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Public Restrooms and Queueing

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by

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Author's Declaration of Originality

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Abstract

In this research, we consider using unisex restrooms to replace the traditional separate restrooms for males and females. In detail, we will consider that people use restrooms at different rates for different types of service (I and II) , and the two types of facilities in the male restroom will be differentiated. We perform analysis using queueing theory and specifically matrix analytic methods. We also use computer software for simulations. Through numerical examples, we compare the waiting probabilities of male and female under the two different restroom systems.

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CHAPTER 1

Introduction

1.1. Motivation of Research

The traditional public restroom system (in restaurants, schools, gyms) is divided into male and female restrooms. But as awareness of LGBTQ people has increased, there are growing voices suggesting that such a restroom system could cause inconvenience or even harassment for people with special needs, especially those with gender identity and transgender issues. The 2015 U.S. Transgender Survey reported that one in ten respondents has been denied access to a public restroom because of their gender identity or gender expression. As a result, unisex restrooms are becoming increasingly more common in schools, restaurants and other public facilities. Further, with the COVID-19 pandemic, small size restrooms with two facilities, do not meet social distancing guidelines. Thus such restrooms effectively become single facility restrooms.

An article by Slater and Jones (2018, [2]) highlighted that unisex restrooms are actually of benefit for a range of people and situations:

"Parents with children of a different gender; those who care for people of a different gender; some disabled people who have a personal assistant of a different gender; and some people whose gender is questioned in the toilet including some trans and non-binary people (and, to a lesser extent, some cisgender people)." ... "In many cases, it is the need for an all-gender toilet that leads non-disabled people to use the accessible toilet."

Based on this information, we consider the queueing impact of the option to use unisex restrooms instead of the traditional separate restrooms for males and females. Wang et al. (2017, [1]) analyzed public stalls and performance by using a queueing model. They concluded that, by replacing the traditional restroom system with unisex restrooms, the new system is superior to the traditional one. However, their analysis assumed only toilet stalls in the male restroom and the service rates are the same no matter what kind of service (short(I) or long (II)) people need.

In this paper, we will consider a restroom system with separate unisex restrooms to replace the traditional restroom system with separate restrooms for male and female. In detail, we will assume that people use restrooms at a different rate for different types of service, and the two types of equipment in the male restroom will be differentiated. Using queueing theory, we analyze the impact of the unisex restroom system by comparing the probability that one person (male or female) finds no restrooms available when needed.

In the study of queueing theory, the birth-and-death model is one of the standard methods to analyze queueing problems. Particularly, the quasi-birth–death (QBD) process describes a generalization of the birth-death process. Quasi birth and death processes are often analyzed using matrix geometric methods, which are well explained in Nelson (1991, [7]) and He (2014, [6]). We used the matrix geometric methods to solve queueing model with presenting the numerical examples to show the difference of the waiting probability of male and female under the different restroom systems. Furthermore, we use computer software to simulate the queueing model and illustrate how the number of people change graphically.

1.2. Matrix Geometric Methods for Solving QBD process

Neuts (1981, [7]) pioneered Matrix Geometric Methods in stochastic models. Then, Nelson (1991, [8]) present a simple derivation of the matrix geometric solution of Markov processes with a repetitive structure, and it is always shown in the quasi and birth case.

Definition 1.1. *A stochastic process* $\{X(t) : t \geqslant 0\}$ *with discrete state space* $\mathcal{S}$ *is called a continuous-time Markvov chain* $(CTMC)$ *if for all* $t \geqslant 0, s \geqslant 0, i \in \mathcal{S}, j \in \mathcal{S}$,

$$P(X(s+t) = j \mid X(s) = i, \{X(u) : 0 \leqslant u < s\}) = P(X(s+t) = j \mid X(s) = i) = P_{ij}(t)$$

$P_{ij}(t)$ *is the probability that the chain will be in state* j, t *time units from now, given it is in state* i *now.*

According to Osogami (2005, [4]), a quasi birth and death process is defined as follows.

Definition 1.2. *In general, a Quasi-Birth-Death (QBD) process is a continuous Markov chain on the state space* $\{(i, \ell) | 1 \leqslant i \leqslant n_\ell, \ell \geqslant 0\}$*, where the state space can be divided into levels, and level* ℓ *has* n_ℓ *states (phases) for each* ℓ*. In a QBD process, transitions are allowed only to the neighboring levels or within the same level. Thus, a QBD process has a generator matrix of the form:*

$$\mathbf{Q} = \begin{pmatrix} \mathbf{L}^{(0)} & \mathbf{F}^{(0)} & & & \\ \mathbf{B}^{(1)} & \mathbf{L}^{(1)} & \mathbf{F}^{(1)} & & \\ & \mathbf{B}^{(2)} & \mathbf{L}^{(2)} & \mathbf{F}^{(2)} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where submatrix $\mathbf{F}^{(\ell)}$ *encodes (forward) transitions from level (column)* ℓ *to level* $\ell + 1$ *for* $\ell \geqslant 0$*, submatrix* $\mathbf{B}^{(\ell)}$ *encodes (backward) transitions from level* ℓ *to level*

$\ell - 1$ for $\ell \geqslant 1$, and submatrix $\mathbf{L}^{(\ell)}$ encodes (local) transitions within level ℓ for $\ell \geqslant 0$. Specifically, (i, j) element of $\mathbf{F}^{(\ell)}$ is the transition rate from state (i, ℓ), i.e. phase i of level ℓ, to state $(j, \ell + 1)$ for all i, j; (i, j) element of $\mathbf{B}^{(\ell)}$ is the transition rate from state (i, ℓ) to state $(j, \ell - 1)$ for all i, j; (i, j) element of $\mathbf{L}^{(\ell)}$ is the transition rate from state (i, ℓ) to state (j, ℓ) for $i \neq j$.

Most of the rest of this section is from Nelson (1991, [8]) and Stewart (2009,[9]). In the simplest case, these matrices in infinitesimal generator $\boldsymbol{Q}$ are infinite block tridiagonal matrices in which the three diagonal blocks repeat after some initial period. We write such a matrix as

$$\mathbf{Q} = \begin{pmatrix} \mathbf{B_{00}} & \mathbf{B_{01}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots \\ \boldsymbol{B_{10}} & \mathbf{A_1} & \mathbf{A_2} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{A_0} & \mathbf{A_1} & \mathbf{A_2} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{A_0} & \mathbf{A_1} & \mathbf{A_2} & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

where A_0, A_1, A_2 are square matrices with the same dimension; B_{00} is also a square matrix, but the dimension could be different than A_1; B_{01} and B_{10} are matrices whose sizes are defined in accordance with B_{00} and A_1 respectively.
The transitions are permitted only between states of the same level (diagonal blocks), to states in the next higher level (superdiagonal blocks), and to states in the adjacent lower level (subdiagonal blocks), which leads to the block tridiagonal effect.
Then we use the Matrix Geometric Methods to solve the stationary distribution of the infinitesimal generator $\boldsymbol{Q}$ of a QBD process. The stationary distribution is obtained from $\boldsymbol{\pi}\boldsymbol{Q} = 0$. Let $\boldsymbol{\pi}$ be partitioned conformally with $\boldsymbol{Q}$, i.e.

$$\boldsymbol{\pi} = (\boldsymbol{\pi_0}, \boldsymbol{\pi_1}, \boldsymbol{\pi_2}, \cdots)$$

where

$$\boldsymbol{\pi_i} = (\boldsymbol{\pi_{i_1}}, \boldsymbol{\pi_{i_2}}, \cdots)$$

for $i = 0, 1, \ldots$, and $\boldsymbol{\pi_{i_k}}$ is the probability of finding the process in state (i, k) at steady state. This gives the following equations:

$$\boldsymbol{\pi_0 B_{00}} + \boldsymbol{\pi_1 B_{10}} = \boldsymbol{0}$$

$$\boldsymbol{\pi_0 B_{01}} + \boldsymbol{\pi_1 A_1} + \boldsymbol{\pi_2 A_0} = \boldsymbol{0}$$

$$\boldsymbol{\pi_1 A_2} + \boldsymbol{\pi_2 A_1} + \boldsymbol{\pi_3 A_0} = \boldsymbol{0}$$

$$\vdots$$

$$\boldsymbol{\pi_{i-1} A_2} + \boldsymbol{\pi_i A_1} + \boldsymbol{\pi_{i+1} A_0} = \boldsymbol{0}, \quad i = 2, 3, \ldots$$

Since state transitions are between nearest blocks, Nelson (1991,[8]) proposes that if the value of $\boldsymbol{\pi_{i-1}} (i \geqslant 2)$ is given, we could find that the value of $\boldsymbol{\pi_i}$ is a function only of the transition rates between states with $i-1$ queued customers and states with i queued customers. Since these transition rates do not depend upon the value of i, this suggests that there is some constant matrix $\boldsymbol{R}$ such that

$$\boldsymbol{\pi_i} = \boldsymbol{\pi_{i-1} R}, \quad \text{for} \quad i = 2, 3, \ldots \tag{1.2.1}$$

If the subvectors $\boldsymbol{\pi_0}$ and $\boldsymbol{\pi_1}$ and the rate matrix $\boldsymbol{R}$ can be found, then the remaining subvectors of the stationary distribution could be formed using Equation (1.2.1). Returning to

$$\boldsymbol{\pi_{i-1} A_2} + \boldsymbol{\pi_i A_1} + \boldsymbol{\pi_{i+1} A_0} = \boldsymbol{0}, \quad i = 2, 3, \ldots \tag{1.2.2}$$

and substituting from Equation (1.2.1), we obtain, for $i = 2, 3, \ldots$

$$\boldsymbol{\pi_1 R^{i-2} A_2} + \boldsymbol{\pi_1 R^{i-1} A_1} + \boldsymbol{\pi_1 R^i A_0} = \boldsymbol{0}$$

i.e.,

$$\boldsymbol{\pi_1 R^{i-2}} \left(\boldsymbol{A_2} + \boldsymbol{R A_1} + \boldsymbol{R^2 A_0} \right) = \boldsymbol{0}$$

So find R from

$$(\boldsymbol{A_2} + \boldsymbol{R A_1} + \boldsymbol{R^2 A_0}) = \boldsymbol{0} \tag{1.2.3}$$

The simplest way to accomplish this is by successive substitution. From Equation (1.2.3) and using the fact that $\boldsymbol{A_1}$ must be non-singular, we have

$$\boldsymbol{A_2 A_1^{-1}} + \boldsymbol{R} + \boldsymbol{R^2 A_0 A_1^{-1}} = \boldsymbol{0}$$

i.e.,

$$\boldsymbol{R} = -\boldsymbol{A_2 A_1^{-1}} - \boldsymbol{R^2 A_0 A_1^{-1}} = -\boldsymbol{V} - \boldsymbol{R^2 W}$$

where $\boldsymbol{V} = \boldsymbol{A_2 A_1^{-1}}$ and $\boldsymbol{W} = \boldsymbol{A_0 A_1^{-1}}$. This leads to the successive substitution procedure proposed by Neuts, namely,

$$\boldsymbol{R_{(0)}} = \boldsymbol{0;} \quad \boldsymbol{R_{(k+1)}} = -\boldsymbol{V} - \boldsymbol{R^2_{(k)} W}, \quad k = 1, 2, \ldots \tag{1.2.4}$$

Neuts (1981, [7]) has shown that the sequence of matrices $\boldsymbol{R_{(k)}}, k = 0, 1, 2, \ldots$, is non-decreasing and converges to the rate matrix $\boldsymbol{R}$. The process is halted once successive differences are less than a specified tolerance criterion.

Then the only remaining problem is the derivation of π_0 and π_1. The first two equations of $\boldsymbol{\pi Q} = \boldsymbol{0}$ are

$$\boldsymbol{\pi_0 B_{00}} + \boldsymbol{\pi_1 B_{10}} = \boldsymbol{0}$$

$$\boldsymbol{\pi_0 B_{01}} + \boldsymbol{\pi_1 A_1} + \boldsymbol{\pi_2 A_0} = \boldsymbol{0}$$

Replacing $\boldsymbol{\pi_2}$ with $\boldsymbol{\pi_1 R}$ and writing these equations in matrix form, we obtain

$$(\boldsymbol{\pi_0}, \boldsymbol{\pi_1}) \begin{pmatrix} \mathbf{B_{00}} & \mathbf{B_{01}} \\ \mathbf{B_{10}} & \mathbf{A_1} + \mathbf{RA_0} \end{pmatrix} = (\mathbf{0}, \mathbf{0}) \tag{1.2.5}$$

Given the rate matrix $\boldsymbol{R}$ and blocks $\boldsymbol{B_{00}, B_{01}, B_{10}, A_1}$ and $\boldsymbol{A_0}$, this system may be solved to obtain $\boldsymbol{\pi_0}$ and $\boldsymbol{\pi_1}$. Since this is a homogeneous system of equations, the computed solution needs to be normalized so that the components of $\boldsymbol{\pi}$ sum to 1. In other words, we insist that $\boldsymbol{\pi e} = 1$. Thus

$$\begin{aligned} 1 = \boldsymbol{\pi e} &= \boldsymbol{\pi_0 e} + \boldsymbol{\pi_1 e} + \sum_{i=2}^{\infty} \boldsymbol{\pi_i e} \\ &= \boldsymbol{\pi_0 e} + \boldsymbol{\pi_1 e} + \sum_{i=2}^{\infty} \boldsymbol{\pi_1 R^{i-1} e} \\ &= \boldsymbol{\pi_0 e} + \sum_{i=1}^{\infty} \boldsymbol{\pi_1 R^{i-1} e} = \boldsymbol{\pi_0 e} + \sum_{i=0}^{\infty} \boldsymbol{\pi_1 R^{i} e} \end{aligned}$$

This implies the condition

$$\boldsymbol{\pi_0 e} + \boldsymbol{\pi_1} \sum_{i=0}^{\infty} (\boldsymbol{R^i}) \boldsymbol{e} = 1$$

According to Stewart (2009,[9]), "The eigenvalues of R lie inside the unit circle, which means that $(I - R)$ is non-singular." Hence

$$\left(\sum_{i=0}^{\infty} \boldsymbol{R^i} \right) = (\boldsymbol{I} - \boldsymbol{R})^{-1} \tag{1.2.6}$$

This enables us to complete the normalization of the vectors $\boldsymbol{\pi_0}$ and $\boldsymbol{\pi_1}$ by computing

$$\boldsymbol{\alpha} = \boldsymbol{\pi_0 e} + \boldsymbol{\pi_1} (\boldsymbol{I} - \boldsymbol{R})^{-1} \boldsymbol{e} \tag{1.2.7}$$

and dividing the computed subvectors $\boldsymbol{\pi_0}$ and $\boldsymbol{\pi_1}$ by $\boldsymbol{\alpha}$.

