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Filter Base Integration

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by

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Abstract

In this paper we seek to investigate and compare several Kurzweil-Henstock type integrals. We present these integrals as special cases of a more general construction and establish key properties required in order to create similar “well behaved” integrals. After establishing basic results (additivity, monotonicity, etc.) for these integrals, we shift our focus to more interesting questions about them. We consider mainly the following topics: additivity, their relationship to the Lebesgue integral, absolute integrability, their relationships with each other, their relationship with differentiation, possible convergence theorems and the establishment of a Fubini Theorem.

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Bradley Charles Howell

Contents

History and Thesis Outline

In the late 19th century and early 20th century, much work was being done in integration theory in order to integrate more and more complex integrands. This eventually culminated in the works of Lebesgue, which did not solve all of the issues, but which had such useful applications that it was eventually embraced by most mathematicians. One of the main downfalls (sometimes seen as an advantage) of the Lebesgue integral, was that it was an absolute integral, in that a function is integrable if and only if its absolute value is integrable. Researchers began working on this issue in many different ways. For example, Perron attacked the problem using majorant and minorant functions and Denjoy using antiderivatives. This time period lead to a multitude of integrals of various complexity, all having their own strengths and weaknesses.

In the 1950's two mathematicians Ralph Henstock and Jaroslav Kurzweil independently introduced a new integral, having many of the desired properties, while still remaining accessible to even 1st year calculus students. This integral has been appropriately termed the Kurzweil-Henstock integral or the generalized Riemann integral due to its similarities with the Riemann integral. This led to a wave of new integrals and interesting questions of extensions and generalizations.

In this thesis, we attempt to gather many of these integrals as a special case of a more general construction. We also aim to establish their properties, relationships to other well known integrals and their relationships with each other.

The first chapter focuses mainly on introductory material. Basic notation of the paper is established, followed by a quick study of filters. These integrals are naturally represented as the limit of a filter so this quick study will pay dividends further on.

The second chapter begins introducing the general construction. The notion of a base is established, followed by the essential properties of bases that will be needed for the investigation. Bases are then used to define differentiation and integration. The remainder of the chapter is devoted to simple consequences of the definitions and alternate forms of the integral.

The third chapter dives into the core of the material. Particular bases of interest are introduced and briefly compared. Essential properties of these bases are then established followed by a rather long investigation into additivity. This investigation leads to further distinctions between bases, since some lead to additive integrals and others do not. The remainder of the chapter focuses on the study of absolute integrability and the relationship of the integrals produced by our bases to the Lebesgue integral. These studies turn out to be one and the same: we discover that the Lebesgue integrable functions are precisely the absolute integrable functions for these bases. Some of our bases turn out to generate absolute integrals giving us alternate representations for the Lebesgue integral and further distinguishes between bases.

The fourth chapter focuses on bases using the notion of regularity. Roughly speaking, regularity is a measure of "how square an interval is" and these bases require a certain level of "squareness". We look into the affects on differentiation and integration when one adjusts this level of "squareness".

The final chapter answers the question: how do we put together bases in product spaces? One could just mash them together without requiring much structure, but then the integral produced by the resulting base will not relate to the integrals of the original bases. This is where the notion of a product base is introduced, leading to a Fubini Theorem for bases with certain properties. The last thing shown in the thesis is a proof of the Fubini Theorem for the Lebesgue integral. We do this by using our Fubini Theorem and the established relationships between our bases and the Lebesgue integral.

Original contributions in the thesis include severe ironing out of the concept non-overlapping introduced in [**4**], along with multiple corrections throughout his work. A completely revamped proof of Pfeffer's famous counter example for additivity 3.3.9. This example is referenced in many papers in the field, [**4**] and [**3**], but there were errors in the details of the proof [**5**]. An original proof of a patching theorem 3.3.13 (and associated results) partially solving this issue of additivity. Also, in most of the unoriginal content within the thesis there has been a concerted effort in presenting more detail or an alternate approach.

CHAPTER 1

Preliminaries

1.1. Notation

Throughout the paper; the assumed norm in $\mathbb{R}^n$ will be the supremum norm; that is, if $x = (x_1, ..., x_n)$, then $||x|| = \max\{|x_1|, ..., |x_n|\}$; when referring to the distance between two points we use the metric induced by this norm. The open ball of radius ϵ centred at x will be denoted $B(x, \epsilon)$. Similarly, $B(A, \epsilon)$ is the ϵ neighborhood of A. For a set E, we will use $\overline{E}$ or $\mathrm{cl}(E)$, E° or $\mathrm{int}(E)$, and $\mathrm{bd}(E)$ to denote the closure, interior, and boundary of E respectively. Generally, the neighborhood system at x will be denoted $\mathcal{U}_x$ and neighborhoods of x will not be assumed open. The power set of X will be denoted $\mathcal{P}(X)$. Intervals, unless otherwise stated, are assumed to be non-degenerate and compact. The Lebesgue measure of a measurable set E will be denoted $\lambda(E)$. The length of an interval I in $\mathbb{R}$ will be denoted either by $\lambda(I)$ or by $|I|$. For any function $f : X \mapsto \mathbb{R}$, we define $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$.

1.2. Filters

Most of this section was taken from [**11**] with slight modifications to write most of it in terms of filterbases. This material is quite common and could be found in most introductory books on topology.

Definition 1.2.1. Filter

Let X be a space and $\mathcal{F}$ be a non-empty family of non-empty subsets of X. We call $\mathcal{F}$ a filter in X if it satisfies the following properties:

(1) if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$ and

(2) if $F_1 \in \mathcal{F}$ and $F_1 \subseteq F_2$, then $F_2 \in \mathcal{F}$.

If $\mathcal{F}$ is a filter, we call a non-empty subfamily $\mathcal{F}_0$ of $\mathcal{F}$ a filterbase for $\mathcal{F}$ if $\mathcal{F} = \{F \subseteq X : F_0 \subseteq F \text{ for some } F_0 \in \mathcal{F}_0\}$. A family $\mathcal{F}_0$ of non-empty sets is a filterbase for some filter if and only if for any $F_1, F_2 \in \mathcal{F}_0$ there is an $F_3 \in \mathcal{F}_0$ such that $F_3 \subseteq F_1 \cap F_2$.

Notice that in a topological space X for any $x \in X$ the neighborhood system $\mathcal{U}_x$ of x is a filter on X. This will be an important fact moving forward, since it gives a connection between the filters on a space and the topology on a space.

Definition 1.2.2. Finer, coarser

A filterbase $\mathcal{F}_1$ is said to be finer than a filterbase $\mathcal{F}_2$ if for every $F_2 \in \mathcal{F}_2$, there exists an $F_1 \in \mathcal{F}_1$ such that $F_1 \subseteq F_2$. We will denote this by $\mathcal{F}_1 \succcurlyeq \mathcal{F}_2$. For filters, one sees that $\mathcal{F}_1 \succcurlyeq \mathcal{F}_2$ if and only if $\mathcal{F}_1 \supseteq \mathcal{F}_2$.

Definition 1.2.3. Convergence of a filter base, cluster point.

A filter base $\mathcal{F}$ on a topological space X is said to converge to a point $x \in X$, denoted $\mathcal{F} \to x$, if $\mathcal{F}$ is finer than the neighborhood system $\mathcal{U}_x$ at x. A point $x \in X$ is said to be a cluster point of $\mathcal{F}$ if each $F \in \mathcal{F}$ meets each neighborhood of x.

Theorem 1.2.4.

A filterbase $\mathcal{F}_1$ clusters at x if and only if there is a filterbase $\mathcal{F}_2$ that is finer than $\mathcal{F}_1$ which converges to x.

Proof. ($\Rightarrow$) If $\mathcal{F}_1$ has x as a cluster point, then $\mathcal{F}_2 = \{U \cap F : U \in \mathcal{U}_x, F \in \mathcal{F}_1\}$ is a filterbase which is finer than $\mathcal{F}_1$ and converges to x.

($\Leftarrow$) Conversely, if $\mathcal{F}_2$ is finer than $\mathcal{F}_1$ and converges to x, then each $F \in \mathcal{F}_1$ and each $U \in \mathcal{U}_x$ belongs to the filter generated by $\mathcal{F}_2$; hence, $F \cap U \neq \emptyset$. □

We will now show that convergence of filterbases is able to describe topological concepts.

Theorem 1.2.5.

If $E \subseteq X$, then $x \in \overline{E}$ if and only if there is a filter $\mathcal{F}$ with $E \in \mathcal{F}$ and $\mathcal{F} \to x$.

Proof. ($\Rightarrow$)If $x \in \overline{E}$, then $\mathcal{C} = \{E \cap U : U \in \mathcal{U}_x\}$ is a filterbase for a filter that contains E which also converges to x.

($\Leftarrow$) If $E \in \mathcal{F} \to x$, then for $U \in \mathcal{U}_x$ we have $U \cap E \in \mathcal{F}$ so that $U \cap E \neq \emptyset$. Therefore, $x \in \overline{E}$. □

Definition 1.2.6. Image filterbase

If $\mathcal{F}$ is a filterbase on X and $f : X \to Y$, then $f(\mathcal{F})$ is the filterbase $\{f(F) : F \in \mathcal{F}\}$.

Theorem 1.2.7.

Let $f : X \to Y$. Then, f is continuous at x if and only if whenever a filterbase $\mathcal{F} \to x$, then $f(\mathcal{F}) \to f(x)$.

Proof. ($\Rightarrow$) Suppose that f is continuous at x and $\mathcal{F} \to x$. Let V be a neighborhood of $f(x)$. Then there is a neighborhood U of x such that $f(U) \subseteq V$. Since $\mathcal{F} \to x$, there is an $F \in \mathcal{F}$ contained in U, so that $f(F) \subseteq V$. Therefore, $f(\mathcal{F}) \to f(x)$.

($\Leftarrow$) Assume the condition is satisfied. Let $\mathcal{F}$ be the filter of all neighborhoods of x. Then $\mathcal{F} \to x$, thus, $f(\mathcal{F}) \to f(x)$. So for every neighborhood V of $f(x)$, there is a neighborhood U of x, such that $f(U) \subseteq V$. Therefore, f is continuous at x.

$\square$

We will write $\lim_{\mathcal{F}} f$, read "the limit of f following $\mathcal{F}$", for the limit of $f(\mathcal{F})$, when it exists. Notice that this limit may not be unique; however, in most of our applications it will be.

Definition 1.2.8. Cauchy filterbase

Let X be a metric space, a filterbase $\mathcal{F} \in X$ is said to be Cauchy, if for any $\epsilon > 0$ there is an $F \in \mathcal{F}$ such that $\operatorname{diam}(F) < \epsilon$. Note that a filterbase is Cauchy if and only if the generated filter is so.

Theorem 1.2.9.

Every convergent filterbase is Cauchy.

Proof. Let $\epsilon > 0$ and suppose that $\mathcal{F}$ is a filterbase converging to x. Then $B(x, \frac{\epsilon}{3})$ contains a member of $\mathcal{F}$ and $\operatorname{diam}(B(x, \frac{\epsilon}{3})) < \epsilon$. $\square$

Theorem 1.2.10.

A metric space X is complete if and only if every Cauchy filterbase in X converges.

Proof. ($\Rightarrow$) Suppose that X is a metric space and $\mathcal{F}$ is a Cauchy filterbase. Choose a sequence $\{F_n\}$ in $\mathcal{F}$ with $\operatorname{diam}(F_n) < \frac{1}{n}$. Without loss of generality we can assume that each F_n is closed since if $F_n \in \mathcal{F}$, then $\overline{F_n} \in \mathcal{F}$ and $\operatorname{diam}(F_n) = \operatorname{diam}(\overline{F}_n)$. We will also assume that $F_n \subseteq F_{n-1} \subseteq ... \subseteq F_1$. This is possible, since if $F_1, F_2 \in \mathcal{F}$, there is an $F_3 \in \mathcal{F}$ such that $F_3 \subseteq F_1 \cap F_2$. Now since $\{F_n\}$ is a nested sequence of closed sets whose diameters are going to 0 and the space is complete, we have that $\bigcap_{n\in\mathbb{N}} F_n = \{x\}$ for some $x \in X$. We will now show that $\mathcal{F} \to x$. Let $U \in \mathcal{U}_x$. Then, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. Choose an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Then $F_n \subseteq B(x, \epsilon) \subseteq U$. Therefore, $\mathcal{F} \to x$.

($\Leftarrow$) Suppose that every Cauchy filterbase in X is convergent in X. Let $\{x_n\}$ be a Cauchy sequence in X and $\epsilon > 0$. Consider the filterbase $\mathcal{F}$ consisting of the sets $F_m = \{x_n : n \geq m\}$. There exists an $N \in \mathbb{N}$ such that for any $n, m \geq N, d(x_n, x_m) < \epsilon$. Therefore, $\operatorname{diam}(F_N) < 2\epsilon$ so that $\mathcal{F}$ is a Cauchy filterbase. Now, by our assumption, $\mathcal{F} \to x$ for some $x \in X$. So, for any neighborhood U of x there exists an $N \in \mathbb{N}$ such that $F_N \subseteq U$. Therefore, for all $n \geq N, x_n \in U$ so that $x_n \to x$. $\square$

Definition 1.2.11. Limit superior, inferior

Let X be a topological space equipped with a complete order $\leq$. For a filterbase $\mathcal{F}$ in X we define the limit superior as

$$\limsup \mathcal{F} = \inf_{F\in\mathcal{F}} \sup F.$$

Similarly, the $\liminf \mathcal{F}$ is simply $\sup_{F\in\mathcal{F}} \inf F$. Generally, for our purposes we will be considering these definitions applied to images of filterbases.

Proposition 1.2.12.

Let $\mathcal{F}$ be a filterbase. Then, $\liminf \mathcal{F} \leq \limsup \mathcal{F}$.

Proof. Let $F_1, F_2 \in \mathcal{F}$, there is an $F_3 \in \mathcal{F}$ such that $F_3 \subseteq F_1 \cap F_2$. In which case

$$\inf F_1 \leq \inf F_3 \leq \sup F_3 \leq \sup F_2.$$

Taking the infimum of the right side over the $F_2 \in \mathcal{F}$ followed by taking the supremum of the left side over the $F_1 \in \mathcal{F}$ we have the result. $\square$

Proposition 1.2.13.

A filterbase $\mathcal{F}$ in $\mathbb{R}$ converges if and only if $\limsup \mathcal{F} = \liminf \mathcal{F}$, *in which case* $\lim \mathcal{F}$ *is the common value.*

Proof. ($\Rightarrow$) Let $\epsilon > 0$ and suppose that $\lim \mathcal{F} = a$. Then, there exists an $F \in \mathcal{F}$ such that $a - \epsilon \leq \inf F \leq \sup F \leq a + \epsilon$ taking suprema and infima we see that $\limsup \mathcal{F} = \liminf \mathcal{F} = \lim \mathcal{F}$.

($\Leftarrow$) Let $\epsilon > 0$ and suppose that $\liminf \mathcal{F} = \limsup \mathcal{F} = a$. Then, there exists $F_1, F_2 \subseteq \mathcal{F}$ such that $\sup F_1 \leq a + \epsilon$ and $\inf F_2 \geq a - \epsilon$. Since $\mathcal{F}$ is a filterbase, there exists an $F_3 \in \mathcal{F}$ such that $F_3 \subseteq F_1 \cap F_2$, in which case $F_3 \subseteq B(a, \epsilon)$, as required.

$\square$

Proposition 1.2.14.

Let $\mathcal{F}_1$, $\mathcal{F}_2$ be filterbases with $\mathcal{F}_1$ finer than $\mathcal{F}_2$. Then, $\limsup \mathcal{F}_1 \leq \limsup \mathcal{F}_2$ *and* $\liminf \mathcal{F}_1 \geq \liminf \mathcal{F}_2$.

PROOF. Let $F_2 \in \mathcal{F}_2$, then there exists an $F_1 \in \mathcal{F}_1$ such that $F_1 \subseteq F_2$. Therefore, $\sup F_1 \leq \sup F_2$. Taking the infimum of the left side over all $F_1 \in \mathcal{F}_1$ we have that $\limsup \mathcal{F}_1 \leq \sup F_1 \leq \sup \mathcal{F}_2$. Finally, taking the infimum of the right side over all $F_2 \in \mathcal{F}_2$ we have the result. The $\liminf$ case is proved similarly.

$\square$

CHAPTER 2

The Base Framework

In this chapter, we will introduce our most basic notation and our general framework for our integration. After doing this, we will show some equivalent approaches to our integral, many of which have been studied by other mathematicians. The material is primarily found in [**4**] but not from the point of view of filters. Most of these definitions, not necessarily to the same generality, can also be found in [**1**], [**5**] and in [**7**].

2.1. Notation

Let X be some non-empty space and $\mathcal{I}$ some non-empty family of subsets of X, sometimes referred to as "(generalized) intervals". Suppose also that we are given a binary relation on $\mathcal{I}$, which we name non-overlapping and denote $I \perp J$. We will assume that $\mathcal{I}$ and the non-overlapping relation satisfy the following property: given any $I_0, ..., I_n \in \mathcal{I}$ with $I_1, ..., I_n \subseteq I_0$, there are $J_1, ..., J_m \in \mathcal{I}$ such that:

$$I_0 = \bigcup_{i=1}^{n} I_i \cup \bigcup_{j=1}^{m} J_j \tag{1}$$

and for any $i \in \{1, .., n\}, j_1, j_2 \in \{1, ..., m\}$ we have $I_i \perp J_{j_1}$ and $J_{j_1} \perp J_{j_2}$ for $j_1 \neq j_2$. We will also assume that for $I, J, K \in \mathcal{I}$

$$I \perp J \text{ and } K \subseteq I \implies K \perp J; \tag{2}$$

$$I \cap J = \emptyset \implies I \perp J; \tag{3}$$

$$I \perp J \implies \nexists K \in \mathcal{I} \text{ such that } K \subseteq I \cap J. \tag{4}$$

In the topological setting, unless otherwise stated, the term non-overlapping will simply mean that the intersection of the interiors of the sets is empty.

Definition 2.1.1. Base

A non-empty family $\mathcal{B} \subseteq \mathcal{P}(X \times \mathcal{I})$ is called a base on X. For $(x, I) \in X \times \mathcal{I}$, x is referred to as a tag. Thus, any base is a collection of families consisting of "intervals" and their associated tags. Any base containing the empty set will be termed trivial and we assume that all bases given a priori are non-trivial.

Definition 2.1.2. Anchored in E, within E

Let $\mathcal{B}$ be a base, $\beta \in \mathcal{B}$ and $E \subseteq X$. We define:

$$\beta[E] = \{(x, I) \in \beta : x \in E\}, \text{ which we call } \beta \text{ anchored in } E,$$

$$\beta(E) = \{(x, I) \in \beta : I \subseteq E\}, \text{ which we call } \beta \text{ within } E,$$

$$\mathcal{B}[E] = \{\beta[E] : \beta \in \mathcal{B}\}, \text{ which we call } \mathcal{B} \text{ anchored in } E \text{ and}$$

$$\mathcal{B}(E) = \{\beta(E) : \beta \in \mathcal{B}\}, \text{ which we call } \mathcal{B} \text{ within } E.$$

Thus, anchoring in E puts the tags in E and in the other case, the entire "interval" is contained in E.

Definition 2.1.3. Finer base, equivalent base

As with filterbases, a base $\mathcal{B}$ is said to be finer than a base $\mathcal{B}'$ or $\mathcal{B}'$ is coarser than $\mathcal{B}$, if for every $\beta' \in \mathcal{B}'$ there is a $\beta \in \mathcal{B}$ such that $\beta \subseteq \beta'$. We denote this by $\mathcal{B} \succcurlyeq \mathcal{B}'$. If $\mathcal{B} \preccurlyeq \mathcal{B}'$ and $\mathcal{B}' \preccurlyeq \mathcal{B}$, then we will call $\mathcal{B}$ and $\mathcal{B}'$ equivalent which we denote $\mathcal{B} \approx \mathcal{B}'$.

Definition 2.1.4. Filtering

A base $\mathcal{B}$ is filtering if for every $\beta_1, \beta_2 \in \mathcal{B}$, there exists a $\beta \in \mathcal{B}$ such that $\beta \subseteq \beta_1 \cap \beta_2$.

Notice that a filtering base $\mathcal{B}$ is simply a filterbase in the space $\mathcal{P}(X \times \mathcal{I})$. This connection will allow us to use limiting processes.

Definition 2.1.5. $\mathcal{B}$ ignores a point

We say that $\mathcal{B}$ ignores a point $x \in X$ if there is a $\beta \in \mathcal{B}$ such that $\beta[\{x\}] = \emptyset$.

Unless otherwise stated, we will assume that our bases ignore no point and are filtering. This will ensure that each $\mathcal{B}[\{x\}]$ is also a filterbase, so that the definition of derivative 2.1.6 will become meaningful.

Definition 2.1.6. Derivative

Let $F, G : X \times \mathcal{I} \mapsto \mathbb{R}$. We define the $\mathcal{B}$-derivative of F with respect to G as a number $D_{\mathcal{B}}F_G(x)$, if it exists, such that for any $\epsilon > 0$, there exists a $\beta \in \mathcal{B}$ such that if $(x, I) \in \beta[\{x\}]$,

$$\left| D_{\mathcal{B}}F_G(x) - \frac{F(x,I)}{G(x,I)} \right| < \epsilon.$$

Thus,

$$D_{\mathcal{B}}F_G(x) = \lim_{\mathcal{B}[\{x\}]} \frac{F}{G}.$$

Given an "interval" function G we will generally write $G(I)$ rather than $G(x, I)$.

Definition 2.1.7. Lower and upper derivatives

We define the upper and lower derivatives of F with respect to G at x as

$$\overline{D}_{\mathcal{B}}F_G(x) = \limsup_{\mathcal{B}[\{x\}]} \frac{F}{G} \text{ and } \underline{D}_{\mathcal{B}}F_G(x) = \liminf_{\mathcal{B}[\{x\}]} \frac{F}{G}.$$

Since $\mathcal{B}[\{x\}]$ is a filterbase, it follows from 1.2.12 and 1.2.13 that

(1) $\overline{D}_{\mathcal{B}}F_G(x) \geq \underline{D}_{\mathcal{B}}F_G(x)$

(2) $D_{\mathcal{B}}F_G(x)$ exists if and only if $\overline{D}_{\mathcal{B}}F_G(x) = \underline{D}_{\mathcal{B}}F_G(x)$, in which case all three are equal.

An extensive study of these sorts of derivatives can be found in [**9**] and [**10**]. This reference also uses many of the bases introduce further on and even some that aren't introduced.

Corollary 2.1.8.

If $\mathcal{B}$ and $\mathcal{B}'$ are equivalent bases, then $\overline{D}_{\mathcal{B}'}F_G(x) = \overline{D}_{\mathcal{B}}F_G(x)$ and $\underline{D}_{\mathcal{B}}F_G(x) = \underline{D}_{\mathcal{B}'}F_G(x)$. Furthermore, the existence of the derivative with respect to either base implies the existence of the derivative with respect to the other base, in which case they are equal.

Definition 2.1.9. Division, partition

A finite set $\mathcal{D} \subseteq \mathcal{I}$ is called a division if its elements are non-overlapping. A partition (also known as a tagged partition) is a finite family $\pi \in \mathcal{P}(X \times \mathcal{I})$ such that the set $\{I : (x, I) \in \pi\}$ forms a division and, for each I in this set, the x such that $(x, I) \in \pi$ is unique. For π a partition we define $\bigsqcup \pi = \bigcup_{(x,I)\in\pi} I$.

We call $\mathcal{D}$ a division of $I \in \mathcal{I}$ if $\bigcup \mathcal{D} = I$; similarly, we call π a partition of $I \in \mathcal{I}$ if $\bigsqcup \pi = I$.

Let $F : X \times \mathcal{I} \mapsto \mathbb{R}$ and π be a partition. Then, we will write

$$F(\pi) = \sum_{(x,I)\in\pi} F(x, I).$$

This is a generalization of the concept of Riemann sum. Similarly if $H : \mathcal{I} \mapsto \mathbb{R}$ and $\mathcal{D}$ is a division, then we write

$$H(\mathcal{D}) = \sum_{I\in\mathcal{D}} H(I) \text{ and } H(\pi) = \sum_{(x,I)\in\pi} H(I).$$

Definition 2.1.10. Partitioning property

A base $\mathcal{B}$ is said to have the partitioning property if for every $I \in \mathcal{I}$ and $\beta \in \mathcal{B}$ there is a partition $\pi \subseteq \beta$ of I.

Notice that if $\mathcal{B}$ is finer than $\mathcal{B}'$ and $\mathcal{B}$ has the partitioning property, then $\mathcal{B}'$ has the partitioning property.

Proposition 2.1.11.

Let $I_0 \in \mathcal{I}$ and let $\mathcal{B}$ be a base with the partitioning a property. Then, any partition $\pi \subseteq \beta \in \mathcal{B}$ in I_0 can be extended to a partition $\hat{\pi} \subseteq \beta$ of I_0.

Proof. By the assumption (1) on the family $\mathcal{I}$, $I \setminus \bigsqcup \pi = \bigcup_{i=1}^{m} K_i$, where $K_i \in \mathcal{I}$ are non-overlapping with each other and the "intervals" in π. Since $\mathcal{B}$ has the partitioning property there are partitions $\pi \subseteq \beta$ of K_i for $i = 1, ..., m$. Let $\hat{\pi} = \pi \cup \bigcup_{i=1}^{m} \pi_i$. Then, $\hat{\pi} \subseteq \beta$ is a partition of I_0. $\square$

We will use this result freely, generally without reference. In the general theory, all bases will be assumed to have the partitioning property and, as mentioned before, be filtering.

Definition 2.1.12. Henstock integral

Let $I_0 \in \mathcal{I}$ and $F : X \times \mathcal{I} \mapsto \mathbb{R}$. We define the Henstock integral of F with respect to $\mathcal{B}$ over I_0 as a number $(\mathcal{B}) \int_{I_0} F$, if it exists, such that for every $\epsilon > 0$, there exists a $\beta \in \mathcal{B}$ such that for every partition $\pi \subseteq \beta$ of I_0,

$$\left| F(\pi) - (\mathcal{B}) \int_{I_0} F \right| \leq \epsilon.$$

Thus, the integral can be interpreted as the limit of the image under F of the filterbase $\mathcal{F}$ whose elements are of the form $\{\pi \subseteq \beta : \ \pi \text{ a partition of } I_0\}$, $\beta \in \mathcal{B}$. From this fact, we obtain the uniqueness, linearity, and monotonicity of the integral; moreover, the integral exists if and only if the image filterbase $F(\mathcal{F})$ is Cauchy.

If I_0 is the union of a division the $(\mathcal{B}) \int_{I_0} f$ is defined in the same way. This modification will be used in the study of the triangular base 3.1.13.

