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Task Interrupted By A Poisson Process

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TASK INTERRUPTED

BY A

POISSON PROCESS

by

Jarrett Nantais

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Author's Declaration of Originality

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Abstract

In this paper, we consider a task which has a completion time T (if not interrupted), which is a random variable with probability density function (pdf) $f(t)$, $t > 0$. Before it is complete, the task may be interrupted by a Poisson process with rate λ. If that happens, then the task must begin again, with the same completion time random variable T, but with a potentially different realization. These interruptions can reoccur, until eventually the task is finished, with a total time of W. In this paper, we will find the Laplace Transform of W in several special cases.

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CHAPTER 1

Introduction

1.1. Motivation of Research

The purpose of this major paper is to provide an extension of the applications that the Laplace transform brings to the discipline of probability and statistics, particularly in applications of queueing theory (i.e., waiting times, busy periods, etc.). Over the last 70-80 years, researchers have provided us with findings in regards to the intersection of probability theory and the Laplace transform.

The following is taken from Yan (2013).

"Van Dantzig (1949) introduced catastrophes and used probabilities as a method for finding Laplace transforms. Runnenberg (1965) revived and popularized the method, and gave numerous applications. Rade (1972) continued the use of the method with applications in queueing. Kleinrock's (1975) classic book discussed the method and extended its popularity. Roy (1997) used the method to give a probabilistic interpretation of the expression for the Laplace transforms of the busy period of an M/G/1 queue, and Horn (1999) used the probabilistic interpretation to find distributions of order statistics of Erlang random variables".

This paper will give reason to utilize the Laplace transform to solve for the total completion time for a task that is facing an interruption that is interrupted by a Poisson process using a variety of univariate distributions, and to determine notable properties within each case.

1.2. Probability Functions of Random Variables

DEFINITION 1.1. Let X be distributed as a gamma random variable ($Gamma \sim (\alpha, \beta)$) with parameter $\alpha > 0$, $\beta > 0$. Then, the probability density function (pdf) of X is defined by:

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{\frac{-x}{\beta}} & \text{if } x \ > 0 \\ 0 & \text{otherwise} \end{cases}$$

DEFINITION 1.2. Let X be distributed as an exponential random variable with parameter $\lambda > 0$. Then, the probability density function (pdf) of X is defined by:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \ > 0 \\ 0 & \text{otherwise} \end{cases}$$

Note that the exponential distribution is a special case of the Gamma distribution where $\alpha = 1$ and $\beta = \frac{1}{\lambda}$. The exponential distribution plays a significant role in the applications of queueing theory and is important in providing a probabilistic interpretation of the Laplace transform, which will be illustrated later in this chapter.

The corresponding cumulative distribution function (cdf) for an exponentially distributed random variable is

$$F(x) = 1 - e^{-\lambda x}, x \geqslant 0, \lambda \geqslant 0$$

The following is taken from Roy (1997).

"Another important property of the exponential distribution is that it is "memoryless". An interpretation of the memoryless property is that the distribution of the time until the next event from a memoryless process is the same regardless of the time that an observer has already waited for the event to occur".

PROPERTY 1.3. If Y is an exponential random variable, it is memoryless if

$$P(Y > s + t | Y > s) = P(Y > t)$$

PROOF. Using the definition of conditional probability, we get

$$\begin{aligned} P(Y > s + t | Y > s) &= \frac{P(Y > s + t)}{P(Y > s)} \\ &= \frac{e^{\lambda(s+t)}}{e^{\lambda s}} \\ &= e^{-\lambda t} \\ &= P(Y > t) \end{aligned}$$

□

DEFINITION 1.4. Let X be distributed as a Poisson random variable with parameter $\lambda > 0$. Then, the probability mass function (pmf) of X is defined by:

$$f(x) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!} & \text{if } x \ > 0 \\ 0 & \text{otherwise} \end{cases}$$

Using the above pdf's and pmf's, we can note an important result regarding the relationship between the Poisson process and exponential distributions.

PROPERTY 1.5. The time between events is $T \sim exp(\lambda)$ for a Poisson Process - that is $N(t) \sim Poisson(\lambda t)$ on the interval $(0, t)$

PROOF.

$$
\begin{aligned}
F_T(t) &= P(T \leqslant t) \\
&= 1 - P(T \geqslant t) \\
&= 1 - P(N(t) = 0) \\
&= 1 - \frac{(\lambda t)^0 e^{-\lambda t}}{0!} \\
&= 1 - e^{-\lambda t}
\end{aligned}
$$

Using Definition 1.2, note that this is the corresponding cumulative distribution function of an exponential random variable. Therefore, $T \sim exp(\lambda)$. □

1.3. The Laplace Transform

This section defines the Laplace transforms, and their link to probability theory, particularly in queueing models. In addition, some important results derived from the probabilistic interpretation of the Laplace transform will be discussed.

The following description of Laplace transforms is taken from Roy (1997).

"The Laplace transform is an often-used integral transform that is employed in many diverse fields of mathematics. It is particularly well known for its use in solving linear differential equations with constants coefficients. The study of stochastic processes also utilizes Laplace transform in areas such as risk theory, renewal theory and queueing theory".

DEFINITION 1.6. Let $f(x)$ be the pdf of a continuous random variable with positive support. Then, the Laplace Transform is defined by

$$L_x(s) = \int_0^{\infty} e^{-sx} f(x) dx$$

where $s > 0$.

It is important to note that the pdf must have positive support as the Laplace transform will always converge with these conditions. Otherwise, the Laplace transform may diverge and give us no probabilistic interpretations for a given random variable.

From inspection, the integral for the Laplace transform as defined in Definition 1.5 is just the continuous case of a moment generating function, evaluated at $t = -s$.

PROPERTY 1.7. Let X be a random variable with positive support. Further, assume that the first two moments of X exist. Then,

$$E(X) = -L'(0)$$

$$E(X^2) = L''(0)$$

The proof of this follows the fact that $L(s) = M_x(-t)$ as described above.

Many of the important results derived from the Laplace transform take place when investigating the properties of two random variables, X and Y, and when one (or both) of these variables are distributed exponentially. First, we should recall the probability of two random variables in the continuous and discrete cases.

THEOREM 1.3.1. *The following theorem is taken from Roy (1997). Let X be a random variable. Further, let Y be exponentially distributed with rate s, where X and Y independent random variables. Let $f(x)$ and $g(y)$ be the probability density functions of X and Y respectively with positive support. Then,*

$$P(X < Y) = L_x(s)$$

PROOF.

$$
\begin{aligned}
L_x(s) &= \int_0^\infty f(x)e^{-sx}dx \\
&= \int_0^\infty \int_x^\infty f(x)se^{-sy}dydx \\
&= \int_0^\infty \int_x^\infty f(x)g(y)dydx \\
&= P(X < Y)
\end{aligned}
$$

□

The fact above is a milestone in queuing applications, particularly in waiting time, for two random variables X and Y.

The following is taken from Jahan (2008).

"The exponential random variable Y is called the catastrophe. The Laplace transform of a random variable X is the probability that X occurs before the catastrophe. More precisely, the Laplace transform of a probability density function $f(x)$ of a random variable X can be interpreted as the probability that X precedes a catastrophe where the time to the catastrophe is an exponentially distributed random variable Y with rate s, independent of X. Another way of describing the process is in terms of a race. The Laplace transform of a random variable X is the probability that X wins a race against an exponential opponent (i.e., catastrophe) Y." It is worth noting that since the time until the next catastrophe is exponential, the catastrophe process is Poisson and memoryless.

The following two properties are well known.

PROPERTY 1.8. If $X \sim exp(\lambda)$ and $Y \sim exp(s)$, and X and Y are independent random variables, then:

$$L_x(s) = P(X < Y) = P(X occurs first) = \frac{\lambda}{\lambda + s}$$

PROOF.

$$\begin{aligned} P(X < Y) &= \int_0^\infty \int_x^\infty \lambda e^{-\lambda x} s e^{-sy} dy dx \\ &= \int_0^\infty -\lambda e^{-(\lambda+s)x} dx \\ &= \frac{\lambda}{\lambda + s} \end{aligned}$$

□

PROPERTY 1.9. If $X \sim exp(\lambda_1)$ and $Y \sim exp(\lambda_2)$, where X and Y are independent, then $Z = min(X, Y) \sim exp(\lambda_1 + \lambda_2)$.

