A Proof of a Hall-Littlewood Polynomial Formula

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A Proof of a Hall-Littlewood Polynomial Formula

By

Jianbai Xu

A Major Research Paper
Submitted to the Faculty of Graduate Studies
through the Department of Mathematics and Statistics
in Partial Fulfillment of the Requirements for
the Degree of Master of Science
at the University of Windsor

Windsor, Ontario, Canada

2019

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December 16, 2019
Declaration of Originality

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Abstract

This paper proves a Hall-Littlewood polynomial formula in a paper by Ram [Ram06] using a theorem by Schwer [Sch06]. We review materials relating to root systems, affine Weyl groups and affine Hecke algebras that are required to study alcoves, galleries and the Hall-Littlewood polynomials. In order to prove the Hall-Littlewood polynomial formula, we formulate in a special case Schwer’s formula in Theorem 5.5 [Sch06] computing right multiplication of the alcove basis by standard basis elements. We show that Ram’s formula for Hall-Littlewood polynomials in terms of positively folded alcove walks coincides with the formulation of Schwer’s formula in terms of positively folded galleries.
Acknowledgements

I would like to express my most sincere appreciation to my supervisor Dr. Wai Ling Yee. Her consistent support and patience made it possible for me to finish this paper. I would also like to express my gratitude to Dr. Borisov as my department reader. I am very grateful for your precious comments. Finally, I would like to thank my parents and my girlfriend for their continuous support and encouragement throughout years of study. I appreciate all your support!
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Chapter 1

Introduction

1.1 Thesis Problem

Hall-Littlewood polynomials (or Macdonald spherical functions) are of great interest with many applications in the study of representation theory and combinatorics. They are a basis for the algebra of symmetric functions depending on a parameter $q$, first arising in P. Hall’s study of some counting problems in group theory [Hal59] and later defined explicitly by D. Littlewood in [Lit61]. Hall-Littlewood polynomials interpolate between Schur functions at $q = 0$ and monomial symmetric functions when $q = 1$. Therefore they specialize to two well-known bases for the ring of symmetric functions. In [Mac71], Macdonald gives a formula for spherical functions on a $p$-adic Chevalley group generalizing Hall-Littlewood polynomials to all root systems. Schwer in [Sch06] proved a formula for Hall-Littlewood polynomials in terms of positively folded galleries and Ram in [Ram06] proved a formula for Hall-Littlewood polynomials in terms of positively folded alcove walks.

In [Sch06], Schwer showed that right multiplication of an alcove basis element by elements of the standard basis can be computed using positively folded galleries (Theorem 5.5 of [Sch06]). In [Ram06], Ram states a formula for Hall-Littlewood polynomials in terms of positively folded alcove walks (Theorem 4.2 of [Ram06]). We prove that Ram’s
Hall-Littlewood polynomial formula follows from Schwer’s Theorem 5.5.

1.2 Outline

- In chapters 2-4, we review some concepts from root systems, reflection groups and Hecke algebras, as background required for the thesis problem.

- In chapter 5, we discuss a paper by Schwer on galleries.

- In chapter 6, we discuss a paper by Ram on alcoves walks and Hall-Littlewood polynomials.

- In chapter 7, we prove that Ram’s formula for Hall-Littlewood polynomials [Ram06, Theorem 4.2] follows from Schwer’s formula describing right multiplication of an alcove basis element by a standard basis element in terms of positively folded galleries [Sch06, Theorem 5.5].
Chapter 2

Root Systems

2.1 Reflections and root systems

Definition 2.1.1 ([Hum90] p.3, [Hum78] p.42). Let \( E \) be a real euclidean space with positive-definite symmetric bilinear form \((\cdot, \cdot)\). A \textbf{reflection} in \( E \) is an invertible linear operator in \( E \) denoted by \( \sigma \) which sends a nonzero vector \( \alpha \) to its negative \(-\alpha\), fixing the reflecting hyperplane \( H_\alpha \) orthogonal to \( \alpha \). The formula for such a reflection is written as:

\[
\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha.
\]  

(2.1)

Given a vector \( \alpha \in E \ (\alpha \neq 0) \), there exists a reflection \( \sigma_\alpha \) through the \textbf{reflecting hyperplane} \( H_\alpha = \{ \beta \in E \mid (\beta, \alpha) = 0 \} \) (shown in the following example).

\textbf{Note.} Recall that a symmetric real \( n \times n \) matrix is a matrix \( A \) such that \( A = A^T \) where \( A^T \) denotes the transpose of \( A \). We call \( A \) \textbf{positive definite} if \( x^T Ax > 0 \) for all \( x \in \mathbb{R}^n \setminus \{0\} \). The matrix \( A \) is positive definite if and only if the corresponding bilinear form \((x, y) = x^T Ay\) is positive definite.

\textbf{Example.} (Figure 2.1) Let \( E \) be a euclidean space, and \( H_\alpha \) a reflecting hyperplane.
Given a vector $\beta$ in $E$, we have

$$
\sigma_\alpha(\beta) = \beta - 2 \left( \|\beta\| \cos \theta \frac{\alpha}{\|\alpha\|} \right).
$$

Since $(\beta, \alpha) = \|\beta\|\|\alpha\|\cos \theta$ and $(\alpha, \alpha) = \|\alpha\|^2$, therefore

$$
\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{\|\alpha\|} \frac{\alpha}{\|\alpha\|} = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha.
$$

Figure 2.1: Reflection of $\beta$ through $H_\alpha$

**Definition 2.1.2** ([Hum78] p.42, [Hum90] p.6). A finite subset $\Phi$ of a euclidean space $E$ is a **root system** in $E$ if it satisfies the following:

(R1) $\Phi$ spans $E$ and $\vec{0} \notin \Phi$;

(R2) If $\alpha \in \Phi$, then $\pm \alpha$ are the only scalar multiples of $\alpha$ in $\Phi$;

(R3) If $\alpha \in \Phi$, then $\sigma_\alpha(\Phi) = \Phi$;

(R4) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle := \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$. (Note that $\langle \alpha, \beta \rangle$ may not equal $\langle \beta, \alpha \rangle$.)

**Definition 2.1.3.** $\alpha$ is called a **root** if $\alpha \in \Phi$ (Definition 2.1.2).

**Definition 2.1.4.** We define the **rank** of the root system to be $\ell = \dim E$.

**Example.** (Figure 2.2) Given $E$ is a one-dimensional euclidean space where $\ell = 1$, then we have, for $\alpha \in E$ and $\alpha \neq 0$, $\Phi = \{\pm \alpha\}$ is a root system, called $A_1$. 4
Figure 2.2: Root system of type $A_1$

Proof. (R1): $\text{span}\{\pm \alpha\} = \text{span}\{\alpha\} = E$.

(R2): Obviously true. $\pm \alpha$ are the only multiples of $\pm \alpha$ in $\Phi$.

(R3): $\sigma_\alpha(\alpha) = \alpha - \frac{2(\alpha, \alpha)}{(\alpha, \alpha)} \alpha = \alpha - 2\alpha = -\alpha$; $\sigma_\alpha(-\alpha) = -\sigma_\alpha(\alpha) = -(-\alpha) = \alpha$.

(R4) $\langle \alpha, \alpha \rangle = \frac{2(\alpha, \alpha)}{(\alpha, \alpha)} = 2 \in \mathbb{Z}$; $\langle \alpha, -\alpha \rangle = \frac{2(\alpha, -\alpha)}{(-\alpha, -\alpha)} = -2 \in \mathbb{Z}$; $\langle -\alpha, \alpha \rangle = -\langle \alpha, \alpha \rangle = -2 \in \mathbb{Z}$; $\langle -\alpha, -\alpha \rangle = -\langle \alpha, -\alpha \rangle = 2 \in \mathbb{Z}$.

Therefore, $\Phi = \{\pm \alpha\}$ is a root system.

Example. When $\ell = 2$, we can get more than one root system. Figure 2.3 is a diagram of the root system of type $A_2$. See the next example.

Figure 2.3: Root system of type $A_2$

Example. Let $E = \{x \in \mathbb{R}^{\ell+1} \mid x_1 + x_2 + \cdots + x_{\ell+1} = 0\}$. Then we have the root system $\Phi = \{\pm (\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq \ell + 1\}$. (This is a root system of type $A_\ell$.)

Proof. (R1) $|\Phi| = (\ell + 1)\ell$ which is finite.

$E = \text{span}\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_\ell - \varepsilon_{\ell+1}\}$, so $\Phi$ spans $E$.

$\Phi$ does not contain 0, since $1 \leq i < j \leq \ell + 1$.

(Note that $E = \text{span}\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_\ell - \varepsilon_{\ell+1}\}$ is a linearly independent subset of size $\ell$ in the $\ell$-dimensional vector space $E$.)
(R2) It is clear to see that the only scalar multiples of $\varepsilon_i - \varepsilon_j$ in $\Phi$ are $\pm (\varepsilon_i - \varepsilon_j)$.

