January 2020

A Proof of a Hall-Littlewood Polynomial Formula

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By

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A Major Research Paper
Submitted to the Faculty of Graduate Studies
through the Department of Mathematics and Statistics
in Partial Fullfillment of the Requirements for
the Degree of Master of Science
at the University of Windsor

Windsor, Ontario, Canada

2019

A Proof of a Hall-Littlewood Polynomial Formula

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December 16, 2019

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Abstract

This paper proves a Hall-Littlewood polynomial formula in a paper by Ram [Ram06] using a theorem by Schwer [Sch06]. We review materials relating to root systems, affine Weyl groups and affine Hecke algebras that are required to study alcoves, galleries and the Hall-Littlewood polynomials. In order to prove the Hall-Littlewood polynomial formula, we formulate in a special case Schwer's formula in Theorem 5.5 [Sch06] computing right multiplication of the alcove basis by standard basis elements. We show that Ram's formula for Hall-Littlewood polynomials in terms of positively folded alcove walks coincides with the formulation of Schwer's formula in terms of positively folded galleries.

Acknowledgements

I would like to express my most sincere appreciation to my supervisor Dr. Wai Ling Yee. Her consistent support and patience made it possible for me to finish this paper. I would also like to express my gratitude to Dr. Borisov as my department reader. I am very grateful for your precious comments. Finally, I would like to thank my parents and my girlfriend for their continuous support and encouragement throughout years of study. I appreciate all your support!

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Chapter 1

Introduction

1.1 Thesis Problem

Hall-Littlewood polynomials (or Macdonald spherical functions) are of great interest with many applications in the study of representation theory and combinatorics. They are a basis for the algebra of symmetric functions depending on a parameter q, first arising in P. Hall's study of some counting problems in group theory [Hal59] and later defined explicitly by D. Littlewood in [Lit61]. Hall-Littlewood polynomials interpolate between Schur functions at $q = 0$ and monomial symmetric functions when $q = 1$. Therefore they specialize to two well-known bases for the ring of symmetric functions. In [Mac71], Macdonald gives a formula for spherical functions on a p-adic Chevalley group generalizing Hall-Littlewood polynomials to all root systems. Schwer in [Sch06] proved a formula for Hall-Littlewood polynomials in terms of positively folded galleries and Ram in [Ram06] proved a formula for Hall-Littlewood polynomials in terms of positively folded alcove walks.

In [Sch06], Schwer showed that right multiplication of an alcove basis element by elements of the standard basis can be computed using positively folded galleries (Theorem 5.5 of [Sch06]). In [Ram06], Ram states a formula for Hall-Littlewood polynomials in terms of positively folded alcove walks (Theorem 4.2 of [Ram06]). We prove that Ram's

Hall-Littlewood polynomial formula follows from Schwer's Theorem 5.5.

1.2 Outline

- In chapters 2-4, we review some concepts from root systems, reflection groups and Hecke algebras, as background required for the thesis problem.
- In chapter 5, we discuss a paper by Schwer on galleries.
- In chapter 6, we discuss a paper by Ram on alcoves walks and Hall-Littlewood polynomials.
- In chapter 7, we prove that Ram's formula for Hall-Littlewood polynomials [Ram06, Theorem 4.2] follows from Schwer's formula describing right multiplication of an alcove basis element by a standard basis element in terms of positively folded galleries [Sch06, Theorem 5.5].

Chapter 2

Root Systems

2.1 Reflections and root systems

Definition 2.1.1 ([Hum90] p.3, [Hum78] p.42)**.** Let E be a real euclidean space with positive-definite symmetric bilinear form $(\cdot,\cdot)$. A **reflection** in E is an invertible linear operator in E denoted by σ which sends a nonzero vector α to its negative $-\alpha$, fixing the reflecting hyperplane H_α orthogonal to α. The formula for such a reflection is written as:

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha. \tag{2.1}$$

Given a vector $\alpha \in E$ ($\alpha \neq 0$), there exists a reflection σ_α through the **reflecting hyperplane** $H_\alpha = \{\beta \in E \mid (\beta,\alpha) = 0\}$ (shown in the following example).

Note. Recall that a symmetric real $n \times n$ matrix is a matrix A such that $A = A^T$ where A^T denotes the transpose of A. We call A **positive definite** if $x^T Ax > 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$. The matrix A is positive definite if and only if the corresponding bilinear form $(x,y) = x^T Ay$ is positive definite.

Example. (Figure 2.1) Let E be a euclidean space, and H_α a reflecting hyperplane.

Given a vector β in E, we have

$$\sigma_\alpha(\beta) = \beta - 2\left(\|\beta\| \, cos\theta \frac{\alpha}{\|\alpha\|} \right).$$

Since $(\beta, \alpha) = \|\beta\|\|\alpha\| cos\theta$ and $(\alpha, \alpha) = \|\alpha\|^2$, therefore

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{\|\alpha\|} \frac{\alpha}{\|\alpha\|} = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha.$$

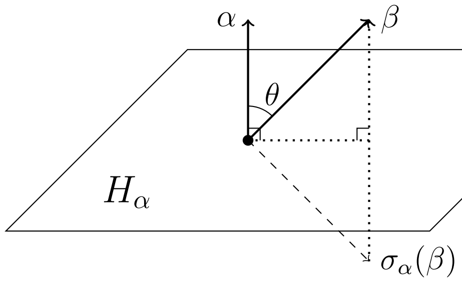


Figure 2.1: Reflection of β through H_α

Definition 2.1.2 ([Hum78] p.42, [Hum90] p.6)**.** A finite subset Φ of a euclidean space E is a **root system** in E if it satisfies the following:

(R1) Φ spans E and $\vec{0} \notin \Phi$;

(R2) If $\alpha \in \Phi$, then $\pm\alpha$ are the only scalar multiples of α in Φ;

(R3) If $\alpha \in \Phi$, then $\sigma_\alpha(\Phi) = \Phi$;

(R4) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle := \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$. (Note that $\langle \alpha, \beta \rangle$ may not equal $\langle \beta, \alpha \rangle$.)

Definition 2.1.3. α is called a **root** if $\alpha \in \Phi$ (Definition 2.1.2).

Definition 2.1.4. We define the **rank** of the root system to be $\ell = \dim E$.

Example. (Figure 2.2) Given E is a one-dimensional euclidean space where $\ell = 1$, then we have, for $\alpha \in E$ and $\alpha \neq 0$, $\Phi = \{\pm\alpha\}$ is a root system, called A_1.

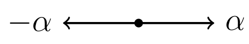


Figure 2.2: Root system of type A_1

Proof. (R1): $span\{\pm\alpha\} = span\{\alpha\} = E$.

(R2): Obviously true. $\pm\alpha$ are the only multiples of $\pm\alpha$ in Φ.

(R3): $\sigma_\alpha(\alpha) = \alpha - \frac{2(\alpha,\alpha)}{(\alpha,\alpha)}\alpha = \alpha - 2\alpha = -\alpha$; $\sigma_\alpha(-\alpha) = -\sigma_\alpha(\alpha) = -(-\alpha) = \alpha$.

(R4) $\langle\alpha,\alpha\rangle = \frac{2(\alpha,\alpha)}{(\alpha,\alpha)} = 2 \in \mathbb{Z}$; $\langle\alpha,-\alpha\rangle = \frac{2(\alpha,-\alpha)}{(-\alpha,-\alpha)} = -2 \in \mathbb{Z}$; $\langle-\alpha,\alpha\rangle = -\langle\alpha,\alpha\rangle = -2 \in \mathbb{Z}$; $\langle-\alpha,-\alpha\rangle = -\langle\alpha,-\alpha\rangle = 2 \in \mathbb{Z}$.

Therefore, $\Phi = \{\pm\alpha\}$ is a root system. □

Example. When $\ell = 2$, we can get more than one root system. Figure 2.3 is a diagram of the root system of type A_2. See the next example.

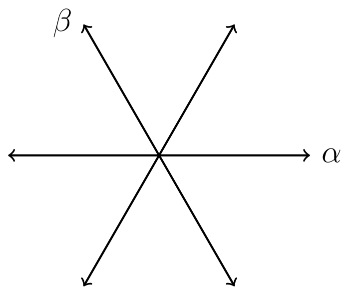


Figure 2.3: Root system of type A_2

Example. Let $E = \{x \in \mathbb{R}^{\ell+1} \mid x_1 + x_2 + \cdots + x_{\ell+1} = 0\}$. Then we have the root system $\Phi = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq \ell + 1\}$. (This is a root system of type A_ℓ.)

Proof. (R1) $|\Phi| = (\ell + 1)\ell$ which is finite.

$E = span\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_\ell - \varepsilon_{\ell+1}\}$, so Φ spans E.

Φ does not contain 0, since $1 \leq i < j \leq \ell + 1$.

(Note that $E = span\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_\ell - \varepsilon_{\ell+1}\}$ is a linearly independent subset of size ℓ in the ℓ-dimensional vector space E.)

(R2) It is clear to see that the only scalar multiples of $\varepsilon_i - \varepsilon_j$ in Φ are $\pm(\varepsilon_i - \varepsilon_j)$.

