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**An acceptable consistency-based framework for group decision making with
intuitionistic preference relations**

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An acceptable consistency-based framework for group decision making with intuitionistic preference relations

Abstract

This article studies acceptable consistency of intuitionistic preference relations (IPRs) and examines how to aggregate individual IPRs into a collective judgment in a group decision making (GDM) context. A consistency index is first introduced to measure the consistency level, thereby defining acceptable consistency for IPRs. If a decision-maker (DM) is unwilling or unavailable to revise his/her judgment for an IPR with unacceptable consistency, an autonomous approach is developed to improve its consistency to an acceptable level. The acceptably consistent IPRs are subsequently aggregated into a group opinion by using an induced ordered weighted averaging (IOWA) operator. A procedure is then proposed to solve GDM problems with IPRs. An illustrative example is presented to demonstrate the effectiveness and applicability of the proposed approach.

Keywords: Intuitionistic preference relation (IPR); Acceptable consistency; Weighted averaging; Aggregation; Group decision making

1. Introduction

Group decision making (GDM) is a decision process involving multiple decision-makers (DMs) and its aim is to derive a ranking for or select the best one(s) from a finite set of feasible alternatives. Each DM often expresses his/her judgment as a preference relation via pairwise comparison over decision alternatives. A classical preference relation introduced by Saaty (1980) is the multiplicative preference relation in the analytic hierarchy process (AHP), which satisfies multiplicative reciprocity.

A preference value in a classical multiplicative preference relation denotes a precise ratio to which an alternative is superior to another. Due to complexity and uncertainty of decision problems, a DM is often unable to furnish a crisp preference of one alternative over another. To deal with such vagueness, the concept of fuzzy preference relations was put forward where the preference values are additively reciprocal fuzzy numbers and represent the membership degree of one alternative being preferred to another (Herrera-Viedma et al., 2004). To further characterize a DM's hesitation in offering his/her fuzzy

preference value, intuitionistic fuzzy sets (IFSs) (Atanassov, 1986) were employed for modelling preference (Xu, 2007b; Xu 2013; Xu and Cai 2014). Along this line, Xu (2007b) introduced the concept of intuitionistic preference relations (IPRs), in which preference ratings are characterized by intuitionistic fuzzy numbers (IFNs) (Xu & Yager, 2006), and the hesitancy margin of a DM's judgment can thus be conveniently characterized by an intuitionistic fuzzy index.

Recent research started addressing consistency and priority derivation of IPRs. By employing IFN operations (Xu, 2007a), Xu (2007b) defined multiplicative transitivity of IPRs, and proposed an approach to aggregate intuitionistic preference information based on an intuitionistic fuzzy weighted averaging operator (IFWA). Xu et al. (2011) noted the deficiency of the multiplicative transitivity condition in Xu (2007b), and introduced a new definition of multiplicative consistent IPRs. Based on this multiplicative consistency, an iterative algorithm was developed by Xu and Xia (2014) to improve the consistency level of an inconsistent IPR. Xu and Liao (2014) developed an intuitionistic fuzzy AHP method. By establishing an equivalent representation between an IPR and an interval fuzzy preference relation, Gong et al. (2009) presented an alternative definition for multiplicative consistent IPRs, and put forward two goal programming models to derive priority weights. Gong et al. (2011) further introduced an additive consistency definition for IPRs and proposed a goal programming and a least squares model to obtain priority weights for IPRs. Given that the additive consistency conditions in Gong et al. (2011) are defined without accounting for transitivity among three or more DM's judgment data and the resulting matrix from the consistency conversion formulas therein does not always yield an IPR, Wang (2013) employed membership degrees of intuitionistic judgments to define additive transitivity of IPRs and investigated the derivation of intuitionistic fuzzy priority weights based on IPRs by establishing goal programming models for both individual and group decision situations.

A critical issue in GDM with IPRs is to aggregate individual preferences into a group judgment. To this end, different intuitionistic aggregation operators have been developed. For instance, Xu (2007a) first introduced operational rules for IFNs and, then, put forth intuitionistic fuzzy weighted averaging (IFWA), intuitionistic fuzzy ordered weighted averaging (IFOWA), and intuitionistic fuzzy hybrid averaging (IFHA) operators. Su et al.

(2012) defined an induced generalized intuitionistic fuzzy ordered weighted averaging (IG-IFOWA) operator. Wei (2010) proposed an induced intuitionistic fuzzy ordered weighted geometric (IIFOWG) operator by extending the traditional induced ordered weighted averaging (IOWA) operator (Yager and Filev, 1999). A common feature of these aggregation operators is their unbalanced treatment of IFNs' membership and non-membership degrees. As such, these aggregation operators do not necessarily possess the desired monotonicity in terms of the ranking order based on score and accuracy functions (Beliakov et al., 2011). Therefore, when these operators are applied to aggregate individual preferences, they sometimes yield a group IPR with low consistency even if all individual IPRs possess high consistency.

Another challenge in GDM with IPRs is that DMs often provide preference relations without sufficient consistency. It is understandable that unacceptably inconsistent preference relations tend to result in unreasonable ranking of alternatives. In this case, the most reliable way is to return the inconsistent judgments to the DMs for an update. But this process can be tedious and time-consuming. In case that the DM is unwilling or unavailable to revise his/her judgments, it is necessary to first improve the consistency of the preference relation to an acceptable level. To address these issues, this paper first defines a consistency index to measure the consistency level of IPRs. By directly employing the membership information in an inconsistent IPR, a method is proposed to construct an additively consistent IPR. This constructed consistent IPR is then employed as a benchmark to rectify the consistency of the original IPR to an acceptable level. By employing the weighted average of the consistency level of the resulting acceptably consistent IPR and the mean absolute deviations (MADs) between the original and resulting IPRs as an order inducing value of the IOWA operator, the individual acceptably consistent IPRs are subsequently aggregated into a group acceptably consistent IPR. A process and algorithm are further proposed to solve GDM problems with IPRs where the weights of DMs are unknown.

The remainder of this paper is organized as follows. Section 2 reviews basic concepts of IFSs and consistent IPRs. A consistency index and acceptable consistency are defined for IPRs in Section 3. Section 4 addresses how to improve consistency of IPRs. In Section 5, an approach is developed for solving GDM problems with IPRs. Section 6

114 presents a practical example to illustrate the proposed models. Finally, concluding
115 remarks are given in Section 7.

116 2. Preliminaries

117 This section presents basic concepts with regard to IFSs and consistent IPRs.

118 *Definition 2.1* (Atanassov, 1986) Let Z be a fixed set, an IFS A in Z is defined as an
119 object in the following form:

$$120 \quad A = \{ \langle z, \mu_A(z), \nu_A(z) \rangle \mid z \in Z \} \quad (2.1)$$

121 where the functions $\mu_A(z)$ and $\nu_A(z)$ denote the membership and non-membership
122 degrees of element z to set A , respectively, such that $0 \leq \mu_A(z) \leq 1, 0 \leq \nu_A(z) \leq 1,$
123 $\mu_A(z) + \nu_A(z) \leq 1, \forall z \in Z$.

124 In addition, $\pi_A(z) = 1 - \mu_A(z) - \nu_A(z)$ is referred to as the hesitancy degree of z to A ,
125 and the pair $(\mu_A(z), \nu_A(z))$ is often called an IFN (Xu & Yager, 2006). For simplicity, for
126 a given z and IFS A , an IFN is denoted by $\tilde{\alpha} = (\mu, \nu)$, where $0 \leq \mu, \nu \leq 1$ and $\mu + \nu \leq 1$.

127 Chen and Tan (1996) defined a score function to calculate the score of $\tilde{\alpha}$ as

$$128 \quad S(\tilde{\alpha}) = \mu - \nu \quad (2.2)$$

129 Later, Hong and Choi (2000) introduced an accuracy function to measure the accuracy
130 degree of $\tilde{\alpha}$ as

$$131 \quad H(\tilde{\alpha}) = \mu + \nu \quad (2.3)$$

132 Therefore, the hesitancy degree of $\tilde{\alpha}$ can be equivalently evaluated as

$$133 \quad \pi(\tilde{\alpha}) = 1 - H(\tilde{\alpha}) \quad (2.4)$$

134 It is obvious that $-1 \leq S(\tilde{\alpha}) \leq 1, 0 \leq H(\tilde{\alpha}) \leq 1$ and $0 \leq \pi(\tilde{\alpha}) \leq 1$. The higher the $H(\tilde{\alpha})$,
135 the lower the hesitancy degree of $\tilde{\alpha}$. Especially, if $\pi(\tilde{\alpha}) = 0$, then $H(\tilde{\alpha}) = 1$ and
136 $\nu = 1 - \mu$, and $\tilde{\alpha}$ is degenerated to a fuzzy number μ .

137 On the basis of the aforesaid score and accuracy functions, Xu and Yager (2006)
138 proposed a comparison method for two IFNs $\tilde{\alpha}_1 = (\mu_1, \nu_1)$ and $\tilde{\alpha}_2 = (\mu_2, \nu_2)$ as follows:

139 If $S(\tilde{\alpha}_1) < S(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1$ is smaller than $\tilde{\alpha}_2$, denoted by $\tilde{\alpha}_1 < \tilde{\alpha}_2$;

140 If $S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2)$, then

141 If $H(\tilde{\alpha}_1) < H(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1$ is smaller than $\tilde{\alpha}_2$, denoted by $\tilde{\alpha}_1 < \tilde{\alpha}_2$;

If $H(\tilde{\alpha}_1) = H(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 = \tilde{\alpha}_2$.

Base on the theory of IFSs, Xu (2007b) introduced IPRs to characterize DMs' pairwise judgments with hesitancy and uncertainty.