Definition 2.1.13. Upper, lower Henstock integrals

For $F : X \times \mathcal{I} \mapsto \mathbb{R}$ we define the upper and lower Henstock integrals on I_0 to be

$$(\mathcal{B}) \overline{\int}_{I_0} F = \inf_{\beta \in \mathcal{B}} \sup_{\pi \subseteq \beta} F(\pi) \text{ and } (\mathcal{B}) \underline{\int}_{I_0} F = \sup_{\beta \in \mathcal{B}} \inf_{\pi \subseteq \beta} F(\pi),$$

where π runs over partitions of $I_0 \in \mathcal{I}$. The upper and lower integrals can be seen as the $\limsup$ and $\liminf$ of the image filterbase mentioned in the previous definition.

Proposition 2.1.14.

The integral $(\mathcal{B})\int_{I_0} F$ *exists if and only if* $(\mathcal{B})\overline{\int}_{I_0} F = (\mathcal{B})\underline{\int}_{I_0} F$, *in which case they are all equal.*

Proof. Again, this is a special case of proposition 1.2.13. $\square$

Proposition 2.1.15.

If $\mathcal{B}$ *is finer than* $\mathcal{B}'$, *then*

$$(\mathcal{B}')\overline{\int}_{I_0} F \geq (\mathcal{B})\overline{\int}_{I_0} F \text{ and } (\mathcal{B})\underline{\int}_{I_0} F \geq (\mathcal{B}')\underline{\int}_{I_0} F.$$

Proof. This is just a special case of proposition 1.2.14 $\square$

Corollary 2.1.16.

If $\mathcal{B}$ *is finer than* $\mathcal{B}'$ *and* F *is* $\mathcal{B}'$ *integrable on* I_0, *then* F *is also* $\mathcal{B}$ *integrable on* I_0.

Corollary 2.1.17.

If $\mathcal{B}$ *and* $\mathcal{B}'$ *are equivalent, then* $(\mathcal{B}')\overline{\int}_{I_0} F = (\mathcal{B})\overline{\int}_{I_0} F$ *and* $(\mathcal{B}')\underline{\int}_{I_0} F = (\mathcal{B})\underline{\int}_{I_0} F$. *Furthermore, if* F *is integrable with respect to either of the bases, then it is integrable with respect to the other base and to the same value.*

From now on, if there is only one base $\mathcal{B}$ involved, we often suppress it in the notation. If we say that F is Henstock integrable, it will be understood to be with respect to this base.

PROPOSITION 2.1.18.

A function $F : X \times \mathcal{I}$ is Henstock integrable on $I_0 \in \mathcal{I}$ if and only if F is Henstock integrable on every "subinterval" of I_0.

PROOF. ($\Leftarrow$) Is immediate.

($\Rightarrow$) Suppose that F is Henstock integrable on $I_0 \in \mathcal{I}$. Let $\epsilon > 0$ and $I \in \mathcal{I}$ with $I \subseteq I_0$. There exists a $\beta \in \mathcal{B}$ such that for any partition $\pi \subseteq \beta$ of I_0,

$$|F(\pi) - \int_{I_0} F| \leq \epsilon.$$

Let $\pi_1, \pi_2 \subseteq \beta$ be partitions of I. We may extend these partitions in the same way to form partitions $\hat{\pi}_1, \hat{\pi}_2 \subseteq \beta$ of I_0. Therefore,

$$\begin{aligned} |F(\pi_1) - F(\pi_2)| &= |F(\hat{\pi}_1) - F(\hat{\pi}_2)| \\ &\leq |F(\hat{\pi}_1) - \int_{I_0} F| + |\int_{I_0} F - F(\hat{\pi}_2)| \\ &\leq 2\epsilon. \end{aligned}$$

Thus the filterbase used in the definition of the Henstock integral over I is Cauchy and must therefore converge. □

PROPOSITION 2.1.19.

Let $F : X \times \mathcal{I} \to \mathbb{R}$, $I_0 \in \mathcal{I}$ and $\mathcal{B}$ be a base with the partitioning property. If F is Henstock integrable on I_0 and $\mathcal{D}$ is a division in I_0 with $\bigcup \mathcal{D} \in \mathcal{I}$. Then,

$$\int_{\bigcup \mathcal{D}} F = \sum_{I \in \mathcal{D}} \int_I F.$$

Proof. By (2.1.18), F is integrable over subintervals of I_0. We may as well assume $I_0 = \bigcup \mathcal{D}$, that is, that $\mathcal{D} = \{I_1, ..., I_m\}$ is a division of I_0.

Let $\epsilon > 0$. For each $i = 0, 1, \ldots, m$, there is a $\beta_i \in \mathcal{B}$ such that if $\pi \subseteq \beta_i$ is a partition, then

$$|\sum_{(x,I)\in\pi} F(x, I) - \int_{I_i} F| \leq \epsilon/m.$$

Since $\mathcal{B}$ is filtering, we can replace these by one β. For $i = 1, ..., m$ let $\pi_i \subseteq \beta$ be a partition of I_i. Notice that now $\pi = \bigcup_{i=1}^m \pi_i \subseteq \beta$ is a partition of I_0 contained in β_0 and $F(\pi) = \sum_{i=1}^m F(\pi_i)$. Thus,

$$\begin{aligned}
&|\int_{I_0} F - \sum_{I\in\mathcal{D}} \int_I F| \\
\leq &|\int_{I_0} F - F(\pi)| + |\sum_{i=1}^m \int_{I_i} F - F(\pi_i)| \\
\leq &2\epsilon.
\end{aligned}$$

□

Theorem 2.1.20. *Saks Henstock lemma*

Let F be a Henstock integrable function on $I_0 \in \mathcal{I}$. For $\epsilon > 0$, let $\beta_\epsilon \in \mathcal{B}$ be such that if $\pi \subseteq \beta_\epsilon$ is a partition of I_0,

$$|F(\pi) - \int_{I_0} F| \leq \epsilon.$$

Then, for any partition $\pi \subseteq \beta_\epsilon(I_0)$,

$$|\sum_{(x,I)\in\pi} F(x, I) - \int_I F| \leq \epsilon.$$

Proof. Let $\alpha > 0$ and $\pi_0 \subseteq \beta_\epsilon(I_0)$ be a partition. By the fundamental assumptions (1) on $\mathcal{I}$, $I_0 \setminus \bigsqcup \pi_0$ can be written as the union of $K_1, ..., K_m \in \mathcal{I}$, where the K_i are non-overlapping with each other and the "intervals" in π_0. By 2.1.18, F is integrable on each K_i and since $\mathcal{B}$ is filtering, there is a $\beta_1 \in \mathcal{B}$ such that if $\pi \subseteq \beta_1$ is a partition of K_i,

$$|F(\pi) - \int_{K_i} F| \leq \frac{\alpha}{m} \text{ and } \beta_1 \subseteq \beta_\epsilon.$$

Find partitions $\pi_i \subseteq \beta_1$ of K_i for $i = 1, .., m$. Let $\hat{\pi} = \pi \cup \bigcup_{i=1}^{m} \pi_i$. Then $\hat{\pi} \subseteq \beta$ and is a partition of I_0. Therefore, using 2.1.19,

$$\begin{aligned}
&|\sum_{(x,I)\in\pi} F(x,I) - \int_I F| \\
=&|\sum_{(x,I)\in\hat{\pi}} F(x,I) - \sum_{i=1}^{m} F(\pi_i) - \int_{I_0} F + \sum_{i=1}^{m} \int_{K_i} F| \\
\leq&|\sum_{(x,I)\in\hat{\pi}} F(x,I) - \int_{I_0} F| + \sum_{i=1}^{m} |F(\pi_i) - \int_{K_i} F| \\
&\leq \epsilon + \alpha.
\end{aligned}$$

Since α was chosen arbitrarily the result follows. □

Corollary 2.1.21.

Let F be Henstock integrable on $I_0 \in \mathcal{I}$. For $\epsilon > 0$, let $\beta_\epsilon \in \mathcal{B}$ be such that if $\pi \subseteq \beta_\epsilon$ is a partition of I_0,

$$|F(\pi) - \int_{I_0} F| \leq \epsilon.$$

Then, for any partition $\pi \subseteq \beta_\epsilon(I_0)$,

$$\sum_{(x,I)\in\pi} |F(x,I) - \int_I F| \leq 2\epsilon.$$

Proof. Let $\pi \subseteq \beta_\epsilon(I_0)$ be a partition. We define

$$\pi^+ = \{(x, I) \in \pi : F(x, I) - \int_I F \geq 0\} \text{ and}$$
$$\pi^- = \{(x, I) \in \pi : F(x, I) - \int_I F < 0\}.$$

By the Saks Henstock lemma 2.1.20 we have

$$\sum_{(x,I)\in\pi^+} |F(x, I) - \int_I F| = |\sum_{(x,I)\in\pi^+} (F(x, I) - \int_I F)| \leq \epsilon \text{ and}$$
$$\sum_{(x,I)\in\pi^-} |F(x, I) - \int_I F| = |\sum_{(x,I)\in\pi^-} (F(x, I) - \int_I F)| \leq \epsilon.$$

Therefore,

$$\sum_{(x,I)\in\pi} |F(x, I) - \int_I F| = \sum_{(x,I)\in\pi^+} |F(x, I) - \int_I F| + \sum_{(x,I)\in\pi^-} |F(x, I) - \int_I F|$$
$$\leq 2\epsilon.$$

□

2.2. Alternate Definitions of the Henstock Integral

Definition 2.2.1. Variation of F over β

Let $\mathcal{B}$ be a base and $F : X \times \mathcal{I} \mapsto \mathbb{R}$. The variation of F over $\beta \in \mathcal{B}$ is defined as

$$V(F, \beta) = \sup_{\pi\subseteq\beta} |F|(\pi)$$

where π runs over all partitions contained in β.

Similarly, we define the variation of F over $\mathcal{B}$ to be:

$$V(F, \mathcal{B}) = \inf_{\beta\in\mathcal{B}} V(F, \beta) = \inf_{\beta\in\mathcal{B}} \sup_{\pi\subseteq\beta} |F|(\pi),$$

where π once again runs over partitions contained in β.

Definition 2.2.2. Variational measure, inner variation

The variational measure $\mathcal{B}$-VF is defined on all subsets E of X by

$$\mathcal{B}\text{-}VF(E) = V(F, \mathcal{B}[E]),$$

which we call the variation of E.

The inner variation $\mathcal{B}$-V_F is defined on all elements I of $\mathcal{I}$ by

$$\mathcal{B}\text{-}V_F(I) = V(F, \mathcal{B}(I)),$$

which we call the inner variation of I.

Note the distinction in bracketing types between the definition of variational measure (VF) and inner variation of F (V_F). The first uses square brackets, which forces tags to be inside the set E. The second definition uses round brackets meaning that the sets associated with the tags are contained inside I. Often times, if the base in use is clear, we will simply write $VF(E)$ rather than $\mathcal{B}$-$VF(E)$ and $V_F(I)$ rather than $\mathcal{B}$-$V_F(I)$.

Proposition 2.2.3.

Let $E_1, E_2 \subseteq X$ and $F : X \times \mathcal{I} \to \mathbb{R}$. Then,

(1) *if $E_1 \subseteq E_2 \subseteq X$, $VF(E_1) \leq VF(E_2)$ (monotonicity),*

(2) *$VF(E_1 \cup E_2) \leq VF(E_1) + VF(E_2)$.*

Proof. The first statement is immediate. Thus, we may assume for the second that that E_1 and E_2 are disjoint. Let $\beta_1 \in \mathcal{B}$ such that $V(F, \beta_1[E_1]) \leq VF(E_1) + \epsilon$

and $\beta_2 \in \mathcal{B}$ such that $V(F, \beta_2[E_2]) \leq VF(E_2) + \epsilon$. Since $\mathcal{B}$ is filtering, we may choose a $\beta \in \mathcal{B}$ such that $\beta \subseteq \beta_1 \cap \beta_2$. Then, for each partition $\pi \subseteq \beta$, since we are summing over disjoint sets,

$$|F|(\pi[E_1 \cup E_2]) = |F|(\pi[E_1]) + |F|(\pi[E_2]) \leq VF(E_1) + VF(E_2) + 2\epsilon.$$

Taking the supremum over all such $\pi \subseteq \beta$ and then taking the infimum over all $\beta \in \mathcal{B}$ gives the result.

$\square$

Definition 2.2.4. Pointwise character

A base $\mathcal{B}$ is said to have pointwise character if, whenever $\beta_x \in \mathcal{B}$, for each $x \in X$, there is one $\beta \in \mathcal{B}$ such that $\beta[\{x\}] \subseteq \beta_x$ for every $x \in X$.

Definition 2.2.5. σ-local character

A base $\mathcal{B}$ is said to have σ-local character if for any sequence $\{X_n\}$ of disjoint subsets of X, and any sequence $\{\beta_n\}_{n \in \mathbb{N}} \subseteq \mathcal{B}$ there is a $\beta \in \mathcal{B}$ such that $\beta[X_n] \subseteq \beta_n[X_n]$ for each $n \in \mathbb{N}$.

The pointwise character of a base is used to gather properties that occur locally and transform them into properties that occur globally.

Proposition 2.2.6.

Any base of pointwise character is of σ-local character.

Proof. Let $\{X_n\}$ be a sequence of disjoint subsets of X and for each $n \in \mathbb{N}$ let $\beta_n \in \mathcal{B}[X_n]$. Let $\beta_x = \beta_n[\{x\}]$ for each $x \in X_n$ and by pointwise character choose a $\beta \in \mathcal{B}$ such that for all $x \in X$, $\beta[\{x\}] \subseteq \beta_x$. Then $\beta[X_n] = \bigcup_{x \in X_n} \beta[\{x\}] \subseteq \bigcup_{x \in X_n} \beta_n[\{x\}] = \beta_n$.

$\square$

Proposition 2.2.7.

Suppose $\mathcal{B}$ is of σ-local character and that $F : X \times \mathcal{I} \mapsto \mathbb{R}$. Then for any sequence $\{E_n\}$ of subsets of X and $E_0 \subseteq \bigcup_{n \in \mathbb{N}} E_n$,

$$VF(E_0) \leq \sum_{n=1}^{\infty} VF(E_n).$$

Thus for $\mathcal{B}$ of σ-local character, VF becomes an outer measure.

Proof. By the monotonicity of VF, we may assume that the E_n are disjoint. Let $\epsilon > 0$. For any $n \in \mathbb{N}$ there is a $\beta_n \in \mathcal{B}$ such that $V(F, \beta_n[E_n]) \leq VF(E_n) + \frac{\epsilon}{2^n}$. Since $\mathcal{B}$ is of σ-local character there is a $\beta \in \mathcal{B}$ such that $\beta[E_n] \subseteq \beta_n$ for all $n \in \mathbb{N}$. Let $\pi \subseteq \beta[E_0]$ be a partition, then

$$\begin{aligned} |F|(\pi) &= \sum_{n=1}^{\infty} |F|(\pi[E_n]) \\ &\leq \sum_{n=1}^{\infty} V(F, \beta_n[E_n]) \\ &\leq \sum_{n=1}^{\infty} VF(E_n) + \frac{\epsilon}{2^n} \\ &= \sum_{n=1}^{\infty} VF(E_n) + \epsilon. \end{aligned}$$

Taking the supremum over all $\pi \subseteq [E_0]$ on the left side, followed by the infimum over all $\beta \in \mathcal{B}$ we have the result. $\square$

Definition 2.2.8. Additive, subadditive, superadditive, 2-additive

Let $H : \mathcal{I} \mapsto \mathbb{R}$. Recall that for a division $\mathcal{D}$, $H(\mathcal{D}) = \sum_{I \in \mathcal{D}} H(I)$. Then we say that

(1) H is additive if $H(\mathcal{D}) = H(I)$ for any $I \in \mathcal{I}$ and any division $\mathcal{D}$ of I,

(2) H is subadditive if $H(\mathcal{D}) \geq H(I)$ for any $I \in \mathcal{I}$ and any division $\mathcal{D}$ of I and

(3) H is superadditive if $H(\mathcal{D}) \leq H(I)$ for any $I \in \mathcal{I}$ and any division $\mathcal{D}$ of I.

We will denote the family of all additive functions by $\mathcal{A}$, the family of all subadditive functions by $\underline{\mathcal{A}}$ and the family of all superadditive functions by $\overline{\mathcal{A}}$. When restricting our attention to functions inside an "interval" I_0, we will write $\mathcal{A}(I_0)$ for the family of additive functions on I_0. Similar conventions will be used for superadditive and subadditive functions. A function H will be called 2-additive if it is additive for all divisions containing only 2 elements.

Definition 2.2.9. Variationally equivalent

Let $F_1, F_2 : X \times \mathcal{I} \mapsto \mathbb{R}$, we say that F_1 and F_2 are variationally equivalent on $I_0 \in \mathcal{I}$ if, for every $\epsilon > 0$ there is a $\beta \in \mathcal{B}$ and a superadditive function $\varphi : \mathcal{I} \mapsto \mathbb{R}^+$ such that $\varphi(I_0) < \epsilon$ and for every $(x, I) \in \beta(I_0)$,

$$|F_1(x, I) - F_2(x, I)| \leq \varphi(I).$$

We say that F_1, F_2 are variationally equivalent if they are variationally equivalent on every $I_0 \in \mathcal{I}$. We denote this by $F_1 \approx F_2$.

Proposition 2.2.10.

For $F_1, F_2 : X \times \mathcal{I} \mapsto \mathbb{R}$,

$$F_1 \approx F_2 \text{ on } I_0 \iff V(F_1 - F_2, \mathcal{B}(I_0)) = 0.$$

Proof. ($\Rightarrow$) Let $\epsilon > 0$, choose a $\beta \in \mathcal{B}$ and a $\varphi : \mathcal{I} \mapsto \mathbb{R}^+$ as in the definition of variationally equivalent 2.2.9. Let $\pi \subseteq \beta(I_0)$ be a partition. Then,

$$\sum_{(x,I)\in\pi} |F_1 - F_2|(x, I) \leq \sum_{(x,I)\in\pi} \varphi(I) \leq \varphi(I_0) < \epsilon.$$

($\Leftarrow$) Suppose $V(F_1 - F_2, \mathcal{B}(I_0)) = 0$ and let $\epsilon > 0$. Choose a $\beta \in \mathcal{B}$ such that $V(F_1 - F_2, \beta(I_0)) \leq V(F_1 - F_2, \mathcal{B}(I_0)) + \epsilon = \epsilon$, then $I \mapsto V(F_1 - F_2, \beta(I))$ is the required function. Superadditivity stems from the fact that if $\pi_1 \subseteq \beta$ is partition in I_1 and $\pi_2 \subseteq \beta$ is a partition in I_2, where I_1 and I_2 are non-overlapping elements of $\mathcal{I}$. Then, $\pi_1 \cup \pi_2$ is a partition in $I_1 \cup I_2$. Thus if $I_1 \cup I_2 \in \mathcal{I}$,

$$(F_1 - F_2)(\pi_1) + (F_1 - F_2)(\pi_2) = F(\pi_1 \cup \pi_2) \leq V(F_1 - F_2, \beta(I_1 \cup I_2)).$$

Taking the supremum of all such partitions $\pi_1 \subseteq \beta$ and $\pi_2 \subseteq \beta$ we have superadditivity.

$\square$

Lemma 2.2.11.

Suppose $H_1, H_2 : \mathcal{I} \mapsto \mathbb{R}$ are additive functions and $H_1 \approx H_2$ with respect to a base $\mathcal{B}$ with the partitioning property. Then $H_1 = H_2$.

Proof. Let $\epsilon > 0$ and $I_0 \in \mathcal{I}$. There is a $\beta \in \mathcal{B}$ and a superadditive function φ such that $\varphi(I_0) < \epsilon$ and for each $(x, I) \in \beta(I_0)$, $|H_1(I) - H_2(I)| \leq \varphi(I)$. Let $\pi \subseteq \beta$

be a partition of I_0. Then,

$$\begin{aligned}
|H_1(I_0) - H_2(I_0)| &\leq \sum_{(x,I)\in\pi} |H_1(I) - H_2(I)| \\
&\leq \sum_{(x,I)\in\pi} \varphi(I) \\
&\leq \varphi(I_0) \\
&< \epsilon.
\end{aligned}$$

□

DEFINITION 2.2.12. Variational integral

If there exists an additive function that is variationally equivalent to F, then it is called the variational integral of F.

COROLLARY 2.2.13.

By the previous lemma if $\mathcal{B}$ has the partitioning property, then the variational integral of $F : X \times \mathcal{I} \mapsto \mathbb{R}$ is uniquely defined.

This theorem was shown in [**4**] and is very convenient. It shows equivalent definitions for the Henstock integral. The most handy of which is the equivalence involving the variation.

THEOREM 2.2.14. *Alternate definitions of the Henstock integral*

Let $\mathcal{B}$ have the partitioning property and be filtering. Let $F : X \times \mathcal{I} \mapsto \mathbb{R}$ and $I_0 \in \mathcal{I}$. Then the following are equivalent:

(1) *F is Henstock integrable on I_0;*

(2) *For every* $\epsilon > 0$ *there exists a* $\beta \in \mathcal{B}(I_0)$ *such that for every* $I \subseteq I_0, I \in \mathcal{I}$ *and for every partition* $\pi \subseteq \beta$ *of* I,

$$\left| \int_I F - F(\pi) \right| < \epsilon;$$

(3) *There is an additive function* H *such that* $V(H - F, \mathcal{B}(I_0)) = 0$;

(4) *There is an additive function* $H \approx F$ *on* I_0;

(5) *For every* $\epsilon > 0$ *there is a* $\beta \in \mathcal{B}$, *an* $A \in \overline{\mathcal{A}}$ *and a* $B \in \underline{\mathcal{A}}$ *such that*

$$A(I_0) - B(I_0) < \epsilon$$

and for every $(x, I) \in \beta(I_0)$,

$$A(I) \geq F(x, I) \geq B(I).$$

Proof. (1) $\iff$ (2) Follows immediately from the Saks Henstock lemma 2.1.20 and proposition 2.1.19.

(3) $\iff$ (4) Follows from proposition 2.2.10.

(2) $\Rightarrow$ (3) Let $\epsilon > 0$ and define $H(I) = \int_I F$ for any $I \in \mathcal{I}$ with $I \subseteq I_0$. The fact that this function is additive was done in proposition 2.1.19. And the fact that $V(H - F, \mathcal{B}(I_0)) = 0$ was done in corollary 2.1.21.

(4) $\Rightarrow$ (5) Suppose that $H \in \mathcal{A}(I_0)$ such that $H \approx F$ on I_0. Let $\epsilon > 0$, choose a $\beta \in \mathcal{B}(I_0)$ and $\varphi \in \overline{\mathcal{A}}$ so that for all $(x, I) \in \beta$,

$$|H(I) - F(x, I)| \leq \varphi(I) \text{ and } \varphi(I_0) < \epsilon.$$

For $I \subseteq I_0$ we define $A(I) = \varphi(I) + H(I)$ and $B(I) = -\varphi(I) + H(I)$. Then clearly $A \in \overline{\mathcal{A}}, B \in \underline{\mathcal{A}}$ and $A(I_0) - B(I_0) \leq 2\epsilon$.

Finally, for $(x, I) \in \beta$,

$$A(I) = \varphi(I) + H(I) \geq F(x, I) \geq H(I) - \varphi(I) = B(I).$$

(5) $\Rightarrow$ (3) Let $\overline{\mathcal{A}}_F$ be the set of all superadditive functions A on $\mathcal{I}(I_0)$ for which there exists a $\beta \in \mathcal{B}(I_0)$ such that for all $(x, I) \in \beta$, $A(I) \geq F(x, I)$. Put

$$\overline{H}(J) = \inf\{A(J) : A \in \overline{\mathcal{A}}_F\}.$$

Let $\underline{\mathcal{A}}_F$ be the set of all subadditive functions B on $\mathcal{I}(I_0)$ for which which there exists a $\beta \in \mathcal{B}(I_0)$ such that for all $(x, I) \in \beta$, $B(I) \leq F(x, I)$. Put

$$\underline{H}(J) = \sup\{B(J) : B \in \underline{\mathcal{A}}_F\}.$$

Then, for each J, $\overline{H}(J) = \underline{H}(J)$ so let $H(J)$ be their common value. Then H is well defined and additive.

Let $\epsilon > 0, \beta \in \mathcal{B}$ and A, B be as shown as in (5). For $(x, I) \in \beta(I_0)$, either $H(I) \geq F(x, I)$, in which case,

$$A(I) \geq H(I) \geq F(x, I) \geq B(I),$$

so that $|H(I) - f(x, I)| < \epsilon$, or $F(x, I) > H(I)$, so that

$$A(I) \geq F(x, I) \geq H(I) \geq B(I)$$

and again $|H(I) - F(x, I)| < \epsilon$. Summing over any partition $\pi \subseteq \beta(I_0)$ yields

$$|H - F|(\pi) \leq |A - B|(I) \leq |A - B|(I_0) \leq \epsilon.$$

Therefore, $V(H - f, \mathcal{B}(I_0)) = 0$.

(3) $\Rightarrow$ (2) Let $\epsilon > 0$ and choose a $\beta \in \mathcal{B}(I_0)$ so that $V(H - F, \beta(I_0)) < \epsilon$. Let $I \subseteq I_0$ and let $\pi \subseteq \beta$ be a partition of I,

$$|H(I) - F(\pi)| = |(H - F)(\pi)| < \epsilon.$$

Thus $\int_I F$ exists and is equal to $H(I)$ and (2) holds.

$\square$

2.3. Condition for Integrability

LEMMA 2.3.1.

Let $\mathcal{B}$ be of pointwise character and have the partitioning property. Let $I_0 \in \mathcal{I}$ and let φ be a non-negative additive function defined on sub"intervals" of I_0. Then for $f : X \to \mathbb{R}, H \in \mathcal{A}(I_0), F(x, I) = f(x)\varphi(I)$ and

$$E = \{x \in I_0 : D_{\mathcal{B}} H_\varphi(x) = f(x)\},$$

we have

$$V(H - F, \mathcal{B}(I_0)[E]) = 0.$$

PROOF. Let $\epsilon > 0$, then for every $x \in E$ there exists a $\beta_x \in \mathcal{B}$ such that for $(x, I) \in \beta_x[\{x\}]$,

$$|\frac{H(I)}{\varphi(I)} - f(x)| < \epsilon.$$

i.e. $|H(I) - F(x, I)| < \epsilon\varphi(I)$. Let $\beta \in \mathcal{B}(I_0)$ such that for every $x \in E$,

$$\beta[\{x\}] \subseteq \beta_x[\{x\}].$$

Let $\pi \subseteq \beta[E]$ be a partition, then

$$|H - F|(\pi) = \sum_{(x,I)\in\pi} |H - F|(x, I) < \sum_{(x,I)\in\pi} \epsilon\varphi(I) \le \epsilon\varphi(I_0).$$

Therefore, $V(H - F, \beta(I_0)[E]) \le \epsilon\varphi(I_0)$. □

Recall that $\mathcal{B}(I_0)$ is the set of $\beta \in \mathcal{B}$ restricted to have their "interval" inside of I_0 and that $\mathcal{B}[E]$ is the set of $\beta \in \mathcal{B}$ restricted to have their tags inside of E. We use both definitions in the following propositions simultaneously. Another thing to note in the following proposition is that even though the integration is restricted to be inside the "interval" I_0 the differentiation is not. The "intervals" used in the limit of the derivative could end up extending outside of I_0. This extending outside of I_0 could occur since we do not know that if we move further along the filter generated by our base that we necessarily shrink the size of the "intervals".