(R3) Let $\alpha = \varepsilon_i - \varepsilon_j$ and $\varepsilon_m - \varepsilon_n \in \Phi$. Since $\sigma_{(-\alpha)} = \sigma_\alpha$ and $\sigma_\alpha(-(\varepsilon_m - \varepsilon_n)) = -\sigma_\alpha(\varepsilon_m - \varepsilon_n)$, WLOG, we assume $i < j$ and $m < n$.

$$\begin{aligned}
\sigma_\alpha(\varepsilon_m - \varepsilon_n) &= \varepsilon_m - \varepsilon_n - \frac{2(\varepsilon_m - \varepsilon_n, \varepsilon_i - \varepsilon_j)}{(\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j)}(\varepsilon_i - \varepsilon_j) \\
&= \varepsilon_m - \varepsilon_n - (\delta_{m,i} - \delta_{n,i} - \delta_{m,j} + \delta_{n,j})(\varepsilon_i - \varepsilon_j) \\
&= \begin{cases}
\varepsilon_m - \varepsilon_n & if\ m \neq i, j\ and\ n \neq i, j \\
\varepsilon_i - \varepsilon_n - (\varepsilon_i - \varepsilon_j) = \varepsilon_j - \varepsilon_n & if\ m = i\ and\ n \neq i, j \\
\varepsilon_j - \varepsilon_n + (\varepsilon_i - \varepsilon_j) = \varepsilon_i - \varepsilon_n & if\ m = j\ and\ n \neq i, j \\
\varepsilon_m - \varepsilon_i + (\varepsilon_i - \varepsilon_j) = \varepsilon_m - \varepsilon_j & if\ m \neq i, j\ and\ n = i \\
\varepsilon_m - \varepsilon_j - (\varepsilon_i - \varepsilon_j) = \varepsilon_m - \varepsilon_i & if\ m \neq i, j\ and\ n = j \\
\varepsilon_i - \varepsilon_j - 2(\varepsilon_i - \varepsilon_j) = -(\varepsilon_i - \varepsilon_j) & if\ m = i\ and\ n = j
\end{cases}
\end{aligned}$$

Thus we have shown that $\sigma_\alpha(\Phi) = \Phi$.

(R4) We compute

$$\langle \varepsilon_m - \varepsilon_n, \varepsilon_i - \varepsilon_j \rangle = \frac{2(\varepsilon_m - \varepsilon_n, \varepsilon_i - \varepsilon_j)}{(\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j)} = \delta_{m,i} - \delta_{n,i} - \delta_{m,j} + \delta_{n,j} \in \mathbb{Z}.$$

Therefore, $\Phi = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq \ell + 1\}$ is a root system. $\square$

Note. From the definition above, (R4) limits possible angles between α and β.

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)} \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \frac{2\|\alpha\|\|\beta\| cos\theta}{\|\beta\|^2} \frac{2\|\beta\|\|\alpha\| cos\theta}{\|\alpha\|^2} = 4cos^2\theta \in \mathbb{Z}$$

Since $cos^2\theta \in [0, 1]$, then $4cos^2\theta \in [0, 4]$. $cos^2\theta = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$

$cos^2\theta = 0 \quad \Rightarrow \quad \theta = 90°$

$cos^2\theta = 1/4 \quad \Rightarrow \quad \theta = 60°\ or\ 120°$

$$cos^2\theta = 1/2 \quad \Rightarrow \quad \theta = 45^\circ \; or \; 135^\circ$$
$$cos^2\theta = 3/4 \quad \Rightarrow \quad \theta = 30^\circ \; or \; 150^\circ$$
$$cos^2\theta = 1 \quad \Rightarrow \quad \theta = 0^\circ \; or \; 180^\circ$$

Definition 2.1.5 ([Hum90] p.39)**.** Φ is a root system in E. $\alpha^\vee$ is the **coroot** of a root α if it satisfies

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}.$$

Note. $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ is also a root system of E (proved in Exercise 2 [Hum78, p.46]).

2.2 Simple roots and Weyl chambers

Definition 2.2.1 ([Hum78] p.47)**.** A subset $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of Φ is a **base** of E if Δ is a basis of E and every $\beta = \sum_{i=1}^{\ell} k_i\alpha_i \in \Phi$ satisfies the k_i are all nonnegative or all nonpositive integers.

Proposition 2.2.2 ([Hum78] p.48)**.** Φ *has a base.*

Definition 2.2.3 ([Hum78] p.47)**.** The roots belonging to a base Δ are called **simple roots**. If all integral coefficients k_i in $\beta = \sum_{i=1}^{\ell} k_i\alpha_i$ are nonnegative, we call β a **positive root** ($\beta \succ 0$). If all integral coefficients k_i are nonpositive, we call β a **negative root** ($\beta \prec 0$). We denote the collection of all positive roots by Φ^+, all negative roots by Φ^-, where $\Phi^- = -\Phi^+$.

Example. In the previous example, we have the root system of type A_ℓ: $\Phi = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq \ell + 1\}$. Thus we can show that $\Delta = \{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq \ell\}$ is a base for Φ. Moreover, with the choice of the base Δ, we get $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq \ell + 1\}$ and $\Phi^- = \{-\varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq \ell + 1\}$.

Proof. Since $E = \{x \in \mathbb{R}^{\ell+1} \mid x_1 + x_2 + \cdots + x_{\ell+1} = 0\}$ is ℓ-dimensional and $\Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_\ell - \varepsilon_{\ell+1}\}$ is a linearly indpendent subset of E of size ℓ, then Δ

is a basis of E. Moreover, $\varepsilon_i - \varepsilon_j$ is a positive integer linear combination of $\varepsilon_\ell - \varepsilon_{\ell+1}$ $(1 \leq i < j \leq \ell + 1)$:

$$\varepsilon_i - \varepsilon_j = (\varepsilon_i - \varepsilon_{i+1}) + (\varepsilon_{i+1} - \varepsilon_{i+2}) + \cdots + (\varepsilon_{j-2} - \varepsilon_{j-1}) + (\varepsilon_{j-1} - \varepsilon_j).$$

And also,

$$-\varepsilon_i + \varepsilon_j = (-\varepsilon_i + \varepsilon_{i+1}) + (-\varepsilon_{i+1} + \varepsilon_{i+2}) + \cdots + (-\varepsilon_{j-2} + \varepsilon_{j-1}) + (-\varepsilon_{j-1} + \varepsilon_j).$$

Therefore, Δ is a base for Φ. □

Definition 2.2.4 ([Hum78] p.52)**.** Let Φ be a root system. We call Φ **irreducible** if $\Phi \neq \Phi_1 \cup \Phi_2$ such that $(\phi_1, \phi_2) = 0$ for all $\phi_1 \in \Phi_1$ and $\phi_2 \in \Phi_2$.

Definition 2.2.5 ([Hum78] p.47)**.** Using the notation of Definition 2.2.1, we define the **height** of a root β as $ht(\beta) = \sum_{i=1}^{\ell} k_i$.

Proposition 2.2.6 ([Hum78] p.47)**.** *Let $\lambda, \mu \in E$. We have a partial order $\mu \prec \lambda$ on E if and only if $\lambda - \mu$ is a sum of α $(\alpha \in \Delta)$.*

Definition 2.2.7 ([Hum78] p.52)**.** Let Φ be an irreducible root system. There is a unique **highest root** $\beta \in \Phi$ such that for all $\alpha \in \Phi$, $ht(\alpha) < ht(\beta)$ for $\alpha \neq \beta$ (β is maximal relative to the partial order $\prec$).

Definition 2.2.8 ([Hum78] p.48)**.** For $\gamma \in E$, $\gamma \notin H_\alpha$ for all $\alpha \in \Phi$, let $\Phi^+(\gamma) = \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\}$ be the set of roots lying on one side of the hyperplane orthogonal to γ. Since in a euclidean space E, the union of the finitely many hyperplanes H_α $(\alpha \in \Phi)$ cannot exhaust E, we call $\gamma \in E$ **regular** if $\gamma \in E - \bigcup_{\alpha \in \Phi} H_\alpha$. Otherwise, γ is **singular**.

Definition 2.2.9 ([Hum78] p.48)**.** If γ is regular, then $\Phi = \Phi^+(\gamma) \bigcup -\Phi^+(\gamma)$. Then we call $\alpha \in \Phi^+(\gamma)$ **decomposable** if $\alpha = \beta_1 + \beta_2$ for some $\beta_i \in \Phi^+(\gamma)$, and **indecomposable** otherwise.

Proposition 2.2.10 ([Hum78] p.48)**.** *Let $\gamma \in E$ be regular. Then the set $\Delta(\gamma)$ of all indecomposable roots in $\Phi^+(\gamma)$ is a base of Φ and every base is of this form.*

Definition 2.2.11 ([Hum78] p.49)**.** The hyperplanes H_α ($\alpha \in \Phi$) partition E into finitely many regions. The connected components of $E - \bigcup_{\alpha \in \Phi} H_\alpha$ are called **Weyl chambers**.

Proposition 2.2.12 ([Hum78] p.49)**.** *Each regular $\gamma \in E$ belongs to precisely one Weyl chamber of E, denoted as $\mathfrak{C}(\gamma)$.*

Proposition 2.2.13 ([Hum78] p.49)**.** *If $\mathfrak{C}(\gamma) = \mathfrak{C}(\gamma')$, then γ and γ' lie on the same side of each hyperplane $H_\alpha(\alpha \in \Phi)$. Thus we have $\Phi^+(\gamma) = \Phi^+(\gamma')$ and $\Delta(\gamma) = \Delta(\gamma')$.*

Definition 2.2.14 ([Hum78] p.49)**.** By proposition 2.2.10 and proposition 2.2.13, Weyl chambers are in one-to-one correspondence with bases. Write $\mathfrak{C}(\Delta) = \mathfrak{C}(\gamma)$ if $\Delta = \Delta(\gamma)$. We call $\mathfrak{C}(\Delta) = \mathfrak{C}(\gamma)$ the **fundamental Weyl chamber** relative to Δ.

2.3 Weyl groups

Definition 2.3.1 ([Hum78] p.51)**.** The **Weyl group** W of the root system Φ is the group generated by the reflections σ_α for $\alpha \in \Phi$.

Proposition 2.3.2 ([Hum78] p.51)**.** *W is generated by the simple reflections σ_α for $\alpha \in \Delta$.*

Example. Root system of type A_1: $\Phi = \{\pm\alpha\}$. The Weyl group of type A_1: $W_{A_1} = \langle\sigma_\alpha\rangle = \{1, \sigma_\alpha\}$.

Proposition 2.3.3 ([Hum78] p.51)**.** *Let Δ be a base of Φ and W be the Weyl group. Then W acts transitively on bases (i.e., if Δ' is another base of Φ, then $\sigma(\Delta) = \Delta'$ for some $\sigma \in W$).*

Definition 2.3.4 ([Hum90] p.118)**.** Let T be the set of reflections in W. We write $w' \to w$ if $w = w's$ for some $s \in T$ with $\ell(w) > \ell(w')$. We define $w' < w$ if there exists a sequence

$w' = w_0 \to w_1 \to ... \to w_n = w$. Thus we call the partial order $w' \leq w$ **Bruhat order**. (Note: a partial order is a binary relation that is reflexive, antisymmetric and transitive.)