An IPR is represented by a matrix $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ on a set of n alternatives, $X = \{x_1, x_2, \dots, x_n\}$, each element $\tilde{r}_{ij} = (\mu_{ij}, \nu_{ij})$ is an IFN. μ_{ij} and ν_{ij} denote, respectively, the membership and non-membership degrees to which x_i is preferred to x_j , such that

$$0 \leq \mu_{ij} \leq 1, 0 \leq \nu_{ij} \leq 1, \mu_{ij} + \nu_{ij} \leq 1, \mu_{ij} = \nu_{ji}, \nu_{ij} = \mu_{ji}, \mu_{ii} = \nu_{ii} = 0.5 \quad i, j = 1, 2, \dots, n \quad (2.5)$$

By employing the membership degrees in $\tilde{R}$, Wang (2013) put forward the following additive consistency definition for IPRs.

Definition 2.2 (Wang, 2013) Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IPR with $\tilde{r}_{ij} = (\mu_{ij}, \nu_{ij})$. $\tilde{R}$ is additively consistent if the following additive transitivity is satisfied

$$\mu_{ij} + \mu_{jk} + \mu_{ki} = \mu_{kj} + \mu_{ji} + \mu_{ik} \quad \text{for all } i, j, k = 1, 2, \dots, n \quad (2.6)$$

According to (2.5) and (2.6), if $\tilde{R}$ is additively consistent, then

$$\nu_{ij} + \nu_{jk} + \nu_{ki} = \nu_{kj} + \nu_{ji} + \nu_{ik} \quad \text{for all } i, j, k = 1, 2, \dots, n \quad (2.7)$$

Based on the score function $S(\cdot)$ and Definition 2.2, Wang (2013) provided the following judgment method to tell whether an IPR is consistent.

Lemma 2.1 (Wang, 2013) Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IPR. $\tilde{R}$ is additively consistent if and only if

$$S(\tilde{r}_{ij}) = S(\tilde{r}_{ik}) - S(\tilde{r}_{jk}) \quad \text{for all } i, j, k = 1, 2, \dots, n. \quad (2.8)$$

Without causing confusion, an additively consistent IPR will be referred to as a consistent IPR hereafter.

3. Consistency measure

As per Lemma 2.1, for a consistent IPR $\tilde{R}$, the score of $\tilde{r}_{ij}$ derived by directly comparing x_i and x_j should be equal to the difference between the scores of intuitionistic preference values obtained by comparing them with an intermediate alternative x_k ($k \neq i, j$). However, if $\tilde{R}$ is inconsistent, the scores of preference values

provided by a DM will not satisfy (2.8), and the difference between $S(\tilde{r}_{ik})$ and $S(\tilde{r}_{jk})$ may not belong to the interval $[-1, 1]$. In other words, there exist some deviations between $S(\tilde{r}_{ij})$ and $S(\tilde{r}_{ik}) - S(\tilde{r}_{jk})$ for some $k = 1, 2, \dots, n, k \neq i, j$. Therefore, the mean absolute deviations of all $n-2$ values $S(\tilde{r}_{ik}) - S(\tilde{r}_{jk})$ with respect to $S(\tilde{r}_{ij})$ can be computed as

$$\varepsilon_{ij} = \frac{1}{n-2} \sum_{k=1, k \neq i, j}^n |S(\tilde{r}_{ij}) - S(\tilde{r}_{ik}) + S(\tilde{r}_{jk})| \quad (3.1)$$

As per (2.2) and (2.5), we have $S(\tilde{r}_{ik}) = \mu_{ik} - v_{ik} = v_{ki} - \mu_{ki} = -S(\tilde{r}_{ki})$. Thus, (3.1) can be rewritten as

$$\varepsilon_{ij} = \frac{1}{n-2} \sum_{k=1, k \neq i, j}^n |S(\tilde{r}_{ij}) + S(\tilde{r}_{jk}) + S(\tilde{r}_{ki})| \quad (3.2)$$

As $-1 \leq S(\tilde{r}_{ij}) \leq 1$ for all $i, j = 1, 2, \dots, n$, we have $0 \leq |S(\tilde{r}_{ij}) + S(\tilde{r}_{jk}) + S(\tilde{r}_{ki})| \leq 3$. Therefore, the consistency level of an IPR is introduced as follows.

Definition 3.1 A consistency index of an IPR $\tilde{R}$ is defined as

$$CI(\tilde{R}) = 1 - \frac{1}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n |S(\tilde{r}_{ij}) + S(\tilde{r}_{jk}) + S(\tilde{r}_{ki})| \quad (3.3)$$

It is obvious that $0 \leq CI(\tilde{R}) \leq 1$. If $CI(\tilde{R}) = 1$, then $\varepsilon_{ij} = 0$ for all $i, j = 1, 2, \dots, n$, i.e. $\tilde{r}_{ij}$ is consistent with respect to the remaining judgment information and $\tilde{R}$ is consistent; otherwise, $\tilde{R}$ is inconsistent, and the smaller the $CI(\tilde{R})$, the more inconsistent the $\tilde{R}$.

As per (2.2), we have $S(\tilde{r}_{ij}) + S(\tilde{r}_{jk}) + S(\tilde{r}_{ki}) = \mu_{ij} - v_{ij} + \mu_{jk} - v_{jk} + \mu_{ki} - v_{ki}$. Since $v_{ij} = \mu_{ji}$ in $\tilde{R}$ for all $i, j = 1, 2, \dots, n$, (3.3) can be rewritten as

$$CI(\tilde{R}) = 1 - \frac{1}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n |\mu_{ij} + \mu_{jk} + \mu_{ki} - \mu_{ji} - \mu_{kj} - \mu_{ik}| \quad (3.4)$$

The motivations to introduce this consistency measure are twofold: Firstly, there has been no such an index measure introduced for additive consistency of IPRs. Although a similar measure was proposed for multiplicative consistency (Xu and Liao, 2014; Xu and Xia, 2014), as explained at the end of this section, their consistency definition suffers from an undesirable property of being sensitive to alternative permutations. Secondly,

this measure is a natural extension of Herrera-Viedma et al. (2007)'s consistency index for a fuzzy preference relation to the IPR case: if $\mu_{ij} + v_{ij} = 1, \forall i, j = 1, 2, \dots, n$, i.e., $\tilde{r}_{ij} = (\mu_{ij}, v_{ij})$ is reduced to an ordinary fuzzy number, as per the intuitionistic reciprocal property of $\mu_{ji} = v_{ij}$, we have $\mu_{ij} - \mu_{ji} = \mu_{ij} - v_{ij} = \mu_{ij} - (1 - \mu_{ij}) = 2\mu_{ij} - 1$ for all $i, j = 1, 2, \dots, n$, and thus $CI(\tilde{R}) = 1 - \frac{2}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n |\mu_{ij} + \mu_{jk} + \mu_{ki} - 1.5|$, which is equivalent to Herrera-Viedma et al. (2007)'s consistency index for a fuzzy preference relation $R = (\mu_{ij})_{n \times n}$.

In solving a GDM problem, the IPRs provided by DMs should be checked for their consistency levels. A moderator may give a consistency threshold t for these IPRs. Given this consideration, the following definition is introduced for acceptably consistent IPRs.

Definition 3.2 Let $0 < t \leq 1$ be a consistency threshold and $\tilde{R}$ be an IPR. If $CI(\tilde{R}) \geq t$, then $\tilde{R}$ is referred to as an acceptably consistent IPR; otherwise, $\tilde{R}$ is said to have an unacceptable consistency level.

Xu et al. (2011) extended the uninorm-function-based equation proposed by Chiclana et al. (2009) to define multiplicative consistency for IPRs. Based on this consistency and the distance-threshold, Xu and Liao (2014) introduced the notion of acceptable multiplicatively consistent IPRs by using the distance between the provided IPR and the constructed consistent IPR. Xu and Xia (2014) developed three iterative algorithms to improve consistency of IPRs. Recently, Wang (2015) pointed out that Xu et al. (2011)'s consistency of IPRs is not robust to alternative permutations. One can verify that Xu and Liao (2014)'s acceptable multiplicative consistency and Xu and Xia (2014)'s iterative algorithms are sensitive to alternative permutations as well. This means that contradictory consistency results may be obtained for the same intuitionistic judgments with only alternatives being re-labelled in a different order.

On the other hand, one can easily prove that acceptable consistency under Definition 3.2 is robust to alternative permutations and does not suffer from the aforesaid problem.

Definition 3.3 Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n} = ((\mu_{ij}, v_{ij}))_{n \times n}$ and $\tilde{R}' = (\tilde{r}'_{ij})_{n \times n} = ((\mu'_{ij}, v'_{ij}))_{n \times n}$ be any two IPRs, the mean absolute deviation (MAD) between $\tilde{R}$ and $\tilde{R}'$ is defined as:

$$MAD(\tilde{R}, \tilde{R}') = \frac{1}{2n(n-1)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n (|\mu_{ij} - \mu'_{ij}| + |v_{ij} - v'_{ij}|) \quad (3.5)$$

Obviously, $0 \leq MAD(\tilde{R}, \tilde{R}') \leq 1$ and $MAD(\tilde{R}, \tilde{R}') = MAD(\tilde{R}', \tilde{R})$. The lower the value $MAD(\tilde{R}, \tilde{R}')$, the closer $\tilde{R}$ is to $\tilde{R}'$. Especially, if $MAD(\tilde{R}, \tilde{R}') = 0$, $\tilde{R}$ is the same as $\tilde{R}'$.

As $v_{ij} = \mu_{ji}$ in $\tilde{R}$ and $v'_{ij} = \mu'_{ji}$ in $\tilde{R}'$ for all $i, j = 1, 2, \dots, n$, (3.5) can be rewritten as

$$MAD(\tilde{R}, \tilde{R}') = \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n |\mu_{ij} - \mu'_{ij}| \quad (3.6)$$

As per (3.4) and (3.6), it is possible to calculate the consistency level of an IPR and the MAD value between two IPRs by using only the membership degrees. Therefore, to improve consistency of an unacceptably consistent IPR $\tilde{R}$, a sensible approach is to adjust its membership degrees.

4. An approach to improving consistency of IPRs

For an IPR with unacceptable consistency, the most reliable approach is to return it to the DM for an update. However, sometimes, the DM is unwilling or unavailable to revise his/her judgments. In this case, it is sensible to introduce an automated procedure to improve consistency of the IPR to an acceptable level. A guiding principle for this type of automated procedure is to retain the DM's original judgment as much as possible. We shall introduce such an approach below.

