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Prioritization and aggregation of intuitionistic preference relations: A multiplicative- transitivity-based transformation from intuitionistic judgment data to priority weights

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Abstract

This article proposes a goal programming framework for deriving intuitionistic fuzzy weights from intuitionistic preference relations (IPRs). A new multiplicative transitivity is put forward to define consistent IPRs. By analyzing the relationship between intuitionistic fuzzy weights and multiplicative consistency, a transformation formula is introduced to convert normalized intuitionistic fuzzy weights into multiplicative consistent IPRs. By minimizing the absolute deviation between the original judgment and the converted multiplicative consistent IPR, two linear goal programming models are developed to obtain intuitionistic fuzzy weights from IPRs for both individual and group decisions. In the context of multicriteria decision making (MCDM) with a hierarchical structure, a linear program is established to obtain a unified criterion weight vector, which is then used to aggregate local intuitionistic fuzzy weights into global priority weights for final alternative ranking. Two numerical examples are furnished to show the validity and applicability of the proposed models.

Keywords: Intuitionistic preference relation (IPR), Multiplicative consistency, Intuitionistic fuzzy weight, Aggregation, Linear programming

1. Introduction

As a popular tool for tackling decision situations involving multiple and often conflicting criteria, the analytic hierarchy process (AHP) [21] has been widely applied in different contexts such as choice, ranking, and forecasting [10]. The original AHP is conceived to deal with crisp pairwise judgments furnished by the decision-maker (DM) or the analyst. However, with rapid development of information technology, the amount of data has been growing at exponential paces for decades. How to make sense of structured and unstructured big data has presented many challenges to the academics and practitioners. It is understandable that, in many cases, only imprecise judgments can be extracted from messy raw data. To further process the vague decision input, various fuzzy AHP methods have been developed based on the fuzzy set theory and hierarchical structure analysis [2, 3, 5, 8, 20, 26, 30, 46]. With these new developments, different preference relations have been introduced to characterize vague and uncertain judgment information, such as interval multiplicative preference relations [22], interval fuzzy preference relations [40], intuitionistic multiplicative preference relations [39], and intuitionistic preference relations (IPRs) [41].

Based on interval multiplicative preference relations, a number of prioritization approaches have been developed to obtain interval weights, such as goal programming models [29, 31], an eigenvector method-based nonlinear programming model [32], and consistency-test-based methods [18]. For interval fuzzy preference relations, Xu and Chen [45] introduce additive and multiplicative consistency based on normalized crisp weights and establish linear programming (LP) models to derive interval weights. Liu et al. [19] use a convex combination approach to define additive consistent interval fuzzy preference relations and put forward an algorithm to obtain interval weights based on a transformation formula between interval fuzzy and interval multiplicative preference relations. Wang and Li [34] employ interval arithmetic to define additive consistent, multiplicative consistent and weakly transitive interval fuzzy preference relations, and develop goal programming models to derive interval weights for both individual and group decisions. In addition, some approaches have been devised to aggregate local interval weights into global interval weights for MCDM problems with a hierarchical structure. For instance, Bryson and Mobolurin [4] propose a pair of LP models to aggregate local interval weights for each alternative, in which the lower and upper bounds of interval criterion weights are treated as constraints. Wang et al. [31] establish two nonlinear programming models to obtain the lower and upper bounds of a global interval weight, in which local interval weights are multiplicative and criterion weights are treated as decision variables for each alternative.

When evaluating an alternative or criterion, a DM often faces massive and messy raw data in a dynamic environment, which may well present conflicting signals to the DM. In this case, it is reasonable to expect that the DM provide his/her membership assessments with hesitancy [9]. To characterize this hesitation, Atanassov [1] introduced intuitionistic fuzzy sets (IFSs) by explicitly considering nonmembership where the sum of membership and nonmembership does not necessarily add up to 1. Since its inception, IFSs have been widely applied to decision modeling [6, 7, 11-17, 23, 24, 27, 28, 33, 35-39, 41-44, 47, 48]. For instance, Szmidt and Kacprzyk [23] conceive an IPR as a fuzzy preference matrix and a hesitancy matrix, and employ a fuzzy majority rule to aggregate individual IPRs into a group fuzzy preference relation. Xu [41] adopts intuitionistic fuzzy numbers (IFNs) to define IPRs, and introduces multiplicative consistency and weak transitivity for IPRs by employing IFN operations [43]. Subsequently, based on the relationships among multiplicative consistent interval fuzzy preference relations, interval weights, and IPRs, Gong et al. [13] put forward another multiplicative consistency definition for IPRs and investigate how to derive interval priority weights by establishing goal programming models. In the context of additive IPRs, Gong et al. [12] introduce an additive consistency definition and develop a goal program and a least squares model to obtain intuitionistic fuzzy weights for an IPR. Wang [33] points out that the additive consistency transformation formulas in [12] do not always convert normalized priority weights into an IPR, and the consistency therein is defined in an indirect manner. As such, Wang [33] employs membership degrees in an IPR to define new additive transitivity conditions and investigates how to derive intuitionistic fuzzy weights by establishing goal programming models for both individual and group decision situations. In addition, Xu [44] develops an error-analysis-based approach to obtain interval priority weights from any IPR.

It is well known that the definitions of consistency and prioritization play an important role in MCDM with preference relations. A literature review shows that Gong et al. [13] handle multiplicative consistency of IPRs in an indirect manner. The definition therein is based on the converted membership intervals and the associated interval priority weights rather than the DM's original pairwise judgments. Although Xu [41] defines multiplicative consistency by using the DM's original IPR judgments, a close examination reveals that such a multiplicative consistent IPR is technically nonexistent (See a further analysis in Section 3). Furthermore, little work has been carried out to aggregate local intuitionistic fuzzy weights into global priority weights in MCDM with a hierarchical structure. This paper is concerned with IPRs based on multiplicative transitivity. By directly employing the DM's intuitionistic judgment information, a new multiplicative consistency definition is proposed for IPRs. When all intuitionistic judgments are degenerated to fuzzy numbers, the multiplicative transitivity conditions are reduced to those of fuzzy reference relations proposed by Tanino [25]. Based on the relationship between intuitionistic fuzzy weights and multiplicative consistency, a transformation formula is introduced to convert normalized intuitionistic fuzzy weights into multiplicative consistent IPRs. For any IPR, a linear goal program is developed to obtain its intuitionistic fuzzy weights. This approach is then extended to group decision situations. In order to aggregate local intuitionistic fuzzy weights into global ones in MCDM with a hierarchical structure, a linear program is devised to determine a unified criterion weight vector, which is subsequently used to synthesize individual intuitionistic fuzzy weights into a global priority weight for each alternative.

The rest of the paper is organized as follows. Section 2 furnishes a brief review on multiplicative consistent fuzzy preference relations, IPRs, and comparison of IFNs. Section 3 defines multiplicative consistent IPRs and shows how to transform normalized intuitionistic fuzzy weights into a multiplicative consistent IPR. In Section 4, goal-programming-based intuitionistic fuzzy weight generation approaches are developed based on individual and group IPRs. Aggregation of local intuitionistic fuzzy weights is investigated in Section 5. Two illustrative examples, consisting of a comparative study with existing approaches and an MCDM problem with a hierarchical structure, are presented in Section 6 to demonstrate the validity and practicality of the proposed models. The paper concludes with some remarks in Section 7.

2. Preliminaries

For an MCDM problem with a finite set of alternatives, let $X = \{x_1, x_2, ..., x_n\}$ be the set of *n* alternatives. In eliciting his/her preference over alternatives, a DM often utilizes a pairwise comparison technique, yielding a fuzzy preference relation $R = (r_{ij})_{n\times n}$, where r_{ij} denotes a fuzzy preference degree of alternative x_i over x_j such that

$$0 \le r_{ij} \le 1, r_{ij} + r_{ji} = 1, r_{ii} = 0.5 \qquad \text{for all } i, j = 1, 2, ..., n \tag{2.1}$$

$r_{ij} > 0.5$ indicates that x_i is preferred to x_j and the greater the r_{ij}, the stronger alternative x_i is superior to x_j. $r_{ij} < 0.5$ signifies that x_j is preferred to x_i and the smaller the r_{ij}, the stronger the preference is. $r_{ij} = 0.5$ shows the DM's indifference between x_i and x_j. In particular, $r_{ij} = 1$ indicates that x_i is absolutely preferred to x_j, $r_{ij} = 0$ implies x_j is absolutely preferred to x_i.

Tanino [25] proposes a multiplicative consistency definition for fuzzy preference relations and introduces the following transitivity conditions.

Definition 2.1 [25] A fuzzy preference relation $R = (r_{ij})_{n \times n}$ is called multiplicative consistent if it satisfies

$$\frac{r_{ik}}{r_{ki}} \frac{r_{kj}}{r_{jk}} = \frac{r_{ij}}{r_{ji}} \qquad \text{for all } i, j, k = 1, 2, ..., n \tag{2.2}$$

As $r_{ij} = 1 - r_{ji}$ for all i, j = 1, 2, …, n, one can obtain

$$\frac{r_{ij}}{r_{ji}} \frac{r_{jk}}{r_{kj}} \frac{r_{ki}}{r_{ik}} = \frac{r_{ik}}{r_{ki}} \frac{r_{kj}}{r_{jk}} \frac{r_{ji}}{r_{ij}} \qquad \text{for all } i, j, k = 1, 2, ..., n \tag{2.3}$$

It has been found that, for a fuzzy preference relation $R = (r_{ij})_{n \times n}$, if there exists a weight vector $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ such that

$$r_{ij} = \frac{\omega_i}{\omega_i + \omega_j} \qquad \text{for all } i, j = 1, 2, ..., n \tag{2.4}$$

where $\sum_{i=1}^{n} \omega_i = 1$ and $\omega_i \geq 0$ for $i = 1, 2, ..., n$, then R is multiplicative consistent [42].

In the presence of uncertainty and vagueness in real-world decision situations, DMs often experience hesitancy in offering their fuzzy preference judgments. To characterize this hesitation, Atanassov [1] generalizes the classic fuzzy sets by introducing the notion of intuitionistic fuzzy sets (IFSs), which furnishes a convenient vehicle to accommodate the DMs' hesitation in their judgment.

Let Z be a fixed nonempty universe set, an IFS A in Z is an object given by

$$A = \{< z, \mu_A(z), \nu_A(z) > | z \in Z\} \tag{2.5}$$

where $\mu_A : Z \to [0,1]$, $\nu_A : Z \to [0,1]$ such that $0 \leq \mu_A(z) + \nu_A(z) \leq 1$, $\forall z \in Z$.

$\mu_A(z)$ and $\nu_A(z)$ denote, respectively, the membership and nonmembership degree of element z to set A. In addition, for each IFS A in Z, $\pi_A(z) = 1 - \mu_A(z) - \nu_A(z)$ is called the intuitionistic fuzzy index of A, representing the hesitation degree of z to A. Obviously,

$0 \le \pi_A(z) \le 1$. If $\pi_A(z) = 0$, for every $z \in Z$, then $\nu_A(z) = 1 - \mu_A(z)$, indicating that A is reduced to a fuzzy set, $A^{'} = \{< z, \mu_A(z) > | z \in Z\}$.

For an IFS A and a given z, the pair $(\mu_A(z), \nu_A(z))$ is called an IFN [41, 43]. For convenience, the pair $(\mu_A(z), \nu_A(z))$ is often denoted by (μ, v), where $\mu, v \in [0,1]$ and $\mu + v \le 1$.

Definition 2.2 [41] An IPR $\tilde{R}$ on X is an intuitionistic fuzzy set on the product set $X \times X$ characterized by a judgment matrix $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ with $\tilde{r}_{ij} = (\mu_{ij}, v_{ij})$, where (μ_{ij}, v_{ij}) indicates the intuitionistic preference degree of alternative x_i over x_j such that

$$0 \le \mu_{ij} + v_{ij} \le 1, \mu_{ij} = v_{ji}, v_{ij} = \mu_{ji}, \mu_{ii} = v_{ii} = 0.5 \qquad i, j = 1, 2, ..., n \tag{2.6}$$

For an IFN $\tilde{\alpha} = (\mu, v)$, its score function is defined as [6],

$$S(\tilde{\alpha}) = \mu - v \tag{2.7}$$

where $S(\tilde{\alpha}) \in [-1,1]$, and its accuracy function is defined as [15]

$$H(\tilde{\alpha}) = \mu + v \tag{2.8}$$

where $H(\tilde{\alpha}) \in [0,1]$. The score function can be loosely treated as the net degree of belonging to a certain set and the accuracy function measures the total amount of non-hesitant information included in the intuitionistic judgment. As such, the score and accuracy functions are often used as a basis to compare two IFNs. By taking a prioritized sequence of these two functions, Xu [41] devises the following approach to comparing any two IFNs.