Proposition 2.3.2.

Let $\mathcal{B}$ be of pointwise character and have the partitioning property, $I_0 \in \mathcal{I}$, φ a non-negative additive function defined on subsets of I_0, $f : X \to \mathbb{R}, H \in \mathcal{A}(I_0)$, and $F(x, I) = f(x)\varphi(I)$. Suppose for $E = \{x \in I_0 : D_{\mathcal{B}}H_\varphi(x) = f(x)\}$ we have

$$V(H - F, \mathcal{B}(I_0)[I_0 \setminus E]) = 0.$$

Then F is Henstock integrable on I_0 and H is its Henstock integral.

Proof. From the previous simple calculations we know that

$$V(H - F, \mathcal{B}(I_0)) \le V(H - F, \mathcal{B}(I_0)[I_0 \setminus E]) + V(H - F, \mathcal{B}(I_0)[E])$$

$$= 0 + 0$$

$$= 0.$$

Therefore from the equivalent definition of the Henstock integral the result follows.

$\square$

CHAPTER 3

Examples of Bases and Basic Results

3.1. Introduction of Relevant Bases

For this chapter $\mathcal{I}$ will be the family of compact, non-degenerate intervals in $\mathbb{R}^n$ and recall that λ denotes Lebesgue measure. Often times the function $F : X \times \mathcal{I} \to \mathbb{R}$ by $F(x, I) = f(x)\lambda(I)$ will be denoted $F = f\lambda$. Unless otherwise stated f and F are assumed to be functions where $f : \mathbb{R}^n \to \mathbb{R}$ and $F : \mathbb{R}^n \times \mathcal{I} \to \mathbb{R}$.

The bulk of the material in this chapter is taken from [**4**] and [**5**]. Most of these bases are quite common and can be found in most of the literature in the field. For example [**1**], [**3**], [**4**], [**6**], [**7**], [**8**] and [**9**] all cover these bases in some way shape or form. The Riemannian base and refining base were added in order to give the reader some familiar material to compare the other bases to. An extensive 1-dimensional study of the gauge bases is shown in [**1**] and is suggested for an introduction to the topic. The 2-dimensional case is studied in [**4**] a little less extensively.

Definition 3.1.1. f is $\mathcal{B}$-integrable, $(\mathcal{B})\int_{I_0} f d\lambda$

Let $I_0 \in \mathcal{I}, f\colon I_0 \to \mathbb{R}$ and $F(x, I) = f(x)\lambda(I)$. If $(\mathcal{B})\int_{I_0} F$ exists, we will write $(\mathcal{B})\int_{I_0} f d\lambda$ or often times simply $(\mathcal{B})\int_{I_0} f$ for that integral and say that f is $\mathcal{B}$-integrable on I_0.

Definition 3.1.2. Riemannian base, constant gauge base

We call the base $\mathcal{B}_{\text{riem}} = \{\beta_\delta : \delta \in \mathbb{R}, \delta > 0\}$ the Riemannian base or constant gauge base, where $\beta_\delta = \{(x, I) \in \mathbb{R}^n \times \mathcal{I} : x \in I \text{ and } \operatorname{diam}(I) \leq \delta\}$.

Notice that a function $F : \mathbb{R}^n \times \mathcal{I} \to \mathbb{R}$ is $\mathcal{B}_{\text{riem}}$-integrable on I_0 if, there exists a real number $(\mathcal{B}_{\text{riem}}) \int_{I_0} F$ such that for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any partition π whose mesh is less than δ,

$$\left| F(\pi) - (\mathcal{B}_{\text{riem}}) \int_{I_0} F \right| < \epsilon.$$

Therefore, when considering real valued functions f on $\mathbb{R}^n$ the $\mathcal{B}_{\text{riem}}$-integral of $F = f\lambda$ is exactly that of the Riemann integral of f.

It is clear that this base is filtering and the fact that it has the partitioning property will be shown in Cousin's theorem 3.2.2.

Proposition 3.1.3.

The Riemannian base is not of σ-local character.

Proof. Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of disjoint subsets of $\mathbb{R}^n$ and $\beta_{\frac{1}{n}} \in \mathcal{B}_{\text{riem}}$ for each $n \in \mathbb{N}$. Now if there is a $\beta_m \in \mathcal{B}_{\text{riem}}$ such that $\beta_m[X_n] \subseteq \beta_{\frac{1}{n}}[X_n]$ for each $n \in \mathbb{N}$. Then, $m \leq \frac{1}{n}$ for each $n \in \mathbb{N}$, so that $m \leq 0$. But no such β_m exists in $\mathcal{B}_{\text{riem}}$.

$\square$

Definition 3.1.4. Refinement base

We call the base $\mathcal{B}_{\text{ref}} = \{\beta_{\mathcal{D}} : \mathcal{D} \text{ is a division}\}$ the refinement base, where $\beta_{\mathcal{D}} = \{(x, I) \in \mathbb{R}^n \times \mathcal{I} : \text{there is some } J \in \mathcal{D} \text{ such that } I \subseteq J \text{ and } x \in I\}$.

For a real valued function f on $\mathbb{R}^n$, $\mathcal{B}_{\mathrm{ref}}$-integrability of f on I_0 just says that there exists an $A \in \mathbb{R}$ such that for any $\epsilon > 0$ there exists a division $\mathcal{D}$ such that for any partition π of I_0 that is finer than $\mathcal{D}$,

$$|f\lambda(\pi) - A| < \epsilon.$$

Notice that this is the refining integral for intervals, which is equivalent to the Riemann integral.

It is clear that this base is filtering and we will show in Cousin's theorem 3.2.2 that is has the partitioning property. It is also easy to see that it is not of σ-local character. Indeed, let $X_n = [\frac{1}{n+1}, \frac{1}{n}]$ and let $\mathcal{D}_n$ be a division of $[0, 1]$ containing X_n. Then for any $\beta_{\mathcal{D}} \in \mathcal{B}_{\mathrm{ref}}$ with $\beta_{\mathcal{D}}[X_n] \subseteq \beta_{\mathcal{D}_n}$ for each $n \in N$, we have that $\mathcal{D}$ must have an infinite number of elements.

Definition 3.1.5. Gauge

A function $\delta : \mathbb{R}^n \mapsto (0, \infty)$ is called a gauge.

We begin now to introduce bases that are directed by gauges. When speaking of these bases at the same time we will refer to them loosely as the gauge bases.

Definition 3.1.6. Kurzweil bases

We define the Kurzweil bases as:

$$\mathcal{B}_1 = \{\beta_\delta : \delta \text{ is a gauge}\}, \text{ where } \beta_\delta = \{(x, I) \in \mathbb{R}^n \times \mathcal{I} : x \in I \subseteq B(x, \delta(x))\},$$

$$\mathcal{B}_1^\diamond = \{\beta_\delta^\diamond : \delta \text{ is a gauge}\}, \text{ where } \beta_\delta^\diamond = \{(x, I) \in \mathbb{R}^n \times \mathcal{I} : I \subseteq B(x, \delta(x))\} \text{ and}$$

$$\widetilde{\mathcal{B}}_1 = \{\widetilde{\beta}_\delta : \delta \text{ is a gauge}\}, \text{ where } \widetilde{\beta}_\delta = \{(x, I) \in \beta_\delta : x \text{ is a vertex of } I\}.$$

We call them the Kurzweil base, the weak Kurzweil base and the modified Kurzweil base respectively.

Definition 3.1.7. Regularity, R(I)

Let I be a an interval in $\mathbb{R}^n$. The regularity of I is defined to be

$$R(I) = \frac{\text{length of the minimal side of I}}{\text{length of the maximal side of I}}.$$

Notice that for, $n = 1$, the regularity is always 1 so this notion generally is not considered in the one dimensional case.

Often times for measurable sets I an alternate definition of regularity is used, the alternate definition being

$$R(I) = \frac{\lambda(I)}{\operatorname{diam}(I)^n}.$$

We will now show that these two definitions are essentially equivalent when it comes down to intervals. To compare them, we denote the first definition by R_1 and the second by R_2.

Proposition 3.1.8.

For I an interval in $\mathbb{R}^n$ we have that $R_1(I) \geq R_2(I)$ and $R_1(I)^n \leq R_2(I)$.

Proof. Let S be the length of the minimal side of I and L the length of the maximal side of I. Then,

$$R_1(I) = \frac{S}{L} \geq \frac{\lambda(I)}{L^n} = \frac{\lambda(I)}{\operatorname{diam}(I)^n} = R_2(I).$$

Also,

$$R_1(I)^n = \left(\frac{S}{L}\right)^n \leq \frac{\lambda(I)}{\operatorname{diam}(I)^n} = R_2(I).$$

$\square$

It will become clear that for the purpose of integration and differentiation these two notions lead to equivalent theories.

Definition 3.1.9. Vitali covering

Let $\rho : \mathbb{R}^n \to (0,1)$. Let $\mathcal{V} \subseteq \mathcal{I}$, and let $E \subseteq \mathbb{R}^n$. If, for every $x \in E$, there exists a sequence $\{V_k\}$ from V such that $\text{diam}(V_k) \to 0$, $R(V_k) \geq \rho(x)$ and $x \in V_k$, we say that $\mathcal{V}$ is a ρ-Vitali cover of E. Generally the ρ in consideration is clear and we simply call $\mathcal{V}$ a Vitali cover of E.

Theorem 3.1.10. *Vitali covering theorem*

Let $\mathcal{V}$ be a Vitali covering of a set $E \subseteq \mathbb{R}^n$. Then there exists a countable family $\{V_k\}$ of sets chosen from $\mathcal{V}$ such that $V_i \cap V_j = \emptyset$, for $i \neq j$ and $\lambda(E \setminus \bigcup_{i=1}^{\infty} V_i) = 0$.

Proof. This was proved in [**8**] p.109.

$\square$

Definition 3.1.11. Kempisty fixed bases

Let $r \in (0,1)$. We define the Kempisty fixed bases as:

$$\mathcal{B}^r = \{\beta_\delta^r : \delta \text{ is a gauge}\}, \text{ where } \beta_\delta^r = \{(x,I) \in \beta_\delta : R(I) \geq r\} \text{ and}$$

$$\mathcal{B}^{r^\diamond} = \{\beta_\delta^{r^\diamond} : \delta \text{ is a gauge}\}, \text{ where } \beta_\delta^{r^\diamond} = \{(x,I) \in \beta_\delta^\diamond : R(I) \geq r\}.$$

We call these the Kempisty fixed r-base and the weak Kempisty fixed r-base respectively. Here we call r a regulator of the base. Often times we will simply say that a function is r-integrable or r-differentiable rather than $\mathcal{B}^r$-integrable or

$\mathcal{B}^r$-differentiable. Notice that different values of r will produce different bases. The effects of this change of r on differentiability and integrability of functions will be studied later in 4.1 and in 4.2 respectively.

DEFINITION 3.1.12. Kempisty bases

Let $\rho : \mathbb{R}^n \to (0, 1)$. We define the Kempisty bases as:

$$\mathcal{B}^\rho = \{\beta_\delta^\rho : \delta \text{ is a gauge}\}, \text{ where } \beta_\delta^\rho = \{(x, I) \in \beta_\delta : R(I) \geq \rho(x)\} \text{ and}$$

$$\mathcal{B}^{\rho^\diamond} = \{\beta_\delta^{\rho^\diamond} : \delta \text{ is a gauge}\}, \text{ where } \beta_\delta^{\rho^\diamond} = \{(x, I) \in \beta_\delta^\diamond : R(I) \geq \rho(x)\}.$$

We call these the Kempisty ρ-base and the weak Kempisty ρ-base respectively. Here we call ρ a regulator of the base. Again notice that different values of ρ lead to different bases.

DEFINITION 3.1.13. Triangular bases

We define the Triangular bases as:

$$\mathcal{T} = \{\tau_\delta : \delta \text{ is a gauge}\}, \text{ where } \tau_\delta = \{(x, I) \in \mathbb{R}^2 \times T : x \in I \subseteq B(x, \delta(x))\} \text{ and}$$

$$\mathcal{T}^\diamond = \{\tau_\delta^\diamond : \delta \text{ is a gauge}\}, \text{ where } \tau_\delta^\diamond = \{(x, I) \in \mathbb{R}^2 \times T : I \subseteq B(x, \delta(x))\},$$

where T is the family of triangles in $\mathbb{R}^2$. We call them the triangular base and the weak triangular base respectively.

It is clear that each of these bases produce different integrals for functions of the form $F(x, I)$. Consider simply any function $F(x, I)$ that is $\lambda(I)$ for the I in consideration in the base and 0 otherwise. Most authors fail to mention this fact in the general setting and begin working out considerably longer solutions in the case where $F(x,I) = f(x)\lambda(I)$. It is nice to be able to show quickly that these bases each produce unique integrals. We present one such example and leave the rest to the reader.

Proposition 3.1.14.

There is a function $F : \mathbb{R}^n \times \mathcal{I} \mapsto \mathbb{R}$ that is $\mathcal{B}^r$-integrable but not $\mathcal{B}_1$-integrable on an interval I_0.

Proof. Consider the function $F(x,I)$ where $F(x,I) = \lambda(I)$ for any r-regular interval I and 0 otherwise. We first show that F is $\mathcal{B}^r$-integrable. Let $\epsilon > 0$, δ be any gauge and let $\pi \subseteq \beta^r_\delta$ be a partition of I_0. Then,

$$\begin{aligned}
&|F(\pi) - \lambda(I_0)| \\
=&|\sum_{(x,I)\in\pi} F(x,I) - \lambda(I_0)| \\
=&|\sum_{(x,I)\in\pi} \lambda(I) - \lambda(I_0)| \\
=&|\lambda(I_0) - \lambda(I_0)| \\
=&0.
\end{aligned}$$

Therefore, F is $\mathcal{B}^r$-integrable on I_0 to 0.

We now show that F is not $\mathcal{B}_1$-integrable on I_0. By proposition 3.2.6 for any gauge δ there is a partition $\pi \subseteq \beta_\delta$ of I_0 such that for any $(x,I) \in \pi$, $R(I) < r$. For such a partition π,

$$F(\pi)$$

$$
\begin{aligned}
&= \sum_{(x,I)\in\pi} F(x,I) \\
&= \sum_{(x,I)\in\pi} 0 \\
&=0.
\end{aligned}
$$

We have already shown that for any gauge δ there is a partition π of I_0 for which $F(\pi) = \lambda(I_0)$. Therefore, F cannot be $\mathcal{B}_1$-integrable on I_0 since the associated filter will not be Cauchy. $\square$

More interesting questions of integrability arise in the traditional setting where $F(x, I)$ is of the form $F(x, I) = f(x)\lambda(I)$. We however, differ this treatment until later in order to present some basic properties of these bases.

PROPOSITION 3.1.15.

Let $p : \mathbb{R}^n \to \mathbb{R}$ and $0 < r < 1$ with $p(x) \geq r$ for every $x \in \mathbb{R}^n$. Then,

(1) *All bases are finer than their weak counterparts,*

(2) *$\widetilde{\mathcal{B}}_1$ is finer than $\mathcal{B}_1$,*

(3) *$\mathcal{B}_{riem} \preccurlyeq \mathcal{B}_1 \preccurlyeq \mathcal{B}^r \preccurlyeq \mathcal{B}^\rho$ and*

(4) *$\mathcal{B}_1^\diamond \preccurlyeq \mathcal{B}^{r^\diamond} \preccurlyeq \mathcal{B}^{\rho^\diamond}$.*

PROOF. Simply look at which family of intervals or triangles contains the other families.

$\square$

PROPOSITION 3.1.16.

All of the gauge bases are filtering.

Proof. Simply notice that for any gauges δ_1, δ_2 and any two regulators r_1, r_2 that if (x, I) is $\min\{\delta_1, \delta_2\}$-fine and $\max\{r_1, r_2\}$-regular, then it is also δ_1, δ_2-fine and r_1, r_2-regular.

□

Proposition 3.1.17.

All of the previous gauge bases are of local character.

Proof. Let δ_x be a gauge for each $x \in \mathbb{R}^n$, then letting $\delta(x) = \delta_x(x)$ we see that if (x, I) is δ-fine, then it is also δ_x-fine. Similarly, if ρ_x is a regulator for each $x \in \mathbb{R}^n$ then setting $\rho(x) = \rho_x(x)$ we see that if (x, I) is ρ regular, then it is also ρ_x regular.

□

Proposition 3.1.18.

All of the previous gauge bases ignore no point.

Proof. Given $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and a gauge δ we have $(x, [x_1, x_1 + \frac{\delta(x)}{2n}] \times [x_2, x_2 + \frac{\delta(x)}{2n}] \times ... \times [x_n, x_n + \frac{\delta(x)}{2n}]) \in \alpha_\delta$ for non triangular bases.

For the triangular base we have $(x, A) \in \tau_\delta$ where A is the triangle with vertices $x, (x_1 + \frac{\delta(x)}{2}, x_2)$ and $(x_1, x_2 + \frac{\delta(x)}{2})$.

□

3.2. Partitioning Property

The existence of ρ-regular partitions using intervals for any interval was shown in [**5**]. We use this to show the existence of triangular partitions partitions of any interval. After this we do a slight variation of Pfeffer's argument in [**5**] to show the existence of partitions whose elements are all non ρ-regular.

LEMMA 3.2.1.

Let A be an interval and let $0 < r < 1$. Then there is a division $\mathcal{D}$ of A such that $R(D) > r$ for each $D \in \mathcal{D}$.

PROOF. Let $A = \prod_{i=1}^{n}[a_i, b_i]$, let $c_i = b_i - a_i$ for $i = 1, ..., n$ and choose a real number ϵ with $1 > \epsilon > 0$ and $(\frac{1-\epsilon}{1+\epsilon}) > r$. Find integers $p_i \geq 1$, such that $|\frac{c_1}{c_i}\frac{p_i}{p_1} - 1| \leq \epsilon$ and divide the i-th side of A into p_i equal intervals of length $\frac{c_i}{p_i}$ for $i = 1, .., n$. This induces a division on A consisting of intervals I with

$$\begin{aligned} R(I) &= \frac{\inf_k \frac{c_k}{p_k}}{\sup_i \frac{c_i}{p_i}} \\ &= \frac{\inf_k \frac{c_k p_1}{p_k c_1}}{\sup_i \frac{c_i p_1}{p_i c_1}} \\ &\geq \frac{1-\epsilon}{1+\epsilon}. \end{aligned}$$

This choice is possible. Indeed,

$$\left|\frac{c_1 p_i}{c_i p_1} - 1\right| \leq \epsilon \qquad \Longleftrightarrow \qquad (1-\epsilon)\frac{c_i}{c_1} \leq \frac{p_i}{p_1} \leq (1+\epsilon)\frac{c_i}{c_1}.$$

The intervals $U_i = \big((1-\epsilon)\frac{c_i}{c_1}, (1+\epsilon)\frac{c_i}{c_1}\big)$ here have positive length $\frac{2c_i}{c_1}$. Choose $p_1 \in \mathbb{N}$ with $\frac{1}{p_1} < \min_i \frac{c_i}{c_1}$. Then, for each $i = 1, .., n$, there is a natural number p_i such that $\frac{p_i}{p_1} \in U_i$, as required. □

THEOREM 3.2.2. *Cousin's theorem*

Let I_0 be an interval, δ a gauge and let $\rho : \mathbb{R}^n \to (0, 1)$ be a regulator. Then there is a δ-fine, ρ-regular partition π of I_0.

PROOF. Suppose that I is a compact interval that is not δ-fine, ρ-regular partitionable. First notice that any interval can be cut into pieces whose diameters are less than any fixed size. Putting this fact together with the previous lemma we can form a division $\mathcal{D}_1$ of I such that for any $D \in \mathcal{D}_1$ we have that $R(D) > 1 - 2^{-1}$ and $\operatorname{diam}(D) < 2^{-1}$. Now if every $D \in \mathcal{D}_1$ is δ-fine, ρ-regular partitionable, then I would be also. Hence, there is a $D_1 \in \mathcal{D}_1$ that is not δ-fine, ρ-regular partitionable. We can find a division $\mathcal{D}_2$ of D_1 such that for any $D \in \mathcal{D}_2$ we have that $R(D) > 1 - 2^{-2}$ and $\operatorname{diam}(D) < 2^{-2}$. Once again if every $D \in \mathcal{D}_2$ is δ-fine, ρ-regular partitionable, then D_1 would be also. Hence, there is a $D_2 \in \mathcal{D}_2$ that is not δ-fine, ρ-regular partitionable. Continue the process inductively.

The result is a sequence of nested closed intervals $\{D_i\}_{i\in\mathbb{N}}$ that are not δ-fine, ρ-regular partitionable for which $R(D_i) > 1 - 2^i$ and $\operatorname{diam}(D_i) < 2^{-i}$. Thus $\bigcap_{i=1}^{\infty} D_i = \{x_0\}$ and there exists an $N \in \mathbb{N}$ such that $R(D_N) > \rho(x_0)$ and $\operatorname{diam}(D_N) < \delta(x_0)$. So we have that $\{(x_0, D_N)\}$ is a δ-fine, ρ-regular partition of D_N, a contradiction.

$\square$

COROLLARY 3.2.3.

All of the previous gauge bases other than the triangular bases and the modified Kurzweil base have the partitioning property.

PROPOSITION 3.2.4.

The triangular bases and $\widetilde{\mathcal{B}}_1$ *have the partitioning property.*

PROOF. Given a gauge δ, use Cousin's theorem to choose find a partition π in β_δ. Replacing each $(x, I) \in \pi$ by the 4 triangles determined by x and the vertices of I, retaining x as the tag, yields a partition in τ_δ. Of course, if any of the triangles are degenerate we do not place them in the partition.

Similarly, once π has been chosen in β_δ, replace each $(x, I) \in \pi$ by the same tag x and the at most 2^n non-degenerate intervals obtained by intersecting I with the half-spaces determined by the $(n-1)$-dimensional hyperplanes through x that are normal to the coordinate axes. This yields a partition in $\widetilde{\beta}_\delta$.

□

It will be useful in some instances to have the existence of a partition that is δ-fine whose elements are all not ρ-regular.

Lemma 3.2.5.

Let $I_0 \in \mathcal{I}$ and $0 < r < 1$. There is a division $\mathcal{D}$ of I_0 for which $R(D) < r$ and $\operatorname{diam}(D) < \frac{\operatorname{diam}(I_0)}{2}$ *for any $D \in \mathcal{D}$.*

Proof. First cut I_0 into 2^n equal sized pieces by cutting it with an $(n-1)$-dimensional planes that cross its center. The resulting intervals all have diameter equal to $\frac{\operatorname{diam}(I_0)}{2}$. Say J is one of these resulting intervals. Now we use $(n-1)$-dimensional planes perpendicular to the shortest side of J to cut J into k pieces of equal size. Then, the regularity of these pieces is $R(J)/k$. Choosing k large enough ensures that the resulting pieces will have regularity less than r. Doing this for each J and amassing the resulting intervals we have our required division.

□

Proposition 3.2.6.

For any $I_0 \in \mathcal{I}$, any gauge δ and any regulator $\rho : \mathbb{R}^n \mapsto \mathbb{R}$ there is a δ-fine partition of I_0 for which $R(I) < \rho(x)$ for any $(x, I) \in \pi$.

Proof. Suppose that $I_0 \in \mathcal{I}$ did not have the desired property and let $\epsilon > 0$. There is a division $\mathcal{D}_1$ of I_0 for which $R(D) < \frac{1}{2}$ and $\text{diam}(D) < \frac{\text{diam}(I_0)}{2}$ for any $D \in \mathcal{D}_1$. Now if each $D \in \mathcal{D}_1$ has the desired property then so would I_0. Suppose that $D_1 \in \mathcal{D}_1$ were such a D without the desired property. Then there is a division $\mathcal{D}_2$ of D_1 for which $R(D) < \frac{1}{2^2}$ and $\text{diam}(D) < \frac{\text{diam}(I_0)}{2^2}$ for any $D \in \mathcal{D}_2$. As before there is a $D_2 \in \mathcal{D}_2$ that does not have the desired property. Continue the process inductively.

The result is a family of nested compact intervals $\{D_i\}_{i \in \mathbb{N}}$ that do not have the desired property for which $\text{diam}(D_i) < \frac{\text{diam}(I_0)}{2^i}$ and $R(D_i) < \frac{1}{2^i}$. Then $\bigcap_{i=1}^{\infty} D_i = \{x\}$. Choose a $k \in \mathbb{N}$ such that $\frac{\text{diam}(I_0)}{2^k} < \delta(x)$ and $\frac{1}{2^k} < \rho(x)$. Then, $\{(x, D_k)\}$ is a δ-fine partition of D_k for which $R(I) < \rho(x)$ for any (x, I) in the partition. This is a contradiction.

$\square$

3.3. Additivity

We now ask the question: "If f is integrable with respect to one of our bases on two intervals, will it be integrable on their union?". Surprisingly, for certain bases, the answer is not always yes. A nice introduction to this and an intuitive feel as to why not can be found in [**3**]. The traditional example of why not 3.3.9 can be found in [**5**]; with a few gaps in the arguments. Most of the remaining material in this section will be found in [**4**]. We also present some interesting results on how this "additivity" will hold for overlapping intervals in 3.3.12 and 3.3.13.

DEFINITION 3.3.1. $\mathcal{B}$ is additive

We will say that $\mathcal{B}$ is additive if V_F is 2-additive for any $F\colon X \times \mathcal{I} \to \mathbb{R}$.

DEFINITION 3.3.2. Additive in the sense of Henstock

In the topological setting, where two sets are non-overlapping if the intersection of their interiors is empty, we will say that $\mathcal{B}$ is additive in the sense of Henstock if for any $\beta_1 \in \mathcal{B}(I)$ and $\beta_2 \in \mathcal{B}(\mathbb{R}^n \setminus I^\circ)$, there is a $\beta \in \mathcal{B}$ such that $\beta \subseteq \beta_1 \cup \beta_2$.

PROPOSITION 3.3.3.

$\widetilde{\mathcal{B}}_1$ is additive in the sense of Henstock.

PROOF. Let $I \in \mathcal{I}$, $\widetilde{\beta}_{\delta_1} \in \widetilde{\mathcal{B}}_1(I)$ and $\widetilde{\beta}_{\delta_2} \in \widetilde{\mathcal{B}}_1(\mathbb{R}^n \setminus I^\circ)$. Define

$$\delta(x) = \begin{cases} \min\{\delta_1(x), \delta_2(x), \frac{\operatorname{dist}(x, \operatorname{bd}(I))}{2}\} \text{ for } x \notin \operatorname{bd}(I), \\ \\ \min\{\delta_1(x), \delta_2(x), \frac{\operatorname{dist}(x,E)}{2}\} \text{ for } x \in \operatorname{bd}(I), \end{cases}$$

where E is the set formed by removing from the boundary of I, all sides of I that contain x. For $(x, J) \in \widetilde{\beta}_\delta$ if $x \in I^\circ$, then $(x, J) \in \widetilde{\beta}_{\delta_1}(I)$ and if $x \in \mathbb{R}^n \setminus I$, then $(x, J) \in \widetilde{\beta}_{\delta_2}(\mathbb{R}^n \setminus I^\circ)$. By the definition of δ and since x is a vertex of J we have that for $x \in \operatorname{bd}(I)$, $J \subseteq I$ or $J \subseteq \mathbb{R}^n \setminus I^\circ$. Therefore, $(x, J) \in \widetilde{\beta}_{\delta_1} \cup \widetilde{\beta}_{\delta_2}$.

