 & 0.0997 & 0 & 0.0881 & 0 & 0.0672 \\ 0.0753 & 0.173 & 0.0672 & 0.1828 & 0.0545 & 0.1532 \\ 0 & 0.0753 & 0 & 0.0672 & 0 & 0.0545 \end{bmatrix}$$

So $f_{11}^{(4)} = 0.1152$.

Method 3

Based on the transition matrix P in method 2, we build a matrix P^* by adding an absorbing state i'^* and putting all the probabilities to state i' to the i'^* column, so entries in the i' column are 0s. Then the (i, i'^*) entry of the $(P^*)^n - (P^*)^{n-1}$ is the first return probability to state i at n-th step.

In our example from state 1 to state 1, P^* is a 7×7 matrix with state space $\{1, 1', 1'^*, 2, 2', 3, 3'\}$ where $1'^*$ is the absorbing state.

$$P^* = \begin{bmatrix} 0 & 0 & 0.2 & 0 & 0.4 & 0 & 0.4 \\ 0 & 0 & 0.2 & 0 & 0.4 & 0 & 0.4 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0.3 & 0 & 0.4 & 0 \\ 0 & 0 & 0.3 & 0 & 0.3 & 0 & 0.4 \\ 0.5 & 0 & 0 & 0.4 & 0 & 0.1 & 0 \\ 0 & 0 & 0.5 & 0 & 0.4 & 0 & 0.1 \end{bmatrix}$$

To get $f_{11}^{(1)}$, we take the $(1,3)$ entry of

$$P^* = \begin{bmatrix} 0 & 0 & 0.2 & 0 & 0.4 & 0 & 0.4 \\ 0 & 0 & 0.2 & 0 & 0.4 & 0 & 0.4 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0.3 & 0 & 0.4 & 0 \\ 0 & 0 & 0.3 & 0 & 0.3 & 0 & 0.4 \\ 0.5 & 0 & 0 & 0.4 & 0 & 0.1 & 0 \\ 0 & 0 & 0.5 & 0 & 0.4 & 0 & 0.1 \end{bmatrix}$$

Therefore $f_{11}^{(1)} = 0.2$.

To get $f_{11}^{(2)}$, we take the $(1,3)$ entry of

$$(P^*)^2 - P^* = \begin{bmatrix} 0 & 0 & 0.32 & 0 & -0.12 & 0 & -0.2 \\ 0 & 0 & 0.32 & 0 & -0.12 & 0 & -0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.01 & 0 & 0.06 & -0.05 & 0.12 & -0.24 & 0.12 \\ 0 & 0 & 0.29 & 0 & -0.05 & 0 & -0.24 \\ -0.33 & 0 & 0.1 & -0.24 & 0.2 & 0.07 & 0.2 \\ 0 & 0 & 0.17 & 0 & -0.24 & 0 & 0.07 \end{bmatrix}$$

Therefore $f_{11}^{(2)} = 0.32$.

To get $f_{11}^{(3)}$, we take the $(1,3)$ entry of

$$(P^*)^3 - (P^*)^2 = \begin{bmatrix} 0 & 0 & 0.184 & 0 & -0.116 & 0 & -0.068 \\ 0 & 0 & 0.184 & 0 & -0.116 & 0 & -0.068 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.135 & 0 & 0.154 & -0.111 & 0.08 & -0.044 & 0.056 \\ 0 & 0 & 0.155 & 0 & -0.111 & 0 & -0.044 \\ -0.037 & 0 & 0.194 & -0.044 & 0.008 & -0.089 & -0.032 \\ 0 & 0 & 0.133 & 0 & -0.044 & 0 & -0.089 \end{bmatrix}$$

Therefore $f_{11}^{(3)} = 0.184$.

To get $f_{11}^{(4)}$, we take the $(1,3)$ entry of

$$(P^*)^4-(P^*)^3=\begin{bmatrix} 0 & 0 & 0.1152 & 0 & -0.062 & 0 & -0.0532 \\ 0 & 0 & 0.1152 & 0 & -0.062 & 0 & -0.0532 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0553 & 0 & 0.179 & -0.0509 & -0.0076 & -0.0488 & -0.0164 \\ 0 & 0 & 0.0997 & 0 & -0.0509 & 0 & -0.0488 \\ -0.0577 & 0 & 0.173 & -0.0488 & -0.0252 & -0.0265 & -0.0148 \\ 0 & 0 & 0.0753 & 0 & -0.0488 & 0 & -0.0265 \end{bmatrix}$$

Therefore $f_{11}^{(4)}=0.1152$.

With three different methods, we obtain the same results, meaning that the methods we mentioned are reasonable and reliable.

CHAPTER 3

K-th Passage Probability Between Two States

In this chapter, we consider the second, third, and generally k-th passage probabilities. By modifying the original transition matrix P for the second and third passage, we can simplify the second and third passage probability to first passage probability, thus we are able to apply the same methods to compute the probabilities. Also, we use the example from state 1 to state 3 when we talking about state i to state j and from state 1 to state 1 when we talking about state i to state i to illustrate the process. Last we conclude the methods for k-th passage case.

3.1. Second Passage Probability

Define $s_{ij}^{(n)}$ to be the probability that given an initial state i, the process enters state j for the second time on step n. There are two situations for second passage probability, from state i to a different state j, $s_{ij}^{(n)}$, or from state i to itself, $s_{ii}^{(n)}$.

3.1.1. From State i to State j (i≠j). Second passage probability describes the probability from state i to state j for the second time at a given step k. It seems we cannot calculate the second passage probability easily because it is not as straightforward as the first passage probability, but actually there exists a relationship between them and we can still apply three similar methods to obtain it.

Method 1

Like the first passage probability, if we consider all the possible situations where the process enters state j exactly twice at step n, which means the total number of steps it needed to reach j from i for the first time and return to j for the other time is n, we can have

$$s_{ij}^{(n)} = \sum_{k=1}^{n-1} f_{ij}^{(n-k)} f_{jj}^{(k)}$$

Applying this method to our example, we have

$s_{13}^{(1)} = 0$
$s_{13}^{(2)} = f_{13}^{(1)} f_{33}^{(1)} = 0.04$
$s_{13}^{(3)} = f_{13}^{(2)} f_{33}^{(1)} + f_{13}^{(1)} f_{33}^{(2)} = 0.024 + 0.144 = 0.168$
$s_{13}^{(4)} = f_{13}^{(3)} f_{33}^{(1)} + f_{13}^{(2)} f_{33}^{(2)} + f_{13}^{(1)} f_{33}^{(3)} = 0.0144 + 0.0864 + 0.0864 = 0.1872$
In particular, $s_{13}^{(1)} = 0$ because we cannot reach j twice in only one step.

Method 2

To obtain the second passage probability in a matrix version, we need to construct a new Markov Chain whose first passage probability gives the second passage probability of the original Markov Chain. In order to do this, we replicate the states, so now the state space is $\{1, 1', 2, 2', ..., i, i', ..., j, j', ...\}$. For the original states other than destination state j, the transition probabilities between them remains the same because they represent the first transition. For the replicated states, they could enter any other replicated states with the transition probabilities the same with the original chain, since they represent the second transition. For state j, the process starting from it could only go to the replicated states with corresponding transition probabilities, because it is the destination of the first transition and the beginning of the second transition. Then using the same logic as in section 2.1.1, we can calculate the second passage probability ([4]).

When we considering state 3 as the destination, the new matrix P is a enlarged 6×6 matrix with state space $\{1, 1', 2, 2', 3, 3'\}$.

$$P = \begin{bmatrix} 0.2 & 0 & 0.4 & 0 & 0.4 & 0 \\ 0 & 0.2 & 0 & 0.4 & 0 & 0.4 \\ 0.3 & 0 & 0.3 & 0 & 0.4 & 0 \\ 0 & 0.3 & 0 & 0.3 & 0 & 0.4 \\ 0 & 0.5 & 0 & 0.4 & 0 & 0.1 \\ 0 & 0.5 & 0 & 0.4 & 0 & 0.1 \end{bmatrix}$$

The corresponding matrix P_0 is P with the 6-th column replaced by 0s.

$$P_0 = \begin{bmatrix} 0.2 & 0 & 0.4 & 0 & 0.4 & 0 \\ 0 & 0.2 & 0 & 0.4 & 0 & 0 \\ 0.3 & 0 & 0.3 & 0 & 0.4 & 0 \\ 0 & 0.3 & 0 & 0.3 & 0 & 0 \\ 0 & 0.5 & 0 & 0.4 & 0 & 0 \\ 0 & 0.5 & 0 & 0.4 & 0 & 0 \end{bmatrix}$$

To get $s_{13}^{(1)}$, we take the $(1,6)$ entry of

$$P_0^0 P = \begin{bmatrix} 0.2 & 0 & 0.4 & 0 & 0.4 & 0 \\ 0 & 0.2 & 0 & 0.4 & 0 & 0.4 \\ 0.3 & 0 & 0.3 & 0 & 0.4 & 0 \\ 0 & 0.3 & 0 & 0.3 & 0 & 0.4 \\ 0 & 0.5 & 0 & 0.4 & 0 & 0.1 \\ 0 & 0.5 & 0 & 0.4 & 0 & 0.1 \end{bmatrix}$$

So $s_{13}^{(1)} = 0$.

To get $s_{13}^{(2)}$, we take the $(1,6)$ entry of

$$P_0^1 P = \begin{bmatrix} 0.16 & 0.2 & 0.2 & 0.16 & 0.24 & 0.04 \\ 0 & 0.16 & 0 & 0.2 & 0 & 0.24 \\ 0.15 & 0.2 & 0.21 & 0.16 & 0.24 & 0.04 \\ 0 & 0.15 & 0 & 0.21 & 0 & 0.24 \\ 0 & 0.22 & 0 & 0.32 & 0 & 0.36 \\ 0 & 0.22 & 0 & 0.32 & 0 & 0.36 \end{bmatrix}$$

So $s_{13}^{(2)} = 0.04$.

To get $s_{13}^{(3)}$, we take the $(1,6)$ entry of

$$P_0^2 P = \begin{bmatrix} 0.092 & 0.208 & 0.124 & 0.224 & 0.144 & 0.168 \\ 0 & 0.092 & 0 & 0.124 & 0 & 0.144 \\ 0.093 & 0.208 & 0.123 & 0.224 & 0.144 & 0.168 \\ 0 & 0.093 & 0 & 0.123 & 0 & 0.144 \\ 0 & 0.14 & 0 & 0.184 & 0 & 0.216 \\ 0 & 0.14 & 0 & 0.184 & 0 & 0.216 \end{bmatrix}$$

So $s_{13}^{(3)} = 0.168$.

To get $s_{13}^{(4)}$, we take the $(1,6)$ entry of

$$P_0^3 P = \begin{bmatrix} 0.0556 & 0.1808 & 0.074 & 0.208 & 0.0864 & 0.1872 \\ 0 & 0.0556 & 0 & 0.074 & 0 & 0.0864 \\ 0.0555 & 0.1808 & 0.0741 & 0.208 & 0.0864 & 0.1872 \\ 0 & 0.0555 & 0 & 0.0741 & 0 & 0.0864 \\ 0 & 0.0832 & 0 & 0.1112 & 0 & 0.1296 \\ 0 & 0.0832 & 0 & 0.1112 & 0 & 0.1296 \end{bmatrix}$$

So $s_{13}^{(4)} = 0.1872$.

Method 3

Once again we add an absorbing state as in section 2.1.1, but this time since j' is the final destination, we add j'^* as the absorbing state.

