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D'Alembert Functions on Groups

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D'ALEMBERT FUNCTIONS ON GROUPS

by

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Abstract

This major paper is devoted to the study of pre-d'Alembert functions and d'Alembert functions on groups.

In this paper, we first study additive and multiplicative Cauchy equations and the sine addition formula on groups. Then we discuss some properties of pre-d'Alembert functions on groups. In particular, we characterize when a pre-d'Alembert function is abelian, and furthermore get the general form of abelian pre-d'Alembert functions on groups. Finally we achieve our goal: we obtain the structure of the solution by group representation theory.

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Contents

List of Notations

$\mathbb{C}$	The complex numbers
$\mathbb{C}^\star$	The complex numbers without 0
$C(S)$	The continuous complex-valued functions on S
$\mathbb{C}^S$	The vector space of functions on S
G	A group
$GL_n(\mathbb{R})$	General linear group of $n \times n$
$[G, G]$	Commutator subgroup of G
H_3	Heisenberg group
$\mathcal{L}(V)$	The linear maps from V to V
M	A monoid
$M(n \times n, \mathbb{C})$	The complex $n \times n$ matrices
$\mathbb{Q}$	The rational numbers
$\mathbb{R}$	The real numbers
$\mathbb{R}^+$	The positive real numbers
$\mathbb{R}^\star$	The real numbers without 0
S	A semigroup
$\mathbb{Z}$	The integers
$\mathbb{Z}^+$	The positive integers

CHAPTER 1

Introduction

Mathematicians began to study functional equations long time ago. Functional equations, as the name implies, are a type of equations in which one or more unknowns are functions. A well-known equation, called the additive Cauchy quuation, is given as follows:

$$f(x+y) = f(x) + f(y) \qquad \text{for all } x, y \in \mathbb{R}, \tag{1.0.1}$$

where the only unknown is $f : \mathbb{R} \to \mathbb{C}$. It is easy to see that $f(x) = \alpha x$ is a solution of Equation (1.0.1), where α is any complex number.

Here is another example (the sine addition equation)

$$f(x+y) = f(x)g(y) + g(x)f(y) \qquad \text{for all } x, y \in \mathbb{R}, \tag{1.0.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{C}$ are two unknowns. Since

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \qquad \text{for all } x, y \in \mathbb{R},$$

we get one solution of Equation (1.0.2): $f = \sin$, $g = \cos$.

The one is *d'Alembert's functional equation*

$$f(x+y) + f(x-y) = 2f(x)f(y) \qquad \text{for all } x, y \in \mathbb{R}, \tag{1.0.3}$$

where the only unknown is $f : \mathbb{R} \to \mathbb{C}$. Clearly, one solution of Equation (1.0.3) is $f(x) = \cos x$. That is why d'Alembert's functional equation is sometimes called the *the cosine equation*.

We should mention that no matter Equation (1.0.1), Equation (1.0.2) or Equation (1.0.3), the solution given above is not the only one.

For Equations (1.0.1), (1.0.2) and (1.0.3), we can replace $(\mathbb{R}, +)$ by group, with $x + y$ by xy.

In this major paper, we shall consider the structures for some functional equations such as the pre-d'Alembert functional equation and the d'Alembert functional equation on a arbitrary group.

In Chapter 2 we provide some preliminaries we will use in this paper. Section 2.1 gives some basic definitions such as commutator groups and Heisenberg groups. Section 2.2 and 2.3 contain the definitions and some properties of the additive Cauchy equation and the multiplicative Cauchy equation. Section 2.4 shows a theorem about the structure of the solutions of the sine addition formula on a topological group. Section 2.5 lists the main results of group representation theorey we will use in this paper.

In Chapter 3, Sections 3.1 and 3.2 introduce the definiton annd properties of the pre-d'Alembert functional equation on groups. In Section 3.3 we show the structure of abelian solutions of pre-d'Alembert function on groups, and give some methods to check when a pre-d'Alembert function is abelian.

Chapter 4 focuses on the d'Alembert's functional equation.

In Chapter 5, Sections 5.1 and 5.2 give the structure of continuous, non-abelian solutions of pre-d'Alembert functions on groups via group representation theory. Section 5.3 gives the statement of Davison's structure theorem. Section 5.4 shows there exist non-abelian d'Alembert functions on groups.

CHAPTER 2

Preliminaries

In this chapter, we give some basic definitions used in this paper. We study the definitions and some properties of additive Cauchy equations, the multiplicative Cauchy equation and the sine addition formula. We also recall some theorems in the group representation theory we will use later.

2.1. Some Basic Definitions

Definition 2.1.1. A *group* $(G, \circ)$ is a set G with a binary operation $\circ$ on G satisfying the following axioms:
(i) $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in G$, *i.e.*, $\circ$ is *associative*;
(ii) there exists an element e in G, called an *identity* of G, such that for all $x \in G$ we have $x \circ e = e \circ x = x$;
(iii) for each $x \in G$ there is an element x^{-1} of G, called an *inverse* of a, such that $x \circ x^{-1} = x^{-1} \circ x = e$.

A group $(G, \circ)$ is called *abelian* if $x \circ y = y \circ x$ for all $x, y \in G$. Usually we simply write $x \circ y$ with xy.

Definition 2.1.2. A *semigroup* is a set with an associative binary opeartion.

Definition 2.1.3. A *monoid* is a semigroup with an identity.

Definition 2.1.4. A subset H of a group G is called a *subgroup* if it satisfying the following properties:
(i) if $x, y \in H$, then $xy \in H$;
(ii) $e \in H$;
(iii) if $x \in H$, then $x^{-1} \in H$.

DEFINITION 2.1.5. Let G be a group. The *commutator* between $x \in G$ and $y \in G$ is $[x, y] := xyx^{-1}y^{-1}$.

The *commutator subgroup* $[G, G]$ is the subgroup generated by all the commutators of the group G.

EXAMPLE 2.1.6. (i) $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are groups under $+$ with $e = 0$ and $x^{-1} = -x$.
(ii) $\mathbb{Z}^+$ and $\mathbb{R}^+$ are semigroups under $+$.
(iii) $\mathbb{R}^\star$ is a monoid under $\times$ with $e = 1$.
(iv) For each $n \in \mathbb{Z}^+$, let $GL_n(\mathbb{R})$ be the set of all $n \times n$ matrices whose entries from $\mathbb{R}$ and whose determinant is nonzero, *i.e.*,

$$GL_n(\mathbb{R}) = \{A \mid A \text{ is an } n \times n \text{ matrix with entries from } \mathbb{R} \text{ and } \det(A) \neq 0\}.$$

Then $GL_n(\mathbb{R})$ is a group under matrix multiplication with e being the $n \times n$ identity matrix and A^{-1} being the inverse of A. $GL_n(\mathbb{R})$ is called the *general linear group of degree n under* $\mathbb{R}$.
(v)

$$H_3 := \left\{ [x, y, z] := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\},$$

is called a *Heisenberg group* under the matrix multiplication with e being the 3×3 identity matrix, and the inverse being

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x & xy - z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}.$$

DEFINITION 2.1.7. Let $(G_1, \circ)$ and $(G_2, \star)$ be groups. A map $\phi : G_1 \to G_2$ such that

$$\phi(a \circ b) = \phi(a) \star \phi(b) \qquad \text{for all } a, b \in G_1$$

is called a *homomorphism*.

Definition 2.1.8. A *topological group* G is a topological space which is also a group such that the maps $G \times G \to G$, $(x, y) \mapsto xy$ and $G \to G$, $x \mapsto x^{-1}$ are continuous.

2.2. Additive Cauchy Equations

Definition 2.2.1. Let $(S, \circ)$ be a semigroup, and $(H,+)$ an abelian group. An *additive Cauchy equation* is in the form

$$f(xy) = f(x) + f(y) \qquad \text{for all } x, y \in S, \tag{2.2.1}$$

where $f : S \to H$ is the unknown. A solution is a function $f : S \to H$ satisfying Equation (2.2.1). Such an f is called an *additive function* from S to H.

If $S = (\mathbb{R}, +)$ and $H = (\mathbb{C}, +)$, then $f(x) = \alpha x$ is a solution of Equation (2.2.1), where α is any complex number. Here f is linear, hence continuous. The following generalizes this.

Lemma 2.2.2. *Let V be a vector space over $\mathbb{R}$, and let $f : V \to \mathbb{C}$ be a solution of Equation (2.2.1).*
(i) $f(qx) = qf(x)$ for all $q \in \mathbb{Q}$ and $x \in V$, i.e., f is $\mathbb{Q}$-linear.
(ii) If V is a topological vector space and f is continuous at a point then f is continuous.
(iii) If V is a topological vector space and f is continuous, then f is linear.

Proof. (i) First we show $f(nx) = nf(x)$, for all $n \in \mathbb{N}, x \in V$. The statement is obvious when $n = 1$. Assume $f(kx) = kf(x)$ when $n = k, x \in V$. Let $n = k + 1$. Then since f is a solution of Equation (2.2.1), we have

$$f((k + 1)x) = f(kx + x) = f(kx) + f(x) = kf(x) + f(x) = (k + 1)f(x).$$

By induction we get $f(nx) = nf(x)$, for $n \in \mathbb{Z}^+, x \in V$. Take $x = 0$ in Equation (2.2.1), we have $f(0) = 0$, then $f(nx) = nf(x)$ for $n = 0$. Let $y = -x$ in Equation (2.2.1), we have $f(-x) = -f(x)$. Similarly, we have $f(-nx) = -nf(x)$, for $n \in$

$\mathbb{Z}^+, x \in V$. Hence we get $f(nx) = nf(x)$, for all $n \in \mathbb{Z}, x \in V$. Let $q \in \mathbb{Q}$, then $q = \frac{b}{a}$ for $a \in \mathbb{N}$ and $b \in \mathbb{Z}$. We have

$$af\left(\frac{b}{a}x\right) = f\left(a\frac{b}{a}x\right) = f(bx) = bf(x),$$

which means $f(\frac{b}{a}x) = \frac{b}{a}f(x)$. Hence $f(qx) = qf(x)$ for $q \in \mathbb{Q}$ and $x \in V$.

(ii) Assume f is continuous at $x_0 \in V$. This means if $x \to x_0$, then $f(x) \to f(x_0)$. For an arbitrary $x' \in V$, if $x \to x'$, then we have $x - x' + x_0 \to x_0$. Since f is continuous at x_0 and f is a solution of Equation (2.2.1)

$$\begin{aligned} & f(x - x' + x_0) \to f(x_0) \Rightarrow f(x - x') + f(x_0) \to f(x_0) \\ \Rightarrow & f(x) - f(x') + f(x_0) \to f(x_0) \Rightarrow f(x) - f(x') \to 0 \\ \Rightarrow & f(x) \to f(x'). \end{aligned}$$

Hence f is continuous on V since x' is arbitrary.

(iii) Since the rational numbers are dense in $\mathbb{R}$, for aribrary $r \in \mathbb{R}$, there exists a sequence $\{q_n\} \in \mathbb{Q}$ such that $\lim_{n\to\infty} q_n = r$. Thus by (i),

$$f(rx) = \lim_{n\to\infty} f(q_n x) = \lim_{n\to\infty} q_n f(x) = rf(x).$$

Thus f is linear. □

Corollary 2.2.3. *Any continuous solution of Equation (2.2.1) from $\mathbb{R}^n$ to $\mathbb{C}$ is a linear function. Thus each such one consists in taking inner product with a vector $\alpha \in \mathbb{C}^n$: $f(x) = \langle \alpha, x \rangle$ for all $x \in \mathbb{R}^n$.*

However, not all additive functions on $\mathbb{R}$ are continuous. Before we talk about this, we shall introduce the definition of *Hamel basis*. As well as an important proposition of it.

Definition 2.2.4. A *Hamel basis* for a vector space V over a field F is a linearly independent subset $\{v_i | i \in I\}$ of V that spans V. So any $v \in V$ can in exactly one way be written as $v = \sum_{i\in I} q_i(v)v_i$, where only finitely many of the coefficients $q_i(v)$

are different from zero. The functions $q_i : V \to F$, $i \in I$, are called the *coefficient functions*. Note that $q_i(v_i) = 1$ and that q_i is additive for all $i \in I$.

PROPOSITION 2.2.5. *Every nonzero vector space has a Hamel basis.*

PROOF. Refer [2, Proposition 3.13]. □

PROPOSITION 2.2.6. *There exists a discontinuous additive function $f : \mathbb{R} \to \mathbb{R}$.*

PROOF. Select a Hamel basis $\{v_i | i \in I\}$ for $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$. Consider a coefficent function q_{i_0}:$\mathbb{R} \to \mathbb{Q} \subseteq \mathbb{R}$. Assume q_{i_0} is continuous. As the image of connected set, $q_{i_0}(\mathbb{R})$ will be a interval. But no intervals would contain just rational numbers since rational numbers are dense in $\mathbb{R}$, except it is degentrate just one-point interval. Then q_{i_0} will be constant. For an element v_{i_0} in the Hamel basis, $q_{i_0}(0) = q_{i_0}(v_{i_0})$. But in fact, $q_{i_0}(0) = 0$ while $q_{i_0}(v_{i_0}) = 1$, a contradiction. Take $f = q_{i_0}$, we finish this proof. □

EXAMPLE 2.2.7. Let H_3 be the Heisenberg group and $f : H_3 \to \mathbb{C}$ be a continuous additive function. Let $x = y = e$ in Equation (2.2.1) we get that $f(e) = 0$. For any commutator $[g, h] = ghg^{-1}h^{-1}$ in an arbitrary group G, we have

$$\begin{aligned} f([g,h]) &= f(ghg^{-1}h^{-1}) \\ &= f(gh) + f(g^{-1}h^{-1}) \\ &= f(g) + f(h) + f(g^{-1}) + f(h^{-1}) \\ &= f(g) + f(g^{-1}) + f(h) + f(h^{-1}) \\ &= f(gg^{-1}) + f(hh^{-1}) \\ &= f(e) + f(e) \\ &= 0. \end{aligned}$$

We find that additive functions vanish on commutators and so on the commutator subgroup. Thus by Example 2.1.6(v), we get

$$[H_3, H_3] = \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So

$$\begin{aligned} f\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} &= f\left(\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= f\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + f\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Here the maps $x \mapsto f\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $y \mapsto f\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ are continuous additive functions from $\mathbb{R}$ to $\mathbb{C}$.

By Corollary 2.2.3, we have the form

$$f\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \alpha x + \beta y \qquad \text{for all } x, y \in \mathbb{R},$$

with

$$\alpha = f\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{C}, \qquad \beta = f\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{C}.$$

Conversely, any functions on H_3 of this form is a continuous additive function from H_3 to $\mathbb{C}$.

