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Multiplier Hopf Algebras and Duality

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by

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Abstract

In this thesis, we study and apprehend Hopf algebras, multiplier Hopf algebras, and their dualities.

A Hopf algebra (A, Δ) is a unital algebra A with an algebraic homomorphism $\Delta : A \to A \otimes A$ and other structures (including the counit ϵ and the antipode S). In the finite-dimensional case, we can construct the dual $(\hat{A}, \hat{\Delta})$ of (A, Δ), which is also a Hopf algebra, and prove that (A, Δ) is isomorphic to its bidual $(\hat{\hat{A}}, \hat{\hat{\Delta}})$, the dual of $(\hat{A}, \hat{\Delta})$.

If we drop the assumption that A is unital and allow Δ to have values in the multiplier algebra $M(A \otimes A)$ of $A \otimes A$, we end with a multiplier Hopf algebra. If the coopposite algebra $(A, \Delta)^{cop}$ of (A, Δ) is also a multiplier Hopf algebra, we say that (A, Δ) is regular, for which we can involve non-zero left (right) invariant linear functionals, called left (right) integrals. It is proved that if left (right) integrals exist, they are faithful and unique up to a scalar, and that left integrals and right integrals can be connected to each other through the antipode. For a regular multiplier Hopf algebra (A, Δ) with integrals, we can construct its dual $(\hat{A}, \hat{\Delta})$ with the help of integrals. In this case, $(\hat{A}, \hat{\Delta})$ is again a regular multiplier Hopf algebra, on which integrals can also be constructed, and it turns out that the bidual $(\hat{\hat{A}}, \hat{\hat{\Delta}})$ is canonically isomorphic to (A, Δ).

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Contents

CHAPTER 1

Introduction

An algebra is a linear space A over a field $\Bbbk$ equipped with an associative multiplication $m : A \otimes A \to A$. If we reverse the arrow, we get a comultiplication $\Delta : A \to A \otimes A$, which is coassociative in the sense that

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta.$$

This idea yields the concept of a Hopf algebra, which is a unital algebra A (over $\mathbb{C}$) with a counit $\epsilon : A \to \mathbb{C}$ and an antipode $S : A \to A$ as well.

Let G be a finite group and let A be the algebra of all complex functions on G with pointwise operations. Then (A, Δ) is a Hopf algebra with the structure maps

$$\Delta(f)((x, y)) = f(xy), \quad \epsilon(f) = f(e), \quad \text{and} \quad S(f)(x) = f(x^{-1}),$$

where $f \in A$, $x, y \in G$, and e is the unit of G. In Chapter 3, we collect various definitions and properties related to general Hopf algebras.

In Chapter 4, we see that the duality and biduality of Hopf algebras behave very nicely in the finite-dimensional case. More precisely, if (A, Δ) is a finite-dimensional Hopf algebra, then its dual (A', Δ') is also a Hopf algebra, and the bidual (A'', Δ'') of (A, Δ) is isomorphic to (A, Δ) as Hopf algebras. In the infinite-dimensional case, this is no longer true. However, we still can find a subspace A^0 of A', called the restricted dual of A, such that $\Delta_{A^0} = m'|_{A^0} : A^0 \to A^0 \otimes A^0$. (A^0, Δ_{A^0}) is a Hopf algebra, and the map $A \times A^0 \to \mathbb{C}$, $(a, f) \mapsto f(a)$ is a dual pairing of Hopf algebras.

If we drop the assumption that A is unital and allow Δ to have values in the multiplier algebra $M(A \otimes A)$ of $A \otimes A$, then we obtain naturally a multiplier Hopf algebra. In a multiplier Hopf algebra (A, Δ), the maps $T_1, T_2 : A \otimes A \to A \otimes A$ defined by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \qquad \text{and} \qquad T_2(a \otimes b) = (a \otimes 1)\Delta(b)$$

are required to be bijective, and they play very important roles. Some basic properties on multiplier Hopf algebras are discussed in Chapter 5. A multiplier Hopf algebra (A, Δ) is regular if the corresponding coopposite algebra $(A, \Delta)^{cop}$ is also a multiplier Hopf algebra. In Section 5.5, we show that a multiplier Hopf algebra (A, Δ) is regular if and only if the antipode S of A is a bijection on A.

Let G be a group and let $\mathbb{C}(G)$ be the algebra of all complex functions on G with pointwise operations. Let A denote the subalgebra $\mathbb{C}_{fin}(G)$ of $\mathbb{C}(G)$ consisting of complex functions on G with finite support. Then $M(A) \cong \mathbb{C}(G)$, $M(A \otimes A) \cong \mathbb{C}(G \times G)$, and (A, Δ) is a regular multiplier Hopf algebra, where

$$\Delta(f)((x, y)) = f(xy) \qquad (f \in A,\ x, y \in G).$$

Moreover, the linear functional $\phi : A \to \mathbb{C},\ f \mapsto \sum_{x \in G} f(x)$ is left and right invariant in the sense that

$$(id \otimes \phi)\Delta(f) = (\phi \otimes id)\Delta(f) = \phi(f)1_G \qquad \text{for all } f \in A.$$

On a general regular multiplier Hopf algebra (A, Δ), we consider non-zero left/right invariant linear functionals, called left/right integrals. In Chapter 6, we prove that they are faithful and unique up to a scalar. We show that left invariance and right invariance of integrals can transfer to each other via the antipode S. In Sections 6.3 and 6.4, we see that for a general regular multiplier Hopf algebra (A, Δ) with a left integral ϕ and a right integral ψ, there exists a modular element $\delta \in M(A)$ and modular automorphisms σ and σ' of A such that for all $a \in A$, we have

$$\phi(S(a)) = \phi(a\delta), \quad \phi(a\cdot) = \phi(\cdot\sigma(a)), \quad \text{and} \quad \psi(a\cdot) = \psi(\cdot\sigma'(a)).$$

In Chapter 7, we consider the duality and biduality of regular multiplier Hopf algebras, which are very satisfying with the help of integrals. For a given regular multiplier Hopf algebra (A, Δ) with integrals, we can construct the dual $(\hat{A}, \hat{\Delta})$, which is also a multiplier Hopf algebra with integrals. Then we show that the dual $(\hat{\hat{A}}, \hat{\hat{\Delta}})$ of $(\hat{A}, \hat{\Delta})$ is canonically isomorphic to (A, Δ). Furthermore, for an algebraic quantum group, which is a multiplier Hopf $*$-algebra with a positive left integral and a positive right integral, its dual $(\hat{A}, \hat{\Delta})$ is still an algebraic quantum group, and its bidual $(\hat{\hat{A}}, \hat{\hat{\Delta}})$ is isomorphic to (A, Δ) as algebraic quantum groups.

The main references of this thesis are [**7**], [**8**], and [**9**]. The thesis is organized following some basic definitions and main results in these references. Furthermore, based on my own understanding, more details, including proofs, remarks, and corollaries, are added so that the theory about Hopf algebras and multiplier Hopf algebras presented in the thesis is more complete and readable for a beginner working in this area.

CHAPTER 2

Preliminary – tensor products

This chapter introduces the basic definitions and properties of tensor products from a purely algebraic viewpoint. Most results are from [**5**].

2.1. Tensor products of vector spaces

Let X, Y, and Z be vector spaces over $\mathbb{C}$. Recall that a map $\phi : X \times Y \to Z$ is *bilinear* if

$$\phi(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 \phi(x_1, y) + \alpha_2 \phi(x_2, y)$$

and

$$\phi(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 \phi(x, y_1) + \beta_2 \phi(x, y_2)$$

for all $x, x_1, x_2 \in X$, $y, y_1, y_2 \in Y$, and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$. We denote by $B(X \times Y, Z)$ the vector space of all bilinear maps from the product $X \times Y$ to Z; when $Z = \mathbb{C}$, we simply denote $B(X \times Y, Z)$ by $B(X \times Y)$.

The tensor product $X \otimes Y$ of vector spaces X and Y can be constructed as a space of linear functionals on $B(X \times Y)$ in the following way: for all $x \in Y$ and $y \in Y$, we denote by $x \otimes y$ the functional on $B(X \times Y)$ given by evaluation at the point (x, y); that is,

$$(x \otimes y)(\phi) = \langle \phi, x \otimes y \rangle = \phi(x, y) \qquad (\phi \in B(X \times Y)).$$

The *tensor product* $X \otimes Y$ is the subspace of the dual $B(X \times Y)'$ spanned by these elements. Thus, a typical tensor in $X \otimes Y$ has the form

$$u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i, \tag{2.1.1}$$

where $n \in \mathbb{N}, \lambda_i \in \mathbb{C}, x_i \in X$, and $y_i \in Y$. Now, the action of u on $B(X \times Y)$ is given by

$$u(\phi) = \langle \phi, \sum_{i=1}^{n} \lambda_i x_i \otimes y_i \rangle = \sum_{i=1}^{n} \lambda_i \phi(x_i, y_i).$$

We need to notice that the representation of u is not unique. This can be easily seen from the following properties of tensor products.