In our example from state 1 to state 3, P^* is a 7×7 matrix with state space $\{1, 1', 2, 2', 3, 3', 3'^*\}$ where $3'^*$ is the absorbing state.

$$P^* = \begin{bmatrix} 0.2 & 0 & 0.4 & 0 & 0.4 & 0 & 0 \\ 0 & 0.2 & 0 & 0.4 & 0 & 0 & 0.4 \\ 0.3 & 0 & 0.3 & 0 & 0.4 & 0 & 0 \\ 0 & 0.3 & 0 & 0.3 & 0 & 0 & 0.4 \\ 0 & 0.5 & 0 & 0.4 & 0 & 0 & 0.1 \\ 0 & 0.5 & 0 & 0.4 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

To get $s_{13}^{(1)}$, we take the $(1,7)$ entry of

$$P^* = \begin{bmatrix} 0.2 & 0 & 0.4 & 0 & 0.4 & 0 & 0 \\ 0 & 0.2 & 0 & 0.4 & 0 & 0 & 0.4 \\ 0.3 & 0 & 0.3 & 0 & 0.4 & 0 & 0 \\ 0 & 0.3 & 0 & 0.3 & 0 & 0 & 0.4 \\ 0 & 0.5 & 0 & 0.4 & 0 & 0 & 0.1 \\ 0 & 0.5 & 0 & 0.4 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore $s_{13}^{(1)} = 0$.

To get $s_{13}^{(2)}$, we take the $(1,7)$ entry of

$$(P^*)^2 - P^* = \begin{bmatrix} -0.04 & 0.2 & -0.2 & 0.16 & -0.16 & 0 & 0.04 \\ 0 & -0.04 & 0 & -0.2 & 0 & 0 & 0.24 \\ -0.15 & 0.2 & -0.09 & 0.16 & -0.16 & 0 & 0.04 \\ 0 & -0.15 & 0 & -0.09 & 0 & 0 & 0.24 \\ 0 & -0.28 & 0 & -0.08 & 0 & 0 & 0.36 \\ 0 & -0.28 & 0 & -0.08 & 0 & 0 & 0.36 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore $s_{13}^{(2)} = 0.04$.

To get $s_{13}^{(3)}$, we take the $(1,7)$ entry of

$$(P^*)^3 - (P^*)^2 = \begin{bmatrix} -0.068 & 0.008 & -0.076 & 0.064 & -0.096 & 0 & 0.168 \\ 0 & -0.068 & 0 & -0.076 & 0 & 0 & 0.144 \\ -0.057 & 0.008 & -0.087 & 0.064 & -0.096 & 0 & 0.168 \\ 0 & -0.057 & 0 & -0.087 & 0 & 0 & 0.144 \\ 0 & -0.08 & 0 & -0.136 & 0 & 0 & 0.216 \\ 0 & -0.08 & 0 & -0.136 & 0 & 0 & 0.216 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore $s_{13}^{(3)} = 0.168$.

To get $s_{13}^{(4)}$, we take the $(1,7)$ entry of

$$(P^*)^4 - (P^*)^3 = \begin{bmatrix} -0.0364 & -0.0272 & -0.05 & -0.016 & -0.0576 & 0 & 0.1872 \\ 0 & -0.0364 & 0 & -0.05 & 0 & 0 & 0.0864 \\ -0.0375 & -0.0272 & -0.0489 & -0.016 & -0.0576 & 0 & 0.1872 \\ 0 & -0.0375 & 0 & -0.0489 & 0 & 0 & 0.0864 \\ 0 & -0.0568 & 0 & -0.0728 & 0 & 0 & 0.1296 \\ 0 & -0.0568 & 0 & -0.0728 & 0 & 0 & 0.1296 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore $s_{13}^{(4)} = 0.1872$.

With three different methods, we obtain the same results, meaning that the methods we mentioned are reasonable and reliable.

3.1.2. From State i to State i. Like the extension of first passage probability to second passage probability in section 3.1.1, we can develop the methods for second passage probability from first passage probability under the circumstances of a process returning to state i.

Method 1

Substitute $j = i$ in $s_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(n-k)} f_{jj}^{(k)}$, we get the formula for this special return case.

$$s_{ii}^{(n)} = \sum_{k=1}^{n} f_{ii}^{(n-k)} f_{ii}^{(k)}$$

Applying this method to our example, we have

$s_{11}^{(1)} = 0$

$s_{11}^{(2)} = f_{11}^{(1)} f_{11}^{(1)} = 0.04$

$s_{11}^{(3)} = f_{11}^{(2)} f_{11}^{(1)} + f_{11}^{(1)} f_{11}^{(2)} = 0.064 + 0.064 = 0.128$

$s_{11}^{(4)} = f_{11}^{(3)} f_{11}^{(1)} + f_{11}^{(2)} f_{11}^{(2)} + f_{11}^{(1)} f_{11}^{(3)} = 0.0368 + 0.1024 + 0.0368 = 0.176$

In particular, $s_{11}^{(1)} = 0$ because we cannot reach state 1 twice in only one step.

Method 2

When computing first passage probability, we replicate the states twice, so in order to obtain second passage probability, we need to replicate the states by three times. Thus now the state space is $\{1, 1', 1'', 2, 2', 2'', ..., i, i', i'', ...\}$. And we take the (i, i'') entry of the matrix $P_0^{n-1}P$ as the result.

When we considering state 1 as the destination, the new matrix P is a enlarged 9×9 matrix with state space $\{1, 1', 1'', 2, 2', 2'', 3, 3', 3''\}$.

$$P = \begin{bmatrix} 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 \\ 0 & 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 \\ 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.4 \\ 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0.1 \end{bmatrix}$$

The corresponding matrix P_0 is P with the third column replaced by 0s.

$$P_0 = \begin{bmatrix} 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0.4 \\ 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0.4 \\ 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0.1 \end{bmatrix}$$

To get $s_{11}^{(1)}$, we take the $(1,3)$ entry of

$$P_0^0 P = \begin{bmatrix} 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 \\ 0 & 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 \\ 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.4 \\ 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0.1 \end{bmatrix}$$

So $s_{11}^{(1)} = 0$.

To get $s_{11}^{(2)}$, we take the $(1,3)$ entry of

$$P_0^1 P = \begin{bmatrix} 0 & 0.32 & 0.04 & 0 & 0.28 & 0.08 & 0 & 0.2 & 0.08 \\ 0 & 0 & 0.32 & 0 & 0 & 0.28 & 0 & 0 & 0.2 \\ 0 & 0 & 0.32 & 0 & 0 & 0.28 & 0 & 0 & 0.2 \\ 0.29 & 0.06 & 0 & 0.25 & 0.12 & 0 & 0.16 & 0.12 & 0 \\ 0 & 0.29 & 0.06 & 0 & 0.25 & 0.12 & 0 & 0.16 & 0.12 \\ 0 & 0 & 0.29 & 0 & 0 & 0.25 & 0 & 0 & 0.16 \\ 0.17 & 0.1 & 0 & 0.16 & 0.2 & 0 & 0.17 & 0.2 & 0 \\ 0 & 0.17 & 0.1 & 0 & 0.16 & 0.2 & 0 & 0.17 & 0.2 \\ 0 & 0 & 0.17 & 0 & 0 & 0.16 & 0 & 0 & 0.17 \end{bmatrix}$$

So $s_{11}^{(2)} = 0.04$.

To get $s_{11}^{(3)}$, we take the $(1,3)$ entry of

$$P_0^2P = \begin{bmatrix} 0 & 0.184 & 0.128 & 0 & 0.164 & 0.184 & 0 & 0.132 & 0.168 \\ 0 & 0 & 0.184 & 0 & 0 & 0.164 & 0 & 0 & 0.132 \\ 0 & 0 & 0.184 & 0 & 0 & 0.164 & 0 & 0 & 0.132 \\ 0.155 & 0.154 & 0.012 & 0.139 & 0.2 & 0.024 & 0.116 & 0.176 & 0.024 \\ 0 & 0.155 & 0.154 & 0 & 0.139 & 0.2 & 0 & 0.116 & 0.176 \\ 0 & 0 & 0.155 & 0 & 0 & 0.139 & 0 & 0 & 0.116 \\ 0.133 & 0.194 & 0.02 & 0.116 & 0.208 & 0.04 & 0.081 & 0.168 & 0.04 \\ 0 & 0.133 & 0.194 & 0 & 0.116 & 0.208 & 0 & 0.081 & 0.168 \\ 0 & 0 & 0.133 & 0 & 0 & 0.116 & 0 & 0 & 0.081 \end{bmatrix}$$

So $s_{11}^{(3)} = 0.128$.

To get $s_{11}^{(4)}$, we take the $(1,3)$ entry of

$$P_0^3P = \begin{bmatrix} 0 & 0.1152 & 0.176 & 0 & 0.102 & 0.196 & 0 & 0.0788 & 0.164 \\ 0 & 0 & 0.1152 & 0 & 0 & 0.102 & 0 & 0 & 0.0788 \\ 0 & 0 & 0.1152 & 0 & 0 & 0.102 & 0 & 0 & 0.0788 \\ 0.0997 & 0.179 & 0.05 & 0.0881 & 0.1924 & 0.0784 & 0.0672 & 0.1596 & 0.0736 \\ 0 & 0.0997 & 0.179 & 0 & 0.0881 & 0.1924 & 0 & 0.0672 & 0.1596 \\ 0 & 0 & 0.0997 & 0 & 0 & 0.0881 & 0 & 0 & 0.0672 \\ 0.0753 & 0.173 & 0.0708 & 0.0672 & 0.1828 & 0.1056 & 0.0545 & 0.1532 & 0.0976 \\ 0 & 0.0753 & 0.173 & 0 & 0.0672 & 0.1828 & 0 & 0.0545 & 0.1532 \\ 0 & 0 & 0.0753 & 0 & 0 & 0.0672 & 0 & 0 & 0.0545 \end{bmatrix}$$

So $s_{11}^{(4)} = 0.176$.

Method 3

This time again we add an absorbing state i''^* to the transition matrix P, to which other states can move taking the place of i''. The (i, i''^*) entry represents the probability of second return occurs at step n.

In our example from state 1 to state 1, P^* is a 10×10 matrix with state space $\{1, 1', 1'', 1''*, 2, 2', 2'', 3, 3', 3''\}$ where $1''*$ is the absorbing state.

$$P^* = \begin{bmatrix} 0 & 0.2 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.4 \\ 0.5 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0.1 \end{bmatrix}$$

To get $s_{11}^{(1)}$, we take the $(1, 4)$ entry of

$$P^* = \begin{bmatrix} 0 & 0.2 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.4 \\ 0.5 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0.1 \end{bmatrix}$$

Therefore $s_{11}^{(1)} = 0$.

To get $s_{11}^{(2)}$, we take the $(1,4)$ entry of

$$(P^*)^2 - P^* = \begin{bmatrix} 0 & 0.12 & 0 & 0.04 & 0 & -0.12 & 0.08 & 0 & -0.2 & 0.08 \\ 0 & 0 & 0 & 0.32 & 0 & 0 & -0.12 & 0 & 0 & -0.2 \\ 0 & 0 & 0 & 0.32 & 0 & 0 & -0.12 & 0 & 0 & -0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.01 & 0.06 & 0 & 0 & -0.05 & 0.12 & 0 & -0.24 & 0.12 & 0 \\ 0 & -0.01 & 0 & 0.06 & 0 & -0.05 & 0.12 & 0 & -0.24 & 0.12 \\ 0 & 0 & 0 & 0.29 & 0 & 0 & -0.05 & 0 & 0 & -0.24 \\ -0.33 & 0.1 & 0 & 0 & -0.24 & 0.2 & 0 & 0.07 & 0.2 & 0 \\ 0 & -0.33 & 0 & 0.1 & 0 & -0.24 & 0.2 & 0 & 0.07 & 0.2 \\ 0 & 0 & 0 & 0.17 & 0 & 0 & -0.24 & 0 & 0 & 0.07 \end{bmatrix}$$

Therefore $s_{11}^{(2)} = 0.04$.