2.3. The Multiplicative Cauchy Equation

Definition 2.3.1. Let G be group. A *group character* on G is a homomorphism of G into $\mathbb{C}^\star$. A character χ is said to be *unity*, if $|\chi(x)| = 1$ for all $x \in G$.

Definition 2.3.2. Let G be a group. The *multiplicative Cauchy equation* is the functional equation

$$\chi(xy) = \chi(x)\chi(y) \qquad \text{for all } x, y \in G, \tag{2.3.1}$$

where $\chi : G \to \mathbb{C}$ is the unknown.

Lemma 2.3.3. *Let G be a group,and let $\chi : G \to \mathbb{C}$ be non-zero multiplicative function.*
(i) χ is a character on G, and $\chi(e) = 1$.
(ii) χ is identically 1 on the commutator subgroup $[G, G]$.

Proof. (i) Fix an arbitrary $x_0 \in G$ such that $\chi(x_0) \neq 0$. Put $x = x_0$ and $y = e$ in Equation (2.3.1), we have $\chi(e) = 1$. Then put $y = x^{-1}$ in Equation (2.3.1), $\chi(e) = \chi(xx^{-1}) = \chi(x)\chi(x^{-1}) = 1$. We have $\chi(x) \neq 0$ for any $x \in G$, so $\chi(x) \in \mathbb{C}^\star$. Thus χ is a character on G.
(ii) For an element $xyx^{-1}y^{-1}$ in the commutator subgroup $[G, G]$, $x \in G$ and $y \in G$, we have

$$\begin{aligned}\chi(xyx^{-1}y^{-1}) &= \chi(xy)\chi(x^{-1}y^{-1}) = \chi(x)\chi(y)\chi(x^{-1})\chi(y^{-1})\\ &= \chi(x)\chi(x^{-1})\chi(y)\chi(y^{-1}) = \chi(xx^{-1})\chi(yy^{-1})\\ &= \chi(e)\chi(e) = 1.\end{aligned}$$

Since any element g in $[G, G]$ has the form of $xyx^{-1}y^{-1}$, $x \in G$ and $y \in G$, so we have $\chi(g) = 1$. □

Example 2.3.4. It is easy to see that

$$\chi_\lambda(x) := e^{\lambda x}, \qquad x \in \mathbb{R},$$

where λ is any complex number, is a continuous character on the group $(\mathbb{R}, +)$.

Conversely, for any continuous character χ on $(\mathbb{R}, +)$, there exists exactly one $\lambda \in \mathbb{C}$ such that $\chi = \chi_\lambda$.

Proof. It is obvious that χ_λ is a continuous character on $\mathbb{R}$. To show character χ on $(\mathbb{R}, +)$ is in the form of χ_λ for some λ. First we would show χ is differentiable. Since $\chi(0) = 1$, we have an $\varepsilon > 0$ such that $\int_0^\varepsilon \chi(y)dy \neq 0$. Integrating Equation (2.3.1) over $[0, \varepsilon]$ with respect to y and change variables:

$$\int_x^{x+\varepsilon} \chi(u)du = \chi(x) \int_0^\varepsilon \chi(y)dy.$$

Since χ is continuous, the left side is differentiable by the first fundamental theorem of calculus. Thus χ is differentiable since $\int_0^\varepsilon \chi(y)dy \neq 0$.
Differential Equation (2.3.1) with respect to y and then let $y = 0$, we have

$$\chi'(x) = \chi'(0)\chi(x) \qquad \text{for } x \in \mathbb{R}.$$

Solve this differential equation to get

$$\chi(x) = Ae^{\chi'(0)x} \qquad \text{for } x \in \mathbb{R}, \text{ where A is a constant complex number.}$$

Let $x = 0$ to give $A = 1$. □

Example 2.3.5. We extend above to the group $(\mathbb{R}^n, +)$, and obtain that on $(\mathbb{R}^n, +)$ the continuous characters are in the form of

$$\chi_\lambda(x) := e^{\langle \lambda, x \rangle}, \qquad x \in \mathbb{R}^n,$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{C}^n$, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $\langle \lambda, x \rangle := \lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n$.

Proof. Similar to Example 2.3.4, we just need to prove χ is in the form of $e^{\langle \lambda, x \rangle}$ for $x \in \mathbb{R}^n$. Notice that

$$\chi(x) = \chi(x_1, x_2, ..., x_n)$$

$$= \chi((x_1, 0, ..., 0) + (0, x_2, ..., 0) + ... + (0, 0, ..., x_n))$$

$$= \chi(x_1, 0, ..., 0)\chi(0, x_2, ..., 0)...\chi(0, 0, ..., x_n).$$

Consider $x_1 \mapsto \chi(x_1, 0, ..., 0)$. It is a continuous character. Then by (i), there is a $\lambda_1 \in \mathbb{C}$ such that $\chi(x_1, 0, ..., 0) = e^{\lambda_1 x_1}$ for $x_1 \in \mathbb{R}$. Similarly for the other factors. Then we can get $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{C}$ such that

$$\begin{aligned}\chi(x) =& \chi(x_1, x_2, ..., x_n)\\ &= e^{\lambda_1 x_1} e^{\lambda_2 x_2} ... e^{\lambda_n x_n}\\ &= e^{\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n}\\ &= e^{\langle \lambda, x \rangle}.\end{aligned}$$

□

EXAMPLE 2.3.6. Let us consider the continuous characters on Heisenberg group H_3. For $x, y, z \in \mathbb{R}$

$$\chi\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \chi\left(\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$= \chi\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\chi\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\chi\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Lemma 2.3.3(ii) we know that

$$\chi\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 \qquad \text{for all } z \in \mathbb{R}.$$

Thus

$$\chi\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} = \chi\begin{pmatrix}1 & x & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\chi\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}.$$

Here the maps $x \mapsto \chi\begin{pmatrix}1 & x & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ and $y \mapsto \chi\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}$ are continuous characters on $(\mathbb{R}, +)$. So according to Example 2.3.4, there exist unique $\lambda_1, \lambda_2 \in \mathbb{C}$ such that

$$\chi\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} = e^{\lambda_1 x}e^{\lambda_2 y} = e^{\lambda_1 x+\lambda_2 y} \qquad \text{for all} \begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} \in H_3.$$

THEOREM 2.3.7. *Let S be a semigroup and $n \in \mathbb{N}$. Let $\chi_1, \chi_2, ..., \chi_n \colon S \to \mathbb{C}$ be n different multiplicative functions, and let $a_1, a_2, ..., a_n \in \mathbb{C}$. Let $f := a_1\chi_1 + a_2\chi_2 + ... + a_n\chi_n$.*
(i) If $f = 0$, then $a_1\chi_1 = a_2\chi_2 = ... = a_n\chi_n = 0$, $n \in \mathbb{N}$.
(ii) The set of non-zero distinct multiplicative functions is a linearly independent subset of the complex-valued funtions on S.

PROOF. (i) It is trivially true when $n = 1$. Assume it is true for $n \geq 1$. Then for $n + 1$,

$$0 = a_1\chi_1(x) + a_2\chi_2(x) + ... + a_n\chi_n(x) + a_{n+1}\chi_{n+1}(x). \tag{2.3.2}$$

Then replace x by xy in Equation (2.3.2) to get

$$0 = a_1\chi_1(xy) + a_2\chi_2(xy) + ... + a_n\chi_n(xy) + a_{n+1}\chi_{n+1}(xy). \tag{2.3.3}$$

Since $\chi_1, \chi_2, ..., \chi_n, \chi_{n+1}$ are multiplicative, Equation (2.3.3) can be written as

(2.3.4)

$$0 = a_1\chi_1(x)\chi_1(y) + a_2\chi_2(x)\chi_2(y) + ... + a_n\chi_n(x)\chi_n(y) + a_{n+1}\chi_{n+1}(x)\chi_{n+1}(y).$$

Multiply Equation (2.3.2) by $\chi_{n+1}(y)$,

(2.3.5)

$$0 = a_1\chi_1(x)\chi_{n+1}(y) + a_2\chi_2(x)\chi_{n+1}(y) + ... + a_n\chi_n(x)\chi_{n+1}(y) + a_{n+1}\chi_{n+1}(x)\chi_{n+1}(y).$$

Subtract Equation (2.3.5) from Equation (2.3.4) to give

(2.3.6)

$$0 = a_1\chi_1(x)[\chi_1(y) - \chi_{n+1}(y)] + a_2\chi_2(x)[\chi_2(y) - \chi_{n+1}(y)] + ... + a_n\chi_n(x)[\chi_n(y) - \chi_{n+1}(y)].$$

By the assumption, every term is 0 in Equation (2.3.6). Consider the last term $a_n\chi_n(x)[\chi_n(y) - \chi_{n+1}(y)]$. There must be a $y_0 \in S$ such that $\chi_n(y_0) - \chi_{n+1}(y_0) \neq 0$ since χ_n and χ_{n+1} are different. Thus $a_n\chi_n = 0$. Similarly $a_i x_i = 0$ for all $1 \leq n \leq n-1$, thus $a_{n+1}\chi_{n+1} = 0$.
(ii) is an immediate consequence of (i). ☐

Corollary 2.3.8. *Let S be a semigroup. Let $\chi_1, \chi_2, \chi_3, \chi_4 : S \to \mathbb{C}$ be multiplicative functions. If $\chi_1 + \chi_2 = \chi_3 + \chi_4$, then there are only two possibilities: $\chi_1 = \chi_3$ and $\chi_2 = \chi_4$, or $\chi_1 = \chi_4$ and $\chi_2 = \chi_3$.*

Proof. $\chi_1 + \chi_2 = \chi_3 + \chi_4$ means $\chi_1 + \chi_2 + (-1)\chi_3 + (-1)\chi_4 = 0$. Then by Theorem 2.3.7, if $\chi_1, \chi_2, \chi_3, \chi_4$ are different, $\chi_1 = \chi_2 = (-1)\chi_3 = (-1)\chi_4 = 0$, a contradiction. Thus at least two of them are the same. It is sufficient to consider the three cases below:
case (i): $\chi_1 = \chi_2$;
case (ii): $\chi_1 = \chi_3$;
case (iii): $\chi_1 = \chi_4$.
For case(i), it turns to $2\chi_1 + (-1)\chi_3 + (-1)\chi_4 = 0$. Then again by Theorem 2.3.7, if χ_1, χ_3, χ_4 are different, $2\chi_1 = (-1)\chi_3 = (-1)\chi_4 = 0$, a contradiction. Thus at least two of χ_1, χ_3, χ_4 are the same. By easy computations it turns out $\chi_1 = \chi_3 = \chi_4$ hence $\chi_1 = \chi_2 = \chi_3 = \chi_4$, a contradiction. For case (ii), it means $\chi_2 = \chi_4$. And for case (iii), it means $\chi_2 = \chi_3$. ☐

Corollary 2.3.8 shows a decomposition $\chi_1 + \chi_2$ with just two terms is unique up to interchange of χ_1 and χ_2.

Corollary 2.3.9. *(Artin) The set of all distinct characters on a group G is a linearly independent subset of the vector space of all complex-valued functions on G.*

Proof. This is an immediate consequence of Theorem 2.3.7(ii). $\square$

2.4. The Sine Addition Formula

The sine addition formula will play an important role in the later studies of the pre-d'Alembert functional equation and the d'Alembert functional equation.

Definition 2.4.1. The *sine addition formula* on a semigroup S is the functional equation

$$f(xy) = f(x)g(y) + f(y)g(x) \qquad \text{for all } x, y \in S,$$

where $f, g : S \to \mathbb{C}$ are unknowns.

Definition 2.4.2. Let S be a semigroup, X a set and $f : S \to X$ a function. f is said to be *abelian* if and only if $f(x_1x_2...x_n) = f(x_{\pi(1)}x_{\pi(2)}...x_{\pi(n)})$ for all $x_1, x_2, ..., x_n \in S$, all permutations π of n elements and all $n = 1, 2, 3, ...$

f is *non-abelian* if it is not abelian.

Lemma 2.4.3. *Let M be a monoid. Then a function $f : M \to X$ is abelian if and only if f satisfies Kannappan's condition, i.e. $f(xyz) = f(xzy)$ for all $x, y, z \in S$.*

Proof. If f is abelian, it is obvious that it satisfies Kannappan's condition. Conversely, if f satisfies Kannappan's condition, let $x = e$, then we have $f(yz) = f(zy)$. Thus f is central. Then for all $x, y, z \in M$, we have

$$\begin{aligned} f(xyz) = f(yzx) = f(zxy) & \qquad \text{(by } f \text{ is central)} \\ = f(xzy) = f(yxz) = f(zyx) & \qquad \text{(by Kannappan's condition).} \end{aligned}$$

Thus by above identities, $f(x_1x_2x_3) = f(x_{\pi(1)}x_{\pi(2)}x_{\pi(3)})$ for every permutation π on $\{1,2,3\}$. This general case can be proved by induction. Hence f is abelian. $\square$

THEOREM 2.4.4. *Let S be a topological group. Let $f, g \in C(S)$ satisfy the sine addition formula*

$$f(xy) = f(x)g(y) + f(y)g(x) \qquad \text{for all } x, y \in S, \tag{2.4.1}$$

where $f \neq 0$. Then we have the following:
(i) There exists a constant $\alpha \in \mathbb{C}$ such that

$$g(xy) = g(x)g(y) + \alpha^2 f(x)f(y) \qquad \text{for all } x, y \in S. \tag{2.4.2}$$

(ii) $\chi_1 := g + \alpha f$ and $\chi_2 := g - \alpha f$ are the only continuous muliplicative functions such that

$$g = \frac{\chi_1 + \chi_2}{2}.$$

(iii) Both f and g are abelian functions.