PROPOSITION 2.1.1. *Let $X \otimes Y$ be the tensor product of vector spaces X and Y. Then for all $x, x_1, x_2 \in X$, $y, y_1, y_2 \in Y$, and $\lambda \in \mathbb{C}$, we have*

(i) $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$;
(ii) $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$;
(iii) $\lambda(x \otimes y) = (\lambda x) \otimes y = x \otimes (\lambda y)$;
(iv) $x \otimes 0 = 0 \otimes y = 0 \otimes 0$.

From Proposition 2.1.1(iii), we see that the typical description of elements u of $X \otimes Y$ given by (2.1.1) can be written in the form

$$u = \sum_{i=1}^{n} x_i \otimes y_i.$$

PROPOSITION 2.1.2. *Let X and Y be vector spaces.*

(i) *Let E and F be linear independent subsets of X and Y, respectively. Then $\{x \otimes y : x \in E, y \in F\}$ is a linearly independent subset of $X \otimes Y$.*
(ii) *If $\{e_i, i \in I\}$ and $\{e'_j, j \in J\}$ are bases for X and Y, respectively, then $\{e_i \otimes e'_j, (i,j) \in I \times J\}$ is a base for $X \otimes Y$.*

COROLLARY 2.1.3. *If X and Y are both finite-dimensional spaces, then*

$$dim\ (X \otimes Y) = dim\ (X)\ dim\ (Y).$$

In particular, dim $(X \otimes \mathbb{C}) =$ dim $(\mathbb{C} \otimes X) =$ dim (X). In fact, as given below, regardless of dimensional considerations, the tensor products $X \otimes \mathbb{C}$ and $\mathbb{C} \otimes X$ are both isomorphic to X.

PROPOSITION 2.1.4. *If X is a vector space, then we have*

$$X \otimes \mathbb{C} \xrightarrow{\cong} X \ \text{via}\ x \otimes \lambda \mapsto \lambda x \quad \text{and} \quad \mathbb{C} \otimes X \xrightarrow{\cong} X \ \text{via}\ \lambda \otimes x \mapsto \lambda x.$$

For every non-zero tensor $u \in X \otimes Y$, there exists a smallest $n \in \mathbb{N}$ such that

$$u = \sum_{i}^{n} x_i \otimes y_i$$

for some $x_i \in X$ and $y_i \in Y$. Then it is clear that the sets $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$ are each linearly independent. The natural number n is called the *rank* of u. Tensors of rank one are often referred to as *elementary tensor.*

To determine whether two tensors are the same, we always use the following proposition.

PROPOSITION 2.1.5. *Let X and Y be two vector spaces and $u \in X \otimes Y$ in the form of $u = \sum_{i=1}^n x_i \otimes y_i$. Then the following statements are equivalent.*

(i) $u = 0$;
(ii) $\sum_{i=1}^n f(x_i)g(y_i) = 0$ *for all* $f \in X'$ *and* $g \in Y'$;
(iii) $\sum_{i=1}^n f(x_i)y_i = 0$ *for all* $f \in X'$;
(iv) $\sum_{i=1}^n g(y_i)x_i = 0$ *for all* $g \in Y'$.

2.2. Tensor products and linearization

The primary purpose of tensor products is to linearize bilinear maps.

Let $\phi : X \times Y \to \mathbb{C}$ be a bilinear functional. We recall that each tensor $u \in X \otimes Y$ acts as a linear functional on $B(X \times Y)$. So, we can define a map

$$\tilde{\phi} : X \otimes Y \to \mathbb{C},\ u \mapsto u(\phi).$$

Now $\tilde{\phi} : X \otimes Y \to \mathbb{C}$ is linear. Furthermore, we have

$$\tilde{\phi}(x \otimes y) = \phi(x, y) \qquad \text{for all } x \in X, y \in Y.$$

On the other hand, if Φ is a linear functional on $X \otimes Y$, then

$$\Phi(x \otimes y) = (\Phi \circ \otimes)(x, y) \qquad \text{for all } x \in X, y \in Y,$$

where $\otimes : X \times Y \to X \otimes Y,\ (x, y) \mapsto x \otimes y$. Obviously, $\Phi \circ \otimes : X \times Y \to \mathbb{C}$ is a bilinear map and $\widetilde{\Phi \circ \otimes} = \Phi$. In summary, we have the linear isomorphism

$$B(X \times Y) \cong (X \otimes Y)',\ \phi \mapsto \tilde{\phi}.$$

The same result also holds for more general bilinear map spaces. More precisely, we have the following proposition.

PROPOSITION 2.2.1. *For every bilinear map $\phi : X \times Y \to Z$, there exists a unique linear map $\tilde{\phi} : X \otimes Y \to Z$ such that*

$$\phi(x, y) = \tilde{\phi}(x \otimes y) \qquad \text{for all } x \in X, y \in Y;$$

that is, the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\phi"] \arrow[d, "\otimes"'] & Z \\ X \otimes Y \arrow[ur, "\tilde{\phi}"'] & \end{tikzcd}$$

commutes. Moreover, $B(X \times Y, Z) \cong L(X \otimes Y, Z)$ via $\phi \mapsto \tilde{\phi}$, where $L(X \otimes Y, Z)$ denotes the vector space of all the linear maps from $X \otimes Y$ to Z.

PROPOSITION 2.2.2. *(Uniqueness theorem) Let X, Y, and Z be vector spaces. Suppose that there exists a bilinear map $\phi : X \times Y \to Z$ with the property that for every vector space W and every bilinear map $\psi : X \times Y \to W$, there exists a unique linear map $L : Z \to W$ such that $\psi = L \circ \phi$. Then there is an isomorphism $\Phi : X \otimes Y \to Z$ such that $\Phi(x \otimes y) = \phi(x, y)$ for all $x \in X, y \in Y$.*

REMARK 2.2.3. (i) The tensor products $X \otimes Y$ and $Y \otimes X$ are canonically isomorphic.

(ii) Given four vector spaces X, X', Y, and Y', let $S : X \to X'$ and $T : Y \to Y'$ be linear maps. Then we can get a linear map $S \otimes T : X \otimes Y \to X' \otimes Y'$ by

$$(S \otimes T)(x \otimes y) = S(x) \otimes T(y) \qquad (x \in X, y \in Y).$$

CHAPTER 3

Hopf algebras

This chapter gives a brief introduction to Hopf algebras. The main references about this chapter are [**1**], [**6**], and [**7**]. In order to introduce Hopf algebras, which is a unital algebra and simultaneously a counital coalgebra, let us begin with coalgebras and then bialgebras.

3.1. Coalgebras and bialgebras

Let A be an algebra over $\mathbb{C}$. Then A is a vector space with the multiplication

$$m:\ A \times A \to A,\ (a,b) \mapsto ab,$$

which is associative in the sense that

$$(ab)c = a(bc) \qquad \text{for all } a,b,c \in A.$$

Obviously, $m : A \times A \to A$ is a bilinear map. From Proposition 2.2.1, it can be considered as a linear map

$$m:\ A \otimes A \to A,\ a \otimes b \mapsto ab.$$

Then the associativity can be expressed as

$$m \circ (m \otimes id_A) = m \circ (id_A \otimes m);$$

that is, the square

$$\begin{array}{ccc} A & \xleftarrow{m} & A \otimes A \\ m\uparrow & & \uparrow id\otimes m \\ A \otimes A & \xleftarrow[m\otimes id]{} & A \otimes A \otimes A \end{array}$$

commutes.

If A is unital with a unit 1_A, the linear map

$$\eta:\ \mathbb{C} \to A,\ \lambda \mapsto \lambda 1_A,$$

is called the unit map of A, which satisfies that

$$m \circ (\eta \otimes id_A) = id_A = m \circ (id_A \otimes \eta);$$

that is, the diagram

$$\begin{tikzcd}
A \otimes A \arrow[r, "m"] & A & A \otimes A \arrow[l, "m"'] \\
\mathbb{C} \otimes A \arrow[u, "\eta\otimes id"] & A \arrow[u, equal] \arrow[l, "\cong"] \arrow[r, "\cong"'] & A \otimes \mathbb{C} \arrow[u, "id\otimes\eta"']
\end{tikzcd}$$

commutes.