To use this matrix geometric method for solving the stationary distribution, the QBD process has to be ergodic/stable, which means the drift to higher-numbered levels must be strictly less than the drift to lower levels. To solve this stability condition, Stewart (2009,[9]) has generalized following method for a QBD process. Let the stationary distribution of the infinitesimal generator $\boldsymbol{A} = \boldsymbol{A_0} + \boldsymbol{A_1} + \boldsymbol{A_2}$ be denoted by $\boldsymbol{f_A}$. The following condition must hold for the ergodicity/stability:

$$\boldsymbol{f_A A_2 e} < \boldsymbol{f_A A_0 e} \tag{1.2.8}$$

"Recall that the elements of $\boldsymbol{A_2}$ move the process up to a higher-numbered level while those of $\boldsymbol{A_0}$ move it down a level. Indeed, it is from this condition that Neuts shows that the spectral radius of $\boldsymbol{R}$ is strictly less than 1 and consequently that the matrix $\boldsymbol{I} - \boldsymbol{R}$ is non-singular."

CHAPTER 2

The Queueing Model for Different Restroom Types

We consider the traditional restroom system as one female restroom with two toilet stalls and one male restroom with a urinal and a toilet stall. We now propose a new system by using three unisex restrooms with one toilet stall each. In this chapter, we establish the corresponding queueing model and QBD process for each restroom type (female, male, and unisex) and use the matrix geometric methods to solve for stationary distribution.

In our case, we assume males and females have the same rate to complete short as well as long service (I and II). All interarrival times and all service times are assumed to be exponentially distributed.

2.1. Female Restroom Queueing Model

We consider an unlimited waiting space system with arrival rate as constant λ and the probability of coming for short service is p. Thus the arrival rate of people coming for short service (I) is $\lambda_1 = \lambda p$ and for long service (II) is $\lambda_2 = \lambda(1-p)$. The short service rate is μ_1 and the long service rate is μ_2.

For specific state (a,b,c) in this queueing model, we set

a = the number of toilet stalls being used for short service = 0,1,2;

b = the number of toilet stalls being used for long service = 0,1,2;

c = the number of people waiting in the queue = 0,1,2, ...

Note: $a + b \leqslant 2$

For example, state (1,0,0) means that one person using the short service in the restroom and no one is waiting outside. And state (1,1,1) means that there are two people in the restroom and one is using the short service and another is using the long service and one person waiting in the queue.

We order states by the number of people in this queueing model and their service type, i.e., (0,0,0), (1,0,0), (0,1,0), (2,0,0), (1,1,0), (0,2,0), (2,0,1), (1,1,1), (0,2,1), (2,0,2), (1,1,2), (0,2,2), ... And we divide every three states into a level.

The process diagram is shown as below.

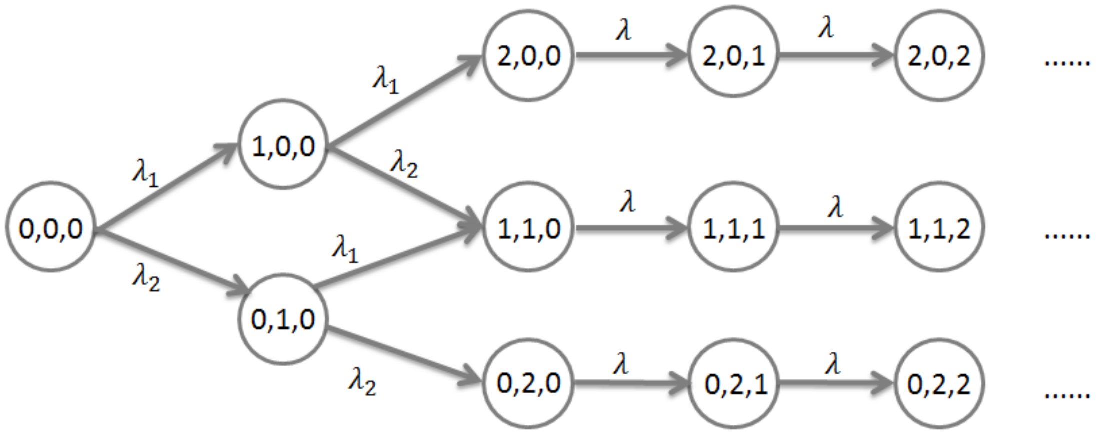
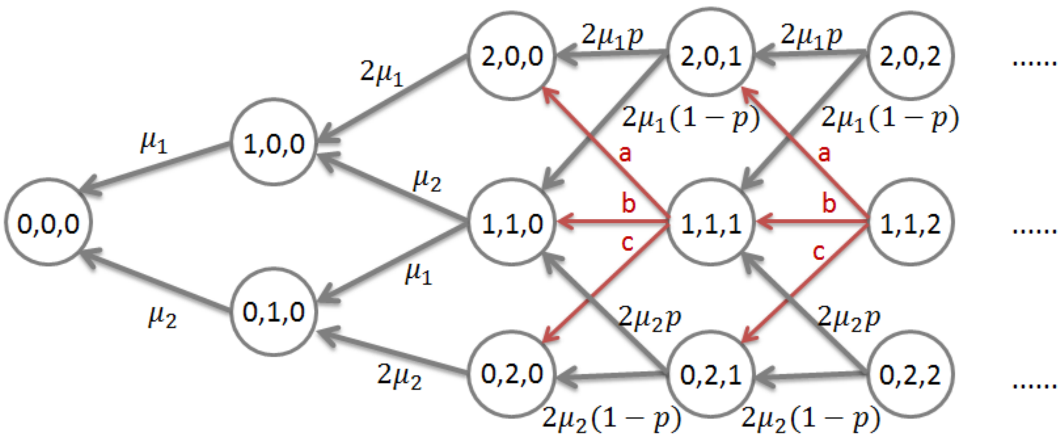


FIGURE 2.1. The In Process Diagram of Female Restroom



$$a = \mu_2 p, \qquad b = \mu_1 p + \mu_2(1-p), \qquad c = \mu_1(1-p)$$

FIGURE 2.2. The Out Process Diagram of Female Restroom

The rate matrix for this queueing model is similar to the rate matrix in quasi birth and death processes (1.2), except for the dimensions of matrices. And it is shown below

$$\mathbf{Q} = \begin{pmatrix} \mathbf{B_{00}} & \mathbf{B_{01}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{B_{10}} & \mathbf{A_1} & \mathbf{A_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{A_0} & \mathbf{A_1} & \mathbf{A_2} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{A_0} & \mathbf{A_1} & \mathbf{A_2} & \mathbf{0} & \cdots \\ & & & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

with blocks only on tridiagonal and the blocks start repeating in the third row as $\mathbf{A_0}; \mathbf{A_1}; \mathbf{A_2}$ on tridiagonal with other entries zero's. Besides, $\mathbf{B_{00}}; \mathbf{B_{01}}; \mathbf{B_{10}}; \mathbf{A_0}; \mathbf{A_1}; \mathbf{A_2}$ are all square matrices with dimension 3×3. Note the summation of each row of the transition matrix $\mathbf{Q}$ is 0 and the summation of all the probabilities is 1. According to the process diagram, the matrices are defined as follows.

$$\mathbf{B_{00}} = \begin{array}{c} \\ 000 \\ 100 \\ 010 \end{array} \begin{array}{c} \begin{array}{ccc} 000 & 100 & 010 \end{array} \\ \begin{pmatrix} -\lambda & \lambda_1 & \lambda_2 \\ \mu_1 & -(\mu_1 + \lambda) & 0 \\ \mu_2 & 0 & -(\mu_2 + \lambda) \end{pmatrix} \end{array}, \quad \mathbf{B_{01}} = \begin{array}{c} \\ 000 \\ 100 \\ 010 \end{array} \begin{array}{c} \begin{array}{ccc} 200 & 110 & 020 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ \lambda_1 & \lambda_2 & 0 \\ 0 & \lambda_1 & \lambda_2 \end{pmatrix} \end{array},$$

$$\mathbf{B_{10}} = \begin{array}{c} \\ 200 \\ 110 \\ 020 \end{array} \begin{array}{c} \begin{array}{ccc} 000 & 100 & 010 \end{array} \\ \begin{pmatrix} 0 & 2\mu_1 & 0 \\ 0 & \mu_2 & \mu_1 \\ 0 & 0 & 2\mu_2 \end{pmatrix} \end{array}$$

and

$$\mathbf{A_1} = \begin{array}{c} \\ 200 \\ 110 \\ 020 \end{array} \begin{array}{c} \begin{array}{ccc} 200 & 110 & 020 \end{array} \\ \begin{pmatrix} -(2\mu_1 + \lambda) & 0 & 0 \\ 0 & -(\mu_1 + \mu_2 + \lambda) & 0 \\ 0 & 0 & -(2\mu_2 + \lambda) \end{pmatrix} \end{array},$$

$$\mathbf{A_2} = \begin{array}{c} \\ 200 \\ 110 \\ 020 \end{array} \begin{array}{c} \begin{array}{ccc} 201 & 111 & 021 \end{array} \\ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \end{array},$$

$$\mathbf{A_0} = \begin{array}{c} \\ 201 \\ 111 \\ 021 \end{array} \begin{array}{c} \begin{array}{ccc} 200 & 110 & 020 \end{array} \\ \begin{pmatrix} 2\mu_1 p & 2\mu_1(1-p) & 0 \\ \mu_2 p & \mu_1 p + \mu_2(1-p) & \mu_1(1-p) \\ 0 & 2\mu_2 p & 2\mu_2(1-p) \end{pmatrix} \end{array}$$

The matrix $\mathbf{Q}$ obviously has the correct QBD structure.

Then the stationary distribution of $\boldsymbol{Q}$ is obtained from $\boldsymbol{\pi Q} = \mathbf{0}, \quad \boldsymbol{\pi e} = 1$. Let $\boldsymbol{\pi}$ be partitioned conformally with $\boldsymbol{Q}$, i.e.

$$\boldsymbol{\pi} = (\boldsymbol{\pi_0}, \boldsymbol{\pi_1}, \boldsymbol{\pi_2}, \cdots)$$

where

$$\boldsymbol{\pi_i} = (\boldsymbol{\pi_{i_1}}, \boldsymbol{\pi_{i_2}}, \boldsymbol{\pi_{i_3}})$$

for $i = 0, 1, \cdots$,and $\boldsymbol{\pi_{i_k}} (k = 1, 2, 3)$ is the probability of finding the queueing model in corresponding state at steady state. This gives the following equations

$$\boldsymbol{\pi_0 B_{00}} + \boldsymbol{\pi_1 B_{10}} = \mathbf{0}$$

$$\boldsymbol{\pi_0 B_{01}} + \boldsymbol{\pi_1 A_1} + \boldsymbol{\pi_2 A_0} = \mathbf{0}$$

$$\boldsymbol{\pi_1 A_2} + \boldsymbol{\pi_2 A_1} + \boldsymbol{\pi_3 A_0} = \mathbf{0}$$

$$\vdots$$

$$\boldsymbol{\pi_{i-1} A_2} + \boldsymbol{\pi_i A_1} + \boldsymbol{\pi_{i+1} A_0} = \mathbf{0}, \quad i = 2, 3, 4, \ldots$$

Now we write down the general form of equations after the block matrices start repeating.

$$\boldsymbol{\pi_{i-1}A_2} + \boldsymbol{\pi_i A_1} + \boldsymbol{\pi_{i+1}A_0} = \mathbf{0}, \quad i = 2, 3, 4, \ldots$$

and the linear equations for solving the boundary states

$$(\boldsymbol{\pi_0}, \boldsymbol{\pi_1}) \begin{pmatrix} \mathbf{B_{00}} & \mathbf{B_{01}} \\ \mathbf{B_{10}} & \mathbf{A_1} + \mathbf{RA_0} \end{pmatrix} = (\mathbf{0}, \mathbf{0}).$$

By using the matrix geometric methods explained in chapter 1, we have the following iterative procedure to solve for the constant matrix $\boldsymbol{R}$

$$\boldsymbol{R}_{(\mathbf{0})} = \mathbf{0}; \quad \boldsymbol{R}_{(\boldsymbol{k}+\mathbf{1})} = -\boldsymbol{V} - \boldsymbol{R}^{\mathbf{2}}_{(\boldsymbol{k})}\boldsymbol{W}, \quad k = 1, 2, \ldots.$$

where $\boldsymbol{V} = \boldsymbol{A_2 A_1^{-1}}$ and $\boldsymbol{W} = \boldsymbol{A_0 A_1^{-1}}$.

At last, we need to check the stability/ergodicity condition of this queueing model by using Equation(1.2.8), which is

$$\boldsymbol{f_A A_2 e} < \boldsymbol{f_A A_0 e}$$

In this case, the infinitesimal generator matrix is

$$\boldsymbol{A} = \boldsymbol{A_0} + \boldsymbol{A_1} + \boldsymbol{A_2} = \begin{pmatrix} -2\mu_1(1-p) & 2\mu_1(1-p) & 0 \\ \mu_2 p & -[\mu_2 p + \mu_1(1-p)] & \mu_1(1-p) \\ 0 & 2\mu_2 p & -2\mu_2 p \end{pmatrix}$$

Solving these two equations

$$\boldsymbol{f_A A} = \mathbf{0}, \quad \boldsymbol{f_A e} = 1$$

will give us

$$\boldsymbol{f_A} = (\frac{\mu_2^2 p^2}{M}, \frac{2\mu_1\mu_2 p(1-p)}{M}, \frac{\mu_1^2(1-p)^2}{M})$$

where $M = [\mu_2 p + \mu_1(1-p)]^2$

Then

$$\boldsymbol{f_A A_2 e} = (\frac{\mu_2^2 p^2}{M}, \frac{2\mu_1\mu_2 p(1-p)}{M}, \frac{\mu_1^2(1-p)^2}{M}) \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \boldsymbol{e} = \lambda$$

and

$$\boldsymbol{f_A A_0 e} = (\frac{\mu_2^2 p^2}{M}, \frac{2\mu_1\mu_2 p(1-p)}{M}, \frac{\mu_1^2(1-p)^2}{M}) \begin{pmatrix} 2\mu_1 p & 2\mu_1(1-p) & 0 \\ \mu_2 p & \mu_1 p + \mu_2(1-p) & \mu_1(1-p) \\ 0 & 2\mu_2 p & 2\mu_2(1-p) \end{pmatrix} \boldsymbol{e}$$

$$= \frac{2\mu_1\mu_2[\mu_2 p + \mu_1(1-p)]}{M} = \frac{2\mu_1\mu_2}{\mu_2 p + \mu_1(1-p)}$$

where $M = [\mu_2 p + \mu_1(1-p)]^2$

Plug in the stability condition, which implies that

$$\lambda < \frac{2\mu_1\mu_2}{\mu_2 p + \mu_1(1-p)} \qquad (2.1.1)$$

For numerical examples, if the parameters λ, μ_1, μ_2 and p meet above inequality, the process is stable.