Proposition 2.3.5 ([Hum90] Proposition 5.7)**.** *For $w \in W$ and $\alpha \in \Phi^+$, we have $ws_\alpha < w \Leftrightarrow w\alpha < 0$ and $ws_\alpha > w \Leftrightarrow w\alpha > 0$.*

2.4 Weights

Definition 2.4.1 ([Hum78] p.67)**.** For $\lambda \in E$, if $\langle \lambda, \alpha \rangle = \dfrac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$ for all $\alpha \in \Phi$, then we call λ a **weight**. The collection of all weights is denoted by Λ.

Definition 2.4.2 ([Hum78] p.67)**.** $\lambda \in \Lambda$ is **dominant** if $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ are all nonnegative for all $\alpha \in \Phi^+$, **strongly dominant** if $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ are all positive for all $\alpha \in \Phi^+$. We denote the set of all dominant weights as Λ^+ .

Definition 2.4.3 ([Hum78] p.67)**.** If $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$, then the vectors $\alpha_i^\vee = \dfrac{2\alpha_i}{(\alpha_i, \alpha_i)}$ $(1 \leq i \leq \ell)$ form a basis of E. Thus we let $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ be the dual basis $\left(\lambda_i, \dfrac{2\alpha_j}{(\alpha_j, \alpha_j)}\right) = (\lambda_i, \alpha_j^\vee) = \delta_{ij}$. Since $\langle \lambda_i, \alpha \rangle = \dfrac{2(\lambda_i, \alpha)}{(\alpha, \alpha)}$ are all nonnegative for $\alpha \in \Delta$, the λ_i are dominant weights, called **fundamental dominant weights**.

Note. $\sigma_i \lambda_j = \lambda_j - \delta_{ij}\alpha_i$ where $\sigma_i = \sigma_{\alpha_i}$.

Example. Consider the root system of the type A_2: $\{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_3)\}$ with simple roots $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2 - \varepsilon_3$. We can show that $\lambda_1 = \dfrac{1}{3}(2\alpha_1 + \alpha_2)$ and $\lambda_2 = \dfrac{1}{3}(\alpha_1 + 2\alpha_2)$.

Proof.

$$
\begin{aligned}
\langle \lambda_1, \alpha_1 \rangle &= 2\frac{(\frac{2\alpha_1+\alpha_2}{3}, \alpha_1)}{(\alpha_1, \alpha_1)} \\
&= \frac{2}{3}\frac{(2(\varepsilon_1 - \varepsilon_2) + \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_2)}{(\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2)} \\
&= \frac{1}{3}(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_2) \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\langle \lambda_1, \alpha_2 \rangle &= 2\frac{(\frac{2\alpha_1+\alpha_2}{3}, \alpha_2)}{(\alpha_2, \alpha_2)} \\
&= \frac{2}{3}\frac{(2(\varepsilon_1 - \varepsilon_2) + \varepsilon_2 - \varepsilon_3, \varepsilon_2 - \varepsilon_3)}{(\varepsilon_2 - \varepsilon_3, \varepsilon_2 - \varepsilon_3)} \\
&= \frac{1}{3}(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3, \varepsilon_2 - \varepsilon_3) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\langle \lambda_2, \alpha_1 \rangle &= 2\frac{(\frac{\alpha_1+2\alpha_2}{3}, \alpha_1)}{(\alpha_1, \alpha_1)} \\
&= \frac{2}{3}\frac{(\varepsilon_1 - \varepsilon_2 + 2(\varepsilon_2 - \varepsilon_3), \varepsilon_1 - \varepsilon_2)}{(\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2)} \\
&= \frac{1}{3}(\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3, \varepsilon_1 - \varepsilon_2) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\langle \lambda_2, \alpha_2 \rangle &= 2\frac{(\frac{\alpha_1+2\alpha_2}{3}, \alpha_2)}{(\alpha_2, \alpha_2)} \\
&= \frac{2}{3}\frac{(\varepsilon_1 - \varepsilon_2 + 2(\varepsilon_2 - \varepsilon_3), \varepsilon_2 - \varepsilon_3)}{(\varepsilon_2 - \varepsilon_3, \varepsilon_2 - \varepsilon_3)} \\
&= \frac{1}{3}(\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3, \varepsilon_2 - \varepsilon_3) \\
&= 1
\end{aligned}
$$

Thus we have proved $\langle \lambda_i, \alpha_j \rangle = \delta_{i,j}$. □

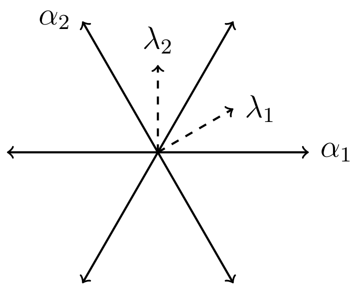


Figure 2.4: Fundamental dominant weights of type A_2

Chapter 3

Affine Weyl Groups

3.1 Affine hyperplanes

Definition 3.1.1. We recall that a **hyperplane** is a subspace with codimension one relative to the space where it is embedded.

Example. In a three-dimensional euclidean space, a hyperplane is two-dimensional, where the codimension is the difference one. A hyperplane in a two-demensional euclidean space will be a line which is one dimensional.

Definition 3.1.2 ([Ram06] p.138)**.** Let α be a root in $\Phi \subset E$ (Definition 2.1.2). We define the **affine hyperplane** $H_{\alpha,k}$ as:

$$H_{\alpha,k} := \{\lambda \in E \mid (\lambda, \alpha^\vee) = k\}. \tag{3.1}$$

Note ([Hum90] p.87)**.** $H_{\alpha,k} = H_{-\alpha,-k}$ and $H_{\alpha,0} = H_\alpha$.

Proposition 3.1.3 ([Hum90] p.88)**.** *$wH_{\alpha,k} = H_{w\alpha,k}$ for $w \in W$.*

Proof.

$$\begin{aligned} wH_{\alpha,k} &= \{w\lambda : (\lambda, \alpha^\vee) = k\} \\ &= \{w\lambda : (w\lambda, w\alpha^\vee) = k\} \\ &= H_{w\alpha,k} \end{aligned}$$

□

Definition 3.1.4 ([Hum90] p.88)**.** We define the **reflection** $s_{\alpha,k}$ through $H_{\alpha,k}$ as

$$s_{\alpha,k}(\lambda) := \lambda - ((\lambda, \alpha^\vee) - k)\alpha. \tag{3.2}$$

Proposition 3.1.5 ([Hum90] p.88)**.** $ws_{\alpha,k}w^{-1} = s_{w\alpha,k}$ *for* $w \in W$*.*

Definition 3.1.6 ([Ram06] p.138)**.** For $\lambda \in E$, we define $t(\lambda)$ as the **translation** that sends μ to $\lambda + \mu$ for all $\mu \in E$.

Note ([Ram06] p.138)**.** $s_{\alpha,k} = t(k\alpha)s_\alpha = s_\alpha t(-k\alpha)$.

Example ([Hum90] p.88)**.** If $\lambda \in E$ satisfies $(\lambda, \alpha^\vee) \in \mathbb{Z}$ for all roots α, then
(a) $t(\lambda)H_{\alpha,k} = H_{\alpha,k+(\lambda,\alpha^\vee)}$
(b) $t(\lambda)s_{\alpha,k}t(-\lambda) = s_{\alpha,k+(\lambda,\alpha^\vee)}$.

Proof. (a) We need to show that $t(\lambda)H_{\alpha,k} \subset H_{\alpha,k+(\lambda,\alpha^\vee)}$ and $t(\lambda)H_{\alpha,k} \supset H_{\alpha,k+(\lambda,\alpha^\vee)}$.
$\subseteq$: Let $\mu \in t(\lambda)H_{\alpha,k}$. Then $\mu = \lambda + \nu$ where $\nu \in H_{\alpha,k}$. Thus $(\mu, \alpha^\vee) = (\lambda, \alpha^\vee) + (\nu, \alpha^\vee) = (\lambda, \alpha^\vee) + k$, so $\mu \in H_{\alpha,k+(\lambda,\alpha^\vee)}$. Thererfore $t(\lambda)H_{\alpha,k} \subset H_{\alpha,k+(\lambda,\alpha^\vee)}$.
$\supseteq$: Let $\mu \in H_{\alpha,k+(\lambda,\alpha^\vee)}$. Then $(\mu, \alpha^\vee) = k + (\lambda, \alpha^\vee)$. Let $\mu = \nu + \lambda$. Then $(\mu, \alpha^\vee) = (\nu, \alpha^\vee) + (\lambda, \alpha^\vee) = k + (\lambda, \alpha^\vee)$. Thus $(\nu, \alpha^\vee) = k$ so that $\nu \in H_{\alpha,k}$. Therefore $\mu \in t(\lambda)H_{\alpha,k}$ and so $H_{\alpha,k+(\lambda,\alpha^\vee)} \subset t(\lambda)H_{\alpha,k}$.
Thus we have shown that $t(\lambda)H_{\alpha,k} = H_{\alpha,k+(\lambda,\alpha^\vee)}$.

(b)

$$\begin{aligned}
t(\lambda)s_{\alpha,k}t(-\lambda)(\mu) &= t(\lambda)(s_{\alpha,k}(\mu-\lambda)) \\
&= t(\lambda)(\mu-\lambda-((\mu-\lambda,\alpha^\vee)-k))\alpha \\
&= t(\lambda)(\mu-\lambda-((\mu,\alpha^\vee)-(\lambda,\alpha^\vee)-k))\alpha \\
&= \mu-((\mu,\alpha^\vee)-k-(\lambda,\alpha^\vee))\alpha \\
&= s_{\alpha,k+(\lambda,\alpha^\vee)}(\mu)
\end{aligned}$$

Therefore $t(\lambda)s_{\alpha,k}t(-\lambda) = s_{\alpha,k+(\lambda,\alpha^\vee)}$. □

3.2 Affine Weyl groups

Definition 3.2.1 ([Hum90] p.88, [Ram06] p.138)**.** The **affine Weyl group** W_a is the group generated by the reflections $s_{\alpha,k}$ for $\alpha \in \Phi$ and $k \in \mathbb{Z}$. Note that $wt(\lambda)w^{-1} = t(w\lambda)$ [Ram06, p.139]. Thus

$$W_a = \Lambda_r \rtimes W = \{t(\lambda)w : \lambda \in \Lambda_r, w \in W\} \text{ where } \Lambda_r = \sum_{\alpha\in\Phi^+} \mathbb{Z}\alpha. \tag{3.3}$$

3.3 Alcoves

Definition 3.3.1 ([Hum90] p.89)**.** An **alcove** is a connected component of

$$E - \bigcup_{\substack{\alpha\in\Phi^+,\\ k\in\mathbb{Z}}} H_{\alpha,k} \tag{3.4}$$

where $H_{\alpha,k}$ are affine hyperplanes.