PROOF. (i) Replace y with yz in Equation (2.4.1) to have

$$f(xyz) = f(x)g(yz) + f(yz)g(x) = f(x)g(yz) + f(y)g(z)g(x) + f(z)g(x)g(y). \tag{2.4.3}$$

Replace x with xy and y with z, respectively in Equation (2.4.1) to have

$$f(xyz) = f(xy)g(z) + f(z)g(xy) = f(x)g(y)g(z) + f(y)g(x)g(z) + f(z)g(xy). \tag{2.4.4}$$

Let Equation (2.4.3) subtract Equation (2.4.4) and simplify it to get

$$f(x)[g(yz) - g(y)g(z)] = f(z)[g(xy) - g(x)g(y)]. \tag{2.4.5}$$

Replace x with y and y with z, respectively in Equation (2.4.5) to get

$$f(x)[g(yz) - g(y)g(z)] = f(z)[g(yz) - g(yx)g(z)]. \tag{2.4.6}$$

Multiply both sides of Equation (2.4.5) by $f(z)$

$$f(x)f(z)[g(yz) - g(y)g(z)] = f(z)^2[g(xy) - g(x)g(y)]. \tag{2.4.7}$$

Substitude $f(z)[g(yz) - g(y)g(z)]$ in Equation (2.4.7) by the left side of Equation (2.4.6). We have

$$f(x)f(y)[g(zz) - g(z)g(z)] = f(z)^2[g(xy) - g(x)g(y)]. \tag{2.4.8}$$

Since $f \neq 0$, then we have $z_0 \in S$ such that $f(z_0) \neq 0$. Let $z = z_0$ in Equation (2.4.8) and simlify it to give

$$g(xy) = g(x)g(y) + \frac{g(z_0z_0) - g(z_0)g(z_0)}{f(z_0)^2}f(x)f(y).$$

Let $\alpha^2 = \frac{g(z_0z_0)-g(z_0)g(z_0)}{f(z_0)^2}$ thus we get the conclusion.
(ii) Multiply α to Equation (2.4.1) and add the result to Equation (2.4.2)

$$(g + \alpha f)(xy) = (g + \alpha f)(x)(g + \alpha f)(y).$$

Similarly, multiply $-\alpha$ to Equation (2.4.1) and add the result to Equation (2.4.2)

$$(g - \alpha f)(xy) = (g - \alpha f)(x)(g - \alpha f)(y).$$

Let $\chi_1 := g + \alpha f$ and $\chi_2 := g - \alpha f$, it is obvious they are multiplicative from above. And $g = \frac{\chi_1+\chi_2}{2}$. The uniqueness has been proved in Corollary 2.3.8.
(iii) By Equation (2.4.1) and Equation (2.4.2), it is clear that $f(xy) = f(yx)$ and $g(xy) = g(yx)$ for all $x, y \in S$. *i.e.* f and g are central.
Replace y by yz in Equation (2.4.2), we have

$$\begin{aligned} g(xyz) &= g(x)g(yz) + \alpha^2 f(x)f(yz) = g(x)g(zy) + \alpha^2 f(x)f(zy) \\ &= g(xzy). \end{aligned}$$

Thus g satisfies Kannappan's condition. By Lemma 2.4.3, g is abelian.
For f, if $\chi_1 \neq \chi_2$, then $f = \frac{1}{2\alpha}(\chi_1 - \chi_2)$, satisfies Kannappan's condition. By Lemma 2.4.3, f is abelian.

If $\chi_1 = \chi_2 = g$, let $\chi = g$. Then Equation (2.4.1) becomes

$$f(xy) = f(x)\chi(y) + f(y)\chi(x). \tag{2.4.9}$$

Since f is central, apply Equation (2.4.9) to have

$$\begin{aligned} f(xyz) &= f(yzx) \\ &= f(yz)\chi(x) + f(x)\chi(yz) \\ &= f(zy)\chi(x) + f(x)\chi(zy) \\ &= f(xzy). \end{aligned}$$

Thus f is also abelian. □

2.5. Group Representations

Definition 2.5.1. Let S be a semigroup and V a vector space. A *representation* of S on V is a map $\pi : S \to \mathcal{L}(V)$ such that

$$\pi(xy) = \pi(x)\pi(y) \qquad \text{for all } x, y \in S.$$

V is called the *representation space* of π. The dimension of the representation π is $\dim \pi{:=}\dim V$.

Definition 2.5.2. Let π be a representation of S on V. A subspace W of V is said to be *invariant* under π, if $\pi(x)W \subseteq W$ for all $x \in S$. In this case, let $\pi|_W$ denote the representation $x \mapsto \pi(x) : W \to W$ of S on W. $\pi|_W$ is said to be a *subrepresentation* of π.

Definition 2.5.3. Let S be a semigroup, π and π' be two representations of G. Then π and π' are said to be *equivalent* if there exists an invertible matrix X such that $X\pi(g)X^{-1} = \pi'(g)$ for all $g \in S$.

Definition 2.5.4. A representation π of S on V is *algebraically irreducible*, if 0 and V are the only invariant subspaces of V under π.

Theorem 2.5.5. *(Maschke). Let (π, V) be a representation of finite group G. Then V is a direct sum of irreducible representations.*

Proof. Refer to [10, Corollary 1.6] or [17, Theorem 2]. $\square$

Definition 2.5.6. Let π be a representation of S on a finite-dimensioal vector space. Its *character* is the function $\chi_\pi : S \to \mathbb{C}$ defined by $\chi_\pi(x) := tr(\pi(x)), x \in S$.

Lemma 2.5.7. *Let π be a finite-dimensional representation of a semigroup S. Then χ_π is a finite sum of characters of irreducible representations. The sum of the dimensions of these irreducible representations is dim π.*

Proof. Refer to [18, Lemma E.9]. $\square$

Definition 2.5.8. Let S be a semigroup. The *right regular representation* R of S on $\mathbb{C}^S$ is defined by $R(x) : \mathbb{C}^S \to \mathbb{C}^S$ by $[R(x)f](y) := f(yx)$ for any $x, y \in S$. Similarly, the *left regular antirepresentation* L' of S on $\mathbb{C}$ is denfined by $L' : \mathbb{C}^S \to \mathbb{C}^S$ by $[L'(x)f](y) := f(xy)$ for any $x, y \in S$.

It is obvious that $L'(xy) = L'(y)L'(x)$ for all $x, y \in S$. That means L' is not a representation. If S is a group then $L(x) := L'(x^{-1})$ is a representation of S on $\mathbb{C}^S$. A map of the form $R(x)$ resp. $L'(x)$, is called a *right translation*, resp. *left translaiton.*

Theorem 2.5.9. *Let $\pi_1, \pi_2, ..., \pi_n$ be algebraically irreducible representations of a semigroup S such that no two of them are algebraically equivalent. Then we have the following.*
(i) $C(\pi_1) + C(\pi_2) + ... + C(\pi_n)$ form a direct sum in $\mathbb{C}^S$.
(ii) In particular, if $\pi_1, \pi_2, ..., \pi_n$ are finite-dimensional, then their characters $\chi_{\pi_1}, \chi_{\pi_2}, ..., \chi_{\pi_n}$ span a direct sum:

$$\mathbb{C}_{\chi_{\pi_1}} \oplus \mathbb{C}_{\chi_{\pi_2}} \oplus ... \oplus \mathbb{C}_{\chi_{\pi_n}} \subseteq C(\pi_1) \oplus C(\pi_2) \oplus ... \oplus C(\pi_n)$$

Proof. Refer to [14, Proposition 2 of Chapter 8]. $\square$

CHAPTER 3

The Pre-d'Alembert Functional Equation on Groups

In this chapter, we study some properties of the solutions to the pre-d'Alembert functional equation on groups. We characterize when a solution is abelian using several derived functions, and get the general form of an abelian solution via symmetrized sine addition formula.

3.1. Definitions and an Example

Definition 3.1.1. Let S be a semigroup. The *pre-d'Alembert functional equation* is in the form

$$g(xyz) + g(xzy) = 2g(x)g(yz) + 2g(y)g(xz) + 2g(z)g(xy) - 4g(x)g(y)g(z) \tag{3.1.1}$$

for all $x, y, z \in S$, where $g : S \to \mathbb{C}$ is the unknown.

Definition 3.1.2. Let M be a monoid. A solution of the pre-d'Alembert functional equation is called a *pre-d'Alembert function* if $g(e) = 1$, where e is the identity element of M.

Example 3.1.3. The function $g(x) := \frac{1}{2}\operatorname{tr} x$, $x \in M(2\times 2, \mathbb{C})$, is a pre-d'Alembert function.

Proof. It is clear that

$$g(I) = g\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\operatorname{tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

If $g = \frac{1}{2}\operatorname{tr}$ satisfis Equation (3.1.1) for any $x, y, z \in M(2 \times 2, \mathbb{C})$, then so does for $x, y, z + \alpha I$, with α is any complex number. There must exist a unique α', such that $\operatorname{tr}(z + \alpha' I) = 0$. So we can substitute z with $z + \alpha' I$ and so $\operatorname{tr} z = 0$. Similarly, we can also assume $\operatorname{tr} x = 0$ and $\operatorname{tr} y = 0$. Put them in Equation (3.1.1). Therefore it suffices to show $\operatorname{tr}[x(yz + zy)] = 0$ with $\operatorname{tr} x = \operatorname{tr} y = \operatorname{tr} z = 0$. After some easy computations, we see that $yz + zy$ is in the form of βI, for some complex number β, since y and z are 2×2 matrices of trace 0. Thus $\operatorname{tr}[x(yz+zy)] = \operatorname{tr}(x(\beta I)) = \beta \operatorname{tr}(xI) = \beta \operatorname{tr} x = 0$, as $\operatorname{tr} x = 0$. □

This example will play an important role when we find all pre-d'Alembert functions.

LEMMA 3.1.4. *Let $g : M \to \mathbb{C}$ be a solution of the pre-d'Alembert functional equaiton on a monoid M with identity element e. There are only three possibilities for $g(e)$: $g(e) = 0$, $g(e) = \frac{1}{2}$, $g(e) = 1$. Furthermore, if $g(e) = \frac{1}{2}$ then $2g : M \to \mathbb{C}$ is multiplicative.*

PROOF. Let $x = y = z = e$ in Equation (3.1.1), we have

$$2g(e) = 6g^2(e) - 4g^3(e).$$

Solve this equation we get $g(e) = 0$, $g(e) = \frac{1}{2}$ or $g(e) = 1$.
If $g(e) = \frac{1}{2}$, let $z = e$ in Equation (2.1.1). We have $g(xy) = 2g(y)g(x)$, thus $2g(xy) = 2g(x)2g(y)$. Thus $2g$ is multiplicative.

□

3.2. Properties of Pre-d'Alembert Functions

DEFINITION 3.2.1. Let S be a semigroup. The *symmetrized sine addition formula* is in the form

$$w(xy) + w(yx) = 2w(x)g(y) + 2w(y)g(x) \qquad \text{for all } x, y \in S, \tag{3.2.1}$$

where $w, g : S \to \mathbb{C}$ are unknowns.

DEFINITION 3.2.2. Let $g : S \to \mathbb{C}$ be a given function. In what follows, we associate an arbitrary function $g : S \to \mathbb{C}$ four functions.
(i) For $x \in S$, $g_x : S \to \mathbb{C}$ is defined by $g_x(y) := g(xy) - g(x)g(y)$ for all $y \in S$.

(ii) $d = d_g : S \to \mathbb{C}$ is defined by $d(x) := 2g(x)^2 - g(x^2)$ for all $x \in S$.

(iii) $\widetilde{\triangle} : S \times S :\to \mathbb{C}$ is defined by

$$\widetilde{\triangle}(x,y) := 2d(x)d(y) + g(xy^2x) + g(yx^2y) - 4g(xy)g(yx) \qquad \text{for all } x, y \in S.$$

(iv) $\triangle : S \times S \to \mathbb{C}$ is defined by

$$\triangle(x,y) := \frac{1}{2}[g(x^2y^2) - g((xy)^2)] \qquad \text{for all } x, y \in S.$$

For (i), we find that g is multiplicative if and only if $g_x = 0$ for all $x \in S$. For (ii), $d(x)$ is important when we investigate the pre-d'Alembert function and the symmetrized sine addition formula in Chapter 5 later. For (iii) and (iv), $\widetilde{\triangle} = 4\triangle$ if g is a pre-d'Alembert function (Lemma 3.2.3). Moreover, g is abelian if and only if $\triangle = 0$ (Lemma 3.3.2).

Lemma 3.2.3. *Let $g : S \to \mathbb{C}$ be a solution of the pre-d'Alembert functional equation. Then we have the following properties,*

(i) g is central, i.e. $g(xy) = g(yx)$ for all $x, y \in S$.

(ii) the pair (g_x, g) satisfies the symmetrized sine addition formula for each $x \in S$.

(iii) $g_x(y^2) = 2g_x(y)g(y)$ for all $x, y \in S$.

(iv) $d : S \to \mathbb{C}$ is multiplicative.

(v) $\frac{\widetilde{\triangle}}{4} = \triangle(x,y) = g_x(x)g_y(y) - g_x(y)^2$ for all $x, y \in S$.

Proof. (i) Switch y and z in Equation (3.1.1) and subtract Equation (3.1.1) from the resulted equation. Then simplify it to get

$$g(x)[g(yz) - g(zy)] = 0.$$

If there is $x \in S$ such that $g(x) \neq 0$, then $g(yz) - g(zy) = 0$ for all $y, z \in S$. Then g is central. If $g(x) = 0$ for all $x \in G$, then $g = 0$ is also central.

(ii) Since $g(yz) = g(zy)$, we can treat Equation (3.1.1) as follows:

$$g(xyz) + g(xzy) = 2g(x)g(yz) + 2g(y)g(xz) + 2g(z)g(xy) - 4g(x)g(y)g(z)$$

$$g(xyz) + g(xzy) - 2g(x)g(yz) = 2g(y)g(xz) + 2g(z)g(xy) - 4g(x)g(y)g(z)$$

$$\Rightarrow \quad g(xyz) - g(x)g(yz) + g(xzy) - g(x)g(zy)$$
$$=2g(y)g(xz) - 2g(x)g(y)g(z) + 2g(z)g(xy) - 2g(x)g(y)g(z)$$

$$\Rightarrow \quad [g(xyz) - g(x)g(yz)] + [g(xzy) - g(x)g(zy)]$$
$$=[2g(xy)g(z) - 2g(x)g(y)g(z) + [2g(xz)g(y) - 2g(x)g(y)g(z)].$$

That is

(3.2.2) $$g_x(yz) + g_x(zy) = 2g_x(y)g(z) + 2g_x(z)g(y),$$

which is the form of the symmetrized sine addition formula with $w = g_x$.

(iii) Let $z = y$ in Equation (3.2.2).