2.2. Male Restroom Queueing Model

Except for the assumption that there are one urinal and one toilet stall in the male restroom, we also assume a male could use either types of equipment to complete short service but would only use the toilet stall for short service when the urinal is unavailable. And to simplify the queueing model, we assume that no more than one person with long service need will be waiting for the toilet stall. When a toilet stall is available, people with long service need always go first.

Because of these restricted assumptions, this queueing model of male restroom is not a Markov chain, just an approximation.

Similar to the female case, we consider an unlimited waiting space system with arrival rate as constant λ, and the probability of coming for short service is p. Again, the arrival rate of people coming for short service is $\lambda_1 = \lambda p$ and for long service is $\lambda_2 = \lambda(1-p)$. And the short service rate is μ_1 and the long service rate is μ_2.

For specific state (A,B,C,D) in this queueing model, we set

A (urinal use) = 0 (empty), 1 (used)

B (toilet stall use) = 0 (empty), a (used for short service), b(used for long service)

C = the number of male waiting in the queue = $0, 1, 2, \ldots$

D = the number of male in the queue waiting for long service = $0, 1$ (maximum=1)

For example, state (1,a,0,0) means that two people are using the short service in the restroom but one is using the urinal and the other is using the toilet stall and no one is waiting outside. State (1,b,2,1) means that there are two people in the restroom and one is using the urinal for short service and the other is using the toilet stall for long service and another two people are waiting in the queue while one of them is specifically waiting for the toilet stall for long service.

We order states by the number of people in this queueing model and their service type, i.e., (0,0,0,0), (1,0,0,0), (0,a,0,0), (0,b,0,0), (1,a,0,0), (0,a,1,1,), (1,b,0,0), (0,b,1,1), (1,a,1,0), (1,a,1,1), (1,b,1,0), (1,b,1,1), (1,a,2,0), ...

The process diagram is shown as below.

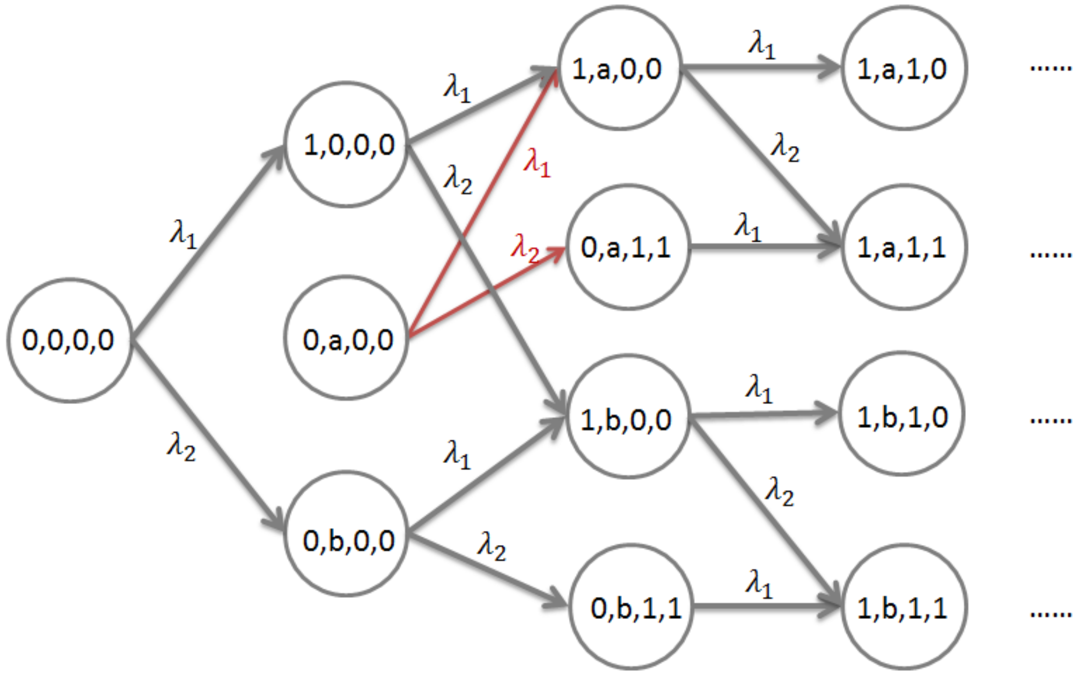


FIGURE 2.3. The In Process Diagram of Male Restroom

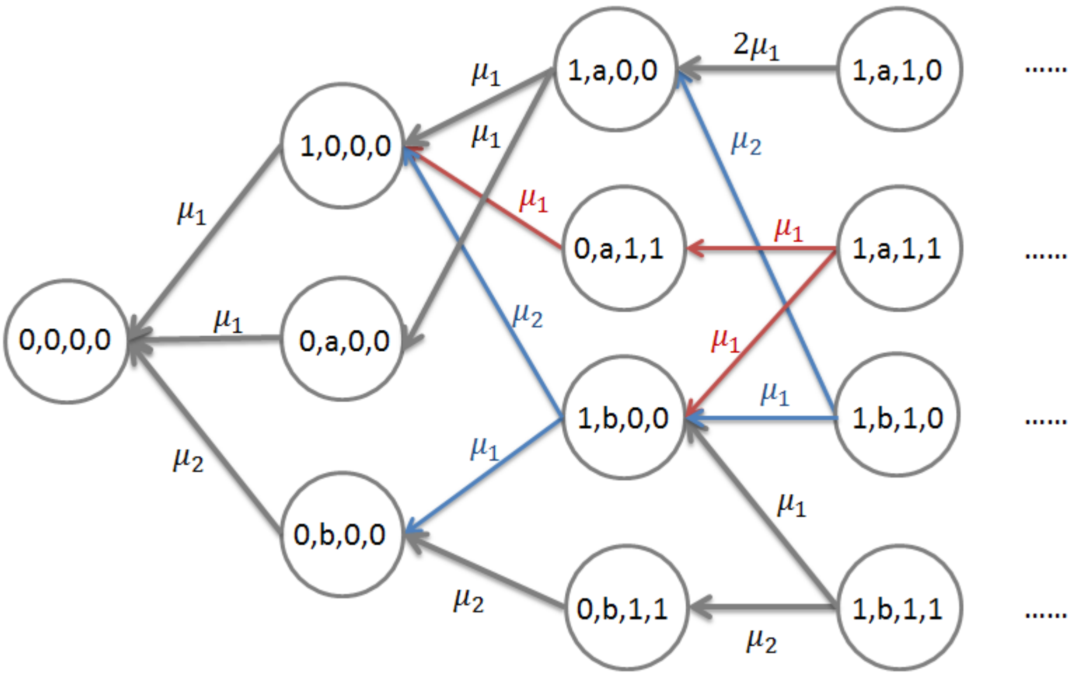


FIGURE 2.4. The Out Process Diagram of Male Restroom

This rate matrix structure is similar to the one of the female restroom case, except for the different boundary matrix and the dimensions of matrices. It is shown below

$$\mathbf{Q} = \begin{pmatrix} \mathbf{B_{00}} & \mathbf{B_{01}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{B_{10}} & \mathbf{B_{11}} & \mathbf{A_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{A_0} & \mathbf{A_1} & \mathbf{A_2} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{A_0} & \mathbf{A_1} & \mathbf{A_2} & \mathbf{0} & \cdots \\ & & & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

with blocks only on tridiagonal and the the blocks start repeating in the third row as $\mathbf{A_0}; \mathbf{A_1}; \mathbf{A_2}$ on tridiagonal with other entries zero's. Besides, $\mathbf{B_{00}}; \mathbf{B_{01}}; \mathbf{B_{10}}; \mathbf{B_{11}}$; $\mathbf{A_0}; \mathbf{A_1}; \mathbf{A_2}$ are all square matrices with dimension 4×4. Note the summation of each row of the transition matrix $\mathbf{Q}$ is 0 and the summation of all the probabilities is 1.

According to the process diagram example, the matrices are defined as follows.

$$\mathbf{B_{00}} = \begin{array}{c} \\ 0000 \\ 1000 \\ 0a00 \\ 0b00 \end{array} \begin{array}{c} \begin{array}{cccc} 0000 & 1000 & 0a00 & 0b00 \end{array} \\ \begin{pmatrix} -\lambda & \lambda_1 & 0 & \lambda_2 \\ \mu_1 & -(\lambda+\mu_1) & 0 & 0 \\ \mu_1 & 0 & -(\lambda+\mu_1) & 0 \\ \mu_2 & 0 & 0 & -(\lambda+\mu_2) \end{pmatrix} \end{array},$$

$$\mathbf{B_{01}} = \begin{array}{c} \\ 0000 \\ 1000 \\ 0a00 \\ 0b00 \end{array} \begin{array}{c} \begin{array}{cccc} 1a00 & 0a11 & 1b00 & 0b11 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & \lambda_2 & 0 \\ \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_1 & \lambda_2 \end{pmatrix} \end{array},$$

$$\mathbf{B_{10}} = \begin{array}{c} \\ 1a00 \\ 0a11 \\ 1b00 \\ 0b11 \end{array} \begin{array}{c} \begin{array}{cccc} 0000 & 1000 & 0a00 & 0b00 \end{array} \\ \begin{pmatrix} 0 & \mu_1 & \mu_1 & 0 \\ 0 & 0 & 0 & \mu_1 \\ 0 & \mu_2 & 0 & \mu_1 \\ 0 & 0 & 0 & \mu_2 \end{pmatrix} \end{array},$$

$$\mathbf{B_{11}} = \begin{array}{c} \\ 1a00 \\ 0a11 \\ 1b00 \\ 0b11 \end{array} \begin{array}{c} \begin{array}{cccc} 1a00 & 0a11 & 1b00 & 0b11 \end{array} \\ \begin{pmatrix} -(\lambda+2\mu_1) & 0 & 0 & 0 \\ 0 & -(\lambda_1+\mu_1) & 0 & 0 \\ 0 & 0 & -(\lambda+\mu_1+\mu_2) & 0 \\ 0 & 0 & 0 & -(\lambda_1+\mu_2) \end{pmatrix} \end{array}$$

and

$$\mathbf{A_2} = \begin{array}{c} \\ 1a00 \\ 0a11 \\ 1b00 \\ 0b11 \end{array} \begin{array}{c} \begin{array}{cccc} 1a10 & 1a11 & 1b10 & 1b11 \end{array} \\ \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & \lambda_2 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} \end{array},$$

$$\mathbf{A_0} = \begin{array}{c} \\ 1a10 \\ 1a11 \\ 1b10 \\ 1b11 \end{array} \begin{array}{c} \begin{array}{cccc} 1a00 & 0a11 & 1b00 & 0b11 \end{array} \\ \begin{pmatrix} 2\mu_1 & 0 & 0 & 0 \\ 0 & \mu_1 & \mu_1 & 0 \\ \mu_2 & 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2 & \mu_1 \end{pmatrix} \end{array},$$

$$\mathbf{A_1} = \begin{array}{c} \\ 1a10 \\ 1a11 \\ 1b10 \\ 1b11 \end{array} \begin{array}{c} \begin{array}{cccc} 1a10 & 1a11 & 1b10 & 1b11 \end{array} \\ \begin{pmatrix} -(\lambda + 2\mu_1) & 0 & 0 & 0 \\ 0 & -(\lambda_1 + 2\mu_1) & 0 & 0 \\ 0 & 0 & -(\lambda + \mu_1 + \mu_2) & 0 \\ 0 & 0 & 0 & -(\lambda_1 + \mu_1 + \mu_2) \end{pmatrix} \end{array}$$

The matrix $\mathbf{Q}$ obviously has the correct QBD structure.

Then the stationary distribution of $\boldsymbol{Q}$ is obtained from $\boldsymbol{\pi Q} = \mathbf{0}, \quad \boldsymbol{\pi e} = 1$. Let $\boldsymbol{\pi}$ be partitioned conformally with $\boldsymbol{Q}$, i.e.

$$\boldsymbol{\pi} = (\boldsymbol{\pi_0}, \boldsymbol{\pi_1}, \boldsymbol{\pi_2}, \cdots)$$

where

$$\boldsymbol{\pi_i} = (\boldsymbol{\pi_{i_1}}, \boldsymbol{\pi_{i_2}}, \boldsymbol{\pi_{i_3}}, \boldsymbol{\pi_{i_4}})$$

for $i = 0, 1, \cdots$,and $\boldsymbol{\pi_{i_k}}(k = 1, 2, 3, 4)$ is the probability of finding the process in corresponding state at steady state. This gives the following equations

$$\boldsymbol{\pi_0 B_{00}} + \boldsymbol{\pi_1 B_{10}} = \mathbf{0}$$

$$\boldsymbol{\pi_0 B_{01}} + \boldsymbol{\pi_1 B_{11}} + \boldsymbol{\pi_2 A_0} = \mathbf{0}$$

$$\boldsymbol{\pi_1 A_2} + \boldsymbol{\pi_2 A_1} + \boldsymbol{\pi_3 A_0} = \mathbf{0}$$

$$\vdots$$

$$\boldsymbol{\pi_{i-1} A_2} + \boldsymbol{\pi_i A_1} + \boldsymbol{\pi_{i+1} A_0} = \mathbf{0}, \quad i = 2, 3, 4, \ldots$$

Now we write down the general form of equations after the block matrices start repeating.

$$\boldsymbol{\pi_{i-1} A_2} + \boldsymbol{\pi_i A_1} + \boldsymbol{\pi_{i+1} A_0} = \mathbf{0}, \quad i = 2, 3, 4, \ldots$$

and the linear equations for solving the boundary states

$$(\boldsymbol{\pi_0}, \boldsymbol{\pi_1}) \begin{pmatrix} \mathbf{B_{00}} & \mathbf{B_{01}} \\ \mathbf{B_{10}} & \mathbf{B_{11}} + \mathbf{RA_0} \end{pmatrix} = (\mathbf{0}, \mathbf{0}).$$

By using the matrix geometric methods explained in chapter 1, we have the following iterative procedure to solve for the constant matrix $\boldsymbol{R}$

$$\boldsymbol{R_{(0)}} = \mathbf{0}; \quad \boldsymbol{R_{(k+1)}} = -\boldsymbol{V} - \boldsymbol{R^2_{(k)} W}, \quad k = 1, 2, \ldots.$$

where $\boldsymbol{V} = \boldsymbol{A_2 A_1^{-1}}$ and $\boldsymbol{W} = \boldsymbol{A_0 A_1^{-1}}$.