PROOF.

$$\begin{aligned} F_z(z) &= P(Z \leqslant z) \\ &= P(min(X, Y) \leqslant z) \\ &= 1 - P(min(X, Y) > z) \\ &= 1 - P(X > z, Y > z) \\ &= 1 - P(X > z)P(Y > z) \\ &= 1 - e^{-\lambda_1 z} e^{-\lambda_2 z} \\ &= 1 - e^{-(\lambda_1 + \lambda_2)z} \end{aligned}$$

The above result is the corresponding cumulative distribution function for an exponential random variable Z with the rate $\lambda_1 + \lambda_2$. Therefore, the desired result is trivial.

$\square$

CHAPTER 2

Statement of Problem

2.1. Statement of Problem

In this paper, we consider a task which has a completion time T (if not interrupted), which is a random variable with probability density function (pdf) $f(t)$, $t > 0$. Before it is complete, the task may be interrupted by a Poisson process with rate λ. If that happens, then the task must begin again, with the same completion time random variable T, but with a potentially different realization. These interruptions can reoccur, until eventually the task is finished, with a total time of W. In this paper, we will find the Laplace Transform of W in general and in several special cases.

In practical terms, let us consider a pedestrian attempting to cross a road, with traffic coming at some rate. If there is no traffic, then the pedestrian will walk uninterrupted and will complete crossing the road after some time. If traffic is present, then the task can only be completed if the pedestrian crosses walks in a sufficiently large gap in the traffic. Of course, in reality, the pedestrian will not cross if the gap is too short as the pedestrian would be hit. We could pretend that if the pedestrian gets hit by the traffic, the pedestrian bounces up and jumps back to the side of the road. For our purposes, we assume that the task must start over by the pedestrian starting back on the one side of the road. Further, in order to derive our result, we assume there is also some catastrophe at rate s which prevents the task from ever being completed. In this case, an example might be if the road is permanently closed and it is impossible for the road to be crossed.

Another example assumes a diver wants to reach a certain depth in the ocean. However, there is a phone that rings randomly above sea level and the diver must surface and pick it up every time it rings. Then, the diver would have to start over to achieve the goal of reaching a certain depth in the ocean. The catastrophe in this case could mean that an emergency occurs and if that happens before the completion of the task, the diver never completes the task.

Another example is a person adding a list of numbers mentally. If interrupted, the person must begin the addition all over again. We are interested in the distribution of the total time to complete the task.

In all cases, there is two possible scenarios that may occur during the task. The task can either be interrupted, or not interrupted. In the case that the task is not interrupted, then any interruption (or catastrophe) will occur after the time it takes to complete the task. In the case that an interruption occurs, then that implies that the interruption (or catastrophe) occurred before the task was completed. However, in order for the task to eventually be completed, the interruption must come before the catastrophe. Then, given the above scenarios, we assume that the task must start over. By combining these two cases, the problem of finding the total time until task completion will be studied in this paper, and we will derive explicit formulas for task completion times assuming various statistical distributions.

2.2. General Theorem

Suppose we want to complete a task where a Poisson process may cause an interruption in the task's progression, and then the task must start over. Then, using Property 1.5, we can assume that the time until an interruption occurs is exponentially distributed with rate λ.

Let T be the time to complete the task without interruption. Let W be the total time to complete the task including interruptions. Formally, we have a sequence of independent T_i, $i = 1, 2, \ldots$. Let U_i be a sequence of independent times until the next interruption. Let $V_i = min\{T_i, U_i\}$. Then $W = \sum_{i=1}^{N} V_i$ where N is a geometric random variable with probability of success equal to $P(T < U)$. The goal is to find an explicit formula for the Laplace Transform for the random variable W, with the corresponding probability density function $g(w)$. The following theorem is new.

THEOREM 2.2.1. *Let $f(t)$ be a probability density function for some continuous random variable T, which is the time to complete the task uninterrupted, with positive support on $[a, b]$, for $0 \leqslant a \leqslant b$.. Further, assume an interruption Poisson process of rate λ. Let W be the total time to complete the task, including interruptions. Then, the Laplace Transform for any the total completion time W is given by*

$$L_w(s) = \frac{\int_a^b e^{-(\lambda+s)t} f(t)dt}{1 - \frac{\lambda}{\lambda+s}\int_a^b (1 - e^{-(\lambda+s)t}) f(t)dt}$$

where $0 \leqslant a \leqslant b \leqslant \infty$.

PROOF. Assume that there exists a catastrophe random variable Y, which is an exponential random variable with rate s. Then, using Property 1.9, the rate of the Poisson process interruption coupled with the catastrophe will occur at a total rate of $\lambda + s$, assuming these events occur independently. Intuitively, we start with the following:

$L_W(s) = \int_a^b f(t)$ P(Poisson and catastrophe are later than t)$dt+$
$\int_a^b f(t)$ P(Poisson and catastrophe are early)P(Poisson beats catastrophe)$L_W(s)dt$

If the task beats the Poisson process interruption and the catastrophe, then we only need the first integrand. The probability of this event is $e^{-(\lambda+s)t}$, which is found using Property 1.9. However, we also have the possibility of either the Poisson interruption or the catastrophe coming before the task completes, which has a probability $1-e^{-(\lambda+s)t}$, and the Poisson process interruption occurs before the catastrophe, which has a probability of $\frac{\lambda}{\lambda+s}$ also found using Property 1.9. Then, we must multiply this integrand by the Laplace Transform, $L_W(s)$, as we must restart the task over again when an interruption occurs based on the memoryless property of exponential random variables.

Solving for $L_W(s)$, we get:

$$\begin{aligned}
L_W(s) &= \int_a^b e^{-(\lambda+s)t} f(t)dt + \int_a^b (1-e^{-(\lambda+s)t})\frac{\lambda}{\lambda+s} f(t)L_w(s)dt \\
&= L_W(s) - \int_a^b (1-e^{-(\lambda+s)t})\frac{\lambda}{\lambda+s} f(t)L_w(s)dt = \int_a^b e^{-(\lambda+s)t} f(t)dt \\
&= L_W(s)(1 - \int_a^b (1-e^{-(\lambda+s)t})\frac{\lambda}{\lambda+s} f(t)dt) = \int_a^b e^{-(\lambda+s)t} f(t)dt \\
&= L_W(s) = \frac{\int_a^b e^{-(\lambda+s)t} f(t)dt}{1-\frac{\lambda}{\lambda+s}\int_a^b (1-e^{-(\lambda+s)t}) f(t)dt}
\end{aligned}$$

And the result is as desired. □

This theorem will be relevant to all univariate continuous distributions with positive support throughout this major paper. Using this theorem allows for easier computation for the Laplace Transform.

The preceding theorem is a foundation for determining properties in various applications of tasks with interruption. For example, consider our example of crossing the road at some rate. The Laplace Transform, as derived above, can give us features such as the expected total time to cross the road, the variability in total time, and the probability of crossing the road uninterrupted.

CHAPTER 3

The Uniform (a, b) Case

3.1. The Laplace Transform for the $a \neq b$ Case

This section will consider a model with an uninterrupted task completion time T uniformly distributed on some interval (a, b) where a and b are distinct and $a < b$. The following theorem is new. Let W be the total completion time, including all interruption times.

THEOREM 3.1.1. *Let $T \sim Uniform(a, b), a < b$ be the uninterrupted task time. Then the Laplace Transform for the total task W time (including interruptions) with a Poisson process interruption at rate λ is given by*

$$L_W(s) = \frac{\left(-\mathrm{e}^{-(\lambda+s)a} + \mathrm{e}^{-(\lambda+s)b}\right)(\lambda + s)}{\lambda\left(-\mathrm{e}^{-(\lambda+s)a} + \mathrm{e}^{-(\lambda+s)b}\right) + s(\lambda + s)(a - b)}$$