An algebra homomorphism of two algebras A and B is a linear map $E : A \to B$ satisfying $E \circ m_A = m_B \circ (E \otimes E)$; that is, the square

$$\begin{tikzcd}
A & A \otimes A \arrow[l, "m_A"'] \arrow[d, "E\otimes E"] \\
B \arrow[from=u, "E"'] & B \otimes B \arrow[l, "m_B"]
\end{tikzcd}$$

commutes. If A and B are both unital algebras, then E is call unital when $E \circ \eta_A = \eta_B$; that is, the square

$$\begin{tikzcd}
A \arrow[d, "E"'] & \mathbb{C} \arrow[l, "\eta_A"'] \arrow[d, equal] \\
B & \mathbb{C} \arrow[l, "\eta_B"]
\end{tikzcd}$$

commutes.

Reversing arrows in the above descriptions of algebras, we obtain the following definitions on coalgebras.

Definition 3.1.1. A *coalgebra* (over $\mathbb{C}$) is a vector space A equipped with a linear map $\Delta : A \to A \otimes A$, called the *comultiplication* or *coproduct*, which is *coassociative* in the sense that

$$(\Delta \otimes id_A) \circ \Delta = (id_A \otimes \Delta) \circ \Delta;$$

that is, the square

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "id\otimes\Delta"] \\
A \otimes A \arrow[r, "\Delta\otimes id"'] & A \otimes A \otimes A
\end{tikzcd} \tag{3.1.1}$$

commutes.

In this case, a linear map $\epsilon : A \to \mathbb{C}$ is a *counit* for (A, Δ) if

$$(\epsilon \otimes id_A) \circ \Delta = id_A = (id_A \otimes \epsilon) \circ \Delta;$$

that is, the diagram

$$\begin{tikzcd} A \otimes A \arrow[d, "\epsilon\otimes id"'] & A \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, equal] & A \otimes A \arrow[d, "id\otimes\epsilon"] \\ \mathbb{C} \otimes A \arrow[r, "\cong"'] & A & A \otimes \mathbb{C} \arrow[l, "\cong"] \end{tikzcd} \tag{3.1.2}$$

commutes. A coalgebra is called *counital* if it has a counit.

A *coalgebra morphism* of two coalgebras (A, Δ_A) and (B, Δ_B) is a linear map $F : A \to B$ satisfying $\Delta_B \circ F = (F \otimes F) \circ \Delta_A$; that is, the square

$$\begin{tikzcd} A \arrow[r, "\Delta_A"] \arrow[d, "F"'] & A \otimes A \arrow[d, "F\otimes F"] \\ B \arrow[r, "\Delta_B"'] & B \otimes B \end{tikzcd} \tag{3.1.3}$$

commutes. If A and B have counits ϵ_A and ϵ_B, respectively, then the map F is called *counital* when $\epsilon_B \circ F = \epsilon_A$; that is, we have the commutative square

$$\begin{tikzcd} A \arrow[r, "\epsilon_A"] \arrow[d, "F"'] & \mathbb{C} \arrow[d, equal] \\ B \arrow[r, "\epsilon_B"'] & \mathbb{C}. \end{tikzcd} \tag{3.1.4}$$

If (A, Δ) is a coalgebra and the comultiplication Δ is understood, we freely speak of A itself as a coalgebra.

Remark 3.1.2. (i) Every coalgebra has at most one counit. Indeed, if ϵ_1 and ϵ_2 are counits for a coalgebra (A, Δ_A), then

$$\epsilon_1 = \epsilon_1 \circ (id_A \otimes \epsilon_2) \circ \Delta = (\epsilon_1 \otimes \epsilon_2) \circ \Delta = \epsilon_2 \circ (\epsilon_1 \otimes id_A) \circ \Delta = \epsilon_2.$$

(ii) Given coalgebras (A, Δ_A) and (B, Δ_B), we can construct the following new coalgebras.

Coopposite coalgebra. Let $\Sigma : A \otimes A \to A \otimes A,\ a \otimes b \mapsto b \otimes a$ be the flip map. Then $(A, \Delta_A)^{cop} := (A, \Sigma \circ \Delta_A)$ is a coalgebra, called the *coopposite coalgebra* of (A, Δ_A). Evidently, a linear map $\epsilon : A \to \mathbb{C}$ is a counit for (A, Δ_A) if and only if it is a counit for $(A, \Delta_A)^{cop}$. The coalgebra (A, Δ_A) is called *cocommutative* if $(A, \Delta_A)^{cop} = (A, \Delta_A)$; that is, if $\Sigma \circ \Delta_A = \Delta_A$.

Direct sum. Define

$$\Delta_{A\oplus B} : A \oplus B \xrightarrow{\Delta_A \oplus \Delta_B} (A \otimes A) \oplus (B \otimes B) \xrightarrow{\zeta^{\oplus}} (A \oplus B) \otimes (A \oplus B),$$

where $\zeta^{\oplus}((a_1 \otimes a_2) \oplus (b_1 \otimes b_2)) = (a_1 \oplus b_1) \otimes (a_2 \oplus b_2)$. Then $(A \oplus B, \Delta_{A\oplus B})$ is a coalgebra. If A and B possess counits ϵ_A and ϵ_B, respectively, then the map $(a, b) \mapsto \epsilon_A(a) + \epsilon_B(b)$ is the counit for $(A \oplus B, \Delta_{A\oplus B})$.

Tensor product. Define

$$\Delta_{A\otimes B} : A \otimes B \xrightarrow{\Delta_A \otimes \Delta_B} A \otimes A \otimes B \otimes B \xrightarrow{\zeta^{\otimes}} A \otimes B \otimes A \otimes B,$$

where $\zeta^{\otimes}(a_1 \otimes a_2 \otimes b_1 \otimes b_2) = a_1 \otimes b_1 \otimes a_2 \otimes b_2$. Then $(A \otimes B, \Delta_{A\otimes B})$ is a coalgebra. If A and B possess counits ϵ_A and ϵ_B, respectively, then the map $a \otimes b \mapsto \epsilon_A(a)\epsilon_B(b)$ is the counit for $(A \otimes B, \Delta_{A\otimes B})$.

For calculations in coalgebras, the *Sweedler notation* (or the Sigma notation) described below is very useful and convenient.

NOTATION 3.1.3. Let (A, Δ) be a coalgebra and $a \in A$. Then $\Delta(a) \in A \otimes A$ can be written in the form $\Delta(a) = \sum_i a_{1,i} \otimes a_{2,i}$, where $a_{1,i}, a_{2,i} \in A$. To simplify the notation, we suppress the summation index i and write

$$\Delta(a) = \sum_i a_{1,i} \otimes a_{2,i} =: \sum a_{(1)} \otimes a_{(2)}.$$

Here, the subscripts "(1)" and "(2)" indicate the orders of the factors in the tensor product; thus for example, $\Sigma(\Delta(a)) = \sum a_{(2)} \otimes a_{(1)}$. We extend the notation to iterated applications of Δ as follows. Since Δ is coassociative, the elements

$$(id_A \otimes \Delta)(\Delta(a)) = \sum a_{(1)} \otimes \Delta(a_{(2)}) = \sum a_{(1)} \otimes (a_{(2)})_{(1)} \otimes (a_{(2)})_{(2)}$$

and

$$(\Delta \otimes id_A)(\Delta(a)) = \sum \Delta(a_{(1)}) \otimes a_{(2)} = \sum (a_{(1)})_{(1)} \otimes (a_{(1)})_{(2)} \otimes a_{(2)}$$

are equal. So, we can write

$$(id_A \otimes \Delta)(\Delta(a)) = (\Delta \otimes id_A)(\Delta(a)) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}.$$

More generally, consider the map $\Delta^{(n)} : A \to A^{\otimes n+1}$, inductively defined by

$$\Delta^{(0)} := id_A \quad \text{and} \quad \Delta^{(n)} := (\Delta^{(n-1)} \otimes id_A) \circ \Delta \quad (n \geq 1).$$

By coassociativity, every map $A \to A^{\otimes n+1}$ that is obtained by n successive application of Δ to one factor of the intermediate tensor product $A^{\otimes n+1}$ coincides with $\Delta^{(n)}$. We write

$$\Delta^{(n)}(a) =: \sum a_{(1)} \otimes \cdots \otimes a_{(n+1)}.$$

EXAMPLE 3.1.4. Let (A, Δ_A) and (B, Δ_B) be coalgebras. Then for all $a \in A$ and $b \in B$, we have

$$\Delta_{A\oplus B}((a, b)) = \sum (a_{(1)}, b_{(1)}) \otimes (a_{(2)}, b_{(2)});$$
$$\Delta_{A\otimes B}(a \otimes b) = \sum a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)}.$$

Next, in vector spaces, we consider algebra structure and coalgebra structure that are compatible in a natural sense.