(R3) Let $\alpha = \varepsilon_i - \varepsilon_j$ and $\varepsilon_m - \varepsilon_n \in \Phi$. Since $\sigma(-\alpha) = \sigma_\alpha$ and $\sigma(-\varepsilon_m - \varepsilon_n) = -\sigma_\alpha(\varepsilon_m - \varepsilon_n)$, WLOG, we assume $i < j$ and $m < n$.

\[
\sigma_\alpha(\varepsilon_m - \varepsilon_n) = \varepsilon_m - \varepsilon_n - \frac{2(\varepsilon_m - \varepsilon_n, \varepsilon_i - \varepsilon_j)}{(\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j)} (\varepsilon_i - \varepsilon_j)
\]
\[
= \varepsilon_m - \varepsilon_n - (\delta_{m,i} - \delta_{n,i} - \delta_{m,j} + \delta_{n,j})(\varepsilon_i - \varepsilon_j)
\]
\[
= \begin{cases} 
\varepsilon_m - \varepsilon_n & \text{if } m \neq i, j \text{ and } n \neq i, j \\
\varepsilon_i - \varepsilon_n - (\varepsilon_i - \varepsilon_j) = \varepsilon_j - \varepsilon_n & \text{if } m = i \text{ and } n \neq i, j \\
\varepsilon_j - \varepsilon_n + (\varepsilon_i - \varepsilon_j) = \varepsilon_i - \varepsilon_n & \text{if } m = j \text{ and } n \neq i, j \\
\varepsilon_m - \varepsilon_i + (\varepsilon_i - \varepsilon_j) = \varepsilon_m - \varepsilon_j & \text{if } m \neq i, j \text{ and } n = i \\
\varepsilon_m - \varepsilon_j - (\varepsilon_i - \varepsilon_j) = \varepsilon_m - \varepsilon_i & \text{if } m \neq i, j \text{ and } n = j \\
\varepsilon_i - \varepsilon_j - 2(\varepsilon_i - \varepsilon_j) = -(\varepsilon_i - \varepsilon_j) & \text{if } m = i \text{ and } n = j
\end{cases}
\]

Thus we have shown that $\sigma_\alpha(\Phi) = \Phi$.

(R4) We compute

\[
\langle \varepsilon_m - \varepsilon_n, \varepsilon_i - \varepsilon_j \rangle = \frac{2(\varepsilon_m - \varepsilon_n, \varepsilon_i - \varepsilon_j)}{(\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j)} = \delta_{m,i} - \delta_{n,i} - \delta_{m,j} + \delta_{n,j} \in \mathbb{Z}.
\]

Therefore, $\Phi = \{\pm (\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq \ell + 1\}$ is a root system. □

**Note.** From the definition above, (R4) limits possible angles between $\alpha$ and $\beta$.

\[
\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = \frac{2(\alpha, \beta)}{\langle \beta, \beta \rangle} \frac{2(\beta, \alpha)}{\langle \alpha, \alpha \rangle} = \frac{2\|\alpha\|\|\beta\|\cos \theta}{\|\beta\|^2} \frac{2\|\beta\|\|\alpha\|\cos \theta}{\|\alpha\|^2} = 4\cos^2 \theta \in \mathbb{Z}
\]

Since $\cos^2 \theta \in [0, 1]$, then $4\cos^2 \theta \in [0, 4]$. $\cos^2 \theta = \left\{ 0, 1, \frac{1}{2}, \frac{3}{4}, 1 \right\}$

- $\cos^2 \theta = 0 \quad \Rightarrow \quad \theta = 90^\circ$
- $\cos^2 \theta = 1/4 \quad \Rightarrow \quad \theta = 60^\circ \text{ or } 120^\circ$
\[
\cos^2 \theta = \frac{1}{2} \Rightarrow \theta = 45^\circ \text{ or } 135^\circ
\]
\[
\cos^2 \theta = \frac{3}{4} \Rightarrow \theta = 30^\circ \text{ or } 150^\circ
\]
\[
\cos^2 \theta = 1 \Rightarrow \theta = 0^\circ \text{ or } 180^\circ
\]

**Definition 2.1.5** ([Hum90] p.39). \( \Phi \) is a root system in \( E \). \( \alpha^\vee \) is the coroot of a root \( \alpha \) if it satisfies

\[
\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}.
\]

**Note.** \( \Phi^\vee = \{ \alpha^\vee \mid \alpha \in \Phi \} \) is also a root system of \( E \) (proved in Exercise 2 [Hum78, p.46]).

### 2.2 Simple roots and Weyl chambers

**Definition 2.2.1** ([Hum78] p.47). A subset \( \Delta = \{ \alpha_1, \ldots, \alpha_\ell \} \) of \( \Phi \) is a base of \( E \) if \( \Delta \) is a basis of \( E \) and every \( \beta = \sum_{i=1}^{\ell} k_i \alpha_i \in \Phi \) satisfies the \( k_i \) are all nonnegative or all nonpositive integers.

**Proposition 2.2.2** ([Hum78] p.48). \( \Phi \) has a base.

**Definition 2.2.3** ([Hum78] p.47). The roots belonging to a base \( \Delta \) are called simple roots. If all integral coefficients \( k_i \) in \( \beta = \sum_{i=1}^{\ell} k_i \alpha_i \in \Phi \) are nonnegative, we call \( \beta \) a positive root (\( \beta \succ 0 \)). If all integral coefficients \( k_i \) are nonpositive, we call \( \beta \) a negative root (\( \beta \prec 0 \)). We denote the collection of all positive roots by \( \Phi^+ \), all negative roots by \( \Phi^- \), where \( \Phi^- = -\Phi^+ \).

**Example.** In the previous example, we have the root system of type \( A_\ell \): \( \Phi = \{ \pm (\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq \ell + 1 \} \). Thus we can show that \( \Delta = \{ \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq \ell \} \) is a base for \( \Phi \). Moreover, with the choice of the base \( \Delta \), we get \( \Phi^+ = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq \ell + 1 \} \) and \( \Phi^- = \{ -\varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq \ell + 1 \} \).

**Proof.** Since \( E = \{ x \in \mathbb{R}^{\ell+1} \mid x_1 + x_2 + \cdots + x_{\ell+1} = 0 \} \) is \( \ell \)-dimensional and \( \Delta = \{ \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_\ell - \varepsilon_{\ell+1} \} \) is a linearly independent subset of \( E \) of size \( \ell \), then \( \Delta \)
is a basis of $E$. Moreover, $\varepsilon_i - \varepsilon_j$ is a positive integer linear combination of $\varepsilon_\ell - \varepsilon_{\ell+1}$ $(1 \leq i < j \leq \ell + 1)$:

$$\varepsilon_i - \varepsilon_j = (\varepsilon_i - \varepsilon_{i+1}) + (\varepsilon_{i+1} - \varepsilon_{i+2}) + \cdots + (\varepsilon_{j-2} - \varepsilon_{j-1}) + (\varepsilon_{j-1} - \varepsilon_j).$$

And also,

$$-\varepsilon_i + \varepsilon_j = (-\varepsilon_i + \varepsilon_{i+1}) + (-\varepsilon_{i+1} + \varepsilon_{i+2}) + \cdots + (-\varepsilon_{j-2} + \varepsilon_{j-1}) + (-\varepsilon_{j-1} + \varepsilon_j).$$

Therefore, $\Delta$ is a base for $\Phi$. \hfill \Box

**Definition 2.2.4** ([Hum78] p.52). Let $\Phi$ be a root system. We call $\Phi$ **irreducible** if $\Phi \neq \Phi_1 \cup \Phi_2$ such that $(\phi_1, \phi_2) = 0$ for all $\phi_1 \in \Phi_1$ and $\phi_2 \in \Phi_2$.

**Definition 2.2.5** ([Hum78] p.47). Using the notation of Definition 2.2.1, we define the **height** of a root $\beta$ as $ht(\beta) = \sum_{i=1}^{\ell} k_i$.

**Proposition 2.2.6** ([Hum78] p.47). Let $\lambda, \mu \in E$. We have a partial order $\mu \prec \lambda$ on $E$ if and only if $\lambda - \mu$ is a sum of $\alpha$ ($\alpha \in \Delta$).

**Definition 2.2.7** ([Hum78] p.52). Let $\Phi$ be an irreducible root system. There is a unique **highest root** $\beta \in \Phi$ such that for all $\alpha \in \Phi$, $ht(\alpha) < ht(\beta)$ for $\alpha \neq \beta$ ($\beta$ is maximal relative to the partial order $\prec$).

**Definition 2.2.8** ([Hum78] p.48). For $\gamma \in E$, $\gamma \notin H_\alpha$ for all $\alpha \in \Phi$, let $\Phi^+(\gamma) = \{ \alpha \in \Phi \mid (\gamma, \alpha) > 0 \}$ be the set of roots lying on one side of the hyperplane orthogonal to $\gamma$. Since in a euclidean space $E$, the union of the finitely many hyperplanes $H_\alpha$ ($\alpha \in \Phi$) cannot exhaust $E$, we call $\gamma \in E$ **regular** if $\gamma \in E - \bigcup_{\alpha \in \Phi} H_\alpha$. Otherwise, $\gamma$ is **singular**.

**Definition 2.2.9** ([Hum78] p.48). If $\gamma$ is regular, then $\Phi = \Phi^+(\gamma) \cup -\Phi^+(\gamma)$. Then we call $\alpha \in \Phi^+(\gamma)$ **decomposable** if $\alpha = \beta_1 + \beta_2$ for some $\beta_i \in \Phi^+(\gamma)$, and **indecomposable** otherwise.
**Proposition 2.2.10** ([Hum78] p.48). Let $\gamma \in E$ be regular. Then the set $\Delta(\gamma)$ of all indecomposable roots in $\Phi^+(\gamma)$ is a base of $\Phi$ and every base is of this form.

**Definition 2.2.11** ([Hum78] p.49). The hyperplanes $H_\alpha$ ($\alpha \in \Phi$) partition $E$ into finitely many regions. The connected components of $E - \bigcup_{\alpha \in \Phi} H_\alpha$ are called **Weyl chambers**.

**Proposition 2.2.12** ([Hum78] p.49). Each regular $\gamma \in E$ belongs to precisely one Weyl chamber of $E$, denoted as $C(\gamma)$.