$\square$

PROPOSITION 3.3.4.

All of the other gauge bases introduced are not additive in the sense of Henstock.

PROOF. Suppose that $\beta \subseteq \beta_1(I) \cup \beta_2(\mathbb{R}^n \setminus I)$, where $\beta_1, \beta_2 \in \mathcal{B}$ and $I \in \mathcal{I}$. Then for $(x, J) \in \beta$ with $x \in J^\circ \cap \mathrm{bd}(I)$, we have $J \cap I^\circ \neq \emptyset$ and $J \cap (\mathbb{R}^n \setminus I)^\circ \neq \emptyset$. Therefore, $(x, J) \notin \beta$ a contradiction.

□

PROPOSITION 3.3.5.

If $\mathcal{B}$ is filtering and additive in the sense of Henstock, then it is additive.

PROOF. Let $I_1, I_2 \in \mathcal{I}$ with I_1, I_2 non-overlapping and $I_1 \cup I_2 \in \mathcal{I}$. Let $\alpha \in \mathcal{B}$, then $V(F, \alpha(I_1) \cup \alpha(I_2)) = V(F, \alpha(I_1)) + V(F, \alpha(I_2))$. Indeed, if $\pi \subseteq \alpha(I_1) \cup \alpha(I_2)$ is a partition, then

$$
\begin{aligned}
|F|(\pi) &= |F|(\pi(I_1)) + |F|(\pi(I_2)) \\
&\leq V(F, \alpha(I_1)) + V(F, \alpha(I_2)).
\end{aligned}
$$

Taking the supremum over all partitions $\pi \subseteq \alpha(I_1) \cup \alpha(I_2)$,

$$V(F, \alpha(I_1) \cup \alpha(I_2)) \leq V(F, \alpha(I_1)) + V(F, \alpha(I_2)).$$

For the reverse inequality let $\pi_1 \subseteq \alpha(I_1)$ and $\pi_2 \subseteq \alpha(I_2)$ be partitions. Then, $\pi_1 \cup \pi_2 \subseteq \alpha(I_1) \cup \alpha(I_2)$ and is a partition, so that

$$|F|(\pi_1) + |F|(\pi_2) = |F|(\pi_1 \cup \pi_2) \leq V(F, \alpha(I_1) \cup \alpha(I_2)).$$

Taking the supremum of the left over all partitions $\pi_1 \subseteq \alpha(I_1)$ followed by the supremum over all partitions $\pi_2 \subseteq \alpha(I_2)$. Therefore, $V(F, \alpha(I_1) \cup \alpha(I_2)) = V(F, \alpha(I_1)) +$

$V(F, \alpha(I_2))$. Therefore,

$$\begin{aligned} V(F, \alpha(I_1 \cup I_2)) &= V(F, \alpha(I_1)) + V(F, \alpha(I_2)) \\ &\geq V(F, \mathcal{B}(I_1)) + V(F, \mathcal{B}(I_2)). \end{aligned}$$

Taking the infimum over all $\alpha \in \mathcal{B}$ of the left,

$$V(F, \mathcal{B}(I_1 \cup I_2)) \geq V(F, \mathcal{B}(I_1)) + V(F, \mathcal{B}(I_2)).$$

Now to show the reverse inequality. Let $\epsilon > 0$ and choose $\hat{\alpha}_1, \hat{\alpha}_2 \in \mathcal{B}$ such that

$$V(F, \hat{\alpha}_1(I_1)) \leq V(F, \mathcal{B}(I_1)) + \epsilon \text{ and}$$

$$V(F, \hat{\alpha}_2(I_2)) \leq V(F, \mathcal{B}(I_2)) + \epsilon.$$

Choose any $\alpha_1, \alpha_2 \in \mathcal{B}$ such that

$$\alpha_1 \subseteq \hat{\alpha}_1(I_1) \cup \hat{\alpha}_1(X \setminus I_1^\circ) \text{ and}$$

$$\alpha_2 \subseteq \hat{\alpha}_2(I_2) \cup \hat{\alpha}_2(X \setminus I_2^\circ).$$

Finally, choose an $\alpha \in \mathcal{B}$ such that $\alpha \subseteq \alpha_1 \cap \alpha_2$.

<u>Claim $\alpha(I_1 \cup I_2) = \alpha(I_1) \cup \alpha(I_2)$</u>

It is clear that $\alpha(I_1) \cup \alpha(I_2) \subseteq \alpha(I_1 \cup I_2)$. Suppose that $\alpha(I_1 \cup I_2) \nsubseteq \alpha(I_1) \cup \alpha(I_2)$. Then, there exists an $(x, I) \in \alpha(I_1 \cup I_2)$ such that $I \nsubseteq I_1$ and $I \nsubseteq I_2$. Now

$$\begin{aligned} \alpha &\subseteq \alpha_1 \cap \alpha_2 \\ &\subseteq \big[\hat{\alpha}_1(I_1) \cup \hat{\alpha}_1(X \setminus I_1^\circ)\big] \cap \big[\hat{\alpha}_2(I_2) \cup \hat{\alpha}_2(X \setminus I_2^\circ)\big]. \end{aligned}$$

Since $I \nsubseteq I_1$ and $I \nsubseteq I_2$, we have that $I \subseteq X \setminus I_1^\circ \cap X \setminus I_2^\circ = X \setminus (I_1^\circ \cup I_2^\circ)$ but $I \subseteq I_1 \cup I_2$. Therefore, $I \subseteq \mathrm{bd}(I_1 \cup I_2)$ but I is non-degenerate. This is a contradiction so the claim holds.

Therefore,

$$
\begin{aligned}
V(F, \alpha(I_1 \cup I_2)) &= V(F, \alpha(I_1) \cup \alpha(I_2)) \\
&= V(F, \alpha(I_1)) + V(F, \alpha(I_2)) \\
&\leq V(F, \alpha_1(I_1)) + V(F, \alpha_2(I_2)) \\
&\leq V(F, \hat{\alpha}_1(I_1)) + V(F, \hat{\alpha}_2(I_2)) \\
&\leq V(F, \mathcal{B}(I_1)) + V(F, \mathcal{B}(I_2)) + 2\epsilon.
\end{aligned}
$$

Taking the infimum on the left over all $\alpha \in \mathcal{B}$ we have the result.

$\square$

Corollary 3.3.6.

$\widetilde{\mathcal{B}}_1$ is additive.

Proposition 3.3.7.

Suppose that $\mathcal{B}$ is additive in the sense of Henstock, has the partitioning property, and is filtering. Let F be integrable on two non-overlapping $I_1, I_2 \in \mathcal{I}$ for which $I_1 \cup I_2 \in \mathcal{I}$. Then F is integrable on $I_1 \cup I_2$.

Proof. Let H be an additive function with $V(H - F, \mathcal{B}(I_1)) = 0$ and $V(H - F, \mathcal{B}(I_2)) = 0$. We may assume that the H's are the same since I_1 and I_2 are non-overlapping and since $\mathcal{B}$ is additive in the sense of Henstock. Indeed, any "interval" can be partitioned into pieces inside I_1, pieces inside I_2 and pieces outside both. After doing this we simply work additively on those pieces. But since $\mathcal{B}$ is additive we have that $V(H-F,\mathcal{B}(I_1\cup I_2)) = V(H-F,\mathcal{B}(I_1)) + V(H-F,\mathcal{B}(I_2)) = 0$. Therefore, F is integrable on $I_1 \cup I_2$.

□

Corollary 3.3.8.

Let F be $\widetilde{\mathcal{B}}_1$-integrable on two non-overlapping $I_1, I_2 \in \mathcal{I}$ for which $I_1 \cup I_2 \in \mathcal{I}$. Then, F is $\widetilde{\mathcal{B}}_1$-integrable on $I_1 \cup I_2$.

The often cited example in the following theorem was given by Pfeffer [**5**]. The proof had some flaws which we have corrected. Pfeffer often times refers to this result as a lack of "additivity".

Theorem 3.3.9.

There is a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is $\mathcal{B}^r$-integrable on two non-overlapping intervals but that is not integrable on their union.

Proof. Let $A = [0,1]\times[0,1], B = [-1,0]\times[0,1], A^n_+ = [3\cdot 2^{-n-1}, 4\cdot 2^{-n-1}] \times [0, 2^{-2n}], A^n_- = [0,2^{-2n}] \times [3\cdot 2^{-n-1}, 4\cdot 2^{-n-1}]$ and $f:\mathbb{R}^2 \to \mathbb{R}$ defined to be $\frac{2^{3n+1}}{n}$ on $A^n_+, -\frac{2^{3n+1}}{n}$ on A^n_- and 0 elsewhere. Let $0 < r < 1$, $r > \epsilon > 0$, $\alpha > 0$ with $\alpha < \frac{\epsilon}{2}$ and let $k \in \mathbb{N}$ with $\frac{k+2}{k^2+1} < \frac{\epsilon}{2}$. We will first show that f is $\mathcal{B}^r$-integrable on A. Choose a gauge δ such that the following conditions are satisfied:

$$\delta(0,0) < 2^{-k^2}, \tag{1}$$

$$\delta(x) < |x| \text{ for } x \neq (0,0), \tag{2}$$

$$\delta(x) < \text{dist}(x, A_+^n) \text{ for } x \notin A_+^n \cup \{(0,0)\}, \tag{3}$$

$$\delta(x) < \text{dist}(x, A_-^n) \text{ for } x \notin A_-^n \cup \{(0,0)\}, \tag{4}$$

$$|f\lambda(\pi[A_+^n])| < \frac{1}{n} + \frac{\alpha}{2^n}, \tag{5}$$

$$|f\lambda(\pi[A_-^n])| < \frac{1}{n} + \frac{\alpha}{2^n}, \tag{6}$$

where π is any δ-fine partition. The second last condition is made possible since for any $n \in \mathbb{N}$, f is Riemann integrable on an interval I_n^+ containing A_+^n that also does not intersect any other $A_\pm^m$. Notice that the Riemann integral of f over I_+^n is $\frac{1}{n}$. So there exists a positive number δ_n such that for any partition π of I_+^n that is δ_n-fine, $|f\lambda(\pi) - \frac{1}{n}| < \frac{\alpha}{2^n}$. Since $f \geq 0$ on all of I_+^n it follows that for any partition π in I_n^+, $f\lambda(\pi) < \frac{1}{n} + \frac{\alpha}{2^n}$. Ensuring now that $\delta < \delta_n$ on I_+^n we have condition (5). A similar process can be used to ensure condition (6). Due to condition (5) for any δ-fine partition π tagged in A_+^n we have $\sum_{(x,I)\in\pi} f(x)\lambda(I \cap A \setminus A_+^n) \leq \frac{\alpha}{2^n}$. Otherwise, π could be extended to a δ-fine partition π_1 that covered A_+^n, in which case,

$$\begin{aligned}
|f\lambda(\pi_1[A_+^n])| &= |\sum_{(x,I)\in\pi_1[A_+^n]} f(x)\lambda(I)| \\
&= |\sum_{(x,I)\in\pi_1[A_+^n]} f(x)\lambda(I \cap A_+^n) + f(x)\lambda(I \cap A \setminus A_+^n)| \\
&= \frac{1}{n} + \sum_{(x,I)\in\pi_1[A_+^n]} f(x)\lambda(I \cap A \setminus A_+^n) \\
&> \frac{1}{n} + \frac{\alpha}{2^n}.
\end{aligned}$$

However, this contradicts condition (5). A similar argument can be used over the A_-^n using condition (6).

Let π be a δ-fine, r-regular partition of A. Condition (3) will ensure that if $(x, I) \in \pi$ with $I \cap A_+^n \neq \emptyset$, then $x \in A_+^n \cup \{(0,0)\}$. Also, due to condition (3), if $(x, I) \in \pi$ with $x \in A_+^n$, then $I \cap A_+^m = \emptyset$ for any $m \neq n$ and $I \cap A_-^m = \emptyset$ for any $m \in \mathbb{N}$. Similar conclusions can be drawn about the A_-^n using condition (4). Condition (2) ensures $\pi[\{(0,0)\}] = \{((0,0), I_0)\}$ for some $I_0 \in \mathcal{I}$. We will assume that the longer side of I_0 is along the y-axis. Let L be the length of the longer side of I_0 and S be the length of its shorter side. Then,

$$
\begin{aligned}
&|f\lambda(\pi)| \\
=&|\sum_{n\in\mathbb{N}} f\lambda(\pi[A_+^n]) + f\lambda(\pi[A_-^n])| \\
=&|\sum_{n\in\mathbb{N}} \Bigg[\sum_{(x,I)\in\pi[A_+^n]} [f(x)\lambda(I\cap A_+^n) + f(x)\lambda(I \cap A \setminus A_+^n)] + \sum_{(x,I)\in\pi[A_-^n]} [f(x)\lambda(I\cap A_-^n) \\
&\quad + f(x)\lambda(I\cap A\setminus A_-^n)]\Bigg]| \\
\leq&|\sum_{n\in\mathbb{N}} \left[\sum_{(x,I)\in\pi[A_+^n]} f(x)\lambda(I\cap A_+^n) + \sum_{(x,I)\in\pi[A_-^n]} f(x)\lambda(I\cap A_-^n)\right]| \\
&\quad + \sum_{n\in\mathbb{N}} |\sum_{(x,I)\in\pi[A_+^n]} f(x)\lambda(I\cap A\setminus A_+^n)| + \sum_{n\in\mathbb{N}} |\sum_{(x,I)\in\pi[A_-^n]} f(x)\lambda(I\cap A\setminus A_-^n)| \\
\leq&|\sum_{n\in\mathbb{N}} \left[\sum_{(x,I)\in\pi[A_+^n]} f(x)\lambda(I\cap A_+^n) + \sum_{(x,I)\in\pi[A_-^n]} f(x)\lambda(I\cap A_-^n)\right]| + 2\sum_{n\in\mathbb{N}} \frac{\alpha}{2^n} \\
\leq&|\sum_{n\in\mathbb{N}} \left[\sum_{(x,I)\in\pi[A_+^n]} f(x)\lambda(I\cap A_+^n) + \sum_{(x,I)\in\pi[A_-^n]} f(x)\lambda(I\cap A_-^n)\right]| + 2\alpha \\
\leq&|\sum_{n\in\mathbb{N}} \left[\sum_{(x,I)\in\pi[A_+^n]} f(x)\lambda(I\cap A_+^n) + \sum_{(x,I)\in\pi[A_-^n]} f(x)\lambda(I\cap A_-^n)\right]| + \epsilon. \qquad (7)
\end{aligned}
$$

The Riemann sum comprises only of the pieces of the partition tagged inside the $A^n_\pm$. Also, for any piece of the partition, say (x, I) tagged in some $A^n_\pm$, we can split its contribution into two pieces, the piece inside $A^n_\pm$ and the piece outside of $A^n_\pm$. We have shown that the contribution of the parts hanging over the $A^n_\pm$ is negligible and we focus our attention to the parts inside the $A^n_\pm$. Notice that if A^n_+ is contained in the interval I_0, then so is A^n_-, since its longer side is along the y-axis; similarly, if $A^n_+ \perp I_0$, then $A^n_- \perp I_0$ also. In both cases, the contribution of A^n_+ and A^n_- cancel each other. Thus, we need only consider $\{n : \frac{3}{2^{n+1}} < L \text{ and } \frac{4}{2^{n+1}} > S\}$, the set of those n for which A^n_+ is not entirely covered by I_0 and for which their counterpart A^n_- is partially covered. If this set is not empty, put $a = \sup\{n \in \mathbb{N} : 4 \cdot 2^{-n-1} > S\}$ and $b = \inf\{n \in \mathbb{N} : 3 \cdot 2^{-n-1} < L\}$. Notice that by condition (1), $L < 2^{-k^2} = 2^{-(k^2+1)+1}$, so that $b \geq k^2 + 1$. Then by regularity, $\epsilon < r < \frac{S}{L} \leq \frac{2^{-a+1}}{3 \cdot 2^{-b-1}} \leq 2^{b-a+1}$. Therefore,

$$2^{b-a+1} > \epsilon > \frac{k+2}{k^2+1} > \frac{1}{k} > 2^{-k}; \text{ hence,}$$

$$k + 1 > a - b.$$

Thus,

$$\begin{aligned}
&\left|\sum_{n\in\mathbb{N}} \left[\sum_{(x,I)\in\pi[A^n_+]} f(x)\lambda(I \cap A^n_+) + \sum_{(x,I)\in\pi[A^n_-]} f(x)\lambda(I \cap A^n_-) \right]\right| \\
&\leq \sum_{i=b}^{a} \frac{1}{i} \\
&\leq \sum_{i=k^2+1}^{k^2+1+(a-b)} \frac{1}{i} \\
&\leq \sum_{i=k^2+1}^{k^2+k+2} \frac{1}{i}
\end{aligned}$$

$$\leq \frac{k+2}{k^2+1}$$
$$< \frac{\epsilon}{2}.$$

Combining this with (7) we have that $|f\lambda(\pi)| \leq \frac{3\epsilon}{2}$. Therefore, f is $\mathcal{B}^r$-integrable on A and $(\mathcal{B}^r)\int_A f d\lambda = 0$. It is also easy to show that f is $\mathcal{B}^r$-integrable on B, using the fact that f is non-zero only on a set of 0 Lebesgue measure.

We now show that f is not $\mathcal{B}^r$-integrable on $A \cup B$. Let $\alpha > 0$ and let δ be a gauge on $A \cup B$. Find a natural number k such that $2^{-k+1} < \delta((0,0))$. Let $C_0 = [0, 2^{-k+1}] \times [0, 2^{-k+1}]$ and $D_0 = [2^{-2k} - 2^{-k+1}, 2^{-2k}] \times [0, 2^{-k+1}]$. Now find δ-fine, r-regular, partitions $P = \{((0,0), C_0), ..., (x_p, C_p)\}$ and $Q = \{((0,0), D_0), ..., (y_t, D_t)\}$ of $A \cup B$ such that $(f\lambda)(P) = 0$ and $(f\lambda)(Q) \geq \sum_{j=1}^{t} (\mathcal{B}^r)\int_{D_j} f d\lambda - \alpha = \sum_{n=k}^{2k-1} \frac{1}{n} - \alpha \geq \frac{k}{2k-1} - \alpha > \frac{1}{2} - \alpha$. This can be done quite easily by ensuring that the partitions are symmetric along the line $y = x$ in the appropriate areas. For P and Q we ensure the symmetry outside of the square C_0 but still within A. For Q we build the partition in such a way that inside $[0, 2^{-k+1}] \times [0, 2^{-k+1}]$ the contribution to the Riemann sum is greater than $\sum_{n=k}^{2k-1} \frac{1}{n}$. This can be done by tagging any intervals covering any A_+^n inside $C_0 \setminus D_0$ with points from A_+^n. We must also ensure that the intervals touching the y-axis are sufficiently small so that their contributions inside B are no bigger than α. Choosing α less than $\frac{1}{4}$, $|f\lambda(P) - f\lambda(Q)| = \frac{1}{2} - \alpha \geq \frac{1}{4}$. Therefore, $(\mathcal{B}^r)\int_{A \cup B} f d\lambda$ does not exist since the Cauchy condition for integrability fails. $\square$

Comparing theorem 3.3.9 with corollary 3.3.8 we obtain:

Corollary 3.3.10.

The integral produced by $\widetilde{\mathcal{B}}_1$ is not the same as the integral produced by $\mathcal{B}^r$ or $\mathcal{B}^\rho$. This is true even when considering only functions of the form $F(x,I) = f(x)\lambda(I)$.

Corollary 3.3.11.

The function of theorem 3.3.9 is not $\mathcal{B}_1$-integrable. Thus the $\mathcal{B}^r$ integral is more general than the $\mathcal{B}_1$ integral, even when considering only functions of the form $F(x,I) = f(x)\lambda(I)$.

Proof. Throughout the proof we use the same notation as in the previous theorem. Suppose that F were $\mathcal{B}_1$-integrable on A. Then it would have to integrate to 0 by the proof of the previous theorem. Let $\epsilon > 0$ and choose a gauge δ such that for any δ-fine partition $\pi \subseteq \beta_\delta$ of A, $|\sum_{(x,I)\in\pi} f(x)\lambda(I)| < \epsilon$.

Now $\delta(0,0) > 3 \cdot 2^{-n-1}$ for some $n \in \mathbb{N}$. Let $I = [0, 2^{-2n}] \times [0, 3 \cdot 2^{-n-1}]$. Now consider how many A_+^k fall in $[0, 3 \cdot 2^{-n-1}] \times [0, 3 \cdot 2^{-n-1}] \setminus I$. For such A_+^k we would need the right end points of their projections onto the x-axis to be less than $3 \cdot 2^{-n-1}$ and the left end points of their projections onto the x-axis to be greater than 2^{-2n}. Now $3 \cdot 2^{-a-1} > 2^{-2n}$ so if $a < 2n$, then A_+^a falls to the right of 2^{-2n}. Also $2^{-b+1} < 2^{-b}$ and $2^{-n} < 3 \cdot 2^{-n-1}$ so for $b < n$ we have that A_+^b falls to the left of $3 \cdot 2^{-n-1}$. Thus there are at least n such A_+^k. But if we look at the volume that each one of those A_+^k produce we see that

$$\sum_{k=n}^{2n} \frac{1}{k} \geq \sum_{k=n}^{2n} \frac{1}{2n} = \frac{1}{2}.$$

Thus choosing a partition δ-fine partition π of I_0 that contains $((0,0), I)$, that is built symmetrically outside the box $[0, 3\cdot 2^{-n-1}] \times [0, 3\cdot 2^{-n-1}]$ and such that for each piece of the partition touching one of the A_+^k that fall in $[0, 3\cdot 2^{-n-1}] \times [0, 3\cdot 2^{-n-1}] \setminus I$ that piece is tagged in A_+^k. Then we see that $(f\lambda)(\pi) \geq \frac{1}{2}$. Thus f is not $\mathcal{B}_1$-integrable on A.

$\square$

So, the $\mathcal{B}^r$-integral is not "additive" but what sort of conditions can we impose to ensure this sort of "additivity"?

Proposition 3.3.12.

Let $I_1, I_2 \in \mathcal{I}$ with $I_1 \cup I_2 \in \mathcal{I}$ and I_1, I_2 overlapping; i.e., $I_1^\circ \cap I_2^\circ \neq \emptyset$. If $f : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{B}^r$-integrable on I_1, I_2, then f is $\mathcal{B}^r$-integrable on $I_1 \cup I_2$.

Proof. We will assume that $I_1 \not\subseteq I_2$ and that $I_2 \not\subseteq I_1$ or the result holds trivially. We will also assume that I_1 is to the left of I_2. The left most face of I_2 will be denoted A and the right most face of I_1 will be denoted B.

Let $\epsilon > 0$. Define H to be the unique additive function on those $I \in \mathcal{I}$ contained in $I_1 \cup I_2$ for which $H(I) = (\mathcal{B}^r) \int_I f d\lambda$ when $I \subseteq I_1$ or $I \subseteq I_2$. Choose a gauge δ such that:

(1) $\delta(x) < \operatorname{dist}(x, A)$ for $x \notin A$,

(2) $\delta(x) < \operatorname{dist}(x, B)$ for $x \notin B$,

(3) for any partition $\pi \subseteq \beta_\delta^r$, $\sum_{(x,I)\in\pi(I_i)} |f(x)\lambda(I) - H(I)| < \frac{\epsilon}{2}$ for $i = 1, 2$.

Let $\pi \subseteq \beta_\delta^r(I_1 \cup I_2)$ be a partition, let $\pi_1 = \pi(I_1)$ and let $\pi_2 = \pi \setminus \pi_1$. For any $(x, I) \in \pi_1$ we have $I \subseteq I_1$ and for any $(x, I) \in \pi_2$ we have $I \subseteq I_2$. Therefore,

$$\begin{aligned}
&\sum_{(x,I)\in\pi} |f(x)\lambda(I) - H(I)| \\
=&\sum_{(x,I)\in\pi_1} |f(x)\lambda(I) - H(I)| + \sum_{(x,I)\in\pi_2} |f(x)\lambda(I) - H(I)| \\
<&\frac{\epsilon}{2} + \frac{\epsilon}{2} \\
=&\epsilon.
\end{aligned}$$

Thus, H is an additive function for which $V(f\lambda - H, \mathcal{B}^r(I_0)) = 0$ giving the result.

□

Corollary 3.3.13. *Patching theorem*

Let $I_1, I_2, I_3 \in \mathcal{I}$, I_3 overlapping each of I_1 and I_2 and let $I_1 \cap I_2 \subseteq I_3$. If f is $\mathcal{B}^r$-integrable on I_1, I_2 and I_3 and $I_1 \cup I_2 \in \mathcal{I}$. Then, f is $\mathcal{B}^r$-integrable on $I_1 \cup I_2$.

Proof. Since the overlapping case was done, we assume that I_1 and I_2 are non-overlapping. We may redefine I_3 to be $I_3 \cap (I_1 \cup I_2) \in \mathcal{I}$ since it has all of the same properties. Then I_1 and I_3 are overlapping, f is $\mathcal{B}^r$-integrable on I_1 and on I_3. So by the previous proposition f is $\mathcal{B}^r$-integrable on $I_1 \cup I_3 \in \mathcal{I}$. Applying the previous theorem again to the intervals $I_1 \cup I_3$ and I_2, we have that f is $\mathcal{B}^r$-integrable on $I_1 \cup I_2 \cup I_3$. But $I_1 \cup I_2 \cup I_3 = I_1 \cup I_2$, so that the result holds.

□

Proposition 3.3.14.

A function f is $\widetilde{\mathcal{B}}_1$-integrable $\iff$ f is $\mathcal{B}_1$-integrable, in which case the integrals are equal.

Proof. Notice that any partition $\pi \subseteq \beta_\delta$ can be converted to a partition $\hat{\pi} \subseteq \widetilde{\beta}_\delta$ for which $f\lambda(\pi) = f\lambda(\hat{\pi})$. We do this by taking each $(x, I) \in \pi$ and cutting I into at most 2^n intervals having x as a vertex and using x as their tag. So the image filterbases generating the integrals for both of these bases are identical. Therefore, their integrals are equal.

$\square$

Corollary 3.3.15.

Let $f\lambda$ be $\mathcal{B}_1$-integrable on two non-overlapping $I_1, I_2 \in \mathcal{I}$ for which $I_1 \cup I_2 \in \mathcal{I}$. Then $f\lambda$ is $\mathcal{B}_1$-integrable on $I_1 \cup I_2$.