At last, we need to check the stability/ergodicity condition of this queueing model by using Equation(1.2.8), which is

$$\boldsymbol{f_A A_2 e} < \boldsymbol{f_A A_0 e}$$

In this case, the infinitesimal generator matrix is

$$\boldsymbol{A} = \boldsymbol{A_0} + \boldsymbol{A_1} + \boldsymbol{A_2} = \begin{pmatrix} -\lambda_2 & \lambda_2 & 0 & 0 \\ 0 & -\mu_1 & \mu_1 & 0 \\ \mu_2 & 0 & -(\lambda_2+\mu_2) & \lambda_2 \\ 0 & 0 & \mu_2 & -\mu_2 \end{pmatrix}$$

Solving these two equations

$$\boldsymbol{f_A A} = \boldsymbol{0}, \quad \boldsymbol{f_A e} = 1$$

will give us

$$\boldsymbol{f_A} = \left(\frac{\mu_1\mu_2^2}{M}, \frac{\lambda_2\mu_2^2}{M}, \frac{\lambda_2\mu_1\mu_2}{M}, \frac{\lambda_2^2\mu_1}{M}\right)$$

where $M = \mu_1\mu_2^2 + \lambda_2\mu_2^2 + \lambda_2\mu_1\mu_2 + \lambda_2^2\mu_1$

Plug in the stability condition, which implies that

$$(2.2.1)\quad 2\mu_1^2\mu_2^2 + (2\lambda_2 - \lambda_1)\mu_1\mu_2^2 + \lambda_2\mu_1^2\mu_2 + \lambda_2^2\mu_1^2 - \lambda_1\lambda_2\mu_2^2 - \lambda_1\lambda_2\mu_1\mu_2 - \lambda_1\lambda_2^2\mu_1 > 0$$

For numerical examples, if the parameters λ, λ_1, λ_2, μ_1, and μ_2 meet above inequality, the process is stable.

2.3. Unisex restroom model

As we mentioned at the beginning of this chapter, we propose a new restroom system with three unisex restrooms with one toilet stall each. The gender neutral model is similar to the female case. We consider an unlimited waiting space system with arrival rate as constant λ, and the probability of people coming for short service is p. Again, the arrival rate of people coming for short service is $\lambda_1 = \lambda p$ and for long service is $\lambda_2 = \lambda(1-p)$. The short service rate is μ_1 and the long service rate is μ_2.

For specific state (a,b,c) in this queueing model, we set
a = the number of restroom being used for short service = 0,1,2,3;
b = the number of restroom being used for long service = 0,1,2,3;
c = the number of people waiting in the queue = 0,1,2, ...
Note: $a + b \leqslant 3$

For example, state (3,0,0) means that three people are using the unisex restrooms for short service and no one is waiting outside. And state (1,2,1) means that three people are using the unisex restrooms and one is using for short service and the others are using for long service and one person waiting in the queue.
We order states by the number of people in this queueing model and their service type, i.e., (0,0,0), (1,0,0), (0,1,0), (2,0,0), (1,1,0), (0,2,0), (3,0,0), (2,1,0), (1,2,0), (0,3,0), (3,0,1), (2,1,1), (1,2,1), (0,3,1), ...
The process diagram is shown as below.

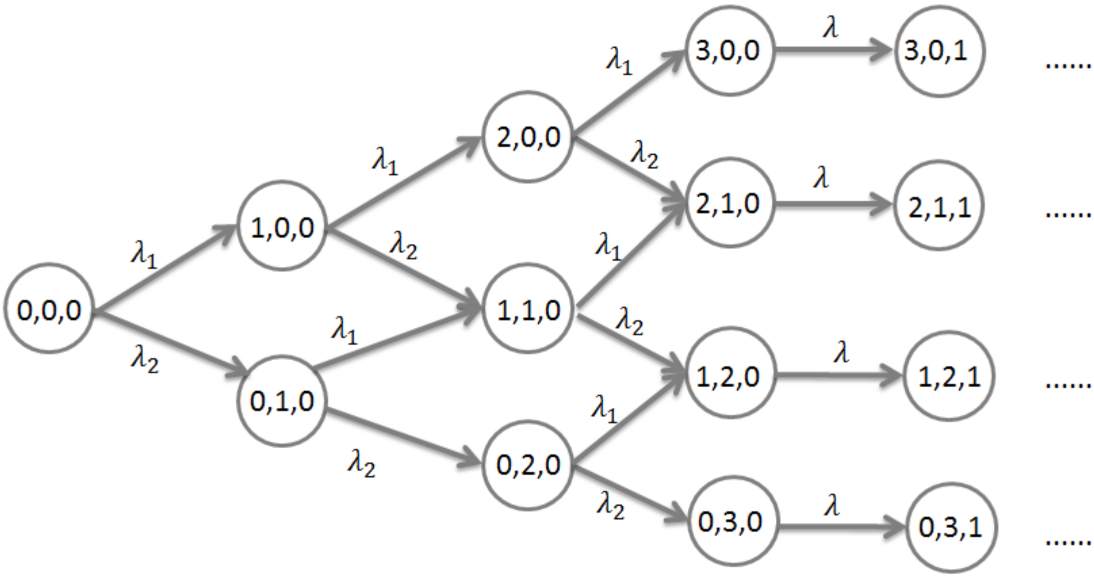
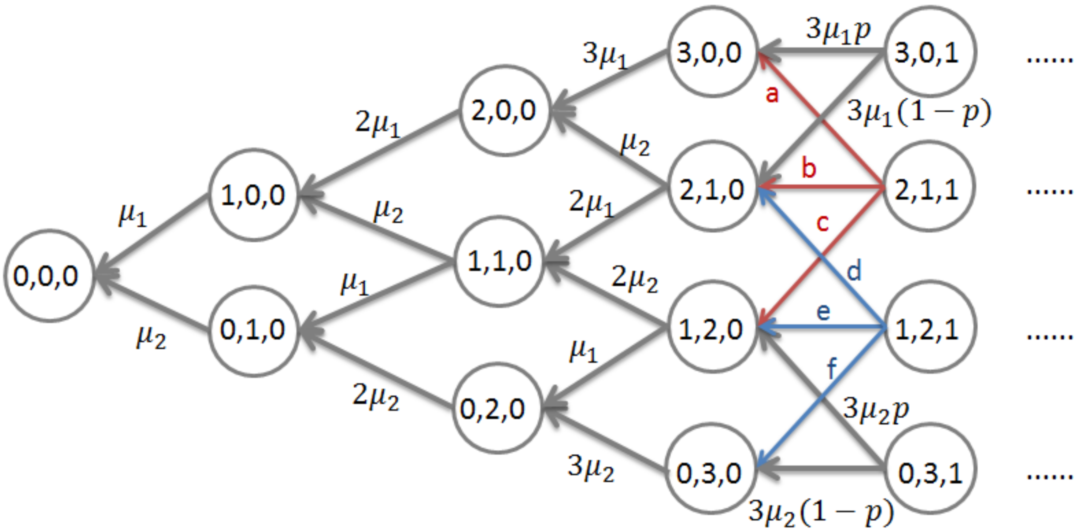


FIGURE 2.5. The In Process Diagram of Unisex Restroom



$$a = \mu_2 p, \qquad b = 2\mu_1 p + \mu_2(1-p), \qquad c = 2\mu_1(1-p)$$
$$d = 2\mu_2 p, \qquad e = \mu_1 p + 2\mu_2(1-p), \qquad f = \mu_1(1-p)$$

FIGURE 2.6. The Out Process Diagram of Unisex Restroom

This rate matrix structure is similar to the one of the female restroom case, except for the different boundary matrix and the dimensions of matrices. It is shown below

$$
\mathbf{Q} = \begin{pmatrix}
\mathbf{B_{00}} & \mathbf{B_{01}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
\mathbf{B_{10}} & \mathbf{B_{11}} & \mathbf{B_{12}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
\mathbf{0} & \mathbf{B_{21}} & \mathbf{A_1} & \mathbf{A_2} & \mathbf{0} & \mathbf{0} & \cdots \\
\mathbf{0} & \mathbf{0} & \mathbf{A_0} & \mathbf{A_1} & \mathbf{A_2} & \mathbf{0} & \cdots \\
 & & & \ddots & \ddots & \ddots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
$$

with blocks only on tridiagonal and the the blocks start repeating in the 4th row as $\mathbf{A_0}; \mathbf{A_1}; \mathbf{A_2}$ on tridiagonal with other entries zero's. Besides, $\mathbf{B_{00}}; \mathbf{B_{01}}; \mathbf{B_{10}}; \mathbf{B_{11}}$ are square matrices with dimension 3×3, $\mathbf{B_{12}}$ is matrix with dimension 3×4, $\mathbf{B_{21}}$ is matrix with dimension 4×3, and $\mathbf{A_0}; \mathbf{A_1}; \mathbf{A_2}$ are square matrices with dimension 4×4. Note the summation of each row of the transition matrix $\mathbf{Q}$ is 0 and the summation of all the probabilities is 1.

According to the process diagram example, the matrices are defined as following

$$
\mathbf{B_{00}} = \begin{array}{c} \\ 000 \\ 100 \\ 010 \end{array} \begin{array}{c} \begin{array}{ccc} 000 & 100 & 010 \end{array} \\ \begin{pmatrix} -\lambda & \lambda_1 & \lambda_2 \\ \mu_1 & -(\mu_1 + \lambda) & 0 \\ \mu_2 & 0 & -(\mu_2 + \lambda) \end{pmatrix} \end{array}, \quad
\mathbf{B_{01}} = \begin{array}{c} \\ 000 \\ 100 \\ 010 \end{array} \begin{array}{c} \begin{array}{ccc} 200 & 110 & 020 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ \lambda_1 & \lambda_2 & 0 \\ 0 & \lambda_1 & \lambda_2 \end{pmatrix} \end{array},
$$

$$
\mathbf{B_{10}} = \begin{array}{c} \\ 200 \\ 110 \\ 020 \end{array} \begin{array}{c} \begin{array}{ccc} 000 & 100 & 010 \end{array} \\ \begin{pmatrix} 0 & 2\mu_1 & 0 \\ 0 & \mu_2 & \mu_1 \\ 0 & 0 & 2\mu_2 \end{pmatrix} \end{array},
$$

$$\mathbf{B_{11}} = \begin{array}{c} \\ 200 \\ 110 \\ 020 \end{array} \begin{array}{c} \begin{array}{ccc} 200 & 110 & 020 \end{array} \\ \begin{pmatrix} -(2\mu_1+\lambda) & 0 & 0 \\ 0 & -(\mu_1+\mu_2+\lambda) & 0 \\ 0 & 0 & -(2\mu_2+\lambda) \end{pmatrix} \end{array},$$

$$\mathbf{B_{12}} = \begin{array}{c} \\ 200 \\ 110 \\ 020 \end{array} \begin{array}{c} \begin{array}{cccc} 300 & 210 & 120 & 030 \end{array} \\ \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 & \lambda_2 \end{pmatrix} \end{array},$$

$$\mathbf{B_{21}} = \begin{array}{c} \\ 300 \\ 210 \\ 120 \\ 030 \end{array} \begin{array}{c} \begin{array}{ccc} 200 & 110 & 020 \end{array} \\ \begin{pmatrix} 3\mu_1 & 0 & 0 \\ \mu_2 & 2\mu_1 & 0 \\ 0 & 2\mu_2 & \mu_1 \\ 0 & 0 & 3\mu_2 \end{pmatrix} \end{array}$$

and

$$\mathbf{A_1} = \begin{array}{c} \\ 300 \\ 210 \\ 120 \\ 030 \end{array} \begin{array}{c} \begin{array}{cccc} 300 & 210 & 120 & 030 \end{array} \\ \begin{pmatrix} -(3\mu_1+\lambda) & 0 & 0 & 0 \\ 0 & -(2\mu_1+\mu_2+\lambda) & 0 & 0 \\ 0 & 0 & -(\mu_1+2\mu_2+\lambda) & 0 \\ 0 & 0 & 0 & -(3\mu_2+\lambda) \end{pmatrix} \end{array},$$

$$\mathbf{A_2} = \begin{array}{c} \\ 300 \\ 210 \\ 120 \\ 030 \end{array} \begin{array}{c} \begin{array}{cccc} 301 & 211 & 121 & 031 \end{array} \\ \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \end{array},$$

$$\mathbf{A_0} = \begin{array}{c} \\ 301 \\ 211 \\ 121 \\ 031 \end{array} \begin{array}{c} \begin{array}{cccc} 300 & 210 & 120 & 030 \end{array} \\ \begin{pmatrix} 3\mu_1 p & 3\mu_1(1-p) & 0 & 0 \\ \mu_2 p & 2\mu_1 p + \mu_2(1-p) & 2\mu_1(1-p) & 0 \\ 0 & 2\mu_2 p & \mu_1 p + 2\mu_2(1-p) & \mu_1(1-p) \\ 0 & 0 & 3\mu_2 p & 3\mu_2(1-p) \end{pmatrix} \end{array}$$

The matrix $\mathbf{Q}$ obviously has the correct QBD structure.