For a given IPR $\tilde{R} = ((\mu_{ij}, v_{ij}))_{n \times n}$, let

$$\hat{\mu}_{ij} = \frac{1}{2n} \left(\sum_{l=1}^n (\mu_{il} - \mu_{jl}) - \sum_{l=1}^n (\mu_{li} - \mu_{lj}) \right) + 0.5(\mu_{ij} + \mu_{ji}) \quad i, j = 1, 2, \dots, n \quad (4.1)$$

Then, the following results are obtained.

Theorem 4.1 Let $\tilde{R} = ((\mu_{ij}, v_{ij}))_{n \times n}$ be an IPR, and $\hat{\mu}_{ij}$ ($i, j = 1, 2, \dots, n$) be defined by

(4.1), then

(i) $\hat{\mu}_{ii} = 0.5 \quad \forall i = 1, 2, \dots, n$.

(ii) $0 \leq \hat{\mu}_{ij} + \hat{\mu}_{ji} = \mu_{ij} + \mu_{ji} \leq 1 \quad \forall i, j = 1, 2, \dots, n$.

(iii) $\hat{\mu}_{ij} + \hat{\mu}_{jk} + \hat{\mu}_{ki} = \hat{\mu}_{kj} + \hat{\mu}_{ji} + \hat{\mu}_{ik} \quad \forall i, j, k = 1, 2, \dots, n$.

Proof. (i) and (ii) can be directly obtained from (4.1). As for (iii), by (4.1), we have

$$\begin{aligned}
& \hat{\mu}_{ij} + \hat{\mu}_{jk} + \hat{\mu}_{ki} = 0.5(\mu_{ij} + \mu_{ji}) + 0.5(\mu_{jk} + \mu_{kj}) + 0.5(\mu_{ki} + \mu_{ik}) \\
& = 0.5(\mu_{kj} + \mu_{jk}) + 0.5(\mu_{ji} + \mu_{ij}) + 0.5(\mu_{ik} + \mu_{ki}) = \hat{\mu}_{kj} + \hat{\mu}_{ji} + \hat{\mu}_{ik}
\end{aligned}$$

Thus, the proof of Theorem 4.1 is completed ■

Denote a matrix by $\hat{\hat{R}} = ((\hat{\mu}_{ij}, \hat{\mu}_{ji}))_{n \times n}$, where $\hat{\mu}_{ij}$ ($i, j = 1, 2, \dots, n$) is defined by (4.1).

If $\hat{\mu}_{ij} \geq 0$ for all $i, j = 1, 2, \dots, n$, then we have $0 \leq \hat{\mu}_{ij} \leq 1$ as $0 \leq \hat{\mu}_{ij} + \hat{\mu}_{ji} = \mu_{ij} + \mu_{ji} \leq 1$ for all $i, j = 1, 2, \dots, n$. Therefore, as per Definition 2.2 and Theorem 4.1, the following corollary is derived.

Corollary 4.1 If $\hat{\mu}_{ij} \geq 0$ for all $i, j = 1, 2, \dots, n$, then $\hat{\hat{R}} = ((\hat{\mu}_{ij}, \hat{\mu}_{ji}))_{n \times n}$ is a consistent IPR.

If the derived matrix $\hat{\hat{R}} = ((\hat{\mu}_{ij}, \hat{\mu}_{ji}))_{n \times n}$ contains at least one element $\hat{\mu}_{i_0 j_0} < 0$, where $1 \leq i_0 \neq j_0 \leq n$, $\hat{\hat{R}}$ is not an IPR. In this case, $\hat{\mu}_{ij}$ in $\hat{\hat{R}}$ should be converted into membership degrees between 0 and 1 by employing an appropriate transform function. Based on the formula furnished by Herrera-Viedma et al. (2004), the converted values are defined as

$$\hat{\mu}'_{ij} = \frac{\hat{\mu}_{ij} + a}{1 + 2a} \quad i, j = 1, 2, \dots, n \quad (4.2)$$

where

$$a = \begin{cases} 0, & \text{if } \hat{\mu}_{ij} \geq 0, \forall i, j = 1, 2, \dots, n \\ \max\{|\hat{\mu}_{ij}| \mid \hat{\mu}_{ij} < 0, i, j = 1, 2, \dots, n\}, & \text{Otherwise} \end{cases} \quad (4.3)$$

Theorem 4.2 Let $\tilde{R} = ((\mu_{ij}, v_{ij}))_{n \times n}$ be an IPR, and the elements in the matrix

$\hat{\hat{R}}' = ((\hat{\mu}'_{ij}, \hat{\mu}'_{ji}))_{n \times n}$ be defined by (4.2), then $\hat{\hat{R}}'$ is a consistent IPR and

$$(1 - \hat{\mu}'_{ij} - \hat{\mu}'_{ji}) = \frac{1}{1 + 2a} (1 - \mu_{ij} - \mu_{ji}).$$

Proof. As $\tilde{R} = ((\mu_{ij}, v_{ij}))_{n \times n}$ is an IPR, it follows from (2.5) that $0 \leq \mu_{ij} \leq 1$ and

$$0 \leq \mu_{ij} + \mu_{ji} \leq 1.$$

267 As per (4.3), we have $\hat{\mu}_{ij} + a \geq 0$. By Theorem 4.1, one can obtain $\hat{\mu}_{ii} = 0.5$
 268 and $0 \leq \hat{\mu}_{ij} + \hat{\mu}_{ji} = \mu_{ij} + \mu_{ji} \leq 1$. It is confirmed that $0 \leq \hat{\mu}_{ij} + a \leq 1 - \hat{\mu}_{ji} + a =$
 269 $1 + 2a - (\hat{\mu}_{ji} + a) \leq 1 + 2a$. Thus, it follows from (4.2) that $\hat{\mu}_{ii}' = \frac{\hat{\mu}_{ii} + a}{1 + 2a} = \frac{0.5 + a}{1 + 2a} = 0.5$,
 270 $0 \leq \hat{\mu}_{ij}' = \frac{\hat{\mu}_{ij} + a}{1 + 2a} \leq 1$ and $0 \leq \hat{\mu}_{ij}' + \hat{\mu}_{ji}' = \frac{\hat{\mu}_{ij} + \hat{\mu}_{ji} + 2a}{1 + 2a} \leq 1$. Therefore, $\hat{R}' = ((\hat{\mu}_{ij}', \hat{\mu}_{ji}'))_{n \times n}$ is
 271 an IPR.

272 On the other hand, $\hat{\mu}_{ij}' + \hat{\mu}_{jk}' + \hat{\mu}_{ki}' = \frac{\hat{\mu}_{ij} + a}{1 + 2a} + \frac{\hat{\mu}_{jk} + a}{1 + 2a} + \frac{\hat{\mu}_{ki} + a}{1 + 2a} = \frac{\hat{\mu}_{ij} + \hat{\mu}_{jk} + \hat{\mu}_{ki} + 3a}{1 + 2a}$ and
 273 $\hat{\mu}_{kj}' + \hat{\mu}_{ji}' + \hat{\mu}_{ik}' = \frac{\hat{\mu}_{kj} + a}{1 + 2a} + \frac{\hat{\mu}_{ji} + a}{1 + 2a} + \frac{\hat{\mu}_{ik} + a}{1 + 2a} = \frac{\hat{\mu}_{kj} + \hat{\mu}_{ji} + \hat{\mu}_{ik} + 3a}{1 + 2a}$. As per Theorem 4.1, we
 274 have $\hat{\mu}_{ij}' + \hat{\mu}_{jk}' + \hat{\mu}_{ki}' = \hat{\mu}_{kj}' + \hat{\mu}_{ji}' + \hat{\mu}_{ik}'$. According to Definition 2.2, $\hat{R}'$ is consistent.

275 By (4.2), one can get $(1 - \hat{\mu}_{ij}' - \hat{\mu}_{ji}') = 1 - \frac{\hat{\mu}_{ij} + \hat{\mu}_{ji} + 2a}{1 + 2a} = \frac{1}{1 + 2a} (1 - \hat{\mu}_{ij} - \hat{\mu}_{ji})$.

276 According to Theorem 4.1, we have $(1 - \hat{\mu}_{ij}' - \hat{\mu}_{ji}') = \frac{1}{1 + 2a} (1 - \mu_{ij} - \mu_{ji})$. ■

277 *Theorem 4.3* If $\tilde{R} = ((\mu_{ij}, \nu_{ij}))_{n \times n}$ is a consistent IPR, then $\hat{\tilde{R}}' = \tilde{R}$.

278 *Proof.* As $\tilde{R}$ is consistent, by (2.6) we have $\mu_{ij} - \mu_{ji} = \mu_{il} - \mu_{jl} - (\mu_{li} - \mu_{lj})$ for all
 279 $i, j, l = 1, 2, \dots, n$. It follows that

$$280 \quad \sum_{l=1}^n (\mu_{il} - \mu_{jl}) - \sum_{l=1}^n (\mu_{li} - \mu_{lj}) = \sum_{l=1}^n (\mu_{il} + \mu_{lj} - \mu_{jl} - \mu_{li}) = \sum_{l=1}^n (\mu_{ij} - \mu_{ji}) = n(\mu_{ij} - \mu_{ji})$$

281 According to (4.1), one can obtain $\hat{\mu}_{ij} = 0.5(\mu_{ij} - \mu_{ji}) + 0.5(\mu_{ij} + \mu_{ji}) = \mu_{ij} \geq 0$. As per

282 (4.3), one gets $a = 0$. Therefore, we have $\hat{\mu}_{ij}' = \frac{\hat{\mu}_{ij} + a}{1 + 2a} = \hat{\mu}_{ij} = \mu_{ij}$ for all $i, j = 1, 2, \dots, n$. It

283 is thus verified that $\hat{\tilde{R}}' = \tilde{R}$. ■

284 Theorem 4.2 furnishes an approach to construct consistent IPRs based on inconsistent
 285 input. For any inconsistent IPR $\tilde{R} = ((\mu_{ij}, \nu_{ij}))_{n \times n}$, if $a = 0$, the hesitancy degree of each

286 IFN in the constructed consistent IPR $\hat{\tilde{R}}'$ is the same as that of the corresponding original

judgment in $\tilde{R}$; if $a > 0$, the hesitancy degree of each IFN in $\tilde{R}$ is narrowed down by a unified proportion $1/(1 + 2a)$. Generally speaking, the requirement of complete consistency comes at a cost as the MAD between the original and the constructed consistent IPRs tends to be large. If the DM is willing to relax his/her consistency requirement and accept limited inconsistency, the rectified IPR with this acceptable consistency will understandably have a smaller MAD from the DM's original judgment. Based on this line of thinking, the following weighted averaging scheme is put forward to combine the membership degrees in the original and the rectified consistent IPRs $\tilde{R}$ and $\hat{\tilde{R}}$:

$$\mu_{ij}(\lambda) = (1 - \lambda)\mu_{ij} + \lambda\hat{\mu}_{ij}' \quad (4.4)$$

where $\lambda \in [0, 1]$, μ_{ij} is the membership degree in the original IPR $\tilde{R}$, and $\hat{\mu}_{ij}'$ is defined by (4.2) for all $i, j = 1, 2, \dots, n$.