Let $\tilde{\alpha}_1 = (\mu_1, v_1)$ and $\tilde{\alpha}_2 = (\mu_2, v_2)$ be two IFNs,

if $S(\tilde{\alpha}_1) < S(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1$ is smaller than $\tilde{\alpha}_2$, denoted by $\tilde{\alpha}_1 < \tilde{\alpha}_2$;

if $S(\tilde{\alpha}_1) > S(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1$ is greater than $\tilde{\alpha}_2$, denoted by $\tilde{\alpha}_1 > \tilde{\alpha}_2$;

otherwise,

if $H(\tilde{\alpha}_1) < H(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1$ is smaller than $\tilde{\alpha}_2$, denoted by $\tilde{\alpha}_1 < \tilde{\alpha}_2$;

if $H(\tilde{\alpha}_1) > H(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1$ is greater than $\tilde{\alpha}_2$, denoted by $\tilde{\alpha}_1 > \tilde{\alpha}_2$;

otherwise $\tilde{\alpha}_1 = \tilde{\alpha}_2$.

Based on the aforesaid score function, Wang [33] proposes a new definition of weak transitivity for IPRs, and shows that additive consistent IPRs are always weakly transitive.

Definition 2.3 [33] Let $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$ be an IPR, $\tilde{R}$ is weakly transitive if $S(\tilde{r}_{ik})\geq 0$ and $S(\tilde{r}_{kj})\geq 0$ imply $S(\tilde{r}_{ij})\geq 0,$ for all $i, j, k=1,2, ..., n$.

3. Multiplicative consistency of intuitionistic preference relations

This section employs the original intuitionistic judgment information to introduce a new multiplicative consistency definition for IPRs. It is first shown that multiplicative consistent IPRs under this definition are always weakly transitive, and a transformation formula is then put forward to convert normalized intuitionistic fuzzy weights into multiplicative consistent IPRs.

As per Definition 2.2, we have $0\leq \mu_{ij}\leq 1$. If $\mu_{ij}>0.5$, then $\frac{1}{1-\mu_{ij}}-1=\frac{\mu_{ij}}{1-\mu_{ij}}>1$; if $\mu_{ij}=0.5$, then $\frac{\mu_{ij}}{1-\mu_{ij}}=1$; if $\mu_{ij}<0.5$, then $0\leq\frac{\mu_{ij}}{1-\mu_{ij}}<1$. Similarly, if $v_{ij}>0.5$, then $\frac{1}{1-v_{ij}}-1=\frac{v_{ij}}{1-v_{ij}}>1$; if $v_{ij}=0.5$, then $\frac{v_{ij}}{1-v_{ij}}=1$; if $v_{ij}<0.5$, then $0\leq\frac{v_{ij}}{1-v_{ij}}<1$. Therefore, (μ_{ij},v_{ij}) denotes that alternative x_i is preferred to x_j with a multiplicative degree of $\frac{\mu_{ij}}{1-\mu_{ij}}$, and alternative x_i is non-preferred to x_j with a multiplicative degree of $\frac{v_{ij}}{1-v_{ij}}$. As $v_{ij}=\mu_{ji}$ for all *i*, *j* = 1, 2, …, *n*, we have $\frac{v_{ij}}{1-v_{ij}}=\frac{\mu_{ji}}{1-\mu_{ji}}$.

Based on the aforesaid analysis, multiplicative consistency of an IPR can be defined as follows.

Definition 3.1 An IPR $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$ with $\tilde{r}_{ij}=(\mu_{ij},v_{ij})$ is called multiplicative consistent if it satisfies

$$\left(\frac{\mu_{ij}}{1-\mu_{ij}}\right)\left(\frac{\mu_{jk}}{1-\mu_{jk}}\right)\left(\frac{\mu_{ki}}{1-\mu_{ki}}\right)=\left(\frac{\mu_{ik}}{1-\mu_{ik}}\right)\left(\frac{\mu_{kj}}{1-\mu_{kj}}\right)\left(\frac{\mu_{ji}}{1-\mu_{ji}}\right)\quad \text{for all } i, j, k=1,2,...,n \qquad (3.1)$$

The idea of the multiplicative consistency condition (3.1) can be graphically illustrated in Figure 1.

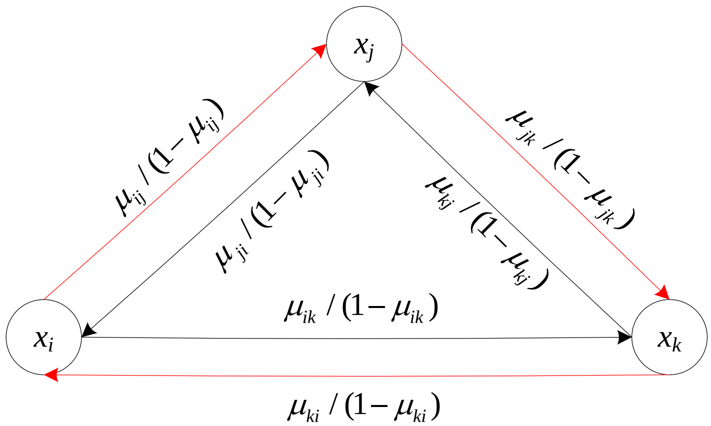

Figure 1. Illustration of the multiplicative transitivity condition

If all IFNs $\tilde{r}_{ij} = (\mu_{ij}, v_{ij})$ are reduced to fuzzy numbers, i.e., $\mu_{ij} + v_{ij} = 1$ for all *i*, *j* = 1, 2, …, *n*, then the IPR $\tilde{R}$ is equivalent to a fuzzy preference relation $R = (r_{ij})_{n\times n}$ with $r_{ij} = \mu_{ij}$ and Eq. (3.1) is degraded to Eq. (2.3) .

As $\mu_{ij} = v_{ji}, v_{ij} = \mu_{ji}$ for all *i*, *j* = 1, 2, …, *n*, from (3.1), one can obtain

$$\left(\frac{v_{ij}}{1-v_{ij}}\right)\left(\frac{v_{jk}}{1-v_{jk}}\right)\left(\frac{v_{ki}}{1-v_{ki}}\right) = \left(\frac{v_{ik}}{1-v_{ik}}\right)\left(\frac{v_{kj}}{1-v_{kj}}\right)\left(\frac{v_{ji}}{1-v_{ji}}\right) \text{ for all } i, j, k = 1, 2, ..., n \quad (3.2)$$

It is worth noting that the multiplicative consistency conditions given by Xu [41] (See Eq. (8) on page 2366) are inappropriate. As per Xu [41], an IPR $\tilde{R}$ is multiplicative consistent if $\tilde{r}_{ij} = \tilde{r}_{ik} \otimes \tilde{r}_{kj}$ for all $i, j, k = 1, 2, ..., n$, where $\otimes$ is a multiplicative operator between two IFNs. According to the IFN operational rules defined by Xu [41] (See Definition 4 on page 2366), one has $\mu_{ij} = \mu_{ik}\mu_{kj}$ and $\mu_{ik} = \mu_{ij}\mu_{jk}$. Hence, $\mu_{ij} = \mu_{ik}\mu_{kj} = \mu_{ij}\mu_{jk}\mu_{kj} \Rightarrow \mu_{kj}\mu_{jk} = \mu_{kj}v_{kj} = 1$. However, this is impossible given that $0 \le \mu_{kj}, v_{kj} \le 1$ and $\mu_{kj} + v_{kj} \le 1$.

From Definitions 2.3 and 3.1, we have the following theorem.

Theorem 3.1 Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ be an IPR, if $\tilde{R}$ is multiplicative consistent, then $\tilde{R}$ is weakly transitive.

Proof. Since $\tilde{R}$ is multiplicative consistent, by Definition 3.1, we have

$$(1-\mu_{ij})(1-\mu_{ki})(1-\mu_{jk})\mu_{ji}\mu_{ik}\mu_{kj} = (1-\mu_{kj})(1-\mu_{ik})(1-\mu_{ji})\mu_{jk}\mu_{ki}\mu_{ij} \qquad \forall i, j, k = 1, 2, ..., n .$$

Note that $\forall i, j = 1, 2, ..., n$, $\mu_{ij} = v_{ji}, v_{ij} = \mu_{ji}$. The aforesaid equation can be rewritten as

$$(1-\mu_{ij})(1-v_{ik})(1-v_{kj})v_{ij}\mu_{ik}\mu_{kj} = (1-\mu_{kj})(1-\mu_{ik})(1-v_{ij})v_{kj}v_{ik}\mu_{ij} \tag{3.3}$$

Meanwhile, for $\forall i, j, k = 1, 2, ..., n$, one can obtain

$$\begin{aligned}
&(1-\mu_{ij})(1-v_{ik})(1-v_{kj})v_{ij}\mu_{ik}\mu_{kj} = \mu_{ik}\mu_{kj}(1-v_{ik})(1-v_{kj})(v_{ij} - v_{ij}\mu_{ij}) \\
&= \mu_{ik}\mu_{kj}(1-v_{ik})(1-v_{kj})(v_{ij} - \mu_{ij} + \mu_{ij}(1-v_{ij})) \\
&= \mu_{ik}\mu_{kj}(1-v_{ik})(1-v_{kj})(v_{ij} - \mu_{ij}) + \mu_{ik}\mu_{kj}\mu_{ij}(1-v_{ik})(1-v_{kj})(1-v_{ij})
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
&(1-\mu_{kj})(1-\mu_{ik})(1-v_{ij})v_{kj}v_{ik}\mu_{ij} = \mu_{ij}(1-v_{ij})(v_{ik} - v_{ik}\mu_{ik})(v_{kj} - v_{kj}\mu_{kj}) \\
&= \mu_{ij}(1-v_{ij})(v_{ik} - \mu_{ik} + \mu_{ik}(1-v_{ik}))(v_{kj} - \mu_{kj} + \mu_{kj}(1-v_{kj})) \\
&= \mu_{ij}(1-v_{ij})[(v_{ik} - \mu_{ik})(v_{kj} - \mu_{kj}) + (v_{ik} - \mu_{ik})\mu_{kj}(1-v_{kj}) \\
&\qquad + \mu_{ik}(1-v_{ik})(v_{kj} - \mu_{kj})] + \mu_{ik}\mu_{kj}\mu_{ij}(1-v_{ik})(1-v_{kj})(1-v_{ij}) \\
&= [\mu_{ij}v_{kj}(1-v_{ij})(1-\mu_{kj})(v_{ik} - \mu_{ik}) + \mu_{ij}\mu_{ik}(1-v_{ij})(1-v_{ik})(v_{kj} - \mu_{kj})] \\
&\qquad + \mu_{ik}\mu_{kj}\mu_{ij}(1-v_{ik})(1-v_{kj})(1-v_{ij})
\end{aligned} \tag{3.5}$$

It follows from (3.3), (3.4) and (3.5) that

$$\begin{aligned}
&\mu_{ik}\mu_{kj}(1-v_{ik})(1-v_{kj})(v_{ij} - \mu_{ij}) \\
&= \mu_{ij}v_{kj}(1-v_{ij})(1-\mu_{kj})(v_{ik} - \mu_{ik}) + \mu_{ij}\mu_{ik}(1-v_{ij})(1-v_{ik})(v_{kj} - \mu_{kj})
\end{aligned} \tag{3.6}$$

According to (2.7), if $S(\tilde{r}_{ik}) \geq 0$ and $S(\tilde{r}_{kj}) \geq 0$, we get $v_{ik} - \mu_{ik} \leq 0$ and $v_{kj} - \mu_{kj} \leq 0$, $\forall i, j, k \in \{1, 2, \ldots, n\}$. On the other hand, for $\forall i, j = 1, 2, ..., n$, we have $0 \leq \mu_{ij} \leq 1$ and $0 \leq v_{ij} \leq 1$. These lead to

$$\mu_{ij}v_{kj}(1-v_{ij})(1-\mu_{kj})(v_{ik} - \mu_{ik}) + \mu_{ij}\mu_{ik}(1-v_{ij})(1-v_{ik})(v_{kj} - \mu_{kj}) \leq 0$$

As per (3.6), it is certified that $\mu_{ik}\mu_{kj}(1-v_{ik})(1-v_{kj})(v_{ij} - \mu_{ij}) \leq 0$, implying $(v_{ij} - \mu_{ij}) \leq 0$, or equivalently, $S(\tilde{r}_{ij}) \geq 0$, the proof of Theorem 3.1 is thus completed. ■

From Definition 2.2, we know that $\tilde{r}_{ij}$ denotes the intuitionistic fuzzy preference degree of alternative x_i to x_j. $\tilde{r}_{ij} = (1, 0)$ indicates that x_i is absolutely better than x_j, $\tilde{r}_{ij} = (0, 1)$ implies that x_j is preferred to x_i without any uncertainty or hesitation, and $\tilde{r}_{ij} = (0.5, 0.5)$ means that the DM is indifferent between x_i and x_j. As the preference values in $\tilde{R}$ are furnished as IFNs, it is sensible to expect that the priority weights derived from $\tilde{R}$ be IFNs rather than crisp values.