**Proposition 2.2.13** ([Hum78] p.49). If $C(\gamma) = C(\gamma')$, then $\gamma$ and $\gamma'$ lie on the same side of each hyperplane $H_\alpha$ ($\alpha \in \Phi$). Thus we have $\Phi^+(\gamma) = \Phi^+(\gamma')$ and $\Delta(\gamma) = \Delta(\gamma')$.

**Definition 2.2.14** ([Hum78] p.49). By proposition 2.2.10 and proposition 2.2.13, Weyl chambers are in one-to-one correspondence with bases. Write $C(\Delta) = C(\gamma)$ if $\Delta = \Delta(\gamma)$. We call $C(\Delta) = C(\gamma)$ the **fundamental Weyl chamber** relative to $\Delta$.

### 2.3 Weyl groups

**Definition 2.3.1** ([Hum78] p.51). The **Weyl group** $W$ of the root system $\Phi$ is the group generated by the reflections $\sigma_\alpha$ for $\alpha \in \Phi$.

**Proposition 2.3.2** ([Hum78] p.51). $W$ is generated by the simple reflections $\sigma_\alpha$ for $\alpha \in \Delta$.

**Example.** Root system of type $A_1$: $\Phi = \{\pm \alpha\}$. The Weyl group of type $A_1$: $W_{A_1} = \langle \sigma_\alpha \rangle = \{1, \sigma_\alpha\}$.

**Proposition 2.3.3** ([Hum78] p.51). Let $\Delta$ be a base of $\Phi$ and $W$ be the Weyl group. Then $W$ acts transitively on bases (i.e., if $\Delta'$ is another base of $\Phi$, then $\sigma(\Delta) = \Delta'$ for some $\sigma \in W$).

**Definition 2.3.4** ([Hum90] p.118). Let $T$ be the set of reflections in $W$. We write $w' \rightarrow w$ if $w = w's$ for some $s \in T$ with $\ell(w) > \ell(w')$. We define $w' < w$ if there exists a sequence
$w' = w_0 \rightarrow w_1 \rightarrow ... \rightarrow w_n = w$. Thus we call the partial order $w' \leq w$ Bruhat order. (Note: a partial order is a binary relation that is reflexive, antisymmetric and transitive.)

**Proposition 2.3.5** ([Hum90] Proposition 5.7). For $w \in W$ and $\alpha \in \Phi^+$, we have $ws_\alpha < w \iff w\alpha < 0$ and $ws_\alpha > w \iff w\alpha > 0$.

### 2.4 Weights

**Definition 2.4.1** ([Hum78] p.67). For $\lambda \in E$, if $\langle \lambda, \alpha \rangle = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$ for all $\alpha \in \Phi$, then we call $\lambda$ a weight. The collection of all weights is denoted by $\Lambda$.

**Definition 2.4.2** ([Hum78] p.67). $\lambda \in \Lambda$ is dominant if $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ are all nonnegative for all $\alpha \in \Phi^+$, strongly dominant if $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ are all positive for all $\alpha \in \Phi^+$. We denote the set of all dominant weights as $\Lambda^+$.

**Definition 2.4.3** ([Hum78] p.67). If $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$, then the vectors $\alpha^\vee_1 = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$ (1 ≤ $i$ ≤ $\ell$) form a basis of $E$. Thus we let $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ be the dual basis $\left(\lambda_i, -\frac{2\alpha_j}{(\alpha_j, \alpha_j)}\right) = (\lambda_i, \alpha^\vee_j) = \delta_{ij}$. Since $\langle \lambda_i, \alpha \rangle = \frac{2(\lambda_i, \alpha)}{(\alpha, \alpha)}$ are all nonnegative for $\alpha \in \Delta$, the $\lambda_i$ are dominant weights, called fundamental dominant weights.

**Note.** $\sigma_i \lambda_j = \lambda_j - \delta_{ij} \alpha_i$ where $\sigma_i = \sigma_{\alpha_i}$.

**Example.** Consider the root system of the type $A_2$: $\{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_3)\}$ with simple roots $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2 - \varepsilon_3$. We can show that $\lambda_1 = \frac{1}{3}(2\alpha_1 + \alpha_2)$ and $\lambda_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2)$.

**Proof.**

\[
\langle \lambda_1, \alpha_1 \rangle = 2\left(\frac{2\alpha_1 + \alpha_2}{3}, \alpha_1 \right) = \frac{2}{3} (2(\varepsilon_1 - \varepsilon_2) + \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_2) = \frac{1}{3} (2\varepsilon_1 - \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_2) = 1
\]
\[ \langle \lambda_1, \alpha_2 \rangle = 2 \frac{(2\alpha_1 + \alpha_2, \alpha_2)}{(\alpha_2, \alpha_2)} = \frac{2}{3} \frac{(2(\epsilon_1 - \epsilon_2) + \epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_3)}{(\epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_3)} = \frac{1}{3} (2\epsilon_1 - \epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_3) = 0 \]

\[ \langle \lambda_2, \alpha_1 \rangle = 2 \frac{(\alpha_1 + 2\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)} = \frac{2}{3} \frac{(\epsilon_1 - \epsilon_2 + 2(\epsilon_2 - \epsilon_3), \epsilon_1 - \epsilon_2)}{(\epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_2)} = \frac{1}{3} (\epsilon_1 + \epsilon_2 - 2\epsilon_3, \epsilon_1 - \epsilon_2) = 0 \]

\[ \langle \lambda_2, \alpha_2 \rangle = 2 \frac{(\alpha_1 + 2\alpha_2, \alpha_2)}{(\alpha_2, \alpha_2)} = \frac{2}{3} \frac{(\epsilon_1 - \epsilon_2 + 2(\epsilon_2 - \epsilon_3), \epsilon_2 - \epsilon_3)}{(\epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_3)} = \frac{1}{3} (\epsilon_1 + \epsilon_2 - 2\epsilon_3, \epsilon_2 - \epsilon_3) = 1 \]

Thus we have proved \( \langle \lambda_i, \alpha_j \rangle = \delta_{i,j} \).

Figure 2.4: Fundamental dominant weights of type \( A_2 \)
Chapter 3

Affine Weyl Groups

3.1 Affine hyperplanes

Definition 3.1.1. We recall that a hyperplane is a subspace with codimension one relative to the space where it is embedded.

Example. In a three-dimensional euclidean space, a hyperplane is two-dimensional, where the codimension is the difference one. A hyperplane in a two-dimensional euclidean space will be a line which is one dimensional.

Definition 3.1.2 ([Ram06] p.138). Let $\alpha$ be a root in $\Phi \subset E$ (Definition 2.1.2). We define the affine hyperplane $H_{\alpha,k}$ as:

$$H_{\alpha,k} := \{ \lambda \in E \mid (\lambda, \alpha^\vee) = k \}. \quad (3.1)$$

Note ([Hum90] p.87). $H_{\alpha,k} = H_{-\alpha,-k}$ and $H_{\alpha,0} = H_{\alpha}.$

Proposition 3.1.3 ([Hum90] p.88). $wH_{\alpha,k} = H_{w\alpha,k}$ for $w \in W.$
Proof.

\[ wH_{\alpha,k} = \{ w\lambda : (\lambda, \alpha^\vee) = k \} \]
\[ = \{ w\lambda : (w\lambda, w\alpha^\vee) = k \} \]
\[ = H_{w\alpha,k} \]

**Definition 3.1.4** ([Hum90] p.88). We define the reflection \( s_{\alpha,k} \) through \( H_{\alpha,k} \) as

\[ s_{\alpha,k}(\lambda) := \lambda - ((\lambda, \alpha^\vee) - k)\alpha. \quad (3.2) \]

**Proposition 3.1.5** ([Hum90] p.88). \( ws_{\alpha,k}w^{-1} = s_{w\alpha,k} \) for \( w \in W \).

**Definition 3.1.6** ([Ram06] p.138). For \( \lambda \in E \), we define \( t(\lambda) \) as the translation that sends \( \mu \) to \( \lambda + \mu \) for all \( \mu \in E \).

**Note** ([Ram06] p.138). \( s_{\alpha,k} = t(k\alpha)s_{\alpha} = s_{\alpha}t(-k\alpha). \)

**Example** ([Hum90] p.88). If \( \lambda \in E \) satisfies \( (\lambda, \alpha^\vee) \in \mathbb{Z} \) for all roots \( \alpha \), then

(a) \( t(\lambda)H_{\alpha,k} = H_{\alpha,k+(\lambda,\alpha^\vee)} \)
(b) \( t(\lambda)s_{\alpha,k}t(-\lambda) = s_{\alpha,k+(\lambda,\alpha^\vee)}. \)

**Proof.** (a) We need to show that \( t(\lambda)H_{\alpha,k} \subset H_{\alpha,k+(\lambda,\alpha^\vee)} \) and \( t(\lambda)H_{\alpha,k} \supset H_{\alpha,k+(\lambda,\alpha^\vee)}. \)

\( \subseteq \): Let \( \mu \in t(\lambda)H_{\alpha,k} \). Then \( \mu = \lambda + \nu \) where \( \nu \in H_{\alpha,k} \). Thus \( (\mu, \alpha^\vee) = (\lambda, \alpha^\vee) + (\nu, \alpha^\vee) = (\lambda, \alpha^\vee) + k \), so \( \mu \in H_{\alpha,k+(\lambda,\alpha^\vee)}. \) Therefore \( t(\lambda)H_{\alpha,k} \subset H_{\alpha,k+(\lambda,\alpha^\vee)}. \)