3.4. Absolute Integrability

One might wonder which of these bases produces an absolute integral in the traditional setting. This was studied in [**4**] and is presented a little more concisely here.

Definition 3.4.1.

Let $\mathcal{B}$ be a base, $\beta \in \mathcal{B}$ and let $\pi_1, \pi_2 \subseteq \beta$ be partitions of $I_0 \in \mathcal{I}$. Let $\mathcal{D}_1, \mathcal{D}_2$ be the respective divisions associated with these partitions by dropping their tags. Let $\mathcal{D}$ be a division of I_0 refining both $\mathcal{D}_1$ and $\mathcal{D}_2$ with $\mathcal{D} \subseteq \mathcal{I}$. Then we define $\pi^1 = \{(x, I) \colon I \in \mathcal{D}$ and there exists $(x_1, I_1) \in \pi_1$ such that $x = x_1, I \subseteq I_1\}$ and we call π^1 a π_2-refinement of π_1.

If $\mathcal{B}$ does not use all of $\mathcal{I}$ and only uses a certain subset then we assume that $\mathcal{D}$ is contained in this subset instead. For example when dealing with $\mathcal{B}^r$ we will assume that $\mathcal{D}$ is contained in the family of r-regular intervals.

DEFINITION 3.4.2. Refining base

We will say that a base $\mathcal{B}$ is refining if for any $\beta \in \mathcal{B}$ and partitions $\pi_1, \pi_2 \subseteq \beta$ of $I_0 \in \mathcal{I}$ and for any π_2-refinement of π_1, say π^1, we have that $\pi^1 \subseteq \beta$.

PROPOSITION 3.4.3.

$\mathcal{B}_1^\diamond, \mathcal{B}^{r^\diamond}, \mathcal{T}^\diamond$ are refining and $\mathcal{B}_1, \widetilde{\mathcal{B}}_1, \mathcal{B}^r, \mathcal{B}^\rho, \mathcal{T}$ are not refining.

PROOF. Let $\pi_1, \pi_2 \subseteq \beta \in \mathcal{B}$ where $\mathcal{B}$ is a base from those listed above. Also let π^1 be some π_2-refinement of π_1. Then for $(x, I) \in \pi^1$, we may have that $x \notin I$, it follows that $\mathcal{B}_1, \widetilde{\mathcal{B}}_1, \mathcal{B}^r, \mathcal{B}^\rho, \mathcal{T}$ are not refining. Now since there is an $(x, I_1) \in \pi_1$ with $I \subseteq I_1 \subseteq B(x, \delta(x))$, we have that $(x, I) \in \beta$ for $\beta \in \mathcal{B}_1^\diamond, \mathcal{B}^{r^\diamond}, \mathcal{T}^\diamond$.

$\square$

LEMMA 3.4.4.

Let $I_0 \in \mathcal{I}$, $f\colon I_0 \to \mathbb{R}$ and let $\mathcal{B}$ be a base. Then, the following are sufficient conditions for f to be $\mathcal{B}$-integrable on I_0.

(1) *For every $\epsilon > 0$, there exists a $\beta \in \mathcal{B}$ such that if $\pi_1, \pi_2 \subseteq \beta$ are partitions of I_0 and $\mathcal{D}$ is a refining division,*

$$\left| \sum_{(x_1,I_1)\in\pi_1} \sum_{(x_2,I_2)\in\pi_2} \sum_{I\in\mathcal{D}, I\subseteq I_1, I\subseteq I_2} (f(x_1) - f(x_2))\lambda(I) \right| < \epsilon.$$

(2) *For every* $\epsilon > 0$*, there exists a* $\beta \in \mathcal{B}$ *such that if* $\pi_1, \pi_2 \subseteq \beta$ *are partitions of* I_0 *and* $\mathcal{D}$ *is a refining division,*

$$\sum_{(x_1,I_1)\in\pi_1} \sum_{(x_2,I_2)\in\pi_2} \sum_{I\in\mathcal{D}, I\subseteq I_1, I\subseteq I_2} |f(x_1) - f(x_2)|\lambda(I) < \epsilon.$$

By a refining division we simply mean that $\mathcal{D}$ *refines the divisions associated with* π_1 *and* π_2*.*

Proof. It is clear that (2) implies (1), so it suffices to show that (1) holds. The fact that (1) implies integrability follows immediately from:

$$\begin{aligned}
&\left| \sum_{(x_1,I_1)\in\pi_1} f(x_1)\lambda(I_1) - \sum_{(x_2,I_2)\in\pi_2} f(x_2)\lambda(I_2) \right| \\
=&\left| \sum_{(x_1,I_1)\in\pi_1} \sum_{(x_2,I_2)\in\pi_2} \sum_{I\in\mathcal{D}, I\subseteq I_1, I\subseteq I_2} (f(x_1) - f(x_2))\lambda(I) \right| \\
<&\epsilon.
\end{aligned}$$

Therefore, we are dealing with a Cauchy filter in a complete space and we have convergence. □

Lemma 3.4.5.

If the second condition holds from the previous lemma, then $|f|$ *is also* $\mathcal{B}$*-integrable.*

Proof. Keeping the same notation as in the previous lemma,

$$\begin{aligned}
&\left| \sum_{(x,I)\in\pi_1} |f|(x)\lambda(I) - \sum_{(x,I)\in\pi_2} |f|(x)\lambda(I) \right| \\
=&\left| \sum_{(x_1,I_1)\in\pi_1} \sum_{(x_2,I_2)\in\pi_2} \sum_{I\in\mathcal{D}, I\subseteq I_1, I\subseteq I_2} (|f|(x_1) - |f|(x_2))\lambda(I) \right|
\end{aligned}$$

$$\leq \sum_{(x_1,I_1)\in\pi_1} \sum_{(x_2,I_2)\in\pi_2} \sum_{I\in\mathcal{D},I\subseteq I_1,I\subseteq I_2} |f(x_1)-f(x_2)|\lambda(I)$$

$$<\epsilon.$$

$\square$

THEOREM 3.4.6.

Let $\mathcal{B}$ be a refining base and $f\colon I_0 \to \mathbb{R}$. Then the following are equivalent:

(1) *f is $\mathcal{B}$-integrable;*

(2) *the second condition of lemma 3.4.4 holds;*

(3) *$|f|$ is $\mathcal{B}$-integrable;*

(4) *f^+ and f^- are $\mathcal{B}$-integrable.*

PROOF. (1) $\implies$ (2) Suppose that f is integrable, let $\epsilon > 0$ and choose a $\beta \in \mathcal{B}$ by the definition of integrability. Let $\pi_1, \pi_2 \subseteq \beta$ be partitions of I_0 and $\pi^1, \pi^2 \subseteq \beta$ be refinements generated by the same division $\mathcal{D}$. We define σ_+ to be the set of (x, I) such that there exists $(x_1, I_1) \in \pi^1$ and $(x_2, I_2) \in \pi^2$ with $I_1 = I_2 = I$ and $x = x_1$ if $f(x_1) - f(x_2) \geq 0$ and $x = x_2$ otherwise. Similarly, we define σ_- to be the set of (x, I) such that there exists $(x_1, I_1) \in \pi^1$ and $(x_2, I_2) \in \pi^2$ with $I_1 = I_2 = I$ and $x = x_1$ if $f(x_1) - f(x_2) < 0$ and $x = x_2$ otherwise. Since $\mathcal{B}$ is refining,

$$\sum_{(x_1,I_1)\in\pi_1} \sum_{(x_2,I_2)\in\pi_2} \sum_{I\in\mathcal{D},I\subseteq I_1,I\subseteq I_2} |f(x_1)-f(x_2)|\lambda(I) = |f\lambda(\sigma_+) - f\lambda(\sigma_-)| < 2\epsilon.$$

(2) $\implies$ (1) This was done in lemma 3.4.4.

(1) $\iff$ (3) f is integrable if and only if the second condition of lemma 3.4.4 holds which happens if and only if $|f|$ is integrable.

(1) $\iff$ (4) Since $f = f^+ - f^-$ and $|f| = f^+ + f^-$ the result follow immediately.

$\square$

Corollary 3.4.7.

If $\mathcal{B}$ is one of the bases $\mathcal{B}_1^\diamond, \mathcal{B}^{r^\diamond}, \mathcal{T}^\diamond$, a function $f : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{B}$ integrable on $I_0 \in \mathcal{I}$ if and only if $|f|$ is $\mathcal{B}$ integrable on I_0.

Notice that the use of Lebesgue measure in this section was of no importance. We could have had the same results with any additive function on $\mathcal{I}$.

3.5. Relation to the Lebesgue Integral

Naturally, now that we have established that some of these integrals are absolute (in the traditional setting), we begin to question their relation to the Lebesgue integral. It turns out that all of these integrals are extensions of the Lebesgue integral and the family of absolutely integrable functions for any of these integrals is exactly the Lebesgue integrable functions.

Following [**5**] we first show that the indefinite integral of a function f is differentiable almost everywhere and equal to f on these places.

Proposition 3.5.1.

Let f be $\mathcal{B}^r$-integrable on I_0 and $F(I)$ be its indefinite integral. Then $D_{\mathcal{B}^\rho} F_\lambda(x) = f(x)$ a.e. on I_0.

PROOF. Let $E_1 = \{x \in I_0 : D_{\mathcal{B}^\rho} F_\lambda(x) = f(x)\}$ and $E = I_0 \setminus E_1$. For $x \in E$ there exists an $\epsilon_x > 0$ such that for any $\beta_\delta^\rho \in \mathcal{B}^\rho$ there is some $(x, I) \in \beta_\delta^\rho[\{x\}]$ with

$$\left|\frac{F(I)}{\lambda(I)} - f(x)\right| \geq \epsilon_x.$$

Define $E_n = \{x \in E : \epsilon_x \geq \frac{1}{n}\}$ for $n = 2, ...$ Choose an $\epsilon \in (0, \frac{1}{n})$ and find a $\beta_\delta^\rho \in \mathcal{B}^\rho$ such that for any partition $\pi \subseteq \beta_\delta^\rho$ of I_0,

$$\sum_{(x,I)\in\pi} |f(x)\lambda(I) - F(I)| < \frac{\epsilon}{n}.$$

Let $\mathcal{R} = \{I \in \mathcal{I} : (x_I, I) \in \beta_\delta[E_n], R(I) \geq \rho(x_I), I \subseteq I_0$ and $|\frac{F(I)}{\lambda(I)} - f(x_I)| \geq \frac{1}{n}\}$. Then $\mathcal{R}$ is a Vitali cover of E_n, so by the Vitali covering theorem 3.1.10, there exists a sequence $\{I_k\}$ of non-overlapping intervals in $\mathcal{R}$ such that

$$\lambda(E_n \setminus \bigcup_{k\in\mathbb{N}} I_k) = 0.$$

There is a $k \in \mathbb{N}$ such that $\lambda(E_n \setminus \bigcup_{i=1}^k I_i) < \epsilon$. Now $\{(x_{I_1}, I_1), ..., (x_{I_k}, I_k)\}$ can be extended to some partition $\pi \subseteq \beta_\delta^\rho$ of I_0. Therefore,

$$\frac{1}{n}\lambda(I_i) \leq |F(I_i) - f(x_{I_i})\lambda(I_i)| \implies \sum_{i=1,..,k} \lambda(I_i) \leq n \sum_{i=1,...,k} |F(I_i) - f(x_{I_i})\lambda(I_{I_i})| \leq \epsilon.$$

Thus $\lambda(E_n) = 0$ for every $n \geq 2$, so that $\lambda(E) = 0$.

□

LEMMA 3.5.2.

The arbitrary union of non-degenerate intervals is measurable.

PROOF. Let $\mathcal{H}$ be a family of non-degenerate intervals. For any $x \in \bigcup \mathcal{H}$ there is a cube containing x with arbitrarily small diameter contained in $\bigcup \mathcal{H}$. Therefore, by the Vitali covering theorem 3.1.10, there is a countable family $\mathcal{D}$ of cubes inside $\mathcal{H}$ such that $\lambda(\bigcup \mathcal{H} \setminus \mathcal{D}) = 0$. Since, each $D \in \mathcal{D}$ is measurable, $\bigcup \mathcal{H}$ can be expressed as the union of a measurable set and a null set. Therefore, $\bigcup \mathcal{H}$ is measurable.

□

We now establish measurability of derivatives as in [**8**].

THEOREM 3.5.3.

The $\mathcal{B}^\rho$-derivative (with respect to Lebesgue measure) of any function is measurable if it exists.

PROOF. We will actually show that the upper derivative is measurable but since the derivative is equal to the upper derivative the result will be shown. Let $F : \mathbb{R}^n \times \mathcal{I} \to \mathbb{R}$, for the remainder of this proof we will denote the upper derivative of F simply by $\bar{F}$. Let $a \in \mathbb{R}$ and let $Q = \{x \in \mathbb{R} : \bar{F}(x) > a\}$. Now if $x \in Q$, there exists a number $\alpha > 0$ and a sequence $\{I_n^x\} \subseteq \mathcal{I}$ such that $\lambda(I_n^x) \to 0$, $x \in I_n^x$ and $\frac{F(I_n^x)}{\lambda(I_n^x)} \geq a + \alpha$. Let $Q_{h,k} = \bigcup_{x \in Q} \bigcup I_n^x$ where the second union is over all I_n^x for which $\operatorname{diam}(I_n^x) < \frac{1}{h}$ and $\frac{F(x, I_n^x)}{\lambda(I_n^x)} \geq a + \frac{1}{h}$. Then $Q_{h,k}$ is the union of intervals and is therefore measurable. Notice that $Q = \bigcup_{h \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} Q_{h,k}$, so that Q is measurable. □

COROLLARY 3.5.4.

Each $\mathcal{B}^\rho$-integrable function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is measurable.

PROOF. Let f be a $\mathcal{B}^\rho$-integrable function with indefinite integral F. Then $D_{\mathcal{B}^\rho} F(x) = f(x)$ a.e. and $D_{\mathcal{B}^\rho} F$ is measurable, so that f is measurable. □

Finally we establish the equivalence of the integral generated by the base $\mathcal{B}_1^\diamond$ and the Lebesgue integral. Half of the proof follows [**5**]. Most of the work is done once the connection between the Lebesgue integral and gauges is introduced via semi-continuity.

THEOREM 3.5.5.

For $f : \mathbb{R}^n \to \mathbb{R}$ and $I_0 \in \mathcal{I}$, f is $\mathcal{B}_1^\diamond$-integrable on $I_0 \iff f$ is Lebesgue integrable on I_0, in which case the integrals are equal.

PROOF. ($\Leftarrow$) Suppose that f is Lebesgue integrable on $I_0 \in \mathcal{I}$. There are functions g and h on I_0, which are upper and lower semi-continuous respectively and for which $g \leq f \leq h$ and $\int_{I_0} h - g d\lambda < \frac{\epsilon}{2}$. Find a gauge δ on I_0 such that $g(y) \leq g(x) + \frac{\epsilon}{2\lambda(I_0)}$ and $h(y) \geq h(x) - \frac{\epsilon}{2\lambda(I_0)}$ for any $x, y \in I_0$ with $\text{dist}(x, y) < \delta(x)$. Let $\pi = \{(x_1, I_1), ..., (x_n, I_n)\}$ be a δ-fine partition of I_0. Then,

$$g_{|I_i} - \frac{\epsilon}{2\lambda(I_0)} \leq g(x_i) \text{ and } h_{|I_i} + \frac{\epsilon}{2\lambda(I_0)} \geq h(x_i),$$

and integrating over I_i,

$$\int_{I_i} g d\lambda - \frac{\epsilon\lambda(I_i)}{2\lambda(I_0)} \leq g(x_i)\lambda(I_i) \leq f(x_i)\lambda(I_i) \leq h(x_i)\lambda(I_i) \leq \int_{I_i} h d\lambda + \frac{\epsilon\lambda(I_i)}{2\lambda(I_0)}.$$

Subtracting $\int_{I_i} f d\lambda$ and noticing that $g \leq f \leq h$,

$$\left| f(x_i)\lambda(I_i) - \int_{I_i} f d\lambda \right| \leq \frac{\epsilon\lambda(I_i)}{2\lambda(I_0)} + \int_{I_i} h - g d\lambda.$$

Summing over all i,

$$\left| f\lambda(\pi) - \int_{I_0} f d\lambda \right|$$

$$\leq \sum_{i=1}^{p} \left| f(x_i)\lambda(I_i) - \int_{I_i} f d\lambda \right|$$

$$\leq \frac{\epsilon}{2} + \int_{I_0} h - g d\lambda$$

$$\leq \epsilon.$$

($\Rightarrow$) If f is $\mathcal{B}_1^{\diamond}$-integrable then both f^+ and f^- are $\mathcal{B}_1^{\diamond}$-integrable. Then they are $\mathcal{B}^{\rho}$-integrable so that they are both measurable. Now for any $M \geq 0$, M is $\mathcal{B}_1^{\diamond}$-integrable. Notice that $\min\{f^+, M\} = \frac{1}{2}(f^+ + M + |f^+ - M|)$ and since $\mathcal{B}_1^{\diamond}$ is an absolute integral we have that $\min\{f^+, M\}$ is $\mathcal{B}_1^{\diamond}$-integrable. But this is a measurable function that is positive and is bounded above by a Lebesgue integrable function, namely M, so that it is Lebesgue integrable. Thus we can form an increasing sequence of Lebesgue integrable functions converging to f^+, so that by the monotone convergence theorem f^+ is Lebesgue integrable. Indeed,

$$\int_{I_0} f^+ d\lambda = \lim \int_{I_0} \min\{f^+, M\} d\lambda = \lim(\mathcal{B}_1^{\diamond}) \int_{I_0} \min\{f^+, M\} d\lambda \leq (\mathcal{B}_1^{\diamond}) \int_{I_0} f^+ d\lambda < \infty.$$

Similarly we can show that f^- is Lebesgue integrable. But, $f = f^+ - f^-$ so that f is Lebesgue integrable. $\square$

Notice that if $f \geq 0$ and is integrable with respect to ANY of the non-triangular gauge bases that the previous theorem holds.

Corollary 3.5.6.

For any of the non-triangular gauge bases presented, the family of absolutely integrable functions with respect to the base is precisely the Lebesgue integrable functions.

From this it is easy to take results from the Lebesgue integral and apply them to the integrals generated by the non-triangular gauge bases. For example, we may now use the monotone convergence and dominated convergence theorems for the integrals generated by the non-triangular gauge bases.

PROPOSITION 3.5.7.

For $f : I_0 \mapsto \mathbb{R}$,

$$(\mathcal{B}_1^\diamond)\overline{\int}_{I_0} f d\lambda = (\mathcal{B}^{r^\diamond})\overline{\int}_{I_0} f d\lambda$$

and

$$(\mathcal{B}_1^\diamond)\underline{\int}_{I_0} f d\lambda = (\mathcal{B}^{r^\diamond})\underline{\int}_{I_0} f d\lambda.$$

PROOF. We prove one of the statements and the other is similar.

Since $\mathcal{B}^{r^\diamond}$ is finer than $\mathcal{B}_1^\diamond$, it suffices to show that

$$(\mathcal{B}_1^\diamond)\overline{\int}_{I_0} f d\lambda \leq (\mathcal{B}^{r^\diamond})\overline{\int}_{I_0} f d\lambda.$$

We assume that $(\mathcal{B}_1^\diamond)\overline{\int}_{I_0} f d\lambda > -\infty$ or else the result follows trivially. Let $\epsilon > 0$ and choose a $\beta_\delta^{r^\diamond} \in \mathcal{B}^{r^\diamond}$ such that for any partition $\pi_2 \subseteq \beta_\delta^{r^\diamond}$ of I_0,

$$(\mathcal{B}^{r^\diamond})\overline{\int}_{I_0} f d\lambda + \epsilon \geq f\lambda(\pi_2).$$

Let $\beta_\delta^\diamond \in \mathcal{B}_1^\diamond$ and $\pi_1 \subseteq \beta_\delta^\diamond$ be a partition of I_0. Now for any $(x, I) \in \pi_1$, I can be divided into a finite number of r-regular intervals. Keeping the same tag x for each of these intervals and doing this for each element of π_1, we can construct a new partition of I_0 whose Riemann sum is identical to that of π_1 and is contained in $\beta_\delta^{r^\diamond}$. So we have that

$$(\mathcal{B}^{r^\diamond})\overline{\int}_{I_0} f d\lambda + \epsilon \geq \sum_{(x,I)\in\pi_1} f\lambda(\pi_1).$$

Taking the supremum over all $\pi_1 \subseteq \beta_\delta^\diamond$ followed by the infimum over all $\beta_\delta^\diamond \in \mathcal{B}_1^\diamond$ we have the result.

□

PROPOSITION 3.5.8.

For $f : I_0 \mapsto \mathbb{R}$,

$$(\mathcal{B}_1^\diamond)\overline{\int}_{I_0} f d\lambda = (\mathcal{B}^{\rho^\diamond})\overline{\int}_{I_0} f d\lambda$$

and

$$(\mathcal{B}_1^\diamond)\underline{\int}_{I_0} f d\lambda = (\mathcal{B}^{\rho^\diamond})\underline{\int}_{I_0} f d\lambda.$$

PROOF. This proof is nearly identical to proposition 3.5.7.

□

This is one of the tougher equivalences to show since an exact cutting procedure is not used. This was presented in [**4**] on page 32.

PROPOSITION 3.5.9.

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $I_0 \in \mathcal{I}$. Then, f is $\mathcal{B}_1^\diamond$-integrable on I_0 if and only if it is $\mathcal{T}^\diamond$-integrable on I_0. In which case both integrals are equal.

PROOF. Let $I_0 \in \mathcal{I}$ and $f : I_0 \mapsto \mathbb{R}$. Since $\mathcal{B}_1^\diamond$ and $\mathcal{T}^\diamond$ produce absolute integrals we may assume that $f \geq 0$. Following a similar process to proposition 3.5.7 it is easy to see that

$$(\mathcal{T}^{\diamond})\underline{\int}_{I_0} f d\lambda \leq (\mathcal{B}_1^{\diamond})\underline{\int}_{I_0} f d\lambda \leq (\mathcal{B}_1^{\diamond})\overline{\int}_{I_0} f d\lambda \leq (\mathcal{T}^{\diamond})\underline{\int}_{I_0} f d\lambda.$$

So we need only show that if f is $\mathcal{B}_1^{\diamond}$-integrable then it is also $\mathcal{T}^{\diamond}$-integrable since the other side is trivial.

Suppose that f is $\mathcal{B}_1^{\diamond}$-integrable and let $\epsilon > 0$. Choose a $\beta_\delta^{\diamond} \in \mathcal{B}_1^{\diamond}$ such that for every partition $\pi_1 \subseteq \beta_\delta^{\diamond}$ in I_0,

$$\left| \sum_{(x,I)\in\pi_1} f(x)\lambda(I) - (\mathcal{B}_1^{\diamond})\int_I f d\lambda \right| < \frac{\epsilon}{4}.$$

Now since f is $(\mathcal{B}_1^{\diamond})$-integrable it is also Lebesgue integrable so there exists an $\eta > 0$ such that for $I \in \mathcal{I}$ with $\lambda(I') < \eta$ and $I' \subseteq I_0$,

$$(\mathcal{B}_1^{\diamond})\int_{I'} f d\lambda = \int_{I'} f d\lambda < \frac{\epsilon}{4}.$$

Let $\pi_3 \subseteq \tau_3^*$ be a partition of I_0. Now for any $(x, T) \in \pi_3$, we can find a finite number of intervals $I_1^x, ..., I_{k_x}^x$ contained in T° that are non-overlapping such that if s is the number of elements in π_3,

$$f(x)\lambda(T \setminus \bigcup_{i=1}^{k_x} I_i^x) < \frac{\epsilon}{4s}$$

and

$$\lambda(T \setminus \bigcup_{i=1}^{k} I_i^x) < \frac{\eta}{s}.$$

Let $\pi_1' = \bigcup_{(x,I)\in\pi_3} \bigcup_{i=1}^{k_x} \{(x, I_i^x)\}$, then $\pi_1' \subseteq \beta_\delta^{\diamond}$ and

$$\lambda(I_0 \setminus (\bigsqcup \pi_1')^\circ) \geq \sum_{(x,I)\in\pi_1'} \lambda(I \setminus \bigcup_{i=1}^{k_x} I_i^x) > s\frac{\eta}{s} = \eta.$$

Choose a partition $\pi_1'' \subseteq \beta_\delta^\diamond$ of $I_0 \setminus (\bigsqcup \pi_1')^\circ$. Then,

$$
\begin{aligned}
&|\sum_{(x,I)\in\pi_3} f(x)\lambda(I) - (\mathcal{B}_1^\diamond)\int_{I_0} f d\lambda| \\
&\leq |\sum_{(x,I)\in\pi_3} f(x)\lambda(I) - \sum_{(x,I)\in\pi_1'} f(x)\lambda(I)| + |\sum_{(x,I)\in\pi_1'} f(x)\lambda(I) - (\mathcal{B}_1^\diamond)\int_{I_0} f d\lambda| \\
&\leq \frac{\epsilon}{4} + |\sum_{(x,I)\in\pi_1'} f(x)\lambda(I) + \sum_{(x,I)\in\pi_1''} f(x)\lambda(I) - \sum_{(x,I)\in\pi_1''} f(x)\lambda(I) - (\mathcal{B}_1^\diamond)\int_{I_0} f d\lambda| \\
&\leq \frac{\epsilon}{4} + |\sum_{(x,I)\in\pi_1'\bigcup\pi_1''} f(x)\lambda(I) - (\mathcal{B}_1^\diamond)\int_{I_0} f d\lambda| + |\sum_{(x,I)\in\pi_1''} f(x)\lambda(I)| \\
&\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + |(\mathcal{B}_1^\diamond)\int_{I_0\setminus(\bigsqcup\pi_1')^\circ} f d\lambda| + |\sum_{(x,I)\in\pi_1''} f(x)\lambda(I) - (\mathcal{B}_1^\diamond)\int_I f d\lambda| \\
&\leq \epsilon.
\end{aligned}
$$

Therefore, f is $\mathcal{T}^\diamond$-integrable on I_0.

$\square$

THEOREM 3.5.10.

For $I_0 \in \mathcal{I}$ and $f : I_0 \mapsto \mathbb{R}$ the following are equivalent

(1) *f is Lebesgue integrable on I_0;*

(2) *f is $\mathcal{B}_1^\diamond$-integrable on I_0;*

(3) *f is $\mathcal{B}^{r^\diamond}$-integrable on I_0;*

(4) *f is $\mathcal{B}^{\rho^\diamond}$-integrable on I_0;*

(5) *f is $\mathcal{T}^\diamond$-integrable on I_0;*

Where all of the integrals are equal if they exist.

PROOF. This result is just a collection of previous results.

□

Corollary 3.5.11.

If f is Lebesgue integrable, then it is also $\mathcal{B}$-integrable where $\mathcal{B}$ is any of the gauge bases presented in this chapter. Thus all of the gauge integrals introduced are simply extensions of the Lebesgue integral.

Proposition 3.5.12.