Denote a normalized intuitionistic fuzzy priority weight vector by $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_n)^T = ((\omega_1^\mu, \omega_1^\nu), (\omega_2^\mu, \omega_2^\nu), ..., (\omega_n^\mu, \omega_n^\nu))^T$ with [33]

$$\omega_i^\mu, \omega_i^\nu \in [0,1],\ \omega_i^\mu + \omega_i^\nu \le 1,\ \sum_{\substack{j=1 \\ j \ne i}}^{n} \omega_j^{\mu} \le \omega_i^{\nu},\ \omega_i^{\mu} + n - 2 \ge \sum_{\substack{j=1 \\ j \ne i}}^{n} \omega_j^{\nu} \quad i = 1, 2, ..., n, \quad (3.7)$$

where $\tilde{\omega}_i = (\omega_i^\mu, \omega_i^\nu)$ (i = 1, 2, …, n) are IFNs and represent the membership and nonmembership degrees of alternative x_i as per a fuzzy concept of "importance".

Let

$$\tilde{t}_{ij} = (t_{ij}^\mu, t_{ij}^\nu) = \begin{cases} (0.5, 0.5) & i = j \\ \left(\dfrac{\omega_i^\mu}{1 + \omega_i^\mu - \omega_j^\nu}, \dfrac{\omega_j^\mu}{1 + \omega_j^\mu - \omega_i^\nu} \right) & i \ne j \end{cases}$$

(3.8)

then we have the following results.

Theorem 3.2 Let $\tilde{T} = (\tilde{t}_{ij})_{n \times n}$ be a matrix defined by (3.8), then $\tilde{T}$ is a multiplicative consistent IPR.

Proof. It is apparent that, for all $i, j = 1, 2, ..., n$, $t_{ji}^\mu = t_{ij}^\nu$ and $t_{ji}^\nu = t_{ij}^\mu$. As $\omega_i^\mu, \omega_i^\nu \in [0,1]$, we have $0 \le \dfrac{\omega_i^\mu}{1 + \omega_i^\mu - \omega_j^\nu} \le 1$ and $0 \le \dfrac{\omega_j^\mu}{1 + \omega_j^\mu - \omega_i^\nu} \le 1$. Moreover, since $\omega_i^\mu + \omega_i^\nu \le 1$ for all i = 1, 2, …, n, it follows that

$$\omega_i^\mu \omega_j^\mu \le (1 - \omega_i^\nu)(1 - \omega_j^\nu)$$

$$1 + \frac{\omega_j^\mu}{1 - \omega_i^\nu} \le 1 + \frac{1 - \omega_j^\nu}{\omega_i^\mu}$$

$$\frac{\omega_i^\mu}{1 + \omega_i^\mu - \omega_j^\nu} \le \frac{1 - \omega_i^\nu}{1 + \omega_j^\mu - \omega_i^\nu} = 1 - \frac{\omega_j^\mu}{1 + \omega_j^\mu - \omega_i^\nu}$$

Therefore, we have $\dfrac{\omega_i^\mu}{1 + \omega_i^\mu - \omega_j^\nu} + \dfrac{\omega_j^\mu}{1 + \omega_j^\mu - \omega_i^\nu} \le 1$. As per Definition 2.2, $\tilde{T}$ is an IPR.

On the other hand, since

$$\left(\frac{t_{ij}^\mu}{1 - t_{ij}^\mu} \right) \left(\frac{t_{jk}^\mu}{1 - t_{jk}^\mu} \right) \left(\frac{t_{ki}^\mu}{1 - t_{ki}^\mu} \right) = \left(\frac{\omega_i^\mu}{1 - \omega_j^\nu} \right) \left(\frac{\omega_j^\mu}{1 - \omega_k^\nu} \right) \left(\frac{\omega_k^\mu}{1 - \omega_i^\nu} \right) = \frac{\omega_i^\mu \omega_j^\mu \omega_k^\mu}{(1 - \omega_i^\nu)(1 - \omega_j^\nu)(1 - \omega_k^\nu)}$$

and

$$\left(\frac{t_{ik}^{\mu}}{1-t_{ik}^{\mu}}\right)\left(\frac{t_{kj}^{\mu}}{1-t_{kj}^{\mu}}\right)\left(\frac{t_{ji}^{\mu}}{1-t_{ji}^{\mu}}\right)=\left(\frac{\omega_i^{\mu}}{1-\omega_k^{\nu}}\right)\left(\frac{\omega_k^{\mu}}{1-\omega_j^{\nu}}\right)\left(\frac{\omega_j^{\mu}}{1-\omega_i^{\nu}}\right)=\frac{\omega_i^{\mu}\omega_j^{\mu}\omega_k^{\mu}}{(1-\omega_i^{\nu})(1-\omega_j^{\nu})(1-\omega_k^{\nu})}$$

By Definition 3.1, $\tilde{T}$ is multiplicative consistent. ■

From (3.8), it is easy to verify that IPR $\tilde{T}=(\tilde{t}_{ij})_{n\times n}$ is equivalent to a fuzzy preference relation if all intuitionistic fuzzy weights $\tilde{\omega}_i=(\omega_i^{\mu},\omega_i^{\nu})$ ($i=1,2,...,n$) are degenerated to classical fuzzy weights, i.e., $\omega_i^{\nu}=1-\omega_i^{\mu}$. In this case, (3.8) is reduced to (2.4), corresponding to the multiplicative consistency condition for fuzzy preference relations.

The following corollary can be directly derived from Theorem 3.2.

Corollary 3.1 For an IPR $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$, if there exists a normalized intuitionistic fuzzy weight vector $\tilde{\omega}=(\tilde{\omega}_1,\tilde{\omega}_2,\cdots,\tilde{\omega}_n)^T$ such that

$$\tilde{r}_{ij}=(\mu_{ij},\nu_{ij})=\begin{cases}(0.5,0.5) & i=j\\ \left(\dfrac{\omega_i^{\mu}}{1+\omega_i^{\mu}-\omega_j^{\nu}},\dfrac{\omega_j^{\mu}}{1+\omega_j^{\mu}-\omega_i^{\nu}}\right) & i\neq j\end{cases} \tag{3.9}$$

then $\tilde{R}$ is multiplicative consistent.

4. Goal programming models for generating intuitionistic fuzzy weights

Base on the aforesaid multiplicative transitivity, this section develops goal programs for deriving intuitionistic fuzzy weights from individual and group IPRs.

4.1 An individual decision model with IPRs

As per Corollary 3.1, for an IPR $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$, if there exists a normalized intuitionistic fuzzy weight vector $\tilde{\omega}=(\tilde{\omega}_1,\tilde{\omega}_2,\cdots,\tilde{\omega}_n)^T$ with $\tilde{\omega}_i=(\omega_i^{\mu},\omega_i^{\nu})$, $\omega_i^{\mu},\omega_i^{\nu}\in[0,1]$, $\omega_i^{\mu}+\omega_i^{\nu}\leq 1$, $\sum_{j=1,j\neq i}^{n}\omega_j^{\mu}\leq\omega_i^{\nu}$ and $\omega_i^{\mu}+n-2\geq\sum_{j=1,j\neq i}^{n}\omega_j^{\nu}$ for $i=1,2,\ldots,n$, such that

$$\mu_{ij}(1+\omega_i^{\mu}-\omega_j^{\nu})=\omega_i^{\mu} \tag{4.1}$$

$$\nu_{ij}(1+\omega_j^{\mu}-\omega_i^{\nu})=\omega_j^{\mu} \tag{4.2}$$

then $\tilde{R}$ is multiplicative consistent. By Theorem 3.1, $\tilde{R}$ is also weakly transitive. However, in real-world decision situations, it is often a challenge for a DM to furnish a consistent IPR, especially when a large number of alternatives are involved. In this case, (4.1) and (4.2) will not hold. To handle these situations with inconsistent decision input, (4.1) and (4.2) will have to be relaxed by allowing some deviations. Priority weights will then be derived by minimizing the absolute deviation from a multiplicative consistent IPR. Based on this idea, the following deviation variables are introduced:

$$\varepsilon_{ij} = \mu_{ij}(1+\omega_i^{\mu} - \omega_j^{v}) - \omega_i^{\mu},\ i, j = 1, 2, ..., n,\ j \neq i \tag{4.3}$$

$$\xi_{ij} = v_{ij}(1+\omega_j^{\mu} - \omega_i^{v}) - \omega_j^{\mu},\ i, j = 1, 2, ..., n,\ j \neq i \tag{4.4}$$

The smaller the sum of the absolute deviations, the closer the $\tilde{R}$ is to a multiplicative consistent IPR. As $\mu_{ij} = v_{ji}$ and $v_{ij} = \mu_{ji}$, one has $\varepsilon_{ij} = \mu_{ij}(1+\omega_i^{\mu} - \omega_j^{v}) - \omega_i^{\mu} = v_{ji}(1+\omega_i^{\mu} - \omega_j^{v}) - \omega_i^{\mu} = \xi_{ji}$ for all $i, j = 1, 2, ..., n, j \neq i$. Therefore, the following nonlinear programming model is established for deriving intuitionistic fuzzy weights:

$$\begin{aligned} &\min \quad J = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(|\varepsilon_{ij}| + |\xi_{ij}|) \\ &s.t. \begin{cases} \mu_{ij}(1+\omega_i^{\mu} - \omega_j^{v}) - \omega_i^{\mu} - \varepsilon_{ij} = 0, & i = 1, 2, ..., n-1,\ j = i+1, ..., n \\ v_{ij}(1+\omega_j^{\mu} - \omega_i^{v}) - \omega_j^{\mu} - \xi_{ij} = 0, & i = 1, 2, ..., n-1,\ j = i+1, ..., n \\ 0 \leq \omega_i^{\mu} \leq 1, 0 \leq \omega_i^{v} \leq 1, \omega_i^{\mu} + \omega_i^{v} \leq 1, & i = 1, 2, ..., n \\ \sum_{j=1, j\neq i}^{n}\omega_j^{\mu} \leq \omega_i^{v}, \omega_i^{\mu} + n - 2 \geq \sum_{j=1, j\neq i}^{n}\omega_j^{v} & i = 1, 2, ..., n \end{cases} \end{aligned} \tag{4.5}$$

where the first two lines represent the relaxed multiplicative consistent conditions from (4.3) and (4.4) and the remaining constraints ensure that the derived weights constitute a normalized intuitionistic fuzzy weight vector $\tilde{\omega}$.