\( \supseteq \): Let \( \mu \in H_{\alpha,k+(\lambda,\alpha^\vee)}. \) Then \( (\mu, \alpha^\vee) = k + (\lambda, \alpha^\vee) \). Let \( \mu = \nu + \lambda \). Then \( (\mu, \alpha^\vee) = (\nu, \alpha^\vee) + (\lambda, \alpha^\vee) = k + (\lambda, \alpha^\vee) \). Thus \( (\nu, \alpha^\vee) = k \) so that \( \nu \in H_{\alpha,k}. \) Therefore \( \mu \in t(\lambda)H_{\alpha,k} \) and so \( H_{\alpha,k+(\lambda,\alpha^\vee)} \subset t(\lambda)H_{\alpha,k}. \)

Thus we have shown that \( t(\lambda)H_{\alpha,k} = H_{\alpha,k+(\lambda,\alpha^\vee)}. \)
\[ t(\lambda)s_{\alpha,k}t(-\lambda)(\mu) = t(\lambda)(s_{\alpha,k}(\mu - \lambda)) \]
\[ = t(\lambda)(\mu - \lambda - ((\mu - \lambda, \alpha^\vee) - k))\alpha \]
\[ = t(\lambda)(\mu - \lambda - ((\mu, \alpha^\vee) - (\lambda, \alpha^\vee) - k))\alpha \]
\[ = \mu - ((\mu, \alpha^\vee) - k - (\lambda, \alpha^\vee))\alpha \]
\[ = s_{\alpha,k + (\lambda, \alpha^\vee)}(\mu) \]

Therefore \( t(\lambda)s_{\alpha,k}t(-\lambda) = s_{\alpha,k + (\lambda, \alpha^\vee)} \).

\[ \square \]

### 3.2 Affine Weyl groups

**Definition 3.2.1** ([Hum90] p.88, [Ram06] p.138). The affine Weyl group \( W_a \) is the group generated by the reflections \( s_{\alpha,k} \) for \( \alpha \in \Phi \) and \( k \in \mathbb{Z} \). Note that \( wt(\lambda)w^{-1} = t(w\lambda) \) [Ram06, p.139]. Thus

\[ W_a = \Lambda_r \rtimes W = \{ t(\lambda)w : \lambda \in \Lambda_r, w \in W \} \text{ where } \Lambda_r = \sum_{\alpha \in \Phi^+} \mathbb{Z}\alpha. \] (3.3)

### 3.3 Alcoves

**Definition 3.3.1** ([Hum90] p.89). An alcove is a connected component of

\[ E - \bigcup_{\alpha \in \Phi^+, k \in \mathbb{Z}} H_{\alpha,k} \] (3.4)

where \( H_{\alpha,k} \) are affine hyperplanes.

**Note** ([Hum90] p.90). The affine Weyl group acts transitively on the collection of all alcoves.

**Definition 3.3.2.** Let \( A_\circ \) be the alcove with walls \( H_\alpha \) (\( \alpha \in \Delta \)) and \( H_{\tilde{\alpha},1} \), where \( \tilde{\alpha}^\vee \) is
the highest root in $\Phi^\vee$.

**Proposition 3.3.3** ([Ram06] p.139). $W_a$ is in bijection with the alcoves of $E$ via $w \leftrightarrow wA_o$.

**Proposition 3.3.4** ([Hum90] Proposition 4.3.1). Let $S_a = \{s_\alpha : \alpha \in \Delta\} \cup \{s_{\tilde{\alpha},1}\}$. Then $W_a = \langle t : t \in S_a \rangle$.

**Definition 3.3.5** ([Hum90] p.89, [Ram06] p.139). Since $W$ normalizes the translation group corresponding to $\Lambda$, we may define the extended affine Weyl group

$$W_e = \Lambda \rtimes W$$

(3.5)

**Definition 3.3.6** ([Ram06] p.139, [Yee19] Definition 5.5). We define the difference between $W_e$ and $W_a$ to be the group

$$\Omega = W_e/W_a \cong \Lambda/\Lambda_r.$$  

(3.6)

$\Omega$ is isomorphic to the stabilizer of $A_o$ in $W_e$. The stabilizer is isomorphic to $\Lambda/\Lambda_r$ via sending $g$ to the coset $g(0) + \Lambda_r$.

**Definition 3.3.7** ([Sch06] p.6). $W_e \cong \Omega \ltimes W_a$. For $\nu \in W_e$, we define the length of $\nu$ as $\ell(w)$ where $\nu = wg$ for $w \in W_a$ and $g \in \Omega$.

### 3.4 Counting hyperplanes

**Definition 3.4.1** ([Hum90] p.12, p.91). For all $w \in W_a$, $w$ can be written as a product of simple reflections

$$w = s_1s_2 \cdots s_k$$

where the $s_i$ belong to $S_a$. We call such an expression a reduced expression when $k$ is minimal. Then $k$ is called the length of $w$, written as $\ell(w) = k$. 

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Definition 3.4.2 ([Hum90] p.91). Let $H_{\alpha,k}$ be a hyperplane where $\alpha \in \Phi^+$. Then $H^+_\alpha$ and $H^-_{\alpha,k}$ denote the affine half-spaces

$$H^+_{\alpha,k} = \{ \lambda \in E \mid (\lambda, \alpha^\vee) > k \} \quad \text{and} \quad H^-_{\alpha,k} = \{ \lambda \in E \mid (\lambda, \alpha^\vee) < k \}. \quad (3.7)$$

Definition 3.4.3 ([Hum90] p.91). Let $H_{\alpha,k}$ be a hyperplane. $H_{\alpha,k}$ separates alcoves $A$ and $B$ if $A, B$ belong to different half-spaces defined by $H_{\alpha,k}$.

Lemma 3.4.4 ([Hum78] p.93, [Hum90] p.92). Let $w = s_1s_2\cdots s_k$ be a reduced expression in $W_a (w \neq 1)$. Then setting $H_i$ to be the affine hyperplane corresponding to $s_i$, the following $k$ hyperplanes $H_1, s_1H_2, s_1s_2H_3, \ldots, s_1s_2\cdots s_{k-1}H_k$ are all distinct and form the set of affine hyperplanes separating $A_o$ and $wA_o$.

3.5 Bruhat order

Definition 3.5.1 ([Hum90] p.118). Let $w \in W_a, T$ be the set of all reflections in $W_a$ where $T = \bigcup_{w \in W_a} wS_aw^{-1}$. We write $w' \rightarrow w$ if $w = w't$ for some $t \in T$ with $\ell(w) > \ell(w')$. Define $w' \leq w$ if there exists a sequence $w' = w_0 \rightarrow w_1 \rightarrow \ldots \rightarrow w_n = w$. Then $\leq$ is a partial order called **Bruhat order**.

Remark ([Hum90] p.119). Definition 2.3.4 and Definition 3.5.1 have a one-sided appearance. Note that if we let $w = w't$ with $\ell(w) > \ell(w')$ and $t \in T$, then $w = (w't(w')^{-1})w'$ where $w't(w')^{-1} \in T$.

Proposition 3.5.2 ([Hum90] p.119). $v < w$ if and only if $v^{-1} < w^{-1}$.

Proof. $\Rightarrow$: Suppose $v < w$. Then there exists a sequence $v = w_0 \rightarrow w_1 \rightarrow \cdots \rightarrow w_k = w$ where $w_i = w_{i-1}t_i$ ($t_i \in T$ the set of reflections in $W_a$). Then $w_i^{-1} = t_iw_{i-1}^{-1} = w_{i-1}^{-1}(w_it_iw_{i-1}^{-1})$ where $w_{i-1}t_iw_{i-1}^{-1} \in T$. Thus $w_{i-1}^{-1} \rightarrow w_i^{-1}$. We have $v^{-1} = w_0^{-1} \rightarrow w_1^{-1} \rightarrow \ldots \rightarrow w_k^{-1} = w^{-1}$. Therefore, $v^{-1} < w^{-1}$. $\Leftarrow$: Suppose $v^{-1} < w^{-1}$. Then there exists a sequence $v^{-1} = w_0^{-1} \rightarrow w_1^{-1} \rightarrow \cdots \rightarrow w_k^{-1} = w^{-1}$ where $w_i^{-1} = w_{i-1}^{-1}(w_it_iw_{i-1}^{-1})$ and $t_i \in T$. Then $w = (w't'(w')^{-1})w'$ where $w't'(w')^{-1} \in T$. Thus $v < w$. Therefore, $v^{-1} < w^{-1}$. $\square$
\[ \cdots \rightarrow w_k^{-1} = w^{-1}. \] Therefore, \( v < w \Rightarrow v^{-1} < w^{-1}. \)

\[ \Leftrightarrow \text{ We already have } v < w \Rightarrow v^{-1} < w^{-1}. \] Apply the sentence again, we have \( v^{-1} < w^{-1} \Rightarrow (v^{-1})^{-1} < (w^{-1})^{-1} \) which is \( v^{-1} < w^{-1} \Rightarrow v < w. \) Therefore, \( v < w \) if and only if \( v^{-1} < w^{-1}. \) \( \Box \)
Chapter 4

Hecke Algebras and Affine Hecke Algebras

4.1 Hecke algebras

Definition 4.1.1 ([Yee19] Definition 5.1). Let $\mathbb{K}$ be the field of fractions of $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. The Hecke algebra $\mathcal{H}$ is the $\mathbb{K}$-algebra with $\mathbb{K}$-basis $\{T_w\}_{w \in W}$ such that for all $s \in S$ and $w \in W$,

$$T_s T_w = T_{sw} \quad \text{if } \ell(sw) > \ell(w)$$

$$T_s T_w = q T_{sw} + (q - 1) T_w \quad \text{if } \ell(sw) < \ell(w).$$

(4.1) (4.2)

We define

$$\check{T}_w = q^{-\frac{1}{2}\ell(w)} T_w.$$ 

(4.3)