Let $f : I_0 \mapsto \mathbb{R}$ for $I_0 \in \mathcal{I}$ we have,

$$(\mathcal{T})\overline{\int}_{I_0} f d\lambda \geq (\mathcal{B}_1)\overline{\int}_{I_0} f d\lambda \geq (\mathcal{B}^r)\overline{\int}_{I_0} f d\lambda$$

and

$$(\mathcal{T})\underline{\int}_{I_0} f d\lambda \leq (\mathcal{B}_1)\underline{\int}_{I_0} f d\lambda \leq (\mathcal{B}^r)\underline{\int}_{I_0} f d\lambda.$$

Proof. The relation between $\mathcal{B}_1$ and $\mathcal{B}^r$ is clear since $\mathcal{B}^r$ is finer than $\mathcal{B}_1$. So we need only show the relation between $\mathcal{T}$ and $\mathcal{B}_1$. We show one of the results, the other is similar.

Let $\epsilon > 0$ and choose a gauge δ such that for any partition $\pi_3 \subseteq \tau_\delta$ we have $(\mathcal{T})\overline{\int}_{I_0} f d\lambda + \epsilon \geq \sum_{(x,T)\in\pi_3} f(x)\lambda(I)$. Let $\pi_1 \in \beta_\delta$ be a partition of I_0. Now for each $(x, I) \in \pi_1$ we cut I into at most four non-overlapping triangles tagged at x. Putting all of these triangles along with their tags in a set we have made a partition $\pi_3 \subseteq \tau_\delta$ which has the same Riemann sum as π_1. Therefore,

$$(\mathcal{T})\overline{\int}_{I_0} f d\lambda + \epsilon \geq \sum_{(x,I)\in\pi_1} f(x)\lambda(I).$$

Taking the supremum over all such partitions $\pi_1 \in \beta_\delta$ followed by an infimum over all $\beta_\delta \in \mathcal{B}_1$ we see that the result holds. □

Corollary 3.5.13.

If a function f is $\mathcal{T}$-integrable, then it is also $\mathcal{B}^r$-integrable, $\mathcal{B}^\rho$-integrable and $\mathcal{B}_1$-integrable.

Theorem 3.5.14.

For any of the gauge bases, the family of absolutely integrable functions with respect to the base is precisely the Lebesgue integrable functions.

Proof. This follows immediately from 3.5.6, 3.5.10 and the previous corollary.

□

Now we show that the triangular base does not produce the same integral as the ones produced by the rectangular bases. This is example is outlined in [**4**]; however, no solution is presented. Rather than using rotation invariance as in the reference we opt for the more direct route giving a full solution.

Theorem 3.5.15.

The bases $\mathcal{B}_1$ and $\mathcal{T}$ produce different integrals even in the classical setting.

Proof. Let $I_0 = [0,1] \times [0,1]$, $a_n = 1 - 2^{-n}$ for $n = 0, ...$ and for $n \in \mathbb{N}$ let $K_n = [a_{n-1}, a_n] \times [a_{n-1}, a_n]$ and $L_n = \{(u,v) \in K_n : v \leq u\}$.

Now for each $n \in \mathbb{N}$ construct a function $f_n : K_n \mapsto \mathbb{R}$ such that

(1) f_n is continuous on K_n, and $f_n = 0$ on $\mathrm{bd}(K_n)$;

(2) $f_n \geq 0$ on L_n;

(3) $f_n(u,v) = -f_n(v,u)$ for $(u,v) \in K_n$;

(4) $\int_{L_n} f_n(u,v)dudv = \frac{1}{n}$ under Lebesgue integration;

(5) $f_n = 0$ on $I_0 \setminus K_n$.

We define $f(u,v) = \sum_{i=1}^{\infty} f_i(u,v)$.

Claim f is not Lebesgue integrable on I_0

Indeed, $f^+ \geq \sum_{k=1}^{n} f_k^+$ so that $\int_{I_0} f^+ d\lambda \geq \sum_{k=1}^{n} \int_{I_0} f_k^+ d\lambda = \sum_{k=1}^{n} \frac{1}{k} \to \infty$. Using the same method we see that $\int_{I_0} f^- d\lambda \geq \infty$ also.

Claim f is $\mathcal{B}_1$-integrable on I_0

Let $\epsilon > 0$ and find an $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{\epsilon}{2}$. Choose a gauge δ on I_0 such that:

(1) $\delta(x) < \operatorname{dist}(x,(1,1))$, for $x \neq (1,1)$ to tag (1,1);

(2) $\delta(x) < \operatorname{dist}(x, \operatorname{bd}(K_n))$, for $x \notin \operatorname{bd}(K_n)$ to force tags in K_n;

(3) $\delta(1,1) < \frac{1}{2^m}$;

(4) $\delta(x)$ small enough so that for any partition $\pi \subseteq \beta_\delta[K_n]$, $|\sum_{(x,I)\in\pi}(f(x)\lambda(I) - \int_I f d\lambda)| < \frac{\epsilon}{2^{n+1}}$.

Condition (4) is possible since f is Lebesgue integrable on $B(K_n, \eta)$ where eta is a positive small number. Let $\pi \subseteq \beta_\delta$ be a partition of I_0, then we have that $((1,1),J) \in \pi$ for some interval J. Now this interval J cuts through some K_n, let b be the smallest such integer where this occurs.

Remembering that our partition could have an error up to $\frac{\epsilon}{2^{b+1}}$ in approximating the volume over $K_b \setminus J$ and that $\delta((1,1)) < \frac{1}{m}$, we know that $|\sum_{(x,I)\in\pi[K_n]} f(x)\lambda(I)| \leq \frac{1}{m} + \frac{\epsilon}{2^{b+1}} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2^{b+1}}$. Therefore,

$$
\begin{aligned}
|\sum_{(x,I)\in\pi} f(x)\lambda(I)| &= |\sum_{(x,I)\in\pi[\bigcup_{k=1}^{b} K_k]} f(x)\lambda(I)| \\
&\leq |\sum_{(x,I)\in\pi[K_b]} f(x)\lambda(I)| + |\sum_{(x,I)\in\pi[\bigcup_{k=1}^{b-1} K_k]} f(x)\lambda(I)|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2^{b+1}} + \Big| \sum_{(x,I)\in\pi[\bigcup_{k=1}^{b-1} K_k]} f(x)\lambda(I)\Big| \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2^{b+1}} + \sum_{k=1}^{b-1} \Big| \sum_{(x,I)\in\pi[K_k]} f(x)\lambda(I)\Big| \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2^{b+1}} + \sum_{k=1}^{b-1} \frac{\epsilon}{2^{k+1}} \\
&\le \epsilon.
\end{aligned}
$$

Therefore, f is $\mathcal{B}_1$-integrable on I_0.

Claim f is not $\mathcal{T}$-integrable on I_0

Let $\epsilon > 0$ and δ be any gauge. By the work done earlier, if f is $\mathcal{T}$-integrable on I_0, then its integral must be 0. Choose an $n_0 \in \mathbb{N}$ such that $\frac{1}{2^{n_0-1}} < \delta(1,1)$. Let A be the triangle with corners $(1,1)$, (a_{n_0}, a_{n_0}) and $(a_{n_0}, 1)$. Then, $((1,1), A) \in \tau_\delta$. Choose $l, m \in \mathbb{N}$ such that $\sum_{i=l}^{m} \frac{1}{i} > \epsilon$. For $i = l, ..., m$ find a partition $\pi_i \subset \tau_\delta$ of L_i, with $f\lambda(\pi_i) > \frac{1}{i} - \frac{\epsilon}{2^{i+1}}$. This is possible since f is Lebesgue integrable on L_i. Let $\pi = \{((1,1), A)\} \cup \bigcup_{i=l}^{m} \pi_i$. We extend π to a partition $\hat{\pi} \subset \tau_\delta$ of I_0 by partitioning symmetrically outside of $[a_{n_0}, 1] \times [a_{n_0}, 1]$ and however we would like for the rest of $[a_{n_0}, 1] \times [a_{n_0}, 1]$. Then,

$$
\begin{aligned}
|f\lambda(\hat{\pi})| &\ge |f\lambda(\pi)| \\
&= \sum_{i=l}^{m} f\lambda(\pi_i) \\
&> \sum_{i=l}^{m} \frac{1}{i} - \frac{\epsilon}{2^{i+1}} \\
&> \frac{\epsilon}{2}.
\end{aligned}
$$

Therefore, f is not $\mathcal{T}$-integrable on I_0.

$\square$

Notice that this example carries over for the bases $\mathcal{B}^r$ and $\mathcal{B}^\rho$.

Corollary 3.5.16.

Even in the traditional setting, $\mathcal{T}$ *provides a different integral than* $\mathcal{B}_1$*,* $\mathcal{B}^r$ *and* $\mathcal{B}^\rho$*.*

Corollary 3.5.17.

$\mathcal{B}_1$*,* $\mathcal{B}^r$ *and* $\mathcal{B}^\rho$ *provide integrals that are different from that of the Lebesgue integral.*

CHAPTER 4

Fixed Regularity Investigation

In this chapter we investigate the effects of changing the regularity r in Kempisty fixed regularity bases. The main questions being, does changing the regularity really affect the integrabiliy or differentiability of functions? The material here was presented in [**2**]. We have tried to include a little more direction and detail in the proof.

4.1. Fixed Regularity Differentiability

It is clear that for two different regularities we can make functions that are differentiable with respect to Lebesgue measure under one regularity but not the other. For example, if we are given two real numbers r_1 and r_2 with $0 < r_1 < r_2 < 1$ we can make the function $F : \mathcal{I} \mapsto \mathbb{R}$ by $F(I) = \lambda(I)$ for r_2-regular intervals and 0 else. This function will be r_2-differentiable at every point to 1 but will fail to be r_1-differentiable anywhere. What sort of conditions can we impose on the function in question to ensure a change in regularity does not alter differentiability? We know that any interval can be subdivided into r-regular pieces, so maybe if the function were additive for any fixed $x \in \mathbb{R}^n$ this could be of use. This approach does lead to a sufficient condition and in fact, we could simply require sub-additivity of the function and achieve the same result.

For the rest of this section we will deviate from our traditional notation in order to simplify the discussion. Often times we will write the integral over a family which is not actually an interval but is a finite union of non-overlapping intervals. This is to be interpreted in the additive sense. We first begin with the notation for the section, let $G : \mathcal{I} \mapsto \mathbb{R}$ be an additive function for fixed $x \in \mathbb{R}^n$. Let $t \in \mathbb{R}^n$, $0 < \alpha < 1$ and $\delta > 0$. Define,

$$\omega = \sup\{|G(I)| : t \in I = [u, v], \alpha\delta \leq v_i - u_i \leq \delta \text{ for all } i\} = \omega(t, \delta, G, \alpha) \text{ and}$$

$$\Omega = \sup\{|G(I)| : I \subseteq B(t, \delta), I = [u, v]\} = \Omega(t, \delta, G, \alpha).$$

Proposition 4.1.1.

Let $n \in \mathbb{N}$. There exists a constant κ such that

$$\omega \leq \Omega \leq \kappa\omega.$$

Proof. Let $Q = \{x \in \mathbb{R}^n : x_i \geq t_i \text{ for } i = 1, ..., n\}$. We estimate $G(I)$ for $I = [a, b] \subseteq B(t, \delta) \cap Q$ with $b_i - a_i \leq (1 - \alpha)\delta$.

Let $c = (c_1, .., c_n) = (b_1 - \delta, .., b_n - \delta)$. Then, $t \in [c, a] \subseteq [c, b] \subseteq B(t, \delta)$. So that,

$$1_{[a_i,b_i)} = 1_{[c_i,b_i)} - 1_{[c_i,a_i)} \text{ and } 1_{I^*} = \sum_H \sigma(H)1_{H^*}$$

where the sum is taken over all intervals $H = [u, v]$ such that $[u_i, v_i] \in \{[c_i, b_i], [c_i, a_i]\}$ for $i = 1, .., n$, the star denotes the corresponding half-open interval i.e., $I^* = [a, b)$ and $\sigma(H) \in \{-1, 1\}$ is chosen appropriately by the inclusion exclusion formula. Notice that the number of summands is less than 2^n (two options n trials). Moreover, $t \in H$ since $c_i = b_i - \delta \leq (t_i + \delta) - \delta = t_i$ and $b_i \geq t_i$. Also, since $b_i - c_i > a_i - c_i =$

$b_i - c_i - (b_i - a_i) \geq b_i - c_i - (1-\alpha)\delta = \alpha\delta$ and $b_i - c_i = \delta$ we have that $|G(H)| \leq \omega$, thus

$$|G(I)| \leq \sum_H |G(H)| \leq 2^n \omega.$$

Now for $[u,v] \subseteq B(t,\delta) \cap Q$ we have $v_i - u_i \leq \delta$. Choose any $m \in \mathbb{N}$ with $m \geq (1-\alpha)^{-1}$, then $\frac{(v_i - u_i)}{m} \leq \delta(1-\alpha)$ and we can cut $[u,v]$ into m^n intervals for which the previous estimate is applicable. Hence,

$$|G[u,v]| \leq m^n 2^n \omega.$$

Now the above arguments could have been used for any one of the orthants in $\mathbb{R}^n$(with t as origin). Therefore, since there are 2^n such orthants,

$$|G(I)| \leq 2^n m^n 2^n \omega$$

for any interval $I \subseteq B(t,\delta)$, hence

$$\Omega \leq (4m)^n \omega.$$

The inequality $\omega \leq \Omega$ is obvious.

$\square$

Corollary 4.1.2.

Suppose that G is α-lipschitzian at $t \in I_0^\circ$.(That is, there is an $\eta > 0$ and $r > 0$ such that $|G(K)| \leq \eta\lambda(K)$ for every interval K with $t \in K \subseteq B(t,r)$ with $R(K) \geq \alpha$). Then the inequality,

$$|G(J)| \leq \kappa\eta(2r_0)^n$$

holds for every interval $J \subseteq B(t, r_0)$ with $r_0 \leq r$. In particular, G is β-lipschitzian at t for any $0 < \beta < 1$.

PROOF. For $\omega = \omega(t, r_0, G, \alpha)$, we have that $\omega \leq \Omega(2r_0)^n$ by α-lipschitzian. Therefore, $|G(J)| \leq \Omega \leq \kappa\omega \leq \kappa\eta(2r_0)^n$ for any interval $J \subseteq B(t, r_0)$. Taking r_0 to be the length of the maximal edge of J, we have that given $0 < \beta < 1$ with $t \in J \subseteq B(t, r)$ and $R(J) \geq \beta$,

$$|G(J)| \leq \kappa\eta(2r_0)^n \leq \kappa\eta\beta^{-n}\lambda(J).$$

□

THEOREM 4.1.3.

Let $0 < \alpha < 1, 0 < \beta \leq 1$. Let F be an additive interval function that is α-differentiable to c at t. Then, F is β-differentiable to c at t.

PROOF. The result is clear for $\alpha \leq \beta$ so we assume that $\beta < \alpha$. Notice that $F - c\lambda$ is α-lipschitzian at t since F has derivative c at t, thus by the previous corollary it is β-lipschitzian there. Thus F is β-differentiable to c at t.

□

4.2. Fixed Regularity Integrability

In this section we wish to show that given any $0 < \alpha < 1$ there is a function f that is α_1-integrable for any $\alpha_1 \geq \alpha$, that is also not α_2-integrable for any $\alpha_2 < \alpha$. We will now describe the construction of the function and prove several lemmas leading to the desired result.

Let $0 < \alpha < 1$ and choose two sequences $\{r_k\}$,$\{a_k\}$ such that $\frac{1}{3} \geq r_k \searrow 0$, $\frac{1}{3} > a_k \searrow 0$ and also $r_0(\alpha^{-1} + a_0) \leq 2$. Let $S_0 = [0, 1]$, remove the concentric open interval T_0 of length $|S_0| - 2r_0|S_0|$, calling the two remaining intervals S_1^1 and S_1^2 (ordered from left to right). We call the two resulting intervals the S intervals of the first order. Now taking the S intervals of the first order and from each we remove a concentric open interval of length $|S_1| - 2r_1|S_1|$, which we denote T_1^1 and T_1^2 and we call the T intervals of the first order. After removing the T intervals of the first order from the S intervals of the first order, we have four intervals S_2^1, S_2^2, S_2^3 and S_2^4 of length $|S_1|r_1$ (once again the ordering is from left to right). We refer to these intervals as the S intervals of the second order. We continue the process inductively. In general, given 2^n intervals $S_n^1, ..., S_n^{2^n}$ remove an open concentric interval T_n^i of length $|S_n| - 2r_n|S_n|$ from each S_n^i leaving the intervals S_n^{2i} and S_n^{2i-1}. The result of which are the intervals $S_{n+1}^1, ..., S_{n+1}^{2^{n+1}}$ of length $r_n|S_n|$, which we call the S intervals of $(n + 1)$-th order.

Clearly there are 2^n n-th order T intervals of length $(1-2r_n)r_0...r_{n-1}$ and similarly, there are 2^n n-th order S intervals of length $r_0...r_{n-1}$. We define $D = \bigcap_{i=0}^{\infty} \bigcup_{k=0}^{2^i} S_i^k$ to be our Cantor set.

Now in order to build our function f we need to pass to higher dimensions, so for a fixed i and $p = (p_1, ..., p_{n-1}) \in \{1, ..., 2^i\}^{n-1}$ we define

$$K_i^p = T_i^{p_1} \times ... \times T_i^{p_{n-1}},$$

$$L_i^p = S_i^{p_1} \times ... \times S_i^{p_{n-1}},$$

$$Q_i^{p+} = K_i^p \times [\alpha^{-1} r_0...r_{i-1}, (\alpha^{-1} + a_i) r_0...r_{i-1}]$$

$$Q_i^{p-} = K_i^p \times [(\alpha^{-1} - a_i)r_0...r_{i-1}, \alpha^{-1}r_0...r_{i-1}]$$

$$Q_i^p = Q_i^{p+} \cup Q_i^{p-}.$$

Often times, out of convenience, we will simply drop the super index p. It is easily seen that there are $2^{(n-1)i}$ intervals K_i, L_i, Q_i^+, Q_i^- and Q_i. For which we know

$$\lambda(K_i) = ((1-2r_i)r_0...r_{i-1})^{n-1},$$

$$\lambda(L_i) = (r_0...r_{i-1})^{n-1},$$

$$\lambda(Q_i) = ((1-2r_i)r_0...r_{i-1})^{n-1}2a_ir_0...r_{i-1} \text{ and}$$

$$\lambda(Q_i^+) = \lambda(Q_i^-) = ((1-2r_i)r_0...r_{i-1})^{n-1}a_ir_0...r_{i-1}.$$

It is clear that $Q_i \subseteq [0,1]^{n-1} \times [0,2] \subseteq [-1,2]^n$, since $(\alpha^{-1} + a_i)r_0...r_{i-1} \leq (\alpha^{-1} + a_0)r_0 \leq 2$. Also, $\alpha^{-1} \geq 1 > \frac{1}{3} \geq a_i$, so that $(\alpha^{-1} - a_i)r_0...r_{i-1} > 0$. Define I_0 to be $[-1,2]^n$. Let $\{c_i\}$ be a sequence decreasing to 0 such that $\sum_{i=0}^{\infty} c_i = +\infty$ and define

$$f(x) = \begin{cases} \frac{c_i}{2^{(n-1)i}\lambda(Q_i^+)} \text{ for } x \in Int(Q_i^+), \\ \frac{-c_i}{2^{(n-1)i}\lambda(Q_i^-)} \text{ for } x \in Int(Q_i^-), \\ 0 \text{ else.} \end{cases}$$

Note that f is Lebesgue integrable over any closed set $H \subseteq [-1,2]^n$ with $H \cap ([0,1]^{n-1} \times \{0\}) = \emptyset$. This is since f is Lebesgue integrable on each Q_i and H would intersect only a finite number of Q_i since $\alpha^{-1}r_0...r_{i-1} \to 0$. We will use this fact in order to prove a few propositions and lemmas that will help clarify the proof.

Proposition 4.2.1.

Let $I_0 \subseteq \mathbb{R}^n$ be a compact interval, $f : I_0 \to \mathbb{R}, S \subseteq I_0$ a closed set and $f(x) = 0$ on S. Assume that for every closed set $H \subseteq I_0$ with $S \cap H = \emptyset$ the integral $\int_H f d\lambda$ exists in the Lebesgue sense, and let us denote its value by $F(H)$. Let $q \in \mathbb{R}, 0 < \alpha < 1$. Then, the following two assertions are equivalent:

(1) *the α-regular integral of f over I_0 exists and is equal to q;*

(2) *for every $\epsilon_0 > 0$ there is a gauge $\delta_0 : S \to (0, \infty)$ such that*

$$|F(I_0 \setminus \bigsqcup \pi) - q| \leq \epsilon_0$$

for any δ_0-fine, α-regular partition π tagged in S for which S is contained in the interior of $\bigsqcup \pi$.

Proof. (1) $\implies$ (2) Let $\epsilon_0 > 0$, $F_1(J) = (\mathcal{B}^\alpha) \int_J f d\lambda$ and choose a gauge δ_0 associated with the integrability of f on I_0 approximating to the level ϵ_0. Then for any α-regular, δ_0-fine partition π anchored in S we have $|F(\bigsqcup \pi)| < \epsilon_0$ by the Saks-Henstock lemma 2.1.20 since $f = 0$ on S. Therefore,

$$|F(I_0 \setminus \bigsqcup \pi) - q| = |F_1(I_0 \setminus \bigsqcup \pi) - F_1(I_0)| = |F_1(\bigsqcup \pi)| \leq \epsilon_0.$$

(2) $\implies$ (1) First recall that integrability in the Lebesgue sense implies α-integrability and both integrals are equal. Let $G_0 = \emptyset$ and $G_l = \{x \in I_0 : d(x, S) > 2^{-l}\}$ for $l = 1, ...$ Also define the functions, $f_j : I_0 \to \mathbb{R}$ by $f_j = f \cdot 1_{G_j}$. Given $\epsilon > 0$, we define $\epsilon_0 = \frac{\epsilon}{2}$ and find a gauge δ_0 by our assumption. Notice that f is integrable on G_j (since its closure satisfies the supposition) so that f_j is integrable on I_0. For $j = 1, ...$ we set $\epsilon_j = \frac{\epsilon}{2^j}$ and find associated gauges δ_j for integrability of f_j on G_j to

the level ϵ_j of approximation. We pull together all approximations by defining

$$\delta(x) = \begin{cases} \delta_0(x) \text{ for } x \in S, \\ \\ \min(\delta_j(x), 2^{-j-1}) \text{ for } x \in G_j \setminus G_{j-1} \text{ and } j = 1, ... \end{cases}$$

Let $\pi = \{(t_1, K_1), ..., (t_m, K_m)\}$ be a δ-fine, α-regular partition of I_0. Now if $(t, K) \in \pi[G_l \setminus G_{l-1}]$ then $K \subseteq G_{l+1}$ due to the definition of G_l and the 2^{-j-1} that appears in the definition of the gauge. Now by the Saks Henstock lemma 2.1.20,

$$\begin{aligned}
&|\sum_{(t,K)\in\pi[G_l\setminus G_{l-1}]} f(t)\lambda(K) - (\mathcal{B}^\alpha)\int_K f d\lambda| \\
=&|\sum_{(t,K)\in\pi[G_l\setminus G_{l-1}]} f(t)\lambda(K) - (\mathcal{B}^\alpha)\int_K f_{l+1} d\lambda| \\
\leq&\epsilon_{l+1} \\
=&\epsilon_0 2^{-l-1}.
\end{aligned}$$

Since $f(x) = 0$ on S,

$$\begin{aligned}
&\Big|\sum_{(t,K)\in\pi} f(t)\lambda(K) - q\Big| \\
=&\Big|\sum_{(t,K)\in\pi[I_0\setminus S]} f(t)\lambda(K) - q\Big| \\
\leq&\Big|\sum_{(t,K)\in\pi[I_0\setminus S]} f(t)\lambda(K) - F(I_0 \setminus \bigsqcup \pi[S])\Big| + \Big|F(I_0 \setminus \bigsqcup \pi[S]) - q\Big| \\
=&\Big|\sum_{l=1}^{\infty} \sum_{(t,K)\in\pi[G_l\setminus G_{l-1}]} f(t)\lambda(K) - F(K)\Big| + \Big|F(I_0 \setminus \bigsqcup \pi[S]) - q\Big| \\
\leq&\Big|\sum_{l=1}^{\infty} \sum_{(t,K)\in\pi[G_l\setminus G_{l-1}]} f(t)\lambda(K) - F(K)\Big| + \epsilon_0
\end{aligned}$$

$$\leq \sum_{l=2}^{\infty} \epsilon_0 2^{-l-1} + \epsilon_0$$

$$= \epsilon.$$

$\square$

LEMMA 4.2.2.

Let $\alpha < \alpha_1 < 1, p \in \mathbb{N}$ such that $\alpha_1 > (\alpha^{-1} - a_p)^{-1}$. Let π be an α_1-regular partition in $[-1,2]^n$ such that for $(t,J) \in \pi$

(1) $t = (t_1, ..., t_{n-1}, 0)$,

(2) $J \subseteq B(t, (\alpha^{-1} - a_p) r_0 ... r_{p-1})$,

(3) $[0,1]^{n-1} \times \{0\} \subseteq \mathrm{int}(\bigsqcup \pi)$

Then, $|\int_{I_0 \setminus \bigsqcup \pi} f d\lambda| \leq 2^{n-1} c_{p+1}$.

PROOF. We will write $F(M)$ instead of $\int_M f d\lambda$ if the integral exists in the sense of Lebesgue. Since $f_{|(\bigcup_i Q_i)^c} = 0$ and the Q are disjoint,

$$F(I \setminus \bigsqcup \pi) = \sum_Q F(Q \setminus \bigsqcup \pi).$$

Clearly the sum could have been taken over all Q such that $F(Q \setminus \bigsqcup \pi) \neq 0$. If Q_i is such a Q, then, $F(Q_i \setminus J) \neq 0$ for some $(t,J) \in \pi$. Indeed, if $F(Q_i \setminus J) = 0$ for every J, then $F(Q_i \cap J) = 0$ for every J since $F(Q_i \cap J) = 0$. Thus, $F(Q_i \cap \bigcup J) = 0$. That is, $F(Q_i \setminus \bigsqcup \pi) = 0$. Now for the $(t,J) \in \pi$ with $F(Q_i \setminus J) \neq 0$ we will set $J = [u_1, v_1] \times ... \times [u_{n-1}, v_{n-1}] \times [w, z]$, notice that since (1) and (3) we have that $w \leq 0$.

Claim $z > (\alpha^{-1} - a_i)r_0...r_{i-1}$

If $z \leq (\alpha^{-1} - a_i)r_0...r_{i-1}$, then $\lambda(Q_i \cap J) = 0$, so that $F(Q_i \setminus J) = F(Q_i) = 0$ a contradiction.