LEMMA 3.1.5. *Let A be a vector space equipped with the structure of an algebra and of a coalgebra. Then the following statements are equivalent.*

(i) *The comultiplication $\Delta : A \to A \otimes A$ is an algebra homomorphism.*
(ii) *The multiplication $m : A \otimes A \to A$ is a coalgebra morphism.*
(iii) *The following diagram commutes:*

$$\begin{tikzcd} A \otimes A \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "m"'] & A \otimes A \otimes A \otimes A \arrow[r, "id\otimes\Sigma\otimes id"] & A \otimes A \otimes A \otimes A \arrow[d, "m\otimes m"] \\ A \arrow[rr, "\Delta"'] & & A \otimes A. \end{tikzcd} \tag{3.1.5}$$

PROOF. Statement (iii) is equivalent to (i) and (ii), because the multiplication and comultiplication of $A \otimes A$ are given by

$$(A \otimes A) \otimes (A \otimes A) \xrightarrow{id\otimes\Sigma\otimes id} A \otimes A \otimes A \otimes A \xrightarrow{m\otimes m} A \otimes A$$

and

$$A \otimes A \xrightarrow{\Delta \otimes \Delta} A \otimes A \otimes A \otimes A \xrightarrow{id \otimes \Sigma \otimes id} (A \otimes A) \otimes (A \otimes A),$$

respectively. □

Definition 3.1.6. A *bialgebra* (over $\mathbb{C}$) is a vector space A equipped with the structure of an algebra and of a coalgebra such that diagram (3.1.5) commutes.

A bialgebra is called *unital* if it is unital as an algebra and the comultiplication is a unital algebra homomorphism; that is, $\Delta_A \circ \eta_A = \eta_{A \otimes A}$. It is called *counital* if it is counital as a coalgebra and the multiplication is a counital coalgebra morphism; that is, $\epsilon_A \circ m_A = \epsilon_{A \otimes A}$.

A *bialgebra morphism* of two bialgebras (A, Δ_A) and (B, Δ_B) is a linear map $F : A \to B$ that is both a algebra homomorphism and a coalgebra morphism. It is called *unital/counital* if it is unital/counital as a map of algebras/coalgebras.

Therefore, a bialgebra A is unital if and only if A is a unital algebra and $\eta_A : \mathbb{C} \to A$ is a bialgebra morphism; a bialgebra A is counital if and only if A is a counital coalgebra and $\epsilon_A : A \to \mathbb{C}$ is a bialgebra morphism.

Remark 3.1.7. (i) Often, bialgebras are assumed to be unital and counital. We explicitly state these assumptions whenever we impose them.

(ii) For a unital/counital bialgebra (A, Δ), the compatibility conditions between the unit and the comultiplication/between the counit and the multiplication amount to the commutativity of the squares

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\eta"] \arrow[d, "\cong"'] & A \arrow[d, "\Delta"] \\ \mathbb{C} \otimes \mathbb{C} \arrow[r, "\eta \otimes \eta"'] & A \otimes A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \otimes A \arrow[r, "\epsilon \otimes \epsilon"] \arrow[d, "m"'] & \mathbb{C} \otimes \mathbb{C} \arrow[d, "\cong"] \\ A \arrow[r, "\epsilon"'] & \mathbb{C}. \end{tikzcd} \tag{3.1.6}$$

In particular, (A, Δ) is counital if and only if the counit ϵ is multiplicative.

(iii) Given bialgebras (A, Δ_A) and (B, Δ_B), we can construct the following new bialgebras.

Opposite and coopposite bialgebra. Reversing the multiplication, the comultiplication, or both of (A, Δ_A), we obtain three new bialgebras. More precisely, denote by A^{op} the opposite algebra of A. Then the pairs $(A, \Delta_A)^{op} := (A^{op}, \Delta_A)$, $(A, \Delta_A)^{cop} := (A, \Sigma \circ \Delta_A)$, and $(A, \Delta_A)^{op,cop} := (A^{op}, \Sigma \circ \Delta_A)$ are bialgebras again.

Direct sum and tensor product. The vector spaces $A \oplus B$ and $A \otimes B$ are bialgebras with respect to the usual algebra structure and the coalgebra structure defined in Remark 3.1.2(ii).

3.2. Hopf algebras

Based on bialgebras, we can define a Hopf algebra, which is one special kind of bialgebras.

Definition 3.2.1. A *Hopf algebra* is a unital and counital bialgebra (A, Δ) with a linear map $S : A \to A$, call the *antipode*, satisfying

$$m \circ (S \otimes id) \circ \Delta = \eta \circ \epsilon = m \circ (id \otimes S) \circ \Delta;$$

that is, the diagram

$$\begin{array}{ccccc} A \otimes A & \xleftarrow{\Delta} & A & \xrightarrow{\Delta} & A \otimes A \\ {\scriptstyle S\otimes id}\downarrow & & \downarrow{\scriptstyle \eta\circ\epsilon} & & \downarrow{\scriptstyle id\otimes S} \\ A \otimes A & \xrightarrow[m]{} & A & \xleftarrow[m]{} & A \otimes A \end{array} \tag{3.2.1}$$

commutes.

A *Hopf algebra morphism* of two Hopf algebras (A, Δ_A) and (B, Δ_B) is a unital and counital bialgebra morphism $F : A \to B$ satisfying $S_B \circ F = F \circ S_A$; that is, we have the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{S_A} & A \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ B & \xrightarrow[S_B]{} & B. \end{array} \tag{3.2.2}$$

We will see in Proposition 3.4.6 that this is automatically satisfied.

Example 3.2.2. (The function algebra of a finite group) Let G be a finite group. We denote by $\mathbb{C}(G)$ the algebra of all complex valued functions on G with pointwise operations. The structure maps of G, that is,

the multiplication,	the inclusion of the unit,	and the inversion
$G \times G \to G,\ (x, y) \mapsto xy,$	$\{e\} \hookrightarrow G,\ e \mapsto e,$	$G \to G,\ x \mapsto x^{-1},$

induce the following algebra homomorphisms:

$$\begin{aligned} \Delta : \mathbb{C}(G) \to \mathbb{C}(G \times G), \qquad & \epsilon : \mathbb{C}(G) \to \mathbb{C}, \qquad & S : \mathbb{C}(G) \to \mathbb{C}(G), \\ \Delta(f)((x,y)) := f(xy); \qquad & \epsilon(f) := f(e); \qquad & S(f)(x) := f(x^{-1}). \end{aligned} \tag{3.2.3}$$

Since G is finite, the tensor product $\mathbb{C}(G) \otimes \mathbb{C}(G)$ can be identified with $\mathbb{C}(G \times G)$. Now, $\mathbb{C}(G)$ is a Hopf algebra with respect to the maps Δ, ϵ, and S.

Let us rewrite the structure of this Hopf algebra in terms of a canonical basis. For each $x \in G$, we define $\delta_x \in \mathbb{C}(G)$ by

$$\delta_x(y) := \delta_{x,y} = \begin{cases} 1, & x = y, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(\delta_x)_{x \in G}$ is a basis of $\mathbb{C}(G)$, and $(\delta_x \otimes \delta_y)_{x,y \in G}$ is a basis of $\mathbb{C}(G) \otimes \mathbb{C}(G)$. It is easy to see that for each $x \in G$, we have

$$\Delta(\delta_x) = \sum_{\substack{y,z \in G \\ yz = x}} \delta_y \otimes \delta_z, \quad \epsilon(\delta_x) = \delta_{x,e}, \quad \text{and} \quad S(\delta_x) = \delta_{x^{-1}}.$$

Remark 3.2.3. (i) In Sweedler notation, the axioms for the counit and the antipode of a Hopf algebra (A, Δ) take the form

$$\sum \epsilon(a_{(1)})a_{(2)} = a = \sum a_{(1)}\epsilon(a_{(2)}) \tag{3.2.4}$$

and

$$\sum S(a_{(1)})a_{(2)} = \eta(\epsilon(a)) = \epsilon(a)1_A = \sum a_{(1)}S(a_{(2)}). \tag{3.2.5}$$

A combination of these axioms yields the following useful formula:

$$\begin{aligned} \sum S(a_{(1)})a_{(2)} \otimes a_{(3)} &= \sum \eta(\epsilon(a_{(1)})) \otimes a_{(2)} \\ &= \sum 1_A \otimes \epsilon(a_{(1)})a_{(2)} = 1_A \otimes a. \end{aligned} \tag{3.2.6}$$

(ii) Evidently, every Hopf algebra is a unital and counital bialgebra. Furthermore, every unital and counital bialgebra (A, Δ) admits at most one antipode that turns it into a Hopf algebra. Indeed, if S_1 and S_2 are antipodes for A, then

$$S_1(a) = \sum \epsilon(a_{(1)})S_1(a_{(2)}) = \sum S_2(a_{(1)})a_{(2)}S_1(a_{(3)}) = \sum S_2(a_{(1)})\epsilon(a_{(2)}) = S_2(a)$$

for all $a \in A$.