Then we get another presentation of $\mathcal{H}$ with a $\mathbb{K}$-basis $\{\check{T}_w\}_{w \in W}$ such that for all $s \in S$ and $w \in W$,

$$\check{T}_s \check{T}_w = \check{T}_{sw} \quad \text{if } \ell(sw) > \ell(w)$$

$$\check{T}_s \check{T}_w = \check{T}_{sw} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \check{T}_w \quad \text{if } \ell(sw) < \ell(w).$$

(4.4) (4.5)
4.2 Affine Hecke algebras

Definition 4.2.1 ([Yee19] Definition 5.7). The affine Hecke algebra $\mathcal{H}_a$ is the $K$-algebra with $K$-basis $\{\tilde{T}_w\}_{w \in W_e}$ and the relations:

$$\tilde{T}_v \tilde{T}_w = \tilde{T}_{vw} \quad \text{if } v, w \in W_e \text{ and } \ell(vw) = \ell(v) + \ell(w) \quad (4.6)$$

$$\tilde{T}_s^2 = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\tilde{T}_s + \tilde{T}_1 \quad s \in S_a. \quad (4.7)$$

$\mathcal{H}_a$ is also the $K$-algebra with $K$-basis $\{T_w\}_{w \in W_e}$ with the relations

$$T_v T_w = T_{vw} \quad \text{if } v, w \in W_e \text{ and } \ell(vw) = \ell(v) + \ell(w) \quad (4.8)$$

$$T_s^2 = (q - 1)T_s + qT_1 \quad s \in S_a. \quad (4.9)$$

Definition 4.2.2 ([Sch06] p.7). We define an involution on $\mathcal{H}$

$$\tilde{\cdot} : \mathcal{H} \to \mathcal{H}$$

such that $T_w = T_{w^{-1}}$ for $w \in W$ and $q^{\frac{1}{2}} = q^{-\frac{1}{2}}$ and an involution on $\mathcal{H}_a$

$$\tilde{\cdot} : \mathcal{H}_a \to \mathcal{H}_a$$

such that $\tilde{T}_w = \tilde{T}_{w^{-1}}$ for $w \in W_e$ and $q^{\frac{1}{2}} = q^{-\frac{1}{2}}$. Then $\tilde{T}_w = \tilde{T}_{w^{-1}}$ for $w \in W_e$. 

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Chapter 5

Schwer's Paper “Galleries, Hall-Littlewood Polynomials and Structure Constants of the Spherical Hecke Algebra”

5.1 Generalized alcoves and affine Weyl groups

Definition 5.1.1 ([Sch06] p.6). Let $\mathcal{A}$ be the set of alcoves. We define the set of generalized alcoves as

$$\tilde{\mathcal{A}} := \{(A, \mu) \in \mathcal{A} \times \Lambda | \mu \in A\}.$$  
(5.1)

There is an embedding $\mathcal{A} \hookrightarrow \tilde{\mathcal{A}}$ via $A \mapsto (A, \mu)$ where $\mu$ is the unique element in $A \cap \Lambda_r$. There is a natural free $W_e$-action on $\tilde{\mathcal{A}}$ by the natural action in the two components. In particular, $\Lambda$ acts on $\tilde{\mathcal{A}}$ by translation in both components.

Note ([Sch06] p.6). The bijection $W_e \rightarrow \tilde{\mathcal{A}}$ extends $W_a \rightarrow \mathcal{A}$.

Let $\lambda \in \Lambda$. Recall from Definition 3.1.6 the definition of $t(\lambda)$. Let $\tau_\lambda = t(\lambda) \in W_e$.

Definition 5.1.2 ([Sch06] p.7). We define $q_\nu = q^{t(\nu)}$ for $\nu \in W_e$. Recall that for $\nu = wg$
where \( w \in W_a \) and \( g \in \Omega, \ell(\nu) = \ell(w) \).

**Lemma 5.1.3** ([Sch06] p.7). Let \( A \) be a generalized alcove. Then it can be written in the form: \( A = \mu + wA_o \) where \( \mu \in \Lambda \) and \( w \in W \). We call \( \mu \) the weight of \( A \) and \( w \) the direction of \( A \). We denote them as \( wt(A) := \mu \) and \( \delta(A) := w \).

**Definition 5.1.4** ([Sch06] p.7). Let \( \lambda \in \Lambda^+ \). We define \( q_\lambda = q_{\frac{1}{2}\ell(\tau_\lambda)} \). (5.2)

**Definition 5.1.5** ([Sch06] p.3). We define \( \rho = \frac{1}{2} \sum_{\alpha^\vee \in (\Phi^\vee)^+} \alpha^\vee \). (5.3)

**Definition 5.1.6** ([Sch06] p.7). Let \( \mu \in \Lambda \). We define \( q_\mu = q_\lambda q_{\lambda' - 1} = q_{\rho^\vee, \mu} \) where \( \mu = \lambda - \lambda' \) for \( \lambda, \lambda' \in \Lambda^+ \).

**Lemma 5.1.7** ([Sch06] Lemma 2.3). \( \ell(\tau_\mu) = 2(\rho^\vee, \mu) \) for \( \mu \in \Lambda^+ \). Thus \( q_\mu = q_{(\rho^\vee, \mu)} \) for \( \mu \in \Lambda^+ \).

*Proof.* \( \ell(w) \) is equal to the number of distinct hyperplanes \( H_{\alpha,k} \) separating \( A_o \) and \( wA_o \) [Ram06, p.144]. Thus \( \ell(\tau_\mu) \) equals the number of hyperplanes separating \( A_o \) and \( \tau_\mu A_o \), where \( \tau_\mu A_o = \mu + A_o \). Thus if we fix \( \alpha \in \Phi^+ \), hyperplanes \( H_{\alpha,k} \) separating \( A_o \) and \( \mu + A_o \) are: \( H_{\alpha,1}, H_{\alpha,2}, ..., H_{\alpha,|\mu|\alpha^\vee} \). Then we get

\[
2(\rho^\vee, \mu) = 2 \left( \frac{1}{2} \sum_{\alpha^\vee \in (\Phi^\vee)^+} \alpha^\vee, \mu \right) = \sum_{\alpha^\vee \in (\Phi^\vee)^+} (\alpha^\vee, \mu) = \ell(\tau_\mu). \tag{5.4}
\]

Therefore, \( \ell(\tau_\mu) = 2(\rho^\vee, \mu) \). \( \square \)

**Lemma 5.1.8.** \( q_\mu = q_\lambda q_{\lambda'}^{-1} = q_{(\rho^\vee, \lambda)} q_{-(\rho^\vee, \lambda')} = q_{(\rho^\vee, \lambda - (\rho^\vee, \lambda') = q_{(\rho^\vee, \lambda - \lambda')} = q_{(\rho^\vee, \mu)} \).}

**Definition 5.1.9** ([Sch06] p.8). Let \( \mu \in \Lambda \). We define \( X_\mu := q_{\mu}^{-1}T_{\tau_\mu}T_{\tau_{\lambda'}}^{-1} \in \mathcal{H}_a \) where \( \mu = \lambda - \lambda' \) for \( \lambda, \lambda' \in \Lambda^+ \). For \( \lambda \in \Lambda^+ \), \( X_\lambda = q_{\lambda}^{-1}T_{\tau_\lambda} \).
5.2 Galleries

Definition 5.2.1 ([Sch06] p.9). A gallery $\sigma$ of type $t = (t_1, ..., t_k)$ with $t_i \in S_a \cup \Omega$ connecting generalized alcoves $A$ and $B$ is a sequence of generalized alcoves: $A = A_0, \ldots, A_k = B$, such that

$$A_{i+1} = \begin{cases} A_i t_{i+1} & (t_{i+1} \in \Omega) \\ A_i \lor A_i t_{i+1} & (t_{i+1} \in S_a) \end{cases}$$

Definition 5.2.2 ([Sch06] p.10). We define the direction of the first generalized alcove $\delta(A_0)$ to be the initial direction, the weight of the last generalized alcove $wt(A_k)$ to be the weight of the gallery, and $\delta(A_k)$ to be the final direction.

Definition 5.2.3 ([Sch06] p.6). Let $A \in A$ and $s \in S_a$ and let $H_{a,k}$ be the hyperplane separating $A$ and $As$. We define $A \prec As$ if $As \subset H_{a,k}^+$, $A \subset H_{a,k}^-$ and $A \succ As$ if $A \subset H_{a,k}^+$, $As \subset H_{a,k}^-$. 

Proposition 5.2.4 ([Sch06] p.6). Let $w \in W$ and $s \in S$. We have $A_w \prec A_{ws}$ if and only if $w > ws$ (relative to Bruhat order).

Definition 5.2.5 ([Sch06] p.10). The gallery $\sigma$ has a positive $s$-direction at $i$ if $t_{i+1} = s, A_{i+1} = A_i s$, where $A_i \prec A_{i+1}$. $\sigma$ has a negative $s$-direction at $i$ if $t_{i+1} = s, A_{i+1} = A_i s$, where $A_i \succ A_{i+1}$. The gallery $\sigma$ is $s$-folded at $i$ if $t_{i+1} = s$ and $A_{i+1} = A_i$. The folding is positive if $A_i \succ A_i s$, and negative if $A_i \prec A_i s$.

Definition 5.2.6 ([Sch06] p.10). A gallery $\sigma$ is called a positively folded gallery if all foldings in $\sigma$ are positive.