Theorem 4.4 Assume that $\mu_{ij}(\lambda)$ is defined by (4.4), then for any $\lambda \in [0, 1]$,

$$\tilde{R}(\lambda) = \left((\mu_{ij}(\lambda), \mu_{ji}(\lambda)) \right)_{n \times n} \text{ is an IPR and } 1 - \mu_{ij}(\lambda) - \mu_{ji}(\lambda) = \left(1 - \frac{2a}{1 + 2a} \lambda \right) (1 - \mu_{ij} - \mu_{ji}).$$

Proof. As $\tilde{R}$ and $\hat{\tilde{R}}$ are two IPRs, according to (2.5), for all $i, j = 1, 2, \dots, n$, we have

$$\begin{aligned} \mu_{ii} &= 0.5, 0 \leq \mu_{ij} \leq 1, \mu_{ij} + \mu_{ji} \leq 1, \hat{\mu}_{ii}' = 0.5, 0 \leq \hat{\mu}_{ij}' \leq 1 \text{ and } \hat{\mu}_{ij}' + \hat{\mu}_{ji}' \leq 1. \text{ Then, for any } \\ \lambda \in [0, 1], \quad \mu_{ii}(\lambda) &= (1 - \lambda)\mu_{ii} + \lambda\hat{\mu}_{ii}' = 0.5, \quad 0 \leq \mu_{ij}(\lambda) = (1 - \lambda)\mu_{ij} + \lambda\hat{\mu}_{ij}' \leq 1 \text{ and } \\ \mu_{ij}(\lambda) + \mu_{ji}(\lambda) &= (1 - \lambda)(\mu_{ij} + \mu_{ji}) + \lambda(\hat{\mu}_{ij}' + \hat{\mu}_{ji}') \leq 1. \text{ Hence, } \tilde{R}(\lambda) \text{ is an IPR.} \end{aligned}$$

As per (4.4) and Theorem 4.2, one can obtain

$$\begin{aligned} 1 - \mu_{ij}(\lambda) - \mu_{ji}(\lambda) &= 1 - ((1 - \lambda)\mu_{ij} + \lambda\hat{\mu}_{ij}' + (1 - \lambda)\mu_{ji} + \lambda\hat{\mu}_{ji}') = 1 - ((1 - \lambda)(\mu_{ij} + \mu_{ji}) + \lambda(\hat{\mu}_{ij}' + \hat{\mu}_{ji}')) \\ &= 1 - \left((1 - \lambda)(\mu_{ij} + \mu_{ji}) + \lambda \left(1 - \frac{1}{1 + 2a} (1 - \mu_{ij} - \mu_{ji}) \right) \right) = 1 - \frac{2a}{1 + 2a} \lambda - \left(1 - \frac{2a}{1 + 2a} \lambda \right) (\mu_{ij} + \mu_{ji}) \\ &= \left(1 - \frac{2a}{1 + 2a} \lambda \right) (1 - \mu_{ij} - \mu_{ji}) \end{aligned}$$

Thus, the proof of Theorem 4.4 is completed ■

$$\text{As } \pi((\mu_{ij}(\lambda), \mu_{ji}(\lambda))) = 1 - \mu_{ij}(\lambda) - \mu_{ji}(\lambda) \text{ and } \pi((\mu_{ij}, v_{ij})) = 1 - \mu_{ij} - v_{ij} = 1 - \mu_{ij} - \mu_{ji},$$

it follows from Theorem 4.2 that if $a = 0$, the hesitancy degree of each IFN in $\tilde{R}(\lambda)$

310 equals that of the corresponding original judgment in $\tilde{R}$; if $a > 0$, the hesitancy degree of
 311 each IFN in $\tilde{R}(\lambda)$ is reduced by a proportion of $\left(1 - \frac{2a}{1+2a}\lambda\right)$, and the smaller the weight
 312 λ , the closer the new hesitancy $\pi((\mu_{ij}(\lambda), \mu_{ji}(\lambda)))$ to the original hesitancy $\pi((\mu_{ij}, v_{ij}))$.

313 *Theorem 4.5.* If $0 \leq \lambda_1 \leq \lambda_2 \leq 1$, then $CI(\tilde{R}(\lambda_1)) \leq CI(\tilde{R}(\lambda_2))$ and $MAD(\tilde{R}(\lambda_1), \tilde{R}) \leq$
 314 $MAD(\tilde{R}(\lambda_2), \tilde{R})$.

315 *Proof.* According to (4.4), we have

$$316 \quad \mu_{ij}(\lambda_1) + \mu_{jk}(\lambda_1) + \mu_{ki}(\lambda_1) - \mu_{ji}(\lambda_1) - \mu_{kj}(\lambda_1) - \mu_{ik}(\lambda_1) = (1 - \lambda_1)(\mu_{ij} + \mu_{jk} + \mu_{ki} - \mu_{ji} - \mu_{kj} - \mu_{ik})$$

$$+ \lambda_1(\hat{\mu}_{ij}' + \hat{\mu}_{jk}' + \hat{\mu}_{ki}' - \hat{\mu}_{ji}' - \hat{\mu}_{kj}' - \hat{\mu}_{ik}')$$

317 As $\hat{\tilde{R}}' = ((\hat{\mu}_{ij}', \hat{\mu}_{ji}'))_{n \times n}$ is a consistent IPR, it follows from Definition 2.2 that

$$318 \quad \hat{\mu}_{ij}' + \hat{\mu}_{jk}' + \hat{\mu}_{ki}' - \hat{\mu}_{ji}' - \hat{\mu}_{kj}' - \hat{\mu}_{ik}' = 0. \text{ Then}$$

$$319 \quad CI(\tilde{R}(\lambda_1)) = 1 - \frac{1}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n |\mu_{ij}(\lambda_1) + \mu_{jk}(\lambda_1) + \mu_{ki}(\lambda_1) - \mu_{ji}(\lambda_1) - \mu_{kj}(\lambda_1) - \mu_{ik}(\lambda_1)|$$

$$= 1 - \frac{1 - \lambda_1}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n |\mu_{ij} + \mu_{jk} + \mu_{ki} - \mu_{ji} - \mu_{kj} - \mu_{ik}|$$

320 Similarly,

$$321 \quad CI(\tilde{R}(\lambda_2)) = 1 - \frac{1 - \lambda_2}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n |\mu_{ij} + \mu_{jk} + \mu_{ki} - \mu_{ji} - \mu_{kj} - \mu_{ik}|$$

322 Therefore, if $0 \leq \lambda_1 \leq \lambda_2 \leq 1$, we have $CI(\tilde{R}(\lambda_1)) \leq CI(\tilde{R}(\lambda_2))$.

323 As per (4.4), $\forall \lambda \in [0, 1]$, we have $|\mu_{ij}(\lambda) - \mu_{ij}| = |(1 - \lambda)\mu_{ij} + \lambda\hat{\mu}_{ij}' - \mu_{ij}| = \lambda|\hat{\mu}_{ij}' - \mu_{ij}|$. By

$$324 \quad (3.6), \text{ one can obtain } MAD(\tilde{R}(\lambda), \tilde{R}) = \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n |\mu_{ij}(\lambda) - \mu_{ij}| = \frac{\lambda}{n(n-1)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n |\hat{\mu}_{ij}' - \mu_{ij}|.$$

325 Thus, if $0 \leq \lambda_1 \leq \lambda_2 \leq 1$, we have $MAD(\tilde{R}(\lambda_1), \tilde{R}) \leq MAD(\tilde{R}(\lambda_2), \tilde{R})$. ■

326 *Theorem 4.6* Let t be an acceptable consistency threshold, $\tilde{R} = ((\mu_{ij}, v_{ij}))_{n \times n}$ and

327 $\hat{\tilde{R}}' = ((\hat{\mu}_{ij}', \hat{\mu}_{ji}'))_{n \times n}$ be the original IPR with unacceptable consistency and the rectified

328 consistent IPR, respectively. If $\frac{t - CI(\tilde{R})}{1 - CI(\tilde{R})} \leq \lambda \leq 1$, then $\tilde{R}(\lambda) = ((\mu_{ij}(\lambda), \mu_{ji}(\lambda)))_{n \times n}$ is an

329 acceptably consistent IPR.