Then the stationary distribution of $\boldsymbol{Q}$ is obtained from $\boldsymbol{\pi Q} = \mathbf{0}, \quad \boldsymbol{\pi e} = 1$. Let $\boldsymbol{\pi}$ be partitioned conformally with $\boldsymbol{Q}$, i.e.

$$\boldsymbol{\pi} = (\boldsymbol{\pi_0}, \boldsymbol{\pi_1}, \boldsymbol{\pi_2}, \cdots)$$

where

$\boldsymbol{\pi_i} = (\boldsymbol{\pi_{i_1}}, \boldsymbol{\pi_{i_2}}, \boldsymbol{\pi_{i_3}})$ for $i = 0, 1$,

$\boldsymbol{\pi_i} = (\boldsymbol{\pi_{i_1}}, \boldsymbol{\pi_{i_2}}, \boldsymbol{\pi_{i_3}}, \boldsymbol{\pi_{i_4}})$ for $i = 2, 3, \cdots$,

and $\boldsymbol{\pi_{i_k}}(k = 1, 2, 3, 4)$ is the probability of finding the process in corresponding state at steady state. This gives the following equations

$$\begin{aligned}\boldsymbol{\pi_0 B_{00}} + \boldsymbol{\pi_1 B_{10}} &= \mathbf{0}\\ \boldsymbol{\pi_0 B_{01}} + \boldsymbol{\pi_1 B_{11}} + \boldsymbol{\pi_2 B_{21}} &= \mathbf{0}\\ \boldsymbol{\pi_1 B_{12}} + \boldsymbol{\pi_2 A_1} + \boldsymbol{\pi_3 A_0} &= \mathbf{0}\\ \boldsymbol{\pi_2 A_2} + \boldsymbol{\pi_3 A_1} + \boldsymbol{\pi_4 A_0} &= \mathbf{0}\\ &\vdots\\ \boldsymbol{\pi_{i-1} A_2} + \boldsymbol{\pi_i A_1} + \boldsymbol{\pi_{i+1} A_0} &= \mathbf{0}, \quad i = 3, 4, 5, \ldots\end{aligned}$$

Now we write down the general form of equations after the block matrices start repeating.

$$\boldsymbol{\pi_{i-1} A_2} + \boldsymbol{\pi_i A_1} + \boldsymbol{\pi_{i+1} A_0} = \mathbf{0}, \quad i = 3, 4, 5, \ldots$$

Except for the repeating block equations, the first three of equations are able to be solved for the boundary probabilities,

$$(\boldsymbol{\pi_0}, \boldsymbol{\pi_1}, \boldsymbol{\pi_2}) \begin{pmatrix} \mathbf{B_{00}} & \mathbf{B_{01}} & \mathbf{0} \\ \mathbf{B_{10}} & \mathbf{B_{11}} & \mathbf{B_{12}} \\ \mathbf{0} & \mathbf{B_{21}} & \mathbf{A_1} + \mathbf{RA_0} \end{pmatrix} = (\mathbf{0}, \mathbf{0}, \mathbf{0})$$

By using the matrix geometric methods explained in chapter 1, we have the following iterative procedure to solve for the constant matrix $\boldsymbol{R}$

$$\boldsymbol{R_{(0)}} = \mathbf{0}; \quad \boldsymbol{R_{(k+1)}} = -\boldsymbol{V} - \boldsymbol{R^2_{(k)}} \boldsymbol{W}, \quad k = 1, 2, \ldots.$$

where $\boldsymbol{V} = \boldsymbol{A_2 A_1^{-1}}$ and $\boldsymbol{W} = \boldsymbol{A_0 A_1^{-1}}$.
At last, we need to check the stability/ergodicity condition of this queueing model by using Equation(1.2.8), which is

$$\boldsymbol{f_A A_2 e} < \boldsymbol{f_A A_0 e}$$

In this case, the infinitesimal generator matrix is

$$\boldsymbol{A} = \boldsymbol{A_0} + \boldsymbol{A_1} + \boldsymbol{A_2}$$

$$= \begin{pmatrix} -3\mu_1(1-p) & 3\mu_1(1-p) & 0 & 0 \\ \mu_2 p & -(\mu_2 p + 2\mu_1(1-p)) & 2\mu_1(1-p) & 0 \\ 0 & 2\mu_2 p & -(2\mu_2 p + \mu_1(1-p)) & \mu_1(1-p) \\ 0 & 0 & 3\mu_2 p & -3\mu_2 p \end{pmatrix}$$

Solving these two equations

$$\boldsymbol{f_A}\boldsymbol{A} = \boldsymbol{0}, \quad \boldsymbol{f_A}\boldsymbol{e} = 1$$

will give us

$$\boldsymbol{f_A} = (\frac{2\mu_2^3 p^3}{M}, \frac{6\mu_1(1-p)\mu_2^2 p^2}{M}, \frac{9\mu_1^2(1-p)^2\mu_2 p}{M}), \frac{3\mu_1^3(1-p)^3}{M})$$

where $M = 2\mu_2^3 p^3 + 6\mu_1(1-p)\mu_2^2 p^2 + 9\mu_1^2(1-p)^2\mu_2 p + 3\mu_1^3(1-p)^3$

Plug in the stability condition, which implies that

$$\lambda < \frac{6\mu_1\mu_2^3 p^2 + 6\mu_1^2\mu_2^2 p(1-p)(3-p) + 9\mu_1^3\mu_2(1-p)^2}{M} \tag{2.3.1}$$

where $M = 2\mu_2^3 p^3 + 6\mu_1(1-p)\mu_2^2 p^2 + 9\mu_1^2(1-p)^2\mu_2 p + 3\mu_1^3(1-p)^3$

For numerical examples, if the parameters λ, μ_1, μ_2 and p meet above inequality, the process is stable.

CHAPTER 3

Numerical Example

In previous chapters, we discuss the Quasi-Birth-Death process, geometric matrix method, and queueing model of three different restroom types. In this chapter, we will go through some numerical examples to have a better understanding of these models.

The parameters are set as $\lambda = 10, p = 0.8, \mu_1 = 12, \mu_2 = 3$. This implies that the arrival rate is 10 and the probability of the arrival person need short service is 0.8. As a result, the arrival rate for short service is 8 and the arrival rate for long service is 2. Then, we set the short service rate is 12 while the long service rate is 3.

3.1. Female Restroom

We substitute these parameters into the rate matrix of the female case, the block matrices are

$$\mathbf{B_{00}} = \begin{pmatrix} -10 & 8 & 2 \\ 12 & -22 & 0 \\ 3 & 0 & -13 \end{pmatrix}, \quad \mathbf{B_{01}} = \begin{pmatrix} 0 & 0 & 0 \\ 8 & 2 & 0 \\ 0 & 8 & 2 \end{pmatrix}, \quad \mathbf{B_{10}} = \begin{pmatrix} 0 & 24 & 0 \\ 0 & 3 & 12 \\ 0 & 0 & 6 \end{pmatrix}$$

and

$$\mathbf{A_0} = \begin{pmatrix} 19.2 & 4.8 & 0 \\ 2.4 & 10.2 & 2.4 \\ 0 & 4.8 & 1.2 \end{pmatrix}, \quad \mathbf{A_1} = \begin{pmatrix} -34 & 0 & 0 \\ 0 & -25 & 0 \\ 0 & 0 & -16 \end{pmatrix}, \quad \mathbf{A_2} = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

By using the equation (2.1.1), we check these parameters satisfy the stability condition.

$$\frac{2\mu_1\mu_2}{\mu_2 p + \mu_1(1-p)} = 15 > 10 = \lambda$$

We would apply these data into R for calculation. And the detailed R code is located in Appendix A.

$$\mathbf{A_1^{-1}} = \begin{pmatrix} -0.0294 & 0 & 0 \\ 0 & -0.04 & 0 \\ 0 & 0 & -0.0625 \end{pmatrix}$$

$$\mathbf{V} = \mathbf{A_2}\mathbf{A_1^{-1}} = \begin{pmatrix} -0.2941 & 0 & 0 \\ 0 & -0.4 & 0 \\ 0 & 0 & -0.625 \end{pmatrix},$$

$$\mathbf{W} = \mathbf{A_0 A_1^{-1}} = \begin{pmatrix} -0.5647 & -0.192 & 0 \\ -0.0706 & -0.408 & -0.150 \\ 0 & -0.192 & -0.075 \end{pmatrix}$$

Now we get this recursive equation:

$$\mathbf{R_{(k+1)}} = -\mathbf{V} - \mathbf{R_{(k)}^2 W} = \begin{pmatrix} 0.2941 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.625 \end{pmatrix} + \boldsymbol{R_{(k)}^2} \begin{pmatrix} 0.5647 & 0.192 & 0 \\ 0.0706 & 0.408 & 0.150 \\ 0 & 0.192 & 0.075 \end{pmatrix}$$

Beginning with $\boldsymbol{R_{(0)}} = \mathbf{0}$ and apply 100 times iterations in R, the limiting matrix $\mathbf{R}$ is obtained as below.

$$\mathbf{R} = \begin{pmatrix} 0.3725 & 0.0269 & 0.0001 \\ 0.0181 & 0.5040 & 0.0382 \\ 0.0005 & 0.0863 & 0.6586 \end{pmatrix}$$

Now we proceed to the boundary conditions.

$$(\boldsymbol{\pi_0}, \boldsymbol{\pi_1}) \begin{pmatrix} \mathbf{B_{00}} & \mathbf{B_{01}} \\ \mathbf{B_{10}} & \mathbf{A_1} + \mathbf{RA_0} \end{pmatrix} = (\mathbf{0}, \mathbf{0}).$$

Then we solve this by setting the first component of the subvectors $\boldsymbol{\pi_0}$ to 1, i.e., $\pi_{0_1} = 1$. By using R, we find

$$(\boldsymbol{\pi_0'}, \boldsymbol{\pi_1'}) = (1, 0.6277, 0.4656, 0.2053, 0.2939, 0.0875)$$

By using equation (1.2.7), the normalization constant is

$$\begin{aligned} \alpha &= \boldsymbol{\pi_0'} \boldsymbol{e} + \boldsymbol{\pi_1'} (\boldsymbol{I} - \boldsymbol{R})^{-1} \boldsymbol{e} \\ &= (1, 0.6277, 0.4656)\boldsymbol{e} + (0.2053, 0.2939, 0.0875)(\boldsymbol{I} - \boldsymbol{R})^{-1} \boldsymbol{e} \\ &= 3.4405 \end{aligned}$$

which allows us to compute

$$(\pi_{0_1}, \pi_{0_2}, \pi_{0_3}) = \boldsymbol{\pi_0} = \boldsymbol{\pi'_0}/\alpha = (0.2907, 0.1824, 0.1353)$$

$$(\pi_{1_1}, \pi_{1_2}, \pi_{1_3}) = \boldsymbol{\pi_1} = \boldsymbol{\pi'_1}/\alpha = (0.0597, 0.0854, 0.0254)$$

By using $\boldsymbol{\pi_k} = \boldsymbol{\pi_{k-1}R}$, we find the following subcomponents of the stationary distribution:

$$\begin{aligned}
\boldsymbol{\pi_2} &= \boldsymbol{\pi_1 R} \\
&= (0.0597, 0.0854, 0.0254)\begin{pmatrix} 0.3725 & 0.0269 & 0.0001 \\ 0.0181 & 0.5040 & 0.0382 \\ 0.0005 & 0.0863 & 0.6586 \end{pmatrix} \\
&= (0.0238, 0.0469, 0.0200)
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\pi_3} &= \boldsymbol{\pi_2 R} \\
&= (0.0238, 0.0469, 0.0200)\begin{pmatrix} 0.3725 & 0.0269 & 0.0001 \\ 0.0181 & 0.5040 & 0.0382 \\ 0.0005 & 0.0863 & 0.6586 \end{pmatrix} \\
&= (0.0097, 0.0260, 0.0150)
\end{aligned}$$

and so on.

Then we could solve the probability that a person must wait for the available facility when he arrives.

$$P_{female}(waiting) = 1 - (\pi_{0_1} + \pi_{0_2} + \pi_{0_3}) = 0.3916$$

By changing the value of λ or μ_1 and μ_2, we could draw the figure on how the waiting probability change by these parameters.

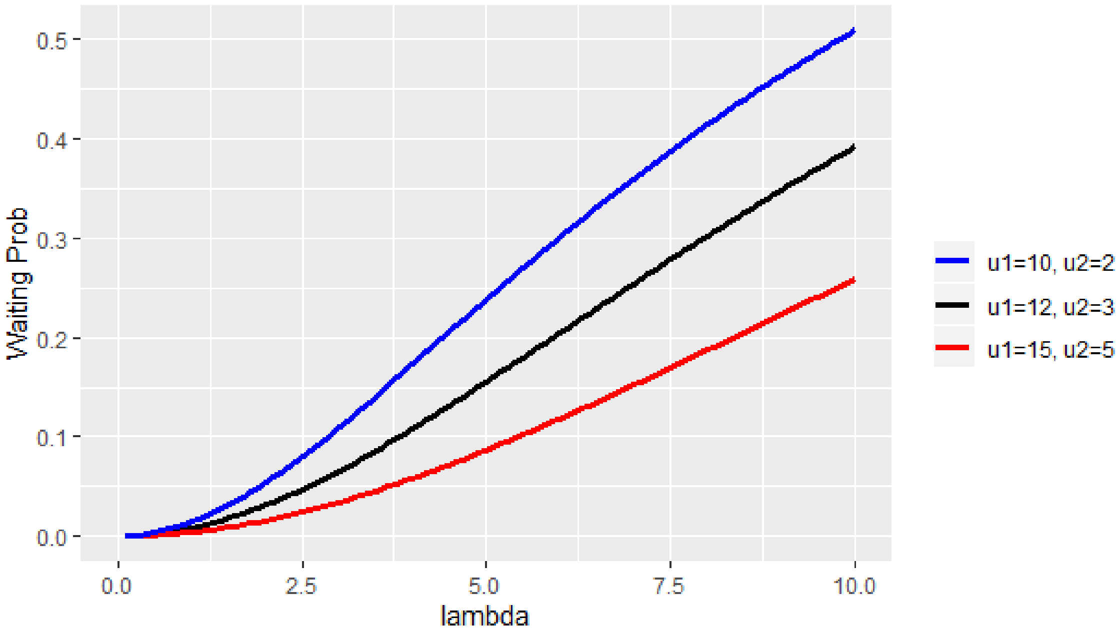


FIGURE 3.1. The Waiting Probability of Female Restroom Queueing Model

3.2. Male Restroom

Now we substitute these parameters into the rate matrix of the male case, the block matrices are

$$\mathbf{B_{00}} = \begin{pmatrix} -10 & 8 & 0 & 2 \\ 12 & -22 & 0 & 0 \\ 12 & 0 & -22 & 0 \\ 3 & 0 & 0 & -13 \end{pmatrix}, \quad \mathbf{B_{01}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 8 & 0 & 2 & 0 \\ 8 & 2 & 0 & 0 \\ 0 & 0 & 8 & 2 \end{pmatrix},$$

$$\mathbf{B_{10}} = \begin{pmatrix} 0 & 12 & 12 & 0 \\ 0 & 0 & 0 & 12 \\ 0 & 3 & 0 & 12 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \mathbf{B_{11}} = \begin{pmatrix} -34 & 0 & 0 & 0 \\ 0 & -32 & 0 & 0 \\ 0 & 0 & -25 & 0 \\ 0 & 0 & 0 & -23 \end{pmatrix}$$

and

$$\mathbf{A_0} = \begin{pmatrix} 24 & 0 & 0 & 0 \\ 0 & 12 & 12 & 0 \\ 3 & 0 & 12 & 0 \\ 0 & 0 & 3 & 12 \end{pmatrix}, \quad \mathbf{A_1} = \begin{pmatrix} -34 & 0 & 0 & 0 \\ 0 & -32 & 0 & 0 \\ 0 & 0 & -25 & 0 \\ 0 & 0 & 0 & -23 \end{pmatrix}$$

$$\mathbf{A_2} = \begin{pmatrix} 8 & 2 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 8 & 2 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