To get $s_{11}^{(3)}$, we take the $(1,4)$ entry of

$$(P^*)^3 - (P^*)^2 = \begin{bmatrix} 0 & -0.14 & 0 & 0.128 & 0 & -0.12 & 0.1 & 0 & -0.07 & 0.09 \\ 0 & 0 & 0 & 0.184 & 0 & 0 & -0.12 & 0 & 0 & -0.07 \\ 0 & 0 & 0 & 0.184 & 0 & 0 & -0.12 & 0 & 0 & -0.07 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.14 & 0.1 & 0 & 0.012 & -0.11 & 0.08 & 0.02 & -0.04 & 0.06 & 0.02 \\ 0 & -0.14 & 0 & 0.154 & 0 & -0.11 & 0.08 & 0 & -0.04 & 0.06 \\ 0 & 0 & 0 & 0.155 & 0 & 0 & -0.11 & 0 & 0 & -0.04 \\ -0.04 & 0.1 & 0 & 0.02 & -0.04 & 0.01 & 0.04 & -0.09 & -0.03 & 0.04 \\ 0 & -0.04 & 0 & 0.194 & 0 & -0.04 & 0.01 & 0 & -0.09 & -0.03 \\ 0 & 0 & 0 & 0.133 & 0 & 0 & -0.04 & 0 & 0 & -0.09 \end{bmatrix}$$

Therefore $s_{11}^{(3)} = 0.128$.

To get $s_{11}^{(4)}$, we take the $(1,4)$ entry of

$$(P^*)^4 - (P^*)^3 = \begin{bmatrix} 0 & -0.07 & 0 & 0.176 & 0 & -0.06 & 0.01 & 0 & -0.05 & 0 \\ 0 & 0 & 0 & 0.115 & 0 & 0 & -0.06 & 0 & 0 & -0.05 \\ 0 & 0 & 0 & 0.115 & 0 & 0 & -0.06 & 0 & 0 & -0.05 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.06 & 0.03 & 0 & 0.05 & -0.05 & -0.01 & 0.05 & -0.05 & -0.02 & 0.05 \\ 0 & -0.06 & 0 & 0.179 & 0 & -0.05 & -0.01 & 0 & -0.05 & -0.02 \\ 0 & 0 & 0 & 0.1 & 0 & 0 & -0.05 & 0 & 0 & -0.05 \\ -0.06 & -0.02 & 0 & 0.071 & -0.05 & -0.03 & 0.07 & -0.03 & -0.01 & 0.06 \\ 0 & -0.06 & 0 & 0.173 & 0 & -0.05 & -0.03 & 0 & -0.03 & -0.01 \\ 0 & 0 & 0 & 0.075 & 0 & 0 & -0.05 & 0 & 0 & -0.03 \end{bmatrix}$$

Therefore $s_{11}^{(4)} = 0.176$.

With three different methods, we obtain the same results, meaning that the methods we mentioned are reasonable and reliable.

3.2. Third Passage Probability

Define $t_{ij}^{(n)}$ to be the probability that given an initial state i, the process enters state j for the third time on step n.

3.2.1. From State i to State j (i≠j). Third passage probability is the extension of the first and second passage probability, so we can use the similar methods to calculate it.

Method 1

To reach a state for a third time, the process must reach it twice before the following first passage, so we have

$$t_{ij}^{(n)} = \sum_{k=1}^{n-2} s_{ij}^{(n-k)} f_{jj}^{(k)}$$

Applying this method to our example, we have

$t_{13}^{(1)} = 0$

$t_{13}^{(2)} = 0$

$t_{13}^{(3)} = s_{13}^{(2)} f_{33}^{(1)} = 0.004$

$t_{13}^{(4)} = s_{13}^{(3)} f_{33}^{(1)} + s_{13}^{(2)} f_{33}^{(2)} = 0.0168 + 0.0144 = 0.0312$

In particular, $t_{13}^{(1)} = 0$ and $t_{13}^{(2)} = 0$ because we cannot reach j thrice in only one step or two steps.

Method 2

To compute the third passage probability, we need to replicate the states thrice to construct a new Markov Chain whose first passage probability is equal to the third passage probability of the original chain. Now the state space is $\{1, 1', 1'', 2, 2', 2'', ..., i, i', i'', ..., j, j', j'', ...\}$. The same as before, all the states except j and j' can move between the states with the same mark. For state j, the process from it can only go to states with one prime mark because it is the destination of the first transition and the beginning of the second transition. Similarly, for state j', the process from it can only go to states with two prime marks because it is the destination of the second transition and the beginning of the third transition.

When we considering state 3 as the destination, the new matrix P is a enlarged 9×9 matrix with state space $\{1, 1', 1'', 2, 2', 2'', 3, 3', 3''\}$.

$$P = \begin{bmatrix} 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 \\ 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.4 \\ 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0.1 \\ 0 & 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0.1 \end{bmatrix}$$

The corresponding matrix P_0 is P with the 9-th column replaced by 0s.

$$P_0 = \begin{bmatrix} 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0 \end{bmatrix}$$

To get $t_{13}^{(n)}$, we take the $(1,9)$ entry of $P_0^{n-1}P$ as in the first and second passage probability.

Method 3

This time we replace the j'' column in P with a zero vector and add j''^* as the absorbing state, which is caused by the fact that j''^* is the final destination.

In our example from state 1 to state 3, P^* is a 10×10 matrix with state space $\{1, 1', 1'', 2, 2', 2'', 3, 3', 3'', 3''^*\}$, where $3''^*$ is the absorbing state.

$$P^* = \begin{bmatrix} 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.4 \\ 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.4 \\ 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

To get $t_{13}^{(n)}$, we take the $(1,10)$ entry of $(P^*)^n - (P^*)^{n-1}$.

3.2.2. From State i to State i. Third passage probability of return to state i at step n can also been obtained by the similar three methods.

Method 1

Substitute $j = i$ in $t_{ij}^{(n)} = \sum_{k=1}^{n} s_{ij}^{(n-k)} f_{jj}^{(k)}$, we get the formula for this special return case.

$$t_{ii}^{(n)} = \sum_{k=1}^{n} s_{ii}^{(n-k)} f_{ii}^{(k)}$$

Applying this method to our example, we have

$t_{11}^{(1)} = 0$

$t_{11}^{(2)} = 0$

$t_{11}^{(3)} = s_{11}^{(2)} f_{11}^{(1)} = 0.008$

$t_{11}^{(4)} = s_{11}^{(3)} f_{11}^{(1)} + s_{11}^{(2)} f_{11}^{(2)} = 0.0256 + 0.0128 = 0.0384$

In particular, $t_{11}^{(1)} = 0$ and $t_{11}^{(2)} = 0$ because we cannot reach state 1 thrice in only one step or two steps.

Method 2

To calculate the third return probability at step n, we replicate the states by four times to get a new chain with state space $\{1, 1', 1'', 1''', 2, 2', 2'', 2''', ..., i, i', i'', i''', ...\}$. Then we use the same method to construct a transition matrix P and a replaced matrix P_0, and the (i, i''') entry of $P_0^{n-1}P$ is the probability of the third recurrence to state i happening at step n.

When we considering state 1 as the destination, the new matrix P is a enlarged 12×12 matrix with state space $\{1, 1', 1'', 1''', 2, 2', 2'', 2''', 3, 3', 3'', 3'''\}$.

$$P = \begin{bmatrix} 0 & 0.2 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.4 \\ 0.3 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.4 \\ 0.5 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.1 \end{bmatrix}$$

The corresponding matrix P_0 is P with the fourth column replaced by 0s.

$$P_0 = \begin{bmatrix} 0 & 0.2 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.4 \\ 0.3 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.4 \\ 0.5 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.1 \end{bmatrix}$$

To get $t_{11}^{(n)}$, we take the $(1,4)$ entry of $P_0^{n-1}P$.

Method 3

Based upon the transition matrix P in method 2, we add i'''^{*} as the absorbing state. The same as before, we construct P^* and take the (i, i'''^{*}) entry as the third passage probability.

In our example from state 1 to state 1, P^* is a 13×13 matrix with state space $\{1, 1', 1'', 1''', 1'''^*, 2, 2', 2'', 2''', 3, 3', 3'', 3'''\}$ where $1'''^*$ is the absorbing state.

$$P^* = \begin{bmatrix} 0 & 0.2 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.4 \\ 0.5 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.1 \end{bmatrix}$$

To get $t_{11}^{(n)}$, we take the $(1, 5)$ entry of $(P^*)^n - (P^*)^{n-1}$.

3.3. K-th Passage Probability

By the analysis about first, second, and third passage probability, we can conclude the general theorem for k-th passage probability via three methods.

3.3.1. From State i to State j (i≠j). First we think about the situation that a process moves from one state, say i, to another different state, say j, at a given n step.

Method 1

Using formula

$$k_{ij}^{(n)} = \sum_{x=1}^{n-k+1} (k-1)_{ij}^{(n-x)} f_{jj}^{(x)}$$

where $k_{ij}^{(n)}$ denote the probability that starting in i, the k-th transition into j occurs at step n.

But for this method, if we want $k_{ij}^{(n)}$, we must first calculate a number of first passage probabilities to (k-1)-th passage probabilities, which is a little sophisticated.

Method 2

The thought behind this method is to construct a new Markov Chain whose first passage probability is the k-th passage probability of the original Markov Chain. So we replicate the state space k times, and build a new transition matrix P with this state space where we link all the states other than states related to j to the corresponding states with the transition probabilities the same with the original chain. And for states related to j, the process starting there could only go to their replicated states with the original transition probability. Then we replace the entries in the final destination column by 0s to get a matrix called P_0. Last we take $(i, j^{k\prime})$ entry of the $P_0^{n-1}P$ as the k-th passage probability. Here $j^{k\prime}$ denotes the state j with k prime marks.

Method 3

Based on the new transition matrix P in method 2, we add an absorbing state about j, and the original probabilities of other states to the final destination actually are the probabilities of other states to the absorbing state. Besides, the vector in the final destination column is zero vector. And we call this matrix P^*. The k-th passage probability is the $(i, j^{k\prime *})$ entry of the $(P^*)^n - (P^*)^{n-1}$.

3.3.2. From State i to State i. By the analysis about first, second, and third passage probability, we can conclude the general theorem for k-th passage probability for a give n step from one state i returning to itself via three methods.

Method 1

Using formula

$$k_{ii}^{(n)} = \sum_{x=1}^{n} (k-1)_{ii}^{(n-x)} f_{ii}^{(x)}$$

where $k_{ii}^{(n)}$ denote the probability that starting in i, the k-th transition into i occurs at step n.

Method 2

In this method, we replicate the state space $k+1$ times, and build a new transition matrix P with this state space where we link all the states other than states related to i to the corresponding states with the transition probabilities the same with the original chain. And for states related to i, the process starting there could only go to their replicated states with the original transition probability. Then we replace the entries in the final destination column by 0s to get a matrix called P_0. Last we take $(i, i^{(k+1)\prime})$ entry of the $P_0^{n-1}P$ as the k-th passage probability.

Method 3

Based on the new transition matrix P in method 2, we add an absorbing state about i, and the original probabilities of other states to the final destination actually are the probabilities of other states to the absorbing state. Besides, the vector in the final destination column is zero vector. And we call this matrix P^*. The k-th passage probability is the $(i, i^{(k+1)\prime *})$ entry of the $(P^*)^n - (P^*)^{n-1}$.

CHAPTER 4

First Passage Probability Between Multiple States

In this chapter, we consider first passage probability between several states, which means the probabilities of a process moving to any one of states or moving to all of some set of states the first time in a given step n, including situations where the process goes to different states and returns to the initial state at the end. We can still solve this problem by our three methods.

4.1. From State i to State j or k (i≠j,i≠k,j≠k)

In this section, we discuss the probability of the transition from state i to state j or state k, where i, j, and k are different states in a discrete Markov Chain. This is to say we count the probability that a process reaches state j and we also count the probability of this process reaching state k. The emphasis of solving this problem is that we can collapse state j and state k into one state, denoted by $(j \vee k)$, because there is no difference between state j and state k if they are taken as destinations. We will use three methods to calculate the first passage probability with the Markov Chain in chapter 2 as an example.

Define $f_{i(j\vee k)}^{(n)}$ to be the first transition from state i to state j or state k occurs at step n. And we use the process from state 1 to state 2 or state 3 as the example to illustrate the methods.

As mentioned above, we collapse state j and state k into state $(j \vee k)$, so we should do some modifications to the transition matrix. Especially, in the new transition matrix P,

$$p_{s(j\vee k)} = p_{sj} + p_{sk}$$

where s is a state other than $(j \vee k)$ in the state space of the Markov Chain.