(iv) Let$z = xy$ in Equation (3.1.1) and we get

(3.2.3) $$g((xy)^2) + g(x^2y^2) = 2g(x)g(yxy) + 2g(y)g(xyx) + 2g(xy)^2 - 4g(x)g(y)g(xy).$$

By the definition of function d, one has

$$d(xy) = 2g(xy)^2 - g((xy)^2).$$

From Equation (3.2.3) and (i), we obtain

$$\begin{aligned} d(xy) &= 2g(xy)^2 - g((xy)^2) \\ &= g(x^2y^2) - 2g(x)g(yxy) - 2g(y)g(xyx) + 4g(x)g(y)g(xy) \\ &= g_{x^2}(y^2) + g(x^2)g(y^2) - 2g(x)g(xy^2) - 2g(y)g(yx^2) + 4g(x)g(y)g(xy) \\ &= g_{x^2}(y^2) + g(x^2)g(y^2) - 2g(x)(g_x(y^2) + g(x)g(y^2)) \\ &- 2g(y)(g_y(x^2) + g(y)g(x^2)) + 4g(x)g(y)g(xy). \end{aligned}$$

With (iii) and the fact that $g_x(y) = g_y(x)$, we have

$$\begin{aligned} d(xy) =& 2g_{x^2}(y)g(y) + g(x^2)g(y^2) - 4g(x)g_x(y)g(y) - 2g(x)^2 g(y^2) - 4g(y)g_x(y)g(x) \\ & - 2g(y^2)g(x^2) + 4g(x)g(y)g(xy) \\ & = g(x^2)g(y^2) - 2g(x)^2 g(y^2) - 2g((y)^2)g(x^2)) + 4g(x)^2 g(y)^2 \\ & = [2g(x)^2 - g(x^2)][2g(y)^2 - g(y^2)] \\ & = d(x)g(y). \end{aligned}$$

(v) Since d is multiplicative by (iii) and g is central, we have

$$\begin{aligned} \widetilde{\triangle}/4 &= d(x)d(y)/2 + [g(xy^2x) + g(yx^2y)]/4 - g(xy)g(yx) \\ &= d(xy)/2 + g(xy^2x)/2 - g(xy)^2 \\ &= [2g(xy)^2 - g((xy)^2)]/2 + +g(xy^2x)/2 - g(xy)^2 \\ &= g(xy)^2 - g((xy)^2)/2 + +g(xy^2x)/2 - g(xy)^2 \\ &= g(x^2y^2)/2 - g(xy)^2/2 \\ &= \triangle(x, y). \end{aligned}$$

Hence we finish the proof of the first identity in (v). To prove the second identity in (v), replace y with yz, and x with xy respectively in Equation (3.1.1) to get

$$w(xyz) + w(yzx) = 2w(x)g(yz) + 2w(yz)g(x), \tag{3.2.4}$$

and

$$w(xyz) + w(zxy) = 2w(xy)g(z) + 2w(z)g(xy). \tag{3.2.5}$$

Replace x with y, y with z and z with x, respectively, in Equation (3.2.4) to get

$$w(yzx) + w(zxy) = 2w(y)g(zx) + 2w(zx)g(y). \tag{3.2.6}$$

Add Equation (3.2.4) with Equation (3.2.6), then use Equation (3.2.5). We obtain

$$\begin{aligned} & w(yzx)+w(xy)g(z)+w(z)g(xy) \\ =& w(x)g(yz)+w(yz)g(x)+w(y)g(zx)+w(zx)g(y). \end{aligned} \tag{3.2.7}$$

Interchange x, y in Equation (3.2.7) and add it to Equation (3.2.7). We have

$$\begin{aligned} & w(yzx)+w(xzy)+[w(xy)+w(yx)]g(z)+w(x)[g(xy)+g(yx)] \\ =& w(x)g(yz)+w(y)g(zx)+g(x)[w(zy)+w(yz)]+w(y)g(zx) \\ &+w(x)g(yz)+g(y)[w(xz)+w(zx)]. \end{aligned} \tag{3.2.8}$$

Replace the terms in square brackets in Equation (3.2.8) using Equation (3.1.1). We have

$$\begin{aligned} & w(yzx)+w(xzy) \\ =& w(x)[g(xy)+g(yx)]+w(y)[g(xz)+g(zx)] \\ &-w(z)[g(xy)+g(yx)-4g(x)g(y)]. \end{aligned} \tag{3.2.9}$$

Let $z = x$ in Equation (3.2.9), subtract two times with $z = y$ in Equation (3.2.9) and use the definition of g_x. Then we obtain

$$\begin{aligned} & w(x^2y)+w(yx^2)-2w(xyx) \\ =& 4w(x)g(x)g(y)+2w(y)g(x^2)-2d(x)w(y)-2w(x)[g(xy)+g(yx)] \\ =& -2w(x)[g(xy)+g(yx)-2g(x)g(y)]+2w(y)[(g(x^2)-d(x))] \\ =& -2w(x)[g_x(y)+g_y(x)]+2w(y)[g(x^2)-2g(x)^2+g(x^2)] \\ =& -2w(x)[g_x(y)+g_y(x)]+4w(y)g_x(x). \end{aligned} \tag{3.2.10}$$

Let $w = g_y$ in Equation (3.2.10) to get

$$g_y(x^2y)+g_y(yx^2)-2g_y(xyx) = -2g_y(x)[g_x(y)+g_y(x)]+4g_y(y)g_x(x). \tag{3.2.11}$$

With the definition of g_y and the fact that g being multiplicative, Equation (3.2.11) can be rewritten as

$$2g(x^2y^2) - 2g((xy)^2) = 4g_y(y)g_x(x) - 4g_x(y).$$

That is exactly the second identity of (v). □

PROPOSITION 3.2.4. *Any pre-d'Alembert function g on a group G satisfies*

$$g(xy) + d(y)g(xy^{-1}) = 2g(x)g(y) \qquad \textit{for all } x, y \in G$$

PROOF. Just replace x by xy^{-1} and z by y respectively in Equation (3.1.1). □

3.3. Abelian Pre-d'Alembert Functions

THEOREM 3.3.1. *Let g be a solution of the pre-d'Alembert functionl equation on a semigroup S. Then g is abelian if and only if g is in the form of $g = \frac{\chi_1+\chi_2}{2}$ with unique multiplicative functions $\chi_1, \chi_2 : S \to \mathbb{C}$.*

PROOF. By Lemma 3.2.3, we have known that the solutions of pre-d'Alembert functionl equation satisfy the symmetrized sine addition formula with $w = g_x$:

$$g_x(yz) + g_x(zy) = 2g_x(y)g(z) + 2g_x(z)g(y). \tag{3.3.1}$$

Since g is abelian and in particular central, we have

$$\begin{aligned} g_x(yz) &= g(xyz) - g(x)g(yz) \\ &= g(xzy) - g(x)g(zy) \\ &= g_x(zy). \end{aligned}$$

Then Equation (3.3.1) can be simplied to

$$g_x(yz) = g_x(y)g(z) + g_x(z)g(y),$$

which has been solved in Theorem 2.4.4. Conversely, when $g = (\chi_1 + \chi_2)/2$, it satisfies Equation (3.1.1) by simple computation.

$$
\begin{aligned}
g(xyz) &= (\chi_1 + \chi_2)(xyz) \\
&= \chi_1(xyz) + \chi_2(xyz) \\
&= \chi_1(x)\chi_1(yz) + \chi_2(x)\chi_2(yz) \\
&= \chi_1(x)\chi_1(y)\chi_1(z) + \chi_2(x)\chi_2(y)\chi_2(z) \\
&= \chi_1(x)\chi_1(z)\chi_1(y) + \chi_2(x)\chi_2(z)\chi_2(y) \\
&= \chi_1(x)\chi_1(zy) + \chi_2(x)\chi_2(zy) \\
&= \chi_1(xzy) + \chi_2(xzy) \\
&= (\chi_1 + \chi_2)(xzy) \\
&= g(xzy).
\end{aligned}
$$

Thus g is an abelian solution by Lemma 2.4.3.

□

This theorem gives the form of the abelian solution of all pre-d'Alembert equations. Thus naturally a question arises: when will a solution be abelian?

LEMMA 3.3.2. *Let g be a solution of the pre-d'alembert functional equation on a semigroups. Then $\triangle = 0$ if and only if g is abelian.*

PROOF. It is clear that if g is abelian, then $g(xyz) = g(xzy)$ for all $x, y, z \in S$. Let $z = xy$ to get $g((xy)^2) = g(x^2y^2)$. This implies $\triangle = 0$ directly.

Conversely, when $\triangle = 0$, to prove g ia abelian, we shall show $g(xyz) = g(xzy)$ for all $x, y, z \in S$ by Lemma 2.4.3.

Replace y with yz and then replace y with zy in Lemma 3.2.3(v) respectively. We have

$$g_x(x)g_{yz}(yz) - g_x(yz)^2 = 0 \qquad \text{for all } x, y, z \in S, \tag{3.3.2}$$

and

$$g_x(x)g_{zy}(zy) - g_x(zy)^2 = 0 \qquad \text{for all } x, y, z \in S. \tag{3.3.3}$$

Since g is central, we have

$$g_{yz}(yz) = g(yzyz) - g(yz)^2 = g(zyzy) - g(zy)^2 = g_{zy}(zy).$$

Then subtract Equation (3.3.2) by Equation (3.3.3) to get

$$g_x(yz) = g_x(zy) \qquad \text{or} \qquad g_x(yz) + g_x(zy) = 0.$$

If $g_x(yz) = g_x(zy)$, then

$$\begin{aligned} 0 = g_x(yz) - g_x(zy) =& g(xyz) - g(x)g(yz) + g(xzy)g(x)g(yz) \\ =& g(xyz) - g(xzy). \end{aligned}$$

Thus g is abelian by Lemma 2.4.3.
If $g_x(yz) + g_x(zy) = 0$, by Lemma 3.2.3(iii) we have

$$2g_x(y)g(z) + 2g_x(z)g(y) = 0. \tag{3.3.4}$$

Interchange x and y to get

$$2g_y(x)g(z) + 2g_y(z)g(x) = 0. \tag{3.3.5}$$

Similarly,

$$2g_z(y)g(x) + 2g_z(x)g(y) = 0. \tag{3.3.6}$$

Add Equation (3.3.5) with Equation (3.3.6), and then substract Equation (3.3.4) to get $g(x)g_y(z) = 0$. So $g_y(z) = 0$ or $g(x) = 0$ for all $x, y, z \in S$.
Case 1: $g_y(z) = 0$. First we need to show that if $g_a(a) = 0$ for a $a \in S$, for any $x \in S$

we have

$$0 = \triangle(x,a) = g_x(x)g_a(a) - g_x(a)^2 = -g_x(a)^2 = -[g(xa) - g(x)g(a)]^2,$$

thus $g(xa) = g(x)g(a)$. Then for any $x, y \in S$, we have $g(xya) = g(xy)g(a) = g(yx)g(a) = g(yxa) = g(xay)$.

So in this case, $0 = \triangle(y,z) = g_y(y)g_z(z) - g_y(z)^2 = g_y(y)g_z(z)$. Thus $g_y(y) = 0$ or $g_z(z) = 0$ which implies $g(xyz) - g(xzy) = 0$.

Case 2: $g(x) = 0$ and we also have $g(y) = 0$ and $g(z) = 0$ by symmetry. We will show the set

$$D := \{(x,y,z) \in S \times S \times S \mid g(xyz) \neq g(xzy)\}$$

is empty by contradiction. With

$$g(xy) \neq 0, \tag{3.3.7}$$

and

$$g(xyz) \neq 0, \tag{3.3.8}$$

otherwise it is the same as Case 1.

Assume there exists an element in D. Let $x_0, y_0, z_0 \in S$ such that

$$g(x_0y_0z_0) \neq g(x_0z_0y_0), \tag{3.3.9}$$

with $g(x_0) = g(y_0) = g(z_0) = 0$.

Replace z_0 by $x_0y_0z_0$ in Equation (3.3.9). If $g(x_0y_0(x_0y_0z_0)) \neq g(x_0(x_0y_0z_0)y_0)$, then $g(x_0y_0z_0) = 0$ which contradicts Equation (3.3.8). So $g(x_0y_0(x_0y_0z_0)) = g(x_0(x_0y_0z_0)y_0)$.

Consider $g(x_0y_0(x_0y_0z_0))$ now, first we should notice that if we put $y = x$ in Equation (3.2.1) and let $w = g_z$ to get

$$g_z(x^2) = 2g_z(x)g(x)$$

then with that

$$\begin{aligned} g((x_0y_0)^2z_0) &= g_{z_0}((x_0y_0)^2) + g(z_0)g((x_0y_0)^2) \\ &= 2g_{z_0}(x_0y_0)g(x_0y_0) \\ &= 2g(x_0y_0z_0)g(x_0y_0) \neq 0 \end{aligned}$$

While

$$\begin{aligned} g(x_0^2y_0z_0y_0) &= g_{y_0z_0y_0}(x_0^2) + g(y_0z_0y_0)g(x_0^2) \\ &= 2g_{y_0z_0y_0}(x_0)g(x_0) + g(z_0y_0^2)g(x_0^2) \\ &= g_{z_0}(y_0^2)g(x_0^2) \\ &= 2g_{z_0}(y_0)g(y_0)g(x_0^2) \\ &= 0. \end{aligned}$$

We get a contradiction. □

Proposition 3.3.3. *A pre-d'Alembert function g on a group G is abelian if and only if $g([x,y]) = 1$ for all $x, y \in G$.*

Proof. By Lemma 3.3.2, we just need to show for any $x, y \in G$, $\triangle = 0$ if and only if $g([x,y]) = 1$.
By Proposition 3.2.4

$$d(y)g(xy^{-1}) = 2g(x)g(y) - g(xy). \tag{3.3.10}$$

Replace x and y by xy and yx in Equation (3.3.10), respectively. Since g is central, we have

$$\begin{aligned} d(xy)g([x,y]) &= d(yx)g(xyx^{-1}y^{-1}) = d(yx)g(xy(yx)^{-1}) \\ &= 2g(xy)g(yx) - g(xyyx) \\ &= 2g(xy)^2 - g(x^2y^2) \\ &= -[g(x^2y^2) - g((xy)^2)] - g((xy)^2) + 2g(xy)^2 \end{aligned}$$

$$= -2\triangle(x,y) + d(xy).$$

That is $\triangle(x,y) = [d(xy) - d(xy)g([x,y])]/2$. Thus $\triangle = 0$ if and only if $d(xy)(1 - g([x,y])) = 0$. Then we have $d(xy) = 0$ or $1 - g([x,y]) = 0$. But $d(xy) \neq 0$, since when we let $x = y = e$, $d(e) = g(e) = 1$. Therefore $\triangle = 0$ if and only if $g([x,y]) = 1$. □

Next we will give a sufficient condition to guarantee that g is abelian. To prove it we need to give the following lemma first.