Remarks 3.1.2(i) and 3.2.3(ii) imply that the counit and the antipode of a Hopf algebra are uniquely determined by the comultiplication. Therefore, we shall refer to a pair (A, Δ) as a Hopf algebra, if A is a Hopf algebra with the comultiplication Δ.

3.3. Convolution

Let (A, Δ) be a coalgebra and B an algebra (over $\mathbb{C}$). Then the space $Hom(A, B)$ of linear maps from A to B carries an important *convolution product*, defined by

$$f * g := m_B \circ (f \otimes g) \circ \Delta_A \quad (f, g \in Hom(A, B));$$

that is,

$$(f * g)(a) := \sum f(a_{(1)})g(a_{(2)}) \quad (f, g \in Hom(A, B),\ a \in A).$$

The convolution is associative because

$$((f * g) * h)(a) = \sum f(a_{(1)})g(a_{(2)})h(a_{(3)}) = (f * (g * h))(a)$$

for all $f, g, h \in Hom(A, B)$ and $a \in A$. Thus $(Hom(A, B), *)$ becomes an algebra, called the *convolution algebra*.

EXAMPLE 3.3.1. (i) Obviously, $A' := Hom(A, \mathbb{C})$ is a convolution algebra. Now

$$f * g = \Delta_A{}'(f \otimes g) \qquad \text{for all } f, g \in A',$$

where $\Delta_A{}' : (A \otimes A)' \to A'$ is defined by

$$\Delta_A{}'(f \otimes g)(a) = (f \otimes g)(\Delta_A(a)) \qquad (f, g \in A',\ a \in A).$$

(ii) Let G be a finite group. For any element $a \in G$, we can define a linear map

$$e_a :\ \mathbb{C}(G) \to \mathbb{C},\ f \mapsto f(a).$$

Then $Hom(\mathbb{C}(G), \mathbb{C}) = \text{span}\{e_a, a \in G\}$, and the convolution on $Hom(\mathbb{C}(G), \mathbb{C})$ is determined by

$$(e_a * e_b)(f) = (e_a \otimes e_b)(\Delta(f)) = \Delta(f)((a, b)) \quad (f \in \mathbb{C}(G),\ a, b \in G).$$

REMARK 3.3.2. (i) In general, the convolution algebra $Hom(A,B)$ need not be unital. However, if (A,Δ) has a counit ϵ_A and B has a unit η_B, then the composition $\eta_B \circ \epsilon_A$ is a unit for the convolution algebra $Hom(A,B)$. Indeed,

$$((\eta_B \circ \epsilon_A) * f)(a) = \sum \epsilon_A(a_{(1)}) 1_B f(a_{(2)}) = \sum f(\epsilon_A(a_{(1)}) a_{(2)}) = f(a)$$

and similarly $(f * (\eta_B \circ \epsilon_A))(a) = f(a)$ for all $f \in Hom(A,B)$ and $a \in A$.

(ii) Let (A,Δ_A) and (D,Δ_D) be coalgebras, and B and C be algebras. Then every algebra homomorphism $F : B \to C$ induces an algebra homomorphism

$$F_* : \; Hom(A,B) \to Hom(A,C), \; f \mapsto F \circ f.$$

Indeed, for all $f, g \in Hom(A,B)$ and $a \in A$,

$$\begin{aligned} ((F_* f) * (F_* g))(a) &= \sum F(f(a_{(1)})) F(g(a_{(2)})) = F(\sum f(a_{(1)}) g(a_{(2)})) \\ &= F((f * g)(a)) = F_*(f * g)(a). \end{aligned}$$

Likewise, if $G : D \to A$ is a coalgebra morphism, then

$$G^* : \; Hom(A,B) \to Hom(D,B), \; g \mapsto g \circ G$$

is an algebra homomorphism.

Using the convolution product on $Hom(A,A)$, we characterize Hopf algebras among bialgebras as follows.

PROPOSITION 3.3.3. *Let $S : A \to A$ be a linear map on a unital and counital bialgebra (A,Δ). Then diagram (3.2.1) commutes if and only if S is inverse to the identity map id_A of A in the convolution algebra $Hom(A,A)$; that is,*

$$S * id_A = \eta \circ \epsilon = id_A * S.$$

Therefore, the bialgebra (A,Δ) can be equipped with the structure of a Hopf algebra if and only if id_A is invertible in the convolution algebra $Hom(A,A)$.

Let us turn to another important convolution type of product. Assume that (A,Δ) is a coalgebra. For each $a \in A$ and $f \in A'$, we define

$$f * a := (id_A \otimes f)(\Delta(a)) = \sum a_{(1)} f(a_{(2)}) \tag{3.3.1}$$

and

$$a * f := (f \otimes id_A)(\Delta(a)) = \sum f(a_{(1)})a_{(2)}. \tag{3.3.2}$$

Then for all $f, g \in A'$ and $a \in A$, we have

$$\begin{aligned} f * (g * a) &= \sum a_{(1)} f(a_{(2)}) g(a_{(3)}) = (f * g) * a; \\ (a * f) * g &= \sum f(a_{(1)}) g(a_{(2)}) a_{(3)} = a * (f * g); \\ (f * a) * g &= \sum g(a_{(1)}) a_{(2)} f(a_{(3)}) = f * (a * g); \\ f(g * a) &= \sum f(a_{(1)}) g(a_{(2)}) = (f * g)(a); \\ f(a * g) &= \sum g(a_{(1)}) f(a_{(2)}) = (g * f)(a). \end{aligned}$$

(Cf. Example 3.3.1(i).)

Based on the products given in (3.3.1) and (3.3.2), we can show the following properties for coalgebras and Hopf algebras.

PROPOSITION 3.3.4.

(i) *If* (A, Δ) *is a coalgebra with a counit* ϵ, *then*

$$\epsilon * a = a = a * \epsilon \qquad \text{for all } a \in A.$$

(ii) *If* (A, Δ) *is a Hopf algebra with a counit* ϵ *and an antipode* S, *then*

$$S * a = \eta(\epsilon(a)) = a * S \qquad \text{for all } a \in A.$$

3.4. Properties of the antipode

The antipode of a Hopf algebra has several fundamental relations that are not obvious from the definition. We will discuss these relations in this section.

First of all, to some extent, the antipode of a Hopf algebra behaves like the inversion of a group. For example, the inversion of a group and the antipode of a Hopf algebra are both antimultiplicative. Moreover, the latter is anticomultiplicative as well.

PROPOSITION 3.4.1. *Let* S *be the antipode of a Hopf algebra* (A, Δ). *Then* $S : (A, \Delta) \to (A, \Delta)^{op,cop}$ *is a unital and counital bialgebra morphism; that is, the following holds:*

(i) $S \circ m = m \circ \Sigma \circ (S \otimes S)$;

(ii) $S \circ \eta = \eta$;

(iii) $\Delta \circ S = (S \otimes S) \circ \Sigma \circ \Delta$;

(iv) $\epsilon \circ S = \epsilon$.

Equivalently, for all $a, b \in A$, we have

(i)' $S(ab) = S(b)S(a)$;

(ii)' $S(1_A) = 1_A$;

(iii)' $\sum S(a)_{(1)} \otimes S(a)_{(2)} = \sum S(a_{(2)}) \otimes S(a_{(1)})$;

(iv)' $\epsilon(S(a)) = \epsilon(a)$.