Similar to the treatment in Wang and Li [34], let

$$\varepsilon_{ij}^{-} \triangleq \frac{|\varepsilon_{ij}| - \varepsilon_{ij}}{2} \quad \text{and} \quad \varepsilon_{ij}^{+} \triangleq \frac{|\varepsilon_{ij}| + \varepsilon_{ij}}{2},\ i = 1, 2, ..., n-1,\ j = i+1, ..., n, \tag{4.6}$$

$$\xi_{ij}^{-} \triangleq \frac{|\xi_{ij}| - \xi_{ij}}{2} \quad \text{and} \quad \xi_{ij}^{+} \triangleq \frac{|\xi_{ij}| + \xi_{ij}}{2},\ i = 1, 2, ..., n-1,\ j = i+1, ..., n. \tag{4.7}$$

It is trivial to verify that $\varepsilon_{ij} = \varepsilon_{ij}^{+} - \varepsilon_{ij}^{-}$, $|\varepsilon_{ij}| = \varepsilon_{ij}^{+} + \varepsilon_{ij}^{-}$, $\varepsilon_{ij}^{+} \cdot \varepsilon_{ij}^{-} = 0$, $\xi_{ij} = \xi_{ij}^{+} - \xi_{ij}^{-}$, $|\xi_{ij}| = \xi_{ij}^{+} + \xi_{ij}^{-}$, and $\xi_{ij}^{+} \cdot \xi_{ij}^{-} = 0$ for $i = 1, 2, ..., n-1, j = i+1, ..., n$. Then, the optimization model (4.5) can be linearized as:

$$\min \quad J=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\varepsilon_{ij}^{+}+\varepsilon_{ij}^{-}+\xi_{ij}^{+}+\xi_{ij}^{-})$$
$$s.t.\begin{cases}\mu_{ij}(1+\omega_i^{\mu}-\omega_j^{v})-\omega_i^{\mu}-\varepsilon_{ij}^{+}+\varepsilon_{ij}^{-}=0, & i=1,2,...,n-1,\ j=i+1,...,n\\ v_{ij}(1+\omega_j^{\mu}-\omega_i^{v})-\omega_j^{\mu}-\xi_{ij}^{+}+\xi_{ij}^{-}=0, & i=1,2,...,n-1,\ j=i+1,...,n\\ 0\le\omega_i^{\mu}\le 1, 0\le\omega_i^{v}\le 1, \omega_i^{\mu}+\omega_i^{v}\le 1, & i=1,2,...,n\\ \sum_{j=1, j\ne i}^{n}\omega_j{}^{\mu}\le\omega_i{}^{v}, \omega_i{}^{\mu}+n-2\ge\sum_{j=1, j\ne i}^{n}\omega_j{}^{v}, & i=1,2,...,n\\ \varepsilon_{ij}^{+}\ge 0, \varepsilon_{ij}^{-}\ge 0, \xi_{ij}^{+}\ge 0, \xi_{ij}^{-}\ge 0 & i=1,2,...,n-1,\ j=i+1,...,n\end{cases} \tag{4.8}$$

Solving (4.8) yields an optimal intuitionistic fuzzy weight vector $\tilde{\omega}^{*}=(\tilde{\omega}_1^{*},\tilde{\omega}_2^{*},\cdots,\tilde{\omega}_n^{*})^T$ $=((\omega_1^{\mu*},\omega_1^{v*}),(\omega_2^{\mu*},\omega_2^{v*}),\cdots,(\omega_n^{\mu*},\omega_n^{v*}))^T$ for $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$.

If the optimal objective function value $J^{*}=0$, one can obtain $\varepsilon_{ij}^{+}=\varepsilon_{ij}^{-}=\xi_{ij}^{+}=\xi_{ij}^{-}=0$. This implies that $\tilde{R}$ can be expressed as (3.9) by the optimal intuitionistic fuzzy weight vector $\tilde{\omega}^{*}$. According to Corollary 3.1, $\tilde{R}$ is multiplicative consistent.

4.2 A group decision model with IPRs

Considering an IPR-based group decision problem with an alternative set $X=\{x_1,x_2,...,x_n\}$ and a group of p DMs $\{d_1,d_2,...,d_p\}$. Each DM d_k $(k=1,2,...,p)$ provides an IPR $\tilde{R}^{k}=(\tilde{r}_{ij}^{k})_{n\times n}=((\mu_{ij}^{k},v_{ij}^{k}))_{n\times n}$ to express his/her preference on alternative set X. Let $\lambda=(\lambda_1,\lambda_2,...,\lambda_p)^T$ be the DMs' weight vector, satisfying $\sum_{k=1}^{p}\lambda_k=1$ and $\lambda_k\ge 0$ for $k=1,2,...,p$.

In a group decision problem, different DMs typically have different subjective preferences, it is hard, if not impossible, to get a unified intuitionistic fuzzy weight vector $\tilde{\omega}=(\tilde{\omega}_1,\tilde{\omega}_2,\cdots,\tilde{\omega}_n)^T$ such that the elements in $\tilde{R}^{k}$ $(k=1,2,...,p)$ can all be expressed as (3.9). In other words, the following conditions of multiplicative transitivity generally cannot be met for all DMs.

$$\mu_{ij}^{k}(1+\omega_i^{\mu}-\omega_j^{v})=\omega_i^{\mu}, i=1,2,...,n,\ j=i+1,...,n, k=1,2,...,p \tag{4.9}$$

$$v_{ij}^{k}(1+\omega_j^{\mu}-\omega_i^{v})=\omega_j^{\mu}\ , i=1,2,...,n,\ j=i+1,...,n, k=1,2,...,p \tag{4.10}$$

Similar to the treatment in Section 4.1, the following goal program is established to find a unified intuitionistic fuzzy priority vector for the group of IPRs. This modeling principle is to minimize the weighted sum of the absolute deviations between the original IPRs and a multiplicative consistent IPR associated with the unified weight vector.

$$\min \quad J=\sum_{k=1}^{p}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\lambda_k(|\varepsilon_{ij}^k|+|\xi_{ij}^k|)$$
$$s.t.\begin{cases}\mu_{ij}^k(1+\omega_i^\mu-\omega_j^v)-\omega_i^\mu-\varepsilon_{ij}^k=0, & i=1,2,...,n,\ j=i+1,...,n,k=1,2,...,p\\ v_{ij}^k(1+\omega_j^\mu-\omega_i^v)-\omega_j^\mu-\xi_{ij}^k=0, & i=1,2,...,n,\ j=i+1,...,n,k=1,2,...,p\\ 0\le\omega_i^\mu\le 1, 0\le\omega_i^v\le 1, \omega_i^\mu+\omega_i^v\le 1, & i=1,2,...,n\\ \sum_{\substack{j=1\\ j\ne i}}^{n}\omega_j{}^\mu\le\omega_i{}^v, \omega_i^\mu+n-2\ge\sum_{\substack{j=1\\ j\ne i}}^{n}\omega_j{}^v & i=1,2,...,n\end{cases} \tag{4.11}$$

Let

$$\varepsilon_{ij}^{k-}\triangleq\frac{|\varepsilon_{ij}^k|-\varepsilon_{ij}^k}{2} \text{ and } \varepsilon_{ij}^{k+}\triangleq\frac{|\varepsilon_{ij}^k|+\varepsilon_{ij}^k}{2},\ i=1,2,...,n-1,\ j=i+1,...,n,k=1,2,...,p, \tag{4.12}$$

$$\xi_{ij}^{k-}\triangleq\frac{|\xi_{ij}^k|-\xi_{ij}^k}{2} \text{ and } \xi_{ij}^{k+}\triangleq\frac{|\xi_{ij}^k|+\xi_{ij}^k}{2},\ i=1,2,...,n-1,\ j=i+1,...,n,k=1,2,...,p. \tag{4.13}$$

Then $\varepsilon_{ij}^k, |\varepsilon_{ij}^k|, \xi_{ij}^k$ and $|\xi_{ij}^k|$ can be expressed as $\varepsilon_{ij}^k=\varepsilon_{ij}^{k+}-\varepsilon_{ij}^{k-}$, $|\varepsilon_{ij}^k|=\varepsilon_{ij}^{k+}+\varepsilon_{ij}^{k-}$, $\xi_{ij}^k=\xi_{ij}^{k+}-\xi_{ij}^{k-}$ and $|\xi_{ij}^k|=\xi_{ij}^{k+}+\xi_{ij}^{k-}$ for $i=1,2,...,n-1,\ j=i+1,...,n,k=1,2,...,p.$

Accordingly, (4.11) can be linearized as the following goal program:

$$\min \quad J=\sum_{k=1}^{p}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\lambda_k(\varepsilon_{ij}^{k+}+\varepsilon_{ij}^{k-}+\xi_{ij}^{k+}+\xi_{ij}^{k-})$$
$$s.t.\begin{cases}\mu_{ij}^k(1+\omega_i^\mu-\omega_j^v)-\omega_i^\mu-\varepsilon_{ij}^{k+}+\varepsilon_{ij}^{k-}=0, & i=1,2,...,n,\ j=i+1,...,n,k=1,2,...,p\\ v_{ij}^k(1+\omega_j^\mu-\omega_i^v)-\omega_j^\mu-\xi_{ij}^{k+}+\xi_{ij}^{k-}=0, & i=1,2,...,n,\ j=i+1,...,n,k=1,2,...,p\\ 0\le\omega_i^\mu\le 1, 0\le\omega_i^v\le 1, \omega_i^\mu+\omega_i^v\le 1, & i=1,2,...,n\\ \sum_{\substack{j=1\\ j\ne i}}^{n}\omega_j{}^\mu\le\omega_i{}^v, \omega_i^\mu+n-2\ge\sum_{\substack{j=1\\ j\ne i}}^{n}\omega_j{}^v, & i=1,2,...,n\\ \varepsilon_{ij}^{k+}\ge 0, \varepsilon_{ij}^{k-}\ge 0, \xi_{ij}^{k+}\ge 0, \xi_{ij}^{k-}\ge 0 & i=1,2,...,n,\ j=i+1,...,n,k=1,2,...,p\end{cases} \tag{4.14}$$

Given that $\mu_{ij}^k(1+\omega_i^\mu-\omega_j^v)-\omega_i^\mu-\varepsilon_{ij}^{k+}+\varepsilon_{ij}^{k-}=0$, $v_{ij}^k(1+\omega_j^\mu-\omega_i^v)-\omega_j^\mu-\xi_{ij}^{k+}+\xi_{ij}^{k-}=0$ and $\sum_{k=1}^{p}\lambda_k=1$, it is easy to verify that

$$\begin{aligned}&\left(\sum_{k=1}^{p}\lambda_k\mu_{ij}^k\right)\left(1+\omega_i^\mu-\omega_j^v\right)-\omega_i^\mu-\sum_{k=1}^{p}\lambda_k\varepsilon_{ij}^{k+}+\sum_{k=1}^{p}\lambda_k\varepsilon_{ij}^{k-}=0\\ &\left(\sum_{k=1}^{p}\lambda_k v_{ij}^k\right)\left(1+\omega_j^\mu-\omega_i^v\right)-\omega_j^\mu-\sum_{k=1}^{p}\lambda_k\xi_{ij}^{k+}+\sum_{k=1}^{p}\lambda_k\xi_{ij}^{k-}=0\end{aligned} \tag{4.15}$$

Denote $\hat{\varepsilon}_{ij}^{+} \triangleq \sum_{k=1}^{p} \lambda_k \varepsilon_{ij}^{k+}, \hat{\varepsilon}_{ij}^{-} \triangleq \sum_{k=1}^{p} \lambda_k \varepsilon_{ij}^{k-}, \hat{\xi}_{ij}^{+} \triangleq \sum_{k=1}^{p} \lambda_k \xi_{ij}^{k+}$ and $\hat{\xi}_{ij}^{-} \triangleq \sum_{k=1}^{p} \lambda_k \xi_{ij}^{k-}$, then (4.14) can be simplified as the following linear program.

$$\min \quad J = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\hat{\varepsilon}_{ij}^{+} + \hat{\varepsilon}_{ij}^{-} + \hat{\xi}_{ij}^{+} + \hat{\xi}_{ij}^{-})$$
$$s.t. \begin{cases} \left(\sum_{k=1}^{p} \lambda_k \mu_{ij}^{k}\right)\left(1 + \omega_i^{\mu} - \omega_j^{v}\right) - \omega_i^{\mu} - \hat{\varepsilon}_{ij}^{+} + \hat{\varepsilon}_{ij}^{-} = 0, \quad i = 1, 2, ..., n, \; j = i+1, ..., n \\ \left(\sum_{k=1}^{p} \lambda_k v_{ij}^{k}\right)\left(1 + \omega_j^{\mu} - \omega_i^{v}\right) - \omega_j^{\mu} - \hat{\xi}_{ij}^{+} + \hat{\xi}_{ij}^{-} = 0, \quad i = 1, 2, ..., n, \; j = i+1, ..., n \\ 0 \le \omega_i^{\mu} \le 1, 0 \le \omega_i^{v} \le 1, \omega_i^{\mu} + \omega_i^{v} \le 1, \quad i = 1, 2, ..., n \\ \sum_{\substack{j=1 \\ j \ne i}}^{n} \omega_j^{\mu} \le \omega_i^{v}, \omega_i^{\mu} + n - 2 \ge \sum_{\substack{j=1 \\ j \ne i}}^{n} \omega_j^{v}, \quad i = 1, 2, ..., n \\ \hat{\varepsilon}_{ij}^{+} \ge 0, \hat{\varepsilon}_{ij}^{-} \ge 0, \hat{\xi}_{ij}^{+} \ge 0, \hat{\xi}_{ij}^{-} \ge 0 \quad i = 1, 2, ..., n, \; j = i+1, ..., n \end{cases} \tag{4.16}$$

Solving this model, one can obtain a unified intuitionistic fuzzy weight vector $\tilde{\omega}^{*} = (\tilde{\omega}_1^{*}, \tilde{\omega}_2^{*}, \cdots, \tilde{\omega}_n^{*})^T = ((\omega_1^{\mu*}, \omega_1^{v*}), (\omega_2^{\mu*}, \omega_2^{v*}), \cdots, (\omega_n^{\mu*}, \omega_n^{v*}))^T$ for the group of IPRs $\tilde{R}^k = (\tilde{r}_{ij}^k)_{n \times n} = ((\mu_{ij}^k, v_{ij}^k))_{n \times n}$ ($k = 1, 2, ..., p$).