Definition 5.2.7 ([Sch06] p.10). Let $\sigma$ be a positively folded gallery of type $t$. We denote: the number of positive $s$-directions as $m_s(\sigma)$; the number of positive $s$-folds as $n_s(\sigma)$; the number of negative $s$-directions as $o_s(\sigma)$.
**Definition 5.2.8** ([Sch06] p.10). Let $\sigma$ be a positively folded gallery of type $t$. We define

$$L_\sigma = \prod_{s \in S_\sigma} q^{n_s(\sigma)}(q - 1)^{n_s(\sigma)}.$$  \hspace{1cm} (5.5)

**Definition 5.2.9** ([Sch06] p.14). Let $A \in \tilde{A}$. Define

$$X_A = q^{-\text{wt}(A)}q^{\delta(A)}X_{\text{wt}(A)}\tilde{T}_{\delta(A)}.$$  \hspace{1cm} (5.6)

**Proposition 5.2.10** ([Sch06] p.14). The set $\{X_A\}_{A \in \tilde{A}}$ is a basis of $\mathcal{H}_a$, called the alcove basis.

**Definition 5.2.11** ([Sch06] p.15). Let $A, B$ be generalized alcoves and $\Gamma^+_t(A, B)$ be the set of all positively folded galleries of type $t$ connecting $A$ and $B$. Then we define

$$L_t(A, B) = \sum_{\sigma \in \Gamma^+_t(A, B)} L_\sigma.$$  \hspace{1cm} (5.7)

**Theorem 5.2.12** ([Sch06] Theorem 5.5). For $\nu \in W_e$, let $t$ be the type of a minimal gallery connecting $A_0$ and $A_\nu$. For $A \in \tilde{A}$ and $\nu \in W_e$, we have

$$X_A T_\nu = \sum_{B \in \tilde{A}} L_t(A, B)X_B.$$  \hspace{1cm} (5.8)
Chapter 6

Ram’s Paper “Alcove Walks, Hecke Algebras, Spherical Functions Crystals and Column Strict Tableaux”

6.1 Alcove walks

Definition 6.1.1 ([Ram06] p.139). We denote the walls of the dominant Weyl chamber as $H_{\alpha_1}, H_{\alpha_2}, \ldots, H_{\alpha_n}$ and extend this so that $H_{\alpha_0}, H_{\alpha_1}, H_{\alpha_2}, \ldots, H_{\alpha_n}$ are the walls of $A_0$ with corresponding reflections $s_0, s_1, \ldots, s_n$.

Proposition 6.1.2 ([Ram06] p.139). $g \in \Omega$ acts on $A_0$ by an automorphism which gives a permutation of the walls $H_{\alpha_0}, H_{\alpha_1}, H_{\alpha_2}, \ldots, H_{\alpha_n}$, hence a permutation of $0, 1, \ldots, n$, so that $gs_i g^{-1} = s_{g(i)}$ for $g \neq 1$.

Proposition 6.1.3 ([Ram06] p.139). The extended affine Weyl group $W_e$ acts freely on $\Omega \times E$, and we have $W_e \rightarrow \tilde{A}, w \mapsto w^{-1}A_0$, so that $g^{-1}A_0$ is in the same place as $A_0$ except on the $g^{th}$ “sheet” of $\Omega \times E$. 

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Definition 6.1.4 ([Ram06] p.139, p.141). Number the alcove walls in a $W_e$-equivariant way: the numbering of the walls of $wA_o$ is the $w$ image of the numbering of the walls of $A_o$ for $w \in W_e$.

Definition 6.1.5 ([Yee19] p.286). An alcove path from $A$ to $B$ is a sequence of alcoves $A = A_0 \xrightarrow{R_1} A_1 \xrightarrow{R_2} \cdots \xrightarrow{R_\ell} A_\ell = B$ where for $1 \leq i \leq \ell$, $A_{i-1}$ and $A_i$ are adjacent. $R_i$ is the affine reflection corresponding to the affine hyperplane separating $A_{i-1}$ and $A_i$: $A_i = R_iA_{i-1}$.

Definition 6.1.6 ([Ram06] p.143). Recall that $\mathbb{K}$ is the field of fractions of $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. Let $g \in \Omega$, $g(i)$ be the index such that $gs_i g^{-1} = s_{g(i)}$. The walls of $A_o$ are labelled $0, 1, 2, \ldots, n$. The alcove walk algebra $A$ is the $\mathbb{K}$-algebra with generators $g \in \Omega$ and for $1 \leq i \leq n$

\[\begin{array}{cccc}
- & + & - & + & - & + & - & + \\
\text{positive } i\text{-crossing} & (c_i^+) & \text{negative } i\text{-crossing} & (c_i^-) & \text{positive } i\text{-fold} & (f_i^+) & \text{negative } i\text{-fold} & (f_i^-) \\
\end{array}\]

with relations

$\begin{align*}
    c_i^+ &= c_i^- + f_i^+ \\
    c_i^- &= c_i^+ + f_i^-
\end{align*}$

and

\[
g\begin{pmatrix}
- & + \\
\end{pmatrix} = g\begin{pmatrix}
g(i) \\
- & + \\
\end{pmatrix} g, \\
g\begin{pmatrix}
- & + \\
\end{pmatrix} = g\begin{pmatrix}
g(i) \\
- & + \\
\end{pmatrix} g.
\]
\[
g \left( \begin{array}{cc}
 -i & \\
 \downarrow & \\
 \end{array} \right) = \left( \begin{array}{cc}
 g(i) & \\
 \downarrow & \\
 \end{array} \right) g, \quad g \left( \begin{array}{cc}
 -i & \\
 \downarrow & \\
 \end{array} \right) = \left( \begin{array}{cc}
 g(i) & \\
 \downarrow & \\
 \end{array} \right) g.
\]

**Definition 6.1.7** ([Ram06] p.143). An alcove walk is a word in the generators of \( \mathcal{A} \) such that

(a) the tail of the first step is in the fundamental alcove \( A_\circ \)

(b) at every step, the head of each arrow and the tail of the next arrow are in the same alcove.

**Definition 6.1.8** ([Ram06] p.144). The type of an alcove walk \( p \) is the sequence of labels on the arrows, corresponding to the folds and wall crossings of the walk.

**Definition 6.1.9** ([Ram06] p.153). An alcove walk \( p \) is positively folded if all foldings in \( p \) are positive.

**Definition 6.1.10** ([Ram06] p.149). Let \( p \) be an alcove walk from \( wA_\circ \) to \( wt(p) + \varphi(p)A_\circ \) where \( w \in W \). We define \( \iota(p) = w \) as the initial direction of \( p \), \( wt(p) \) as the weight of \( p \), and \( \varphi(p) \) as the final direction of \( p \).

**Definition 6.1.11** ([Ram06] p.145). The affine Hecke algebra \( \mathcal{H}_a \) is the quotient of the alcove walk algebra by the relations

\[
c_i^+ = (c_i^-)^{-1}, \quad f_i^+ = q^{\frac{1}{2}} - q^{-\frac{1}{2}}, \quad f_i^- = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})
\]

and we say nonfolded alcove walks \( p = p' \) if they have the same ending alcove. (Note that the affine Hecke algebra only remembers the ending alcove of a walk.)

**Definition 6.1.12** ([Ram06] p.146). Let \( \mathcal{H}_a \) be the affine Hecke algebra. We define \( \mathcal{T}_{w^{-1}}^{-1} \) as the image in \( \mathcal{H}_a \) of a minimal length alcove walk from \( A_\circ \) to \( wA_\circ \) and \( X^\lambda \) as the image in \( \mathcal{H}_a \) of a minimal length alcove walk from \( A_\circ \) to \( \tau_\lambda A_\circ \) for \( w \in W \) and \( \lambda \in \Lambda \).
Proposition 6.1.13 ([Ram06] Proposition 3.2). $\mathcal{H}_a$ coincides with the usual definition of the affine Hecke algebra.

Proposition 6.1.14 ([Ram06] Proposition 3.2).

\[ X^\lambda X^\mu = X^{\lambda+\mu} = X^\mu X^\lambda \quad \text{for } \lambda, \mu \in \Lambda. \quad (6.1) \]

### 6.2 Hall-Littlewood polynomials

**Definition 6.2.1** ([Ram06] p.151). We define an element $\tilde{1}_0$ in $\mathcal{H}_a$ by

\[ \tilde{T}_{w^{-1}}^{-1} \tilde{1}_0 = q^{-\frac{1}{2} \ell(w)} \tilde{1}_0 \quad (6.2) \]

for $w \in W$.

An explicit formula for $\tilde{1}_0$ is

\[ \tilde{1}_0 = \frac{1}{W_0(q^{-1})} \sum_{w \in W} q^{-\frac{1}{2} \ell(w)} \tilde{T}_{w^{-1}}^{-1} \quad (6.3) \]

where $W_0(t) = \sum_{w \in W} t^{\ell(w)}$ is the Poincaré polynomial of $W$.

**Definition 6.2.2** ([Ram06] p.151, [Yee19] p.283). We define $\mathbb{K}[\Lambda] = \text{span}\{X^\mu : \mu \in \Lambda\}$.

**Proposition 6.2.3** ([Ram06] p.151). $\{X^\mu \tilde{1}_0 : \mu \in \Lambda\}$ is a basis of $\mathcal{H}_a \tilde{1}_0$. Then there exists a vector space isomorphism $\mathbb{K}[\Lambda] \to \mathcal{H}_a \tilde{1}_0$, $f \mapsto f \tilde{1}_0$.