LEMMA 3.3.4. *Let G be a group which is generated by its squares. Let N be a subgroup of G such that $[G,G] \subseteq N$. Let g be a solution of the pre-d'Alembert functional equation on G such that $g|_N$ is abelian. Let $x_0 \in G$ and put $N_0 := \langle N, x_0\rangle$. Then $g|_{N_0}$ is abelian.*

PROOF. We shall use Lemma 3.3.2 to show that $g|_{N_0}$ is abelian. First we would show that N is normal. Since $[G,G] \subseteq N$, we have $gng^{-1}n^{-1}$ is in N for any $n \in N, g \in G$. If we let $n' = gng^{-1}n^{-1}$, then we have

$$gng^{-1} = gng^{-1}n^{-1}n = n'n \in N \qquad \text{(as } N \text{ is closed under multiplication).}$$

This means N is normal.
Then let us consider about the gengeral form of elements in N_0. Notice that an arbitrary element of $N_0 := \langle N, x_0\rangle$ has the form $n_1{x_0}^{i_1}n_2{x_0}^{i_2}...n_k{x_0}^{i_k}$ where $n_i \in N$, $i_1, i_2, ..., i_n$ are integers. Since N is normal, we have

$$\begin{aligned} {x_0}^{i_1}n_2{x_0}^{i_2} &= {x_0}^{i_1}n_2{x_0}^{-i_1}{x_0}^{i_1}{x_0}^{i_2} \\ &= n_2'{x_0}^{i_1}{x_0}^{i_2} \qquad \text{(here } n_2' := {x_0}^{i_1}n_2{x_0}^{-i_1} \in N\text{)} \\ &= n_2'{x_0}^{i_1+i_2}. \end{aligned}$$

Continuing this process gives

$$\begin{aligned} n_1{x_0}^{i_1}n_2{x_0}^{i_2}...n_k{x_0}^{i_k} &= n_1n_2'{x_0}^{i_1+i_2}{x_0}^{i_3}...n_k{x_0}^{i_k} \\ &= n_1n_2'n_3'...n_k{x_0}^{i_1+i_2+...i_k} \end{aligned}$$

for some $n_2', ..., n_k' \in N$. Since $n_1n_2'n_3'...n_k \in N$ and ${x_0}^{i_1+i_2+...i_k} \in \langle x_0 \rangle$, the above shows that any element in N_0 is in the form nx_0^α for some $n \in N$ and $\alpha \in \mathbb{Z}$. Then by Lemma 3.3.2 we only need to compute $g([n_1x_1, n_2x_2])$ where $n_1, n_2 \in N$, $x_1, x_2 \in \langle x_0 \rangle$. By Theorem 3.3.1, we have $g|_N = \frac{\chi_1+\chi_2}{2}$, where $\chi_1, \chi_2 : N \to \mathbb{C}^\star$ are characters on N. For a character γ on N and $x \in G$, let $x \cdot \gamma$ denote the character on N by $(x \cdot \gamma)(n) := \gamma(x^{-1}nx), n \in N$. Then $x \in G$ for any $n \in N$ we have

$$\begin{aligned} g(n) &= \frac{\chi_1(n) + \chi_2(n)}{2} \\ &= \frac{\chi_1(x^{-1}nx) + \chi_2(x^{-1}nx)}{2} \qquad \text{(since } g \text{ is central)} \\ &= \frac{(x \cdot \chi_1)(n) + (x \cdot \chi_2)(n)}{2}. \end{aligned}$$

This means $\chi_1 + \chi_2 = x \cdot \chi_1 + x \cdot \chi_2$. By Corallary 2.3.8, there are two cases:
(i) $\chi_1 = x \cdot \chi_1$ and $\chi_2 = x \cdot \chi_2$.
(ii) $\chi_1 = x \cdot \chi_2$ and $\chi_2 = x \cdot \chi_1$.
For (ii), one can easily get $\chi_2 = x^2 \cdot \chi_2$ and $\chi_1 = x^2 \cdot \chi_1$. Since G is generated by squares, from (i) and (ii) we conclude that $x \cdot \chi_1 = \chi_1$ and $x \cdot \chi_2 = \chi_2$ for all $x \in G$. Since χ_1 is multiplicative, we have

$$\begin{aligned} \chi_1([n_1x_1, n_2x_2]) &= \chi_1(n_1x_1n_2x_2(n_1x_1)^{-1}(n_2x_2^{-1}) \\ &= \chi_1(n_1x_1n_2x_2x_1^{-1}n_1^{-1}x_2^{-1}n_2^{-1}) \\ &= \chi_1(n_1x_1n_2(x_1^{-1}x_1)x_2x_1^{-1}n_1^{-1}x_1x_2^{-1}x_1^{-1}x_1x_2x_1^{-1}x_2^{-1}n_2^{-1}) \\ &= \chi_1(n_1(x_1n_2x_1^{-1})(x_1x_2x_1^{-1})n_1^{-1}x_1x_2^{-1}x_1^{-1}(x_1x_2x_1^{-1}x_2^{-1})n_2^{-1}) \\ &= \chi_1(n_1(x_1n_2x_1^{-1})(x_1x_2x_1^{-1})n_1^{-1}(x_1x_2x_1^{-1})^{-1}(x_1x_2x_1^{-1}x_2^{-1})n_2^{-1}) \\ &= \chi_1(n_1)\chi_1(x_1n_2x_1^{-1})\chi_1((x_1x_2x_1^{-1})n_1^{-1}(x_1x_2x_1^{-1})^{-1})\chi_1([x_1, x_2])\chi_1(n_2^{-1}) \\ &= \chi_1(n_1)(x_1^{-1} \cdot \chi_1)(n_2)(x_1x_2x_1^{-1} \cdot \chi_1)(n_1^{-1})\chi_1([x_1, x_2])\chi_1(n_2^{-1}) \\ &= \chi_1(n_1)\chi_1(n_2)\chi_1(n_1^{-1})\chi_1([x_1, x_2])\chi_1(n_2^{-1}) \\ &= \chi_1([x_1, x_2]). \end{aligned}$$

By the same way, $\chi_2([n_1x_1, n_2x_2]) = \chi_2([x_1, x_2])$. Then we have:

$$g([n_1x_1, n_2x_2]) = \frac{\chi_1 + \chi_2}{2}([n_1x_1, n_2x_2]) = \frac{\chi_1 + \chi_2}{2}([x_1, x_2]) = 1,$$

since x_1 and x_2 commute. Thus we finish the proof. $\square$

THEOREM 3.3.5. *Let G be a group which is generated by its squares. Let g be a solution of the pre-d'Alembert functional equation on G such that $g|_{[G,G]}$ is abelian. Then g is abelian.*

PROOF. By Lemma 3.1.4, there are three cases for $g(e)$:
(i) $g(e) = 0$. Let $y = z = e$ in Equation (3.1.1). Then $g = 0$. Clearly g is abelian.
(ii) $g(e) = \frac{1}{2}$ and $2g$ is multiplicative by Lemma 3.1.4. Thus for any $x, y, z \in G$,

$$g(xyz) = 2g(xy)g(z) = 4g(x)g(y)g(z) = 4g(x)g(z)g(y) = 2g(xz)g(y) = g(xzy).$$

This means g is abelian.
(iii) $g(e) = 1$. In this case, g is a pre-d'Alembert function. Suppose that $\mathcal{N}$ is the collection of all the subgroups N of G, such that $[G, G] \subseteq N$, and $g|_N$ is abelian. We want to show that there exists a maximal element $N_{max} \in \mathcal{N}$ by the help of Zorn's lemma. Clearly, $\mathcal{N}$ is non-empty as $[G, G] \in \mathcal{N}$. Let $N_1, N_2 \in \mathcal{N}$, and define

$$N_1 \preceq N_2 :\Leftrightarrow N_1 \subseteq N_2.$$

Then $(\mathcal{N}, \preceq)$ is a partial order set. Let $\mathcal{C} = \{N_\alpha : \alpha \in I\}$ be a chain in $\mathcal{N}$, where I is an index set. Then $\mathcal{S} := \underset{\alpha \in I}{\cup} N_\alpha$ is an upper bound of $\mathcal{C}$. Clearly $S \subseteq G$. Since arbitrary $N_\alpha \subseteq \mathcal{S}$ is a subgroup of G, then N_α is closed under products and inverses, so does $\mathcal{S}$. Hence $\mathcal{S}$ is a subgroup of G. And it is obvious that $[G, G] \subseteq N_\alpha \subseteq S$. For any $N_\alpha \subseteq S$, $g|_{N_\alpha}$ is abelian. Therefore by the definition of S, $g|_S$ is abelian. By Zorn's lemma, $\mathcal{N}$ has a maximal element N_{max}. We claim that $N_{max} = G$. Otherwise, there is an $x_0 \in G \backslash N_{max}$. Then apply Lemma 3.3.4 to $\langle N_{max}, x_0 \rangle$ to get that $N_{max} \subsetneq \langle N_{max}, x_0 \rangle \in \mathcal{N}$. This contradicts the property that N_{max} is maximal. Therefore $N_{max} = G$ and $g|_G = g$ is abelian. $\square$

CHAPTER 4

Abelian D'Alembert Functions on Groups

In this chapter, we study the d'Alembert functional equation on groups. Our focus here is abelian solutions. We get the general form of abelian d'Alembert functions on groups.

4.1. The D'Alembert Functional Equation

Definition 4.1.1. Let S be a semigroup. A map $\iota : S \to S$ is called an *involution* in S, if $\iota(xy) = \iota(y)\iota(x)$ and $\iota(\iota(x)) = x$ for all $x, y \in S$.

Definition 4.1.2. Let S be a semigroup. The *d'Alembert's functional equation* with involution $\iota : S \to S$ is the functional equation in the form

$$g(xy) + g(x\iota(y)) = 2g(x)g(y) \qquad \text{for all } x, y, \in S, \tag{4.1.1}$$

where $g : S \to \mathbb{C}$ is the unknown. If $g \neq 0$, it is called a *d'Alembert function.*

If $S = \mathbb{R}$ and $\iota(x) = -x$ for $x \in \mathbb{R}$, then Equation (4.1.1) becomes

$$g(x + y) + g(x - y) = 2g(x)g(y) \qquad \text{for all } x, y \in \mathbb{R}. \tag{4.1.2}$$

We can easily see that there is a trivial solution $g = 0$, so we will not consider about this case later. And another obvious solution is $g = \cos$. So this equation is sometimes also called the *cosine functional equation.*

4.2. Classical d'Alembert Functions

Definition 4.2.1. Let G be a group. The *classical d'Alembert's functional equation* is the functional equation in the form

$$g(xy) + g(xy^{-1}) = 2g(x)g(y) \qquad \text{for all } x, y \in G, \tag{4.2.1}$$

where $g : G \to \mathbb{C}$ is the unknown.

If $g \neq 0$, it is called a *classical d'Alembert function.*

Example 4.2.2. Let $g : \mathbb{R} \to \mathbb{C}$ be a continuous solution of d'Alembert's classical functional equation on $\mathbb{R}$:

$$g(x+y) + g(x-y) = 2g(x)g(y) \qquad \text{for all } x, y \in \mathbb{R}. \tag{4.2.2}$$

In this example we will find the form of g. Since g is continuous, then $g \in C^\infty(\mathbb{R})$ by [11, Theorem 4.14 Chapter I]. So it is differentiable. We take the second derivatives of both sides with respect to y. Let $y = 0$. Then $g'' = g''(0)g$.
If $g''(0) = 0$, then $g'' = 0$. So g should be in the form of $g(x) = Ax + B$. Then substitute it to Equation (4.2.2) to get $g = 1$.
If $g''(0) \neq 0$, we assume $g''(0) = \alpha^2 (\neq 0)$. Solve this differential equation $g'' = \alpha^2 g$. Then we get $g(x) = Ae^{\alpha x} + Be^{-\alpha x}$, where $x \in \mathbb{R}, A, B \in \mathbb{C}$. Replacing y by $-y$ in Equation (4.2.2), we have $Ae^{\alpha x} + Be^{-\alpha x} = Ae^{-\alpha x} + Be^{\alpha x}$ for all $x \in \mathbb{R}$. Hence $A = B$. Let $y = 0$ in Equation (4.2.2). We get $g(0) = 1$ and so $A = \frac{1}{2}$. Thus $g(x) = \frac{e^{\alpha x} + e^{-\alpha x}}{2}$

Before discussing more properties of classical d'Alembert functions, we need to recall (the first kind) *Chebyshev polynomials* :

$$T_0(x) = 1,$$
$$T_1(x) = x,$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

So T_n is polynomial of degree n ($n \geq 0$).

In [5], Davison noted that any classical d'Alembert function g on the group $(\mathbb{Z}, +)$ is given by the values of the Chebyshev polynomials $\{T_n\}_{n=0}^\infty$ at a point of $\mathbb{C}$, *i.e.* $g(n) = T_n(g(1))$. Now we can extend $(\mathbb{Z}, +)$ to any group.

Proposition 4.2.3. *If g is a classical d'Alembert function on a group G. Then $T_n(g(x)) = g(x^n)$ for any $x \in G$ and $n = 0, 1, \ldots$.*

PROOF. Since $T_0(x) = 1$, let $n = 0$, $T_0(g(x)) = g(x^0) = 1$. Similary, $T_1(g(x)) = g(x) = g(x^1)$. We will finish this proof by induction. Assume $T_k(g(x)) = g(x^k)$ for all $0 \le k \le n$. To see $T_{n+1}(g(x))$ from $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, we have

$$\begin{aligned} T_{n+1}(g(x)) &= 2g(x)T_n(g(x)) - T_{n-1}(g(x)) \\ &= 2g(x)g(x^n) - g(x^{n-1}). \end{aligned}$$

Replacing x and y by x^n and x reapectively in Equation (4.2.1), we get $g(x^n x) + g(x^n x^{-1}) = 2g(x^n)g(x)$. Then

$$\begin{aligned} 2g(x)g(x^n) - g(x^{n-1}) &= g(x^n x) + g(x^n x^{-1}) - g(x^{n-1}) \\ &= g(x^{n+1}). \end{aligned}$$

Thus $T_{n+1}(g(x)) = g(x^{n+1})$. Hence we finish the proof. □

4.3. Abelian D'Alembert Functions

Similar to last chapter, we will see that the solution of the d'Alembert functionl equation is also related to the sine addition formula.