PROOF. Recall that the probability density function for a uniform random variable T is $f(t) = \frac{1}{b-a}$ for $0 < a < b < \infty$. Using Theorem 2.1.1, the Laplace Transform can be found. In the Uniform case where $a \neq b, a < b$, the expression would simplify to

$$\begin{aligned}
L_W(s) &= \frac{\int_a^b e^{-(\lambda+s)t}\frac{1}{b-a}dt}{1 - \frac{\lambda}{\lambda+s}\int_a^b(1 - e^{-(\lambda+s)t})\frac{1}{b-a}dt} \\
&= \frac{\mathrm{e}^{-(\lambda+s)t}}{(-b + a)(\lambda + s)}\left(1 - \frac{\lambda}{(b - a)(\lambda + s)}\left(t - \frac{\mathrm{e}^{-(\lambda+s)t}}{-\lambda - s}\right)\right)^{-1} \\
&= \frac{-\mathrm{e}^{-(\lambda+s)a} + \mathrm{e}^{-(\lambda+s)b}}{(-b + a)(\lambda + s)}\left(1 + \frac{\lambda\left(-a\lambda - as + b\lambda + bs + \mathrm{e}^{-b(\lambda+s)} - \mathrm{e}^{-a(\lambda+s)}\right)}{(\lambda + s)^2(-b + a)}\right)^{-1}
\end{aligned}$$

$$= -\frac{\left(\mathrm{e}^{-(\lambda+s)a} - \mathrm{e}^{-(\lambda+s)b}\right)(\lambda+s)^2}{\left(-\mathrm{e}^{-a(\lambda+s)}\lambda + \mathrm{e}^{-b(\lambda+s)}\lambda + s(\lambda+s)(-b+a)\right)(\lambda+s)}$$

$$= \frac{\left(-\mathrm{e}^{-(\lambda+s)a} + \mathrm{e}^{-(\lambda+s)b}\right)(\lambda+s)}{\lambda\left(-\mathrm{e}^{-(\lambda+s)a} + \mathrm{e}^{-(\lambda+s)b}\right) + s(\lambda+s)(a-b)}$$

which completes the proof. □

Although this formula is lengthy for the uniform case, the Laplace Transform simplifies for numerical values of a and b. Using Theorem 3.1.1, we can find important properties in relation to a uniformly distributed task that is facing a Poisson process interruption, and analyze the effects of the Poisson process of rate λ, and how this impedes the completion of a task at hand.

For an example of a numerical computation, assume that $T \sim Uniform(0,1)$, and assume a Poisson process interruption occurring with a rate of 3 per minute. Then the Laplace Transform simplifies to

$$L_W(s) = -\frac{\left(-1 + \mathrm{e}^{-3-s}\right)(3+s)}{s^2 + 3\,s - 3\,\mathrm{e}^{-3-s} + 3}.$$

Recall that we can interpret s as the rate of a catastrophe. Then the probability that the [total] time of the task will complete before a catastrophe is given by $L_W(s)$. Plugging in various rates for the catastrophe (that is $Y \sim exp(s)$) will give the corresponding probabilities that the task will beat the catastrophe. Note that an increase in the rate of catastrophe will decrease the probability of the task beating the catastrophe. i.e. $L_W(s)$ is a decreasing function of s.

3.2. Mean and Variance of the Uniform $a \neq b$ Case

The Laplace Transform in the uniform case is twice differentiable, and the first two moments of the uniform distribution exist. Therefore, it would be beneficial to apply Property 1.7 to determine the expected time to complete a uniformly distributed task and its variance. Notice that utilizing the Laplace Transform provides us with much more simplistic calculations as opposed to the traditional method of using integration techniques to solve for the moments of the random variable, and the summary statistics of the task.

Taking the first two derivatives of the Laplace Transform in Theorem 3.1.1, we get expressions that may seem cumbersome due to the number of parameters involved.

$$L'_W(s) = \frac{A + B + C}{\left(-\lambda\, \mathrm{e}^{-(\lambda+s)a} + \lambda\, \mathrm{e}^{-(\lambda+s)b} + s\,(\lambda+s)\,(-b+a)\right)^2}$$

where:

$$A = -2\,\mathrm{e}^{-(\lambda+s)(a+b)}\lambda + (\lambda+s)^2\,(sa+1)\,(-b+a)\,\mathrm{e}^{-(\lambda+s)a}$$

$$B = -\,(\lambda+s)^2\,(sb+1)\,(-b+a)\,\mathrm{e}^{-(\lambda+s)b}$$

$$C = \lambda(e^{-2(\lambda+s)a} + e^{-2(\lambda+s)b})$$

Let us assume the random variable is $T \sim Uniform(0,1)$. Evaluating these expressions at $s = 0$ will give us the two moments of the Laplace Transform, namely:

$$L'_W(0) = \frac{-2\,\lambda\,\mathrm{e}^{-\lambda} - \lambda^2 + \lambda^2\mathrm{e}^{-\lambda} + \lambda\,\left(1 + \mathrm{e}^{-2\,\lambda}\right)}{\left(-\lambda\, + \lambda\,\mathrm{e}^{-\lambda}\right)^2}$$

and

$$L''_W(0) = \frac{8\,\lambda^2\mathrm{e}^{-\lambda} - \lambda\,\left(2\,\lambda^2 + 4\,\lambda\right)\mathrm{e}^{-2\,\lambda} + 2\,\lambda^3 - 4\,\lambda^2}{\left(-\lambda\,\mathrm{e}^{-\lambda} + \lambda\right)^3}$$

Now assume that the interruption occurs at a rate of 5 per minute (i.e., $\lambda = 5$), then the total expected time to complete the task for T is

$$E(W) = -L'_W(0) = -(-0.8067836550) = 0.8067$$

Therefore, the expected time to complete the task is 0.8067 minutes (approximately 48.4 seconds).

$$\begin{aligned} Var(W) &= E(W^2) - (E(W))^2 \\ &= L''_W(0) - (E(W))^2 = 1.235459077 - (0.8067)^2 = 0.5847. \end{aligned}$$

So the variance of the task completion is 0.5847 and standard deviation is .7647 (approximately 45.9 seconds). These computational results follow directly from Property 1.7 and Theorem 3.1.1.

Increasing the rate of the Poisson process interruption, we can infer that the expected time to complete a task should increase, as there is more of a chance that the task at hand may face an interruption. If we increase λ, for example, $\lambda = 10$ per minute, we get

$$E(W) = -(-0.9000) = 0.9000$$

and

$$Var(W) = 1.60025 - (0.9000)^2 = 0.7903$$

If we let λ be sufficiently large, then we should receive the maximum waiting time for T. If we let $\lambda = 100$ per minute, then

$$E(T) = -(-0.999) = 0.999$$

and

$$Var(T) = 1.996 - (0.999)^2 = 0.998$$

.

We notice that as λ gets larger, then the expected time, and the variance, approach 1. This happens to be the upper bound of the uniform random variable (recall that $T \sim Uniform(0,1)$). Taking the limit of the Laplace Transform, we get

$\lim_{\lambda \to \infty} -L'_w(0) = 1$ when $T \sim Uniform(0,1)$.

This corresponds with our previous calculations. This interesting result poses questions on whether or not there is a general result for a task that is uniformly distributed and whether there is a maximal waiting time on a specified interval.

COROLLARY 3.1. *Let $T \sim Uniform(0,b), b > 0$. The maximum total completion time for a task interrupted by a Poisson process is b.*

The proof follows from our expression from $\lim \lambda \to \infty L'_W(s)$ when $a = 0$ and $s = 0$. It is curious that there should be an upper bound at all.

3.3. The $a = b$ Case

If $a = b$, then the pdf of the Uniform distribution turns into $f(t) = \frac{1}{a-a}$, which is infinite. Therefore, direct integration is not an appropriate method to determine an explicit formula for the Laplace Transform for this particular case. Fortunately, we can approximate the integrals used to determine the Laplace Transform using Riemann sums.

Definition 3.2. The Riemann sum, denoted as S for $f(x)$ with (a,b) partitioned n times is found using

$$S = \sum_{i=1}^{n} f(x_i)\Delta x_i$$

For the purposes of simplicity, we are going to take only one 'sub-interval', so we are not going to partition the function into more than one piece. Therefore, $n = 1$ and we do not need the summation sign. To find $\lim_{b\to a}\int_a^b e^{-(\lambda+s)t} f(t)dt$, we will use a geometric interpretation to find the area underneath $f(t)$ by multiplying the base by the height.