Note ([Hum90] p.90)**.** The affine Weyl group acts transitively on the collection of all alcoves.

Definition 3.3.2. Let $A_\circ$ be the alcove with walls H_α ($\alpha \in \Delta$) and $H_{\tilde{\alpha},1}$, where $\tilde{\alpha}^\vee$ is the highest root in $\Phi^\vee$.

Proposition 3.3.3 ([Ram06] p.139)**.** *W_a is in bijection with the alcoves of E via $w \leftrightarrow wA_\circ$.*

Proposition 3.3.4 ([Hum90] Proposition 4.3.1)**.** *Let $S_a = \{s_\alpha : \alpha \in \Delta\} \cup \{s_{\tilde{\alpha},1}\}$. Then $W_a = \langle t : t \in S_a \rangle$.*

Definition 3.3.5 ([Hum90] p.89, [Ram06] p.139)**.** Since W normalizes the translation group corresponding to Λ, we may define the **extended affine Weyl group**

$$W_e = \Lambda \rtimes W \tag{3.5}$$

Definition 3.3.6 ([Ram06] p.139, [Yee19] Definition 5.5)**.** We define the **difference** between W_e and W_a to be the group

$$\Omega = W_e/W_a \cong \Lambda/\Lambda_r. \tag{3.6}$$

Ω is isomorphic to the stabilizer of $A_\circ$ in W_e. The stabilizer is isomorphic to Λ/Λ_r via sending g to the coset $g(0) + \Lambda_r$.

Definition 3.3.7 ([Sch06] p.6)**.** $W_e \cong \Omega \ltimes W_a$. For $\nu \in W_e$, we define the **length** of ν as $\ell(w)$ where $\nu = wg$ for $w \in W_a$ and $g \in \Omega$.

3.4 Counting hyperplanes

Definition 3.4.1 ([Hum90] p.12, p.91)**.** For all $w \in W_a$, w can be written as a product of simple reflections

$$w = s_1 s_2 \cdots s_k$$

where the s_i belong to S_a. We call such an expression a **reduced expression** when k is minimal. Then k is called the **length** of w, written as $\ell(w) = k$.

Definition 3.4.2 ([Hum90] p.91)**.** Let $H_{\alpha,k}$ be a hyperplane where $\alpha \in \Phi^+$. Then $H^+_{\alpha,k}$ and $H^-_{\alpha,k}$ denote the affine half-spaces

$$H^+_{\alpha,k} = \{\lambda \in E \mid (\lambda, \alpha^\vee) > k\} \text{ and } H^-_{\alpha,k} = \{\lambda \in E \mid (\lambda, \alpha^\vee) < k\}. \tag{3.7}$$

Definition 3.4.3 ([Hum90] p.91)**.** Let $H_{\alpha,k}$ be a hyperplane. $H_{\alpha,k}$ **separates** alcoves A and B if A, B belong to different half-spaces defined by $H_{\alpha,k}$.

Lemma 3.4.4 ([Hum78] p.93, [Hum90] p.92)**.** *Let $w = s_1 s_2 \cdots s_k$ be a reduced expression in W_a ($w \neq 1$). Then setting H_i to be the affine hyperplane corresponding to s_i, the following k hyperplanes*

$$H_1, s_1 H_2, s_1 s_2 H_3, \ldots, s_1 s_2 \cdots s_{k-1} H_k$$

are all distinct and form the set of affine hyperplanes separating $A_\circ$ and $wA_\circ$.

3.5 Bruhat order

Definition 3.5.1 ([Hum90] p.118)**.** Let $w \in W_a$, T be the set of all reflections in W_a where $T = \bigcup_{w \in W_a} w S_a w^{-1}$. We write $w' \to w$ if $w = w't$ for some $t \in T$ with $\ell(w) > \ell(w')$. Define $w' \leq w$ if there exists a sequence $w' = w_0 \to w_1 \to \ldots \to w_n = w$. Then $\leq$ is a partial order called **Bruhat order**.

Remark ([Hum90] p.119). Definition 2.3.4 and Definition 3.5.1 have a one-sided appearance. Note that if we let $w = w't$ with $\ell(w) > \ell(w')$ and $t \in T$, then $w = (w't(w')^{-1})w'$ where $w't(w')^{-1} \in T$.

Proposition 3.5.2 ([Hum90] p.119)**.** *$v < w$ if and only if $v^{-1} < w^{-1}$.*

Proof. $\Rightarrow$: Suppose $v < w$. Then there exists a sequence $v = w_0 \to w_1 \to \cdots \to w_k = w$ where $w_i = w_{i-1} t_i$ ($t_i \in T$ the set of reflections in W_a). Then $w_i{}^{-1} = t_i w_{i-1}^{-1} = w_{i-1}^{-1}(w_i t_i w_{i-1}^{-1})$ where $w_{i-1} t_i w_{i-1}^{-1} \in T$. Thus $w_{i-1}^{-1} \to w_i^{-1}$. We have $v^{-1} = w_0^{-1} \to w_1^{-1} \to$

$\cdots \to w_k^{-1} = w^{-1}$. Therefore, $v < w \Rightarrow v^{-1} < w^{-1}$.

$\Leftarrow$: We already have $v < w \Rightarrow v^{-1} < w^{-1}$. Apply the sentence again, we have $v^{-1} < w^{-1} \Rightarrow (v^{-1})^{-1} < (w^{-1})^{-1}$ which is $v^{-1} < w^{-1} \Rightarrow v < w$. Therefore, $v < w$ if and only if $v^{-1} < w^{-1}$. □

Chapter 4

Hecke Algebras and Affine Hecke Algebras

4.1 Hecke algebras

Definition 4.1.1 ([Yee19] Definition 5.1)**.** Let $\mathbb{K}$ be the field of fractions of $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. The **Hecke algebra** $\mathcal{H}$ is the $\mathbb{K}$-algebra with $\mathbb{K}$-basis $\{T_w\}_{w \in W}$ such that for all $s \in S$ and $w \in W$,

$$T_s T_w = T_{sw} \quad \text{if } \ell(sw) > \ell(w) \tag{4.1}$$

$$T_s T_w = qT_{sw} + (q-1)T_w \quad \text{if } \ell(sw) < \ell(w). \tag{4.2}$$

We define

$$\tilde{T}_w = q^{-\frac{1}{2}\ell(w)} T_w. \tag{4.3}$$

Then we get another presentation of $\mathcal{H}$ with a $\mathbb{K}$-basis $\{\tilde{T}_w\}_{w \in W}$ such that for all $s \in S$ and $w \in W$,

$$\tilde{T}_s \tilde{T}_w = \tilde{T}_{sw} \quad \text{if } \ell(sw) > \ell(w) \tag{4.4}$$

$$\tilde{T}_s \tilde{T}_w = \tilde{T}_{sw} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\tilde{T}_w \quad \text{if } \ell(sw) < \ell(w) \tag{4.5}$$

4.2 Affine Hecke algebras

Definition 4.2.1 ([Yee19] Definition 5.7)**.** The **affine Hecke algebra** $\mathcal{H}_a$ is the $\mathbb{K}$-algebra with $\mathbb{K}$-basis $\{\tilde{T}_w\}_{w\in W_e}$ and the relations:

$$\tilde{T}_v\tilde{T}_w = \tilde{T}_{vw} \quad \text{if } v, w \in W_e \text{ and } \ell(vw) = \ell(v) + \ell(w) \tag{4.6}$$

$$\tilde{T}_s^2 = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\tilde{T}_s + \tilde{T}_1 \quad s \in S_a. \tag{4.7}$$

$\mathcal{H}_a$ is also the $\mathbb{K}$-algebra with $\mathbb{K}$-basis $\{T_w\}_{w\in W_e}$ with the relations

$$T_vT_w = T_{vw} \quad \text{if } v, w \in W_e \text{ and } \ell(vw) = \ell(v) + \ell(w) \tag{4.8}$$

$${T_s}^2 = (q-1)T_s + qT_1 \quad s \in S_a. \tag{4.9}$$

Definition 4.2.2 ([Sch06] p.7)**.** We define an **involution** on $\mathcal{H}$

$$\bar{\cdot} : \mathcal{H} \to \mathcal{H}$$

such that $\overline{T}_w = T_{w^{-1}}^{-1}$ for $w \in W$ and $\overline{q^{\frac{1}{2}}} = q^{-\frac{1}{2}}$ and an **involution** on $\mathcal{H}_a$

$$\bar{\cdot} : \mathcal{H}_a \to \mathcal{H}_a$$

such that $\overline{T}_w = T_{w^{-1}}^{-1}$ for $w \in W_e$ and $\overline{q^{\frac{1}{2}}} = q^{-\frac{1}{2}}$. Then $\overline{\tilde{T}}_w = \tilde{T}_{w^{-1}}^{-1}$ for $w \in W_e$.

Chapter 5

Schwer's Paper "Galleries, Hall-Littlewood Polynomials and Structure Constants of the Spherical Hecke Algebra"

5.1 Generalized alcoves and affine Weyl groups

Definition 5.1.1 ([Sch06] p.6)**.** Let $\mathcal{A}$ be the set of alcoves. We define the set of **generalized alcoves** as

$$\tilde{\mathcal{A}} := \{(A, \mu) \in \mathcal{A} \times \Lambda \mid \mu \in A\}. \tag{5.1}$$

There is an embedding $\mathcal{A} \hookrightarrow \tilde{\mathcal{A}}$ via $A \mapsto (A, \mu)$ where μ is the unique element in $A \cap \Lambda_r$. There is a natural free W_e-action on $\tilde{\mathcal{A}}$ by the natural action in the two components. In particular, Λ acts on $\tilde{\mathcal{A}}$ by translation in both components.