Proof. (i) For all $a, b \in A$, by Remark 3.2.3, we have

$$\begin{aligned} S(a)S(b) &= \sum S(a_{(1)})S(b_{(1)})\epsilon(b_{(2)})\epsilon(a_{(2)}) \\ &= \sum S(a_{(1)})S(b_{(1)})\epsilon(b_{(2)}a_{(2)}) \\ &= \sum S(a_{(1)})S(b_{(1)})b_{(2)}a_{(2)}S(b_{(3)}a_{(3)}) \\ &= \sum \epsilon(b_{(1)})\epsilon(a_{(1)})S(b_{(2)}a_{(2)}) \\ &= \sum \epsilon(b_{(1)}a_{(1)})S(b_{(2)}a_{(2)}) = S(ba). \end{aligned}$$

(ii) This is true since $S(1_A) = S(1_A)1_A = \eta(\epsilon(1_A)) = 1_A$.

(iii) For all $a \in A$,

$$\begin{aligned} \sum S(a_{(2)}) \otimes S(a_{(1)}) &= \sum (S(a_{(2)}) \otimes S(a_{(1)}))\Delta(\eta(\epsilon(a_{(3)}))) \\ &= \sum (S(a_{(2)}) \otimes S(a_{(1)}))\Delta(a_{(3)}S(a_{(4)})) \\ &= \sum (S(a_{(2)})a_{(3)} \otimes S(a_{(1)})a_{(4)})\Delta(S(a_{(5)})) \\ &= \sum (\eta(\epsilon(a_{(2)})) \otimes S(a_{(1)})a_{(3)})\Delta(S(a_{(4)})) \\ &= \sum (1_A \otimes S(a_{(1)})a_{(2)})\Delta(S(a_{(3)})) = \Delta(S(a)). \end{aligned}$$

(iv) For all $a \in A$,

$$\begin{aligned} \epsilon(S(a)) = \epsilon(S(\sum a_{(1)}\epsilon(a_{(2)}))) &= \sum \epsilon(S(a_{(1)}))\epsilon(a_{(2)}) \\ &= \sum \epsilon(S(a_{(1)})a_{(2)}) = (\epsilon \circ \eta \circ \epsilon)(a) = \epsilon(a). \end{aligned}$$

$\square$

REMARK 3.4.2. Proposition 3.4.1(iii)' is also equivalent to

$$S'(f * g) = S'(g) * S'(f) \qquad \text{for all } f, g \in A',$$

where $S' : A' \to A'$ is the adjoint of S (compare with (i)' in Proposition 3.4.1).

Proposition 3.4.1 implies that for every Hopf algebra (A, Δ) with a antipode S, the map $S^2 : A \to A$ is a Hopf algebra morphism. But unlike the inversion of a group, S need not be involutive; that is, S^2 need not be equal to the identity. In fact, S need not even be bijective. If S is bijective, S^{-1} is the antipode of both $(A, \Delta)^{op}$ and $(A, \Delta)^{cop}$. To show this fact, we need the following lemma.

LEMMA 3.4.3. *Assume that (A, Δ) is a Hopf algebra. Let $T : A \to A$ be a linear map. Then the following statements are equivalent.*

(i) *The bialgebra $(A, \Delta)^{op}$ is a Hopf algebra with an antipode T.*
(ii) $m \circ \Sigma \circ (T \otimes id) \circ \Delta = \eta \circ \epsilon = m \circ \Sigma \circ (id \otimes T) \circ \Delta$.
(iii) $\sum a_{(2)} T(a_{(1)}) = \eta(\epsilon(a)) = \sum T(a_{(2)}) a_{(1)}$ *for all* $a \in A$.
(iv) $m \circ (T \otimes id) \circ \Sigma \circ \Delta = \eta \circ \epsilon = m \circ (id \otimes T) \circ \Sigma \circ \Delta$.
(v) *The bialgebra $(A, \Delta)^{cop}$ is a Hopf algebra with an antipode T.*

Note that for every Hopf algebra (A, Δ), the bialgebra $(A, \Delta)^{op,cop}$ is always a Hopf algebra with the same antipode as (A, Δ). Now, we can give the relation between the antipodes of a Hopf algebra and its cooposite Hopf algebra.

PROPOSITION 3.4.4. *For every Hopf algebra (A, Δ), the following statements are equivalent.*

(i) *The antipode S of (A, Δ) is bijective.*
(ii) *The bialgebra $(A, \Delta)^{op}$ is a Hopf algebra.*
(iii) *The bialgebra $(A, \Delta)^{cop}$ is a Hopf algebra.*

If (i)-(iii) *hold, then S^{-1} is the antipode of $(A, \Delta)^{op}$ and $(A, \Delta)^{cop}$.*

PROOF. (i) $\Rightarrow$ [(ii) and (iii)]. Suppose that S is invertible. Then for all $a \in A$, we have

$$\sum a_{(2)} S^{-1}(a_{(1)}) = \sum S^{-1}(a_{(1)} S(a_{(2)})) = S^{-1}(\eta(\epsilon(a))) = \eta(\epsilon(a))$$

and similarly $\sum S^{-1}(a_{(2)})a_{(1)} = \eta(\epsilon(a))$. Therefore, S^{-1} is the antipode of both $(A, \Delta)^{op}$ and $(A, \Delta)^{cop}$ by Lemma 3.4.3.

[(ii) or (iii)] $\Rightarrow$ (i). Let $(A, \Delta)^{op}$ or $(A, \Delta)^{cop}$ be a Hopf algebra with the antipode T. Then

$$\begin{aligned} S(T(a)) &= \sum \epsilon(a_{(2)})S(T(a_{(1)})) = \sum a_{(3)}T(a_{(2)})S(T(a_{(1)})) \\ &= \sum a_{(2)}T(a_{(1)})_{(2)}S(T(a_{(1)})_{(2)}) = \sum a_{(2)}\epsilon(T(a_{(1)})) \\ &= \sum a_{(2)}\epsilon(a_{(1)}) = a \end{aligned}$$

and similarly $T(S(a)) = a$ for all $a \in A$. Therefore, S is bijective. □

Corollary 3.4.5. *For every commutative or cocommutative Hopf algebra* (A, Δ), *we have* $S^2 = id_A$.

Proof. In both cases, S^{-1} and S are antipodes for $(A, \Delta)^{op}$ and $(A, \Delta)^{cop}$. Since the antipode of a Hopf algebra is unique (cf. Remark 3.2.3(ii)), $S = S^{-1}$. □

Proposition 3.4.6. *Assume that* (A, Δ_A) *and* (B, Δ_B) *are Hopf algebras. Let* $F : A \to B$ *be a unital and counital bialgebra morphism. Then* $F \circ S_A = S_B \circ F$; *that is,* F *is a Hopf algebra morphism.*

Proof. Consider the convolution algebra $Hom(A, B)$. For all $a \in A$, we have

$$\begin{aligned} ((S_B \circ F) * F)(a) &= (S_B \otimes id)(F \otimes F)(\Delta(a)) \\ &= (S_B \otimes id)\Delta(F(a)) = \eta_B(\epsilon_B(F(a))), \end{aligned}$$

which implies that $(S_B \circ F) * F = \eta_B \circ \epsilon_B \circ F$. With the same argument, we can show that $F * (F \circ S_A) = F \circ \eta_A \circ \epsilon_A$. Since the unit of $Hom(A, B)$ is $\eta_B \circ \epsilon_A = \eta_B \circ \epsilon_B \circ F = F \circ \eta_A \circ \epsilon_A$, the map F is invertible with respect to the convolution, and the inverse of F is $S_B \circ F = F \circ S_A$. □

3.5. A characterization of Hopf algebras

By Proposition 3.3.3, a Hopf algebra can be characterized as a bialgebra, for which the identity map is invertible with respect to the convolution product. Now we present another characterization of Hopf algebras.

Let (A, Δ) be a unital bialgebra. Consider the linear maps

$$T_1 : \ A \otimes A \to A \otimes A, \ a \otimes b \mapsto \Delta(a)(1_A \otimes b) = \sum a_{(1)} \otimes a_{(2)} b \tag{3.5.1}$$

and

$$T_2 : \ A \otimes A \to A \otimes A, \ a \otimes b \mapsto (a \otimes 1_A)\Delta(b) = \sum ab_{(1)} \otimes b_{(2)}. \tag{3.5.2}$$

These maps will play a central role in the later chapters.