$f(x) = \dfrac{1}{b-a}$ and the base would be $\Delta x = b - a$, since we are measuring the distance from the endpoints. So the integrals needed for the Laplace transform calculations exist and

$$\lim_{b\to a}\int_a^b e^{-(\lambda+s)t} f(t)dt = \frac{1}{b-a}(b-a)e^{-(\lambda+s)a} = e^{-(\lambda+s)a}$$

and

$$\lim_{b\to a}\frac{1}{b-a}\int_a^a (1-e^{-(\lambda+s)t})f(t)dt = \frac{1}{b-a}(b-a)(1-e^{-(\lambda+s)a}) = 1-e^{-(\lambda+s)a}$$

By applying Theorem 2.1.1 and 3.1.1, we have:

$$\begin{aligned} L_W(s) &= \frac{e^{-(\lambda+s)a}}{1-\frac{\lambda}{\lambda+s}(1-e^{-(\lambda+s)a})} \\ &= \frac{e^{-(\lambda+s)a}(\lambda+s)}{e^{-(\lambda+s)a}\lambda+s} \end{aligned}$$

To compute the mean and variance of the $a = b$ case, it is very similar to the $a \neq b$ case above. Using MAPLE,

$$L'_W(s) = -\frac{e^{-a(\lambda+s)}\left(a\lambda\, s + as^2 - e^{-a(\lambda+s)}\lambda + \lambda\right)}{\left(e^{-a(\lambda+s)}\lambda + s\right)^2}$$

$$L''_W(s) = \frac{\left(-\lambda\left(\left(a^2s+2a\right)\lambda+a^2s^2+4sa+2\right)e^{-a(\lambda+s)}+\left(a^2s^2+2sa+2\right)\lambda+a^2s^3\right)e^{-a(\lambda+s)}}{\left(e^{-a(\lambda+s)}\lambda+s\right)^3}$$

so

$$EW) = -L'_w(0) = -\frac{-e^{-\lambda\, a}+1}{e^{-\lambda\, a}\lambda} = \frac{e^{\lambda a}-1}{\lambda}$$

$$\begin{aligned} Var(W) &= L''_W(0) - (L'_W(0))^2 \\ &= \frac{4\,(\lambda\, a+1)^2\, e^{2\,\lambda\, a} + (-8\,\lambda\, a - 8)\, e^{3\,\lambda\, a} + \lambda^3 e^{\lambda\, a} - \lambda^3 + 4\, e^{4\,\lambda\, a}}{\lambda^4} \end{aligned}$$

Note that if you substitute all of the b values into the mean and variance expressions in Section 3.1 with a, then you will receive the same desired mean and variance expressions that are seen above. The expression for $E(W)$ matches the well known result for the expected time to safely cross a one-way highway when vehicles arrive according to a Poisson process.

3.4. Graphical Interpretation of the $Uniform(0,1)$ Distribution, $a \neq b$

The Laplace Transform should be a strictly decreasing function on the interval $(0, \infty)$.

As it is shown, the probability decreases exponentially, and converges to zero when s is relatively large. Information about the moments is obtained for small values of s (close to zero).

For convenience, this graph is for the case where $\lambda = 5$.

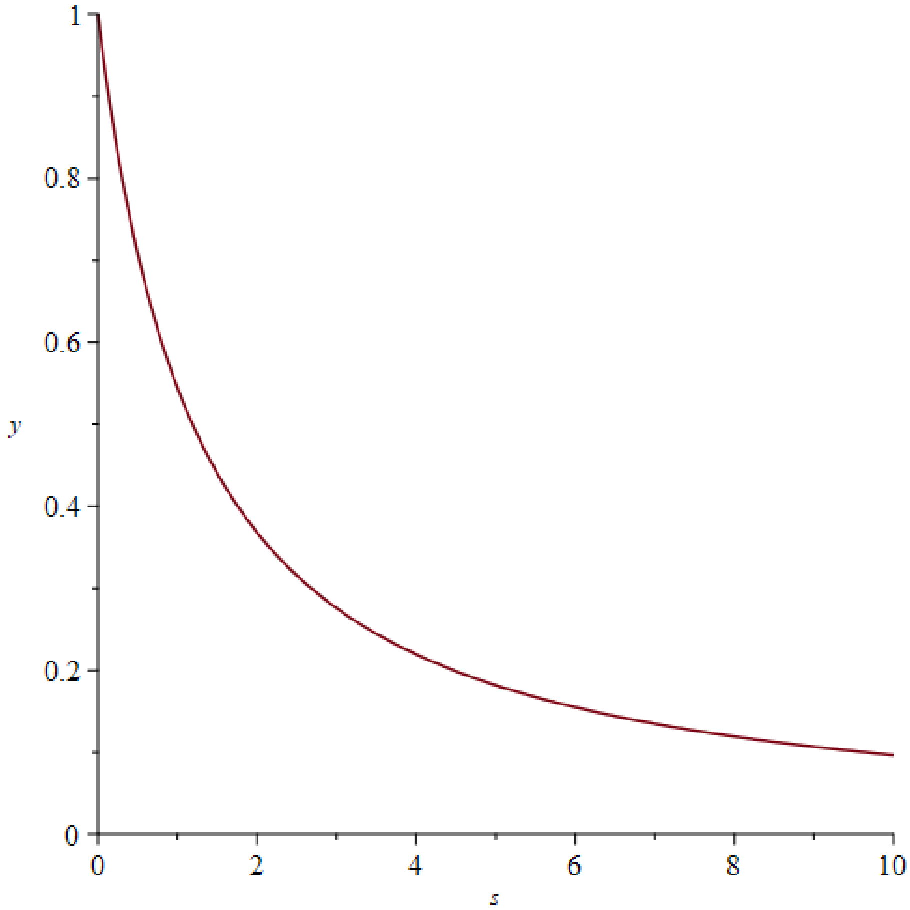

FIGURE 3.1. Laplace Transform for the Uniform(0,1) Distribution for $\lambda = 5$

Another feature of the Laplace Transform graphs is that you can approximate the mean and variance, since both quantities are directly related to the derivatives (i.e., the rate of change) of $L_w(s)$. The following approximation technique was used by Huang (2016).

$$E(X) \approx -\frac{L(0.01) - L(0)}{0.01 - 0}$$

$$Var(X) \approx \frac{\Delta L(0.02) - \Delta L(0.01)}{0.02 - 0.01} - \mu^2$$

where $\Delta L(0.02) = L(0.02) - L(0.01)$ and $\Delta L(0.01) = L(0.01) - L(0)$.

In this case,

$$E(X) = -\frac{0.9919934729 - 1}{0.01 - 0} = 0.8007$$

$$Var(X) = \frac{-0.0078857349 - (- - 0.0080065271)}{0.02 - 0.01} = 1.208 - (0.8007)^2 = 0.5669$$

.

These results have a minor marginal error compared to the true values of the mean and variance seen in Section 3.2.

CHAPTER 4

The Exponential(k) Case

4.1. The Laplace Transform for $T \sim Exp(k)$

This section will consider an uninterrupted exponentially distributed task time, with the conditions of the interruptions as in previous chapters(Poisson process - i.e., exponentially distributed). The fact that the tasks and interruptions come from the same class of distributions provides us with some very interesting results. We will be using the pdf as described in Definition 1.2. The following theorem is new.

Method 1

THEOREM 4.1.1. *Let $T \sim Exp(k), k > 0$ be the uninterrupted completion time random variable. Assume a Poisson process interruption with rate λ. Let W be the total completion time. Then the corresponding Laplace Transform for W is given by*

$$L_w(s) = \frac{k}{s+k}$$

.

PROOF. From Theorem 2.2.1,

$$\begin{aligned} L_W(s) &= \frac{\int_0^\infty \mathrm{e}^{-(\lambda+s)t} k\mathrm{e}^{-kt}\,\mathrm{d}t}{1 - \int_0^\infty \frac{\lambda\left(1-\mathrm{e}^{-(\lambda+s)t}\right)k\mathrm{e}^{-kt}}{\lambda+s}\,\mathrm{d}t} \\ &= \frac{\lim_{t\to\infty} -\frac{k\left(\mathrm{e}^{-kt-\lambda t-st}-1\right)}{k+\lambda+s}}{1 - \lim_{t\to\infty} \frac{(\mathrm{e}^{-\lambda t-st}\mathrm{e}^{-kt}k-k\mathrm{e}^{-kt}-\lambda\mathrm{e}^{-kt}-s\mathrm{e}^{-kt}+\lambda+s)\lambda}{(k+\lambda+s)(\lambda+s)}} \end{aligned}$$

Using properties of the exponential function, we know that $\lim_{x\to\infty} e^{-kx} = 0$ for $k > 0$ and $\lim_{x\to\infty} Ce^{-kx} = 0$ for any collection of constants C.