Note ([Sch06] p.6)**.** The bijection $W_e \to \tilde{\mathcal{A}}$ extends $W_a \to \mathcal{A}$.

Let $\lambda \in \Lambda$. Recall from Definition 3.1.6 the definition of $t(\lambda)$. Let $\tau_\lambda = t(\lambda) \in W_e$.

Definition 5.1.2 ([Sch06] p.7)**.** We define $q_\nu = q^{\ell(\nu)}$ for $\nu \in W_e$. Recall that for $\nu = wg$

where $w \in W_a$ and $g \in \Omega$, $\ell(\nu) = \ell(w)$.

Lemma 5.1.3 ([Sch06] p.7)**.** *Let A be a generalized alcove. Then it can be written in the form: $A = \mu + wA_\circ$ where $\mu \in \Lambda$ and $w \in W$. We call μ the* ***weight*** *of A and w the* ***direction*** *of A. We denote them as $wt(A) := \mu$ and $\delta(A) := w$.*

Definition 5.1.4 ([Sch06] p.7)**.** Let $\lambda \in \Lambda^+$. We define

$$q_\lambda = q^{\frac{1}{2}\ell(\tau_\lambda)}. \tag{5.2}$$

Definition 5.1.5 ([Sch06] p.3)**.** We define

$$\rho^\vee = \frac{1}{2} \sum_{\alpha^\vee \in (\Phi^\vee)^+} \alpha^\vee. \tag{5.3}$$

Definition 5.1.6 ([Sch06] p.7)**.** Let $\mu \in \Lambda$. We define $q_\mu := q_\lambda q_{\lambda'}^{-1}$ where $\mu = \lambda - \lambda'$ for $\lambda, \lambda' \in \Lambda^+$.

Lemma 5.1.7 ([Sch06] Lemma 2.3)**.** *$\ell(\tau_\mu) = 2(\rho^\vee, \mu)$ for $\mu \in \Lambda^+$. Thus $q_\mu = q^{(\rho^\vee,\, \mu)}$ for $\mu \in \Lambda^+$.*

Proof. $\ell(w)$ is equal to the number of distinct hyperplanes $H_{\alpha,k}$ separating $A_\circ$ and $wA_\circ$ [Ram06, p.144]. Thus $\ell(\tau_\mu)$ equals the number of hyperplanes separating $A_\circ$ and $\tau_\mu A_\circ$, where $\tau_\mu A_\circ = \mu + A_\circ$. Thus if we fix $\alpha \in \Phi^+$, hyperplanes $H_{\alpha,k}$ separating $A_\circ$ and $\mu + A_\circ$ are: $H_{\alpha,1}, H_{\alpha,2}, ..., H_{\alpha,(\mu,\alpha^\vee)}$. Then we get

$$2(\rho^\vee, \mu) = 2\left(\frac{1}{2} \sum_{\alpha^\vee \in (\Phi^\vee)^+} \alpha^\vee, \mu\right) = \sum_{\alpha^\vee \in (\Phi^\vee)^+} (\alpha^\vee, \mu) = \ell(\tau_\mu). \tag{5.4}$$

Therefore, $\ell(\tau_\mu) = 2(\rho^\vee, \mu)$. □

Lemma 5.1.8. *$q_\mu = q_\lambda q_{\lambda'}^{-1} = q^{(\rho^\vee,\lambda)} q^{-(\rho^\vee,\lambda')} = q^{(\rho^\vee,\lambda)-(\rho^\vee,\lambda')} = q^{(\rho^\vee,\lambda-\lambda')} = q^{(\rho^\vee,\mu)}$.*

Definition 5.1.9 ([Sch06] p.8)**.** Let $\mu \in \Lambda$. We define $X_\mu := q_\mu^{-1} T_{\tau_\lambda} T_{\tau_{\lambda'}}^{-1} \in \mathcal{H}_a$ where $\mu = \lambda - \lambda'$ for $\lambda, \lambda' \in \Lambda^+$. For $\lambda \in \Lambda^+$, $X_\lambda = q_\lambda^{-1} T_{\tau_\lambda}$.

5.2 Galleries

Definition 5.2.1 ([Sch06] p.9)**.** A **gallery** σ of type $t = (t_1, ..., t_k)$ with $t_i \in S_a \cup \Omega$ connecting generalized alcoves A and B is a sequence of generalized alcoves: $A = A_0, \ldots, A_k = B$, such that

$$A_{i+1} = \begin{cases} A_i t_{i+1} \ (t_{i+1} \in \Omega) \\ A_i \text{ or } A_i t_{i+1} \ (t_{i+1} \in S_a) \end{cases}.$$

Definition 5.2.2 ([Sch06] p.10)**.** We define the direction of the first generalized alcove $\delta(A_0)$ to be the **initial direction**, the weight of the last generalized alcove $wt(A_k)$ to be the **weight** of the gallery, and $\delta(A_k)$ to be the **final direction**.

Definition 5.2.3 ([Sch06] p.6)**.** Let $A \in \mathcal{A}$ and $s \in S_a$ and let $H_{\alpha,k}$ be the hyperplane **separating** A and As. We define $A \prec As$ if $As \subset H^+_{\alpha,k}$, $A \subset H^-_{\alpha,k}$ and $A \succ As$ if $A \subset H^+_{\alpha,k}$, $As \subset H^-_{\alpha,k}$.

Proposition 5.2.4 ([Sch06] p.6)**.** *Let $w \in W$ and $s \in S$. We have $A_w \prec A_{ws}$ if and only if $w > ws$ (relative to Bruhat order).*

Definition 5.2.5 ([Sch06] p.10)**.** The gallery σ has a **positive s-direction** at i if $t_{i+1} = s, A_{i+1} = A_i s$, where $A_i \prec A_{i+1}$. σ has a **negative s-direction** at i if $t_{i+1} = s, A_{i+1} = A_i s$, where $A_i \succ A_{i+1}$. The gallery σ is **s-folded** at i if $t_{i+1} = s$ and $A_{i+1} = A_i$. The folding is **positive** if $A_i \succ A_i s$, and **negative** if $A_i \prec A_i s$.

Definition 5.2.6 ([Sch06] p.10)**.** A gallery σ is called a **positively folded gallery** if all foldings in σ are positive.

Definition 5.2.7 ([Sch06] p.10)**.** Let σ be a positively folded gallery of type t. We denote: the number of positive s-directions as $m_s(\sigma)$; the number of positive s-folds as $n_s(\sigma)$; the number of negative s-directions as $o_s(\sigma)$.

Definition 5.2.8 ([Sch06] p.10)**.** Let σ be a positively folded gallery of type t. We define

$$L_\sigma = \prod_{s \in S_a} q^{m_s(\sigma)} (q-1)^{n_s(\sigma)}. \tag{5.5}$$

Definition 5.2.9 ([Sch06] p.14)**.** Let $A \in \tilde{\mathcal{A}}$. Define

$$X_A = q_{-wt(A)} q_{\delta(A)} X_{wt(A)} \bar{T}_{\delta(A)}. \tag{5.6}$$

Proposition 5.2.10 ([Sch06] p.14)**.** *The set $\{X_A\}_{A \in \tilde{\mathcal{A}}}$ is a basis of $\mathcal{H}_a$, called the alcove basis.*

Definition 5.2.11 ([Sch06] p.15)**.** Let A, B be generalized alcoves and $\Gamma_t^+(A, B)$ be the set of all positively folded galleries of type t connnecting A and B. Then we define

$$L_t(A, B) = \sum_{\sigma \in \Gamma_t^+(A,B)} L_\sigma. \tag{5.7}$$

Theorem 5.2.12 ([Sch06] Theorem 5.5)**.** *For $\nu \in W_e$, let t be the type of a minimal gallery connecting $A_\circ$ and A_ν. For $A \in \tilde{\mathcal{A}}$ and $\nu \in W_e$, we have*

$$X_A T_\nu = \sum_{B \in \tilde{\mathcal{A}}} L_t(A, B) X_B. \tag{5.8}$$

Chapter 6

Ram's Paper "Alcove Walks, Hecke Algebras, Spherical Functions Crystals and Column Strict Tableaux"

6.1 Alcove walks

Definition 6.1.1 ([Ram06] p.139)**.** We denote the walls of the dominant Weyl chamber as $H_{\alpha_1}, H_{\alpha_2}, \ldots, H_{\alpha_n}$ and extend this so that $H_{\alpha_0}, H_{\alpha_1}, H_{\alpha_2}, \ldots, H_{\alpha_n}$ are the walls of $A_\circ$ with corresponding reflections $s_0, s_1, \ldots, s_n$.

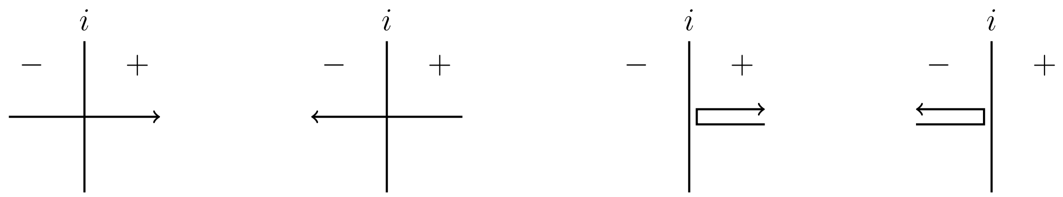
Proposition 6.1.2 ([Ram06] p.139)**.** *$g \in \Omega$ acts on $A_\circ$ by an automorphism which gives a permutation of the walls $H_{\alpha_0}, H_{\alpha_1}, H_{\alpha_2}, \ldots, H_{\alpha_n}$, hence a permutation of $0, 1, \ldots, n$, so that $g s_i g^{-1} = s_{g(i)}$ for $g \neq 1$.*

Proposition 6.1.3 ([Ram06] p.139)**.** *The extended affine Weyl group W_e acts freely on $\Omega \times E$, and we have $W_e \to \tilde{\mathcal{A}}, w \mapsto w^{-1}A_\circ$, so that $g^{-1}A_\circ$ is in the same place as $A_\circ$ except on the g^{th} "sheet" of $\Omega \times E$.*

Definition 6.1.4 ([Ram06] p.139, p.141)**.** Number the alcove walls in a W_e-equivariant way: the numbering of the walls of $wA_\circ$ is the w image of the numbering of the walls of $A_\circ$ for $w \in W_e$.