PROPOSITION 4.3.1. *If $g : S \to \mathbb{C}$ is an abelian solution of the d'Alembert functionl equation on a semigroup S with involution $\iota : S \to S$, then g is in the form of $g = (\chi + \chi \circ \iota)/2$, where $\chi : S \to \mathbb{C}$ is multiplicative.*

PROOF. Replace x and y by xy and z, respectively, in Equation (4.1.1). Then we get

$$g(xyz) + g(xy\iota(z)) = 2g(xy)g(z). \tag{4.3.1}$$

Replacing y with $y\iota(z)$ in Equation (4.1.1) gives

$$g(xy\iota(z)) + g(x\iota(y\iota(z))) = 2g(x)g(y\iota(z)).$$

Thus

$$g(xy\iota(z)) + g(xz\iota(y)) = 2g(x)g(y\iota(z)). \tag{4.3.2}$$

By the definition of the d'Alembert functionl equation, $g(xz\iota(y))$ can also be written as

$$g(xz\iota(y)) = 2g(xz)g(y) - g(xzy). \tag{4.3.3}$$

Put Equation (4.3.3) in Equation (4.3.2) and subtract by Equation (4.3.1). After simplifing it, we obtain

$$\begin{aligned} g(xyz) - g(x)g(yz) &= g(xy)g(z) + g(xz)g(y) - 2g(x)g(y)g(z) \\ &= g(xy)g(z) - g(x)g(y)g(z) + g(xz)g(y) - g(x)g(y)g(z) \\ &= [g(xy) - g(x)g(y)]g(z) + [g(xz) - g(x) - g(z)]g(y) \\ &= g_x(y)g(z) + g_x(z)g(y). \end{aligned} \tag{4.3.4}$$

The left side of the above equation is exactly $g_x(yz)$. This means that Equation (4.3.4) satisfy the sine addition Equation (3.2.1) with $f = g_x$. If $g_x \neq 0$ then by Theorem 2.4.4, the solution g should be in the form $g = (\chi_1 + \chi_2)/2$, with χ_1 and $\chi_2 : S \to \mathbb{C}$ multiplicative functions from S to $\mathbb{C}$. We shall show $\chi_1 = \chi_2 \circ \iota$. Let $g = (\chi_1 + \chi_2)/2$ in Equation (4.1.1). We obtain

$$\chi_1(xy) + \chi_2(xy) + \chi_1(x\iota(y)) + \chi_2(x\iota(y)) = [\chi_1(x) + \chi_2(x)][\chi_1(y) + \chi_2(y)].$$

Simiplify the equation above from the fact that χ_1 and χ_2 are multiplicative, we have

$$\chi_1(x\iota(y)) + \chi_2(x\iota(y)) = \chi_1(x)\chi_2(y) + \chi_2(x)\chi_1(y).$$

Changing the order of items, since χ_1 and χ_2 are multiplicative, we have

$$\chi_1(x)(\chi_1(\iota(y)) - \chi_2(y)) + \chi_2(x)(\chi_2(\iota(y)) - \chi_1(y)) = 0.$$

Since χ_1 and χ_2 are different, $\chi_1(x)(\chi_1(\iota(y)) - \chi_2(y)) = \chi_2(x)(\chi_2(\iota(y)) - \chi_1(y)) = 0$ by Theorem 2.3.6. Since χ_1 and χ_2 can not be zero together, at least $\chi_1(\iota(y)) = \chi_2(y)$

or $\chi_2(\iota(y)) = \chi_1(y)$. No matter which case, we all have $\chi_1 = \chi_2 \circ \iota$. Then let $\chi_1 = \chi$ to get $g = (\chi + \chi \circ \iota)/2$. $\square$

By taking $\iota(x) = x^{-1}$, we get the solutions for the classical d'Alembert functional equation.

COROLLARY 4.3.2. *Let G be a group which is generated by its squares. Let g be a classical d'Alembert function on G such that $g|_{[G,G]}$ is abelian. Then $g = (\chi + \chi^{-1})/2$, where $\chi : G \to \mathbb{C}$ is multiplicative.*

PROOF. First, by the similar argument to the proof of Theorem 3.3.5, we have that g is abelian. Then the rest follows from Proposition 4.3.1. $\square$

CHAPTER 5

Non-Abelian D'Alembert Functions on Groups

In Chapter 4, we have shown the structure for abelian d'Alembert functions on groups. Now we focus on non-abelian d'Alembert functions. Our main goal is to obtain Davison's Structure Theorem.

5.1. Translates of Pre-d'Alembert Functions

As what we discussed in Chapter 2, the way to consider non-abelian solutions of the d'Alembert functional equation, is with the help of harmonic analysis by decomposing complex equations into simple terms based on group representation theory.

Let g be a continuous pre-d'Alembert function on a topological monoid M. That is

$$g(xyz) + g(xzy) = 2g(x)g(yz) + 2g(y)g(xz) + 2g(z)g(xy) - 4g(x)g(y)g(z)$$

for all $x, y, z \in M$. We consider the space of translates of g first:

$$T(g) := \text{span}\{R(x)g | x \in M\} \subseteq C(M),$$

where R is the right regular representation of M. Let

$$W(g) := \{w \in C(M) | w(xy) + w(yx) = 2w(x)g(y) + 2w(y)g(x) \text{ for all } x, y \in M\}.$$

LEMMA 5.1.1. *Let g be a continuous pre-d'Alembert function on a topological monoid M. Then*

(i) $\dim T(g)$*=1,2 or 4;*

(ii) $T(g)$ is invariant under the right regular representation R;

(iii) $R|_{T(g)}$ is a continuous representation of M on $T(g)$;

(iv) $T(g) \subseteq \mathbb{C}g + W(g)$, and the sum on the right is direct.

Proof. (i) Theorem 3.3.1 shows the structure of the solution g when it is abelian: $g = \frac{\chi_1+\chi_2}{2}$ with unique multiplicative functions $\chi_1, \chi_2 : S \to \mathbb{C}$.. Thus $T(g) \subseteq \text{span}\{\chi_1, \chi_2\}$ and so $\dim T(g) \leq 2$. The case $\dim T(g) = 4$ arises when g is non-abelian. We will discuss it in Lemma 5.2.3 below.

(ii) By the definition of R, for any $\lambda R(x)g \in T(g)$ with $\lambda \in \mathbb{C}$

$$[R(y)(\lambda R(x)g)](z) = \lambda R(x)g(zy) = \lambda g(zyx) = [\lambda R(yx)g](z) \qquad \text{for all } x, y, z \in M.$$

So, $R(y)(\lambda R(x)g) = \lambda R(yx)g \in T(g)$. Thus $T(g)$ is invariant under R.

(iii) We will use the notations in Appendix B. $R|_{T(g)}$ is clearly a representation of $T(g)$ by (ii). Consider the special matrix-coefficients $x \mapsto \langle ev_y, R(x)f\rangle = f(yx)$ where $y \in G$ and $f \in T(g)$. They are continuous, because G is a topological group and $f \in C(G)$. Since they span the space of all matrix-coefficients (see Lemma B.2), $R|_{T(g)}$ is continuous because each matrix-coefficient which is the coordinates of $R|_{T(g)}$ is continuous.

(iv) First we prove the inclusion. By Lemma 3.2.3(ii), we have for all $x \in M$

$$\begin{aligned} g_x \in W(g) &\Rightarrow g(xy) - g(x)g(y) \in W(g) \\ &\Rightarrow g(yx) - g(x)g(y) \in W(g) \qquad \text{(since } g \text{ is central)} \\ &\Rightarrow [R(x)g](y) - g(x)g(y) \in W(g) \qquad \text{(by the definition of } R) \\ &\Rightarrow R(x)g - g(x)g \in W(g) \\ &\Rightarrow T(g) \in \mathbb{C}g + W(g) \qquad \text{(by the definition of } T(g)). \end{aligned}$$

To show the sum is direct, let λg with $\lambda \neq 0$ such that $\lambda g \in W(g)$. Put $y = 0$ in the definition of $W(g)$. Then we have $2g(x) = 4g(x)$, for all $x \in G$, which gives $g(x) = 0$ for all $x \in G$. Thus $g = 0$. $\square$

5.2. Non-abelian Pre-d'Alembert Functions

Let M be a topological monoid, and $g \in C(M)$ be a continuous pre-d'Alembert function. Last section we figure out the relationship between $T(g)$ and $W(g)$. Now we shall apply it to the case when g is non-abelian, thus $\triangle(a,b) \neq 0$ for some $a, b \in M$. Recall the symmetrized sine addition formula

$$w(xy) + w(yx) = 2w(x)g(y) + 2w(y)g(x) \qquad \text{for all } x, y \in M, \tag{5.2.1}$$

where $w, g \in C(S)$.

In this section, we assume that $g \in C(M)$ is non-abelian.

LEMMA 5.2.1. *Let (w, g) be a solution of Equation (5.2.1) on M, and let $a, b \in M$, $\widetilde{\triangle}(a,b) \neq 0$. Then*

$$w(a) = w(b) = 0 \qquad \Rightarrow \qquad w = 0.$$

PROOF. First we need to make $\widetilde{\triangle}(a,b)$ appear. To do this we need to use some of the previous formulas.
Letting $y = x$ in Equation (3.2.8) and by the definition of d give

$$w(xzx) = d(x)w(z) + w(x)[g(xz) + g(zx)]. \tag{5.2.2}$$

Replace z by yzy in Equation (3.2.8) to get

$$w(xyzyx) = d(x)w(yzy) + w(x)[g(xyxy) + g(yzyx)]. \tag{5.2.3}$$

Change the order of x and y to give

$$w(yzzxy) = d(y)w(xzx) + w(y)[g(yxyx) + g(xzxy)]. \tag{5.2.4}$$

Add Equation (5.2.3) and Equation (5.2.4) and replace terms in the form $w(xzx)$ by Equation (5.2.2). We obtain

$$\begin{aligned} &w(xyzyx) + w(yzzxy) = \\ &w(x)[g(xyzy) + g(yzyx) + (g(xz) + g(zx))d(y)] \\ &+ w(y)[g(yxzx) + g(xzxy) + (g(yz) + g(zy))d(x)] \\ &+ 2w(z)d(x)d(y). \end{aligned} \tag{5.2.5}$$

For the left side of Equation (5.2.5), by the help of Equation (3.2.8) we replace x with yx and y with xy respectively, and use the definition of $\widetilde{\triangle}(x,y)$ to get

$$
\begin{aligned}
& w(yx)[g(xyz)+g(zxy)]+w(xy)[g(yxz)+g(zyx)] \\
& =w(x)[g(xyzy)+g(yzyx)+(g(xz)+g(zx))d(y)] \\
& \quad +w(y)[g(yxzx)+g(xzxy)+(g(yz)+g(zy))d(x)] \\
& \quad +w(z)\widetilde{\triangle}(x,y).
\end{aligned} \tag{5.2.6}
$$

Let $x=a$, $y=b$ in Equation (5.2.6). We have

$$
\begin{aligned}
& w(ba)[g(abz)+g(zab)]+w(ab)[g(baz)+g(zba)] \\
& =w(z)\widetilde{\triangle}(a,b).
\end{aligned}
$$

Since (w,g) is a solution of Equation (5.2.1), we also have $w(ba)=0$. Then we have $0=w(z)\widetilde{\triangle}(a,b)$. Hence $w(z)=0$, as $\widetilde{\triangle}(a,b)\neq 0$. That is $w=0$. $\square$

LEMMA 5.2.2. *The three functions*

$$
\begin{aligned}
\omega_1 &:= \frac{g_b(b)}{\triangle(a,b)}g_a - \frac{g_a(b)}{\triangle(a,b)}g_b + \frac{g(b)}{\triangle(a,b)}\frac{g_{ab}-g_{ba}}{2}, \\
\omega_2 &:= \frac{g_a(b)}{\triangle(a,b)}g_a - \frac{g_a(a)}{\triangle(a,b)}g_b + \frac{g(a)}{\triangle(a,b)}\frac{g_{ab}-gba}{2}, \\
\omega_3 &:= -\frac{1}{\triangle(a,b)}\frac{g_{ab}-g_{ba}}{2} = -\frac{1}{\triangle(a,b)}\frac{(R_{ab}-R_{ba})g}{2},
\end{aligned}
$$

in $\operatorname{span}\{g_x | x\in M\}$ *have the properties*

$$
\begin{aligned}
& \omega_1(a)=1,\ \omega_1(b)=0,\ \omega_1(ab)=0, \\
& \omega_2(a)=0,\ \omega_2(b)=1,\ \omega_2(ab)=0, \\
& \omega_3(a)=0,\ \omega_3(b)=0,\ \omega_3(ab)=1.
\end{aligned}
$$

PROOF. We first give some formulas we will use in the proof. Since g is central we have

$$\begin{aligned} g_a(b) &= g(ab) - g(a)g(b) \\ &= g(ba) - g(b)g(a) \\ &= g_b(a), \end{aligned}$$

and

$$\begin{aligned} g_{ab}(a) &= g(aba) - g(ab)g(a) \\ &= g(baa) - g(a)g(ab) \\ &= g_{ba}(a). \end{aligned}$$

Similarly, $g_{ab}(b) = g_{ba}(b)$.

Also

$$\begin{aligned} g_a(ab) &= g(aab) - g(a)g(ab) \\ &= g(aba) - g(a)g(ba) \\ &= g_a(ba). \end{aligned}$$

Similarly, $g_b(ab) = g_b(ba)$.

Now, consider $\omega_1(a)$. The last term will be 0. But $\triangle(a, b)$ is not zero. Then

$$\begin{aligned} \omega_1(a) &= \frac{g_b(b)}{\triangle(a,b)} g_a(a) - \frac{g_a(b)}{\triangle(a,b)} g_b(a) \\ &= \frac{g_b(b)g_a(a) - g_a(b)g_b(a)}{\triangle(a,b)} \\ &= \frac{g_b(b)g_a(a) - g_a(b)g_a(b)}{\triangle(a,b)} \\ &= \frac{\triangle(a,b)}{\triangle(a,b)} \qquad \text{(by definition of } \triangle\text{)} \\ &= 1. \end{aligned}$$

The rest of this lemma can be proved by a similar way. □

Lemma 5.2.3. *(i)* $W(g) = \text{span}\{g_x | x \in M\} = \text{span}\{\omega_1, \omega_2, \omega_3\}$.
(ii) $T(g) = \mathbb{C}g + W(g)$, *where the sum is direct.*
(iii) $\dim T(g) = 4$ *and* $\dim W(g) = 3$.
(iv) $\{f \in T(g) | f$ *is central*$\} = \mathbb{C}g$.