Claim $(\alpha^{-1} + a_i)r_0...r_{i-1} > z$

First off, note that $\alpha^{-1} > 1$ and $a_i < \frac{1}{3}$, so that $\alpha^{-1} - a_i > 0$. So we have that $(\alpha^{-1} - a_i)r_0...r_{i-1} > 0 \geq w$. Now if $z \geq (\alpha^{-1} + a_i)r_0...r_{i-1}$, then the n-th projection of Q_i is contained in $[w, z]$, in which case

$$\lambda(Q_i^+ \cap J) = \lambda(K_1 \cap [u_1, v_1])...\lambda(K_{n-1} \cap [u_{n-1}, v_{n-1}])(\frac{z-w}{2}) = \lambda(Q_i^- \cap J).$$

Therefore, $F(Q_i \cap J) = 0$ and $F(Q_i) = 0$, so that $F(Q_i \setminus J) = 0$ a contradiction.

Now since J is α_1-regular we have that $\alpha_1 \leq R(J) \leq \frac{v_j - u_j}{z-w}$ so that $v_j - u_j \geq \alpha_1(z-w) \geq \alpha_1 z > \alpha_1(\alpha^{-1} - a_i)r_0...r_{i-1}$. Now $\alpha_1(\alpha^{-1} - a_i) > 1$ if and only if $\alpha_1 > (\alpha^{-1} - a_i)^{-1}$, and for $i > p$, $\alpha_1 > (\alpha^{-1} - a_p)^{-1} \geq (\alpha^{-1} - a_i)^{-1}$ in which case $v_j - u_j > \alpha_1(\alpha^{-1} - a_i)r_0...r_{i-1} > r_0...r_{i-1}$.

Claim $i > p$

Due to (2) and a previous claim we have that $(\alpha^{-1} - a_i)r_0...r_{i-1} < z < (\alpha^{-1} - a_p)r_0...r_{p-1}$. Now, if $(\alpha^{-1} - a_j)r_0...r_{j-1}$ is decreasing, then we would have our claim. But,

$$(\alpha^{-1} - a_j)r_0...r_{j-1} - (\alpha^{-1} - a_{j+1})r_0...r_j \geq 0$$

$$\iff r_0...r_{j-1}[\alpha^{-1} - a_j - (\alpha^{-1} - a_{j+1})r_j] \geq 0$$

$$\iff \alpha^{-1} - a_j - (\alpha^{-1} - a_{j+1})r_j \geq 0$$

$$\iff \alpha^{-1} - \alpha^{-1}r_j + a_{j+1}r_j - a_j \geq 0.$$

However,

$$\begin{aligned}
&\alpha^{-1}(1-r_j)+a_{j+1}r_j-a_j\\
\geq&\alpha^{-1}(1-r_j)+0-\frac{1}{3}\\
\geq&\alpha^{-1}(1-\frac{1}{3})-\frac{1}{3}\\
\geq&\frac{2}{3}-\frac{1}{3}\\
=&\frac{1}{3}.
\end{aligned}$$

Therefore, $i>p$.

Now $F(Q_i\setminus J)\neq 0$ so that $[u_1,v_1]\times ...\times[u_{n-1},v_{n-1}]\cap K_i\neq\emptyset$ (for some interval K_i of the i-th order). i.e. $[u_j,v_j]\cap T_i^{p_j}\neq\emptyset$ for $j=1,..,n-1$. Since $|T_i|+2|S_{i+1}|=|S_i|=r_0...r_{i-1}$, $[u_j,v_j]\cap T_i^{p_j}\neq\emptyset$, $|v_j-u_j|>r_0...r_{i-1}$ and since there are S_{i+1}'s on each side of $T_i^{p_j}$, we have that $[u_j,v_j]$ contains at least one interval $S_{i+1}^{q_j}$. Thus

$$S_{i+1}^{q_1}\times ...\times S_{i+1}^{q_{n-1}}\subseteq[u_1,v_1]\times ...\times[u_{n-1},v_{n-1}].$$

Then $[u_1,v_1]\times ...\times[u_{n-1},v_{n-1}]$ contains all intervals K_m of order $m\geq 1$ lying inside $S_{i+1}^{q_1}\times ...\times S_{i+1}^{q_{n-1}}$. i.e. one interval K_{i+1}, 2^{n-1} intervals K_{i+2} (since each S_{i+1} contains two intervals S_{i+2}) and in general $2^{(n-1)(l-1)}$ intervals K_{i+l}. We therefore have that J contains at least $2^{(n-1)(l-1)}$ intervals Q_{i+l} where $l\in\mathbb{N}$. Indeed, $K_{i+l}\subseteq[u_1,v_1]\times ...\times[u_{n-1},v_{n-1}]$, $z>(\alpha^{-1}-a_i)r_0...r_{i-1}>(\alpha^{-1}+a_i)r_0...r_i>(\alpha^{-1}-a_i)r_0...r_i>...>(\alpha^{-1}+a_{i+l})r_0...r_{i+l-1}$ and $w\leq 0<(\alpha^{-1}-a_{i+l})r_0...r_{i+l-1}$. Therefore, $Q_{i+l}=K_{i+l}\times[(\alpha^{-1}-a_{i+l})r_0...r_{i+l-1},(\alpha^{-1}+a_{i+l})r_0...r_{i+l-1}]\subseteq[u_1,v_1]\times ...\times[u_{n-1},v_{n-1}]\times[w,z]$ and there are $2^{(n-1)(l-1)}$ such Q_{i+l} from above. Evidently for these Q_{i+l}, $F(Q_{i+l} \setminus J) = 0$ since $Q_{i+l} \setminus J = \emptyset$.

Let k_l be the number of intervals Q_l of the l-th order such that $F(Q_l \setminus \bigsqcup \pi) \neq 0$. Now by (2) we have that $k_0 = ... = k_p = 0$ since for $j \in \{1, ..., p\}, Q_j \setminus \bigsqcup \pi = Q_j$. Now since $[0,1]^{n-1} \times \{0\} \subseteq \text{int}(\bigsqcup \pi)$ and $(\alpha^{-1} + a_i) r_0 ... r_{i-1} \to 0$, we have that eventually $Q_i \subseteq \text{int}(\bigsqcup \pi)$. So that eventually $Q_i \setminus \bigsqcup \pi = \emptyset$, in which case, $F(Q_i \setminus \bigsqcup \pi) = 0$. Therefore, there exists an $m \in \mathbb{N}$ such that $k_{p+m+1} = k_{p+m+2} = ... = 0$.

Now $k_{p+1} \leq 2^{(n-1)(p+1)}$ since there are only $2^{(n-1)(p+1)}$ intervals Q_{p+1}.

<u>Claim $k_{p+2} \leq 2^{(n-1)(p+2)} - k_{p+1}$</u>

First off, there are $2^{(n-1)(p+2)}$ Q intervals of the $(p+2)$-th order, so that $k_{p+2} \leq 2^{(n-1)(p+2)}$. Now if $F(Q_{p+1} \setminus \bigsqcup \pi) \neq 0$, then $F(Q_{p+1} \setminus J) \neq 0$ for some $(t, J) \in \pi$. In which case from previous calculations we see that J contains at least $2^{(n-1)(1-1)}$ intervals Q_{p+2}. But there are k_{p+1} such Q_{p+1} which are all disjoint, so there at least k_{p+1} intervals Q_{p+2} contained in $\bigsqcup \pi$. For these intervals Q_{p+2} we have that $F(Q_{p+2} \setminus \bigsqcup \pi) = 0$, therefore, $k_{p+2} \leq 2^{(n-1)(p+2)} - k_{p+1}$.

<u>$k_{p+3} \leq 2^{(n-1)(p+3)} - 2^{(n-1)} k_{p+1} - k_{p+2}$</u>

Since there are only $2^{(n-1)(p+3)}$ intervals Q_{p+3}, $k_{p+3} \leq 2^{(n-1)(p+3)}$. Now each Q_{p+1} counting towards the k_{p+1} induces a J containing 2^{n-1} intervals Q_{p+3}. Similarly, each Q_{p+2} contributing to k_{p+2} induces a Q_{p+3} not contributing to the k_{p+3}. Therefore, $k_{p+3} \leq 2^{(n-1)(p+3)} - 2^{(n-1)} k_{p+1} - k_{p+2}$.

Following the same process we see that for $1 \leq l \leq m$, we receive m inequalities of the form

$$k_{p+l} \leq 2^{(n-1)(p+l)} - 2^{(n-1)(l-2)}k_{p+1} - ... - 2^{n-1}k_{p+l-2} - k_{p+l-1}.$$

Now transferring all but the first number in each inequality to the left, we have m inequalities of the form

$$k_{p+l} + 2^{(n-1)(l-2)}k_{p+1} + ... + 2^{n-1}k_{p+l-2} + k_{p+l-1} \leq 2^{(n-1)(p+l)}.$$

Adding all of these inequalities and factoring,

$$k_{p+1}(2 + 2^{n-1} + ... + 2^{(n-1)(m-2)}) + ... + k_{p+m} \leq 2^{(n-1)(p+1)}(1 + 2^{n-1} + ... + 2^{(n-1)(m-1)}).$$

After noticing that these are geometric sums,

$$k_{p+1}(1 + \frac{2^{(n-1)(m-1)} - 1}{2^{(n-1)} - 1}) + ... + k_{p+m-1}(1 + \frac{2^{n-1} - 1}{2^{n-1} - 1}) + k_{p+m} \leq 2^{(n-1)(p+1)}\frac{2^{(n-1)m} - 1}{2^{n-1} - 1}$$

Multiplying both sides by $2^{n-1} - 1$ and noticing that $2^{(n-1)(p+1)}(2^{(n-1)m} - 1) \leq 2^{(n-1)(p+1+m)}$,

$$k_{p+1}(2^{(n-1)(m-1)} - 1) + ... + k_{p+m-1}(2^{n-1} - 1) + (2^{n-1} - 1)\sum_{i=1}^{m} k_{p+i} \leq 2^{(n-1)(p+1+m)}.$$

Now dropping some unwanted terms,

$$k_{p+1}2^{(n-1)(m-1)} + k_{p+2}2^{(n-1)(m-2)} + ... + k_{p+m-1}2^{n-1} + k_{p+m} \leq 2^{(n-1)(p+1+m)}.$$

Finally, dividing both sides by $2^{(n-1)(m-1)}$,

$$k_{p+1} + k_{p+2}2^{-(n-1)} + ... + k_{p+m-1}2^{-(n-1)(m-2)} + k_{p+m}2^{-(n-1)(m-1)} \leq 2^{(n-1)(p+2)}.$$

Notice that by the definition f we have that $|F(Q_{p+l} \setminus \bigsqcup \pi)| \leq \frac{c_{p+l}}{2^{(n-1)(p+l)}}$. Putting together all of our previous calculations,

$$\begin{aligned}
|F(I_0 \setminus \bigsqcup \pi)| &= \Big|\sum_{i=1}^{m} \sum_{j=1}^{2^{(n-1)i}} F(Q_{p+i}^j \setminus \bigsqcup \pi)\Big| \\
&= \Big|\sum_{i=1}^{m} \sum_{j} F(Q_{p+i}^j \setminus \bigsqcup \pi)\Big| \\
&\leq \Big|\sum_{i=1}^{m} \sum_{j} \frac{c_{p+i}}{2^{-(n-1)(p+i)}}\Big| \\
&= \Big|\sum_{i=1}^{m} k_{p+i} \frac{c_{p+i}}{2^{-(n-1)(p+i)}}\Big| \\
&\leq \Big|\sum_{i=1}^{m} k_{p+i} \frac{c_{p+1}}{2^{-(n-1)(p+i)}}\Big| \\
&\leq \Big|c_{p+1} 2^{-(n-1)(p+1)} \sum_{i=1}^{m} k_{p+i} 2^{-(n-1)(i-1)}\Big| \\
&\leq \Big|2^{n-1} c_{p+1}\Big|
\end{aligned}$$

where the sum without indicated bounds is over all Q intervals of the $p+i$-th order contributing to k_{p+i}.

$\square$

Lemma 4.2.3.

Let $0 < \alpha_2 < \alpha < 1$. *For any gauge* δ *on* I_0 *there exists* α_2*-regular,* δ*-fine partitions* π_j *for* $j = 1, 2$ *satisfying:*

(1) *for* $(t, J) \in \pi_j$ *we have* $t = (t_1, ..., t_{n-1}, 0)$,

(2) $[0,1]^{n-1} \times \{0\} \subseteq \text{int}(\bigsqcup \pi_j)$,

(3) $F(I_0 \setminus \bigsqcup \pi_1) \geq 1$ *and*

(4) $F(I_0 \setminus \bigsqcup \pi_2) = 0$.

PROOF. We intend to build our partitions on or containing $[0,1]^{n-1} \times \{0\}$. In light of this throughout the proof we will refer to the height of an interval as its right endpoint in the n-th dimension rather than its actual length along the n-th dimension.

Let δ be a gauge on I_0. In order to find areas where we will have freedom to build larger intervals over the K_i^p, we define

$$W_k = \{w = (w_1, ..., w_{n-1}) \in D^{n-1} : \delta(w, 0) > \frac{1}{k}\} \text{ for } k \in \mathbb{N}.$$

It may seem strange that in order to build over the K_i^p we restrict our attention to $D^{n-1} = \bigcap_{i=1}^{\infty} \bigcup_p L_i^p$. Recall however, that each L_i^p contains a K_i^q. Now D^{n-1} is dense in itself so by the Baire Category theorem there is a W_p that is not nowhere dense in D^{n-1}. Therefore, there exists a $z \in \mathrm{cl}_{D^{n-1}}(W_p)$ and a $w > 0$ such that $D^{n-1} \cap B(z, w) \subseteq \mathrm{cl}_{D^{n-1}}(W_p) \subseteq \mathrm{cl}(W_p)$. Now since $D^{n-1} = \bigcap_{i=0}^{\infty} \bigcup L_i$, $\lambda(L_i) \to 0$ and $D^{n-1} \cap B(z, w) \neq \emptyset$, there exists a $q \in \mathbb{N}$ such that $z \in L_q \subseteq B(z, w)$.

Without loss of generality, we assume that q is chosen such that

$$\frac{1 - 2r_q}{\alpha^{-1}} > \alpha_2 \text{ and } \alpha^{-1} r_0 ... r_{q-1} < \frac{1}{p}.$$

This is all possible since $\frac{1-2r_q}{\alpha^{-1}} \nearrow \alpha, \alpha^{-1} r_0 ... r_{q-1} \searrow 0$ and since for any $i > j$ there is an $L_i \subseteq L_j$. Now since $\sum_{i=0}^{\infty} c_i = \infty$, there is an $m \in \mathbb{N}$ such that

$$c_p + c_{p+1} + ... + c_{p+m} \geq 2^{(n-1)q}.$$

There is an interval K_q of order q such that $K_q \subseteq L_q$, 2^{n-1} intervals K_{q+1} with $K_{q+1} \subseteq L_q$, in general there are $2^{(n-1)j}$ intervals K_{q+j} with $K_{q+j} \subseteq L_q$ for $j = 0, ..., m$.

Now each interval K_{q+j} can be written as

$$K_{q+j} = T_{q+j}^{z_1} \times ... \times T_{q+j}^{z_{n-1}}$$

where $T_{q+j}^{z_i} = (\lambda_{q+j}^{z_i} - \frac{(1-2r_{q+j})r_0...r_{q+j-1}}{2}, \lambda_{q+j}^{z_i} + \frac{(1-2r_{q+j})r_0...r_{q+j-1}}{2})$. We extend K_{q+j} in order to get an element from W_p inside of it. Find positive numbers φ_{q+j} with

$$\frac{r_0...r_{q+j-1}}{2} > \varphi_{q+j} > \frac{(1-2r_{q+j})r_0...r_{q+j-1}}{2}$$

such that all the intervals

$$\widetilde{T}_{q+j}^{z_i} = (\lambda_{q+j}^{z_i} - \frac{(1-r_{q+j})r_0...r_{q+j-1}}{2}, \lambda_{q+j}^{z_i} + \varphi_{q+j})$$

are pairwise disjoint with i fixed and $j = 0, ..., m$. All of this is possible since for $i > j$, $\bar{T}_i \cap \bar{T}_j = \emptyset$ and since $\lambda_{q+j}^{z_i} + \frac{(1-2r_{q+j})r_0...r_{q+j-1}}{2}$ is the right endpoint of $T_{q+j}^{z_i}$. From this process we obtain non-overlapping intervals $H_{q+j} = \mathrm{cl}\,(\widetilde{T}_{q+j}^{z_1} \times ... \times \widetilde{T}_{q+j}^{z_{n-1}}) \subseteq L_{q+j} \subseteq L_q$. This is since $\frac{r_0...r_{q+j-1}}{2} > \varphi_{q+j}$ and $\mathrm{dist}(\lambda_{q+j}^{z_i}, \mathrm{bd}(S_{q+j})) = \frac{r_0...r_{q+j-1}}{2}$.

$\underline{H_{q+j} \cap W_p \neq \emptyset \text{ for } j = 0, ..., m}$

Indeed, $D^{n-1} \cap B(z, w) \subseteq \bar{W}_p$ and $H_{q+j} \subseteq L_q \subseteq B(z, w)$ so if $H_{q+j} \cap D^{n-1} \neq \emptyset$, then we are done. But $H_{q+j} \cap L_{q+j} \neq \emptyset$, in fact, the left corner of L_{q+j} i.e. $(\lambda_{q+j}^{z_1} - \frac{r_0...r_{q+j}}{2}, ..., \lambda_{q+j}^{z_{n-1}} - \frac{r_0...r_{q+j}}{2})$ is contained in the interior of H_{q+j} and there is a sequence of elements in D^{n-1} tending to this corner. Therefore, $H_{q+j} \cap W_p \neq \emptyset$ as required.

Say $\tau_{q+j} \in H_{q+j} \cap W_p$, let $\psi > 0$ and set $J = H_{q+j} \times [-\psi r_0...r_{q+j-1}, \alpha^{-1} r_0...r_{q+j-1}]$ and let (t, J) be included in our partition π_1, where $t = (\tau_{q+j}, 0)$. Notice that the height of J is the height of Q_{q+j}^- but is lower that the height of Q_{q+j}^+. This is how $F(I_0 \setminus \bigsqcup \pi)$ will become large.

We must now estimate the regularity of J (since π was supposed to be α_2-regular). Notice first that

$$\begin{aligned}
&\varphi_{q+j} + \frac{(1-2r_{q+j})r_0...r_{q+j-1}}{2} \\
<&\frac{r_0...r_{q+j-1}}{2} + \frac{(1-2r_{q+j})r_0...r_{q+j-1}}{2} \\
=&(1-r_{q+j})r_0...r_{q+j-1} \\
\leq&r_0...r_{q+j-1} \\
\leq&(\alpha^{-1}+\psi)r_0...r_{q+j-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
R(J) &= \frac{\varphi_{q+j} + \frac{(1-2r_{q+j})}{2}r_0...r_{q+j-1}}{(\alpha^{-1}+\psi)r_0...r_{q+j-1}} \\
&\geq \frac{(1-2r_{q+j})r_0...r_{q+j-1}}{(\alpha^{-1}+\psi)r_0...r_{q+j-1}} \\
&\geq \frac{1-2r_{q+j}}{\alpha^{-1}+\psi}.
\end{aligned}$$

Notice that $J \subseteq B(t, (\alpha^{-1}+\psi)r_0...r_{q+j-1}), (1-2r_q)\alpha > \alpha_2$ and $\alpha^{-1}r_0...r_{q-1} < \frac{1}{p}$, so ψ can be chosen small enough so that $R(J) \geq \alpha_2$ and $J \subseteq B(T, \frac{1}{p}) \subseteq B(t, \delta(t))$.

All the pairs (t, J) constructed up until now form an α_2-regular, δ-fine partition which we will complete in order to form π_1. We wish to complete π_1 in such a way as to have $F(I_0 \backslash \bigsqcup \pi) = F(I_0 \backslash \bigcup J)$, where the union is over all J previously mentioned. In order to do this we will ensure that the height of any interval we add falls outside of $[(\alpha^{-1}-a_i)r_0...r_{i-1}, (\alpha^{-1}+a_i)r_0...r_{i-1}]$. This will ensure that our added intervals intersect an equal portion of Q_i^{p-} and Q_i^{p+}. This will result in $F(I_0 \backslash \bigsqcup \pi)$ being equal to $F(I_0 \setminus \bigcup J)$.

Each J corresponds to an $(n-1)$-dimensional interval H_{q+j}. The set $\mathrm{cl}([0,1]^{n-1}\setminus \bigcup H_{q+j})$ can be written as a finite number of compact intervals. So by Cousin's theorem 3.2.2 we can find a δ-fine, α_2-regular partition of $\mathrm{cl}([0,1]^{n-1}\setminus \bigcup H_{q+j})$ consisting of pairs $(\widetilde{s}, \widetilde{M})$ where $\widetilde{s} = (s_1, ..., s_{n-1})$ and $\widetilde{M}$ are $(n-1)$-dimensional intervals. Consider a particular $\widetilde{M} = [\hat{c}_1, \hat{d}_1] \times ... \times [\hat{c}_{n-1}, \hat{d}_{n-1}]$, we form the n-dimensional interval $M = \widetilde{M} \times [-h, -h + d_1 - c_1]$ for some $h > 0$. We first notice that the regularity of M is the same as the regularity of $\widetilde{M}$. We choose h small enough so that $0 < h < d_1 - c_1$ and $M \subseteq B(s, \delta(s))$ where $s = (\widetilde{s}, 0)$. We will also choose h so that

$$-h + d_1 - c_1 \notin ((\alpha^{-1} - a_i)r_0...r_{i-1}, (\alpha^{-1} + a_i)r_0...r_{i-1})$$

for $i \in \mathbb{N}$. This is possible since $(\alpha^{-1}-a_i)r_0...r_{i-1}$ and $(\alpha^{-1}+a_i)r_0...r_{i-1}$ both converge to 0 and the intervals $[(\alpha^{-1} - a_i)r_0...r_{i-1}, (\alpha^{-1} + a_i)r_0...r_{i-1}]$ are disjoint.

Then all (t, J) and (s, M) form a partition π_1 satisfying conditions (1) and (2).

<u>Claim $F(I_0 \setminus \bigsqcup \pi_1) \geq 1$</u>

Now $F(M) = 0$ for any $(s, M) \in \pi_1$ and for any $p \in \mathbb{N}$ and $j = 0, ..., m$, $Q^p_{q+j}- \subseteq J$ for any $(t, J) \in \pi_1$. Since $\overline{K}^p_{q+j} = H^p_{q+j}$ and the n-th projection of J, $[-\psi r_0...r_{q+j-1}, \alpha^{-1}r_0...r_{q+j-1}]$ contains $[(\alpha^{-1} - a_i)r_0...r_{q+j-1}, \alpha^{-1}r_0...r_{q+j-1}]$. □

Therefore, $F(I_0 \setminus \bigsqcup \pi_1)$ is just the integral over the Q^+_{q+j} that are associated with each $(t, J) \in \pi_1$. i.e.

$$F(I_0 \setminus \bigsqcup \pi_1) = \sum_{i=0}^{m} \sum F(Q^+_{q+i})$$

where the interior sum is taken over the intervals Q^+_{q+j} associated with each $(t, J) \in \pi_1$. The number of such intervals of order $q + i$ is $2^{(n-1)i}$ and we have that

$$\begin{aligned} F(I_0 \setminus \bigsqcup \pi_1) &= c_q 2^{-(n-1)q} + ... + 2^{(n-1)m} c_{q+m} 2^{-(n-1)(q+m)} \\ &= 2^{-(n-1)q}(c_q + ... + c_{q+m}) \\ &\geq 2^{-(n-1)q} 2^{(n-1)q} \\ &= 1. \end{aligned}$$

We now construct π_2, first we make an α_2-regular, δ-fine $(n-1)$-dimensional partition of $[0,1]^{n-1}$ consisting of pairs $(\widetilde{t}, \widetilde{J})$. For an element $(\widetilde{t}, \widetilde{J})$, let d be the length of $\widetilde{J}$'s maximal side and let $0 < h < d$ be such that

$$-d + h \notin ((\alpha^{-1} - a_i) r_0 ... r_{i-1}, (\alpha^{-1} + a_i) r_0 ... r_{i-1}).$$

Let $J = \widetilde{J} \times [-h, -h+d]$ and $t = (t, 0)$. Now, (t, J) is δ-fine and $R(J) = R(\widetilde{J})$ so that (t, J) is α_2-fine. We let π_2 be the collection of all these (t, J). Then, π_2 is δ-fine, α_2-regular and satisfies both (1) and (2). Now to show that $F(I_0 \setminus \bigsqcup \pi_2) = 0$. Indeed, for every Q_i we have that $F(Q_i \setminus \bigsqcup \pi_2) = 0$. This is since, $\lambda(Q_i^+ \setminus \bigsqcup \pi_2) = \lambda(Q_i^+ \setminus \bigsqcup \pi_2)$ and therefore, $F(Q_i^+ \setminus \bigsqcup \pi_2) = -F(Q_i^- \setminus \bigsqcup \pi_2)$. Now, $F(I_0 \setminus \bigsqcup \pi_2) = \sum_{i \in \mathbb{N}} \sum F(Q_i \setminus \bigsqcup \pi_2) = 0$, where the interior sum is over all i-th order intervals Q_i.

Theorem 4.2.4.

Given any $0 < \alpha < 1$, there is a function f that is $\mathcal{B}^{\alpha_1}$-integrable for any $\alpha_1 \geq \alpha$ that is not $\mathcal{B}^{\alpha_2}$-integrable for any $\alpha_2 < \alpha$.

Proof. Consider the function built earlier. Choose a $p \in \mathbb{N}$ such that $\alpha_1 > (\alpha^{-1} - a_p)^{-1}$ and $2^{n-1}c_{p+1} \leq \epsilon_0$ and let $\delta = (\alpha^{-1} - a_p)r_0...r_{p-1}$. We see that lemma 4.2.2 gives us (2) of proposition 4.2.1, which implies $\mathcal{B}^{\alpha_1}$-integrability of f over I_0 for any $\alpha_1 > \alpha$.

Lemma 4.2.3 clearly shows that condition (2) of proposition 4.2.1 cannot hold. Suppose it did, then

$$|F(I_0 \setminus \bigsqcup \pi_1) - F(I_0 \setminus \bigsqcup \pi_2)| \leq |F(I_0 \setminus \bigsqcup \pi_1) - q| + |F(I_0 \setminus \bigsqcup \pi_2) - q|.$$

We have shown that the right side of the inequality can be made arbitrarily small. But we have also shown that the term on the left side of the inequality is larger than 1. Therefore, f cannot be $\mathcal{B}^{\alpha_2}$-integrable on I_0 for any $\alpha_2 < \alpha$. $\square$

CHAPTER 5

Fubini's Theorem and Product Bases

5.1. Product Bases and Basic Results

One might begin to wonder if we can put together two bases to create another base in a higher dimension. One could just cross the elements in the bases together. This would in fact create a base with most of the properties mentioned earlier, however, in most cases the resulting integral will have little to do with the integrals of the two initial bases. We would like to impose more structure on how we put together bases in order to end up with some sort of Fubini theorem. This material was presented in [**4**]. We have added a little more direction in the proof and fixed some minor issues.