This is because the probability of a process moving from one state s to state j or state k is the probability of this state to state j plus that of the initial state to state k and the probability of a process moving from state j or state k to another state s is the sum of p_{js} and p_{ks}. And the value of $p_{(j\vee k)s}$ does not affect the calculation of first passage probability because we do not need to start from this collapsed state, so we can put letters in that row for understanding consideration.

When we considering state 2 or state 3 are the destination, we have $p_{1(2\vee 3)} = p_{12} + p_{13} = 0.8$, $p_{(2\vee 3)1} = a$, and $p_{(2\vee 3)(2\vee 3)} = 1 - a$.

Method 1

If we calculate the probability of a process starting from state i to state j or state k in n steps, the process must go to state j or state k at some step for the first time and stay there until the n-th step. This straightforward idea gives the formula

$$p_{i(j\vee k)}^{(n)} = \sum_{k=1}^{n} p_{(j\vee k)(j\vee k)}^{(n-k)} f_{i(j\vee k)}^{(k)}$$

Applying this method to our example, we have

$p_{1(2\vee 3)}^{(1)} = f_{1(2\vee 3)}^{(1)} = 0.8$

Thus $f_{1(2\vee 3)}^{(1)} = 0.8$.

$p_{1(2\vee 3)}^{(2)} = 0.16 + 0.8(1-a) = f_{1(2\vee 3)}^{(1)} p_{(2\vee 3)(2\vee 3)}^{(1)} + f_{1(2\vee 3)}^{(2)} = 0.8(1-a) + f_{1(2\vee 3)}^{(2)}$

Thus $f_{1(2\vee 3)}^{(2)} = 0.16$.

$p_{1(2\vee 3)}^{(3)} = 0.032 + 0.64a + 0.16(1-a) + 0.8(1-a)^2 = f_{1(2\vee 3)}^{(1)} p_{(2\vee 3)(2\vee 3)}^{(2)} + f_{1(2\vee 3)}^{(2)} p_{(2\vee 3)(2\vee 3)}^{(1)} + f_{1(2\vee 3)}^{(3)} = 0.8(0.8a + (1-a)^2) + 0.16(1-a) + f_{1(2\vee 3)}^{(3)}$

Thus $f_{1(2\vee 3)}^{(3)} = 0.032$.

$p_{1(2\vee 3)}^{(4)} = 0.0064 + 0.256a + 0.032(1-a) + 1.28a(1-a) + 0.16(1-a)^2 + 0.8(1-a)^3 = f_{1(2\vee 3)}^{(1)} p_{(2\vee 3)(2\vee 3)}^{(3)} + f_{1(2\vee 3)}^{(2)} p_{(2\vee 3)(2\vee 3)}^{(2)} + f_{1(2\vee 3)}^{(3)} p_{(2\vee 3)(2\vee 3)}^{(1)} + f_{1(2\vee 3)}^{(4)} = 0.8(0.16a + 1.6a(1-a) + (1-a)^3) + 0.16(0.8a + (1-a)^2) + 0.032(1-a) + f_{1(2\vee 3)}^{(4)}$

Thus $f_{13}^{(4)} = 0.0064$.

Method 2

The $(i,(j\vee k))$ entry of $P_0^{n-1}P$ is the probability we want with specific step n, where P_0 is P with the $(j\vee k)$ column replaced by 0s.

We can write the transition probabilities in the matrix form with state space $\{1,(2\vee 3)\}$.

$$P = \begin{bmatrix} 0.2 & 0.8 \\ a & 1-a \end{bmatrix}$$

The corresponding matrix P_0 is P with the second column replaced by 0s.

$$P_0 = \begin{bmatrix} 0.2 & 0 \\ a & 1-a \end{bmatrix}$$

To get $f_{1(2\vee 3)}^{(1)}$, we take the $(1, 2)$ entry of

$$P_0^0 P = \begin{bmatrix} 0.2 & 0.8 \\ a & b \end{bmatrix}$$

So $f_{1(2\vee 3)}^{(1)} = 0.8$.

To get $f_{1(2\vee 3)}^{(2)}$, we take the $(1, 2)$ entry of

$$P_0^1 P = \begin{bmatrix} 0.04 & 0.16 \\ 0.2a & 0.8a \end{bmatrix}$$

So $f_{1(2\vee 3)}^{(2)} = 0.16$.

To get $f_{1(2\vee 3)}^{(3)}$, we take the $(1, 2)$ entry of

$$P_0^2 P = \begin{bmatrix} 0.008 & 0.032 \\ 0.04a & 0.16a \end{bmatrix}$$

So $f_{1(2\vee 3)}^{(3)} = 0.032$.

To get $f_{1(2\vee 3)}^{(4)}$, we take the $(1, 2)$ entry of

$$P_0^3 P = \begin{bmatrix} 0.0016 & 0.0064 \\ 0.008a & 0.032a \end{bmatrix}$$

So $f_{1(2\vee 3)}^{(4)} = 0.0064$.

Method 3

Based on the transition matrix P in method 2, we add an absorbing state $(j \vee k)^*$ and thus getting a new transition matrix P^*. Whenever the process reaches state $(j \vee k)$, it gets into state $(j \vee k)^*$ immediately and cannot get out. So the entries in $(j \vee k)$ column of the P^* are 0s and the entries in $(j \vee k)^*$ column are probabilities to state $(j \vee k)$. After calculating $(P^*)^n - (P^*)^{n-1}$, we take $(i,(j \vee k)^*)$ entry as the probability of first transition from state i to state j or state k occurs at step n.

In our example from state 1 to state 2 or state 3, P^* is a 3×3 matrix with state space $\{1,(2 \vee 3),(2 \vee 3)^*\}$ where $(2 \vee 3)^*$ is the absorbing state.

$$P^* = \begin{bmatrix} 0.2 & 0 & 0.8 \\ a & 0 & 1-a \\ 0 & 0 & 1 \end{bmatrix}$$

To get $f^{(1)}_{1(2\vee 3)}$, we take the $(1,3)$ entry of

$$P^* = \begin{bmatrix} 0.2 & 0 & 0.8 \\ a & 0 & 1-a \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore $f^{(1)}_{1(2\vee 3)} = 0.8$.

To get $f^{(2)}_{1(2\vee 3)}$, we take the $(1,3)$ entry of

$$(P^*)^2 - P^* = \begin{bmatrix} -0.16 & 0 & 0.16 \\ -0.8a & 0 & 0.8a \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $f^{(2)}_{1(2\vee 3)} = 0.16$.

To get $f^{(3)}_{1(2\vee 3)}$, we take the $(1,3)$ entry of

$$(P^*)^3 - (P^*)^2 = \begin{bmatrix} -0.032 & 0 & 0.032 \\ -0.16a & 0 & 0.16a \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $f^{(3)}_{1(2\vee 3)} = 0.032$.

To get $f^{(4)}_{1(2\vee 3)}$, we take the $(1,3)$ entry of

$$(P^*)^4 - (P^*)^3 = \begin{bmatrix} -0.0064 & 0 & 0.0064 \\ -0.032a & 0 & 0.032a \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $f_{1(2\vee 3)}^{(4)} = 0.0064$.

With three different methods, we obtain the same results, meaning that the methods we mentioned are reasonable and reliable.

4.2. From State i to State i or j (i≠j)

In this section, we discuss the situation that a process moves from state i to state i or state j, in other words, returns to state i or reaches state j. The main difference between this section and section 4.1 lies in the construction of matrices, since in section 4.1, we can easily collapse destination states into a new state in transition matrix since it dose not affect the start state. But in this case, we cannot just do the same thing because if state i and state j were collapsed into $(i \vee j)$, we cannot distinguish whether the process starts at state i or state j. Therefore, we have to modify the methods. Additionally, we use the process from state 1 to state 1 or state 3 as the example. Define $f_{i(i\vee j)}^{(n)}$ to be the probability that the first transition from state i to state i or state j occurs at step n.

As mentioned above, we can first collapse state i and state j into state $(i \vee j)$, but we should do some modifications to the transition matrix and calculation methods. Especially, when we constructing the new transition matrix P, we let

$$p_{s(i\vee j)} = p_{si} + p_{sj}$$

$$p_{(i\vee j)s} = p_{is}$$

where s is a state other than $(i \vee j)$ in the state space of the Markov Chain.

This is because the probability of a process moving from one state s to state i or state j is the probability of this state to state i plus that of the initial state to state j and the probability of a process moving from state i or state j to another state s is the sum of p_{is} and p_{js}, but the process always starts at state i and has no chance to go from j.

When we considering state 1 or state 3 are the destination, we have $p_{2(1\vee 3)} = p_{21} + p_{23} = 0.7$ and $p_{(1\vee 3)2} = p_{12} = 0.4$, so $p_{(1\vee 3)(1\vee 3)} = 0.6$.

Method 1

If we want to calculate the first passage probability step by step, we just put $j = i$ and $k = j$ in the formula in section 4.1 because this is a special case when $j = i$ and $k = j$, and we get

$$p_{i(i\vee j)}^{(n)} = \sum_{k=1}^{n} p_{(i\vee j)(i\vee j)}^{(n-k)} f_{i(i\vee j)}^{(k)}$$

Applying this method to our example where state 1 moves to state 1 or state 3, we have

$p_{1(1\vee 3)}^{(1)} = f_{1(1\vee 3)}^{(1)} = 0.6$

Thus $f_{1(1\vee 3)}^{(1)} = 0.6$.

$p_{1(1\vee 3)}^{(2)} = f_{1(1\vee 3)}^{(1)} p_{(1\vee 3)(1\vee 3)}^{(1)} + f_{1(1\vee 3)}^{(2)} = 0.36 + f_{11}^{(2)} = 0.64$

Thus $f_{1(1\vee 3)}^{(2)} = 0.28$.

$p_{1(1\vee 3)}^{(3)} = f_{1(1\vee 3)}^{(1)} p_{(1\vee 3)(1\vee 3)}^{(2)} + f_{1(1\vee 3)}^{(2)} p_{(1\vee 3)(1\vee 3)}^{(1)} + f_{1(1\vee 3)}^{(3)} = 0.384 + 0.168 + f_{11}^{(3)} = 0.636$

Thus $f_{1(1\vee 3)}^{(3)} = 0.084$.

$p_{1(1\vee 3)}^{(4)} = f_{1(1\vee 3)}^{(1)} p_{(1\vee 3)(1\vee 3)}^{(3)} + f_{1(1\vee 3)}^{(2)} p_{(1\vee 3)(1\vee 3)}^{(2)} + f_{1(1\vee 3)}^{(3)} p_{(1\vee 3)(1\vee 3)}^{(1)} + f_{1(1\vee 3)}^{(4)} = 0.3816 + 0.1792 + 0.0504 + f_{11}^{(4)} = 0.6364$

Thus $f_{11}^{(4)} = 0.0252$.

Method 2

To obtain the first passage probability from state i to state i or j, we need to replicate the states, so now the state space is $\{1, 1', 2, 2', ..., (i \vee j), (i \vee j)', ...\}$. For the original states other than destination state $(i \vee j)$, the transition probabilities between them remains the same. For all the replicated states, they could enter any other replicated states with the transition probabilities the same with the original chain. For state $(i \vee j)$, the process starting from it could only go to the replicated states with corresponding transition probabilities. And the matrix constructed by this procedure is the transition matrix P under this method. Then by replacing $(i \vee j)'$ column in P by 0s to get P_0, we can take $((i \vee j), (i \vee j)')$ entry of the product of $P_0^{n-1}P$ as the probability of first returning to state i or reaching state j at some step n.

When we considering state 1 or state 3 as the destination, the new matrix P is a enlarged 4×4 matrix with state space $\{(1 \vee 3), (1 \vee 3)', 2, 2'\}$.

$$P = \begin{bmatrix} 0 & 0.6 & 0 & 0.4 \\ 0 & 0.6 & 0 & 0.4 \\ 0.7 & 0 & 0.3 & 0 \\ 0 & 0.7 & 0 & 0.3 \end{bmatrix}$$

The corresponding matrix P_0 is P with the second column replaced by 0s.

$$P_0 = \begin{bmatrix} 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0.4 \\ 0.7 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.3 \end{bmatrix}$$

To get $f^{(1)}_{1(1\vee 3)}$, we take the $(1,2)$ entry of

$$P_0^0 P = \begin{bmatrix} 0 & 0.6 & 0 & 0.4 \\ 0 & 0.6 & 0 & 0.4 \\ 0.7 & 0 & 0.3 & 0 \\ 0 & 0.7 & 0 & 0.3 \end{bmatrix}$$

So $f^{(1)}_{1(1\vee 3)} = 0.6$.