**Definition 6.2.4** ([Yee19] p.284). The spherical Hecke algebra is $\tilde{1}_0 \mathcal{H}_a \tilde{1}_0$.

**Definition 6.2.5** ([Ram06] p.151, [Yee19] p.284). The ring of symmetric functions is $\mathbb{K}[\Lambda]^W$. By a theorem of Bernstein, $\mathbb{K}[\Lambda]^W$ is the centre of $\mathcal{H}_a$. There is an isomorphism called the Satake isomorphism

\[ \Phi : \mathbb{K}[\Lambda]^W \to \tilde{1}_0 \mathcal{H}_a \tilde{1}_0, f \mapsto f \tilde{1}_0. \]
**Definition 6.2.6** ([Ram06] p.153). For $\mu \in \Lambda$, the Hall-Littlewood polynomial $P_\mu(X; q^{-1}) \in \mathbb{K}[\Lambda]^W$ is defined by

$$P_\mu(X; q^{-1}) \tilde{1}_0 = \left( \sum_{w \in W_\mu} q^{-\frac{1}{2} \ell(w) \tilde{T}_{w^{-1}}^{-1}} \right) X^\mu \tilde{1}_0$$  \hspace{1cm} (6.4)$$

where $W_\mu$ is the stabilizer of $\mu$ and $W^\mu$ is a set of minimal length coset representatives for the cosets in $W/W_\mu$.

**Theorem 6.2.7.** Fix $\lambda \in \Lambda^+$. Let $p_\lambda = c_1^+ \cdots c_\ell^+$ be a minimal length alcove walk from $A_o$ to $\lambda + A_o$ and $B_q(p_\lambda) = \{ \text{positively folded alcove walks of type } (i_1, \ldots, i_\ell) \text{ beginning at } wA_o \text{ where } w \in W^\lambda \}$. Then

$$P_\lambda(X; q^{-1}) = \sum_{p \in B_q(p_\lambda)} q^{-\frac{1}{2} (\ell(\i(p)) + \ell(\varphi(p)) - f(p))} (1 - q^{-1})^f(p) X^{wt(p)}$$  \hspace{1cm} (6.5)$$

where $\i(p)$ is the alcove where $p$ begins, $wt(p) + \varphi(p)A_o$ is the alcove where $p$ ends and $f(p)$ is the number of folds in $p$.

We will prove that this theorem follows from Schwer’s Theorem 5.5 (Theorem 5.2.12 of this paper).
Chapter 7

Research Problem

In this chapter, we will show that Theorem 6.2.7 follows from Theorem 5.2.12.

Lemma 7.0.1. For $\lambda \in \Lambda^+$, $X^\lambda = \tilde{T}_\tau$ and $\tilde{T}_\tau = q^{-\frac{1}{2}\ell(\tau)}T_\tau$.

Lemma 7.0.2. For $\mu \in \Lambda$, $X_\mu$ (Definition 5.1.9) = $X^\mu$ (Definition 6.1.12).

Proof. Recall that $\mu = \lambda - \lambda' \in \Lambda$ where $\lambda, \lambda' \in \Lambda^+$, $q_\mu = q^{(\rho', \mu)}$ (Lemma 5.1.8) and $X_\mu := q_\mu^{-1}T_\tau T_{\tau'}^{-1}$ (Definition 5.1.9).

$$X_\mu = q^{-(\rho', \mu)}T_\tau T_{\tau'}^{-1}$$

$$= q^{-(\rho', \mu)}(\tilde{T}_\tau q^{\frac{1}{2}\ell(\tau)})(\tilde{T}_{\tau'}^{-1}q^{-\frac{1}{2}\ell(\tau')}) \quad \text{from (4.3)}$$

$$= q^{-(\rho', \mu)}q^\ell(\tau)q^{-\ell(\tau')}/2\tilde{T}_\tau \tilde{T}_{\tau'}^{-1}$$

$$= q^{-(\rho', \mu)}q^{(\rho', \lambda)}q^{-(\rho', \lambda')}\tilde{T}_\tau \tilde{T}_{\tau'}^{-1} \quad \text{from Lemma 5.1.7}$$

$$= q^{-(\rho', \mu)}q^{(\rho', \lambda - \lambda')}\tilde{T}_\tau \tilde{T}_{\tau'}^{-1} \quad \text{(7.1)}$$

$$= q^{-(\rho', \mu)}q^{(\rho', \mu)}\tilde{T}_\tau \tilde{T}_{\tau'}^{-1}$$

$$= \tilde{T}_\tau \tilde{T}_{\tau'}^{-1}$$

$$= X^\lambda (X^{\lambda'})^{-1}$$

$$= X^\mu$$

}$
Now we would like to prove Theorem 6.2.7 using Theorem 5.2.12.

Recall Theorem 6.2.7: Fix $\lambda \in \Lambda^+$. Let $p_\lambda = c_{i_1}^+ \cdots c_{i_\ell}^+$ be a minimal length alcove walk from $A_\circ$ to $\lambda + A_\circ$ and $B_q(p_\lambda) = \{ \text{positively folded alcove walks of type } (i_1, \ldots, i_\ell) \}$ beginning at $wA_\circ$ where $w \in W^\lambda$. Then,

$$P_\lambda(X; q^{-1}) = \sum_{p \in B_q(p_\lambda)} q^{-\frac{1}{2}(\ell(p) + \ell(\varphi(p)) - f(p))}(1 - q^{-1})f(p)X^{wt(p)}$$  \hfill (7.2)

where $\iota(p)$ is the alcove where $p$ begins, $wt(p) + \varphi(p)A$ is the alcove where $p$ ends and $f(p)$ is the number of folds in $p$.

**Proof.** Recall Theorem 5.2.12. Fix $\lambda \in \Lambda^+$ and $w \in W$. We choose $A$ as $A_w \in W_e$ and $\nu$ as $\tau_\lambda$, where $A_w = wA_\circ$. Then $t$ is the type of a minimal gallery connecting $A_\circ$ and $\tau_\lambda A_\circ$ and Theorem 5.2.12 becomes

$$X_{A_w}T_{\tau_\lambda} = \sum_{B \in A} L_t(A_w, B)X_B.$$  \hfill (7.3)

By Lemma 5.1.3, $A$ is of the form $wt(A) + \delta(A)A_\circ$ for unique $wt(A) \in \Lambda$ and $\delta(A) \in W$. Then $wt(A_w) = 0$ and $\delta(A_w) = w$. Recalling Definition 5.2.9, we rewrite $X_{A_w}T_{\tau_\lambda}$ as

$$X_{A_w}T_{\tau_\lambda} = q^{-wt(A_w)q\delta(A_w)}X_{wt(A_w)}T_{\delta(A_w)}T_{\tau_\lambda}$$

$$= q^{\delta(A_w)}T_{\delta(A_w)}T_{\tau_\lambda}$$

$$= q^{-w_1T_wT_{\tau_\lambda}}$$

$$= q^{\ell(w) - 1}T_{w^{-1}}T_{\tau_\lambda}.$$  \hfill (7.4)

From (4.3) and Definition 5.1.9, we get

$$\tilde{T}_w = q^{-\frac{1}{2}\ell(w)}T_w \iff \tilde{T}_{w^{-1}} = q^{-\frac{1}{2}\ell(w)}T_{w^{-1}}$$

$$\iff \tilde{T}_{w^{-1}}^{-1} = T_{w^{-1}}^{-1}q^{\frac{1}{2}\ell(w)}$$

$$\iff T_{w^{-1}}^{-1} = \tilde{T}_{w^{-1}}^{-1}q^{-\frac{1}{2}\ell(w)},$$  \hfill (7.5)
\[ X_\lambda = q_\lambda^{-1} T_{\tau_\lambda} \iff T_{\tau_\lambda} = q_\lambda X_\lambda \]  
\[ \iff T_{\tau_\lambda} = q^{\frac{1}{2}\ell(\tau_\lambda)} X_\lambda. \]  
(7.6)

Thus

\[ X_{A_w} T_{\tau_\lambda} = q^{\ell(w)} T_{w^{-1}}^{-1} q^{\frac{1}{2}\ell(\tau_\lambda)} T_{\tau_\lambda} X_\lambda \]
\[ = q^{\frac{1}{2}\ell(w)} T_{w^{-1}}^{-1} q^{\frac{1}{2}\ell(\tau_\lambda)} X_\lambda \]  
(7.7)

\[ = q^{\frac{1}{2}\ell(w)} T_{w^{-1}}^{-1} q^{\frac{1}{2}\ell(\tau_\lambda)} X_\lambda \]  
by Lemma 7.0.2.