5. Aggregation of intuitionistic fuzzy weights

For an MCDM problem with a hierarchical structure, let $C = \{c_1, c_2, ..., c_m\}$ be the set of upper-level criteria and $X = \{x_1, x_2, ..., x_n\}$ be the set of lower-level alternatives. Suppose the local intuitionistic fuzzy weights for criteria and alternatives have all been obtained using the proposed models in Section 4 as shown in Table 1, where $((\omega_{c_1}^{\mu}, \omega_{c_1}^{v}), (\omega_{c_2}^{\mu}, \omega_{c_2}^{v}), ..., (\omega_{c_m}^{\mu}, \omega_{c_m}^{v}))^T$ is a normalized intuitionistic fuzzy weight vector for criteria $C = \{c_1, c_2, ..., c_m\}$ and $((\omega_{1j}^{\mu}, \omega_{1j}^{v}), (\omega_{2j}^{\mu}, \omega_{2j}^{v}), ..., (\omega_{nj}^{\mu}, \omega_{nj}^{v}))^T$ is a normalized intuitionistic fuzzy weight vector for alternatives $X = \{x_1, x_2, ..., x_n\}$ with respect to the criterion c_j $(j = 1, 2, ..., m)$. According to (3.7), these weights satisfy the following normalization constraints:

$$\sum_{\substack{k=1 \\ k \ne j}}^{m} \omega_{c_k}^{\mu} \le \omega_{c_j}^{v}, \;\; \omega_{c_j}^{\mu} + m - 2 \ge \sum_{\substack{k=1 \\ k \ne j}}^{m} \omega_{c_k}^{v} \quad j = 1, 2, ..., m \tag{5.1}$$

$$\sum_{\substack{k=1 \\ k \ne i}}^{n} \omega_{kj}^{\mu} \le \omega_{ij}^{v}, \;\; \omega_{ij}^{\mu} + n - 2 \ge \sum_{\substack{k=1 \\ k \ne i}}^{n} \omega_{kj}^{v} \quad i = 1, 2, ..., n, \; j = 1, 2, ..., m \tag{5.2}$$

Table 1. Aggregation of intuitionistic fuzzy weights

Alternatives	c_1	c_2	...	c_m	Aggregated intuitionistic fuzzy weights
	$(\omega_{c_1}^{\mu}, \omega_{c_1}^{v})$	$(\omega_{c_2}^{\mu}, \omega_{c_2}^{v})$	...	$(\omega_{c_m}^{\mu}, \omega_{c_m}^{v})$	
x_1	$(\omega_{11}^{\mu}, \omega_{11}^{v})$	$(\omega_{12}^{\mu}, \omega_{12}^{v})$	...	$(\omega_{1m}^{\mu}, \omega_{1m}^{v})$	$(\omega_{x_1}^{\mu}, \omega_{x_1}^{v})$
x_2	$(\omega_{21}^{\mu}, \omega_{21}^{v})$	$(\omega_{22}^{\mu}, \omega_{22}^{v})$	...	$(\omega_{2m}^{\mu}, \omega_{2m}^{v})$	$(\omega_{x_2}^{\mu}, \omega_{x_2}^{v})$
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$
x_n	$(\omega_{n1}^{\mu}, \omega_{n1}^{v})$	$(\omega_{n2}^{\mu}, \omega_{n2}^{v})$	...	$(\omega_{nm}^{\mu}, \omega_{nm}^{v})$	$(\omega_{x_n}^{\mu}, \omega_{x_n}^{v})$

From Table 1, we understand that $\omega_{c_j}^{\mu}$ and $\omega_{c_j}^{v}$ denote the degrees of membership and non-membership of criterion c_j $(j = 1, 2, ..., m)$ as per a fuzzy concept of "importance". It is clear that the lowest importance degree of c_j is $\omega_{c_j}^{\mu}$ and the highest importance degree of c_j is $1 - \omega_{c_j}^{v}$ when all hesitation is attributed to membership. As such, the importance degree of c_j, denoted by w_j, should lie between $\omega_{c_j}^{\mu}$ and $1 - \omega_{c_j}^{v}$. Similarly, ω_{ij}^{μ} and ω_{ij}^{v} give the degrees of membership (or satisfaction) and non-membership (or dissatisfaction) of alternative x_i $(i = 1, 2, ..., n)$ on criterion c_j $(j = 1, 2, ..., m)$.

If $(w_1, w_2, ..., w_m)^T$ is a crisp weight vector normalized to 1, then $0 \le \sum_{j=1}^{m} \omega_{ij}^{\mu} w_j \le 1$, $0 \le \sum_{j=1}^{m} \omega_{ij}^{v} w_j \le 1$ and $\sum_{j=1}^{m} \omega_{ij}^{\mu} w_j + \sum_{j=1}^{m} \omega_{ij}^{v} w_j = \sum_{j=1}^{m} (\omega_{ij}^{\mu} + \omega_{ij}^{v}) w_j \le \sum_{j=1}^{m} w_j = 1$ as $0 \le \omega_{ij}^{\mu} \le 1$, $0 \le \omega_{ij}^{v} \le 1, \omega_{ij}^{\mu} + \omega_{ij}^{v} \le 1$ and $\sum_{j=1}^{m} w_j = 1$. Therefore, for each alternative x_i $(i = 1, 2, ..., n)$, its aggregated value by incorporating criterion weights can be expressed as an IFN $(z_i^{\mu}, z_i^{v}) = (\sum_{j=1}^{m} \omega_{ij}^{\mu} w_j, \sum_{j=1}^{m} \omega_{ij}^{v} w_j)$.

As the aggregated value (z_i^{μ}, z_i^{v}) reflects the overall membership and non-membership degree of alternative x_i to the fuzzy concept of "excellence", the greater the

(z_i^μ, z_i^ν), the better the alternative x_i is. Hence, a reasonable criterion weight vector $(w_1, w_2, ..., w_m)^T$ is to maximize (z_i^μ, z_i^ν).

As per (2.7) and the comparison approach for any two IFNs in Section 2, the optimal membership z_i^μ and non-membership z_i^ν of an aggregated value for alternative x_i can be obtained by solving the following two linear programs:

$$\max \quad z_i^\mu = \sum_{j=1}^{m} \omega_{ij}^\mu w_j$$
$$s.t. \begin{cases} \omega_{c_j}^\mu \le w_j \le 1 - \omega_{c_j}^\nu,\ j = 1, 2, ..., m, \\ \sum_{j=1}^{m} w_j = 1. \end{cases} \tag{5.3}$$

and

$$\min \quad z_i^\nu = \sum_{j=1}^{m} \omega_{ij}^\nu w_j$$
$$s.t. \begin{cases} \omega_{c_j}^\mu \le w_j \le 1 - \omega_{c_j}^\nu,\ j = 1, 2, ..., m, \\ \sum_{j=1}^{m} w_j = 1. \end{cases} \tag{5.4}$$

for each $i = 1, 2, ..., n$.

Solving (5.3) and (5.4) yields optimal solutions $\tilde{W}_i^\mu = (\tilde{w}_{i1}^\mu, \tilde{w}_{i2}^\mu, \cdots, \tilde{w}_{im}^\mu)^T$ and $\tilde{W}_i^\nu = (\tilde{w}_{i1}^\nu, \tilde{w}_{i2}^\nu, \cdots, \tilde{w}_{im}^\nu)^T$ (i = 1, 2, …, n), respectively.

Let

$$\tilde{\omega}_{x_i}^\mu \triangleq \sum_{j=1}^{m} \omega_{ij}^\mu \tilde{w}_{ij}^\mu, \quad \tilde{\omega}_{x_i}^\nu \triangleq \sum_{j=1}^{m} \omega_{ij}^\nu \tilde{w}_{ij}^\nu \tag{5.5}$$

for each $i = 1, 2, ..., n$.

It is obvious that $0 \le \tilde{\omega}_{x_i}^\mu \le 1$ and $0 \le \tilde{\omega}_{x_i}^\nu \le 1$. Since $\omega_{ij}^\mu \le 1 - \omega_{ij}^\nu$, we have $\tilde{\omega}_{x_i}^\mu = \sum_{j=1}^{m} \omega_{ij}^\mu \tilde{w}_{ij}^\mu \le \sum_{j=1}^{m} (1 - \omega_{ij}^\nu) \tilde{w}_{ij}^\mu = 1 - \sum_{j=1}^{m} \omega_{ij}^\nu \tilde{w}_{ij}^\mu$. On the other hand, $\tilde{W}_i^\mu = (\tilde{w}_{i1}^\mu, \tilde{w}_{i2}^\mu, \cdots, \tilde{w}_{im}^\mu)^T$ is an optimal solution of (5.3), it is also a feasible solution of (5.4) as they share the same constraints. Moreover, since $\tilde{W}_i^\nu = (\tilde{w}_{i1}^\nu, \tilde{w}_{i2}^\nu, \cdots, \tilde{w}_{im}^\nu)^T$ is an optimal solution of the minimization problem (5.4), it is thus confirmed that $\tilde{\omega}_{x_i}^\nu = \sum_{j=1}^{m} \omega_{ij}^\nu \tilde{w}_{ij}^\nu \le \sum_{j=1}^{m} \omega_{ij}^\nu \tilde{w}_{ij}^\mu$. These lead to $\tilde{\omega}_{x_i}^{\mu} + \tilde{\omega}_{x_i}^{\nu} \le 1$. Therefore, the optimal aggregated value for alternative x_i ($i = 1, 2, ..., n$) can be computed as an IFN $(\tilde{\omega}_{x_i}^{\mu}, \tilde{\omega}_{x_i}^{\nu})$.

As the criterion weight vectors $\tilde{W}_i^{\mu} = (\tilde{w}_{i1}^{\mu}, \tilde{w}_{i2}^{\mu}, \cdots, \tilde{w}_{im}^{\mu})^T$ and $\tilde{W}_i^{\nu} = (\tilde{w}_{i1}^{\nu}, \tilde{w}_{i2}^{\nu}, \cdots, \tilde{w}_{im}^{\nu})^T$ are independently determined by solving 2*n* linear programs in (5.3) and (5.4), they are generally different for distinct alternatives, i.e., $\tilde{W}_i^{\mu} \ne \tilde{W}_l^{\mu}, \tilde{W}_i^{\nu} \ne \tilde{W}_l^{\nu}$ for $i, l = 1, 2, ..., n$, $l \ne i$. Therefore, based on the different criterion weight vectors for different alternatives, the aggregated values $(\tilde{\omega}_{x_i}^{\mu}, \tilde{\omega}_{x_i}^{\nu})$ ($i = 1, 2, ..., n$) tend not to furnish a fair comparison ground for ranking alternatives or selecting the best alternative(s). To circumvent this problem, it is necessary to derive a unified criterion weight vector for all alternatives. The following procedure is introduced to accomplish this task.