To get $f^{(2)}_{1(1\vee 3)}$, we take the $(1,2)$ entry of

$$P_0^1 P = \begin{bmatrix} 0 & 0.28 & 0 & 0.12 \\ 0 & 0.28 & 0 & 0.12 \\ 0.21 & 0.42 & 0.09 & 0.28 \\ 0 & 0.21 & 0 & 0.09 \end{bmatrix}$$

So $f^{(2)}_{1(1\vee 3)} = 0.28$.

To get $f^{(3)}_{1(1\vee 3)}$, we take the $(1,2)$ entry of

$$P_0^2 P = \begin{bmatrix} 0 & 0.084 & 0 & 0.036 \\ 0 & 0.084 & 0 & 0.036 \\ 0.063 & 0.322 & 0.027 & 0.168 \\ 0 & 0.063 & 0 & 0.027 \end{bmatrix}$$

So $f^{(3)}_{1(1\vee 3)} = 0.084$.

To get $f^{(4)}_{1(1\vee 3)}$, we take the $(1,2)$ entry of

$$P_0^3 P = \begin{bmatrix} 0 & 0.0252 & 0 & 0.0108 \\ 0 & 0.0252 & 0 & 0.0108 \\ 0.0189 & 0.1554 & 0.0081 & 0.0756 \\ 0 & 0.0189 & 0 & 0.0081 \end{bmatrix}$$

So $f^{(4)}_{1(1\vee 3)} = 0.0252$.

Method 3

Based on the transition matrix P in method 2, we build a matrix P^* by adding an absorbing state $(i \vee j)'^*$ and putting all the probabilities to state $(i \vee j)'$ to the $(i \vee j)'^*$ column, so entries in the $(i \vee j)'$ column are 0s. Then the $((i \vee j), (i \vee j)'^*)$ entry of the $(P^*)^n - (P^*)^{n-1}$ is the first transition probability to state i or state j at n-th step.

In our example from state 1 to state 1 or state 3, P^* is a 5×5 matrix with state space $\{(1 \vee 3), (1 \vee 3)', (1 \vee 3)'^*, 2, 2'\}$, where $(1 \vee 3)'^*$ is the absorbing state.

$$P^* = \begin{bmatrix} 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 1 & 0 & 0 \\ 0.7 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0.7 & 0 & 0.3 \end{bmatrix}$$

To get $f^{(1)}_{1(1\vee 3)}$, we take the $(1,3)$ entry of

$$P^* = \begin{bmatrix} 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 1 & 0 & 0 \\ 0.7 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0.7 & 0 & 0.3 \end{bmatrix}$$

Therefore $f^{(1)}_{1(1\vee 3)} = 0.6$.

To get $f^{(2)}_{1(1\vee 3)}$, we take the $(1,3)$ entry of

$$(P^*)^2 - P^* = \begin{bmatrix} 0 & 0 & 0.28 & 0 & -0.28 \\ 0 & 0 & 0.28 & 0 & -0.28 \\ 0 & 0 & 0 & 0 & 0 \\ -0.49 & 0 & 0.42 & -0.21 & 0.28 \\ 0 & 0 & 0.21 & 0 & -0.21 \end{bmatrix}$$

Therefore $f^{(2)}_{1(1\vee 3)} = 0.28$.

To get $f^{(3)}_{1(1\vee 3)}$, we take the $(1,3)$ entry of

$$(P^*)^3 - (P^*)^2 = \begin{bmatrix} 0 & 0 & 0.084 & 0 & -0.084 \\ 0 & 0 & 0.084 & 0 & -0.084 \\ 0 & 0 & 0 & 0 & 0 \\ -0.147 & 0 & 0.322 & -0.063 & -0.112 \\ 0 & 0 & 0.063 & 0 & -0.063 \end{bmatrix}$$

Therefore $f^{(3)}_{1(1\vee 3)} = 0.084$.

To get $f^{(4)}_{1(1\vee 3)}$, we take the $(1,3)$ entry of

$$(P^*)^4 - (P^*)^3 = \begin{bmatrix} 0 & 0 & 0.0252 & 0 & -0.0252 \\ 0 & 0 & 0.0252 & 0 & -0.0252 \\ 0 & 0 & 0 & 0 & 0 \\ -0.0441 & 0 & 0.1554 & -0.0189 & -0.0924 \\ 0 & 0 & 0.0189 & 0 & -0.0189 \end{bmatrix}$$

Therefore $f_{1(1\vee 3)}^{(4)} = 0.0252$.

With three different methods, we obtain the same results, meaning that the methods we mentioned are reasonable and reliable.

4.3. From State i to State j and k (i≠j,i≠k,j≠k)

In this section, we focus on a process moving from state i to states j and k, where i, j, and k are different states. This means the process successes once only if it reaches both states j and k, which includes two cases, one of which is the process first passes state j without reaching state k and then passes state k, and the other of which is the process first passes state k without reaching state j and then passes state j. We use the transition from state 1 to state 2 and state 3 as the example to illustrate the analysis.

Define $f_{i(j\wedge k)}^{(n)}$ to be the first transition from state i to state j and state k occurs at step n.

Method 1

The calculation of $f_{i(j\wedge k)}^{(n)}$ is a little complicated because there exist states that the process cannot reach during the transition, so we have to sum the probability of the two cases, which gives the formula

$$f_{i(j\wedge k)}^{(n)} = \sum_{k=1}^{n} f_{ij\backslash k}^{(n-k)} f_{jk}^{(k)} + f_{ik\backslash j}^{(n-k)} f_{kj}^{(k)}$$

where $f_{ij\backslash k}^{(n)}$ denotes the first passage probability of a transition from state i to state j without reaching state k occurs at step n.

Applying this method to our example, we have

$f_{1(2\wedge 3)}^{(1)} = 0$

$f_{1(2\wedge 3)}^{(2)} = f_{12\backslash 3}^{(1)} f_{23}^{(1)} + f_{13\backslash 2}^{(1)} f_{32}^{(1)} = 0.32$

$f_{1(2\wedge 3)}^{(3)} = f_{12\backslash 3}^{(2)} f_{23}^{(1)} + f_{12\backslash 3}^{(1)} f_{23}^{(2)} + f_{13\backslash 2}^{(2)} f_{32}^{(1)} + f_{13\backslash 2}^{(1)} f_{32}^{(2)} = 0.256$

$f_{1(2\wedge 3)}^{(4)} = f_{12\backslash 3}^{(3)} f_{23}^{(1)} + f_{12\backslash 3}^{(2)} f_{23}^{(2)} + f_{12\backslash 3}^{(1)} f_{23}^{(3)} + f_{13\backslash 2}^{(3)} f_{32}^{(1)} + f_{13\backslash 2}^{(2)} f_{32}^{(2)} + f_{13\backslash 2}^{(1)} f_{32}^{(3)} = 0.1664$

In particular, $f_{1(2\wedge 3)}^{(1)} = 0$ because we cannot reach both state 2 and state 3 in only one step.

Method 2

During the transition, there is some time that the process does not reach state j and state k, and some time that the process dose not reach state j, and some time that the process does not reach state k, so besides the transition matrix P, we need a P_{01} where the j-th and k-th columns are replaced by 0s, a P_{02} where the k-th column are replaced by 0s, and a P_{03} where the j-th column are replaced by 0s. Moreover, like the cases mentioned in method 1, we take

$$f_{i(j\wedge k)}^{(n)} = \sum_{k=0}^{n-2} ((P_{01}^{n-2-k} P_{02})_{ij} (P_{02}^{k} P)_{jk} + (P_{01}^{n-2-k} P_{03})_{ik} (P_{03}^{k} P)_{kj})$$

to get the first passage probability of the transition from state i to state j and state k at step n. In this formula, $(P_{01}P_{02})_{ij}$ represents the (i,j) entry of the product $P_{01}P_{02}$.

When we considering state 2 and state 3 as the destination,

$$P_{01} = \begin{bmatrix} 0.2 & 0 & 0 \\ 0.3 & 0 & 0 \\ 0.5 & 0 & 0 \end{bmatrix}$$

$$P_{02} = \begin{bmatrix} 0.2 & 0.4 & 0 \\ 0.3 & 0.3 & 0 \\ 0.5 & 0.4 & 0 \end{bmatrix}$$

$$P_{03} = \begin{bmatrix} 0.2 & 0 & 0.4 \\ 0.3 & 0 & 0.4 \\ 0.5 & 0 & 0.1 \end{bmatrix}$$

To get $f_{1(2\wedge 3)}^{(1)}$,

$f_{1(2\wedge 3)}^{(1)} = 0$

To get $f_{1(2\wedge 3)}^{(2)}$,

$f_{1(2\wedge 3)}^{(2)} = (P_{02})_{12}(P)_{23} + (P_{03})_{13}(P)_{32} = 0.32$

To get $f_{1(2\wedge 3)}^{(3)}$,

$f_{1(2\wedge 3)}^{(3)} = (P_{01}P_{02})_{12}(P)_{23} + (P_{02})_{12}(P_{02}P)_{23} + (P_{01}P_{03})_{13}(P)_{32} + (P_{03})_{13}(P_{03}P)_{32} = 0.256$

To get $f^{(4)}_{1(2\wedge 3)}$,

$f^{(4)}_{1(2\wedge 3)} = (P_{01}^2 P_{02})_{12}(P)_{23} + (P_{01}P_{02})_{12}(P_{02}P)_{23} + (P_{02})_{12}(P_{02}^2 P)_{23} + (P_{01}^2 P_{03})_{13}(P)_{32} + (P_{01}P_{03})_{13}(P_{03}P)_{32} + (P_{03})_{13}(P_{03}^2 P)_{32} = 0.1664$

Method 3

Based on the transition matrix P, we add an absorbing state j^* to construct P_1 and use the same method to get P_1^*. Besides, we add an absorbing state k^* to construct P_2 and use the same method to get P_2^*. Similarly, we have

$$
\begin{aligned}
f^{(n)}_{i(j\wedge k)} = \sum_{k=1}^{n-1} & (((P_1^*)^{n-k} - (P_1^*)^{n-k-1})_{ij^*}(P_2^k - P_2^{k-1})_{jk^*} \\
& + ((P_2^*)^{n-k} - (P_2^*)^{n-k-1})_{ik^*}(P_1^k - P_1^{k-1})_{kj^*})
\end{aligned}
$$

In our example from state 1 to state 2 and state 3, P_1 is a 4×4 matrix with state space $\{1, 2, 2^*, 3\}$, where 2^* is the absorbing state.

$$
P_1 = \begin{bmatrix} 0.2 & 0 & 0.4 & 0.4 \\ 0.3 & 0 & 0.3 & 0.4 \\ 0 & 0 & 1 & 0 \\ 0.5 & 0 & 0.4 & 0.1 \end{bmatrix}
$$

And the corresponding P_1^* is P_1 with the fourth column replaced by 0s.

$$
P_1^* = \begin{bmatrix} 0.2 & 0 & 0.4 & 0 \\ 0.3 & 0 & 0.3 & 0 \\ 0 & 0 & 1 & 0 \\ 0.5 & 0 & 0.4 & 0 \end{bmatrix}
$$

P_2 is a 4×4 matrix with state space $\{1, 2, 3, 3^*\}$, where 3^* is the absorbing state.

$$
P_2 = \begin{bmatrix} 0.2 & 0.4 & 0 & 0.4 \\ 0.3 & 0.3 & 0 & 0.4 \\ 0.5 & 0.4 & 0 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

And the corresponding P_2^* is P_1 with the second column replaced by 0s.

$$P_2^* = \begin{bmatrix} 0.2 & 0 & 0 & 0.4 \\ 0.3 & 0 & 0 & 0.4 \\ 0.5 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To get $f_{1(2\wedge 3)}^{(1)}$,

$f_{1(2\wedge 3)}^{(1)} = 0$

To get $f_{1(2\wedge 3)}^{(2)}$,

$f_{1(2\wedge 3)}^{(2)} = (P_1^*)_{13}(P_2)_{24} + (P_2^*)_{14}(P_1)_{43} = 0.32$

To get $f_{1(2\wedge 3)}^{(3)}$,

$f_{1(2\wedge 3)}^{(3)} = ({P_1^*}^2 - P_1^*)_{13}(P_2)_{24} + (P_1^*)_{13}({P_2}^2 - P_2)_{24} + ({P_2^*}^2 - P_2^*)_{14}(P_1)_{43} + (P_2^*)_{14}({P_1}^2 - P_1)_{43} = 0.256$

To get $f_{1(2\wedge 3)}^{(4)}$,

$f_{1(2\wedge 3)}^{(4)} = ({P_1^*}^3 - {P_1^*}^2)_{13}(P_2)_{24} + ({P_1^*}^2 - P_1^*)_{13}({P_2}^2 - P_2)_{24} + (P_1^*)_{13}({P_2}^3 - {P_2}^2)_{24} + ({P_2^*}^3 - {P_2^*}^2)_{14}(P_1)_{43} + ({P_2^*}^2 - P_2^*)_{14}({P_1}^2 - P_1)_{43} + (P_2^*)_{14}({P_1}^3 - {P_1}^2)_{43} = 0.1664$

With three different methods, we obtain the same results, meaning that the methods we mentioned are reasonable and reliable.