Proof. (i) It is clear that $w \in W(g)$. By Lemma 5.2.2, w and $w(a)w_1 + w(b)w_2 + w(ab)w_3$ have the same value at a, b, ab repectively. And $w = w(a)w_1 + w(b)w_2 + w(ab)w_3$ by Lemma 5.2.1. Then $W(g) = \text{span}\{\omega_1, \omega_2, \omega_3\}$. Thus $w = w(a)w_1 + w(b)w_2 + w(ab)w_3 \in \text{span}\{g_x | x \in M\}$ by Lemma 5.2.2. Thus $W(g) \subseteq \text{span}\{g_x | x \in M\}$. And Lemma 3.2.3(ii) shows $\text{span}\{g_x | x \in M\} \subseteq W(g)$.
(ii) Lemma 5.2.2 implies that $T(g) \subseteq \mathbb{C}g + W(g)$, and that the sum on the right is direct. The rest we need to show is $T(g) \supseteq \mathbb{C}g + W(g)$. For any $x \in M$, $g(x) = g(xe) = R(e)g(x)$, which means $g = R(e)g \in T(g)$. For any $y \in M$, $g_x(y) = g(xy) - g(x)g(y) = g(yx) - g(y)g(x) = [R(x)g](y) - g(y)g(x)$ since g is central. So $g_x = R(x)g - g(x)g \in T(g) + T(g) \subseteq T(g)$.
(iii) By (i), $\dim W(g) = 3$. Then by (ii), since the sum is direct, $\dim T(g) = 4$.
(iv) $f \in T(g)$ is central means g and w are all central by (ii). We only need to show that if w is central then $W(g) = 0$ since g is central by Lemma 3.2.3(i). When w is central, Equation (5.2.1) can be simplified to $w(xy) = w(x)g(y) + w(y)g(x)$. If $w \neq 0$, then by Theorem 2.4.4, g should be abelian, a contradiction. □

With all the relations, we can continue to study the structure of g.

Lemma 5.2.4. *(i) If* $W \neq \{0\}$ *is an* R*-invariant subspace of* $T(g)$*, then the character* $\chi_{R|_W}$ *of the representation* $R|W$ *is* $\chi_{R|_W} = \dim(W)g$.
(ii) There are no 1-dimensional, R*-invariant subspace of* $T(g)$.

Proof. (i) By Proposition B.0.11(ii), $C(R|_{T_{(g)}}) = T(g)$. Thus $\chi_{R|_W} \in T(g)$ since $\chi_{R|_W}$ is in $C(R|_{T(g)})$. Then by Lemma 5.2.3(iv), $\chi_{R|_W} = \mathbb{C}g$ since $\chi_{R|_W}$ is central. Assumme $\chi_{R|_W} = \alpha g$, with using of e, we get $\alpha = \alpha g(e) = \chi_{R|_W}(e) =$

$\operatorname{tr}((R_W)(e)) \operatorname{tr} I_W = \dim W$.
(ii) Suppose that there is a 1-dimensional, R-invariant subspace of $T(g)$. Hence by (i), $g = \chi_{R|_W}$ is non-abelian. But a 1-dimensional representation character is abelian since characters are multiplicative, a contradiction. □

THEOREM 5.2.5. *Let M be a topological monoid, and $g \in C(M)$ be a continuous, non-abelian pre-d'Alembert function on M. Arbitrarily choose an R-invariant subspace $W \neq \{0\}$ of $T(g)$, such that $\rho := R|_W$ is irreducible. Let χ_ρ denote the character of ρ, and $C(\rho)$ be the space of matrix-cofficients of ρ.*
(i) ρ is an irreducible, continuous representation of M on W.
(ii) $\dim W = 2$.
(iii) $g = \frac{1}{2}\chi_\rho$.
(iv) $d = \det \rho$.

PROOF. (i) ρ is chosen irreducible, and Lemma 5.1.1 shows it is continuous.
(ii) By Proposition B.0.11(ii), we have $C(\rho) = C(R|_W \subseteq C(R|_{T(g)})) = T(g)$. By Lemma 5.2.3(iii), $\dim T(g) = 4$. By Corollary B.0.7, $\dim C(\rho) = (\dim \rho)^2$. So we have the following

$$\dim T(g) \geq \dim C(\rho) = (\dim \rho)^2 = (\dim W)^2 \geq 2^2 = 4 = \dim T(g).$$

Thus $\dim W = 2$.
(iii) (ii) shows $\dim W = 2$. By Lemma 5.2.4(i), we have $g = \frac{1}{2}\chi_\rho$.
(iv) For any 2×2 matrix A, $(\operatorname{tr} A)^2 - \operatorname{tr}(A^2) = 2 \det A$, then by (iii) we have

$$d(x) = 2g(x)^2 - g(x^2) = \frac{1}{2}[(\operatorname{tr} \rho(x))^2 - \operatorname{tr}(\rho(x)^2)] = \det \rho(x).$$

□

5.3. Davison's Structure Theorem

We now summarize the previous results and obtain Davison's structure theorem. Here, M is a topological monoid with a identity element e. The function $d : M \to \mathbb{C}$, given by $d(x) := 2g(x)^2 - g(x^2)$, for $x \in M$. We have proved d is multiplicative in

Lemma 3.2.3. In Example 3.1.3 described the pre-d'Alembert functions on M in terms of characters of 2-dimensional representations of M. Replacing the 2-dimensional vector space W in Section 5.2 by the vector space $\mathbb{C}^2$ we get Davison's structure theorem:

THEOREM 5.3.1. *(Davison) Let M denote a topological monoid, g be a continuous pre-d'Alembert function.*
(i) The continuous pre-d'Alembert functions on M are the functions of the form $g = \frac{1}{2}\chi_\rho = \frac{1}{2}tr\rho$ for some 2-dimensional continuous representations ρ of M on $\mathbb{C}^2$.
Below we let $g = \frac{1}{2}\chi_\rho$, where ρ denotes a 2-dimensional continuous representation of M on $\mathbb{C}^2$.
(ii) $d(x) = \det \rho(x)$ for all $x \in M$.
(iii) g is non-abelian if and only if ρ is irreducible. If g is non-abelian, then ρ is unique up to equivalence of representation.
(iv) If g is abelian, then $g = \frac{\chi_1+\chi_2}{2}$ where $\chi_1, \chi_2 \in C(M)$ are multiplicative functions such that $\chi_1(e) = \chi_2(e) = 1$. The multiplicative functions χ_1 and χ_2 are unique, except that they may be interchanged.

PROOF. The proof of this Theorem has been contained in previous sections. □

5.4. Non-abelian μ-D'Alembert Functions

To begin this section, we introduce a generalization of d'Alembert functions.

DEFINITION 5.4.1. Let S be a semigroup with an involution $\iota : S \to S$. Let $\mu : S \to \mathbb{C}^\star$ be a multiplicative function such that $\mu(x\iota(X)) = 1$ for all $x \in S$. A *μ-d'Alembert function* on G is a non-zero solution $g : S \to \mathbb{C}$ of the *μ-d'Alembert functional equation*

$$g(xy) + \mu(y)g(x\iota(y)) = 2g(x)g(y), \qquad x, y \in S. \tag{5.4.1}$$

When $\mu = 1$ it becomes D'Alembert's functional equation.

Lemma 5.4.2. *If π is a 2-dimensional representation of a group G, then $g := \frac{1}{2}\operatorname{tr}\pi$ is a non-zero solution of*

$$g(xy) + \det(\pi(y))g(xy^{-1}) = 2g(x)g(y), \qquad x, y \in G.$$

Proof. The Cayley-Hamilton Theorem tells us

$$\pi(y)^2 - \operatorname{tr}(\pi(y))\pi(y) + (\det \pi(y))I_2 = 0.$$

Multiply both sides by $\pi(x)\pi(y)^{-1}$ from left, and take the trace we can get the conclusion. ☐

Proposition 5.4.3. *Let S be a semigroup with an involution $\iota : S \to S$. Let $\mu : S \to \mathbb{C}^\star$ be a multiplicative function such that $\mu(x\iota(x)) = 1$ for all $x \in S$. Let $g : S \to \mathbb{C}$ be a solution of Equation (5.4.1), i.e.*

$$g(xy) + \mu(y)g(x\iota(y)) = 2g(x)g(y), \qquad x, y \in S. \tag{5.4.2}$$

Then
(i) $g \circ \iota = g/\mu$;
(ii) g is central;
(iii) g is a solution of the pre-d'Alembert functional equation.

Proof. (i) Replace y by $\iota(y)$ in Equation (5.4.2) to get

$$g(x\iota(y)) + \mu(\iota(y))g(xy) = 2g(x)g(\iota(y)). \tag{5.4.3}$$

Multiply Equation (5.4.3) by $\mu(y)$, since μ is multiplicative we have

$$\begin{aligned} 2\mu(y)g(x)g(\iota(y)) &= \mu(y)g(x\iota(y)) + \mu(y)\mu(\iota(y))g(xy) \\ &= \mu(y)g(x\iota(y)) + \mu(y\iota(y))g(xy) \\ &= \mu(y)g(x\iota(y)) + g(xy) = 2g(x)g(y). \end{aligned}$$

So $\mu(y)g(\iota(y)) = g(y)$ for all $y \in S$. Thus $g \circ \iota = g/\mu$.

(ii) Switch x and y in Equation (5.4.2) and by (i), we have

$$\begin{aligned} g(yx) + \mu(x)g(y\iota(x)) &= 2g(y)g(x) = 2g(x)g(y) = g(xy) + \mu(y)g(x\iota(y)) \\ &= g(xy) + \mu(y)(g \circ \iota)(y\iota(x)) = g(xy) + \frac{\mu(y)}{\mu(y\iota(x))} g(y\iota(x)) \\ &= g(xy) + \mu(x)g(y\iota(x)). \end{aligned}$$

So $g(xy) = g(yx)$ for all $x, y \in S$. Thus g is central.

(iii) Let $x, y, z \in S$, replace x by xy and y by z, respectively, in Equation (5.4.2) to get

$$g(xyz) + \mu(z)g(xy\iota(z)) = 2g(xy)g(z). \tag{5.4.4}$$

Replace y by $y\iota(z)$ in Equation (5.4.2) and change the order to get

$$g(xy\iota(z)) = 2g(x)g(y\iota(z)) - \mu(y\iota(z))g(xz\iota(y)). \tag{5.4.5}$$

Substitute $g(xy\iota(z)$ in Equation (5.4.4) by Equation (5.4.5) and simplify it to get

$$g(xyz) + 2g(x)\mu(z)g(y\iota(z)) = 2g(xy)g(z) + \mu(y)g(xz\iota(y)). \tag{5.4.6}$$

Replace x by xz and y by $\iota(y)$, respectively, in Equation (5.4.2). Change the order to get

$$g(xz\iota(y)) = 2g(xz)g(\iota(y)) - \mu(\iota(y))g(xzy). \tag{5.4.7}$$

Substitute $g(xz\iota(y))$ in Equation (5.4.6) by Equation (5.4.7) and simplify it to get

$$\begin{aligned} g(xyz) + g(xzy) &= 2g(xz)\mu(y)g(\iota(y)) + 2g(xy)g(z) - 2g(x)\mu(z)g(y\iota(z)) \\ &= 2g(xz)\mu(y)g/\mu(y) + 2g(xy)g(z) - 2g(x)\mu(z)g(y\iota(z)) \qquad \text{(by (i))} \\ &= 2g(xz)g(y) + 2g(xy)g(z) - 2g(x)(2g(y)g(z) - g(yz)) \\ &= 2g(xz)g(y) + 2g(xy)g(z) + 2g(x)g(yz) - 4g(x)g(y)g(z). \end{aligned}$$

Thus g is a solution of the pre-d'Alembert functional equation. □

Lemma 5.4.2 shows that there exist non-abelian μ-d'Alembert functions, so the studying of it makes sense.

THEOREM 5.4.4. *Let M be a topological monoid and $\iota : M \to M$ an involution. The non-abelian d'Alembert functions $g \in C(M)$ are the functions of the form $g = \frac{1}{2}\operatorname{tr}\rho$ such that $\rho(\iota(x)) = \operatorname{adj}(\rho(x))$ for all $x \in M$. Where ρ is a continuous, algebraically irreducible represrntation of M on $\mathbb{C}^2$.*

PROOF. By Proposition 5.4.3(iii), we know that g is a pre-d'Alembert function. Then it must have the form in Theorem 5.3.1. Then we would try to show $\rho(\iota(x)) = \operatorname{adj}(\rho(x))$. Put $g = \frac{1}{2}\operatorname{tr}\rho$ in Equation (4.1.1) and take the trace

$$\operatorname{tr}(\{\rho(x)[\rho(y) + \rho(\iota(y)) - \operatorname{tr}(\rho(y))I]\} = 0.$$

Corollary A.0.7 shows $\dim\rho(x)$ cannot be zero. So the only case is $\rho(y) + \rho(\iota(y)) - \operatorname{tr}(\rho(y))I = 0$. Then by the definition of adjugate matrix we have $(\operatorname{tr}\rho(y))I = \rho(y) + \operatorname{adj}(\rho(y))$. We can get $\rho(\iota(x)) = \operatorname{adj}(\rho(x))$ for all $x \in M$.
Conversly, let $g = \frac{1}{2}\operatorname{tr}\rho$ and $\rho(\iota) = \operatorname{adj}(\rho)$. Then for all $x, y \in M$, by Example B.0.7, we have,

$$\begin{aligned} g(xy) + g(x\iota(y)) &= \frac{1}{2}\operatorname{tr}\rho(xy) + \frac{1}{2}\operatorname{tr}\rho(x\iota(y)) \\ &= \frac{1}{2}\operatorname{tr}[\rho(xy) + \rho(x\iota(y))] = \frac{1}{2}\operatorname{tr}[\rho(x)\rho(y) + \rho(x)\rho(\iota(y))] \\ &= \frac{1}{2}\operatorname{tr}[\rho(x)(\rho(y) + \rho(\iota(y)))] = \frac{1}{2}\operatorname{tr}[\rho(x)(\rho(y) + \operatorname{adj}(\rho(y))] \\ &= \frac{1}{2}\operatorname{tr}[\rho(x)\operatorname{tr}\iota(y)I] = \frac{1}{2}\operatorname{tr}[\rho(x)I]\operatorname{tr}\iota(y) \\ &= \frac{1}{2}\operatorname{tr}\rho(x)\operatorname{tr}\iota(y) = 2[\frac{1}{2}\operatorname{tr}\rho(x)][\frac{1}{2}\operatorname{tr}\rho(y)] \\ &= 2g(x)g(y). \end{aligned}$$

Thus g is a continuous d'Alembert function. And g is clearly non-abelian by Theorem 5.3.1(iii). □

Finally we turn our attention to d'Alembert's classical functional equation on groups.