Proof. As per (3.4), we have

$$CI(\tilde{R}) = 1 - \frac{1}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n |\mu_{ij} + \mu_{jk} + \mu_{ki} - \mu_{ji} - \mu_{kj} - \mu_{ik}|$$

On the other hand, from the proof of Theorem 4.5, it can be seen that

$$CI(\tilde{R}(\lambda)) = 1 - \frac{1-\lambda}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n |\mu_{ij} + \mu_{jk} + \mu_{ki} - \mu_{ji} - \mu_{kj} - \mu_{ik}|$$

Then, one gets $CI(\tilde{R}(\lambda)) = CI(\tilde{R}) + \lambda(1 - CI(\tilde{R}))$. Therefore, if $\frac{t - CI(\tilde{R})}{1 - CI(\tilde{R})} \leq \lambda \leq 1$, we

have $CI(\tilde{R}(\lambda)) \geq t$, the proof of Theorem 4.6 is thus completed ■

From Theorem 4.5, it can be seen that $CI(\tilde{R}(\lambda))$ and $MAD(\tilde{R}(\lambda), \tilde{R})$ are both increasing in $\lambda \in [0, 1]$. Based on Theorem 4.6, for a given IPR $\tilde{R}$ with unacceptable consistency and an acceptable consistency threshold t ($t > CI(\tilde{R})$), (4.4) can be employed to improve the consistency of $\tilde{R}$ by setting $\lambda = \lambda_0$, where

$$\lambda_0 = \frac{t - CI(\tilde{R})}{1 - CI(\tilde{R})} \quad (4.5)$$

In this case, the improved IPR $\tilde{R}(\lambda_0)$ has acceptable consistency.

The aforesaid consistency improving process for an unacceptable IPR $\tilde{R} = ((\mu_{ij}, \nu_{ij}))_{n \times n}$ can be summarized as follows:

Step 1. Construct the matrix $\hat{\tilde{R}} = ((\hat{\mu}_{ij}, \hat{\mu}_{ji}))_{n \times n}$ from $\tilde{R}$ as per (4.1).

Step 2. Calculate the value $\hat{\mu}_{ij}'$ ($i, j=1, 2, \dots, n$) to convert $\hat{\tilde{R}}$ into a consistent IPR

$\hat{\tilde{R}}' = ((\hat{\mu}_{ij}', \hat{\mu}_{ji}'))_{n \times n}$ according to (4.2) and (4.3).

Step 3. Compute the weight $\lambda_0 = \frac{t - CI(\tilde{R})}{1 - CI(\tilde{R})}$, where t is an acceptable consistency

threshold given by the moderator, and $CI(\tilde{R})$ is determined by (3.4).

Step 4. Derive the improved acceptably consistent IPR $\tilde{R}(\lambda_0) = ((\mu_{ij}(\lambda_0), \mu_{ji}(\lambda_0)))_{n \times n}$

as per (4.4).

5. An approach to group decision making with IPRs

5.1 Aggregation of IPRs based on an IOWA operator

Yager & Filev (1999) extended the ordered weighted averaging (OWA) (Yager, 1988) operator by introducing order inducing variables, thereby establishing an induced ordered weighted averaging (IOWA) operator, which can be mathematically defined as follows.

Definition 5.1 (Yager & Filev, 1999) An IOWA operator of dimension m is a mapping $IOWA: (\mathfrak{R} \times \mathfrak{R})^m \rightarrow \mathfrak{R}$ such that

$$IOWA_{\omega}(<I_1, a_1>, <I_2, a_2>, \dots, <I_m, a_m>) = \sum_{i=1}^m \omega_i a_{\sigma(i)} \quad (5.1)$$

where $\{<I_1, a_1>, <I_2, a_2>, \dots, <I_m, a_m>\}$ is a set of the OWA pairs, $\omega = (\omega_1, \omega_2, \dots, \omega_m)^T$ is an associated weight vector satisfying $\omega_i \in [0, 1]$ ($i = 1, 2, \dots, m$) and $\sum_{i=1}^m \omega_i = 1$, and σ is a permutation of $\{1, 2, \dots, m\}$ such that $I_{\sigma(i)} \geq I_{\sigma(i+1)}$ for each $i = 1, 2, \dots, m-1$, i.e., $(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(m)})$ is a reordering of $(a_1, a_2, \dots, a_m)$ according to a decreasing order of $\{I_1, I_2, \dots, I_m\}$.

In the pair $<I_i, a_i>$, I_i is often referred to as the order inducing value and a_i as the argument value.

The IOWA operator has been widely used in developing aggregation approaches for different GDM problems. If the IOWA operator is employed to aggregate IPRs, we have the following results.

Theorem 5.1 Let $\tilde{R}^k = ((\mu_{ij}^k, \nu_{ij}^k))_{n \times n}$ ($k = 1, 2, \dots, m$) be m IPRs, and I_k ($k = 1, 2, \dots, m$) be m order inducing values, then the aggregation $\tilde{R}^G = ((\mu_{ij}^G, \nu_{ji}^G))_{n \times n}$ is also an IPR, where

$$\mu_{ij}^G = IOWA_{\omega}(<I_1, \mu_{ij}^1>, <I_2, \mu_{ij}^2>, \dots, <I_m, \mu_{ij}^m>) = \sum_{k=1}^m \omega_k \mu_{ij}^{\sigma(k)} \quad i, j = 1, 2, \dots, n \quad (5.2)$$

Proof. Since $\tilde{R}^k = ((\mu_{ij}^k, \nu_{ij}^k))_{n \times n}$ is an IPR, as per (2.5), we have $\mu_{ii}^k = 0.5, 0 \leq \mu_{ij}^k \leq 1$ and $\mu_{ij}^k + \mu_{ji}^k \leq 1$ for all $i, j = 1, 2, \dots, n, k = 1, 2, \dots, m$.

376 By (5.2), one can obtain $\mu_{ii}^G = \sum_{k=1}^m \omega_k \mu_{ii}^{\sigma(k)} = 0.5 \sum_{k=1}^m \omega_k = 0.5$, $0 \leq \mu_{ij}^G = \sum_{k=1}^m \omega_k \mu_{ij}^{\sigma(k)} \leq 1$

377 and $\mu_{ij}^G + \mu_{ji}^G = \sum_{k=1}^m \omega_k \mu_{ij}^{\sigma(k)} + \sum_{k=1}^m \omega_k \mu_{ji}^{\sigma(k)} = \sum_{k=1}^m \omega_k (\mu_{ij}^{\sigma(k)} + \mu_{ji}^{\sigma(k)}) \leq 1$. Therefore, $\tilde{R}^G$ is an

378 IPR. ■

379 *Theorem 5.2* If $\tilde{R}^k = (\mu_{ij}^k, \nu_{ij}^k)_{n \times n}$ is a consistent IPR for each $k = 1, 2, \dots, m$, then

380 the aggregation $\tilde{R}^G = (\mu_{ij}^G, \mu_{ji}^G)_{n \times n}$ is consistent.

381 *Proof.* As per (5.2), we have $\mu_{ij}^G + \mu_{jk}^G + \mu_{ki}^G = \sum_{l=1}^m \omega_l (\mu_{ij}^{\sigma(l)} + \mu_{jk}^{\sigma(l)} + \mu_{ki}^{\sigma(l)})$ and

382 $\mu_{kj}^G + \mu_{ji}^G + \mu_{ik}^G = \sum_{l=1}^m \omega_l (\mu_{kj}^{\sigma(l)} + \mu_{ji}^{\sigma(l)} + \mu_{ik}^{\sigma(l)})$. Since $\tilde{R}^l = (\mu_{ij}^l, \nu_{ij}^l)_{n \times n}$ is consistent for each

383 $l = 1, 2, \dots, m$, by (2.6), one can obtain $\mu_{ij}^l + \mu_{jk}^l + \mu_{ki}^l = \mu_{kj}^l + \mu_{ji}^l + \mu_{ik}^l$. Therefore,

384 $\mu_{ij}^G + \mu_{jk}^G + \mu_{ki}^G = \mu_{kj}^G + \mu_{ji}^G + \mu_{ik}^G$ for all $i, j, k = 1, 2, \dots, n$. According to Definition 2.2, $\tilde{R}^G$

385 is consistent. ■

386 *Theorem 5.3* Let t be an acceptable consistency threshold, and $\tilde{R}^k = (\mu_{ij}^k, \nu_{ij}^k)_{n \times n}$ (k

387 $= 1, 2, \dots, m$) be m IPRs. If $CI(\tilde{R}^k) \geq t$ for all $k = 1, 2, \dots, m$, then the aggregation

388 $\tilde{R}^G = (\mu_{ij}^G, \mu_{ji}^G)_{n \times n}$ is an acceptably consistent IPR.

389 *Proof.* Since $CI(\tilde{R}^l) \geq t$ for all $l = 1, 2, \dots, m$, as per (3.4), one can obtain

$$390 \quad CI(\tilde{R}^{\sigma(l)}) = 1 - \frac{1}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n \left| \mu_{ij}^{\sigma(l)} + \mu_{jk}^{\sigma(l)} + \mu_{ki}^{\sigma(l)} - \mu_{kj}^{\sigma(l)} - \mu_{ji}^{\sigma(l)} - \mu_{ik}^{\sigma(l)} \right| \geq t.$$

$$391 \quad \text{Thus, } \omega_l - \frac{1}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n \omega_l \left| \mu_{ij}^{\sigma(l)} + \mu_{jk}^{\sigma(l)} + \mu_{ki}^{\sigma(l)} - \mu_{kj}^{\sigma(l)} - \mu_{ji}^{\sigma(l)} - \mu_{ik}^{\sigma(l)} \right| \geq \omega_l t$$

392 for all $l = 1, 2, \dots, m$. Therefore,

$$393 \quad \sum_{l=1}^m \omega_l - \frac{1}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n \sum_{l=1}^m \omega_l \left| \mu_{ij}^{\sigma(l)} + \mu_{jk}^{\sigma(l)} + \mu_{ki}^{\sigma(l)} - \mu_{kj}^{\sigma(l)} - \mu_{ji}^{\sigma(l)} - \mu_{ik}^{\sigma(l)} \right| \geq \sum_{l=1}^m \omega_l t$$

$$394 \quad 1 - \frac{1}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n \sum_{l=1}^m \omega_l \left| \mu_{ij}^{\sigma(l)} + \mu_{jk}^{\sigma(l)} + \mu_{ki}^{\sigma(l)} - \mu_{kj}^{\sigma(l)} - \mu_{ji}^{\sigma(l)} - \mu_{ik}^{\sigma(l)} \right| \geq t$$

395 On the other hand, by (5.2), we have

$$\left| \mu_{ij}^G + \mu_{jk}^G + \mu_{ki}^G - \mu_{kj}^G - \mu_{ji}^G - \mu_{ik}^G \right| = \left| \sum_{l=1}^m \omega_l \left(\mu_{ij}^{\sigma(l)} + \mu_{jk}^{\sigma(l)} + \mu_{ki}^{\sigma(l)} - \mu_{kj}^{\sigma(l)} - \mu_{ji}^{\sigma(l)} - \mu_{ik}^{\sigma(l)} \right) \right| \leq$$

$$\sum_{l=1}^m \omega_l \left| \mu_{ij}^{\sigma(l)} + \mu_{jk}^{\sigma(l)} + \mu_{ki}^{\sigma(l)} - \mu_{kj}^{\sigma(l)} - \mu_{ji}^{\sigma(l)} - \mu_{ik}^{\sigma(l)} \right|$$

This confirms that $CI(\tilde{R}^G) = 1 - \frac{1}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n \left| \mu_{ij}^G + \mu_{jk}^G + \mu_{ki}^G - \mu_{kj}^G - \mu_{ji}^G - \mu_{ik}^G \right| \geq t$.