Therefore, the above expression simplifies to

$$
\begin{aligned}
L_W(s) &= -\frac{k\,(0-1)}{k+\lambda+s}\left(1 - \frac{0-0-0-0+\lambda(\lambda+s)}{(k+\lambda+s)(\lambda+s)}\right)^{-1} \\
&= (\frac{k}{k+\lambda+s})(\frac{s+k}{\lambda+s+k})^{-1} = (\frac{k}{\lambda+s+k})(\frac{\lambda+s+k}{s+k}) \\
&= \frac{k}{s+k}
\end{aligned}
$$

This completes the proof. □

This result is very surprising for many reasons. First and foremost, the Laplace Transform is a function of k and s only. This implies that the time to complete the task does not depend on λ, the interruption rate. Intuitively, if $\lambda \to 0$, then the completion time should only depend on k. However, if λ is large enough, then the probability that the task will face an interruption should increase, and thus increase the overall expected total time to complete the task. This poses questions on the authenticity of the result. In order to help validate this Laplace transform, we consider a case when λ is not zero.

Suppose $\lambda = 1$ and $k = 1$. If our Laplace transform is correct, then the expected time to complete the task is $\frac{1}{k} = 1$. (this result will be derived in the next section). However, we have two possible next event - a completion or an interruption, each at rate 1. So the rate of the minimum of the two events is $1 + 1 = 2$. So, the expected time for either of these events to occur is $\frac{1}{\lambda+k} = \frac{1}{2}$. The probability that first obtain the k event (i.e, no interruption) is $\frac{1}{2}$. If not, we must restart. Now, we again have expected time $\frac{1}{2}$left until for the next event with probability $\frac{1}{2}$ of being the k event. And, the conditional probability for this specific scenario (given that the first event is an interruption) would also be $\frac{1}{2}$. So, the probability for this scenario (interruption, completion) is $(\frac{1}{2})^2 = \frac{1}{4}$. We can obtain the expected total time to completion by summing all possible cases to get :

$$\begin{aligned} P(total) &= (\frac{1}{2})(\frac{1}{2}) + (\frac{1}{2})^2(1) + (\frac{1}{2})^3(\frac{3}{2}) + (\frac{1}{2})^4(2) + ... + (\frac{1}{2})^n(\frac{n}{2}) + \dots \\ &= \sum_{n=1}^{+\infty} \frac{n}{2^{n+1}} \end{aligned}$$

We recognize this as the expected value of a geometric random varaible or, by Wolfram-Alpha, it is determined that

$$\sum_{n=1}^{+\infty} \frac{n}{2^{n+1}} = 1$$

This is consistent with our earlier result, so this suggests that our Laplace Transform is correct.

Method 2

To simplify integral calculations needed for the derivation, we can use the tails of the exponential calculations for simplicity.

$$\int_0^\infty e^{-(\lambda+s)t} k e^{-kt} dt = k \int_0^\infty e^{-(\lambda+s+k)t} dt = k \frac{\lambda+s+k}{\lambda+s+k} \int_0^\infty e^{-(\lambda+s+k)t} dt.$$

Notice that $(\lambda + s + k) \int_0^\infty e^{-(\lambda+s+k)t}$ is the probability density function for the exponential distribution for $T \sim Exp(\lambda + s + k)$, therefore:

$$(\lambda + s + k) \int_0^\infty e^{-(\lambda+s+k)t} dt = 1$$

Therefore,

$$\int_0^\infty e^{-(\lambda+s)t} k e^{-kt} dt = \frac{k}{\lambda+s+k}$$

Further,

$$\int_0^\infty ke^{-kt}(1-e^{-(\lambda+s)t})dt = \int_0^\infty ke^{-kt}dt - \int_0^\infty ke^{-kt}e^{-(\lambda+s)t}dt$$

The first integrand is just simply the density function of $T \sim Exp(k)$, and the second integrand is the same integrand above that is $T \sim Exp(\lambda+s+k)$. So, we get

$$\int_0^\infty ke^{-kt}(1-e^{-(\lambda+s)t})dt = 1 - \frac{k}{\lambda+s+k} = \frac{\lambda+s}{\lambda+s+k}$$

Plugging these evaluated integrals into Theorem 2.1.1 gives the same result in Method 1, that

$$L_W(s) = \frac{k}{s+k}$$

.

4.2. Mean and Variance the Exponential Case

We could simply observe that the Laplace transform tells us immediately that the total completion time is exponentially distributed, with mean $1/k$. Or using similar procedures as for he Uniform case, taking the first two derivatives of the Laplace Transform will give us results for the mean and variance of total time to complete the task. Using $L_w(s)$ from Theorem 4.1.1, then:

$$L'_w(s) = \frac{-k}{(s+k)^2}$$

$$L''_w(s) = \frac{2k}{(s+k)^3}$$

Substituting $s = 0$ and rearranging the formulae in Property 1.7, we get

$$E(W) = -(\frac{-1}{k}) = \frac{1}{k}$$

and

$$Var(T) = E(T^2) - (\frac{1}{k})^2 = L''_w(0) - \frac{1}{k^2} = \frac{2}{k^2} - \frac{1}{k^2} = \frac{1}{k^2}.$$

If one studies the monotonicity of the mean and variance functions, both of them are decreasing in k on the interval $(0, \infty)$, which is sensible since increasing k should increase the task rate, and consequently decrease the expected time to complete the task, as hypothesized.

PROPERTY 4.1. As $k \to \infty$, for fixed s, the Laplace transform tends to 1, so the probability that the task completes before encountering a catastrophe with rate s is one.

PROOF.

$$\begin{aligned}\lim_{k\to\infty} P(W < Y) &= \lim_{k\to\infty} \frac{k}{s+k} \\ &= \lim_{k\to\infty} \frac{1}{\frac{s}{k}+1} \\ &= \frac{1}{0+1} \\ &= 1\end{aligned}$$

□

PROPERTY 4.2. As $k \to \infty$, for fixed s the expected value of total completion time approaches zero with a variance of zero.

PROOF.

$$\lim_{k\to\infty} E(T) = \lim_{k\to\infty} \frac{1}{k} = 0$$

$$\lim_{k\to\infty} Var(T) = \lim_{k\to\infty} \frac{1}{k^2} = 0$$

□

4.3. Graphical Interpretation of the Exponential Distribution

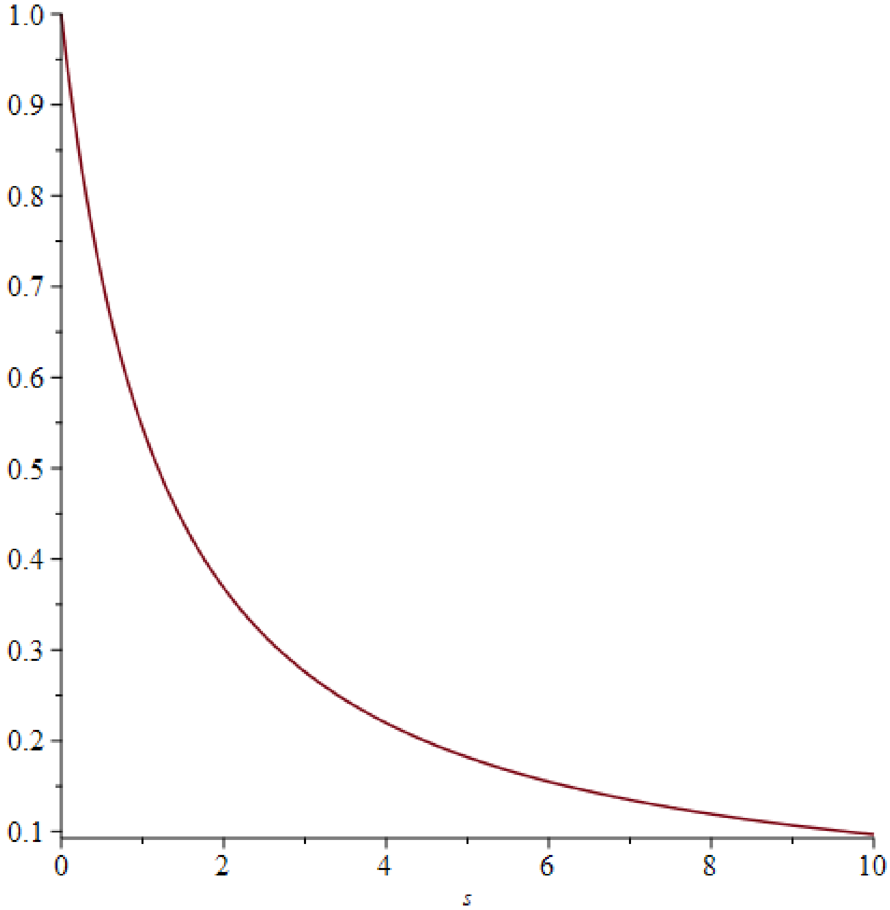

FIGURE 4.1. Laplace Transform for the Exp(1) Distribution for $\lambda = 5$

The graphical properties for the Laplace transform are similar to those of the Uniform case, with the exception that the graph decreases quicker than in the Uniform case. For future chapters, the graphs of the Laplace transform will be omitted due to similarity of the Laplace transform graphs. The numerical approximation of determining the mean and variance of the total waiting time of a task can also be found using the same methods as described in section 3.4.