Lemma 3.5.1. *Let (A, Δ) be a Hopf algebra. Then the maps T_1 and T_2 are bijective.*

Proof. Consider the maps

$$R_1 : A \otimes A \to A \otimes A, \ a \otimes b \mapsto (id \otimes S)(\Delta(a))(1_A \otimes b) = \sum a_{(1)} \otimes S(a_{(2)})b$$

and

$$R_2 : A \otimes A \to A \otimes A, \ a \otimes b \mapsto (a \otimes 1_A)(S \otimes id)(\Delta(b)) = \sum aS(b_{(1)}) \otimes b_{(2)}.$$

For all $a, b \in A$, we have

$$\begin{aligned} R_1(T_1(a \otimes b)) = R_1(\sum a_{(1)} \otimes a_{(2)} b) &= \sum a_{(1)} \otimes S(a_{(2)})a_{(3)} b \\ &= \sum a_{(1)} \otimes \epsilon(a_{(2)})b = a \otimes b \end{aligned}$$

and

$$\begin{aligned} T_1(R_1(a \otimes b)) = T_1(\sum a_{(1)} \otimes S(a_{(2)})b) &= \sum a_{(1)} \otimes a_{(2)} S(a_{(3)})b \\ &= \sum a_{(1)} \otimes \epsilon(a_{(2)})b = a \otimes b. \end{aligned}$$

Therefore, R_1 is inverse to T_1. Similarly, we can show that R_2 is inverse to T_2. □

Remark 3.5.2. For a Hopf algebra (A, Δ), there are some useful relations between the maps T_1 and T_2, the multiplication m, the comultiplication Δ, and the antipode S. From (3.5.1), (3.5.2), and the proof of Lemma 3.5.1, we have

(1) $T_1 = (id \otimes m) \circ (\Delta \otimes id)$;
(2) $T_2 = (m \otimes id) \circ (id \otimes \Delta)$;
(3) $T_1^{-1} = (id \otimes m) \circ (id \otimes S \otimes id) \circ (\Delta \otimes id)$;

(4) $T_2^{-1} = (m \otimes id) \circ (id \otimes S \otimes id) \circ (id \otimes \Delta)$.

Moreover, it is easy to check that

(5) $(\Delta \otimes id) \circ T_1 = (id \otimes T_1) \circ (\Delta \otimes id) : a \otimes b \mapsto \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)} b$;

(6) $T_1 \circ (id \otimes m) = (id \otimes m) \circ (T_1 \otimes id) : a \otimes b \otimes c \mapsto \sum a_{(1)} \otimes a_{(2)} bc$;

(7) $(id \otimes \Delta) \circ T_2 = (T_2 \otimes id) \circ (id \otimes \Delta) : a \otimes b \mapsto \sum ab_{(1)} \otimes b_{(2)} \otimes b_{(3)}$;

(8) $T_2 \circ (m \otimes id) = (m \otimes id) \circ (id \otimes T_2) : a \otimes b \mapsto \sum abc_{(1)} \otimes c_{(2)}$.

As shown below, the converse of Lemma 3.5.1 holds.

Proposition 3.5.3. *Let (A, Δ) be a unital bialgebra. If the maps T_1 and T_2 defined in* (3.5.1) *and* (3.5.2) *are bijective, then (A, Δ) is a Hopf algebra.*

Proof. To show that (A, Δ) is a Hopf algebra, we need to construct a counit and an antipode for (A, Δ). Let us start with the counit. Consider the map

$$E : \; A \to A, \; a \mapsto m(T_1^{-1}(a \otimes 1_A)).$$

For all $a, b \in A$, by equations (1) and (5) in Remark 3.5.2, we have

$$\begin{aligned}(id \otimes E)((a \otimes 1_A)\Delta(b)) &= (a \otimes 1_A)(id \otimes m)(id \otimes T_1^{-1})(\Delta(b) \otimes 1_A) \\ &= (a \otimes 1_A)(id \otimes m)(\Delta \otimes id)(T_1^{-1}(b \otimes 1_A)) \\ &= (a \otimes 1_A)T_1(T_1^{-1}(b \otimes 1_A)) = ab \otimes 1_A.\end{aligned}$$

Since T_2 is surjective, elements of the form $(a \otimes 1_A)\Delta(b)$ span $A \otimes A$. Therefore, the calculation above shows that $E(A) \subseteq \mathbb{C} \cdot 1_A$. Now we can define $\epsilon : A \to \mathbb{C}$ by

$$E(a) = \epsilon(a) \cdot 1_A \quad (a \in A).$$

Let $b \in A$. Then $(id \otimes \epsilon)(\Delta(b)) = b$ by the calculation above. Using equations (2) and (6) in Remark 3.5.2, we find

$$\begin{aligned}(\epsilon \otimes id)(\Delta(b)) &= \sum m(T_1^{-1}(b_{(1)} \otimes 1_A))b_{(2)} \\ &= \sum m((id \otimes m)(T_1^{-1}(b_{(1)} \otimes 1_A) \otimes b_{(2)})) \\ &= \sum m(T_1^{-1}(b_{(1)} \otimes b_{(2)})) = m(b \otimes 1_A) = b.\end{aligned}$$

So, ϵ is the counit on A. Moreover, the previous results and the multiplicativity of Δ imply that

$$\sum \epsilon(a_{(1)})\epsilon(b_{(1)})a_{(2)}b_{(2)}c = abc = \sum \epsilon(a_{(1)}b_{(1)})a_{(2)}b_{(2)}c$$

for all $a, b, c \in A$. Since T_2 is surjective, we can replace $\sum a_{(1)} \otimes b_{(1)} \otimes a_{(2)}b_{(2)}c$ by $a' \otimes b' \otimes 1_A$, where $a', b' \in A$ are arbitrary. Thus we find $\epsilon(a')\epsilon(b') = \epsilon(a'b')$ for all $a', b' \in A$, which implies that (A, Δ) is counital.

Next, we construct the antipode. Consider the map

$$S: \ A \to A, \ a \mapsto (\epsilon \otimes id)(T_1^{-1}(a \otimes 1_A)).$$

Let $a \in A$. From the relation $(id \otimes m) \circ (T_1 \otimes id)^{-1} = T_1^{-1} \circ (id \otimes m)$ (cf. Remark 3.5.2(6)), we deduce

$$\begin{aligned}
\sum S(a_{(1)})a_{(2)} &= \sum (\epsilon \otimes id)(T_1^{-1}(a_{(1)} \otimes 1_A))a_{(2)} \\
&= \sum (\epsilon \otimes id)(T_1^{-1}(a_{(1)} \otimes a_{(2)})) \\
&= (\epsilon \otimes id)(a \otimes 1_A) = \epsilon(a)1_A.
\end{aligned}$$

Since $(\Delta \otimes id) \circ T_1^{-1} = (id \otimes T_1)^{-1} \circ (\Delta \otimes id)$ (cf. Remark 3.5.2(5)) and $(id \otimes \epsilon) \circ \Delta = id$, we have

$$\begin{aligned}
\sum a_{(1)}S(a_{(2)}) &= \sum a_{(1)}(\epsilon \otimes id)(T_1^{-1}(a_{(2)} \otimes 1_A)) \\
&= \sum m((id \otimes \epsilon \otimes id)(id \otimes T_1^{-1})(a_{(1)} \otimes a_{(2)} \otimes 1_A)) \\
&= m((id \otimes \epsilon \otimes id)(\Delta \otimes id)(T_1^{-1}(a \otimes 1_A))) \\
&= m(T_1^{-1}(a \otimes 1_A)) = \epsilon(a)1_A.
\end{aligned}$$

So, S is the antipode on (A, Δ). Therefore, (A, Δ) is a Hopf algebra. $\square$

Combining Lemma 3.5.1 and Proposition 3.5.3, we are ready to present the following important characterization for a unital bialgebra being a Hopf algebra.

THEOREM 3.5.4. *A unital bialgebra is a Hopf algebra if and only if the maps T_1 and T_2 are both bijective.*

3.6. Hopf $*$-algebras

A Hopf $*$-algebra is a Hopf algebra equipped with a conjugate-linear involution that is compatible with the bialgebra structure in a natural way.

DEFINITION 3.6.1. An *involution* on a complex vector space A is a map $* : A \to A,\ a \mapsto a^*$ that is conjugate-linear and involutive in the sense that

$$(a+b)^* = a^* + b^*, \quad (\lambda a)^* = \overline{\lambda} a^*, \quad \text{and} \quad (a^*)^* = a$$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$. A complex vector space with a fixed involution is also called a *$*$-vector space*. A linear map $\phi : A \to B$ of $*$-vector spaces is *$*$-linear* if $\phi(a^*) = \phi(a)^*$ for all $a \in A$.