By using the equation (2.2.1), we check these parameters satisfy the stability condition. Similar to the female case, after calculation in R software, we solve the constant matrix **R**.

$$\mathbf{R} = \begin{pmatrix} 0.2980 & 0.0640 & 0.00206 & 0 \\ 0.0001 & 0.2792 & 0.0381 & 0 \\ 0.0140 & 0 & 0.3964 & 0.0913 \\ 0 & 0 & 0.0253 & 0.4566 \end{pmatrix}$$

Then we proceed to the boundary conditions.

$$(\boldsymbol{\pi_0}, \boldsymbol{\pi_1}) \begin{pmatrix} \mathbf{B_{00}} & \mathbf{B_{01}} \\ \mathbf{B_{10}} & \mathbf{B_{11}} + \mathbf{RA_0} \end{pmatrix} = (\mathbf{0}, \mathbf{0}).$$

Similarly, use the normalization method to solve these equations and get the final answer shown below.

$$\begin{aligned} \boldsymbol{\pi_0} &= (\pi_{0_1}, \pi_{0_2}, \pi_{0_3}, \pi_{0_4}) = (0.2903, 0.1484, 0.0319, 0.1400) \\ \boldsymbol{\pi_1} &= (\pi_{1_1}, \pi_{1_2}, \pi_{1_3}, \pi_{1_4}) = (0.0585, 0.0065, 0.0801, 0.0666) \\ \boldsymbol{\pi_2} &= \boldsymbol{\pi_1 R} = (0.0186, 0.0056, 0.0338, 0.0377) \\ \boldsymbol{\pi_3} &= \boldsymbol{\pi_2 R} = (0.0060, 0.0027, 0.0146, 0.0203) \\ &\dots \end{aligned}$$

The probability that a male come with short service need and no facility available is

$$P_{male_{short}}(waiting) = 1 - (\pi_{0_1} + \pi_{0_2} + \pi_{0_3} + \pi_{0_4}) - (\pi_{1_2} + \pi_{1_4}) = 0.3161$$

And the probability that a male come with long service need and no facility available is

$$P_{male_{long}}(waiting) = 1 - (\pi_{0_1} + \pi_{0_2}) = 0.5612$$

By changing the value of λ or μ_1 and μ_2, we could draw the figure on how the waiting probability change by these parameters.

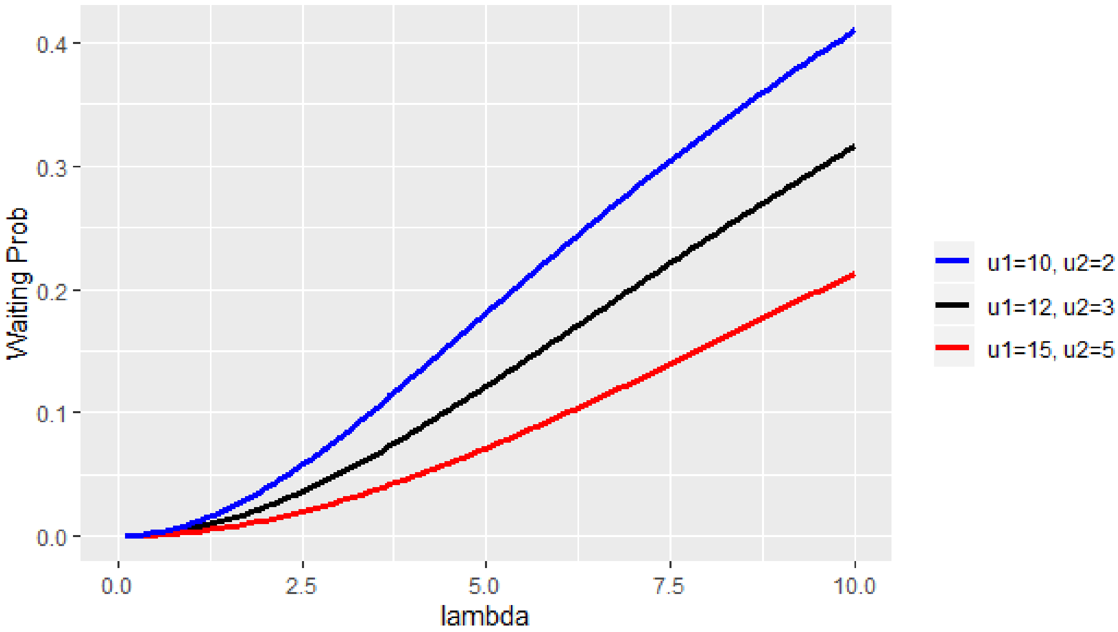


FIGURE 3.2. The Waiting Probability for Short Service of Male Restroom Queueing Model

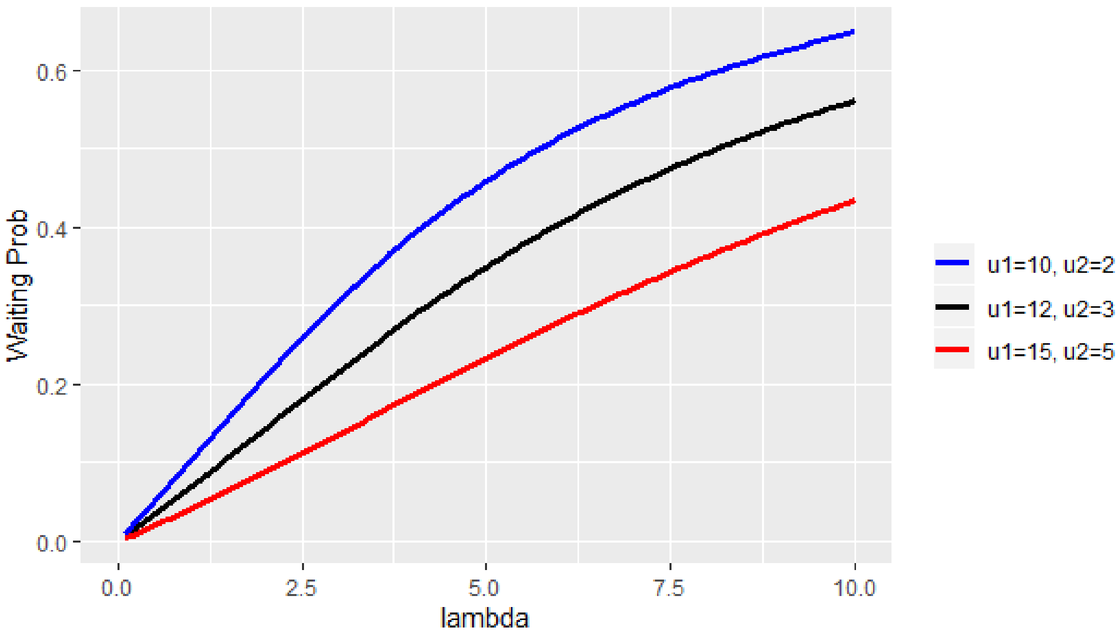


FIGURE 3.3. The Waiting Probability for Long Service of Male Restroom Queueing Model

3.3. Unisex Restroom

Since now both male and female would use these unisex restrooms, we have arrival rate $2 \times \lambda = 20$ in this queueing model.

Now we substitute these parameters into the rate matrix of the unisex case, the block matrices are

$$\mathbf{B_{00}} = \begin{pmatrix} -20 & 16 & 4 \\ 12 & -32 & 0 \\ 3 & 0 & -23 \end{pmatrix}, \quad \mathbf{B_{01}} = \begin{pmatrix} 0 & 0 & 0 \\ 16 & 4 & 0 \\ 0 & 16 & 4 \end{pmatrix},$$

$$\mathbf{B_{10}} = \begin{pmatrix} 0 & 24 & 0 \\ 0 & 3 & 12 \\ 0 & 0 & 6 \end{pmatrix}, \quad \mathbf{B_{11}} = \begin{pmatrix} -44 & 0 & 0 \\ 0 & -35 & 0 \\ 0 & 0 & -22 \end{pmatrix},$$

$$\mathbf{B_{12}} = \begin{pmatrix} 16 & 4 & 0 & 0 \\ 0 & 16 & 4 & 0 \\ 0 & 0 & 16 & 4 \end{pmatrix}, \quad \mathbf{B_{21}} = \begin{pmatrix} 36 & 0 & 0 \\ 3 & 24 & 0 \\ 0 & 6 & 12 \\ 0 & 0 & 9 \end{pmatrix}$$

and

$$\mathbf{A_0} = \begin{pmatrix} 28.8 & 7.2 & 0 & 0 \\ 2.4 & 19.8 & 4.8 & 0 \\ 0 & 4.8 & 10.8 & 2.4 \\ 0 & 0 & 7.2 & 1.8 \end{pmatrix}, \quad \mathbf{A_1} = \begin{pmatrix} -56 & 0 & 0 & 0 \\ 0 & -47 & 0 & 0 \\ 0 & 0 & -38 & 0 \\ 0 & 0 & 0 & -29 \end{pmatrix}$$

$$\mathbf{A_2} = \begin{pmatrix} 20 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 20 \end{pmatrix}$$

By using the equation (2.3.1), we check these parameters satisfy the stability condition. Similar to the previous cases, after calculation in R software, we solve the constant matrix $\mathbf{R}$.

$$\mathbf{R} = \begin{pmatrix} 0.4715 & 0.0346 & 0.0002 & 0 \\ 0.0133 & 0.5559 & 0.0395 & 0.0001 \\ 0.0001 & 0.0433 & 0.6450 & 0.0345 \\ 0 & 0.0011 & 0.1020 & 0.7230 \end{pmatrix}$$

Then we proceed to the boundary conditions.

$$(\boldsymbol{\pi_0}, \boldsymbol{\pi_1}, \boldsymbol{\pi_2}) \begin{pmatrix} \mathbf{B_{00}} & \mathbf{B_{01}} & \mathbf{0} \\ \mathbf{B_{10}} & \mathbf{B_{11}} & \mathbf{B_{12}} \\ \mathbf{0} & \mathbf{B_{21}} & \mathbf{A_1} + \mathbf{RA_0} \end{pmatrix} = (\mathbf{0}, \mathbf{0}, \mathbf{0})$$

Under these equations, the normalization constant α is a little different from the one in equation (1.2.7). In this case, we have

$$\begin{aligned} 1 = \boldsymbol{\pi e} &= \boldsymbol{\pi_0 e} + \boldsymbol{\pi_1 e} + \boldsymbol{\pi_2 e} + \sum_{i=3}^{\infty} \boldsymbol{\pi_i e} \\ &= \boldsymbol{\pi_0 e} + \boldsymbol{\pi_1 e} + \boldsymbol{\pi_2 e} + \sum_{i=3}^{\infty} \boldsymbol{\pi_2 R^{i-2} e} \\ &= \boldsymbol{\pi_0 e} + \boldsymbol{\pi_1 e} + \sum_{i=2}^{\infty} \boldsymbol{\pi_2 R^{i-2} e} = \boldsymbol{\pi_0 e} + \boldsymbol{\pi_1 e} + \sum_{i=0}^{\infty} \boldsymbol{\pi_1 R^{i} e} \\ &= \boldsymbol{\pi_0 e} + \boldsymbol{\pi_1 e} + \boldsymbol{\pi_1} (\boldsymbol{I} - \boldsymbol{R})^{-\mathbf{1}} \boldsymbol{e} \end{aligned}$$

Then we use $\boldsymbol{\alpha} = \boldsymbol{\pi_0 e} + \boldsymbol{\pi_1 e} + \boldsymbol{\pi_1} (\boldsymbol{I} - \boldsymbol{R})^{-\mathbf{1}} \boldsymbol{e}$ to complete the normalization of the vectors $\boldsymbol{\pi_0}$, $\boldsymbol{\pi_1}$ and $\boldsymbol{\pi_2}$. The final results are shown below.

$\boldsymbol{\pi_0} = (\pi_{0_1}, \pi_{0_2}, \pi_{0_3}) = (0.1152, 0.1236, 0.0872)$

$\boldsymbol{\pi_1} = (\pi_{1_1}, \pi_{1_2}, \pi_{1_3}) = (0.0747, 0.1063, 0.0448)$

$\boldsymbol{\pi_2} = (\pi_{2_1}, \pi_{2_2}, \pi_{2_3}, \pi_{2_4}) = (0.0310, 0.0648, 0.0460, 0.0095)$

$\boldsymbol{\pi_3} = \boldsymbol{\pi_2 R} = (0.0155, 0.0391, 0.0332, 0.0084)$

$\boldsymbol{\pi_4} = \boldsymbol{\pi_3 R} = (0.0078, 0.0237, 0.0238, 0.0072)$

$\ldots$

The probability that one person come and no facility available is

$$P_{unisex}(waiting) = 1 - (\pi_{0_1} + \pi_{0_2} + \pi_{0_3}) - (\pi_{1_1} + \pi_{1_2} + \pi_{1_3}) = 0.4483$$

By changing the value of λ or μ_1 and μ_2, we could draw the figure on how the waiting probability change by these parameters.