CHAPTER 5

The Gamma(α, β) Case

5.1. The Laplace Transform for $T \sim Gamma(2,1)$

This section will consider a model where a task, T, is distributed as $Gamma(2,1)$. Using Definition 1.1, the probability density function in this case is $f(t) = te^{-t}$. This is structurally similar to the exponential case seen in Chapter 4, with a linear t term being multiplied by the exponential term, but the results end up changing drastically. These results will be shown in this chapter. The following theorem is new.

THEOREM 5.1.1. *Let $T \sim Gamma(2,1)$ be the time to complete a task without interruption. Let W be the total time to complete the task, which can be interrupted by a Poisson process at rate λ. Then the Laplace Transform for W is given by*

$$L_W(s) = \frac{1}{s^2 + (\lambda+2)s + 1}.$$

PROOF. Using Theorem 2.1.1, the Laplace Transform will be:

$$\begin{aligned} L_W(s) &= \int_0^\infty e^{-(\lambda+s)t} te^{-t} dt + \int_0^\infty (1 - e^{-(\lambda+s)t}) \frac{\lambda}{\lambda+s} te^{-t} L_w(s) dt \\ &= \frac{\int_0^\infty e^{-(\lambda+s)t} te^{-t} dt}{1 - \frac{\lambda}{\lambda+s} \int_0^\infty (1 - e^{-(\lambda+s)t}) te^{-t} dt} \\ &= \frac{(N)}{(D)} \end{aligned}$$

Now,

$$
\begin{aligned}
(N) &= \int_0^\infty e^{-(\lambda+s)t} t e^{-t} dt \\
&= \int_0^\infty t e^{-(\lambda+s+1)t} dt \\
&= \lim_{t\to\infty} \frac{1 + (-1 + t\,(-1 - \lambda - s))\, \mathrm{e}^{-t(1+\lambda+s)}}{(1+\lambda+s)^2}
\end{aligned}
$$

Similar to Chapter 4, we know by L'Hopital's Rule that $\lim_{x\to\infty} xe^{-x} = 0$. This fact, with the other properties of exponential limits in the last chapter, implies that

$$
\lim_{x\to\infty} (-1 + t(-1 - \lambda - s)e^{-t(1+\lambda+s)}) = 0
$$

Therefore,

$$
(N) = \frac{1}{(1+\lambda+s)^2}
$$

Next,

$$
\begin{aligned}
(D) &= 1 + \lim_{t\to\infty} \frac{\lambda \left((-t\lambda - ts - t - 1)\, \mathrm{e}^{-t(\lambda+s+1)} + (\lambda+s+1)^2 (t+1)\, \mathrm{e}^{-t} - (*) \right)}{(\lambda+s+1)^2 (\lambda+s)} \\
&= 1 - \frac{\lambda(\lambda+s)(\lambda+s+2)}{(\lambda+s)(\lambda+s+1)^2} \\
&= \frac{s^2 + (\lambda+2)s + 1}{(\lambda+s+1)^2}
\end{aligned}
$$

with

$$
(*) = (\lambda+s+2)(\lambda+s)
$$

Finally,

$$
\frac{(N)}{(D)} = L_w(s) = \frac{1}{s^2 + (\lambda+2)s + 1}
$$

This completes the proof. □

Clearly, the denominator is quadratic in s. Because the denominator is factorizable into two real roots, we can write the Laplace Transform as a product. This will be discussed further in the chapter.

5.2. The Erlang Case

DEFINITION 5.1. A Gamma (α, β) random variable X is said to be Erlang (n, λ) if $\alpha = n$ is a positive integer, and $\lambda = 1/\beta$.

The following property is well known.

PROPERTY 5.2. If $X_1, X_2, ..X_n$ are i.i.d. exponentially distributed with common rate λ then $X = \sum_{i=1}^{n} X_i$ is Erlang (n, λ).

DEFINITION 5.3. If $X_1, X_2, ..X_n$ are independent and exponentially distributed with rates λ_i for $i = 1, 2, \ldots, n$ then we say that $X = \sum_{i=1}^{n} X_i$ is generalized Erlang and write

$$X = \sum_{i=1}^{n} X_i \sim genEr(\lambda_1 \lambda_2 ... \lambda_n)$$

If $\lambda_1 = \cdots = \lambda_n = \lambda$, then $genEr(\lambda_1 \lambda_2 ... \lambda_n)$ reduces to Erlang (n, λ)

In previous chapters, we have been using the fact we want to predict the probability that the task will be completed before a catastrophe. The following property is taken from Roy (1997)

PROPERTY 5.4. If $X \sim genEr(\lambda_1 \lambda_2 ... \lambda_n)$, then:

$$L_X(s) = \prod_{i=1}^{n} \frac{\lambda_i}{\lambda_i + s}$$

Proof.

$$
\begin{aligned}
L_X(s) &= P(X < Y) \\
&= P(X_1 + X_2 +, ..., +X_n < Y) \\
&= P(X_1 < Y)P(X_2 < Y)...P(X_n < Y) \qquad (Memoryless) \\
&= \prod_{i=1}^{n} P(X_i < Y) \\
&= \prod_{i=1}^{n} \frac{\lambda_i}{\lambda_i + s}
\end{aligned}
$$

□

Corollary 5.5. *If* $X \sim Er(n, \lambda)$, *then:*

$$L_X(s) = (\frac{\lambda}{\lambda + s})^n$$

This corollary follows from the proof of Property 5.4, with using the fact that all of the rates are identical independently distributed (iid).

In words, Property 5.4 implies that the product of the Laplace transforms for independent exponential random variables is the Laplace transform of the sum of the sum of the exponential random variables. This property will be particularly useful for the $Gamma(2, 1)$ Laplace transform seen in Section 5.1.

Recall that the Laplace Transform for the $Gamma(2, 1)$ random variable is

$$L_W(s) = \frac{1}{s^2 + (\lambda + 2)s + 1}.$$

We can factor the denominator to split the Laplace transform into a product of two Laplace transforms, each representing exponentially distributed random variables at different rates.

The following result may be useful.

PROPERTY 5.6. Descartes' Law of Signs states that the number of positive real zeroes in a polynomial function $f(x)$ is the same or less than by an even numbers as the number of changes in the sign of the coefficients. The number of negative real zeroes of the $f(x)$ is the same as the number of changes in sign of the coefficients of the terms of $f(-x)$ or less than this by an even number.

By using Descartes' Law of Signs, our denominator will have no positive roots. Alos, since the discriminant $b^2 - 4ac$ is nonnegative, then there are two real roots (perhaps equal), so they are both negative.

The denominator can be expressed as $(s-a)(s-b)$, where a and b are the solutions to the equation

$$0 = s^2 + (\lambda + 2)s + 1$$

.

So,

$$a, b = \frac{-(\lambda+2) \pm \sqrt{(\lambda+2)^2 - 4(1)(1)}}{2(1)}$$

Therefore, using the form above,

$$a = \frac{-\lambda - 2 + \sqrt{\lambda^2 + 4\lambda}}{2}$$

and

$$b = \frac{-\lambda - 2 - \sqrt{\lambda^2 + 4\lambda}}{2}$$

Let's consider the special case where $\lambda = \frac{1}{2}$. Then, using all of the facts above, we know this case satisfies the equations above. We can then split the Laplace transform as such:

$$L_w(s) = \frac{1}{s^2 + (\frac{5}{2})s + 1} = \frac{\frac{1}{2}}{s + \frac{1}{2}} \frac{2}{s + 2}$$

Using Theorem 4.1.1, we can see that the two products are of the form of the exponential Laplace transform, with the first product being of $Exp(\frac{1}{2})$ and the second product being $Exp(2)$. Using Definition 5.3 coupled with Property 5.4, we can conclude that the r.v. W is distributed as generalized Erlang with rates $2, 0.5$.

5.3. The $Gamma(3, 1)$ Case

The following theorem is new.