Definition 6.1.5 ([Yee19] p.286)**.** An **alcove path** from A to B is a sequence of alcoves $A = A_0 \xrightarrow{R_1} A_1 \xrightarrow{R_2} \cdots \xrightarrow{R_\ell} A_\ell = B$ where for $1 \le i \le \ell$, A_{i-1} and A_i are adjacent. R_i is the affine reflection corresponding to the affine hyperplane separating A_{i-1} and A_i: $A_i = R_i A_{i-1}$.

Definition 6.1.6 ([Ram06] p.143)**.** Recall that $\mathbb{K}$ is the field of fractions of $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. Let $g \in \Omega$, $g(i)$ be the index such that $gs_ig^{-1} = s_{g(i)}$. The walls of $A_\circ$ are labelled $0, 1, 2, \ldots, n$. The **alcove walk algebra** $\mathcal{A}$ is the $\mathbb{K}$-algebra with generators $g \in \Omega$ and for $1 \le i \le n$

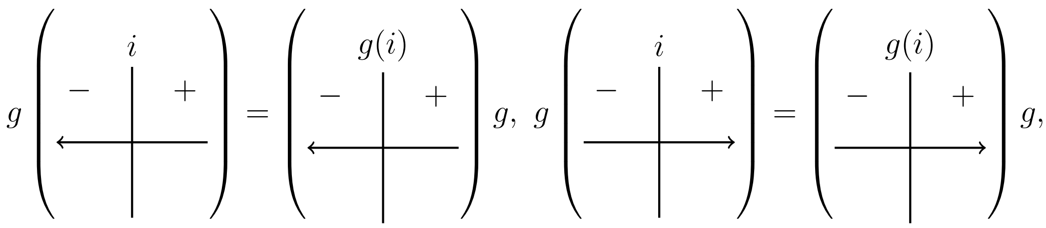


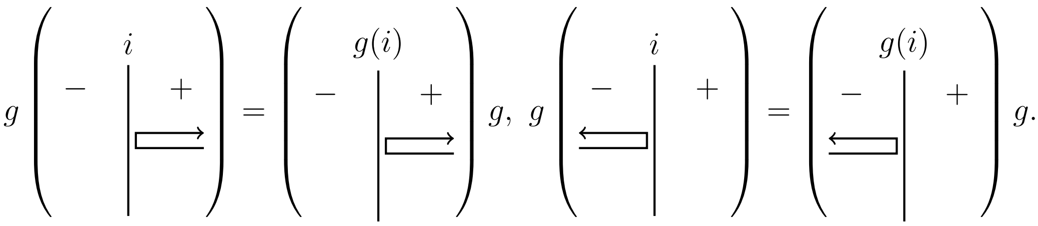
positive i-crossing (c_i^+) | negative i-crossing (c_i^-) | positive i-fold (f_i^+) | negative i-fold (f_i^-)

with relations

$$c_i^+ = c_i^- + f_i^+ \text{ and } c_i^- = c_i^+ + f_i^-$$

and





Definition 6.1.7 ([Ram06] p.143)**.** An **alcove walk** is a word in the generators of $\mathcal{A}$ such that

(a) the tail of the first step is in the fundamental alcove $A_\circ$

(b) at every step, the head of each arrow and the tail of the next arrow are in the same alcove.

Definition 6.1.8 ([Ram06] p.144)**.** The **type** of an alcove walk p is the sequence of labels on the arrows, corresponding to the folds and wall crossings of the walk.

Definition 6.1.9 ([Ram06] p.153)**.** An alcove walk p is **positively folded** if all foldings in p are positive.

Definition 6.1.10 ([Ram06] p.149)**.** Let p be an alcove walk from $wA_\circ$ to $wt(p)+\varphi(p)A_\circ$ where $w \in W$. We define $\iota(p) = w$ as the **initial direction** of p, $wt(p)$ as the **weight** of p, and $\varphi(p)$ as the **final direction** of p.

Definition 6.1.11 ([Ram06] p.145)**.** The **affine Hecke algebra** $\mathcal{H}_a$ is the quotient of the alcove walk algebra by the relations

$$c_i^+ = (c_i^-)^{-1},\ f_i^+ = q^{\frac{1}{2}} - q^{-\frac{1}{2}},\ f_i^- = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$$

and we say nonfolded alcove walks $p = p'$ if they have the same ending alcove. (Note that the affine Hecke algebra only remembers the ending alcove of a walk.)

Definition 6.1.12 ([Ram06] p.146)**.** Let $\mathcal{H}_a$ be the affine Hecke algebra. We define $\tilde{T}_{w^{-1}}^{-1}$ as the image in $\mathcal{H}_a$ of a minimal length alcove walk from $A_\circ$ to $wA_\circ$ and X^λ as the image in $\mathcal{H}_a$ of a minimal length alcove walk from $A_\circ$ to $\tau_\lambda A_\circ$ for $w \in W$ and $\lambda \in \Lambda$.

Proposition 6.1.13 ([Ram06] Proposition 3.2)**.** *$\mathcal{H}_a$ coincides with the usual definition of the affine Hecke algebra.*

Proposition 6.1.14 ([Ram06] Proposition 3.2)**.**

$$X^\lambda X^\mu = X^{\lambda+\mu} = X^\mu X^\lambda \quad \textit{for } \lambda, \mu \in \Lambda. \tag{6.1}$$

6.2 Hall-Littlewood polynomials

Definition 6.2.1 ([Ram06] p.151)**.** We define an element $\tilde{\mathbf{1}}_0$ in $\mathcal{H}_a$ by

$$\tilde{T}_{w^{-1}}^{-1} \tilde{\mathbf{1}}_0 = q^{-\frac{1}{2}\ell(w)} \tilde{\mathbf{1}}_0 \tag{6.2}$$

for $w \in W$.

An explicit formula for $\tilde{\mathbf{1}}_0$ is

$$\tilde{\mathbf{1}}_0 = \frac{1}{W_0(q^{-1})} \sum_{w \in W} q^{\frac{-\ell(w)}{2}} \tilde{T}_{w^{-1}}^{-1} \tag{6.3}$$

where $W_0(t) = \sum\limits_{w \in W} t^{\ell(w)}$ is the Poincaré polynomial of W.

Definition 6.2.2 ([Ram06] p.151, [Yee19] p.283)**.** We define $\mathbb{K}[\Lambda] = span\{X^\mu : \mu \in \Lambda\}$.

Proposition 6.2.3 ([Ram06] p.151)**.** *$\{X^\mu \tilde{\mathbf{1}}_0 : \mu \in \Lambda\}$ is a basis of $\mathcal{H}_a \tilde{\mathbf{1}}_0$. Then there exists a vector space isomorphism $\mathbb{K}[\Lambda] \to \mathcal{H}_a \tilde{\mathbf{1}}_0, f \mapsto f\tilde{\mathbf{1}}_0$.*

Definition 6.2.4 ([Yee19] p.284)**.** The **spherical Hecke algebra** is $\tilde{\mathbf{1}}_0 \mathcal{H}_a \tilde{\mathbf{1}}_0$.

Definition 6.2.5 ([Ram06] p.151, [Yee19] p.284)**.** The **ring of symmetric functions** is $\mathbb{K}[\Lambda]^W$. By a theorem of Bernstein, $\mathbb{K}[\Lambda]^W$ is the centre of $\mathcal{H}_a$. There is an isomorphism called the Satake isomorphism

$$\Phi : \mathbb{K}[\Lambda]^W \to \tilde{\mathbf{1}}_0 \mathcal{H}_a \tilde{\mathbf{1}}_0, f \mapsto f\tilde{\mathbf{1}}_0.$$

Definition 6.2.6 ([Ram06] p.153). For $\mu \in \Lambda$, the **Hall-Littlewood polynomial** $P_\mu(X;q^{-1}) \in \mathbb{K}[\Lambda]^W$ is defined by

$$P_\mu(X;q^{-1})\tilde{\mathbf{1}}_0 = \left(\sum_{w\in W^\mu} q^{-\frac{1}{2}\ell(w)}\tilde{T}^{-1}_{w^{-1}}\right)X^\mu\tilde{\mathbf{1}}_0 \tag{6.4}$$

where W_μ is the stabilizer of μ and W^μ is a set of minimal length coset representatives for the cosets in W/W_μ.

Theorem 6.2.7. *Fix $\lambda \in \Lambda^+$. Let $p_\lambda = c^+_{i_1}\cdots c^+_{i_\ell}$ be a minimal length alcove walk from $A_\circ$ to $\lambda + A_\circ$ and $B_q(p_\lambda) = \{$positively folded alcove walks of type $(i_1,\ldots,i_\ell)$ beginning at $wA_\circ$ where $w \in W^\lambda$ $\}$. Then*

$$P_\lambda(X;q^{-1}) = \sum_{p\in B_q(p_\lambda)} q^{-\frac{1}{2}(\ell(\iota(p))+\ell(\varphi(p))-f(p))}(1-q^{-1})^{f(p)}X^{wt(p)} \tag{6.5}$$

where $\iota(p)$ is the alcove where p begins, $wt(p)+\varphi(p)A_\circ$ is the alcove where p ends and $f(p)$ is the number of folds in p.

We will prove that this theorem follows from Schwer's Theorem 5.5 (Theorem 5.2.12 of this paper).

Chapter 7

Research Problem

In this chapter, we will show that Theorem 6.2.7 follows from Theorem 5.2.12.

Lemma 7.0.1. *For* $\lambda \in \Lambda^+$*,* $X^\lambda = \tilde{T}_{\tau_\lambda}$ *and* $\tilde{T}_{\tau_\lambda} = q^{-\frac{1}{2}\ell(\tau_\lambda)} T_{\tau_\lambda}$.