Thus, the proof of Theorem 5.3 is completed. $\blacksquare$

Theorem 5.1 reveals that a group IPR $\tilde{R}^G$ can be derived by using the IOWA operator to aggregate all individual membership degrees in $\tilde{R}^k$ ($k = 1, 2, \dots, m$). Theorem 5.3 (or 5.2) further verifies that $\tilde{R}^G$ is acceptable (or additive) consistent if all individual IPRs are acceptable (or additive) consistent.

5.2 A procedure for GDM with IPRs

Consider a GDM problem based on IPRs. Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of n feasible alternatives, and $D = \{d_1, d_2, \dots, d_m\}$ be a set of m DMs. To express his/her preference over the alternative set X , each DM $d_k \in D$ ($k = 1, 2, \dots, m$) conducts a pairwise comparison of alternatives and provides his/her assessment as an IPR $\tilde{R}^k = \left((\mu_{ij}^k, \nu_{ij}^k) \right)_{n \times n}$.

Assume that a consistency threshold t is given by the moderator. The first phase of the proposed solution procedure is to judge and improve consistency of the individual IPR $\tilde{R}^k$ ($k = 1, 2, \dots, m$). In this phase, an acceptably consistent IPR $\tilde{R}^k(\lambda_k) = \left((\mu_{ij}^k(\lambda_k), \mu_{ji}^k(\lambda_k)) \right)_{n \times n}$ is derived from $\tilde{R}^k$ based on the given acceptable consistency threshold value t for each $k = 1, 2, \dots, m$. If $\tilde{R}^k$ is acceptably consistent, then $\tilde{R}^k(\lambda_k) = \tilde{R}^k$; otherwise, $\tilde{R}^k(\lambda_k)$ is obtained by employing the proposed consistency improving method in Section 4.

After deriving the acceptably consistent IPR $\tilde{R}^k(\lambda_k) = \left((\mu_{ij}^k(\lambda_k), \mu_{ji}^k(\lambda_k)) \right)_{n \times n}$ ($k = 1, 2, \dots, m$), the next phase is to aggregate $\tilde{R}^k(\lambda_k)$ ($k = 1, 2, \dots, m$) into a group acceptably consistent IPR $\tilde{R}^G = \left((\mu_{ij}^G, \mu_{ji}^G) \right)_{n \times n}$ by an IOWA operator. In this article, the importance

degree of each DM d_k ($k = 1, 2, \dots, m$) is assumed to be unknown. We first determine the order inducing values and, then, their associated weight vector for all individual IPRs.

Chiclana et al. (2007) developed a consistency-based IOWA operator to aggregate individual fuzzy preference relations, in which consistency indices of fuzzy preference relations are used to determine order inducing values of the operator. They also suggested that more importance should be attached to more consistent individual information. Motivated by this idea, we expect that more importance be associated to $\tilde{R}^k(\lambda_k)$ with higher consistency levels and closer to the original judgment information. Therefore, the following weighted averaging scheme is introduced as an order inducing value that accounts for both the consistency level $CI(\tilde{R}^k(\lambda_k))$ and MAD value $MAD(\tilde{R}^k(\lambda_k), \tilde{R}^k)$.

$$I_k = 0.5CI(\tilde{R}^k(\lambda_k)) + 0.5(1 - MAD(\tilde{R}^k(\lambda_k), \tilde{R}^k)) \quad k = 1, 2, \dots, m \quad (5.3)$$

It is obvious that $0 \leq I_k \leq 1$.

Once the order inducing values are determined, the next issue is to calculate the weight vector associated with the order inducing values. Chiclana et al. (2007) devised a formula to calculate the associated weight vector based on a linguistic quantifier for IOWA-based aggregation operators. The formula can be rewritten by the notation in this article as

$$\omega_l = \begin{cases} \left(\frac{I_{\sigma(1)}}{\sum_{k=1}^m I_{\sigma(k)}} \right)^\beta & l = 1 \\ \left(\frac{\sum_{k=1}^l I_{\sigma(k)}}{\sum_{k=1}^m I_{\sigma(k)}} \right)^\beta - \left(\frac{\sum_{k=1}^{l-1} I_{\sigma(k)}}{\sum_{k=1}^m I_{\sigma(k)}} \right)^\beta & l = 2, 3, \dots, m \end{cases} \quad (5.4)$$

where $0 \leq \beta \leq 1$.

By (5.4), it is easy to obtain that $\omega_{\sigma(1)} \geq \omega_{\sigma(2)} \geq \dots \geq \omega_{\sigma(m)}$ if $I_{\sigma(1)} \geq I_{\sigma(2)} \geq \dots \geq I_{\sigma(m)}$. The greater the value I_k , the higher the associated weight ω_k is. If $\beta = 0$, then (5.2) is based on the maximum inducing value aggregation method for IPRs. If $\beta = 1$, then (5.2) is based on the normalized inducing value aggregation method for IPRs.

442 The last phase is to obtain a ranking order for all alternatives or select the best one(s)
 443 based on the aggregated IPR R^G . As R^G is acceptably consistent, the overall assessment
 444 of x_i can be determined as follows:

$$445 \quad \tilde{\alpha}_{x_i} = (\mu_{x_i}, v_{x_i}) = \left(\frac{1}{n-1} \left(\sum_{j=1}^n \mu_{ij}^G - 0.5 \right), \frac{1}{n-1} \left(\sum_{j=1}^n \mu_{ji}^G - 0.5 \right) \right) \quad i = 1, 2, \dots, n \quad (5.5)$$

446 As $0 \leq \mu_{ij}^G \leq 1$ and $\mu_{ij}^G + \mu_{ji}^G \leq 1$ for $i, j = 1, 2, \dots, n$, one can obtain $0 \leq \mu_{x_i}, v_{x_i} \leq 1$
 447 and $\mu_{x_i} + v_{x_i} \leq 1$. Therefore, $\tilde{\alpha}_{x_i}$ given in (5.5) is an IFN, which can be used for ranking
 448 alternatives.

449 Based on the aforesaid analyses, an algorithm for GDM with IPRs is summarized as
 450 follows.

451 Step 1. Calculate the consistency index $CI(\tilde{R}^k)$ for each IPR $\tilde{R}^k = ((\mu_{ij}^k, v_{ij}^k))_{n \times n}$
 452 provided by DM $d_k \in D$ ($k = 1, 2, \dots, m$) as per (3.3) or (3.4).

453 Step 2. For each $k = 1, 2, \dots, m$, if $CI(\tilde{R}^k) \geq t$, then let $\tilde{R}^k(\lambda_k) = \tilde{R}^k$; Otherwise,
 454 return $\tilde{R}^k$ to the DMs for updates. If the DMs are unwilling or unable to provide their
 455 judgment matrices with acceptable consistency, the automated approach in Section 4 is
 456 employed to drive $\tilde{R}^k(\lambda_k)$ with acceptable consistency.

457 Step 3. Compute the MAD value $MAD(\tilde{R}^k(\lambda_k), \tilde{R}^k)$ ($k = 1, 2, \dots, m$) as per (3.6).

458 Step 4. Calculate the order inducing values I_k ($k = 1, 2, \dots, m$) by plugging the
 459 values of $CI(\tilde{R}^k(\lambda_k))$ and $MAD(\tilde{R}^k(\lambda_k), \tilde{R}^k)$ into (5.3).

460 Step 5. Determine the associated weights ω_k ($k = 1, 2, \dots, m$) as per (5.4).

461 Step 6. Use the IOWA operator to aggregate all $\tilde{R}^k(\lambda_k)$ ($k = 1, 2, \dots, m$) into a
 462 group acceptably consistent IPR $\tilde{R}^G = ((\mu_{ij}^G, \mu_{ji}^G))_{n \times n}$, where $\mu_{ij}^G = IOWA_{\omega}(< I_1, \mu_{ij}^1(\lambda_1) >, < I_2, \mu_{ij}^2(\lambda_2) >, \dots, < I_m, \mu_{ij}^m(\lambda_m) >)$, i.e., $\mu_{ij}^G = \sum_{k=1}^m \omega_k \mu_{ij}^{\sigma(k)}(\lambda_{\sigma(k)})$ ($i, j = 1, 2, \dots, n$).

464 Step 7. Determine the overall IFN assessment $\tilde{\alpha}_{x_i}$ for alternative $x_i \in X$ ($i = 1, 2, \dots,$
 465 n) as per (5.5).

Step 8. Compute the values of the score function $S(\tilde{\alpha}_{x_i})$ and accuracy function $H(\tilde{\alpha}_{x_i})$ for each alternative x_i by using (2.2) and (2.3), and rank all alternatives and/or select the best one(s) as per a decreasing order of $\tilde{\alpha}_{x_i}$ ($i = 1, 2, \dots, n$).

Step 9. End.