THEOREM 5.3.1. *Let $T \sim Gamma(3, 1)$ be the task time for an uninterrupted task. Let W be the total task time for a task with potential interruptions from a Poisson process with rate λ. Then the corresponding Laplace Transform for W is*

$$L_W(s) = \frac{1}{(\lambda^2 + (2\,s + 2)\,\lambda + s^2 + 2\,s + 2)\,(2 + \lambda + s)}.$$

The proof is procedurally the same as the $Gamma(2, 1)$ case. The previous result was found using Maple software. After expanding the denominator, it is clearly cubic in s for this particular case, whereas in the $Gamma(2, 1)$ case the denominator is quadratic in s. Recall that it was possible to write the Laplace transform as the product of two exponential random variables, which implies that the W is a sum of two exponential random variables.

It is determined using basic algebra that the denominator only has one real root when $s = -\lambda - 2$. However, from the Descartes' Law of Signs, the quadratic part seen in the Laplace transform will have zero real roots. Due to this, there will be complex roots in the cubic. Therefore, it is not possible to write W as a sum of three exponential random variables.

To determine the mean and variance of the Laplace transform, the same techniques as seen in the $Gamma(2, 1)$ case should be utilized. The calculations are not presented to avoid redundancy.

When comparing the $Gamma(2, 1)$ and $Gamma(3, 1)$ cases, there are questions that arise that should be considered for future research. For example, the pdf of the $Gamma(3, 1)$ case has a quadratic t term, whereas the $Gamma(2, 1)$ case has a linear t term. When calculating the values of each Laplace transform under the same conditions (i.e., rates of interruptions), the $Gamma(3, 1)$ values are numerically less than the $Gamma(2, 1)$ case. Further, recall that the $Gamma(2, 1)$ case has numerically lower values than the $Exp(k)$ case, which did not have a t term. So, the question on which parameters influence the rate of interruption less frequently could be addressed. Further, the degree of the Laplace transform is cubic when $\alpha = 3$ and quadratic when $\alpha = 2$. The question remains as to whether the degree of the denominator is n when $\alpha = n$.

CHAPTER 6

The Weibull(λ,k) Case

DEFINITION 6.1. A random variable X is said to be $Weibull(\lambda, k)$ if the pdf is

$$f(x) = \frac{k}{\lambda}(\frac{x}{\lambda})^{k-1}e^{-(\frac{x}{\lambda})^k}, x > 0.$$

A Weibull random variable is a generalization of an exponential random variable. This section will consider a model where an uninterrupted task, T, is distributed as $Weibull(2, 1)$. Results are presented in this chapter. The following theorem is new.

THEOREM 6.0.1. *Let $T \sim Weibull(2, 1)$ be the completion time of an uninterrupted task. Let W be the total time to complete the task, which can be interrupted by a Poisson process at rate λ. Then the corresponding Laplace Transform for W is*

$$L_w(s) = \frac{\sqrt{\pi}\mathrm{e}^{\frac{(\lambda+s)^2}{4}}(\lambda+s)\,\mathrm{erf}\left(\frac{\lambda}{2}+\frac{s}{2}\right)+2-\sqrt{\pi}\mathrm{e}^{\frac{(\lambda+s)^2}{4}}(\lambda+s)}{\sqrt{\pi}\mathrm{erf}\left(\frac{\lambda}{2}+\frac{s}{2}\right)\mathrm{e}^{\frac{(\lambda+s)^2}{4}}\lambda-\sqrt{\pi}\mathrm{e}^{\frac{(\lambda+s)^2}{4}}\lambda+2}$$

with

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx$$

PROOF. The pdf of the $T \sim Weibull(2, 1)$ distribution is $f(t) = 2te^{-t^2}$. We use Theorem 2.1.1 and write $L_W(s) = \frac{(N)}{(D)}$ as we did with the $Gamma(2, 1)$ example. Using Maple software:

$$(N) = \int_0^\infty 2e^{-(\lambda+s)t}te^{-t^2}dt$$

$$= \frac{\sqrt{\pi}\,(\lambda+s)}{2}\mathrm{e}^{\frac{(\lambda+s)^2}{4}}\mathrm{erf}\left(\frac{\lambda}{2}+\frac{s}{2}\right)+1-\frac{\sqrt{\pi}\,(\lambda+s)}{2}\mathrm{e}^{\frac{(\lambda+s)^2}{4}}$$

$$(D) = 1-\frac{\lambda}{\lambda+s}\int_0^\infty (1-e^{-(\lambda+s)t})2te^{-t^2}dt$$

$$= 1+\frac{\sqrt{\pi}\lambda}{2}\mathrm{e}^{\frac{(\lambda+s)^2}{4}}\mathrm{erf}\left(\frac{\lambda}{2}+\frac{s}{2}\right)-\frac{\sqrt{\pi}\lambda}{2}\mathrm{e}^{\frac{(\lambda+s)^2}{4}}$$

And finally,

$$L_w(s) = \frac{(N)}{(D)} = \frac{\sqrt{\pi}\mathrm{e}^{\frac{(\lambda+s)^2}{4}}(\lambda+s)\,\mathrm{erf}\left(\frac{\lambda}{2}+\frac{s}{2}\right)+2-\sqrt{\pi}\mathrm{e}^{\frac{(\lambda+s)^2}{4}}(\lambda+s)}{\sqrt{\pi}\mathrm{erf}\left(\frac{\lambda}{2}+\frac{s}{2}\right)\mathrm{e}^{\frac{(\lambda+s)^2}{4}}\lambda-\sqrt{\pi}\mathrm{e}^{\frac{(\lambda+s)^2}{4}}\lambda+2}$$

This completes the proof. □

There are some limitations for this Laplace transform due to the inclusion of the error function that is in the expression. If s becomes too large, then we have issues with the Laplace result. This is because as $x \to \infty$, $erf(x) \to 1$ since $\int_0^x e^{-t^2}dt \approx \frac{\sqrt{\pi}}{2}$ for reasonably large values of x. In our case, the error function is a function of λ and s.

A graphical version of $L_W(s)$ is shown here:

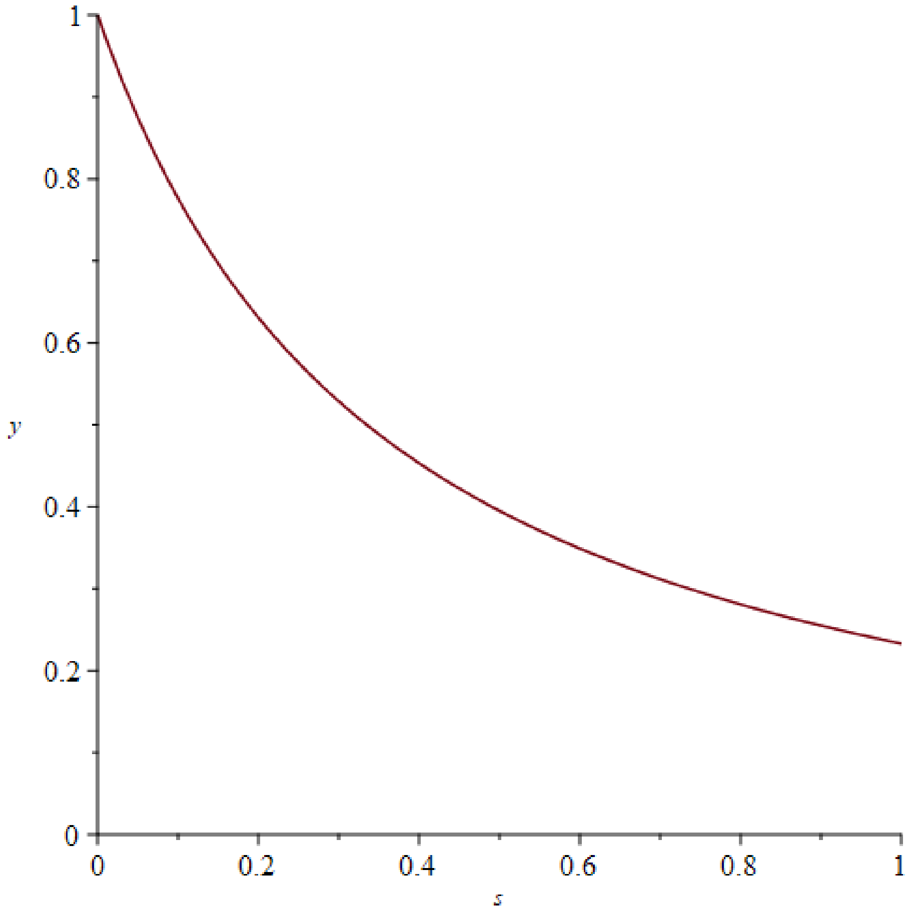

FIGURE 6.1. Laplace Transform for the Weibull Distribution for $\lambda = 5$

The mean and variance of the expected total time to complete the task is done similarly to the uniform, exponential and gamma cases. However, we need the derivative of the erf() function. This can be found using the Fundamental Theorem of Calculus. Beyond, that, there is nothing different about the computations so they are not included to avoid redundancy.