Lemma 7.0.2. *For* $\mu \in \Lambda$*,* X_μ *(Definition 5.1.9)* $= X^\mu$ *(Definition 6.1.12).*

Proof. Recall that $\mu = \lambda - \lambda' \in \Lambda$ where $\lambda, \lambda' \in \Lambda^+$, $q_\mu = q^{(\rho^\vee, \mu)}$ (Lemma 5.1.8) and $X_\mu := q_\mu^{-1} T_{\tau_\lambda} T_{\tau_{\lambda'}}^{-1}$ (Definition 5.1.9).

$$\begin{aligned}
X_\mu &= q^{-(\rho^\vee,\mu)} T_{\tau_\lambda} T_{\tau_{\lambda'}} \\
&= q^{-(\rho^\vee,\mu)} (\tilde{T}_{\tau_\lambda} q^{\frac{1}{2}\ell(\tau_\lambda)}) (\tilde{T}_{\tau_{\lambda'}}^{-1} q^{-\frac{1}{2}\ell(\tau_{\lambda'})}) \qquad \text{from (4.3)} \\
&= q^{-(\rho^\vee,\mu)} q^{\ell(\tau_\lambda)/2} q^{-\ell(\tau_{\lambda'})/2} \tilde{T}_{\tau_\lambda} \tilde{T}_{\tau_{\lambda'}}^{-1} \\
&= q^{-(\rho^\vee,\mu)} q^{(\rho^\vee,\lambda)} q^{-(\rho^\vee,\lambda')} \tilde{T}_{\tau_\lambda} \tilde{T}_{\tau_{\lambda'}}^{-1} \qquad \text{from Lemma 5.1.7} \\
&= q^{-(\rho^\vee,\mu)} q^{(\rho^\vee,\lambda-\lambda')} \tilde{T}_{\tau_\lambda} \tilde{T}_{\tau_{\lambda'}}^{-1} \qquad (7.1) \\
&= q^{-(\rho^\vee,\mu)} q^{(\rho^\vee,\mu)} \tilde{T}_{\tau_\lambda} \tilde{T}_{\tau_{\lambda'}}^{-1} \\
&= \tilde{T}_{\tau_\lambda} \tilde{T}_{\tau_{\lambda'}}^{-1} \\
&= X^\lambda (X^{\lambda'})^{-1} \\
&= X^\mu
\end{aligned}$$

□

Now we would like to prove Theorem 6.2.7 using Theorem 5.2.12.

Recall Theorem 6.2.7: Fix $\lambda \in \Lambda^+$. Let $p_\lambda = c_{i_1}^+ \cdots c_{i_\ell}^+$ be a minimal length alcove walk from $A_\circ$ to $\lambda + A_\circ$ and $B_q(p_\lambda) = \{$positively folded alcove walks of type $(i_1, \ldots, i_\ell)$ beginning at $wA_\circ$ where $w \in W^\lambda$ $\}$. Then,

$$P_\lambda(X; q^{-1}) = \sum_{p \in B_q(p_\lambda)} q^{-\frac{1}{2}(\ell(\iota(p)) + \ell(\varphi(p)) - f(p))}(1 - q^{-1})^{f(p)} X^{wt(p)} \tag{7.2}$$

where $\iota(p)$ is the alcove where p begins, $wt(p) + \varphi(p)A$ is the alcove where p ends and $f(p)$ is the number of folds in p.

Proof. Recall Theorem 5.2.12. Fix $\lambda \in \Lambda^+$ and $w \in W$. We choose A as $A_w \in W_e$ and ν as τ_λ, where $A_w = wA_\circ$. Then t is the type of a minimal gallery connecting $A_\circ$ and $\tau_\lambda A_\circ$ and Theorem 5.2.12 becomes

$$X_{A_w} T_{\tau_\lambda} = \sum_{B \in \tilde{\mathcal{A}}} L_t(A_w, B) X_B. \tag{7.3}$$

By Lemma 5.1.3, A is of the form $wt(A) + \delta(A)A_\circ$ for unique $wt(A) \in \Lambda$ and $\delta(A) \in W$. Then $wt(A_w) = 0$ and $\delta(A_w) = w$. Recalling Definition 5.2.9, we rewrite $X_{A_w} T_{\tau_\lambda}$ as

$$\begin{aligned} X_{A_w} T_{\tau_\lambda} &= q_{-wt(A_w)} q_{\delta(A_w)} X_{wt(A_w)} \bar{T}_{\delta(A_w)} T_{\tau_\lambda} \\ &= q_{\delta(A_w)} \bar{T}_{\delta(A_w)} T_{\tau_\lambda} \\ &= q_w \bar{T}_w T_{\tau_\lambda} \\ &= q^{\ell(w)} T_{w^{-1}}^{-1} T_{\tau_\lambda}. \end{aligned} \tag{7.4}$$

From (4.3) and Definition 5.1.9, we get

$$\begin{aligned} \tilde{T}_w = q^{-\frac{1}{2}\ell(w)} T_w \ &\Leftrightarrow\ \tilde{T}_{w^{-1}} = q^{-\frac{1}{2}\ell(w)} T_{w^{-1}} \\ &\Leftrightarrow\ \tilde{T}_{w^{-1}}^{-1} = T_{w^{-1}}^{-1} q^{\frac{1}{2}\ell(w)} \\ &\Leftrightarrow\ T_{w^{-1}}^{-1} = \tilde{T}_{w^{-1}}^{-1} q^{-\frac{1}{2}\ell(w)}, \end{aligned} \tag{7.5}$$

$$X_\lambda = q_\lambda^{-1} T_{\tau_\lambda} \;\Leftrightarrow\; T_{\tau_\lambda} = q_\lambda X_\lambda$$
$$\Leftrightarrow\; T_{\tau_\lambda} = q^{\frac{1}{2}\ell(\tau_\lambda)} X_\lambda. \tag{7.6}$$

Thus

$$\begin{aligned} X_{A_w} T_{\tau_\lambda} &= q^{\ell(w)} T_{w^{-1}}^{-1} q^{\frac{1}{2}\ell(\tau_\lambda)} X_\lambda \\ &= q^{\frac{1}{2}\ell(w)} \tilde{T}_{w^{-1}}^{-1} q^{\frac{1}{2}\ell(\tau_\lambda)} X_\lambda \\ &= q^{\frac{1}{2}\ell(w)} \tilde{T}_{w^{-1}}^{-1} q^{\frac{1}{2}\ell(\tau_\lambda)} X^\lambda \qquad \text{by Lemma 7.0.2.} \end{aligned} \tag{7.7}$$

Thus

$$q^{\frac{1}{2}\ell(w)} \tilde{T}_{w^{-1}}^{-1} q^{\frac{1}{2}\ell(\tau_\lambda)} X^\lambda = \sum_{B \in \tilde{\mathcal{A}}} L_t(A_w, B) X_B. \tag{7.8}$$

Now, we look at $\sum_{B \in \tilde{\mathcal{A}}} L_t(A_w, B) X_B$.

$$\begin{aligned} q^{\frac{1}{2}\ell(w)} \tilde{T}_{w^{-1}}^{-1} q^{\frac{1}{2}\ell(\tau_\lambda)} X^\lambda &= \sum_{B \in \tilde{\mathcal{A}}} L_t(A_w, B) X_B \\ &= \sum_{B \in \tilde{\mathcal{A}}} \Bigg(\sum_{\sigma \in \Gamma_t^+(A_w, B)} L_\sigma \Bigg) X_B \\ &= \sum_{B \in \tilde{\mathcal{A}}} \Bigg(\sum_{\sigma \in \Gamma_t^+(A_w, B)} \Bigg(\prod_{s \in S_a} q^{m_s(\sigma)} (q-1)^{n_s(\sigma)} \Bigg) \Bigg) X_B \\ &= \sum_{B \in \tilde{\mathcal{A}}} \Bigg(\sum_{\sigma \in \Gamma_t^+(A_w, B)} \Bigg(\prod_{s \in S_a} q^{m_s(\sigma)} (q-1)^{n_s(\sigma)} \Bigg) \Bigg) q_{-wt(B)} q_{\delta(B)} X_{wt(B)} \bar{T}_{\delta(B)} \\ &= \sum_{B \in \tilde{\mathcal{A}}} \Bigg(\sum_{\sigma \in \Gamma_t^+(A_w, B)} \Bigg(\prod_{s \in S_a} q^{m_s(\sigma)+n_s(\sigma)} (1-q^{-1})^{n_s(\sigma)} \Bigg) \Bigg) q_{-wt(B)} q_{\delta(B)} X_{wt(B)} \bar{T}_{\delta(B)} \\ &= \sum_{B \in \tilde{\mathcal{A}}} \Bigg(\sum_{\sigma \in \Gamma_t^+(A_w, B)} \Bigg(\prod_{s \in S_a} q^{m_s(\sigma)+n_s(\sigma)} (1-q^{-1})^{n_s(\sigma)} \Bigg) \Bigg) q_{-wt(B)} q_{\delta(B)} X_{wt(B)} T_{\delta(B)^{-1}}^{-1} \end{aligned} \tag{7.9}$$

Multiplying by the element $\tilde{\mathbf{1}}_0$ (6.2.1) on both sides and simplifying, we get:

$$q^{\frac{1}{2}\ell(w)} \tilde{T}_{w^{-1}}^{-1} q^{\frac{1}{2}\ell(\tau_\lambda)} X^\lambda \tilde{\mathbf{1}}_0 = \sum_{B \in \tilde{\mathcal{A}}} \Bigg(\sum_{\sigma \in \Gamma_t^+(A_w, B)} \Bigg(\prod_{s \in S_a} q^{m_s(\sigma)+n_s(\sigma)} (1-q^{-1})^{n_s(\sigma)} \Bigg) \Bigg) q_{-wt(B)} q_{\delta(B)} X_{wt(B)} T_{\delta(B)^{-1}}^{-1} \tilde{\mathbf{1}}_0 \tag{7.10}$$

$$q^{-\frac{1}{2}\ell(w)}\tilde{T}_{w^{-1}}^{-1}X^{\lambda}\tilde{\mathbf{1}}_0 = \sum_{B\in\tilde{\mathcal{A}}}\Bigg(\sum_{\sigma\in\Gamma_t^+(A_w,B)}\Bigg(\prod_{s\in S_a} q^{m_s(\sigma)+n_s(\sigma)}(1-q^{-1})^{n_s(\sigma)}\Bigg)\Bigg)q^{-\ell(w)-\frac{1}{2}l(\tau_\lambda)}q_{-wt(B)}q_{\delta(B)}$$

$$X_{wt(B)}T_{\delta(B)^{-1}}^{-1}\tilde{\mathbf{1}}_0. \quad (7.11)$$

Since t is the type of a minimal gallery connecting $A_\circ$ and $\tau_\lambda A_\circ$, then

$$\ell(\tau_\lambda) = \sum_{s\in S_a}(m_s(\sigma)+n_s(\sigma)+o_s(\sigma)). \quad (7.12)$$

for all $\sigma \in \Gamma_t^+(A_w, B)$.