THEOREM 5.4.5. *Let G be a topological group and $g \in C(G)$ a non-abelian classical d'Alembert's function. g is in the form $g = \frac{1}{2}tr\rho$ where ρ is a continuous, algebraically irreducible represrntation of G on $\mathbb{C}^2$ such that $\rho(x) \in SL(2,\mathbb{C})$ for all $x \in G$*

PROOF. Theorem 5.3.1(i) shows that. $\square$

APPENDIX A

Matrices

DEFINITION A.0.1. Let $A \in M(n \times n, \mathbb{C})$. The (i,j)-minor of A, denoted M_{ij}, is the *determinant* of the $(n-1) \times (n-1)$ matrix that results from deleting row i and column j of A. The cofactor matrix of A is the $n \times n$ matrix C whose (i,j) entry is the (i,j) cofactor of A, which is the (i,j)-minor times a sign factor:

$$C = ((-1)^{i+j} M_{ij})_{1 \le i,j \le n}.$$

DEFINITION A.0.2. The *adjugate matrix* of A is the transpose of the cofactor matrix C of A,

$$\mathrm{adj}(A) = C^T.$$

PROPOSITION A.0.3. *Let A be a $n \times n$ matrix,* $\mathrm{adj}(A)$ *be the adjugate matrix of A, then*

$$A \, \mathrm{adj}(A) = \det(A) I,$$

where I is the $n \times n$ identity matrix.

DEFINITION A.0.4. The *trace* of a matrix $A = (a_{ij}) \in M(n \times n, \mathbb{C})$, denoted $\mathrm{tr}(A)$, is defined to be $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$.

PROPOSITION A.0.5. *Let A and B are $n \times n$ matrices, and c be a complex number.*
(i) $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$.
(ii) $\mathrm{tr}(cA) = c\,\mathrm{tr}(A)$.
(iii) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Example A.0.6. The adjugate of the 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is $\operatorname{adj}(A) = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$. By direct computations,

$$A \operatorname{adj}(A) = \begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix} = (\det A)I.$$

Here I is the 2×2 identity matrix. And we also have

$$A + \operatorname{adj}(A) = \begin{bmatrix} a_{11}a_{22} + a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22} + a_{12}a_{21} \end{bmatrix} = \operatorname{tr}(A)I.$$

Theorem A.0.7. *(Cayley-Hamilton) Let $A = (a_{ij})$ be an $n \times n$ matrix and I be the $n \times n$ identity matrix, and let p be the characteristic polynomial of A. Then A satisfies its own characteristic polynomial, that is $p(A) = 0$.*

Proof. Refer to [2, Chapter 4, E15] □

APPENDIX B

Matrix-Coefficients of Representations

Let W denote an n-dimensional vector space where $n \geq 1$. Its dual vector space is denoted by $W^\star$, and the value of $w^\star \in W^\star$ on $w \in W$ is denoted by $\langle w^\star, w\rangle$. For any $w \in W$ and $w^\star \in W^\star$ define $w \bigotimes w^\star(v) := \langle w^\star, v\rangle w$ for $v \in W$. If $w_1, w_2, ..., w_n$ is a basis of W and $w_1^\star, w_2^\star, ..., w_n^\star$ is the dual basis of $W^\star$, then $\{w_i \bigotimes w_j^\star | i, j = 1, 2, ..., n\}$ is a basis of $\mathcal{L}(W)$, so any $L \in \mathcal{L}(W)$ can be decomposed by $L = \sum_n^{i,j=1} L_{ij} w_i \bigotimes w_j^\star$, where $L_{ij} \in \mathbb{C}$. So it is obvious that (L_{ij}) is the matrix of the linear operator L with respect to the basis $w_1, w_2, ..., w_n$ of W. Note that $L_{ij} = \langle w_i^\star, Lw_j\rangle$ and $\langle \mathrm{tr}((w_j \bigotimes w_i^\star)L)\rangle = L_{ij}$.

Definition B.0.1. Let W be an n-dimensional vector space where $1 \leq n \leq \infty$. Let X be a set and $F : X \to \mathcal{L}(W)$. A *matrix-coefficient* of F is a function on X of the form $x \mapsto \langle w^\star, F(x)w\rangle, x \in X$, where $w^\star \in W^\star$ and $w \in W$. The vector space of functions on X spanned by the matrix-coefficients of F is called the *space of matrix-coefficients* of F and denoted $C(F)$.

Proposition B.0.2. *Let W denote an n-dimensional vector space where $n \geq 1$. Let X be a set and $F : X \to \mathcal{L}(W)$. Letting $w_1, w_2, ..., w_n$ be a basis of W and $F(x)w_j = \sum_n^{i=1} F_{ij} w_i$, where $F_{ij} : X \to \mathbb{C}$. Then*
(i) each F_{ij}, $i, j = 1, ..., n$, is a matrix-coefficient of F.
(ii) $C(F) = \mathrm{span}\{F_{ij} | i, j = 1, 2, ..., n\}$, so $\dim C(F) \leq n^2$.
(iii) $C(F)$ consists of the functions $x \mapsto tr(AF(x)), x \in X$, where A ranges over $L(W)$.

Proof. (i) It is clear by the definition of matrix-coefficient since $F_{ij} = \mathrm{tr}((w_j \bigotimes w_i^\star)F)$, just in the form of matrix-coefficient.
(ii) It is trivial.

(iii) By(i) and (ii)

$$\begin{aligned} C(F) &= \text{span}\{F_{ij} | i, j = 1, 2, ..., n\} \\ &= \text{span}\{\text{tr}((w_j \bigotimes w_i^\star)F) | i, j = 1, 2, ..., n\} \\ &= \text{span}\{\text{tr}(AF) | A \in \mathcal{L}(W)\} \end{aligned}$$

where $w_1, w_2, ..., w_n$ is a basis of W and $w_1^\star, w_2^\star, ..., w_n^\star$ is the dual basis of $W^\star$. □

LEMMA B.0.3. *Let W denote a finite-dimensional vector space. Let X be a set and $F : X \to \mathcal{L}(W)$. Let furthermore W' be a subspace of W such that $F(x)(W') \subseteq W'$ for all $x \in X$. Define $F' : X \to \mathcal{L}(W')$ by $F'(x)w' := F(x)w'$ for $x \in X$ and $w' \in W'$. Then any matrix-coefficient of F' is a matrix-coefficient of F, and so $C(F') \subseteq C(F)$.*

PROOF. Let $f(x) = \langle \phi, F'(x)w' \rangle, x \in X$, be a matrix-coefficient of F'. So $\phi \in (W')^\star$ and $w' \in W'$. Extending ϕ from the subspace W' of W to a linear functional $w^\star \in W^\star$ on W gives $f(x) = \langle w^\star, F(x)w' \rangle$, which shows that f is a matrix-coefficient of F. □

THEOREM B.0.4. *Let V denote an a finite dimensional vector space. Then there is exactly one topology on V that makes it a Hausdorff topological vector space. Any isomorphism of $\mathbb{C}^n$ onto V is a homeomorphism, when V is equipped with this topology.*

LEMMA B.0.5. *Let W denote a finite-dimensional vector space. Let X be a topological space and $F : X \to \mathcal{L}(W)$. Then F is continuous if and only if each matrix-coefficient of F is continuous.*

PROOF. By Proposition B.0.2, F_{ij} is the coordinates of F on the basis $\{w_j \bigotimes w_i^\star | i, j = 1, 2, ..., n\}$, so F is continuous if and only if every F_{ij} is continuous. □

THEOREM B.0.6. *([14] Burnside 1905). Let W be a finite-dimensional vector space of dimension larger than 1 over an algebraically closed field. If A is an irreducible algebra of linear transformations of W containing I, then $A = \mathcal{L}(W)$.*

Corollary B.0.7. *Let ρ be an irreducible representation of a monoid M on a finite vector space W such that* $\dim W \geq 2$. *Let* $c_{A(x)} := \frac{1}{2}\operatorname{tr}(A\rho(x)), x \in M$. *The map $A \mapsto c_A$ is an isomorphism of $\mathcal{L}(W)$ onto the space $C(\rho)$ of matrix-coefficients of ρ. In particular,* $\dim C(\rho) = (\dim \rho)^2$

Proof. Proposition B.0.2(iii) has showed that $A \mapsto c_A$ is injective. The only we need to show is it is subjective, that is to show if $\operatorname{tr}(A\rho(x)) = 0$ we can get $A = 0$. It is known that $\dim W \geq 2$. Since $\operatorname{span}\{\rho(x)|x \in M\}$ is an algebra containing I, and ρ is irreducible, thus $\operatorname{span}\{\rho(x)|x \in M\} = \mathcal{L}(W)$ by Theorem B.0.6. Then for all $X \in \mathcal{L}(W)$, $\operatorname{tr}(AX) = 0$. Thus $A = 0$. □

Definition B.0.8. Let W be a subspace of the complex-valued functions on a set Y. For any $y \in Y$ define the point-evaluation $ev_y \in W^\star$ by $\langle ev_y, f\rangle := f(y), f \in W$.

Proposition B.0.9. *If W is a finite-dimensional space, and V is a subspace of W and $W \neq V$. Then there exists a nonzero $\phi \in W^\star$ such that for any $v \in V$, $\phi(v) = 0$.*

Proof. Let $v_1, v_2, ..., v_n$ be a basis of V. Since $W \neq V$, $v_1, v_2, ..., v_n$ could be extended to a basis of W, say $v_1, v_2, ..., v_n, v_{n+1}, ..., v_{m+n}, m \geq 1$. Then define $\phi \in W^\star$:

$$\phi(v_i) = \begin{cases} 0, & \text{if} \quad i \neq n+1; \\ 1, & \text{if} \quad i = n+1 \end{cases}$$

Then $\phi \in W^\star$ and $\phi(v) = 0$ for every $v \in V$ but $\phi \neq 0$. □

Lemma B.0.10. *If W is a finite-dimensional subspace of the complex-valued functions on a set Y, then $W^\star = \operatorname{span}\{ev_y|y \in Y\}$. For any $F : X \to \mathcal{L}(W)$ its space of matrix-coefficients $C(F)$ is spanned by the functions $x \mapsto \langle ev_y, F(x)f\rangle$, $x \in X$, where $f \in W$ and $y \in Y$.*

Proof. We have known that $W^\star \subseteq \operatorname{span}\{ev_y|y \in Y\}$, then we only need to show $\operatorname{span}\{ev_y|y \in Y\}$ is not a subspace of $W^\star$. By Proposition B.0.9, there will be a $f \in W^{\star\star}$ such that for any $y \in Y, w^{\star\star}(ev_y) = 0$ with $w^{\star\star} \neq 0$. Consider the natural embedding $i : W \to W^{\star\star}$. i is surjective since there must be some $f \in W$, such that

$w^{\star\star} = i(f)$. But for all $y \in Y, w^{\star\star}(ev_y) = 0$, then we have $f(y) = 0$, thus $f = 0$. But $f \neq 0$ since $i(f) = w^{\star\star} \neq 0$, it is a contradiction. □

PROPOSITION B.0.11. *Let M be a topological monoid. Let W be an R-invariant, finite-dimensional subspace of $C(M)$.*
(i) $R|_W$ is a continuous representation of M on W, and $W \subseteq C(R|_W)$.
(ii) If W is also L-invariant, then $W = C(R|_W)$

PROOF. (i) By Lemma 4.1.1(iii), $R|_W$ is clearly a representation of W. Consider the special matrix-coefficients $x \mapsto \langle ev_y, R(x)f\rangle = f(yx)$ where $y \in G$ and $f \in W$. They are continuous, because G is a topological group and $f \in C(G)$. Since they span the space of all matrix-coefficients (Lemma B.0.10), we refer to Lemma B.0.5 to infer the continuity of $R|_W$. From heve, $R(x)f = f(ex) = f(x)$ we see that any $f \in W$ is a matrix-coefficient, so $W \subseteq C(R|_W)$.
(ii) According to (ii) we shall prove that $W \subset C(R|_W)$. By Lemma B.0.10 the matrix-coefficients $x \mapsto [R(x)f](y)$, where $f \in W$ and $y \in G$, span $C(R|_W)$, so it shows that they belong to W. They are functions on G of the form $L(g)f$, because $[R(x)f](y) = f(yx) = [L(y^{-1})f](x)$. Now, $L(g)f \in W$ for any $g \in G$, because W is L-invariant. □

Bibliography

[1] Aczél János , *Lectures on functional equations and their applications.* Academic Press, (1966).

[2] Artin Michael, *Algebra.* Wiley, Prentice Hall, Englewood Cliffs, NJ (1991).

[3] Bourbaki Nicholas, *Elements of Mathematics. Algebra III.* Springer, (1990).

[4] Davison Thomas M.K., *D'Alembert's functional equation and Chebyshev polynomials.* Annales Academiae Paedagogicae Cracoviensis. 4, Studia Mathematica 1 (2001), 31-38.

[5] Davison Thomas M.K., *D'Alembert's functional equation on topological groups.* Aequationes Mathematicae, 76(1-2), (2008), 33–53.

[6] Dummit David S., Richard M. Foote. *Abstract algebra.* Wiley, Hoboken, NJ (2004).

[7] Fadli B, D. Zeglami and S. Kabbaj, *Solution of several functional equations on abelian groups with involution.* African Mathematical Union and Springer-Verlag GmbH Deutschland (2017).

[8] Folland Gerald B., *A course in abstract harmonic analysis.* CRC Press, Boca Raton (1995).

[9] Friis Peter de Place and Stetkaer Henrik, *On the cosine-sine functional equation on groups.* Aequationes Mathematicae, 64(1-2), (2002), 145–164.

[10] Fulton William and Harris Joe, *Representation theory. A first course.* Springer, (1991).

[11] Hausner Melvin and Schwartz Jacob T., *Lie Groups, Lie Algebras.* Pure and Applied Mathematics, 80. Academic Press, New York-London (1968).

[12] Kannappan Palaniappan, *On quadratic functional equation.* Internat. J. Math.Statist. Sci. 9 (2000), 10–35.

[13] Kannappan Palaniappan, *The Functional Equation* $f(xy) + f(xy - 1) = 2f(x)f(y)$ *for Groups.* Proceedings of the American Mathematical Society, 19(1), (1968), 69–74.

[14] Lam T. Y., *A Theorem of Burnside on Matrix Rings.* The American Mathematical Monthly, 105(7),(1998), 651–653.

[15] Papp F. J., *The D'Alembert Functional Equation.* The American Mathematical Monthly, 92(4), (1985), 273–275.

[16] Penney R. C.and Rukhin A. L., *D'Alembert's Functional Equation on Groups.* American Mathematical Society, Vol. 77, No. 1 (1979), 73-80.

[17] Serre Jean-Pierre and Leonhard L. Scott, *Linear representations of finite groups.* Springer, (1996).

[18] Stetkær Henrik, *Functional Equations on Groups*. World Scientific Publishing Co, Singapore (2013).

[19] Stetkær Henrik, *A variant of d'Alembert's functional equation*. Aequationes Mathematicae, 89(3), (2015), 657-662.

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