4.4. From State i to State i and j (i≠j)

In this section, we talk about one special case of section 4.3, which is to say a process starts at state i, and after n steps, returns to itself or moves to a different state j for certain times. Besides, we take a transition from state 1 to state 1 and 3 as the example.

Define $f_{i(i\wedge j)}^{(n)}$ to be probability that the first transition from state i to state i and state j occurs at step n.

Method 1

When applying the first method, we substitute $j = i$ and $k = j$ in $f_{i(j\wedge k)}^{(n)} = \sum_{k=1}^{n} f_{ij\backslash k}^{(n-k)} f_{jk}^{(k)} + f_{ik\backslash j}^{(n-k)} f_{kj}^{(k)}$ and obtain

$$f_{i(i\wedge j)}^{(n)} = \sum_{k=1}^{n} f_{ii\backslash j}^{(n-k)} f_{ij}^{(k)} + f_{ij\backslash j}^{(n-k)} f_{ji}^{(k)}$$

Applying this method to our example, we have

$f^{(1)}_{1(1\wedge 3)} = 0$

$f^{(2)}_{1(1\wedge 3)} = f^{(1)}_{11\backslash 3}f^{(1)}_{13} + f^{(1)}_{13\backslash 1}f^{(1)}_{31} = 0.28$

$f^{(3)}_{1(1\wedge 3)} = f^{(2)}_{11\backslash 3}f^{(1)}_{13} + f^{(1)}_{11\backslash 3}f^{(2)}_{13} + f^{(2)}_{13\backslash 1}f^{(1)}_{31} + f^{(1)}_{13\backslash 1}f^{(2)}_{31} = 0.244$

$f^{(4)}_{1(1\wedge 3)} = f^{(3)}_{11\backslash 3}f^{(1)}_{13} + f^{(2)}_{11\backslash 3}f^{(2)}_{13} + f^{(1)}_{11\backslash 3}f^{(3)}_{13} + f^{(3)}_{13\backslash 1}f^{(1)}_{31} + f^{(2)}_{13\backslash 1}f^{(2)}_{31} + f^{(1)}_{13\backslash 1}f^{(3)}_{31} = 0.1756$

In particular, $f^{(1)}_{1(1\wedge 3)} = 0$ because we cannot reach both state 1 and state 3 in only one step.

We can also use matrices to compute the probability, but as what we have done many times in previous sections, every time the destination states contain the beginning state, we need to replicate the space state once more, causing the calculation more cumbersome. Consequently, there is no need to perform matrix-version methods here.

CHAPTER 5

Mean Passage Steps

In this chapter, we develop method to calculate the mean steps needed to transit from a state to another state. Let $\mu_{ij}^{(k)}$ be the mean steps of a transition from state i to a different state, j, for k times and similarly, $\mu_{ii}^{(k)}$ be the mean steps of a recurrence from state i to itself for k times.

5.1. Mean First Passage Steps

First we begin with the mean first passage steps $\mu_{ij}^{(1)}$. If we have a process starting at state i and targeting at state j and let it go, it could have two cases. On one hand, it moves directly to state j with probability p_{ij}, thus it only needs one step to state j. On the other hand, it moves to a state other than j, say s, with probability p_{is}. Under such circumstances, it has used one step and needs $\mu_{sj}^{(1)}$ more steps to move to j. Applying this thought, we have system equations

$$\mu_{ij}^{(1)} = \sum_{s\in S, s\neq j} p_{is}(1+\mu_{sj}^{(1)}) + p_{ij}$$

where S is the state space of a Markov Chain.

So if we are interested in $\mu_{ij}^{(1)}$, we also need to calculate $\mu_{sj}^{(1)}$ for each state s. And in order to calculate $\mu_{sj}^{(1)}$, we substitute $s = i$ in the formula and so on. This is the reason why they are called system equations.

For the process moving from state i to state i, we take $j = i$ in it and get a similar formula

$$\mu_{ii}^{(1)} = \sum_{s\in S, s\neq i} p_{is}(1+\mu_{si}^{(1)}) + p_{ii}$$

We can take the mean first passage steps from state 1 to state 3 and from state 3 to state 3 as examples to illustrate the calculation process.

For $\mu_{13}^{(1)}$, we establish

$$\mu_{13}^{(1)} = 0.2\times(1+\mu_{13}^{(1)}) + 0.4\times(1+\mu_{23}^{(1)}) + 0.4$$

Since there is an unknown $\mu_{23}^{(1)}$ in it, we need to solve $\mu_{23}^{(1)}$ as below.

$$\mu_{23}^{(1)} = 0.3 \times (1 + \mu_{13}^{(1)}) + 0.3 \times (1 + \mu_{23}^{(1)}) + 0.4$$

By solving both equations, we obtain $\mu_{13}^{(1)} = 2.5$ and $\mu_{23}^{(1)} = 2.5$.

For $\mu_{33}^{(1)}$, we have

$$\mu_{33}^{(1)} = 0.5 \times (1 + \mu_{13}^{(1)}) + 0.4 \times (1 + \mu_{23}^{(1)}) + 0.1 = 3.25$$

5.2. Mean Second Passage Steps

Now we consider the mean steps of second passage from state i to state j, $\mu_{ij}^{(2)}$. Second passage step is the convolution of two first passage steps since if a process wants to passage state j twice from state i, it should first reach j from i and then reach j from j again, the mean steps needed for second passage is the mean steps needed for first passage plus the mean steps needed for first recurrence. That is to say

$$\mu_{ij}^{(2)} = \mu_{ij}^{(1)} + \mu_{jj}^{(1)}$$

We can also prove this formula mathematically.

Proof.

$$\begin{aligned}
\mu_{ij}^{(2)} &= E(ns_{ij}^{(n)}) \\
&= E(\sum_{s=1}^{n}\sum_{t=1}^{n-s}(s+t)f_{ij}^{(s)}f_{jj}^{(t)}) \\
&= E(\sum_{s=1}^{n}\sum_{t=1}^{n-s}sf_{ij}^{(s)}f_{jj}^{(t)} + \sum_{s=1}^{n}\sum_{t=1}^{n-s}tf_{ij}^{(s)}f_{jj}^{(t)}) \\
&= E(\sum_{s=1}^{n}sf_{ij}^{(s)}\sum_{t=1}^{n-s}f_{jj}^{(t)} + \sum_{t=1}^{n-s}tf_{jj}^{(t)}\sum_{s=1}^{n}f_{ij}^{(s)}) \\
&= E(\sum_{s=1}^{n}sf_{ij}^{(s)}) + E(\sum_{t=1}^{n-s}tf_{jj}^{(t)}) \\
&= \mu_{ij}^{(1)} + \mu_{jj}^{(1)}
\end{aligned}$$

Suppose we want to know the mean steps of a transition from state 1 to state 3 for twice, we should perform $\mu_{13}^{(2)} = \mu_{13}^{(1)} + \mu_{33}^{(1)}$.

Thus $\mu_{13}^{(2)} = \mu_{13}^{(1)} + \mu_{33}^{(1)} = 2.5 + 3.25 = 5.75$.

5.3. Mean Third Passage Steps

For mean third passage steps $mu_{ij}^{(3)}$, it is the convolution of a second passage step and two first passage steps or three first passage steps. If we consider it contains the mean steps needed for first passage from state i to j plus the mean steps needed for twice recurrence from state j to state j, we have

$$\mu_{ij}^{(2)} = \mu_{ij}^{(1)} + 2 \times \mu_{jj}^{(1)}$$

The proof is similar to the proof in section 5.2, so we omit it here.

Applying it to the case that how many steps needed of a process moves from state 1 to state 3 for the third time on average, we obtain

$$\mu_{13}^{(2)} = \mu_{13}^{(1)} + \mu_{33}^{(1)} = 2.5 + 2 \times 3.25 = 9$$

5.4. Mean K-th Passage Steps

As mentioned above, the mean steps of a transition from state i to state j for k times $\mu_{ij}^{(k)}$ is given by

$$\mu_{ij}^{(k)} = \mu_{ij}^{(1)} + (k-1) \times \mu_{jj}^{(1)}$$

And this is also true for the case $j = i$.

Therefore, $\mu_{ij}^{(k)}$ is only determined by $\mu_{ij}^{(1)}$ and $\mu_{jj}^{(1)}$, but which are related to all the probabilities in the transition matrix of a Markov Chain.

5.5. Mean Passage Steps Between Multiple States

If a process goes from state i to states j or k and we are interested in the mean steps needed for the first transition, $\mu_{i(j\vee k)}^{(1)}$, we collapse states j and k into one state $(j \vee k)$ as what we have done in chapter 4, and use the new transition matrix to calculate $\mu_{i(j\vee k)}^{(1)}$ and $\mu_{(j\vee k)(j\vee k)}^{(1)}$, which leads to the result. This is true for the transition between more states.

However, we should pay attention to that there is no relationship between $\mu_{i(j\vee k)}^{(1)}$ and $\mu_{ij}^{(1)}$ and $\mu_{ik}^{(1)}$.

CHAPTER 6

Probability Generating Function of Passage Steps

6.1. Introduction

Suppose we have a Markov chain with n states labeled $1, 2, \ldots, n$.

Define the random variable X_{ij} to be the number of steps needed to move from state i to state j for the first time. We refer to X_{ij} as the first passage time. Define the first passage probability as $f_{ij}(k) = P(X_{ij} = k)$. There are several ways to compute the first passage probabilities. For example, see (Hunter, 1983) and (Kao, 1996). First passage probabilities are important as they can be used to control processes and determine when to implement parameter changes. First passage times are indicators of changes to a system (e.g. climate change) and act as warning signals that some action may be needed.

Suppose we have a probability mass function for a discrete random variable X that takes on value k with probability p_k for $k = 0, 1, \ldots$. Define the probability generating function for X to be $\psi_X(z) = \sum_{k=0}^{\infty} p_k z^k$. (Alfa, 2014, p. 76) gives an expression for the probability generating function of the first passage probabilities from state i to state j as follows.

$$\psi_{ij}(z) = \frac{P_{ij}(z)}{1 - P_{ij}(z)}$$

where $P_{ij}(z) = \sum_{k=1}^{\infty} p_{ij}^{(k)} z^k$. But this is not a closed form since we need the values $p_{ij}^{(k)}$.

The method of collective marks was originated by (van Dantzig, 1949), and discussed in (Runnenburg, 1965) and (Kleinrock, 1975, chapter 7). The method gives a probabilistic interpretation of a probability generating function $\sum_{k=0}^{\infty} p_k z^k$. Let z be the probability that an item is "marked." Then $p_k z^k$ represents the probability that random variable X takes on the value k and each of the k counts is marked. Summing over all k gives the total probability that all items from a single realization of the random variable X are marked. The method can often simplify computation and explain a system in an understandable way.