Also, we show that

$$q_{-wt(B)} = q^{-(wt(B),\rho^\vee)} = q^{\frac{1}{2}(\sum(-m_s(\sigma)+o_s(\sigma))-\ell(\delta(B))+\ell(w))}. \quad (7.13)$$

Proof. It is equivalent to show that $q^{(wt(B),2\rho^\vee)} = q^{\sum(m_s(\sigma)-o_s(\sigma))+\ell(\delta(B))-\ell(w)}$. $(wt(B), 2\rho^\vee)$ is the number of $H_{\alpha,k}$ where $\alpha > 0, k > 0$ minus the number of $H_{\alpha,k}$ where $\alpha > 0, k \leq 0$ separating $A_\circ$ and $A_\circ + wt(B)$, and $\sum_{s\in S_a}(m_s(\sigma) - o_s(\sigma))$ is the number of $H_{\alpha,k}$ where $\alpha > 0, k > 0$ minus the number of $H_{\alpha,k}$ where $\alpha > 0, k \leq 0$ separating $wA_\circ$ and $\delta(B)A_\circ + wt(B)$. Therefore

$$(wt(B), 2\rho^\vee) = \sum_{s\in S_a}(m_s(\sigma) - o_s(\sigma)) + \ell(\delta(B)) - \ell(w) \quad (7.14)$$

since $A_\circ$ and $wA_\circ$ are separated by $\ell(w)$ $H_{\alpha,0}$, and $A_\circ + wt(B)$ and $\delta(B)A_\circ + wt(B)$ are separated by $\ell(\delta(B))$ $H_{\alpha,(\alpha^\vee,wt(B))}$. □

Apply equations 7.12 and 7.13 and simplify:

$$q^{-\frac{1}{2}\ell(w)}\tilde{T}_{w^{-1}}^{-1}X^{\lambda}\tilde{\mathbf{1}}_0 = \sum_{B\in\tilde{\mathcal{A}}}\Bigg(\sum_{\sigma\in\Gamma_t^+(A_w,B)}\Bigg(\prod_{s\in S_a} q^{m_s(\sigma)+n_s(\sigma)}(1-q^{-1})^{n_s(\sigma)}\Bigg)q^{-\frac{1}{2}(m_s(\sigma)+n_s(\sigma)+o_s(\sigma))}\Bigg)$$

$$q^{-\ell(w)}q_{-wt(B)}q_{\delta(B)}X_{wt(B)}T_{\delta(B)^{-1}}^{-1}\tilde{\mathbf{1}}_0 \quad (7.15)$$

$$= \sum_{B\in\tilde{\mathcal{A}}}\Bigg(\sum_{\sigma\in\Gamma_t^+(A_w,B)}\Bigg(\prod_{s\in S_a} q^{\frac{1}{2}(m_s(\sigma)+n_s(\sigma)-o_s(\sigma))}(1-q^{-1})^{n_s(\sigma)}\Bigg)\Bigg)q^{-\ell(w)}q_{-wt(B)}q_{\delta(B)}X_{wt(B)}T^{-1}_{\delta(B)^{-1}} \tilde{\mathbf{1}}_0. \quad (7.16)$$

By (4.3), we can rewrite $\tilde{T}_{\delta(B)} = q^{-\frac{1}{2}\ell(\delta(B))}T_{\delta(B)}$:

$$\begin{aligned}\tilde{T}_{\delta(B)} = q^{-\frac{1}{2}\ell(\delta(B))}T_{\delta(B)} &\Leftrightarrow \tilde{T}_{\delta(B)^{-1}} = q^{-\frac{1}{2}\ell(\delta(B))}T_{\delta(B)^{-1}} \\ &\Leftrightarrow \tilde{T}^{-1}_{\delta(B)^{-1}} = T^{-1}_{\delta(B)^{-1}}q^{\frac{1}{2}\ell(\delta(B))} \qquad (7.17)\\ &\Leftrightarrow T^{-1}_{\delta(B)^{-1}} = \tilde{T}^{-1}_{\delta(B)^{-1}}q^{-\frac{1}{2}\ell(\delta(B))}.\end{aligned}$$

Thus

$$q^{-\frac{1}{2}\ell(w)}\tilde{T}^{-1}_{w^{-1}}X^{\lambda}\tilde{\mathbf{1}}_0 = \sum_{B\in\tilde{\mathcal{A}}}\Bigg(\sum_{\sigma\in\Gamma_t^+(A_w,B)}\Bigg(\prod_{s\in S_a} q^{\frac{1}{2}(m_s(\sigma)+n_s(\sigma)-o_s(\sigma))}(1-q^{-1})^{n_s(\sigma)}\Bigg)\Bigg)$$
$$\prod_{s\in S_a}(q^{\frac{1}{2}(-m_s(\sigma)+o_s(\sigma))})q^{\frac{1}{2}(-\ell(\delta(B))+\ell(w))}q^{\ell(\delta(B))}q^{-\ell(w)}X^{wt(B)}\tilde{T}^{-1}_{\delta(B)^{-1}}q^{-\frac{1}{2}\ell(\delta(B))}\tilde{\mathbf{1}}_0 \quad (7.18)$$

$$\begin{aligned}&= \sum_{B\in\tilde{\mathcal{A}}}\Bigg(\sum_{\sigma\in\Gamma_t^+(A_w,B)}\Bigg(\prod_{s\in S_a} q^{\frac{1}{2}n_s(\sigma)}(1-q^{-1})^{n_s(\sigma)}\Bigg)\Bigg)q^{-\frac{1}{2}\ell(\delta(B))+\frac{1}{2}\ell(w)}q^{\ell(\delta(B))}q^{-\ell(w)}X^{wt(B)}\tilde{T}^{-1}_{\delta(B)^{-1}}q^{-\frac{1}{2}\ell(\delta(B))}\tilde{\mathbf{1}}_0\\ &= \sum_{B\in\tilde{\mathcal{A}}}\Bigg(\sum_{\sigma\in\Gamma_t^+(A_w,B)}\Bigg(\prod_{s\in S_a} q^{\frac{1}{2}n_s(\sigma)}(1-q^{-1})^{n_s(\sigma)}\Bigg)\Bigg)q^{-\frac{1}{2}\ell(\delta(B))+\frac{1}{2}\ell(w)}q^{\ell(\delta(B))}q^{-\ell(w)}X^{wt(B)}q^{-\ell(\delta(B))}\tilde{\mathbf{1}}_0\\ &= \sum_{B\in\tilde{\mathcal{A}}}\Bigg(\sum_{\sigma\in\Gamma_t^+(A_w,B)}\Bigg(\prod_{s\in S_a} q^{\frac{1}{2}n_s(\sigma)}(1-q^{-1})^{n_s(\sigma)}\Bigg)\Bigg)q^{-\frac{1}{2}\ell(\delta(B))-\frac{1}{2}\ell(w)}X^{wt(B)}\tilde{\mathbf{1}}_0\\ &= \sum_{B\in\tilde{\mathcal{A}}}\Bigg(\sum_{\sigma\in\Gamma_t^+(A_w,B)}\Bigg(q^{-\frac{1}{2}(\ell(w)+\ell(\delta(B)))}\prod_{s\in S_a} q^{-n_s(\sigma)}(1-q^{-1})^{n_s(\sigma)}\Bigg)\Bigg)X^{wt(B)}\tilde{\mathbf{1}}_0\end{aligned} \quad (7.19)$$

Now replacing positively folded galleries with positively folded alcove walks in the above formula gives Theorem 6.2.7. Since $\Gamma_t^+(A_w, B)$ is the set of all positively folded galleries of type t starting in $wA_\circ$ (5.2.11) while $B_q(p_\lambda)$ is the set of all positively folded alcove walks of type t which begin at $wA_\circ$ ($w \in W^\lambda$ the minimal length coset representatives of the cosets in W/W_λ for $\lambda \in \Lambda^+$), we may switch $\sum_{w\in W^\lambda}\sum_{B\in\tilde{\mathcal{A}}}\sum_{\sigma\in\Gamma_t^+(A_w,B)}$ to

$\sum_{p \in B_q(p_\lambda)}$. Moreover, $wt(B)$ is equivalent to $wt(p)$; the sum of positive folds $\sum_{s \in S_a} n_s(\sigma)$ $= f(p)$; the final direction $\delta(B) = \varphi(p)$ and $\ell(w) =$ initial direction $\iota(p)$ since the gallery σ starts from $wA_\circ$.

Therefore, we finish the proof showing that

$$\begin{aligned} \sum_{w \in W^\lambda} (q^{-\frac{1}{2}\ell(w)} \tilde{T}_{w^{-1}}^{-1}) X^\lambda \tilde{\mathbf{1}}_0 &= \sum_{p \in B_q(p_\lambda)} q^{-\frac{1}{2}(\ell(\iota(p)) + \ell(\varphi(p)) - f(p))} (1 - q^{-1})^{f(p)} X^{wt(p)} \tilde{\mathbf{1}}_0 \\ P_\lambda(X; q^{-1}) \tilde{\mathbf{1}}_0 &= \sum_{p \in B_q(p_\lambda)} q^{-\frac{1}{2}(\ell(\iota(p)) + \ell(\varphi(p)) - f(p))} (1 - q^{-1})^{f(p)} X^{wt(p)} \tilde{\mathbf{1}}_0 \end{aligned} \tag{7.20}$$

□

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