6. An illustrative example

This section applies the proposed models to a GDM problem concerning the selection of international exchange doctoral students at a Chinese university (adapted from Wang and Li (2012)).

This illustrative example examines the ranking of talented doctoral students for international exchange programs based on the first author's personal experience at a Chinese leading university. The selection process requires a panel of faculty members to evaluate student applications. Without loss of generality, assume that the committee comprises four members d_1, d_2, d_3 and d_4 , and six students x_1, x_2, x_3, x_4, x_5 and x_6 are shortlisted for final consideration. Assume that each expert d_k ($k = 1, 2, 3, 4$) compares each pair of the six candidates and provides his/her assessments as IPRs $\tilde{R}^k = (\mu_{ij}^k, \nu_{ij}^k)_{6 \times 6}$ as shown below:

$$\tilde{R}^1 = \begin{bmatrix} (0.5, 0.5) & (0.45, 0.25) & (0.30, 0.40) & (0.40, 0.45) & (0.75, 0.15) & (0.25, 0.60) \\ (0.25, 0.45) & (0.5, 0.5) & (0.20, 0.35) & (0.30, 0.45) & (0.25, 0.30) & (0.10, 0.55) \\ (0.40, 0.30) & (0.35, 0.20) & (0.5, 0.5) & (0.35, 0.45) & (0.50, 0.20) & (0.30, 0.40) \\ (0.45, 0.40) & (0.45, 0.30) & (0.45, 0.35) & (0.5, 0.5) & (0.65, 0.15) & (0.20, 0.65) \\ (0.15, 0.75) & (0.30, 0.25) & (0.20, 0.50) & (0.15, 0.65) & (0.5, 0.5) & (0.40, 0.45) \\ (0.60, 0.25) & (0.55, 0.10) & (0.40, 0.30) & (0.65, 0.20) & (0.45, 0.40) & (0.5, 0.5) \end{bmatrix}$$

$$\tilde{R}^2 = \begin{bmatrix} (0.5, 0.5) & (0.20, 0.40) & (0.35, 0.30) & (0.45, 0.25) & (0.65, 0.25) & (0.30, 0.55) \\ (0.40, 0.20) & (0.5, 0.5) & (0.65, 0.25) & (0.35, 0.45) & (0.10, 0.55) & (0.35, 0.30) \\ (0.30, 0.35) & (0.25, 0.65) & (0.5, 0.5) & (0.30, 0.45) & (0.60, 0.15) & (0.55, 0.30) \\ (0.25, 0.45) & (0.45, 0.35) & (0.45, 0.30) & (0.5, 0.5) & (0.55, 0.35) & (0.15, 0.55) \\ (0.25, 0.65) & (0.55, 0.10) & (0.15, 0.60) & (0.35, 0.55) & (0.5, 0.5) & (0.45, 0.30) \\ (0.55, 0.30) & (0.30, 0.35) & (0.30, 0.55) & (0.55, 0.15) & (0.30, 0.45) & (0.5, 0.5) \end{bmatrix}$$

$$\tilde{R}^3 = \begin{bmatrix} (0.5, 0.5) & (0.40, 0.55) & (0.05, 0.90) & (0.10, 0.80) & (0.55, 0.15) & (0.45, 0.30) \\ (0.55, 0.40) & (0.5, 0.5) & (0.30, 0.60) & (0.05, 0.85) & (0.10, 0.70) & (0.65, 0.10) \\ (0.90, 0.05) & (0.60, 0.30) & (0.5, 0.5) & (0.95, 0.00) & (0.35, 0.40) & (0.45, 0.25) \\ (0.80, 0.10) & (0.85, 0.05) & (0.00, 0.95) & (0.5, 0.5) & (0.20, 0.60) & (0.60, 0.25) \\ (0.15, 0.55) & (0.70, 0.10) & (0.40, 0.35) & (0.60, 0.20) & (0.5, 0.5) & (0.10, 0.80) \\ (0.30, 0.45) & (0.10, 0.65) & (0.25, 0.45) & (0.25, 0.60) & (0.80, 0.10) & (0.5, 0.5) \end{bmatrix}$$

$$\tilde{R}^4 = \begin{bmatrix} (0.5, 0.5) & (0.10, 0.70) & (0.80, 0.10) & (0.50, 0.40) & (0.60, 0.10) & (0.30, 0.55) \\ (0.70, 0.10) & (0.5, 0.5) & (0.70, 0.10) & (0.30, 0.00) & (0.20, 0.60) & (0.30, 0.45) \\ (0.10, 0.80) & (0.10, 0.70) & (0.5, 0.5) & (0.80, 0.10) & (0.40, 0.35) & (0.35, 0.30) \\ (0.40, 0.50) & (0.00, 0.30) & (0.10, 0.80) & (0.5, 0.5) & (0.60, 0.20) & (0.20, 0.70) \\ (0.10, 0.60) & (0.60, 0.20) & (0.35, 0.40) & (0.20, 0.60) & (0.5, 0.5) & (0.30, 0.45) \\ (0.55, 0.30) & (0.45, 0.30) & (0.30, 0.35) & (0.70, 0.20) & (0.45, 0.30) & (0.5, 0.5) \end{bmatrix}$$

As per (3.3) or (3.4), for $k = 1, 2, 3, 4$, one can determine the consistency index $CI(\tilde{R}^k)$ for each IPR $\tilde{R}^k$ as follows:

$$CI(\tilde{R}^1) = 0.90167, CI(\tilde{R}^2) = 0.84, CI(\tilde{R}^3) = 0.71167, CI(\tilde{R}^4) = 0.785.$$

If an acceptable consistency threshold is set at $t = 0.8$ by the committee, by Definition 3.2, $\tilde{R}^1$ and $\tilde{R}^2$ are deemed to have acceptable consistency. Thus, we have $\tilde{R}^1(\lambda_1) = \tilde{R}^1$ and $\tilde{R}^2(\lambda_2) = \tilde{R}^2$.

On the other hand, the consistency levels of $\tilde{R}^3$ and $\tilde{R}^4$ are unacceptable as they are below the threshold, $CI(\tilde{R}^3) = 0.71167 < t = 0.8$ and $CI(\tilde{R}^4) = 0.785 < t = 0.8$. If the two DMs d_3 and d_4 are unwilling/unavailable to revise their judgments, then $\tilde{R}^3$ and $\tilde{R}^4$ have to be rectified as per the proposed method in Section 4 so that their consistency levels are improved to the acceptable threshold t .

According to (4.1), one can obtain the two transformation matrices $\hat{\tilde{R}}^3 = (\hat{\mu}_{ij}^3, \hat{\mu}_{ji}^3)_{6 \times 6}$ and $\hat{\tilde{R}}^4 = (\hat{\mu}_{ij}^4, \hat{\mu}_{ji}^4)_{6 \times 6}$, where $\hat{\mu}_{ij}^3 > 0$ for all $i, j = 1, 2, \dots, n$ and $\hat{\mu}_{42}^4 = -0.02917 < 0$ with remaining element $\hat{\mu}_{ij}^4 > 0$. As per (4.3), their corresponding values of a are 0 and 0.02917, respectively.

By (4.2), the corresponding consistent IPRs are constructed as follows:

$$\hat{R}^3 = \left((\hat{\mu}_{ij}^3, \hat{\mu}_{ji}^3) \right)_{6 \times 6} = \left((\hat{\mu}_{ij}^3, \hat{\mu}_{ji}^3) \right)_{6 \times 6}$$

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$$\begin{bmatrix} (0.5, 0.5) & (0.46250, 0.48750) & (0.19167, 0.75833) & (0.31250, 0.58750) & (0.25833, 0.44167) & (0.32500, 0.42500) \\ (0.48750, 0.46250) & (0.5, 0.5) & (0.17917, 0.72083) & (0.32500, 0.57500) & (0.32083, 0.47917) & (0.33750, 0.41250) \\ (0.75833, 0.19167) & (0.72083, 0.17917) & (0.5, 0.5) & (0.62083, 0.32917) & (0.56667, 0.18333) & (0.58333, 0.11667) \\ (0.58750, 0.31250) & (0.57500, 0.32500) & (0.32917, 0.62083) & (0.5, 0.5) & (0.44583, 0.35417) & (0.51250, 0.33750) \\ (0.44167, 0.25833) & (0.47917, 0.32083) & (0.18333, 0.56667) & (0.35417, 0.44583) & (0.5, 0.5) & (0.49167, 0.40833) \\ (0.42500, 0.32500) & (0.41250, 0.33750) & (0.11667, 0.58333) & (0.33750, 0.51250) & (0.40833, 0.49167) & (0.5, 0.5) \end{bmatrix}$$

$$\hat{R}^4 = \left((\hat{\mu}_{ij}^4, \hat{\mu}_{ji}^4) \right)_{6 \times 6} =$$

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$$\begin{bmatrix} (0.5, 0.5) & (0.36614, 0.44489) & (0.52756, 0.37795) & (0.58268, 0.32284) & (0.44882, 0.26772) & (0.38583, 0.47244) \\ (0.44489, 0.36614) & (0.5, 0.5) & (0.51968, 0.29134) & (0.33859, 0.00000) & (0.53543, 0.27559) & (0.37795, 0.38583) \\ (0.37795, 0.52756) & (0.29134, 0.51968) & (0.5, 0.5) & (0.50787, 0.39764) & (0.39764, 0.36614) & (0.21654, 0.45276) \\ (0.32284, 0.58268) & (0.00000, 0.33859) & (0.39764, 0.50787) & (0.5, 0.5) & (0.36614, 0.44489) & (0.27953, 0.62598) \\ (0.26772, 0.44882) & (0.27559, 0.53543) & (0.36614, 0.39764) & (0.44489, 0.36614) & (0.5, 0.5) & (0.24803, 0.51575) \\ (0.47244, 0.38583) & (0.38583, 0.37795) & (0.45276, 0.21654) & (0.62598, 0.27953) & (0.51575, 0.24803) & (0.5, 0.5) \end{bmatrix}$$

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According to (4.4) and (4.5), one can establish the following IPRs by the weighted