(5.3) and (5.4) consider one alternative at a time. Generally, X is a non-inferior alternative set with no alternative dominating or being dominated by any other alternative. Hence, when all *n* alternatives are taken into account simultaneously, the contribution to the objective function from each individual alternative should be equally weighted as $1/n$. Therefore, in parallel to (5.3) and (5.4), the following two aggregated linear programs are established.

$$\max \quad z_0^{\mu} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m} \omega_{ij}^{\mu} w_j$$
$$s.t. \begin{cases} \omega_{c_j}^{\mu} \le w_j \le 1 - \omega_{c_j}^{\nu},\ j = 1, 2, ..., m, \\ \sum_{j=1}^{m} w_j = 1. \end{cases} \tag{5.6}$$

and

$$\min \quad z_0^{\nu} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m} \omega_{ij}^{\nu} w_j$$
$$s.t. \begin{cases} \omega_{c_j}^{\mu} \le w_j \le 1 - \omega_{c_j}^{\nu},\ j = 1, 2, ..., m, \\ \sum_{j=1}^{m} w_j = 1. \end{cases} \tag{5.7}$$

The minimization model (5.7) can be converted to an equivalent maximization linear program by multiplying its objective function with –1 as follows.

$$\max \quad z_0^{v} = -\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\omega_{ij}^{v} w_j$$
$$s.t. \begin{cases} \omega_{c_j}^{\mu} \le w_j \le 1-\omega_{c_j}^{v},\ j=1,2,...,m, \\ \sum_{j=1}^{m} w_j = 1. \end{cases} \tag{5.8}$$

Now both (5.6) and (5.8) are maximization models with the same constraints. If the two objectives are equally weighted, they can be combined as a single linear program (5.9) for obtaining a unified criterion weight vector.

$$\max \quad z = \frac{1}{2n}\sum_{i=1}^{n}\sum_{j=1}^{m}(\omega_{ij}^{\mu} - \omega_{ij}^{v}) w_j$$
$$s.t. \begin{cases} \omega_{c_j}^{\mu} \le w_j \le 1-\omega_{c_j}^{v},\ j=1,2,...,m, \\ \sum_{j=1}^{m} w_j = 1. \end{cases} \tag{5.9}$$

Denote the optimal solution of (5.9) by $W^* = (w_1^*, w_2^*, ..., w_m^*)$, and use similar notation as that for (5.5) to define:

$$\omega_{x_i}^{\mu} \triangleq \sum_{j=1}^{m} \omega_{ij}^{\mu} w_j^*,\quad \omega_{x_i}^{v} \triangleq \sum_{j=1}^{m} \omega_{ij}^{v} w_j^* \tag{5.10}$$

As $0 \le \omega_{ij}^{\mu} \le 1, 0 \le \omega_{ij}^{v} \le 1$ and $0 \le \omega_{ij}^{\mu} + \omega_{ij}^{v} \le 1$, it follows that $0 \le \omega_{x_i}^{\mu} \le 1$, $0 \le \omega_{x_i}^{v} \le 1$ and $\omega_{x_i}^{\mu} + \omega_{x_i}^{v} = \sum_{j=1}^{m}(\omega_{ij}^{\mu} + \omega_{ij}^{v}) w_j^* \le \sum_{j=1}^{m} w_j^* = 1$. Therefore, the aggregated value $(\omega_{x_i}^{\mu}, \omega_{x_i}^{v})$ for alternative x_i $(i = 1, 2, ..., n)$ based on the unified weight vector W^* constitutes an IFN.

Theorem 5.1 Assume that IFNs $(\tilde{\omega}_{x_i}^{\mu}, \tilde{\omega}_{x_i}^{v})$ and $(\omega_{x_i}^{\mu}, \omega_{x_i}^{v})$ are defined by (5.5) and (5.10), respectively, then $\tilde{\omega}_{x_i}^{\mu} \ge \omega_{x_i}^{\mu}, \tilde{\omega}_{x_i}^{v} \le \omega_{x_i}^{v}$ $(i = 1, 2, ..., n)$.

*Proof***.** Since (5.3), (5.4) and (5.9) have the same set of constraints, the optimal solution of (5.9), $W^* = (w_1^*, w_2^*, ..., w_m^*)$, is also a feasible solution of (5.3) and (5.4). Furthermore, because $\tilde{W}_i^{\mu} = (\tilde{w}_{i1}^{\mu}, \tilde{w}_{i2}^{\mu}, \cdots, \tilde{w}_{im}^{\mu})^T$ and $\tilde{W}_i^{v} = (\tilde{w}_{i1}^{v}, \tilde{w}_{i2}^{v}, \cdots, \tilde{w}_{im}^{v})^T$ are the optimal solutions of maximization model (5.3) and minimization model (5.4), respectively, it follows that

$$\tilde{\omega}_{x_i}^{\mu} = \sum_{j=1}^{m} \omega_{ij}^{\mu} \tilde{w}_{ij}^{\mu} \ge \sum_{j=1}^{m} \omega_{ij}^{\mu} w_j^* = \omega_{x_i}^{\mu} \text{ and } \tilde{\omega}_{x_i}^{v} = \sum_{j=1}^{m} \omega_{ij}^{v} \tilde{w}_{ij}^{\mu} \le \sum_{j=1}^{m} \omega_{ij}^{v} w_j^* = \omega_{x_i}^{v}.$$ ■

As per (2.7) and Theorem 5.1, we have $S((\tilde{\omega}^{\mu}_{x_i}, \tilde{\omega}^{\nu}_{x_i})) = \tilde{\omega}^{\mu}_{x_i} - \tilde{\omega}^{\nu}_{x_i} \geq \omega^{\mu}_{x_i} - \omega^{\nu}_{x_i} = S((\omega^{\mu}_{x_i}, \omega^{\nu}_{x_i}))$, indicating that, for each alternative x_i ($i = 1, 2, ..., n$), the score value of the aggregated IFN in (5.10) is always smaller than that obtained from individual models (5.3) and (5.4).

Theorem 5.2 Let IFNs $(\omega^{\mu}_{x_i}, \omega^{\nu}_{x_i})$ ($i = 1, 2, ..., n$) be defined by (5.10), then for each $i = 1, 2, ..., n$, $\sum_{\substack{k=1 \\ k \neq i}}^{n} \omega^{\mu}_{x_k} \leq \omega^{\nu}_{x_i}$ and $\omega^{\mu}_{x_i} + n - 2 \geq \sum_{\substack{k=1 \\ k \neq i}}^{n} \omega^{\nu}_{k}$.

Proof. Since $((\omega^{\mu}_{1j}, \omega^{\nu}_{1j}), (\omega^{\mu}_{2j}, \omega^{\nu}_{2j}), ..., (\omega^{\mu}_{nj}, \omega^{\nu}_{nj}))^T$ is a normalized intuitionistic fuzzy weight vector for the *n* alternatives on criterion c_j ($j = 1, 2, ..., m$), as per (5.2), for each $i = 1, 2, ..., n$, we have

$$\left(\sum_{\substack{k=1 \\ k \neq i}}^{n} \omega^{\mu}_{kj} \right) w^*_j \leq \omega^{\nu}_{ij} w^*_j \ (j = 1, 2, ...m) \text{ and } \left(\omega^{\mu}_{ij} + n - 2 \right) w^*_j \geq \left(\sum_{\substack{k=1 \\ k \neq i}}^{n} \omega^{\nu}_{kj} \right) w^*_j \ (j = 1, 2, ...m).$$

As $W^* = (w^*_1, w^*_2, ..., w^*_m)$ is a normalized crisp weight vector, by (5.10), one can obtain

$$\sum_{\substack{k=1 \\ k \neq i}}^{n} \omega^{\mu}_{x_k} = \sum_{\substack{k=1 \\ k \neq i}}^{n} \left(\sum_{j=1}^{m} \omega^{\mu}_{kj} w^*_j \right) = \sum_{j=1}^{m} \left(\left(\sum_{\substack{k=1 \\ k \neq i}}^{n} \omega^{\mu}_{kj} \right) w^*_j \right) \leq \sum_{j=1}^{m} \omega^{\nu}_{ij} w^*_j = \omega^{\nu}_{x_i}$$

and

$$\omega^{\mu}_{x_i} + n - 2 = \sum_{j=1}^{m} \omega^{\mu}_{ij} w^*_j + n - 2 = \sum_{j=1}^{m} (\omega^{\mu}_{ij} + n - 2) w^*_j \geq \sum_{j=1}^{m} \left(\left(\sum_{\substack{k=1 \\ k \neq i}}^{n} \omega^{\nu}_{kj} \right) w^*_j \right) = \sum_{\substack{k=1 \\ k \neq i}}^{n} \left(\sum_{j=1}^{m} \omega^{\nu}_{kj} w^*_j \right) = \sum_{\substack{k=1 \\ k \neq i}}^{n} \omega^{\nu}_{x_k}$$

The proof of Theorem 5.2 is thus completed. ■

Theorem 5.2 demonstrates that the aggregated IFN values derived from model (5.9) are normalized intuitionistic fuzzy weights.

6. Numerical examples

This section presents two numerical examples to illustrate how the proposed models are applied to an individual decision situation with IPRs as well as a group decision problem with a hierarchical structure.

Example 1. Assume that a DM provides the following IPR on an alternative set $X = \{x_1, x_2, x_3, x_4\}$.

$$\tilde{R} = (\tilde{r}_{ij})_{4\times4} = ((\mu_{ij}, v_{ij})_{4\times4} = \begin{bmatrix} (0.5, 0.5) & (1/3, 2/3) & (1/5, 4/5) & (1/4, 3/4) \\ (2/3, 1/3) & (0.5, 0.5) & (1/3, 2/3) & (2/5, 3/5) \\ (4/5, 1/5) & (2/3, 1/3) & (0.5, 0.5) & (4/7, 3/7) \\ (3/4, 1/4) & (3/5, 2/5) & (3/7, 4/7) & (0.5, 0.5) \end{bmatrix}$$

In $\tilde{R}$, the diagonal elements are always (0.5, 0.5), indicating the DM's indifference between any alternative and itself. The cells off the diagonal represent the DM's pairwise comparison result between two alternatives. For instance, $\tilde{r}_{12} = (1/3, 2/3)$ denotes a degree of 1/3 to which alternative x_1 is preferred to x_2, and a degree of 2/3 to which alternative x_1 is non-preferred to x_2. The remaining elements in $\tilde{R}$ can be interpreted in a similar fashion.

By plugging $\tilde{R}$ into (4.8), one can obtain the optimal objective function value $J^* = 0$, and the corresponding optimal intuitionistic fuzzy weight vector as:

$$\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_4)^T = ((0.1, 0.9), (0.2, 0.8), (0.4, 0.6), (0.3, 0.7))^T$$

As $J^* = 0$, $\tilde{R}$ is multiplicative consistent. According to (2.7), one has

$$S(\tilde{\omega}_1) = -0.8, S(\tilde{\omega}_2) = -0.6, S(\tilde{\omega}_3) = -0.2, S(\tilde{\omega}_4) = -0.4$$

Since $S(\tilde{\omega}_3) > S(\tilde{\omega}_4) > S(\tilde{\omega}_2) > S(\tilde{\omega}_1)$, the ranking order of the four alternatives is $x_3 \succ x_4 \succ x_2 \succ x_1$.

Next, Algorithm (I) developed by Xu [41] will be applied to the same IPR $\tilde{R}$ and the ranking result will be compared with our proposed approach.

According to Algorithm (I) (n = 4, m = 1) in [41], a priority vector is obtained as $((0.3312, 0.6688), (0.4919, 0.5081), (0.6543, 0.3457), (0.5889, 0.4111))^T$. Based on the comparison method for IFNs in Section 2, one has $x_3 \succ x_4 \succ x_2 \succ x_1$.

It is worth noting that this priority vector does not satisfy the intuitionistic fuzzy weight normalization condition (3.7) as $\omega_1^\mu + \omega_2^\mu + \omega_3^\mu = 1.4774 > 0.4111 = \omega_4^v$. If the derived priority weight vector is the evaluation result for eliciting final ranking, it does not matter whether it is normalized. However, if this priority weight vector will be used as decision input for further aggregation such as the priority weights for alternatives against criteria in the hierarchical decision structure in Section 5, it is important to normalize the priority weights so that heterogeneous dimension problems can be avoided.

Xu [44] presents an error-analysis-based method to obtain interval priority weights for both consistent and inconsistent IPRs. By employing Eqs. (13) and (15) in [44], an interval priority weight vector is obtained as: $([0.1903, 0.1903), [0.2417, 0.2417], [0.2948, 0.2948], [0.2732, 0.2732])^T$, which is equivalent to an IFN vector:

$$((0.1903, 0.8097), (0.2417, 0.7583), (0.2948, 0.0.7052), (0.2732, 0.7268))^T$$

As per the ranking approach in [44], the four alternatives are ranked as: $x_3 \succ x_4 \succ x_2 \succ x_1$.

Gong et al. [13] propose a linear programming model to derive an interval priority weight vector from IPRs. These interval weights are then used for ranking alternatives.

Using linear program (21) in [13], the optimal interval weight vector is obtained as $([0.1, 0.1], [0.2, 0.2], [0.4, 0.4], [0.3, 0.3])^T$, which can be expressed in an IFN form as $((0.1, 0.9), (0.2, 0.8), (0.4, 0.6), (0.3, 0.7))^T$. According to the IFN comparison method in Section 2, one has $x_3 \succ x_4 \succ x_2 \succ x_1$.

On the other hand, since $\mu_{ij} + v_{ij} = 1$ for all *i*, *j* = 1, 2, 3, 4, $\tilde{R}$ is equivalent to the following fuzzy preference relation.

$$R = (r_{ij})_{4\times 4} = \begin{bmatrix} 0.5 & 1/3 & 1/5 & 1/4 \\ 2/3 & 0.5 & 1/3 & 2/5 \\ 4/5 & 2/3 & 0.5 & 4/7 \\ 3/4 & 3/5 & 3/7 & 0.5 \end{bmatrix}$$

As per Definition 2.1, this is a multiplicative consistent fuzzy preference relation. Next, a comparative study is conducted for the proposed method herein and another approach to generating priority weights for multiplicative consistent fuzzy preference relations in [42].