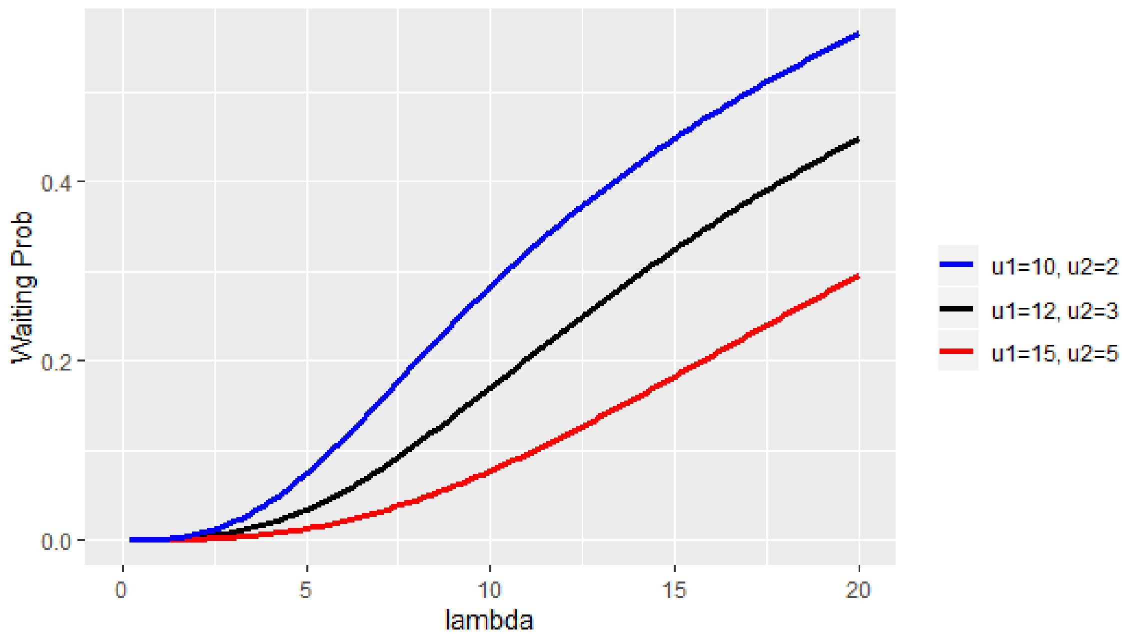


FIGURE 3.4. The Waiting Probability of Unisex Restroom Queueing Model

3.4. Comparison of Different Restroom Systems

To better understand the difference between the traditional and new systems, we changed the ratio of females to investigate the difference in the waiting probability of male and female under different restroom systems. In this case, we still assume that the arrival rate of the unisex restroom queueing model is 20, but we assume different female proportions to solve the corresponding waiting probability. The following table and figure show how the waiting probability change under the different female proportions.

TABLE 3.1. The Waiting Probability under Different Female Proportion

Female Proportion	Female Arrival Rate	Male Arrival Rate	Female Waiting Prob.	Male Waiting Prob. for Short	Male Waiting Prob. for Long
0.25	5	15	0.1551	0.4688	0.6654
0.35	7	13	0.2538	0.4137	0.6312
0.45	9	11	0.3481	0.3507	0.5878
0.50	10	10	0.3916	0.3161	0.5612
0.55	11	9	0.4322	0.2794	0.5304
0.65	13	7	0.5049	0.2013	0.4529
0.75	15	5	0.5668	0.1213	0.3489

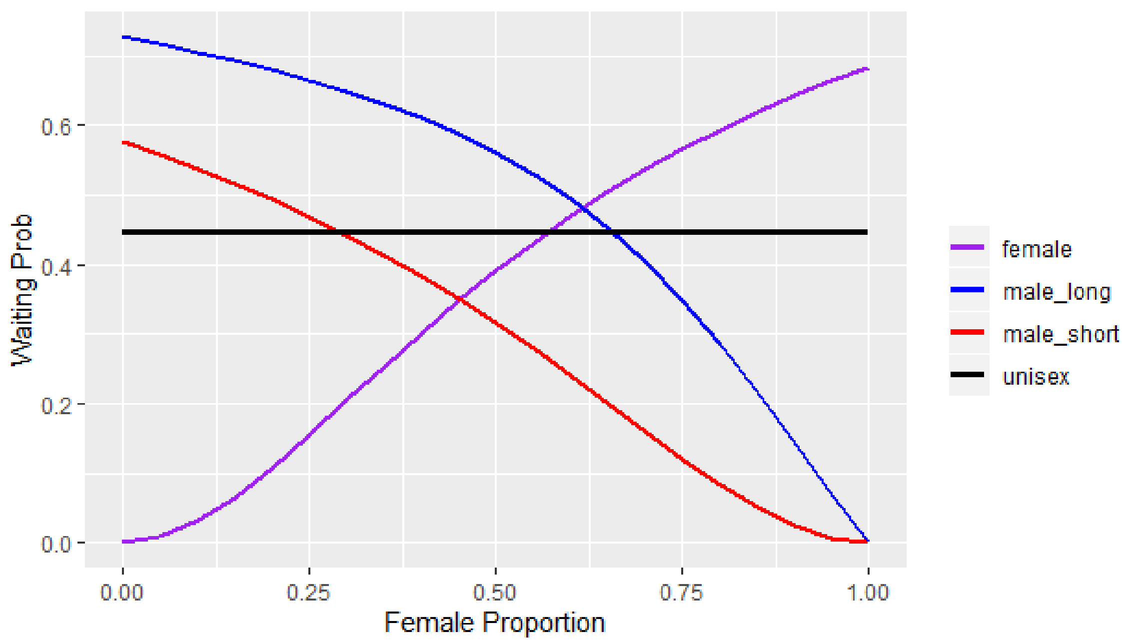


FIGURE 3.5. The Waiting Probability under Different Female Proportion

From above figure, we could make the following conclusions under these parameters setting.

When female proportion is larger than 0.55 approximately, the female queueing would be improved by using the new system. When female proportion is smaller than 0.65 approximately, the male long service queueing would be improved, and when female proportion is smaller than 0.25 approximately, the male short service queueing would be improved.

Overall, when female proportion lies in about $(0.55, 0.65)$ interval, both the male long service queueing and female queueing would be improved by using the unisex restroom system.

In extreme conditions, when the female proportion becomes zero, which means there is no female in the queueing system, then the male queueing would be improved at the most extent, no matter what type of service. When the female proportion becomes 100%, the female queueing would be improved to the most extent by using unisex restrooms.

CHAPTER 4

Queueing Model Simulation

Now we simulate the queueing model of above three restrooms under $\lambda = 10$, $p = 0.8$, $\mu_1 = 12$, $\mu_2 = 3$ by using R.

For the simulation of the male restroom queueing model, different from the assumption we made in the queuing model in chapter 2, now we assume an unlimited waiting space for people with long service need as well and this model for simulation follows FIFO principle.

By using 100 times simulation first, the time-average number of people in three different restroom types is shown in the following pictures.

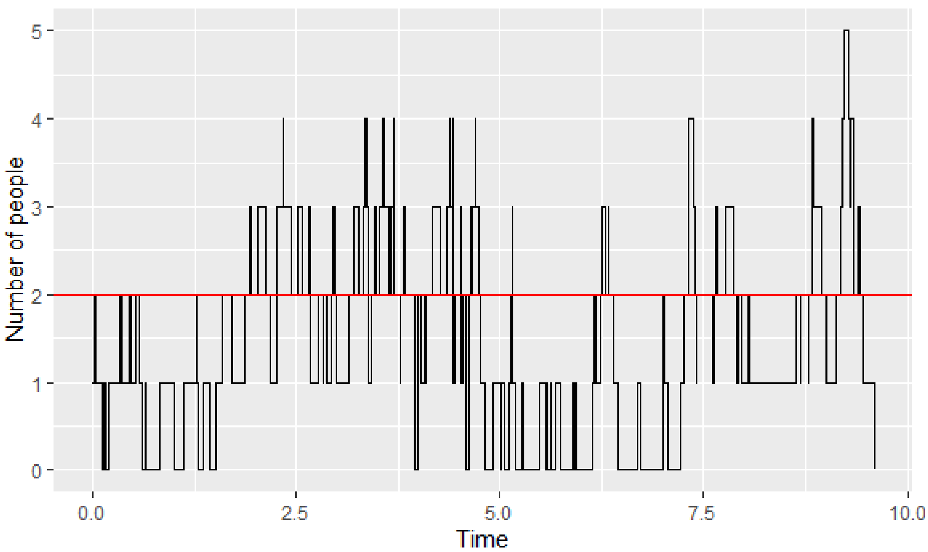


FIGURE 4.1. The Number of People in the Female Restroom Queueing Model

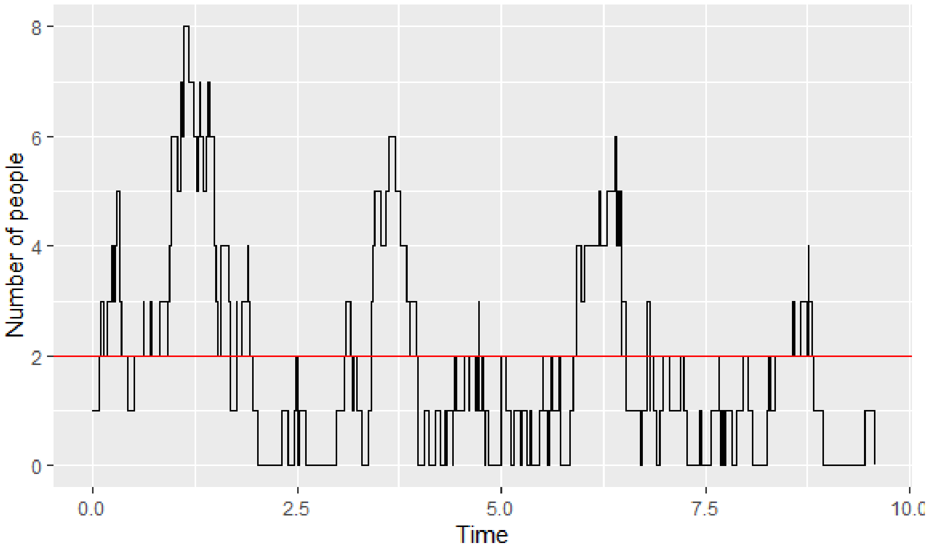


FIGURE 4.2. The Number of People in the Male Restroom Queueing Model

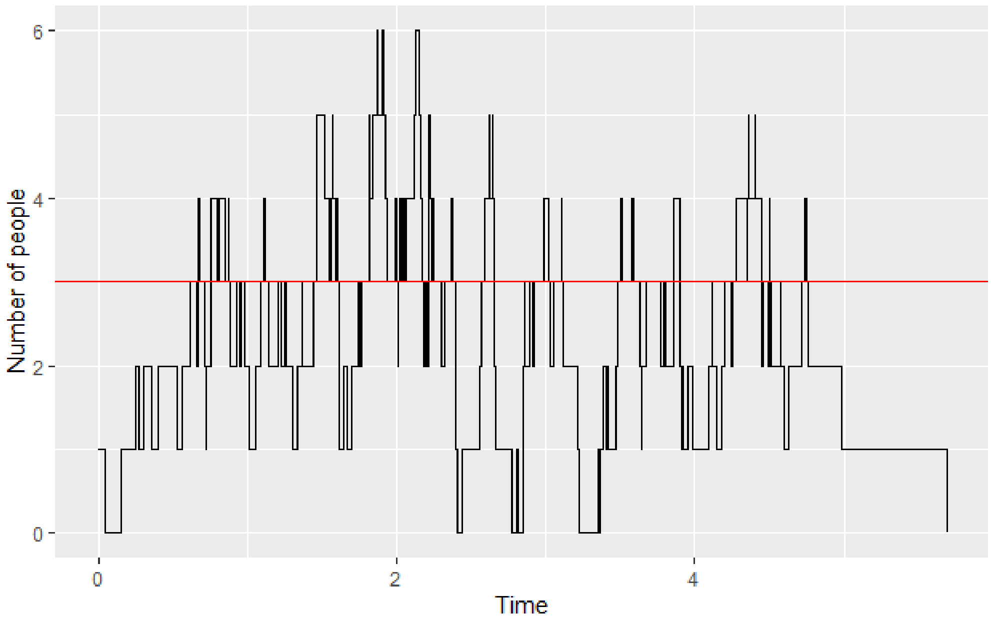


FIGURE 4.3. The Number of People in the Unisex Restroom Queueing Model

By calculating the time proportion of the number of people greater than or equal to 2 or 3 in the queueing model for female (male/unisex) restroom, we can estimate the probability of equipment unavailable in the queueing model, that is, the probability of waiting.

Note that for the male restroom, when the number of people in this queueing model is equal to 2, one person may be using the toilet stall while another person with long service needs is waiting for the toilet stall. In this case, the urinal is available but the toilet stall is not. So in fact, the waiting probability of short service is lower than the waiting probability that we figured out this way, and the waiting probability of long service is higher than that.

To get the corresponding waiting probability for each queueing model, we use 10000 times simulation in this case and the results are shown below.

$$P_{female}(waiting) = 37.87\%$$

$$P_{male}(waiting) = 64.70\%$$

$$P_{unisex}(waiting) = 50.34\%$$

Also, by changing the value of λ or μ_1 and μ_2, we could draw the figure on how the waiting probability change by these parameters. Note we use 1000 times simulation here and the detailed R code is located in Appendix A.