CHAPTER 7

Laplace Inversion

The following theorem from Spiegel (1965) is referenced in Huang (2016).

THEOREM 7.0.1. *Given the Laplace transform* $L(s)$*, the function value* $f(t)$ *can be recovered from the contour integral*

$$f(t) = \frac{1}{2\pi i}\int_{b-\infty}^{b+\infty} e^{st}L(s)\,ds$$

where b *is a real number to right of all singularities of* $L(s)$*, and the contour integral yields the value* 0 *for* $t < 0$.

The following integral is commonly referred to as the Bromwich inversion integral. This theorem is the stepping stone to invert a Laplace transform to find the probability density function. It is very difficult by hand to utilize the Bromwich inversion integral, or other known techniques to solve for the inverse Laplace transform. However, Maple software can be used to solve for the inverted Laplace transform, which will consequently give us the pdf's for the Laplace transform. To use Maple to find the inverse Laplace transform, one must use the with(inttrans): command coupled with the invtrans(LT,s,t) command.

The advantage of using the inverse Laplace transform is that it gives us the density function of the random variable W, which is the expected total time to complete the task. For the Uniform and Weibull distributions, there is no explicit inverse Laplace transform, so the pdf's of these Laplace transform can not be determined from MAPLE. However, explicit solutions were found for the Exponential and Gamma case.

For the exponential case, the pdf from the Laplace transform, by using Maple, is

$$f(w) = ke^{-kw}$$

. This result is interesting because this pdf is simply just the exponential distribution with parameter k.

For the $Gamma(2, 1)$ case, the pdf from the Laplace transform, by using Maple, is

$$f(w) = \frac{2\, \mathrm{e}^{\frac{-1}{2}(\lambda+2)w} \sinh\left(\frac{1}{2}\, w\sqrt{\lambda\ (4+\lambda)}\right)}{\sqrt{\lambda\ (4+\lambda)}}$$

.

This pdf is clearly more complex than that of the exponential case. Further, the pdf does not align with any of the common univariate distributions that are normally used in statistical inference. This pdf also is possibly not ideal as the hyperbolic sine function includes supported with $s < 0$, which is not consistent with our version of Laplace transforms of pdfs with positive support. However, it will be shown that we can write this pdf in a much simpler form.

The graph of the inverse Laplace transform for the exponential case is trivial, as it is just the standard exponential function pdf. The pdf of W resulting from $T \sim$ gamma is shown when $\lambda = 5$.

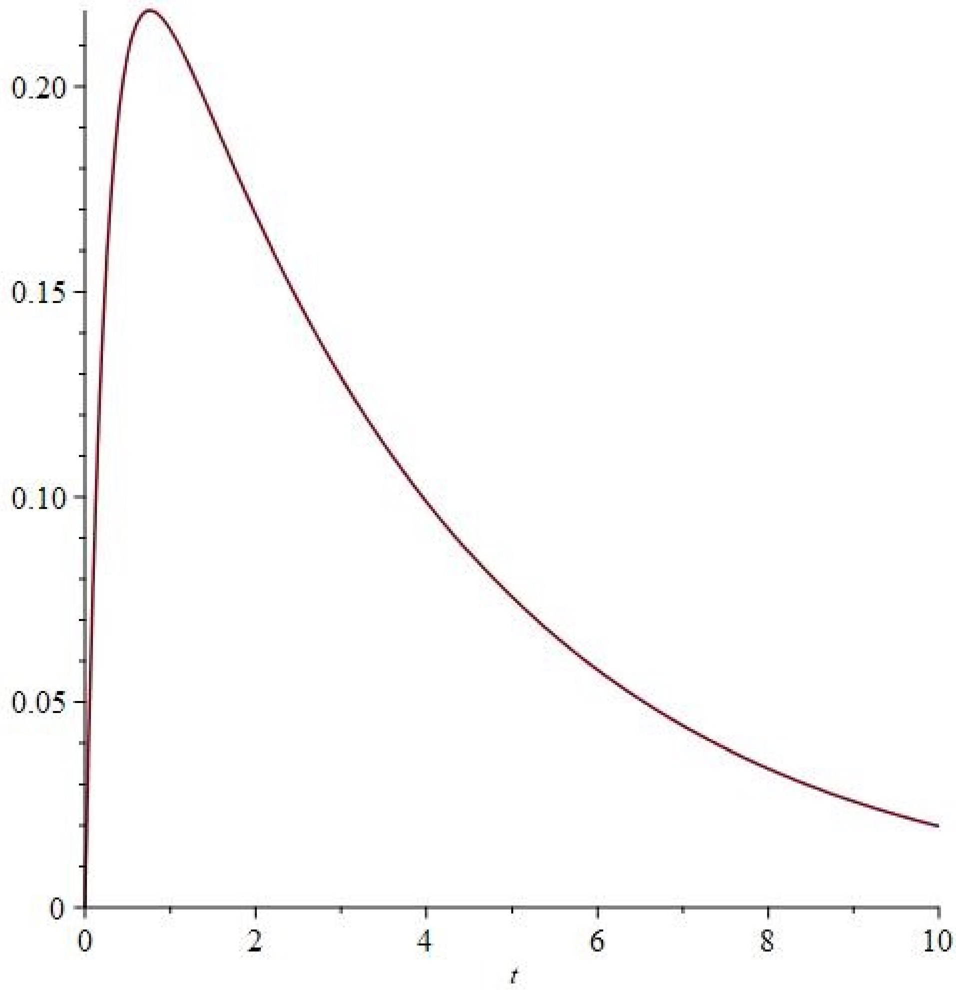

FIGURE 7.1. PDF of Total Task Completion for Gamma(2,1)

As one can see, the probability density function increases rapidly and maximizes its value at $w = 1$, and then decreases rapidly until the density function converges to zero.

For the $Gamma(2, 1)$ case, we can consider the case where $\lambda = \frac{1}{2}$. Clearly, the inverted Laplace transform above for this distribution does not provide us with much information in this form. If we split the original Laplace transform into partial fractions, we can simplify the pdf of W (i.e., the inverse Laplace transform) into something recognizable.

So,

$$L_W(s) = \frac{1}{s^2 + (\frac{5}{2})s + 1} = \frac{A}{s + \frac{1}{2}} + \frac{B}{s + 2}$$

Solving for A and B, we get $A = \frac{2}{3}$ and $B = \frac{-2}{3}$. And then,

$$L_W(s) = \frac{\frac{2}{3}}{s + \frac{1}{2}} + \frac{\frac{-2}{3}}{s + 2}$$

Inverting the Laplace transform, we have:

$$\begin{aligned} f(w) &= \frac{2}{3}L^{-1}(\frac{1}{s + \frac{1}{2}}) - \frac{2}{3}L^{-1}(\frac{1}{s + 2}) \\ &= \frac{4}{3}L^{-1}(\frac{\frac{1}{2}}{s + \frac{1}{2}}) - \frac{1}{3}L^{-1}(\frac{2}{s + 2}) \\ &= \frac{4}{3} * \frac{1}{2}e^{-\frac{1}{2}w} - \frac{1}{3} * 2e^{-2w} \\ &= \frac{2}{3}e^{-\frac{1}{2}w} - \frac{2}{3}e^{-2w} \end{aligned}$$

This is the more useful pdf for W in the $Gamma(2, 1)$ distribution. But it looks quite different from the version given by MAPLE. If one subtracts the pdf found using Maple when $\lambda = \frac{1}{2}$ from the pdf found above, the result is zero. This means the pdf yields the same values for various values of s for any values of λ, implying that the density functions are equivalent! However, in the above case, an interesting result is that the pdf is consistent with the forms that we normally see in the generalized Erlang distribution. So, the pdf of the total expected waiting time for a task distributed as $Gamma(2, 1)$ is generalized Erlang at some particular rate.

The $Gamma(3, 1)$ case is not factorable in its denominator, so unfortunately we cannot write the pdf of W in partial fractions. Further, technology does not give us an explicit form for the inverse Laplace transform. So, the pdf of the total time to complete the task still remains unknown for this case.

It is worth mentioning that there exist excellent Laplace transform inversion approximations and these approximations can be obtained using R or Wolfram Alpha.

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Vita Auctoris

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