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averaging schemes $\mu_{ij}^3(\lambda_3) = (1 - \lambda_3)\mu_{ij}^3 + \lambda_3\hat{\mu}_{ij}^3 = 0.69365\mu_{ij}^3 + 0.30635\hat{\mu}_{ij}^3$ and $\mu_{ij}^4(\lambda_4) =$

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$(1 - \lambda_4)\mu_{ij}^4 + \lambda_4\hat{\mu}_{ij}^4 = 0.9302\mu_{ij}^4 + 0.0698\hat{\mu}_{ij}^4$.

$$\tilde{R}^3(\lambda_3) = \left((\mu_{ij}^3(\lambda_3), \mu_{ji}^3(\lambda_3)) \right)_{6 \times 6}$$

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$$\begin{bmatrix} (0.5, 0.5) & (0.41915, 0.53085) & (0.09340, 0.85660) & (0.16510, 0.73490) & (0.46065, 0.23935) & (0.41171, 0.33829) \\ (0.53085, 0.41915) & (0.5, 0.5) & (0.26298, 0.63702) & (0.13425, 0.76575) & (0.16765, 0.63235) & (0.55427, 0.19573) \\ (0.85660, 0.09340) & (0.63702, 0.26298) & (0.5, 0.5) & (0.84916, 0.10084) & (0.41638, 0.33362) & (0.49085, 0.20915) \\ (0.73490, 0.16510) & (0.76575, 0.13425) & (0.10084, 0.84916) & (0.5, 0.5) & (0.27531, 0.52469) & (0.57319, 0.27681) \\ (0.23935, 0.46065) & (0.63235, 0.16765) & (0.33362, 0.41638) & (0.52469, 0.27531) & (0.5, 0.5) & (0.21999, 0.68001) \\ (0.33829, 0.41171) & (0.19573, 0.55427) & (0.20915, 0.49085) & (0.27681, 0.57319) & (0.68001, 0.21999) & (0.5, 0.5) \end{bmatrix}$$

$$\tilde{R}^4(\lambda_4) = \left((\mu_{ij}^4(\lambda_4), \mu_{ji}^4(\lambda_4)) \right)_{6 \times 6}$$

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$$\begin{bmatrix} (0.5, 0.5) & (0.11857, 0.68220) & (0.78099, 0.11939) & (0.50577, 0.39462) & (0.58945, 0.11170) & (0.30599, 0.54459) \\ (0.68220, 0.11857) & (0.5, 0.5) & (0.68742, 0.11335) & (0.30269, 0.00000) & (0.22340, 0.57737) & (0.20544, 0.44552) \\ (0.11939, 0.78099) & (0.11335, 0.68742) & (0.5, 0.5) & (0.77962, 0.12077) & (0.39984, 0.35113) & (0.34069, 0.31066) \\ (0.39462, 0.50577) & (0.00000, 0.30269) & (0.12077, 0.77962) & (0.5, 0.5) & (0.58368, 0.21709) & (0.20555, 0.69484) \\ (0.11170, 0.58945) & (0.57737, 0.22340) & (0.35113, 0.39984) & (0.21709, 0.58368) & (0.5, 0.5) & (0.29637, 0.45459) \\ (0.54459, 0.30599) & (0.44552, 0.20544) & (0.31066, 0.34069) & (0.69484, 0.20555) & (0.45459, 0.29637) & (0.5, 0.5) \end{bmatrix}$$

By (3.4), one can verify that $CI(\tilde{R}^3(\lambda_3)) = CI(\tilde{R}^4(\lambda_4)) = 0.8 \geq t$. Therefore, we obtain two IPRs $\tilde{R}^3(\lambda_3)$ and $\tilde{R}^4(\lambda_4)$ with acceptable consistency.

As per (3.6), the MAD value $MAD(\tilde{R}^k(\lambda_k), \tilde{R}^k)$ between the rectified consistent IPR $\tilde{R}^k(\lambda_k)$ and the original judgment matrix $\tilde{R}^k$ can be determined for each k ($k = 1, 2, 3, 4$) as follows:

$$MAD(\tilde{R}^1(\lambda_1), \tilde{R}^1) = 0, MAD(\tilde{R}^2(\lambda_2), \tilde{R}^2) = 0, MAD(\tilde{R}^3(\lambda_3), \tilde{R}^3) = 0.06467, MAD(\tilde{R}^4(\lambda_4), \tilde{R}^4) = 0.01063.$$

By plugging the values of $CI(\tilde{R}^k(\lambda_k))$ and $MAD(\tilde{R}^k(\lambda_k), \tilde{R}^k)$ into (5.3), one can obtain the order inducing values as $I_1 = 0.9508, I_2 = 0.92, I_3 = 0.8677$ and $I_4 = 0.8947$.

Since $I_1 > I_2 > I_4 > I_3$, a permutation of $\{1, 2, 3, 4\}$ can be established as $\{1, 2, 4, 3\}$. Based on (5.4), various associated weight vectors are obtained under different β values and shown in Table 1.

Table 1. Associated weight vectors based on different β values

β	ω_1	ω_2	ω_3	ω_4	$\omega_1 - \omega_2$	$\omega_2 - \omega_3$	$\omega_3 - \omega_4$	$\sum_{i=1}^3 (\omega_i - \omega_{i+1})$
0	1	0	0	0	1	0	0	1
0.2	0.765	0.111	0.071	0.053	0.654	0.040	0.018	0.712
0.5	0.512	0.206	0.155	0.127	0.306	0.051	0.028	0.385
0.8	0.342	0.246	0.216	0.196	0.096	0.030	0.020	0.146
1	0.262	0.253	0.246	0.239	0.009	0.007	0.007	0.023

It can be observed from Table 1 that the bigger the β value, the smaller the sum of the differences between ω_i and ω_{i+1} . If the committee feels that it is necessary to differentiate the committee members in terms of the quality of their input judgment data but do not wish to make the difference too wide, a larger β value is better. If a more heterogeneous weight vector is desired, a smaller β value is more appropriate. For instance, by setting $\beta = 1$, one obtains relatively homogeneous weights as $\omega_1 = 0.262$, $\omega_2 = 0.253$, $\omega_3 = 0.246$ and $\omega_4 = 0.239$.

By employing the IOWA operator with the aforesaid permutation σ and the associated weight vector $(\omega_1, \omega_2, \omega_3, \omega_4)^T$, the aggregated IPR $\tilde{R}^G = (\mu_{ij}^G, \mu_{ji}^G)_{n \times n}$ is derived as

$$\tilde{R}^G = \begin{bmatrix} (0.5, 0.5) & (0.29785, 0.46139) & (0.38160, 0.41480) & (0.38253, 0.45387) & (0.61605, 0.18723) & (0.31507, 0.51117) \\ (0.46139, 0.29785) & (0.5, 0.5) & (0.44881, 0.33508) & (0.27370, 0.41476) & (0.18582, 0.51091) & (0.32236, 0.37638) \\ (0.41480, 0.38160) & (0.33508, 0.44881) & (0.5, 0.5) & (0.56234, 0.28556) & (0.48068, 0.25646) & (0.41887, 0.30711) \\ (0.45387, 0.38253) & (0.41476, 0.27370) & (0.28556, 0.56234) & (0.5, 0.5) & (0.51883, 0.30666) & (0.27791, 0.54654) \\ (0.18723, 0.61605) & (0.51091, 0.18582) & (0.25646, 0.48068) & (0.30666, 0.51883) & (0.5, 0.5) & (0.34413, 0.46815) \\ (0.51117, 0.31507) & (0.37638, 0.32236) & (0.30711, 0.41887) & (0.54654, 0.27791) & (0.46815, 0.34413) & (0.5, 0.5) \end{bmatrix}$$

According to (3.4), one can verify that $CI(\tilde{R}^G) = 0.89489 > t = 0.8$. Therefore, $\tilde{R}^G$ is a group IPR with acceptable consistency.

By (5.5), the overall assessment IFNs for the six candidates are determined as:

$$\tilde{\alpha}_{x_1} = (0.39862, 0.40569), \tilde{\alpha}_{x_2} = (0.33842, 0.387), \tilde{\alpha}_{x_3} = (0.44235, 0.33591),$$

$$\tilde{\alpha}_{x_4} = (0.39019, 0.41435), \tilde{\alpha}_{x_5} = (0.32108, 0.45391), \tilde{\alpha}_{x_6} = (0.44187, 0.33567)$$

By using the score function (2.2), one can obtain $S(\tilde{\alpha}_{x_3}) > S(\tilde{\alpha}_{x_6}) > S(\tilde{\alpha}_{x_1}) > S(\tilde{\alpha}_{x_4}) > S(\tilde{\alpha}_{x_2}) > S(\tilde{\alpha}_{x_5})$, leading to a full rank of the six candidates as $x_3 \succ x_6 \succ x_1 \succ x_4 \succ x_2 \succ x_5$.

7. Conclusion

This article investigates GDM problems with IPRs as basic input data. The proposed solution process consists of three phases: (1) Consistency measurement and improvement of individual IPRs; (2) Aggregation of acceptably consistent IPRs; (3) Ranking of alternatives.

Based on the additive consistency definition introduced by Wang (2013), a consistency index is defined to measure an IPR's consistency level. A consistency improving approach is developed to derive an acceptably consistent IPR for an unacceptable IPR if a DM is unwilling or unavailable to revise his/her judgments therein. Given IPR's property of $v_{ij} = \mu_{ji}$, the proposed approach adjusts only the membership degrees in an IFN and the nonmembership degrees are indirectly accommodated. By adopting an order inducing value that considers both the consistency level of the resulting acceptably consistent IPR and the MAD between the original and resulting IPRs, an IOWA-operator-based aggregation approach is devised to aggregate the rectified

individual IPRs into a group judgment. Based on the aggregated acceptably consistent IPR, an overall assessment IFN is obtained for ranking alternatives.

Current research assumes that all IPRs provided by DMs are complete. Further research is needed to handle the cases with missing judgment values. Another possible topic is to consider if the proposed models can be adapted for an even weaker consistency requirement such as weak transitivity.

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