According to Theorem 9 in [42], $R = (r_{ij})_{4\times 4}$ can be transformed into an equivalent multiplicative consistent preference relation $P = (p_{ij})_{4\times 4}$ with $p_{ij} = r_{ij} / r_{ji}$.

$$P = (p_{ij})_{4\times 4} = \begin{bmatrix} 1 & 1/2 & 1/4 & 1/3 \\ 2 & 1 & 1/2 & 2/3 \\ 4 & 2 & 1 & 4/3 \\ 3 & 3/2 & 3/4 & 1 \end{bmatrix}$$

As per Eq. (9) in [42], the priority weight vector derived from P is computed as $W = (1/\sum_{i=1}^{4} p_{i1}, 1/\sum_{i=1}^{4} p_{i2}, 1/\sum_{i=1}^{4} p_{i3}, 1/\sum_{i=1}^{4} p_{i4})^T = (0.1, 0.2, 0.4, 0.3)^T$, which is equivalent to an intuitionistic fuzzy weight vector $((0.1,0.9),(0.2,0.8),(0.4,0.6),(0.3,0.7))^T$. Hence, the ranking of all alternatives is $x_3 \succ x_4 \succ x_2 \succ x_1$.

The intuitionistic fuzzy priority weight vectors and ranking results based on the models in Xu [41, 42, 44], Gong et al. [13] and our approach are summarized in Table 2.

Table 2. A comparative study for the intuitionistic preference relation $\tilde{R}$

Model	Reference	Priority weight vector $(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)^T$	Ranking
Algorithm (I)	Xu [41]	$((0.3312, 0.6688), (0.4919, 0.5081), (0.6543, 0.3457), (0.5889, 0.4111))^T$	$x_3 \succ x_4 \succ x_2 \succ x_1$
Eqs. (13) and (15)	Xu [44]	$((0.1903, 0.8097), (0.2417, 0.7583), (0.2948, 0.0.7052), (0.2732, 0.7268))^T$	$x_3 \succ x_4 \succ x_2 \succ x_1$
(21)	Gong et al. [13]	$((0.1,0.9),(0.2,0.8),(0.4,0.6),(0.3,0.7))^T$	$x_3 \succ x_4 \succ x_2 \succ x_1$
Theorem 9 and Eq. (9)	Xu [42]	$((0.1,0.9),(0.2,0.8),(0.4,0.6),(0.3,0.7))^T$	$x_3 \succ x_4 \succ x_2 \succ x_1$
(4.8)	This article	$((0.1,0.9),(0.2,0.8),(0.4,0.6),(0.3,0.7))^T$	$x_3 \succ x_4 \succ x_2 \succ x_1$

Table 2 demonstrates that the ranking results based on the five different approaches are identical although the priority weight vectors obtained from the models in Xu [41, 44] differ from the results derived from the remaining three methods. For this degenerated fuzzy preference relation, the proposed approach in this article yields the same priority weights as those obtained from the models in Xu [42] and Gong et al. [13]. In our opinion, the difference in the derived priority weight vectors is due to the fact that the models in Xu [41, 44] employ different aggregation schemes and do not incorporate the normalization constraints. Furthermore, Xu's method [42] can only be applied to multiplicative consistent fuzzy preference relations. Compared to the proposed model in this article, the linear program in Gong et al. [13] need more constraints and decision variables.

Example 2. This example is adapted from [47]. Consider a two-level group decision problem with a hierarchical structure. A core enterprise has to select its supply chain partner for spare parts. The partner selection decision is made based on the following five main criteria: product quality (c_1), cost and delivery time (c_2), supplier flexibility and responsiveness (c_3), financial status (c_4), and trust and information sharing (c_5).

The upper-level concern of this core enterprise is to generate a weighting scheme for these five criteria. At the lower level, the selection committee is responsible for assessing spare parts suppliers based on these criterion weights. The hierarchical structure of this supply chain partner selection problem is shown in Fig. 2.

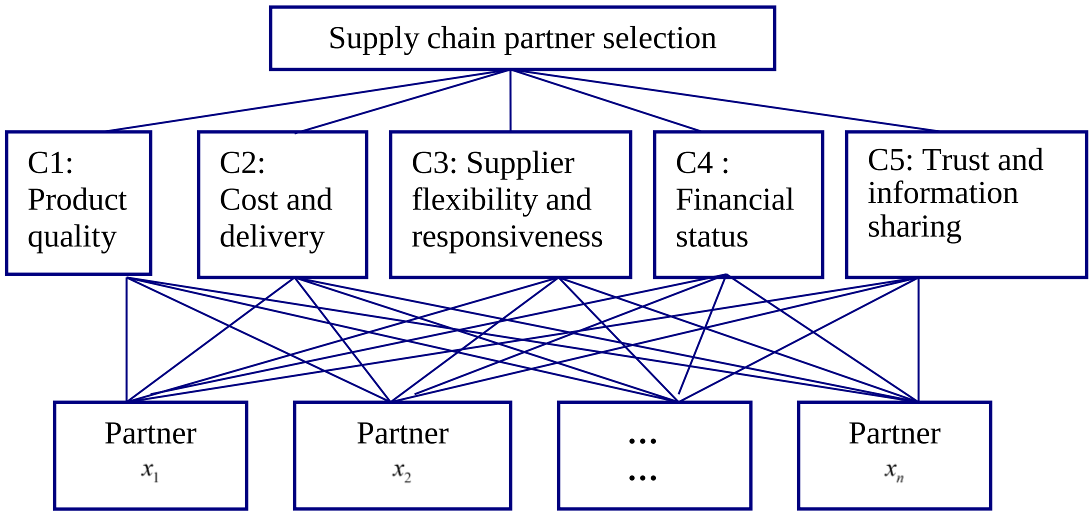

Fig. 2 A hierarchical structure of a supply chain partner selection problem

Assume that an upper level committee consisting of four senior executives is set up to generate a weighting scheme for the five criteria, and the executive weights are 0.4, 0.3, 0.2 and 0.1, respectively. Each executive is required to furnish his/her pairwise comparisons for the five criteria as an IPR as shown in Table 3.

By employing the linear program (4.16), one can get the optimal objective function value $J^* = 0.3491995$, and an optimal criterion weight vector as

$$((0.3026, 0.6468), (0.1987, 0.7508), (0.1222, 0.8273), (0.1255, 0.8311), (0.0910, 0.8935))^T .$$

Based on these criterion weights, five potential suppliers, denoted by x_1, x_2, x_3, x_4 and x_5, are assessed by a lower level committee. Assume that three managers are involved in the assessment and each manager carries the same weight in the partner selection process. The IPR assessments for the five potential partners with respect to each criterion are summarized in Tables 4-8.

Table 3. Intuitionistic preference relations for the four executives on the five criteria

Expert	Criteria	c_1	c_2	c_3	c_4	c_5
#1	c_1	(0.50,0.50)	(0.70,0.20)	(0.65,0.25)	(0.40,0.40)	(0.60,0.25)
	c_2	(0.20,0.70)	(0.50,0.50)	(0.55,0.40)	(0.50,0.45)	(0.70,0.20)
	c_3	(0.25,0.65)	(0.40,0.55)	(0.50,0.50)	(0.65,0.25)	(0.55,0.35)
	c_4	(0.40,0.40)	(0.45,0.50)	(0.25,0.65)	(0.50,0.50)	(0.55,0.40)
	c_5	(0.25,0.60)	(0.20,0.70)	(0.35,0.55)	(0.40,0.55)	(0.50,0.50)
#2	c_1	(0.50,0.50)	(0.60,0.30)	(0.75,0.15)	(0.60,0.30)	(0.70,0.20)
	c_2	(0.30,0.60)	(0.50,0.50)	(0.50,0.30)	(0.55,0.30)	(0.65,0.25)
	c_3	(0.15,0.75)	(0.30,0.50)	(0.50,0.50)	(0.50,0.45)	(0.60,0.30)
	c_4	(0.30,0.60)	(0.30,0.55)	(0.45,0.50)	(0.50,0.50)	(0.55,0.25)
	c_5	(0.20,0.70)	(0.25,0.65)	(0.30,0.60)	(0.25,0.55)	(0.50,0.50)
#3	c_1	(0.50,0.50)	(0.50,0.30)	(0.53,0.35)	(0.65,0.30)	(0.55,0.25)
	c_2	(0.30,0.50)	(0.50,0.50)	(0.50,0.30)	(0.65,0.20)	(0.62,0.30)
	c_3	(0.35,0.53)	(0.30,0.50)	(0.50,0.50)	(0.65,0.30)	(0.60,0.40)
	c_4	(0.30,0.65)	(0.20,0.65)	(0.30,0.65)	(0.50,0.50)	(0.52,0.45)
	c_5	(0.25,0.55)	(0.30,0.62)	(0.40,0.60)	(0.45,0.52)	(0.50,0.50)
#4	c_1	(0.50,0.50)	(0.45,0.52)	(0.55,0.42)	(0.52,0.30)	(0.54,0.25)
	c_2	(0.52,0.45)	(0.50,0.50)	(0.65,0.10)	(0.60,0.25)	(0.52,0.30)
	c_3	(0.42,0.55)	(0.10,0.65)	(0.50,0.50)	(0.65,0.25)	(0.65,0.35)
	c_4	(0.30,0.52)	(0.25,0.60)	(0.25,0.65)	(0.50,0.50)	(0.52,0.25)
	c_5	(0.25,0.54)	(0.30,0.52)	(0.35,0.65)	(0.25,0.52)	(0.50,0.50)

Table 4. IPRs for the five potential partners with respect to c_1

Expert	Candidate	x_1	x_2	x_3	x_4	x_5
#1	x_1	(0.50,0.50)	(0.55,0.30)	(0.46,0.40)	(0.48,0.40)	(0.50,0.30)
	x_2	(0.30,0.55)	(0.50,0.50)	(0.36,0.50)	(0.40,0.50)	(0.60,0.35)
	x_3	(0.40,0.46)	(0.50,0.36)	(0.50,0.50)	(0.42,0.40)	(0.65,0.28)
	x_4	(0.40,0.48)	(0.50,0.40)	(0.40,0.42)	(0.50,0.50)	(0.70,0.25)
	x_5	(0.30,0.50)	(0.35,0.60)	(0.28,0.65)	(0.25,0.70)	(0.50,0.50)
#2	x_1	(0.50,0.50)	(0.65,0.30)	(0.55,0.35)	(0.52,0.32)	(0.55,0.35)
	x_2	(0.30,0.65)	(0.50,0.50)	(0.25,0.60)	(0.35,0.60)	(0.58,0.30)
	x_3	(0.35,0.55)	(0.60,0.25)	(0.50,0.50)	(0.55,0.30)	(0.75,0.20)
	x_4	(0.32,0.52)	(0.60,0.35)	(0.30,0.55)	(0.50,0.50)	(0.80,0.15)
	x_5	(0.35,0.55)	(0.30,0.58)	(0.20,0.75)	(0.15,0.80)	(0.50,0.50)
#3	x_1	(0.50,0.50)	(0.62,0.30)	(0.48,0.40)	(0.45,0.40)	(0.52,0.35)
	x_2	(0.30,0.62)	(0.50,0.50)	(0.30,0.60)	(0.40,0.50)	(0.58,0.32)
	x_3	(0.40,0.48)	(0.60,0.30)	(0.50,0.50)	(0.45,0.50)	(0.62,0.28)
	x_4	(0.40,0.45)	(0.50,0.40)	(0.50,0.45)	(0.50,0.50)	(0.72,0.18)
	x_5	(0.35,0.52)	(0.32,0.58)	(0.28,0.62)	(0.18,0.72)	(0.50,0.50)