In this paper, we use the collective marks method to find the probability generating function for first passage probabilities, in a closed form for a fixed number of states n. We find expressions for moments of the first passage times, by using the system of equations that we develop. We present a method to find probability generating functions of second passage times.

6.2. Computing first passage probabilities

Theorem 6.1 *Let $\psi_{ij}(z)$ be the probability generating function for the first passage random variable from i to j for an n state Markov chain. Then we obtain an equation,*

$$\psi_{ij}(z) = p_{ij}z + \sum_{k:k\neq j} p_{ik}z\psi_{kj}(z).$$

Proof. By the method of collective marks, $\psi_{ij}(z)$ represents the probability that the path starting from i and reaching j for the first time has all of its steps receiving a mark. Here the probability of a step being marked is assumed to be z. The first step may enter state j immediately and this occurs with probability p_{ij}. The probability that the singleton path is marked is z. So $p_{ij}z$ is the probability that the first passage probability consists of 1 step and is marked. Otherwise, the process goes to some other state k with probability p_{ik} and that step is marked with probability z. From the new position k, the process moves to state j eventually with each step being marked with probability generating function $\psi_{kj}(z)$. Summing over all cases gives the result.

Note The equation in our theorem involves the generating functions $\psi_{kj}(z)$ (for all k) and we can get a similar equation for each of these. For fixed j, this will give us a linear system of equations in the variables $\psi_{1j}(z), \ldots, \psi_{nj}(z)$, which can be solved to get any particular first passage generating function desired as a non linear function of z. The coefficients in the system of equations may involve z as well as constants.

Theorem 6.2 *Let $\psi_{13}(z)$ be the probability generating function for the first passage random variable from 1 to 3 for an 3 state Markov chain. Then*

$$\psi_{13}(z) = \frac{p_{13}z + (p_{12}p_{23} - p_{13}p_{22})z^2}{1 - (p_{11} + p_{22})z + (p_{11}p_{22} - p_{12}p_{21})z^2}.$$

Proof. From Theorem 6.1, we have

$$\psi_{13}(z) = p_{11}z\psi_{13}(z) + p_{12}z\psi_{23}(z) + p_{13}z$$

$$\psi_{23}(z) = p_{21}z\psi_{13}(z) + p_{22}z\psi_{23}(z) + p_{23}z$$

Solving this system of two equations in two unknowns gives our result.

Note

(a) A similar result holds for any pair, not just $(1,3)$.

(b) Our method manages to obtain a closed form for the probability generating function of the first passage times for 3 state Markov chains

(c) Theorem 6.2 can be extended to a larger number number of states as we still essentially get a linear system to solve.

(d) Although the system of equations is linear in the $\psi_{ij}(z)$ unknowns, the coefficients involve the variable z, and the resulting expressions are nonlinear functions of z.

Example 6.1

Consider the Markov transition matrix $P = \begin{bmatrix} .3 & .4 & .3 \\ .3 & .3 & .4 \\ .5 & .4 & .1 \end{bmatrix}$ We will compute first passage probability generating functions for $\psi_{13}(z)$, $\psi_{23}(z)$, and $\psi_{33}(z)$. For the first two we use theorem 2, with appropriate changes for $\psi_{23}(z)$, and for the third, we get a separate equation. According to Theorem 2, the probability generating function for the first passage probabilities from state 1 to state 3 is given by

$$\psi_{13}(z) = \frac{.3z + (.4 * .4 - .3 * .3)z^2}{1 - (.3 + .3)z + (.3 * .3 - .4 * .3)z^2} = \frac{.3z + .07z^2}{1 - .6z - .03z^2}$$

We use the "series" command in MAPLE to find the Taylor expansion and get results.

$$\psi_{13}(z) = .3z + .25z^2 + .159z^3 + .1029z^4 + .06651z^5 + 0.04299z^6 + 0.02779z^7 + \ldots$$

This result agrees with other methods.

Similarly, from Theorem 2, we find

$$\psi_{23}(z) = \frac{.4z + (.3 * .3 - .4 * .3)z^2}{1 - (.3 + .3)z + (.3 * .3 - .3 * .4)z^2} = \frac{.4z - .03z^2}{1 - .6z - .03z^2}$$

Finally,

$$\begin{aligned}
\psi_{33}(z) &= p_{33}z + p_{31}z\psi_{13}(z) + \psi_{32}z\psi_{23}(z) = .1z + .5z\psi_{13}(z) + .4z\psi_{23}(z) \\
&= \frac{.1z - .06z^2 - .003z^3 + .15z^2 + .035z^3 + .16z^2 - .012z^3}{1 - .6z - .03z^2} \\
&= \frac{.1z + .25z^2 + .02z^3}{1 - .6z - .03z^2}
\end{aligned}$$

6.3. Moments of first passage times

One can easily find expressions for the moments of first passage probabilities via a system of equations (Hunter, 1983). However, we will use our system of equations to give an alternative method. Theorem 6.2 gives an expression for $\psi_{ij}(z)$ so we can find the moments of the first passage probabilities by simply taking derivatives and evaluating the expressions at $z = 1$, making any additional computations needed. But this explicitly requires solving for $\psi_{ij}(z)$ which can be a somewhat burdensome task as the coefficients of the linear system involve the variable z.

Theorem 6.1 gives an equation for $\psi_{ij}(z)$ involving the probability generating function of first passage times from i to j and since we have similar expressions for $\psi_{kj}(z)$ (for $k \neq j$), we have a system of equations that we can work with. We can take the derivative of the SYSTEM of equations, and then substitute $z = 1$ into the system to create a much more tractable system of equations. Of course, $\psi_{ij}(1) = 1$ and $\psi'_{ij}(1) = \mu_{ij}$ where $\mu_{ij} = E(X_{ij})$, where X_{ij} is the number of steps needed to reach state j from state i for the first time. Also, $\psi^{(2)}_{ij}(1) = E(X_{ij}(X_{ij} - 1))$.

Example 6.2 We use the same 3×3 transition matrix as in Example 6.1 The system of equations from Theorem 1 is

$$\psi_{13}(z) = .3z\psi_{13}(z) + .4z\psi_{23}(z) + .3z$$

$$\psi_{23}(z) = .3z\psi_{13}(z) + .3z\psi_{23}(z) + .4z$$

Also $\psi_{33}(z) = .1z + .5z\psi_{13}(z) + .4z\psi_{23}(z)$.

Taking derivatives gives

$$\psi'_{13}(z) = .3\psi_{13}(z) + .3z\psi'_{13}(z) + .4\psi_{23}(z) + .4z\psi'_{23}(z) + .3$$

$$\psi'_{23}(z) = .3\psi_{13}(z) + .3z\psi'_{13}(z) + .3\psi_{23}(z) + .3z\psi'_{23}(z) + .4$$

and $\psi'_{33}(z) = .1 + .5\psi_{13}(z) + .5z\psi'_{13}(z) + .4\psi_{23}(z) + .4z\psi'_{23}(z)$

Evaluating at $z = 1$ gives

$$\mu_{13} = .3 + .3\mu_{13} + .4 + .4\mu_{23} + .3 = 1 + .3\mu_{13} + .4\mu_{23}$$

$$\mu_{23} = .3 + .3\mu_{13} + .3 + .3\mu_{23} + .4 = 1 + .3\mu_{13} + .3\mu_{23}$$

and $\mu_{33} = .1 + .5 + .5\mu_{13} + .4 + .4\mu_{23} = 1 + .5\mu_{13} + .4\mu_{23}$

Solving these gives $\mu_{13} = 2.97$, $\mu_{23} = 2.70$ and $\mu_{33} = 3.57$.

6.4. Second passage times

Theorem 6.3 *Let Y_{ij} be the random variable representing the number of steps needed to move from i to j for the second time. Then the probability generating function for Y_{ij} is $\psi_{ij}(z)\psi_{jj}(z)$.*

Proof. $Y_{ij} = X_{ij} + X_{jj}$ where X_{ij} is the first passage random variable, so Y_{ij} is just the convolution of two independent random variables. Since the pgf of a convolution is the product of the pgf's of each part, the result follows.

Example 6.3 We will compute the second passage time from state 1 to state 3 in the Markov chain with transition matrix $P = \begin{bmatrix} .3 & .4 & 0.3 \\ .3 & .3 & 0.4 \\ .5 & .4 & 0.1 \end{bmatrix}$ We earlier calculated

$\psi_{13}(z) = \dfrac{.3z + .07z^2}{1 - .6z - .03z^2}$ and $\psi_{33}(z) = \dfrac{.1z + .25z^2 + .02z^3}{1 - .6z - .03z^2}$ so

$\psi_{second}(z) = \dfrac{(.3z + .07z^2)(.1z + .25z^2 + .02z^3)}{(1 - .6z - .03z^2)^2}$. If we expand this (using MAPLE) into a Taylor series, we get

$\psi_{second}(z) = 0.03z^2 + .118z^3 + .1561z^4 + .1522z^5 + .1316z^6 + .1065z^7 + \ldots$

Thus, for example, the probability of moving from 1 to 3 for the second time on step 4 is 0.1561.

In a similar manner,we can obtain higher order passage probabilities.

6.5. Discussion

The use of collective marks is not absolutely necessary to obtain Theorem 6.1, but it does make the proof simpler than other methods. The closed form result of Theorem 6.2 appears to be new. If there are a large number of states, the expressions of Theorem 6.2 would quickly become much more complex. Second passage (or higher order) passage times can be studied by expanding the Markov transition matrix to contain the information about how many passages have occurred but it is much easier to simply view a second passage time as the convolution of two single passage times, as presented here.

CHAPTER 7

Laplace Transforms of Passage Time

In this chapter, we extend discrete Markov Chains to continuous Markov processes by adding a condition that the transition time needed from one state to another state belongs to an exponential distribution with $\lambda = 1$, thus the discrete random variables, passage steps, are extended to a continuous random variable, passage time, in this situation. And we can use Laplace Transforms to compute the mean and variance of the passage time.

Let X be a continuous random variable with non-negative support. The Laplace Transforms of the probability density function (pdf) $f(x)$, denoted by $L_X(s)$, is defined as

$$L_X(s) = \int_0^\infty e^{-sx} f(x)dx$$

Let X and Y be independent random variables. X has non-negative support and pdf $f(x)$, and Y belongs to an exponential distribution with $\lambda = s$. Then

$$L_X(s) = P(X < Y)$$

Proof.

$$\begin{aligned} L_X(s) &= \int_0^\infty e^{-sx} f(x)dx \\ &= \int_0^\infty f(x) \int_x^\infty s e^{-sy} dy dx \\ &= \int_0^\infty \int_x^\infty f(x) g(y) dy dx \\ &= P(X < Y) \end{aligned}$$

Thus the Laplace Transforms of $f(x)$ can be regarded as the probability that X is less than a catastrophe exponential-distributed random variable Y and that is to say the probability that X wins a race against Y ([7]).

If we think about the transition time is exponential distribution with parameter $\lambda = 1$, we can straightforwardly realize the number of steps from one state to another state is equal to the amount of time from the two states, but we can also prove it by Wald's Theorem.

Proof. Let Y, N, and n be the total time, total steps, and mean steps needed from one state to another state respectively, and X_i be the random variable representing transition time. Thus the total time is the sum of transition time for every step, which gives

$$Y = \sum_{i=1}^{N} X_i$$

According to Wald's Theorem,

$$EY = EXEN = n$$

We use the example that processes move from state 1 to state 3 and from state 3 to state 3 for two cases to illustrate how to apply Laplace Transforms to obtain the mean and variance of the passage probability.

Define $N_{ij}^{(k)}$ be the number of time needed from state i to state j for k times.

7.1. Laplace Transforms of First Passage Time

In this section, we compute the mean and variance of the first passage time $N_{ij}^{(1)}$ for $i \neq j$ and $N_{ii}^{(1)}$.