Definition 5.1.1. Product base

Let $\mathcal{B}^1$ be a base in X, $\mathcal{B}^2$ be a base in Y and suppose that they are both of local character. Let $\mathcal{I}^1 \subseteq \mathcal{P}(X)$, $\mathcal{I}^2 \subseteq \mathcal{P}(Y)$ be the corresponding classes of intervals. Let

$$\mathcal{I} = \{I \times J : I \in \mathcal{I}^1, J \in \mathcal{I}^2\}$$

and

$$Z = X \times Y.$$

A family $\mathcal{B} \subseteq \mathcal{P}(Z \times \mathcal{I})$ will be called the product base of $\mathcal{B}^1$ and $\mathcal{B}^2$, denoted $\mathcal{B} = \mathcal{B}^1 \times \mathcal{B}^2$, if for every $\beta \in \mathcal{B}$ there are choice functions

$$\phi_X : X \to \mathcal{B}^2 \text{ by } \phi_X(x) = \beta_x^2 \text{ and}$$

$$\phi_Y : Y \to \mathcal{B}^1 \text{ by } \phi_Y(y) = \beta^1_y,$$

for which $(z, P) \in \beta$ if and only if

$$z = (x, y) \text{ and } P = I \times J$$

where,

$$(x, I) \in \beta^1_y \text{ and } (y, J) \in \beta^2_x.$$

When convenient we will simply denote the choice functions by the images under the map, that is by β^2_x and β^1_y. We will also denote the element in the base by its choice functions, for example $\beta = \beta^1_y \times \beta^2_x$.

PROPOSITION 5.1.2.

Every product base is of local character.

PROOF. Let $\mathcal{B} = \mathcal{B}^1 \times \mathcal{B}^2$ and for each $(a, b) \in Z$ let ${}_{(a,b)}\beta \in \mathcal{B}$, say ${}_{(a,b)}\beta =_{(a,b)} \beta^1_y \times_{(a,b)} \beta^2_x$. Fix any $b \in Y$ and choose $\beta^1_b \in \mathcal{B}^1$ such that $\beta^1_b[\{a\}] \subseteq_{(a,b)} \beta^1_b[\{a\}]$ for each $a \in X$. Do this for each $b \in Y$. Similarly, fix any $a \in X$ and choose a $\beta^2_a \in \mathcal{B}^2$ such that $\beta^2_a[\{b\}] \subseteq_{(a,b)} \beta^2_a[\{b\}]$ for each $b \in Y$. Let $((a, b), I \times J) \in \beta^1_b \times \beta^2_a \in \mathcal{B}$. Then, $(a, I) \in \beta^1_b[\{a\}] \subseteq_{(a,b)} \beta^1_b[\{a\}]$ and $(b, J) \in \beta^2_a[\{b\}] \subseteq_{(a,b)} \beta^2_a[\{b\}]$. Therefore, $((a, b), I \times J) \in_{(a,b)} \beta$.

$\square$

Proposition 5.1.3.

The product of two filtering bases is filtering.

Proof. Let $\mathcal{B} = \mathcal{B}^1 \times \mathcal{B}^2$ and let $\alpha, \beta \in \mathcal{B}$. Suppose that $\beta = \beta^1_y \times \beta^2_x$ and $\alpha = \alpha^1_y \times \alpha^2_x$. Now, since both bases are filtering, there are $\eta^2_x \in \mathcal{B}^2$ and $\eta^1_y \in \mathcal{B}^1$ such that $\eta^2_x \subseteq \beta^2_x \cap \alpha^2_x$ and $\eta^1_y \subseteq \beta^1_y \cap \alpha^1_y$ for each $x \in X$ and $y \in Y$. Let $\eta = \eta^1_y \times \eta^2_x$ and $(z, P) = ((x, y), I \times J) \in \eta$. Then, $(x, I) \in \eta^1_y \subseteq \alpha^1_y \cap \beta^1_y$ and $(y, J) \in \alpha^2_x \cap \beta^2_x \subseteq \eta^2_x$. Therefore, $(z, P) \in \alpha \cap \beta$. $\square$

Definition 5.1.4. Compound partition

Let $\pi^1 = \{(x_1, I_1), ..., (x_n, I_n)\}$ be a partition in X, and for $i = 1, .., n$ let $\pi^2_{x_i} = \{(y^i_1, J^i_1), ..., (y^i_{k_i}, J^i_{k_i})\}$ be a partition in Y. Then,

$$\pi = \bigcup_{(x,I)\in\pi^1} \bigcup_{(y,I)\in\pi^2_x} \{((x, y), I \times J)\} = \{((x_i, y^i_j), I_i \times J^i_j) : i = 1, ..., n, j = 1, ..., k\}$$

is a partition of $X \times Y$, and such a partition will be called a compound partition.

Proposition 5.1.5.

If $\mathcal{B}^1$ and $\mathcal{B}^2$ have the partitioning property and are filtering, then $\mathcal{B} = \mathcal{B}^1 \times \mathcal{B}^2$ has the partitioning property.

Proof. Let $\beta^1_y \times \beta^2_x = \beta \in \mathcal{B}^1$ and $I_0 \times J_0 \in \mathcal{I}$. Fix an $x \in I_0$. Since $\mathcal{B}^2$ has the partitioning property, there exists a partition $\pi^2_x \subseteq \beta^2_x$ of J_0. Say

$$\pi^2_x = \{(y^x_1, J^x_1), ..., (y^x_{k_x}, J^x_{k_x})\}.$$

Now, since $\mathcal{B}^1$ is filtering, for every $x \in I_0$ there is a ${}_x\beta^1 \in \mathcal{B}^1$ such that

$${}_x\beta^1 \subseteq \bigcap_{i=1}^{k_x} \beta^1_{y^x_i}.$$

Since $\mathcal{B}^1$ is of local character, there is a $\beta^1 \in \mathcal{B}^1$ such that

$$\beta^1[\{x\}] \subseteq_x \beta^1[\{x\}]$$

for every $x \in X$. There exists a partition $\pi_1 \subseteq \beta^1$ of I_0 say

$$\pi^1 = \{(x_1, I_1), ..., (x_n, I_n)\}.$$

Then, the compound partition

$$\pi = \{\big((x_i, y_j^{x_i}), I_i \times J_j^{x_i}\big) : i = 1, .., n, j = 1, .., k_{x_i}\}$$

is contained in β and is a partition of $I_0 \times J_0$. □

5.2. Fubini Theorem

THEOREM 5.2.1. *Fubini theorem*

Let $\mathcal{B} = \mathcal{B}^1 \times \mathcal{B}^2, I_0 \in \mathcal{I}^1, J_0 \in \mathcal{I}^2$, $U_1 : I_0 \times \mathcal{I}_1 \mapsto \mathbb{R}$ *and* $U_2 : I_0 \times J_0 \times \mathcal{I}_2 \mapsto \mathbb{R}$. *Define*

$$U : Z \times \mathcal{I} \mapsto \mathbb{R} \text{ by } U\big((x, y), I \times J\big) = U_1(x, I)U_2(x, y, J).$$

Suppose that U *is* $\mathcal{B}$*-integrable on* $I_0 \times J_0$ *and set*

$$T = \{x \in I_0 : U_2(x, \cdot, \cdot) \textit{ is } \mathcal{B}^2 - \textit{integrable}\}.$$

Let,

$$g(x) = \begin{cases} (\mathcal{B}^2) \int_{J_0} U_2(x, \cdot, \cdot), \textit{ for } x \in T, \\ \\ \textit{anything, for } x \notin T. \end{cases}$$

Let $W(x, I) = U_1(x, I)g(x)$ *for* $(x, I) \in I_0 \times \mathcal{I}_1$. *Then,*

(1) $V(U_1, \mathcal{B}^1[I_0 \setminus T]) = 0$,

(2) *W is $\mathcal{B}^1$-integrable and*

(3) $(\mathcal{B}^1)\int_{I_0} W = (\mathcal{B})\int_{I_0\times J_0} U.$

i.e. $(\mathcal{B}^1)\int_{I_0} U_1(x,I)\left[(\mathcal{B}^2\int_{J_0} U_2(x,\cdot,\cdot)\right] = (\mathcal{B})\int_{I_0\times J_0} U.$

Proof. We will first show (1). In an attempt to gain some control over the lack of integrability of $U_2(x,\cdot,\cdot)$ on the points in $I_0 \setminus T$, we define X_n to be the set of $x \in I_0$ such that for all $\beta_2 \in \mathcal{B}^2$ there are partitions $\pi^{2,1}, \pi^{2,2} \subseteq \mathcal{B}^2$ of J_0 such that $|U_2(x,\pi^{2,1}) - U_2(x,\pi^{2,2})| \geq \frac{1}{n}$. i.e. the set of points that fail Cauchyness by more than $\frac{1}{n}$. We then have that $I_0 \setminus T = \bigcup_{n\in\mathbb{N}} X_n$, so it will suffice to show that

$$V(U_1, \mathcal{B}^1[X_n]) = 0 \text{ for each } n \in \mathbb{N}.$$

Our plan is to take a partition anchored in X_n far enough in the filter, extend it to two partitions in $I_0 \times J_0$ for which Cauchyness is satisfied under U_1. However, we will require that the pieces we use to extend fail Cauchyness under U_2. Now since $U = U_1U_2$, it will follow that our original partition must have small variation under U_1.

Fix an $n \in \mathbb{N}$ and let $\epsilon > 0$. Choose a $\beta = \beta_y^1 \times \beta_x^2 \in \mathcal{B}$ such that for every partition $\pi \subseteq \beta$ of $I_0 \times J_0$,

$$\left|(\mathcal{B})\int_{I_0\times J_0} U - U(\pi)\right| \leq \frac{\epsilon}{2}.$$

For $x \in X_n$ we can find partitions $\pi_x^2, \hat{\pi}_x^2 \subseteq \beta_x^2$ of J_0 for which

$$|U_2(x,\pi_x^2) - U_2(x,\hat{\pi}_x^2)| \geq \frac{1}{n}.$$

Since $\mathcal{B}^1$ is filtering we can find a ${}_x\beta^1 \in \mathcal{B}^1$ such that

$$ {}_x\beta^1 \subseteq \alpha_y^1 $$

for each $y \in J_0$ with $(y, J) \in \pi_x^1 \cup \hat{\pi}_x^1$ for some $J \in \mathcal{I}_2$. Now, $\mathcal{B}^1$ is of local character so we can find a $\beta^1 \in \mathcal{B}^1$ such that

$$ \beta^1[\{x\}] \subseteq_x \beta^1[\{x\}] \subseteq \beta_y^1[\{x\}] $$

for each $x \in X_n$.

Let $x \in I_0 \setminus X_n$ and let $\pi_x^2 \subseteq \beta_x^2$ be a partition of J_0, say

$$ \pi_x^2 = \{(y_j^x, J_j^x) : j = 1, ..., m\} \text{ and } \hat{\pi}_x^2 = \pi_x^2. $$

We define both partitions to be the same since we are not interested in what happens on $I_0 \setminus X_n$. Defining them this way will ensure cancellation further on.

As before we can choose a $\gamma \in \beta^1$ such that for every $x \in I_0 \setminus X_n$ and every $y \in J_0$ with $(y, J) \in \pi_x^2 \cup \hat{\pi}_x^2$ for some $J \in \mathcal{I}_2$,

$$ \gamma^1[\{x\}] \subseteq \beta_y^1[\{x\}]. $$

Now bringing everything together, choose a $\phi^1 \in \beta^1$ with

$$ \phi^1 \subseteq \beta^1 \cap \gamma^1. $$

Let $\overline{\pi}^1$ be a partition contained in $\phi^1[X_n]$. Without loss of generality we assume that $\overline{\pi}^1$ is maximal in size, we can do this since if $\overline{\pi}^1 \subseteq \pi^{1*}$, then $|U_1|(\overline{\pi}^1) \leq |U_1|(\pi^{1*})$.

Extend $\overline{\pi}^1$ to a partition $\pi^1 \subseteq \phi^1$ of I_0. By maximality we have that $\overline{\pi}^1 = \pi^1[X_n]$. Suppose that $\pi^1 = \{(x_1, I_1), ..., (x_k, I_k)\}$, then we define the compound partitions

$$p = \{\big((x_i, y_j^{x_i}), I_i \times J_j^{x_i}\big) : i = 1, .., k \text{ and } j = 1, ..., m_{x_i}\} \text{ and}$$

$$\hat{p} = \{\big((x_i, \hat{y}_j^{x_i}), I_i \times \hat{J}_j^{x_i}\big) : i = 1, .., k \text{ and } j = 1, ..., \hat{m}_{x_i}\}.$$

Then $p, \hat{p} \subseteq \beta$, so that $|U(p) - U(\hat{p})| \leq \epsilon$.

Without loss of generality we will assume that

$$\operatorname{sgn}\big(U_1(x_i, I_i)\big) = \operatorname{sgn}\big(U_2(x_i, \pi_{x_i}^2) - U_2(x_i, \hat{\pi}_{x_i}^2)\big)$$

for $x_i \in X_n$ with $U_1(x_i, I_i) \neq 0$. This can be made possible by switching the roles of $\pi_{x_i}^2$ and $\hat{\pi}_{x_i}^2$ if it does not hold.

We then have,

$$\begin{aligned}
|U(p) - U(\hat{p})| &= \left|\sum_{i=1}^{n} U_1(x_i, I_i)\right| \left|\sum_{j=1}^{m_{x_i}} U_2(x_i, j_j^{x_i}, J_j^{x_i}) - \sum_{j=1}^{\hat{m}_{x_i}} U_2(x_i, \hat{y}_j^{x_i}, \hat{J}_j^{x_i})\right| \\
&\geq \sum_{i=1}^{n} |U_1(x_i, I_i)| \frac{1}{n} \\
&\geq |U_1|(\overline{\pi}^1) \frac{1}{n}.
\end{aligned}$$

Therefore, $|U_1|(\overline{\pi}^1) \leq n\epsilon$ and $V(U_1, \mathcal{B}^1[X_n]) = 0$ as required.

We now move onto showing (2). We need to show that for every $\epsilon > 0$, there exists a $\beta^1 \in \mathcal{B}^1$ such that for every partition $\pi^1 \subseteq \beta^1$ of I_0,

$$\left|(\mathcal{B}) \int_{I_0 \times J_0} U - W(\pi^1)\right| \leq \epsilon.$$

Let $\epsilon > 0$ and find an $\alpha \in \mathcal{B}$ such that for every partition $\pi \subseteq \alpha$ of $I_0 \times J_0$,

$$\left|(\mathcal{B}) \int_{I_0 \times J_0} U - U(\pi)\right| \leq \frac{\epsilon}{8}.$$

Thus, for any partitions $\pi, \hat{\pi} \subseteq \alpha$ of $I_0 \times J_0$,

$$|U(\pi) - U(\hat{\pi})| \leq \left|U(\pi) - (\mathcal{B}) \int_{I_0 \times J_0} U\right| + \left|U(\hat{\pi}) - (\mathcal{B}) \int_{I_0 \times J_0} U\right| \leq \frac{\epsilon}{4}.$$

Now $W(x, I) = U_1(x, I)g(x)$ and by (1), $V(U_1, \mathcal{B}^1[I_0 \setminus T]) = 0$. So we aim to gain some control over $g(x)$ for $x \in I_0 \setminus T$. For each $x \in I_0 \setminus T$ choose a partition $\pi_x^2 \subseteq \alpha_x^2$ of J_0. As before find an $\alpha^1 \in \mathcal{B}^1$ such that for every $x \in I_0 \setminus T$,

$$\alpha^1[\{x\}] \subseteq \bigcap_{(y,J) \in \pi_x^2} \alpha_y^1[\{x\}].$$

Set

$$Q_1 = \{x \in I_0 \setminus T : |g(x)| + |U_2(x, \pi_x^2)| \leq 1\}$$

and for $r \in \mathbb{N}$, $r \geq 2$

$$Q_r = \{x \in I_0 \setminus T : r - 1 < |g(x)| + |U_2(x, \pi_x^2)| \leq r\}.$$

Now since $Q_r \subseteq I_0 \setminus T$ and (1),

$$V(U_1, \mathcal{B}^1[Q_r]) = 0.$$

Therefore, for every $r \in \mathbb{N}$ there exists an $\alpha^{1,r} \in \mathcal{B}^1$ such that for every partition $\pi^{1,r} \subseteq \alpha^{1,r}[Q_r]$,

$$|U_1|(\pi^{1,r}) \leq \frac{\epsilon}{r 2^{r+2}}.$$

For $x \in I_0 \setminus T$, let

$$p_x^2 = \tilde{p}_x^2 = \pi_x^2.$$

We define the two partitions to be the same for $x \in I_0 \setminus T$ because they are not needed for the approximations on $I_0 \setminus T$; (1) will be enough. However, the partitions are required in order to use the integrability of U and they are therefore defined. Find a $\beta^1 \in \mathcal{B}^1$ such that for every $r \in \mathbb{N}$ and every $x \in Q_r$,

$$\beta_1[\{x\}] \subseteq \alpha^1[\{x\}] \cap \alpha^{1,r}[\{x\}].$$

Now lets consider $x \in T$. Let $\pi_x^2 \subseteq \alpha_x^2$ be a partition of J_0. Now since T is the set of $x \in I_0$ such that $U_2(x, \cdot, \cdot)$ is integrable and $|U_2(x, \pi_x^2) - g(x)|$ is a non-negative real number, we can find an $\eta^2 \in \mathcal{B}^2$ such that for every partition $\pi^2 \subseteq \eta^2$ of J_0,

$$|U_2(x, \pi^2) - g(x)| < \frac{1}{2}|U_2(x, \pi_x^2) - g(x)|.$$

It should be noted that if the right side is 0 we may skip these approximations and move on to (5).

Choose an $\alpha^1 \in \mathcal{B}^1$, $\alpha^1 \subseteq \beta^1$ such that for $x \in T$,

$$\alpha^1[\{x\}] \subseteq \alpha_y^1[\{x\}]$$

for all $(y, J) \in \pi_x^2 \cup \pi_x^{2,2}$.

Let $\pi^1 \subseteq \alpha^1$ be a partition of I_0. For $x \in T$, $(x, I) \in \pi^1$ with

$$U_1(x, I)(U_2(x, \pi_x^2) - g(x)) > 0$$

we set $p_x^2 = \pi_x^2$ and $\tilde{p}_x^2 = \pi_x^{2,2}$. For all other $x \in T$ we set $p_x^2 = \pi_x^{2,2}$ and $\tilde{p}_x^2 = \pi_x^2$. We do this in order to remove some absolute values further in the proof.

Define the compound partitions

$$p = \bigcup_{(x,I)\in\pi^1} \{\big((x,y), I\times J\big) : (y,J)\in p_x^2\} \text{ and}$$

$$\tilde{p} = \bigcup_{(x,I)\in\pi^1} \{\big((x,y), I\times J\big) : (y,J)\in \tilde{p}_x^2\}.$$

Then, $p, \tilde{p} \subseteq \alpha$ are partitions of $I_0 \times J_0$, so that

$$|U(p) - U(\tilde{p})| \leq \frac{\epsilon}{4}.$$

Therefore,

$$\begin{aligned}
&|U(p) - W(\pi^1)|\\
\leq &|U(p[I_0 \setminus T \times J_0]) - W(\pi^1[I_0 \setminus T])| + |U(p[T \times J_0]) - W(\pi^1[T])|.
\end{aligned}$$

Now since $\pi^1 \subseteq \alpha^1$,

$$\begin{aligned}
&|U(p[I_0 \setminus T \times J_0]) - W(\pi^1[I_0 \setminus T])|\\
=&\left| \sum_{\big((x,y),I\times J\big)\in p[I_0\setminus T\times J_0]} U_1(x,I)U_2(x,y,J) - \sum_{(x,I)\in\pi^1[I_0\setminus T]} U_1(x,I)g(x) \right|\\
=&\left| \sum_{(x,I)\in\pi^1[I_0\setminus T]} \sum_{(y,J)\in p_x^2} U_1(x,I)U_2(x,y,J) - \sum_{(x,I)\in\pi^1[I_0\setminus T]} U_1(x,I)g(x) \right|\\
=&\left| \sum_{(x,I)\in\pi^1[I_0\setminus T]} U_1(x,I) \left(\sum_{(y,J)\in p_x^2} U_2(x,y,J) - g(x) \right) \right|\\
\leq &\sum_{r\in\mathbb{N}} \sum_{(x,I)\in\pi^1[Q_r]} r|U_1(x,I)|\\
\leq &\sum_{r\in\mathbb{N}} \frac{r\epsilon}{r2^{r+2}}
\end{aligned}$$

$$=\frac{\epsilon}{4}.$$

Now for $(x, I) \in \pi^1[T]$, if

$$U_1(x, I)\left(\sum_{(y,J)\in\pi_x^2} U_2(x, y, J) - g(x)\right) > 0,$$

then,

$$\begin{aligned}
0 &< U_1(x, I)(\sum_{(y,J)\in\pi_x^2} U_2(x, y, J) - g(x)) \\
&= U_1(x, I)(\sum_{(y,J)\in p_x^2} U_2(x, y, J) - g(x)) \\
&\le U_1(x, I)\left(\sum_{(y,J)\in\pi_x^2} U_2(x, y, J) - g(x) + U_2(x, p_x^2) - g(x) - 2U_2(x, \tilde{p}_x^2) + 2g(x)\right) \\
&= 2U_1(x, I)\left(U_2(x, p_x^2) - U_2(x, \tilde{p}_x^2)\right).
\end{aligned}$$

However, if

$$U_1(x, I)\left(\sum_{(y,J)\in p_x^2} U_2(x, y, J) - g(x)\right) \le 0$$

then,

$$\begin{aligned}
&|U_1(x, I)(U_2(x, p_x^2) - g(x))| \\
=&|U_1(x, I)||U_2(x, \pi_x^{2,2}) - g(x)| \\
\le&|U_1(x, I)||U_2(x, \pi_x^{2,2}) - g(x)| + |U_1(x, I)|\left(|U_2(x, \pi_x^2) - g(x)| - 2|U_2(x, \pi_x^{2,2}) - g(x)|\right) \\
=&|U_1(x, I)|\left(|U_2(x, \pi_x^2) - g(x)| - |U_2(x, \pi_x^{2,2}) - g(x)|\right) \\
=&- U_1(x, I)(U_2(x, \pi_x^2) - g(x)) - |U_1(x, I)||U_2(x, \pi_x^{2,2}) - g(x)| \\
\le&- U_1(x, I)(U_2(x, \pi_x^2) - g(x)) + U_1(x, I)(U_2(x, \pi_x^{2,2}) - g(x))
\end{aligned}$$

$$=U_1(x,I)(U_2(x,\pi_x^{2,2})-U_2(x,\pi_x^2))$$

$$=U_1(x,I)(U_2(x,p_x^2)-U_2(x,\tilde{p}_x^2).$$

Therefore, for $x \in T$ and $(x,I) \in \pi^1$,

$$|U_1(x,I)(U_2(x,p_x^2)-g(x))| \leq 2U_1(x,I)(U_2(x,p_x^2)-U_2(x,\tilde{p}_x^2)). \tag{5}$$

Also,

$$U(p)-U(\tilde{p}) = \sum_{(x,I)\in\pi^1} U_1(x,I)(U_2(x,p_x^2)-U_2(x,\tilde{p}_x^2)).$$

Therefore,

$$\left|\sum_{((x,y),I\times J)\in p, x\in T} U((x,y),I\times J) - W(\pi^1[T])\right|$$

$$=\left|\sum_{(x,I)\in\pi^1[T]} U_1(x,I)(U_2(x,p_x^2)-g(x))\right|$$

$$\leq \sum_{(x,I)\in\pi^1[T]} 2U_1(x,I)(U_2(x,p_x^2)-U_2(x,\tilde{p}_x^{2,2}))$$

$$=2(U(p)-U(\tilde{p}))$$

$$\leq 2\frac{\epsilon}{4}$$

$$=\frac{\epsilon}{2}.$$

Thus,

$$|U(p)-W(\pi^1)|$$

$$\leq \left|\sum_{((x,y),I\times J)\in p, x\in T} U((x,y),I\times J) - W(\pi^1[T])\right|$$

$$
\begin{aligned}
&+\left|\sum_{((x,y),I\times J)\in p, x\notin T} U((x,y),I\times J)-W(\pi^1[I_0\setminus T])\right|\\
\le&\frac{\epsilon}{2}+\frac{\epsilon}{4}\\
=&\frac{3\epsilon}{4}.
\end{aligned}
$$

And finally,

$$
\begin{aligned}
&\left|(\mathcal{B}\int_{I_0\times J_0} U - W(\pi^1)\right|\\
\le&\left|(\mathcal{B})\int_{I_0\times J_0} U - U(p)\right| + |U(p)-W(\pi^1)|\\
\le&\frac{\epsilon}{8}+\frac{3\epsilon}{4}\\
\le&\epsilon.
\end{aligned}
$$

$\square$

5.3. Corollary to the Fubini Theorem

Let us now look at some more concrete examples of product bases and find some corollaries to the Fubini theorem. It is easily seen that the product of a $\mathcal{B}_1$ base in $\mathbb{R}^m$ and that of a $\mathcal{B}_1$ base in $\mathbb{R}^n$ results in the $\mathcal{B}_1$ base in $\mathbb{R}^{n+m}$. The same holds true for the $\widetilde{\mathcal{B}}_1$ and the $\mathcal{B}_1^\diamond$. This clearly does not hold for bases that use regularity. For example if we take two squares, one of which whose sides are l times the length of the other's sides. Then the product of these squares is an interval whose regularity is $\frac{1}{l}$. Thus, every element of this product base will contain intervals of any regularity.

For the sake of the following corollary $\mathcal{I}^n$ will denote the family of n-th degree non-degenerate, compact intervals. Also we will write $n\mathcal{B}_1$ for the $\mathcal{B}_1$ base in $\mathbb{R}^n$, other bases will follow follow similar conventions.

Corollary 5.3.1.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be $n\mathcal{B}_1^\diamond$-integrable on $I_0 = [a_1, b_1] \times ... \times [a_n, b_n]$ and let $m < n$ be some integer. Choose any m different coordinate directions $\mathbb{R}^n$, without loss of generality we assume the first m. Define the set T to be the set of x in $[a_1, b_1] \times ... \times [a_m, b_m]$ such that $f(x, \cdot)$ is Lebesgue integrable on $[a_{m+1}, b_{m+1}] \times ... \times [a_n, b_n]$. Then,

(1) *$[a_1, b_1] \times ... \times [a_m, b_m] \setminus T$ is a Lebesgue null set;*

(2) *For $g(x) = \int_{[a_{m+1}, b_{m+1}] \times ... \times [a_n, b_n]} f(x, \cdot) d\lambda_m$ on T and arbitrary otherwise, we have*

$$(n\mathcal{B}_1^\diamond) \int_{I_0} f d\lambda = \int_{[a_1, b_1] \times ... \times [a_m, b_m]} g(x) d\lambda_m.$$

Notice that since the weak Kurzweil base provides the same integral as that of Lebesgue, this is in fact the Fubini theorem for the Lebesgue integral.

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