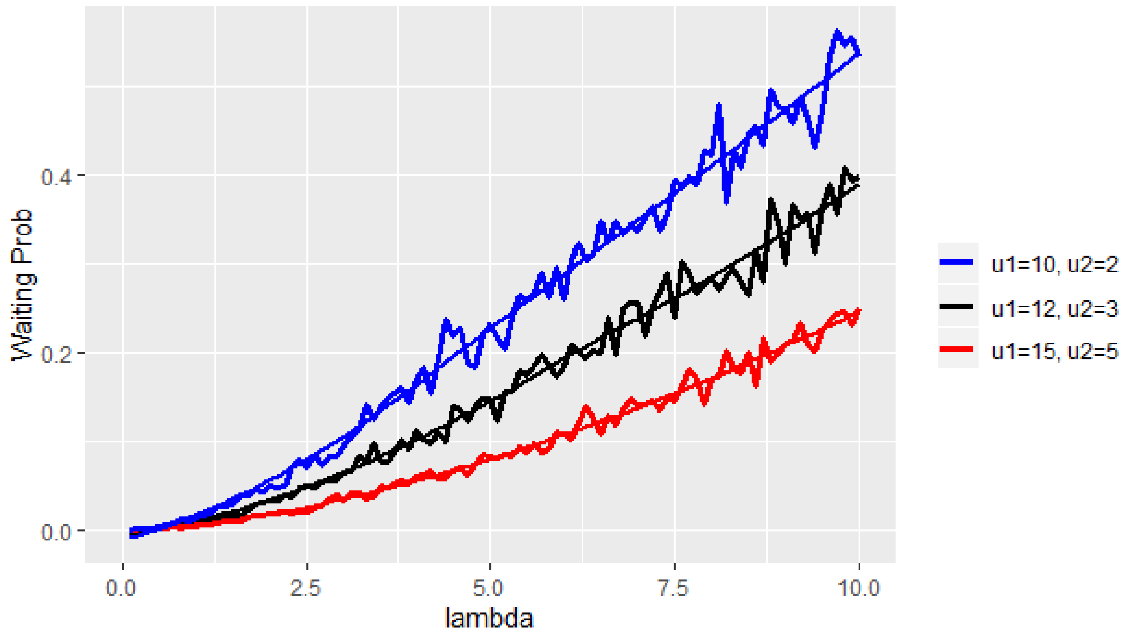


FIGURE 4.4. The Waiting Probability of Female Restroom Queueing Simulation

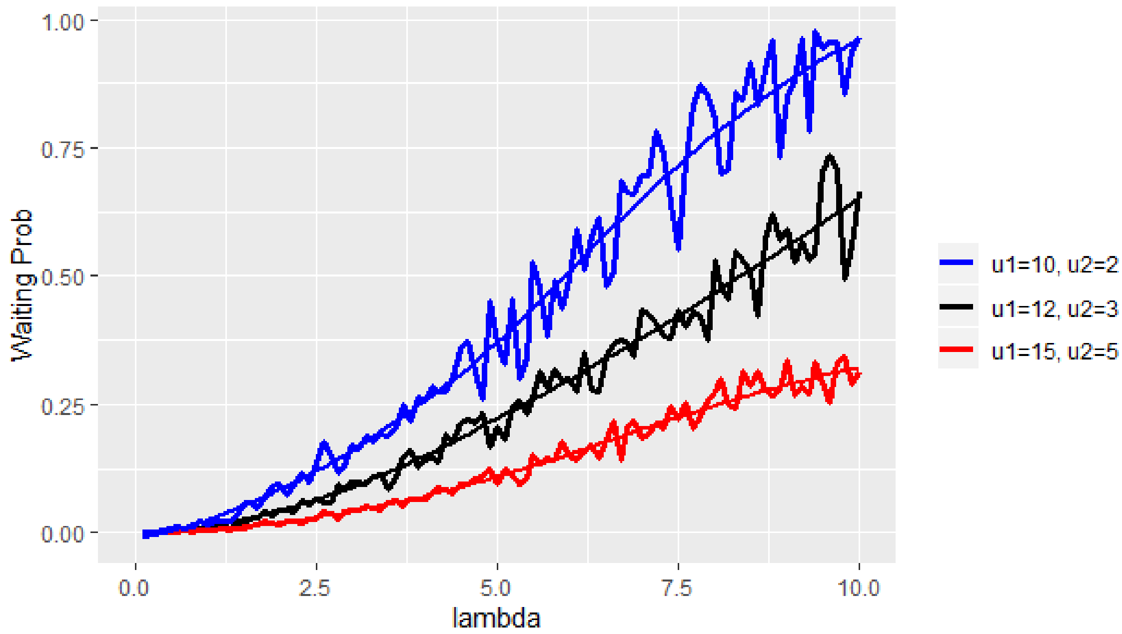


Figure 4.5. The Waiting Probability of Male Restroom Queueing Simulation

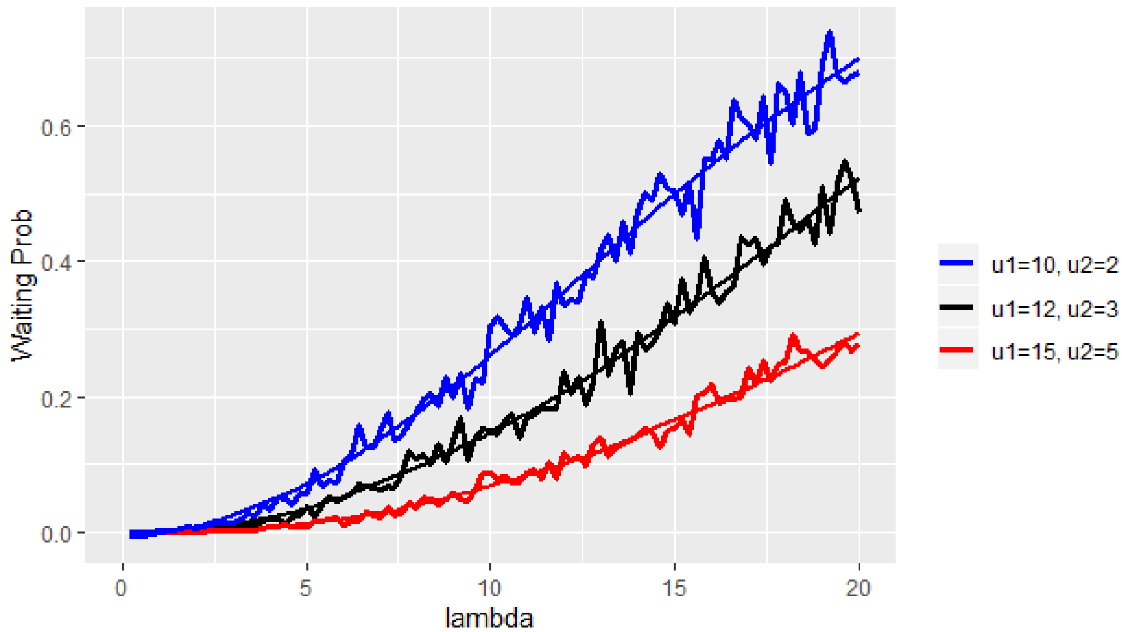


Figure 4.6. The Waiting Probability of Unisex Restroom Queueing Simulation

CHAPTER 5

Discussion

Under the queuing model of Chapter 2 and 3 and the queuing simulation of Chapter 4, we both calculate the waiting probability and show further results in some figures. From the results and pictures shown in previous chapters, we conclude that, under the assumption that male and female has the same arrival rate, using the unisex restrooms at a similar cost could usually improve the long service queueing of male but the queueing of female might worsen slightly.

To investigate the difference in the waiting probability of male and female under different restroom systems, we changed female proportion and calculate the corresponding waiting probability. We found that the male long service queueing and female queueing could be both improved by using the unisex restroom system, if the female proportion lies in a specified interval, which is about $(0.55, 0.65)$ in our case.

More importantly, using the unisex restrooms to replace the traditional ones could decrease the inconvenience of special groups such as transgender people. And separate unisex restroom could satisfy the guidelines of social distance during the current COVID-19 pandemic.

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Appendix

R Code

```
############Female Queueing Model##############

###########parameters setting#################

u1=12 #service rate for short service

u2=3 #service rate for long service

p=0.8 #the probability that coming person has short service need

lambda=10 #arrival rate


lambda1=lambda*p  #arrival rate for short service

lambda2=lambda*(1-p) #arrival rate for long service


#############stability condition###########

2*u1*u2/(u2*p+u1*(1-p))>lambda #TRUE


###########blocks for rate matrix#########

B00=matrix(c(-lambda,lambda1,lambda2,u1,-(u1+lambda),0,u2,0,

-(u2+lambda)),3,3,TRUE);B00

B01=matrix(c(0,0,0,lambda1,lambda2,0,0,lambda1,lambda2),3,3,TRUE);B01

B10=matrix(c(0,2*u1,0,0,u2,u1,0,0,2*u2),3,3,TRUE);B10

A1 =diag(c(-(lambda+2*u1),-(lambda+u1+u2),-(lambda+2*u2)),3,3);A1

A2 =diag(lambda,3,3);A2

A0 =matrix(c(2*u1*p,2*u1*(1-p),0,u2*p,u1*p+u2*(1-p),u1*(1-p),
```

```
0,2*u2*p,2*u2*(1-p)),3,3,TRUE);A0

################Find matrix R##########
A1I=solve(A1) #inverse of A1
V=A2%*%A1I;V
W=A0%*%A1I;W

R=matrix(rep(0,9),3,3);R #start guess
for (i in 1:100){
R=-V-R^2%*%W
}

#########Solve boundary condition##########
K=rbind(cbind(B00,B01),cbind(B10,A1+R%*%A0))
K[,1]=c(1,rep(0,5));K
b=matrix(c(1,rep(0,5)),6,1);b
phi=solve(t(K),b);phi

phi0=phi[1:3,];phi0
phi1=phi[4:6,];phi1
alpha=sum(phi0)+sum(phi1%*%(solve(diag(1,3,3)-R)))
#normalization aonstant
phi=phi/alpha;sum(phi)#check summation of phi0,phi1,phi2
phi0=phi[1:3,];phi0
phi1=phi[4:6,];phi1

phi2=phi1%*%R;phi2
```

```
phi3=phi2%*%R;phi3
#.......

#########probility of waiting
prob=1-sum(phi0);prob

#########Female Matrix Waiting Prob vs Lambda#######
###########parameters setting#################
u1=12 #service rate for short service;10;12;15
u2=3 #service rate for long service;2;3;5
p=0.8 #the probability that coming person has short service need
prob=0 #waiting probability

for(j in 1:100){
lambda=j/10
lambda1=lambda*p  #arrival rate for short service
lambda2=lambda*(1-p) #arrival rate for long service
B00=matrix(c(-lambda,lambda1,lambda2,u1,-(u1+lambda),0,
u2,0,-(u2+lambda)),3,3,TRUE);B00
B01=matrix(c(0,0,0,lambda1,lambda2,0,0,lambda1,lambda2),3,3,TRUE);B01
B10=matrix(c(0,2*u1,0,0,u2,u1,0,0,2*u2),3,3,TRUE);B10
A1 =diag(c(-(lambda+2*u1),-(lambda+u1+u2),-(lambda+2*u2)),3,3);A1
A2 =diag(lambda,3,3);A2
A0 =matrix(c(2*u1*p,2*u1*(1-p),0,u2*p,u1*p+u2*(1-p),u1*(1-p),
0,2*u2*p,2*u2*(1-p)),3,3,TRUE)
A1I=solve(A1) #inverse of A1
V=A2%*%A1I
```

```
W=A0%*%A1I
R=matrix(rep(0,9),3,3)#start guess
for (i in 1:100){
R=-V-R^2%*%W
}
K=rbind(cbind(B00,B01),cbind(B10,A1+R%*%A0))
K[,1]=c(1,rep(0,5))
b=matrix(c(1,rep(0,5)),6,1)
phi=solve(t(K),b)
phi0=phi[1:3,]
phi1=phi[4:6,]
alpha=sum(phi0)+sum(phi1%*%(solve(diag(1,3,3)-R)))
phi=phi/alpha
phi0=phi[1:3,]
prob[j]=1-sum(phi0)
}
lambda=1:100/10
data=data.frame(lambda,prob)
#########################################
library(ggplot2)
x=ggplot(data,aes(x=lambda,y=prob,col='u1=12, u2=3'))+
geom_line(size=1.2)+
scale_colour_manual("",values = c("u1=12, u2=3" = "black",
"u1=15, u2=5" = "red",
"u1=10, u2=2" = "blue"))+
labs(y='Waiting Prob')+
xlim(0,10);x
```

```
#########################################
# y=x+geom_line(data,mapping=aes(x=lambda,y=prob,col='u1=15,
 u2=5'),size=1.2);y
#
# z=y+geom_line(data,mapping=aes(x=lambda,y=prob,col='u1=10,
 u2=2'),size=1.2);z

###############Female Simulation###########
###########parameters setting#################
n=10000  #the number of customers
mu1=12
mu2=3
prob=0.8
lambda=10

service1 <- rexp(n,mu1) #service time
service2 <- rexp(n,mu2)
IA <- rexp(n,lambda) #interarrival time
IA[1] <- 0
A <-cumsum(IA)
#actual arrival times; assume first customer arrives at time 0

#################record the actual service time##############
service_type=0
S=0
p=runif(n,0,1)
for (i in 1:n){
```

```
if (p[i]<prob){
service_type[i]="short"
S[i]=service1[i]        #actual service time
}else{
service_type[i]="long"
S[i]=service2[i]
}
}

############solve for completion time##########
m=0  #record the number of  stalls being used
=rep(0,2) #record the completion time of the current user

W <- rep(0,n)
C <- rep(0,n)
C[1] <- S[1];C[2] <- A[2]+S[2]
for (i in 3:n){
a=i-1
for (j in 1:a){
if (A[i]<C[j]) {
m=m+1
[m]=C[j]
}
if (m<2){
W[i]=0
} else {
W[i]=min()-A[i]
```

```
}
C[i]=A[i]+W[i]+S[i] #completion times C for each customer
m=0
}
}

###########number of people in system and time#######
N <- rep(0,2*n)
order <- seq(1,n,1)
status <- rep(0,2*n)
time <- (c(A,C))
M <- data.frame(time,status=c(rep('arrival',n),rep('leave',n)),
N=c(rep(1,n),rep(-1,n)))
B=M[order(M[,1]),]

time <- B[,1]
N_total <- cumsum(B[,3])

##############record the available situation#######
available=0
n_time=2*n
for (i in 1:n_time){
if (N_total[i]<2){
available[i] <- "YES"
}else{
available[i] <- "NO"}
i=i+1
```

```
}
DF=data.frame(time,N_total,available);DF

###############solve the waiting prob#########
Nava_time=0
for (i in 1:n_time){
if (available[i] == "NO"){
Nava_time <- Nava_time+time[i+1]-time[i]
}else{
Nava_time <- Nava_time
}
}
Nava_time
Nava_prob <- Nava_time/time[n_time];Nava_prob

#More detailed R code please contact chen1hy@uwindsor.ca
```

Vita Auctoris

Xintong Chen was born in Lanxi, Zhejiang, China in 1995. She completed her secondary school education locally at Lanxi No.1 Middle School from 2011-2014. She then went to the Southwest University of Finance and Economics in Chengdu, Sichuan, China and recieved her Bachelor of Economics in Acturial Science. She received her Bachelor of Science in Acturial Science in Heriot-Watt University in UK in 2018. She is currently a Master of Science candidate in Statistics at the University of Windsor, and is expected to graduate in Winter 2021.