In order to establish the Laplace Transforms, we consider the movement of a process intending to transit from state i to state j. On one hand, if the process directly moves to state j with the transition rate p_{ij}, it should win a race against all the states other than j and a catastrophe with exponential distribution with $\lambda = s$. Since the process definitely has somewhere to go after a transition, the total transition rate is $1 + s$, so the Laplace Transforms takes the form of $\frac{p_{ij}}{1+s}$ in this situation. On the other hand, if the process moves to state t other than j with transition rate p_{it}, it need another transition to state j. Under such circumstance, state t wins with probability $\frac{p_{it}}{1+s}$ and the procedure from state t to state j takes Laplace Transforms of $L_{tj}(s)$. Combining these two cases, we obtain the system equations

$$L_{ij}(s) = \sum_{t\in S, t\neq j} \frac{p_{it}}{1+s} L_{tj}(s) + \frac{p_{ij}}{1+s}$$

For the process returning to itself, the transitions are almost the same, and we can just substitute $j = i$ in the system equations above to get the Laplace Transforms

$$L_{ii}(s) = \sum_{t\in S, t\neq i} \frac{p_{it}}{1+s} L_{ti}(s) + \frac{p_{ii}}{1+s}$$

7.1.1. Mean of First Passage Time. If we establish the Laplace Transforms of $f(x)$, take derivative of $L_X(s)$, and put $s = 0$ in it, we can get the negative expected value of the random variable X, which means $L'_X(0) = -EX$.

Proof. First apply Taylor series to the Laplace Transforms $L_X(s)$.

$$\begin{aligned} L_X(s) &= \int_0^\infty e^{-sx} f(x)dx \\ &= \int_0^\infty 1 - sx + \frac{s^2x^2}{2!} - \frac{s^3x^3}{3!} + \cdots + \frac{(-s)^n x^n}{n!} + \cdots \\ &= 1 - sEX + \frac{s^2}{2!}EX^2 - \frac{s^3}{3!}EX^3 + \cdots + \frac{(-s)^n}{n!}EX^n + \cdots \end{aligned}$$

Then we take the derivative of $L_X(s)$ respective to s.

$$L'_X(s) = -EX + sEX^2 - \frac{s^2}{2!}EX^3 + \cdots - \frac{(-s)^{n-1}}{(n-1)!}EX^n + \cdots$$

Substitute s with 0, we obtain the negative mean of X.

$$L'_X(0) = -EX$$

To find the mean time from state 1 to state 3, we set up $L_{13}(s)$.

$$L_{13}(s) = \frac{0.2}{1+s} L_{13}(s) + \frac{0.4}{1+s} L_{23}(s) + \frac{0.4}{1+s}$$

Since we need $L_{23}(s)$ to solve the function, we also set up the Laplace Transforms of the transition from state 2 to state 3, $L_{23}(s)$.

$$L_{23}(s) = \frac{0.3}{1+s} L_{13}(s) + \frac{0.3}{1+s} L_{23}(s) + \frac{0.4}{1+s}$$

By solving both equations, we obtain $L_{13}(s) = \frac{2}{5s+2}$ and $L_{23}(s) = \frac{2}{5s+2}$.

And to obtain the mean time need from state 1 to state 3, we take the negative value of the derivative of $L_{13}(s)$ at $s = 0$.

$$L'_{13}(s) = -\frac{10}{(5s+2)^2}$$

Therefore $EN_{13}^{(1)} = -L'_{13}(0) = 2.5$, which is the same as the mean steps calculated in section 5.1.

We can also establish Laplace Transforms for the process returning to state 3.

$$L_{33}(s) = \frac{0.5}{1+s}L_{13}(s) + \frac{0.4}{1+s}L_{23}(s) + \frac{0.1}{1+s} = \frac{s+4}{10s^2+14s+4}$$

Then take derivative of $L_{33}(s)$ respective to s,

$$L'_{33}(s) = -\frac{5s^2+40s+26}{2(5s^2+7s+2)^2}$$

Thus we obtain the mean time needed from state 3 to state 3, $EN_{33}^{(1)} = -L'_{33}(0) = 3.25$. This result is also consistent with the mean steps in section 5.1.

7.1.2. Variance of First Passage Time. Now we calculate the variance of first passage time. If we take second derivative of $L_X(s)$ and put $s = 0$ in it, we can get the second moment of random variable X, say EX^2. With EX and EX^2, variance can be computed by $Var(x) = EX^2 - (EX)^2$.

Proof. Taking second derivative of $L_X(s)$ in terms of Taylor series gives

$$L''_X(s) = EX^2 - sEX^3 + \cdots + \frac{(-s)^{n-2}}{(n-2)!}EX^n + \cdots$$

And substitute s with 0, we have$L''_X(0) = EX^2$.

In our example from state 1 to state 3, the second derivative of L_{13} is

$$L''_{13}(s) = \frac{100}{(5s+2)^3}$$

Therefore $EN_{13}^{(1)^2} = L''_{13}(0) = 12.5$, and consequently $Var(N_{13}^{(1)}) = EN_{13}^{(1)^2} - (N_{13}^{(1)})^2 = L''_{13}(0) - (-L'_{13}(0))^2 = 12.5 - (2.5)^2 = 6.25$.

As for the transition from state 3 to state 3, taking second derivative of L_{33} yields

$$L''_{33}(s) = \frac{25s^3 + 300s^2 + 390s + 142}{(5s^2 + 7s + 2)^3}$$

In this situation, $E{N_{33}^{(1)}}^2 = L''_{33}(0) = 17.75$, and consequently $Var(N_{33}^{(1)}) = E{N_{33}^{(1)}}^2 - (EN_{33}^{(1)})^2 = L''_{33}(0) - (-L'_{33}(0))^2 = 17.75 - (3.25)^2 = 7.1875$.

7.2. Laplace Transforms of Second Passage Time

In this section, we find out the mean and variance of second passage time, $N_{ij}^{(2)}$, with the example of a process from state 1 to state 3. Like the analysis in section 5.2, if a process reaches state j twice, it should first reach j from i for the first time and then reach j from j for the other time, so the random variable second passage time is the convolution of the first passage time from state i to state j and the first passage time from state j to state j. That means $N_{ij}^{(2)} = N_{ij}^{(1)} + N_{jj}^{(1)}$.

7.2.1. Mean of Second Passage Time. Now we can easily calculate the mean of second passage time with

$$EN_{ij}^{(2)} = E(N_{ij}^{(1)} + N_{jj}^{(1)}) = EN_{ij}^{(1)} + EN_{jj}^{(1)} = -L'_{ij}(0) - L'_{jj}(0)$$

Applying this formula in our example, we have $EN_{13}^{(2)} = EN_{13}^{(1)} + EN_{33}^{(1)} = -L'_{13}(0) - L'_{33}(0) = 2.5 + 3.25 = 5.75$. And this is still consistent with the mean steps in section 5.2.

7.2.2. Variance of Second Passage Time. Since $N_{ij}^{(1)}$ and $N_{jj}^{(1)}$ are independent, the variance of second passage time can be computer by

$$\begin{aligned} Var(N_{ij}^{(2)}) = Var(N_{ij}^{(1)} + N_{jj}^{(1)}) &= Var(N_{ij}^{(1)}) + Var(N_{jj}^{(1)}) \\ &= L''_{ij}(0) - (-L'_{ij}(0))^2 + L''_{jj}(0) - (-L'_{jj}(0))^2 \end{aligned}$$

If we are interested in $Var(N_{13}^{(2)})$, we perform $Var(N_{13}^{(2)}) = Var(N_{13}^{(1)}) + Var(N_{33}^{(1)}) = L''_{13}(0) - (-L'_{13}(0))^2 + L''_{33}(0) - (-L'_{33}(0))^2 = 6.25 + 7.1875 = 13.4375$.

7.3. Laplace Transformss of Third Passage Time

Adding another transition from state j to state j to the second passage from state i to state j produces the third passage from state i to state j, thus the random variable third passage time, $N_{ij}^{(3)}$, is the convolution of the second passage time from state i to state j and the first passage time from state j to state j, or the first passage time from state i to state j and the second passage time from state j to state j, or the first passage time from state i to state j and the first passage time from state j to state j and the other first passage time from state j to state j. Interpret the third explanation mathematically, we have $N_{ij}^{(3)} = N_{ij}^{(1)} + 2N_{jj}^{(1)}$.

7.3.1. Mean of Third Passage Time. Similar with section 7.2, the mean time of third passage is given by

$$EN_{ij}^{(3)} = E(N_{ij}^{(1)} + 2N_{jj}^{(1)}) = EN_{ij}^{(1)} + 2EN_{jj}^{(1)} = -L'_{ij}(0) - 2L'_{jj}(0)$$

If we use the transition from state 1 to state 3 for the third time as the example to calculate the mean third passage time, we have $EN_{13}^{(2)} = EN_{13}^{(1)} + 2EN_{33}^{(1)} = -L'_{13}(0) - 2L'_{33}(0) = 2.5 + 2 \times 3.25 = 9$, the same result with the discrete situation.

7.3.2. Variance of Third Passage Time. The independence between $N_{ij}^{(1)}$ and $N_{jj}^{(1)}$ makes the computation of $Var(N_{ij}^{(3)})$ easily with their relationship.

$$\begin{aligned} Var(N_{ij}^{(3)}) = Var(N_{ij}^{(1)} + 2N_{jj}^{(1)}) &= Var(N_{ij}^{(1)}) + 4Var(N_{jj}^{(1)}) \\ &= L''_{ij}(0) - (-L'_{ij}(0))^2 + 4(L''_{jj}(0) - (-L'_{jj}(0))^2) \end{aligned}$$

If we would like to know the variance of the third passage time from state 1 to state 3, we establish $Var(N_{13}^{(3)}) = Var(N_{13}^{(1)}) + 4Var(N_{33}^{(1)}) = L''_{13}(0) - (-L'_{13}(0))^2 + 4(L''_{33}(0) - (-L'_{33}(0))^2) = 6.25 + 4 \times 7.1875 = 35$.

7.4. Laplace Transformss of K-th Passage Time

In this section, we develop the general theorem for the mean and variance of k-th passage time from state i to state j. Since there must exist such relationship that $N_{ij}^{(k)} = N_{ij}^{(1)} + (k-1)N_{jj}^{(1)}$, the mean and variance of $N_{ij}^{(k)}$ are respectively give by

$$
\begin{aligned}
EN_{ij}^{(k)} &= E(N_{ij}^{(1)} + (k-1)N_{jj}^{(1)}) \\
&= EN_{ij}^{(1)} + (k-1)EN_{jj}^{(1)} \\
&= -L'_{ij}(0) - (k-1)L'_{jj}(0)
\end{aligned}
$$

and

$$
\begin{aligned}
Var(N_{ij}^{(k)}) &= Var(N_{ij}^{(1)} + (k-1)N_{jj}^{(1)}) \\
&= Var(N_{ij}^{(1)}) + (k-1)^2 Var(N_{jj}^{(1)}) \\
&= L''_{13}(0) - (-L'_{13}(0))^2 + (k-1)^2(L''_{33}(0) - (-L'_{33}(0))^2)
\end{aligned}
$$

And this is also true for the case $j = i$.

The formulas are explicit, and this is the reason why we extend discrete Markov Chains to continuous Markov processes with the transition time exponentially distributed with $\lambda = 1$ and the application of Laplace Transforms, i.e., we can easily calculate the mean and variance of any times passage between any two states.

7.5. Laplace Transforms of Passage Time Between Multiple States

Extension could be further expanded to the passage between multiple states. If we are interested in the mean and variance of k-th passage time from state i to states j or k, $N_{i(j\vee k)}^{(k)}$, where j and k do not need to be different with i, we collapse states j and k into one new state $(j \vee k)$ and construct the transition matrix according to the rules mentioned in chapter 4. Then we can apply Laplace Transforms to get the moments and thus the mean and variance. It is the same for the transition from state i to states j or k or some other states.

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Vita Auctoris

Yiping Zhang was born in 1997 in Xinxiang, Henan province, China. She studied Applied Statistics at Zhongnan University of Economics and Law from 2014 to 2018. She is currently a candidate for the Master's degree in Statistics at the University of Windsor and is expected to graduate in December 2019.