Table 5. IPRs for the five potential partners with respect to c_2

Expert	Candidate	x_1	x_2	x_3	x_4	x_5
#1	x_1	(0.50,0.50)	(0.60,0.24)	(0.62,0.30)	(0.58,0.25)	(0.45,0.25)
	x_2	(0.24,0.60)	(0.50,0.50)	(0.34,0.52)	(0.32,0.55)	(0.62,0.32)
	x_3	(0.30,0.62)	(0.52,0.34)	(0.50,0.50)	(0.56,0.28)	(0.60,0.20)
	x_4	(0.25,0.58)	(0.55,0.32)	(0.28,0.56)	(0.50,0.50)	(0.72,0.15)
	x_5	(0.25,0.45)	(0.32,0.62)	(0.20,0.60)	(0.15,0.72)	(0.50,0.50)
#2	x_1	(0.50,0.50)	(0.25,0.50)	(0.30,0.55)	(0.25,0.65)	(0.25,0.45)
	x_2	(0.50,0.25)	(0.50,0.50)	(0.35,0.50)	(0.38,0.48)	(0.38,0.40)
	x_3	(0.55,0.30)	(0.50,0.35)	(0.50,0.50)	(0.46,0.30)	(0.55,0.30)
	x_4	(0.65,0.25)	(0.48,0.38)	(0.30,0.46)	(0.50,0.50)	(0.58,0.20)
	x_5	(0.45,0.25)	(0.40,0.38)	(0.30,0.55)	(0.20,0.58)	(0.50,0.50)
#3	x_1	(0.50,0.50)	(0.30,0.62)	(0.32,0.58)	(0.15,0.70)	(0.40,0.52)
	x_2	(0.62,0.30)	(0.50,0.50)	(0.46,0.54)	(0.36,0.56)	(0.45,0.35)
	x_3	(0.58,0.32)	(0.54,0.46)	(0.50,0.50)	(0.30,0.58)	(0.50,0.40)
	x_4	(0.70,0.15)	(0.56,0.36)	(0.58,0.30)	(0.50,0.50)	(0.58,0.28)
	x_5	(0.52,0.40)	(0.35,0.45)	(0.40,0.50)	(0.28,0.58)	(0.50,0.50)

Table 6. IPRs for the five potential partners with respect to c_3

Expert	Candidate	x_1	x_2	x_3	x_4	x_5
#1	x_1	(0.50,0.50)	(0.35,0.50)	(0.25,0.55)	(0.18,0.65)	(0.35,0.45)
	x_2	(0.50,0.35)	(0.50,0.50)	(0.35,0.58)	(0.27,0.60)	(0.55,0.30)
	x_3	(0.55,0.25)	(0.58,0.35)	(0.50,0.50)	(0.25,0.45)	(0.65,0.25)
	x_4	(0.65,0.18)	(0.60,0.27)	(0.45,0.25)	(0.50,0.50)	(0.40,0.30)
	x_5	(0.45,0.35)	(0.30,0.55)	(0.25,0.65)	(0.30,0.40)	(0.50,0.50)
#2	x_1	(0.50,0.50)	(0.38,0.50)	(0.28,0.55)	(0.18,0.72)	(0.45,0.25)
	x_2	(0.50,0.38)	(0.50,0.50)	(0.38,0.52)	(0.30,0.60)	(0.55,0.45)
	x_3	(0.55,0.28)	(0.52,0.38)	(0.50,0.50)	(0.38,0.52)	(0.40,0.50)
	x_4	(0.72,0.18)	(0.60,0.30)	(0.52,0.38)	(0.50,0.50)	(0.46,0.24)
	x_5	(0.25,0.45)	(0.45,0.55)	(0.50,0.40)	(0.24,0.46)	(0.50,0.50)
#3	x_1	(0.50,0.50)	(0.50,0.40)	(0.52,0.28)	(0.60,0.20)	(0.52,0.38)
	x_2	(0.40,0.50)	(0.50,0.50)	(0.50,0.40)	(0.54,0.36)	(0.40,0.45)
	x_3	(0.28,0.52)	(0.40,0.50)	(0.50,0.50)	(0.56,0.24)	(0.40,0.50)
	x_4	(0.20,0.60)	(0.36,0.54)	(0.24,0.56)	(0.50,0.50)	(0.35,0.55)
	x_5	(0.38,0.52)	(0.45,0.40)	(0.50,0.40)	(0.55,0.35)	(0.50,0.50)

Table 7. IPRs for the five potential partners with respect to c_4

Expert	Candidate	x_1	x_2	x_3	x_4	x_5
#1	x_1	(0.50,0.50)	(0.58,0.32)	(0.36,0.44)	(0.32,0.48)	(0.56,0.34)
	x_2	(0.32,0.58)	(0.50,0.50)	(0.46,0.40)	(0.32,0.58)	(0.65,0.25)
	x_3	(0.44,0.36)	(0.40,0.46)	(0.50,0.50)	(0.48,0.40)	(0.68,0.22)
	x_4	(0.48,0.32)	(0.58,0.32)	(0.40,0.48)	(0.50,0.50)	(0.76,0.14)
	x_5	(0.34,0.56)	(0.25,0.65)	(0.22,0.68)	(0.14,0.76)	(0.50,0.50)
#2	x_1	(0.50,0.50)	(0.45,0.35)	(0.40,0.30)	(0.42,0.46)	(0.56,0.34)
	x_2	(0.35,0.45)	(0.50,0.50)	(0.35,0.55)	(0.38,0.52)	(0.52,0.38)
	x_3	(0.30,0.40)	(0.55,0.35)	(0.50,0.50)	(0.58,0.28)	(0.78,0.12)
	x_4	(0.46,0.42)	(0.52,0.38)	(0.28,0.58)	(0.50,0.50)	(0.72,0.20)
	x_5	(0.34,0.56)	(0.38,0.52)	(0.12,0.78)	(0.20,0.72)	(0.50,0.50)
#3	x_1	(0.50,0.50)	(0.46,0.34)	(0.42,0.48)	(0.35,0.55)	(0.68,0.22)
	x_2	(0.34,0.46)	(0.50,0.50)	(0.48,0.52)	(0.42,0.48)	(0.60,0.30)
	x_3	(0.48,0.42)	(0.52,0.48)	(0.50,0.50)	(0.47,0.43)	(0.74,0.16)
	x_4	(0.55,0.35)	(0.48,0.42)	(0.43,0.47)	(0.50,0.50)	(0.78,0.12)
	x_5	(0.22,0.68)	(0.30,0.60)	(0.16,0.74)	(0.12,0.78)	(0.50,0.50)

Table 8. IPRs for the five potential partners with respect to c_5

Expert	Candidate	x_1	x_2	x_3	x_4	x_5
#1	x_1	(0.50,0.50)	(0.55,0.35)	(0.30,0.60)	(0.40,0.45)	(0.48,0.42)
	x_2	(0.35,0.55)	(0.50,0.50)	(0.20,0.70)	(0.35,0.55)	(0.45,0.50)
	x_3	(0.60,0.30)	(0.70,0.20)	(0.50,0.50)	(0.68,0.22)	(0.75,0.20)
	x_4	(0.45,0.40)	(0.55,0.35)	(0.22,0.68)	(0.50,0.50)	(0.55,0.25)
	x_5	(0.42,0.48)	(0.50,0.45)	(0.20,0.75)	(0.25,0.55)	(0.50,0.50)
#2	x_1	(0.50,0.50)	(0.48,0.40)	(0.30,0.60)	(0.25,0.70)	(0.35,0.52)
	x_2	(0.40,0.48)	(0.50,0.50)	(0.42,0.48)	(0.35,0.55)	(0.55,0.35)
	x_3	(0.60,0.30)	(0.48,0.42)	(0.50,0.50)	(0.46,0.34)	(0.58,0.22)
	x_4	(0.70,0.25)	(0.55,0.35)	(0.34,0.46)	(0.50,0.50)	(0.65,0.25)
	x_5	(0.52,0.35)	(0.35,0.55)	(0.22,0.58)	(0.25,0.65)	(0.50,0.50)
#3	x_1	(0.50,0.50)	(0.56,0.34)	(0.48,0.42)	(0.40,0.50)	(0.32,0.58)
	x_2	(0.34,0.56)	(0.50,0.50)	(0.42,0.48)	(0.26,0.64)	(0.34,0.56)
	x_3	(0.42,0.48)	(0.48,0.42)	(0.50,0.50)	(0.42,0.46)	(0.46,0.44)
	x_4	(0.50,0.40)	(0.64,0.26)	(0.46,0.42)	(0.50,0.50)	(0.58,0.22)
	x_5	(0.58,0.32)	(0.56,0.34)	(0.44,0.46)	(0.22,0.58)	(0.50,0.50)

Similarly, by using model (4.16), a normalized intuitionistic fuzzy weight vector for alternative x_i with respect to criterion c_j (i, j=1, 2, …, 5) can be obtained as shown in columns 1-5 in Table 9, where the first row lists the upper level criterion weights obtained earlier.

Table 9. Intuitionistic fuzzy weights for alternatives under each criterion and the aggregated intuitionistic fuzzy assessments.

Candidate	c_1 (0.3026,0.6468)	c_2 (0.1987,0.7508)	c_3 (0.1222,0.8273)	c_4 (0.1255,0.8311)	c_5 (0.0910,0.8935)	Aggregated intuitionistic fuzzy weights
x_1	(0.2359,0.7007)	(0.1285,0.8124)	(0.1273,0.7968)	(0.1669,0.7482)	(0.1445,0.8111)	(0.1727,0.7621)
x_2	(0.1283,0.8440)	(0.1555,0.8099)	(0.1778,0.8222)	(0.1695,0.8203)	(0.1263,0.8378)	(0.1484,0.8283)
x_3	(0.2040,0.7326)	(0.2059,0.7498)	(0.1778,0.7441)	(0.1726,0.7425)	(0.2271,0.7285)	(0.1985,0.7396)
x_4	(0.1783,0.7584)	(0.2143,0.7351)	(0.1730,0.7296)	(0.2000,0.7155)	(0.2239,0.7317)	(0.1937,0.7396)
x_5	(0.0745,0.9010)	(0.1072,0.8337)	(0.1186,0.8099)	(0.0515,0.8887)	(0.1091,0.8465)	(0.0908,0.8616)

Plugging these normalized intuitionistic fuzzy assessments and criterion weights into (5.9), the following linear program is established.

$$\begin{aligned} &\max \quad z = (-3.1157w_1 - 3.1295w_2 - 3.1281w_3 - 3.1547w_4 - 3.1247w_5)/10 \\ &s.t. \quad \begin{cases} 0.3026 \le w_1 \le 0.3532, 0.1987 \le w_2 \le 0.2492, 0.1222 \le w_3 \le 0.1727, \\ 0.1255 \le w_4 \le 0.1689, 0.091 \le w_5 \le 0.1065, w_1 + w_2 + w_3 + w_4 + w_5 = 1. \end{cases} \end{aligned}$$

Solving this linear program yields an optimal solution as:

$$W^* = (w_1^*, w_2^*, w_3^*, w_4^*, w_5^*)^T = (0.3532,0.2421,0.1727,0.1255,0.1065)^T$$

By applying (5.10), one can obtain the aggregated intuitionistic fuzzy weight $(\omega_{x_i}^{\mu}, \omega_{x_i}^{\nu})$ for each alternative x_i (i=1, 2, …, 5) as shown in the last column of Table 9.

As per (2.7), the score function value is calculated for each aggregated weight as $S((\omega_{x_1}^{\mu}, \omega_{x_1}^{\nu})) = \text{-}0.5894, S((\omega_{x_2}^{\mu}, \omega_{x_2}^{\nu})) = \text{-}0.6799, S((\omega_{x_3}^{\mu}, \omega_{x_3}^{\nu})) = \text{-}0.5411, S((\omega_{x_4}^{\mu}, \omega_{x_4}^{\nu})) = \text{-}0.5459,$ $S((\omega_{x_5}^{\mu}, \omega_{x_5}^{\nu})) = \text{-}0.7708$. By using the IFN comparison method in Section 2, a full ranking of the five potential suppliers is derived as $x_3 \succ x_4 \succ x_1 \succ x_2 \succ x_5$.

7. Conclusions

This article is concerned with individual and group decisions with IPRs. The key modeling idea is to establish a goal programming framework for deriving intuitionistic fuzzy weights. The research starts with introducing an innovative multiplicative consistency definition for IPRs. By examining the inherent link between intuitionistic fuzzy weights and multiplicative consistency of IPRs, a transformation formula is put forward to convert normalized intuitionistic fuzzy weights into multiplicative consistent IPRs. Then deviation variables are defined to gauge the difference between a DM's original judgment and its converted multiplicative consistent IPR, thereby two linear goal programs are proposed to obtain intuitionistic fuzzy weights from IPRs for both individual and group decision problems. Subsequently, a linear program is established to obtain a unified criterion weight vector for MCDM with a hierarchical structure, these weights are then employed to aggregate local intuitionistic fuzzy weights into global priority weights. Finally, two numerical examples are presented to show how the proposed models can be applied.

The research reported in this article can be further extended along a number of lines. For instance, if the DM can accept limited inconsistency, a worthy topic is to examine acceptable multiplicative consistency, thereby developing decision models with acceptable multiplicative consistent IPRs. Another potential research problem is to investigate how to rectify multiplicative inconsistency and improve consistency for IPRs.

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