Thus

\[ q^{\frac{1}{2}\ell(w)} T_{w^{-1}}^{-1} q^{\frac{1}{2}\ell(\tau_\lambda)} X_\lambda = \sum_{B \in \mathcal{A}} L_t(A_w, B) X_B. \]  
(7.8)

Now, we look at \( \sum_{B \in \mathcal{A}} L_t(A_w, B) X_B. \)

\[ q^{\frac{1}{2}\ell(w)} T_{w^{-1}}^{-1} q^{\frac{1}{2}\ell(\tau_\lambda)} X_\lambda = \sum_{B \in \mathcal{A}} L_t(A_w, B) X_B \]
\[ = \sum_{B \in \mathcal{A}} \left( \sum_{A \in \Gamma^+_w(A_w, B)} \sum_{\sigma \in \Gamma^+_w(A_w, B)} \left( \prod_{s \in S_a} q^{m_s(\sigma)}(q - 1)^{n_s(\sigma)} \right) \right) X_B \]
\[ = \sum_{B \in \mathcal{A}} \left( \sum_{A \in \Gamma^+_w(A_w, B)} \left( \prod_{s \in S_a} q^{m_s(\sigma)}(q - 1)^{n_s(\sigma)} \right) q_{-wt(B)} q_{\delta(B)} X_{wt(B)} \right) X_B \]
\[ = \sum_{B \in \mathcal{A}} \left( \sum_{A \in \Gamma^+_w(A_w, B)} \left( \prod_{s \in S_a} q^{m_s(\sigma)}(1 - q^{-1})^{n_s(\sigma)} \right) q_{-wt(B)} q_{\delta(B)} X_{wt(B)} \right) X_B \]
(7.9)

Multiplying by the element \( \tilde{1}_0 \) (6.2.1) on both sides and simplifying, we get:

\[ q^{\frac{1}{2}\ell(w)} T_{w^{-1}}^{-1} q^{\frac{1}{2}\ell(\tau_\lambda)} X_\lambda \tilde{1}_0 = \sum_{B \in \mathcal{A}} \left( \sum_{A \in \Gamma^+_w(A_w, B)} \left( \prod_{s \in S_a} q^{m_s(\sigma)}(1 - q^{-1})^{n_s(\sigma)} \right) q_{-wt(B)} q_{\delta(B)} X_{wt(B)} \right) \]
\[ T_{\delta(B)^{-1}}^{-1} \tilde{1}_0 \]  
(7.10)
\[ q^{-\frac{1}{2}\ell(w)}\tilde{T}_{w^{-1}}X^{\lambda} \hat{1}_0 = \sum_{B \in \tilde{A}} \left( \sum_{\sigma \in \Gamma^+_t(A_w, B)} \left( \prod_{s \in S_\sigma} q^{m_s(\sigma) + n_s(\sigma)} (1 - q^{-1}n_s(\sigma)) \right) q^{-\ell(w) - \frac{1}{2}(\tau_\lambda)} q^{-w(B)q_{\delta(B)}} \right) X_{w(B)}T_{\delta(B)^{-1}}^{-1} \hat{1}_0. \] (7.11)

Since \( t \) is the type of a minimal gallery connecting \( A_0 \) and \( \tau_\lambda A_0 \), then

\[ \ell(\tau_\lambda) = \sum_{s \in S_\sigma} (m_s(\sigma) + n_s(\sigma) + o_s(\sigma)). \] (7.12)

for all \( \sigma \in \Gamma^+_t(A_w, B) \).

Also, we show that

\[ q^{-w(B)} = q^{-w(B)\rho^\nu} = q^{\frac{1}{2}\sum (m_s(\sigma) + o_s(\sigma) - \ell(\delta(B)) + \ell(w))}. \] (7.13)

**Proof.** It is equivalent to show that \( q^{w(B)2\rho^\nu} = q^{\sum (m_s(\sigma) + o_s(\sigma)) + \ell(\delta(B)) - \ell(w)} \). \( w(B), 2\rho^\nu \) is the number of \( H_{\alpha,k} \) where \( \alpha > 0, k > 0 \) minus the number of \( H_{\alpha,k} \) where \( \alpha > 0, k \leq 0 \) separating \( A_0 \) and \( A_0 + w(B) \), and \( \sum_{s \in S_\sigma} (m_s(\sigma) - o_s(\sigma)) \) is the number of \( H_{\alpha,k} \) where \( \alpha > 0, k > 0 \) minus the number of \( H_{\alpha,k} \) where \( \alpha > 0, k \leq 0 \) separating \( wA_0 \) and \( \delta(B)A_0 + w(B) \). Therefore

\[ (w(B), 2\rho^\nu) = \sum_{s \in S_\sigma} (m_s(\sigma) - o_s(\sigma)) + \ell(\delta(B)) - \ell(w) \] (7.14)

since \( A_0 \) and \( wA_0 \) are separated by \( \ell(w) \) \( H_{\alpha,0} \), and \( A_0 + w(B) \) and \( \delta(B)A_0 + w(B) \) are separated by \( \ell(\delta(B)) \) \( H_{\alpha,\nu, w(B)} \). \( \square \)

Apply equations 7.12 and 7.13 and simplify:

\[ q^{-\frac{1}{2}\ell(w)}\tilde{T}_{w^{-1}}X^{\lambda} \hat{1}_0 = \sum_{B \in \tilde{A}} \left( \sum_{\sigma \in \Gamma^+_t(A_w, B)} \left( \prod_{s \in S_\sigma} q^{m_s(\sigma) + n_s(\sigma)} (1 - q^{-1}n_s(\sigma)) \right) q^{\frac{1}{2}(m_s(\sigma) + o_s(\sigma))} \right) \] \[ q^{-\ell(w)} q^{-w(B)q_{\delta(B)}} X_{w(B)}T_{\delta(B)^{-1}}^{-1} \hat{1}_0 \] (7.15)
By (4.3), we can rewrite \( \tilde{T}_{\delta(B)} = q^{-\frac{1}{2} \ell(\delta(B))} T_{\delta(B)} \):

\[
\tilde{T}_{\delta(B)} = q^{-\frac{1}{2} \ell(\delta(B))} T_{\delta(B)} \Leftrightarrow \tilde{T}_{\delta(B)}^{-1} = q^{-\frac{1}{2} \ell(\delta(B))} T_{\delta(B)}^{-1} \\
\Leftrightarrow \tilde{T}_{\delta(B)}^{-1} \Leftrightarrow T_{\delta(B)}^{-1} q^{\frac{1}{2} \ell(\delta(B))} \\
\Leftrightarrow T_{\delta(B)}^{-1} = T_{\delta(B)}^{-1} q^{-\frac{1}{2} \ell(\delta(B))}.
\]

Thus

\[
q^{-\frac{1}{2} \ell(w)} \tilde{T}_{\delta(B)}^{-1} X^{\lambda} \mathbf{i}_0 = \sum_{B \in \mathcal{A}} \left( \sum_{\sigma \in \Gamma^+_{t}(A_w, B)} \left( \prod_{s \in S_n} q^{\frac{1}{2}(m_s(\sigma)+n_s(\sigma)-o_s(\sigma))} \right) \prod_{s \in S_n} q^{-\frac{1}{2}(m_s(\sigma)+n_s(\sigma)-o_s(\sigma))} \left( 1 - q^{-1} n_s(\sigma) \right) \right)
\]

\[
= \sum_{B \in \mathcal{A}} \left( \sum_{\sigma \in \Gamma^+_{t}(A_w, B)} \left( \prod_{s \in S_n} q^{\frac{1}{2} n_s(\sigma)} \right) \prod_{s \in S_n} q^{-\frac{1}{2} n_s(\sigma)} \left( 1 - q^{-1} n_s(\sigma) \right) \right) q^{-\frac{1}{2} \ell(\delta(B)) + \frac{1}{2} \ell(w)} q^{\ell(\delta(B))} q^{-\ell(w)} X^{\omega t(B)} \tilde{T}_{\delta(B)}^{-1} q^{-\frac{1}{2} \ell(\delta(B))} \mathbf{i}_0
\]

\[
= \sum_{B \in \mathcal{A}} \left( \sum_{\sigma \in \Gamma^+_{t}(A_w, B)} \left( \prod_{s \in S_n} q^{\frac{1}{2} n_s(\sigma)} \right) \prod_{s \in S_n} q^{-\frac{1}{2} n_s(\sigma)} \left( 1 - q^{-1} n_s(\sigma) \right) \right) q^{-\frac{1}{2} \ell(\delta(B)) + \frac{1}{2} \ell(w)} q^{-\ell(w)} X^{\omega t(B)} \mathbf{i}_0
\]

\[
= \sum_{B \in \mathcal{A}} \left( \sum_{\sigma \in \Gamma^+_{t}(A_w, B)} \left( \prod_{s \in S_n} q^{\frac{1}{2} (\ell(w)+\ell(\delta(B)))} \prod_{s \in S_n} q^{-\frac{1}{2} (\ell(w)+\ell(\delta(B)))} \left( 1 - q^{-1} n_s(\sigma) \right) \right) X^{\omega t(B)} \mathbf{i}_0
\]

(7.19)

Now replacing positively folded galleries with positively folded alcove walks in the above formula gives Theorem 6.2.7. Since \( \Gamma^+_t(A_w, B) \) is the set of all positively folded galleries of type \( t \) starting in \( wA_0 \) (5.2.11) while \( B_q(p_\lambda) \) is the set of all positively folded alcove walks of type \( t \) which begin at \( wA_0 \) (\( w \in W^\lambda \) the minimal length coset representatives of the cosets in \( W/W_\lambda \) for \( \lambda \in \Lambda^+ \)), we may switch \( \sum_{w \in W^\lambda} \sum_{B \in \mathcal{A}} \sum_{\sigma \in \Gamma^+_t(A_w, B)} \) to
Moreover, $wt(B)$ is equivalent to $wt(p)$; the sum of positive folds $\sum_{s \in S_n} n_s(\sigma) = f(p)$; the final direction $\delta(B) = \varphi(p)$ and $\ell(w) = \text{initial direction} \, \iota(p)$ since the gallery $\sigma$ starts from $wA_0$.

Therefore, we finish the proof showing that

\[
\sum_{w \in W^\lambda} (q^{-\frac{1}{2} \ell(w)} T_{w^{-1}}^{-1}) X^\lambda \tilde{I}_0 = \sum_{p \in B_q(p_\lambda)} q^{-\frac{1}{2} (\ell(\iota(p)) + (\ell(\varphi(p)) - f(p)))} (1 - q^{-1}) f(p) X^{wt(p)} \tilde{I}_0
\]

\[\]

\[P_\lambda(X; q^{-1}) \tilde{I}_0 = \sum_{p \in B_q(p_\lambda)} q^{-\frac{1}{2} (\ell(\iota(p)) + (\ell(\varphi(p)) - f(p)))} (1 - q^{-1}) f(p) X^{wt(p)} \tilde{I}_0 \]  

(7.20)
Bibliography


**Vita Auctoris**

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