A *$*$-algebra* is a complex algebra A equipped with an involution such that $(ab)^* = b^* a^*$ for all $a, b \in A$. A *$*$-coalgebra* is a complex coalgebra (A, Δ), where A is equipped with an involution such that

$$\Delta(a^*) = \sum {a_{(1)}}^* \otimes {a_{(2)}}^* \quad \text{for all } a \in A.$$

A *$*$-bialgebra* is a complex bialgebra (A, Δ), where A is a $*$-algebra and (A, Δ) a $*$-coalgebra. A $*$-bialgebra that is a Hopf algebra is called a *Hopf $*$-algebra*.

A *$*$-algebra/$*$-coalgebra/$*$-bialgebra/Hopf $*$-algebra morphism* is a $*$-linear morphism of the underlying algebras/coalgebras/bialgebras/Hopf algebras. A $*$-algebra morphism is also called a *$*$-homomorphism*.

REMARK 3.6.2. (i) We note that for a $*$-algebra A, the involution reverses the multiplication, and can be considered as a homomorphism $A \to A^{op}$; whereas for a $*$-coalgebra (A, Δ), the involution does not reverse the comultiplication, but is a coalgebra morphism $(A, \Delta) \to (A, \Delta)$.

(ii) Given a $*$-coalgebra (A, Δ) and $a \in A$, the expressions $\sum {a^*}_{(1)} \otimes {a^*}_{(2)}$ and $\sum {a_{(1)}}^* \otimes {a_{(2)}}^*$ coincide; hence, we shortly write $\Delta(a^*) = \sum a^*_{(1)} \otimes a^*_{(2)}$.

(iii) Clearly, if A is a unital $*$-algebra, then $1_A^* = 1_A$, or equivalently, the unit map $\eta : \mathbb{C} \to A$ is $*$-linear.

EXAMPLE 3.6.3. Let G be a finite group. Consider the Hopf algebra $\mathbb{C}(G)$ defined in Example 3.2.2.

(i) Obviously, $f \mapsto \bar{f}$ given by $\bar{f}(x) := \overline{f(x)}$ $(x \in G, f \in \mathbb{C}(G))$ is a involution on $\mathbb{C}(G)$. Then $\mathbb{C}(G)$ with this involution is a Hopf $*$-algebra.

(ii) Define $f \mapsto f^*$ by $f^*(x) := \overline{f(x^{-1})}$ $(x \in G, f \in \mathbb{C}(G))$. Then for all $f, g \in \mathbb{C}(G)$, $x \in G$, and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
(f+g)^*(x) &= \overline{(f+g)(x^{-1})} = \overline{f(x^{-1}) + g(x^{-1})} \\
&= \overline{f(x^{-1})} + \overline{g(x^{-1})} = f^*(x) + g^*(x); \\
(\lambda f)^*(x) &= \overline{\lambda f(x^{-1})} = \bar{\lambda}\overline{f(x^{-1})} = \bar{\lambda} f^*(x); \\
(f^*)^*(x) &= \overline{f^*(x^{-1})} = \overline{\overline{f((x^{-1})^{-1})}} = f(x); \\
(fg)^*(x) &= \overline{(fg)(x^{-1})} = \overline{f(x^{-1})g(x^{-1})} \\
&= \overline{f(x^{-1})}\ \overline{g(x^{-1})} = f^*(x)g^*(x) = g^*(x)f^*(x).
\end{aligned}$$

These imply that $\mathbb{C}(G)$ equipped with $* : f \mapsto f^*$ is a $*$-algebra. However, $\mathbb{C}(G)$ is not a Hopf $*$-algebra, since

$$\begin{aligned}
\Delta(f^*)((x,y)) = f^*(xy) = \overline{f(y^{-1}x^{-1})} &= \overline{\Delta(f)((y^{-1}, x^{-1}))} \\
&= \sum \overline{f_{(1)}(y^{-1})}\ \overline{f_{(2)}(x^{-1})} = \sum f^*_{(1)}(y) f^*_{(2)}(x) \\
&= (\sum f^*_{(2)} \otimes f^*_{(1)})((x,y))
\end{aligned}$$

for all $f \in \mathbb{C}(G)$ and $x, y \in G$, and hence we do not have $\Delta(f^*) = \sum f^*_{(1)} \otimes f^*_{(2)}$ in general.

From Remark 3.6.2(iii), we know that for a unital $*$-coalgebra, the unit map is $*$-linear. In fact, for a counital $*$-algebra, the counit is also $*$-linear.

Proposition 3.6.4. *Let (A, Δ) be a counital $*$-algebra with the counit ϵ. Then ϵ is $*$-linear.*

Proof. Consider the map

$$\epsilon^* : \ A \to \mathbb{C}, \ a \mapsto \overline{\epsilon(a^*)}.$$

For all $a \in A$, we have

$$(id \otimes \epsilon^*)(\Delta(a^*)) = \sum a^*_{(1)} \epsilon^*(a^*_{(2)}) = (\sum a_{(1)} \epsilon(a_{(2)}))^* = a^*$$

and similarly $(\epsilon^* \otimes id)(\Delta(a^*)) = a^*$. So, ϵ^* is a counit of (A, Δ) as well. By uniqueness of the counit, $\epsilon^* = \epsilon$. $\square$

PROPOSITION 3.6.5. *The antipode S of a Hopf $*$-algebra (A, Δ_A) is bijective, and satisfies $S \circ * \circ S \circ * = id$; that is, $S(S(a^*)^*) = a$ for all $a \in A$.*

PROOF. Consider the map

$$S^* : \ A \to A, \ a \mapsto S(a^*)^*.$$

For all $a \in A$, we have

$$\sum a^*_{(2)} S^*(a^*_{(1)}) = \sum a^*_{(2)} S(a_{(1)})^* = (\sum S(a_{(1)}) a_{(2)})^* = \eta(\epsilon(a))^* = \eta(\epsilon(a^*))$$

and likewise $\sum S^*(a^*_{(2)}) a^*_{(1)} = \eta(\epsilon(a^*))$. So, S^* is the antipode for the Hopf algebra $(A, \Delta)^{op}$. By Proposition 3.4.4, we get $S^* = S^{-1}$. $\square$

COROLLARY 3.6.6. *The antipode S of a Hopf $*$-algebra is $*$-linear if and only if it is involutive in the sense that $S^2 = id$.*

PROOF. By the previous proposition, the map $S \circ *$ is invertible, and

$$\begin{aligned} S \text{ is } *\text{-linear} &\iff * \circ S = S \circ *, \\ &\iff (S \circ *) \circ (* \circ S) = (S \circ *) \circ (S \circ *) = id, \\ &\iff S^2 = id. \end{aligned}$$

$\square$

CHAPTER 4

The duality of Hopf algebras

In this chapter, we consider the duality of Hopf algebras, which behaves very nicely in the finite-dimensional case. In the infinite-dimensional case, additional concepts and stronger assumptions are needed. A satisfying duality will be discussed in Chapter 7. The main reference of this chapter is [**7**].

4.1. The duality of finite-dimensional Hopf algebras

Consider a vector space V over $\mathbb{C}$. We denote by $V' = Hom(V, \mathbb{C})$ the linear dual of V, and by $\iota_V : V \to V''$ the canonical embedding given by

$$\iota_V(v)(f) = f(v) \qquad (v \in V, f \in V').$$

Furthermore, we consider $V' \otimes V'$ as a subspace of $(V \otimes V)'$ via the embedding given by

$$(f \otimes g)(v \otimes w) = f(v)g(w) \qquad (f, g \in V',\ v, w \in V).$$

If V is finite-dimensional, then this embedding is also surjective. Indeed, suppose that $\dim V = n$, and $\{e_i \in V : i = 1, 2, \cdots, n\}$ is a basis for V. Then

$$\{e_i \otimes e_j \in V \otimes V : i, j = 1, 2, \cdots, n\},$$

$$\{f_i \in V' : i = 1, 2 \cdots, n\},$$

$$\{f_i \otimes f_j \in V' \otimes V' : i, j = 1, 2, \cdots, n\},$$

and

$$\{F_{i,j} \in (V \otimes V)' : i, j = 1, 2, \cdots, n\}$$

are bases for the spaces $V \otimes V$, V', $V' \otimes V'$, and $(V \otimes V)'$, respectively, where

$$f_i(e_j) = \delta_{i,j} \qquad \text{and} \qquad F_{i,j}(e_k \otimes e_l) = \delta_{i,k}\delta_{j,l}.$$

Then the surjectivity of the embedding $V' \otimes V' \to (V \otimes V)'$ follows from

$$(f_i \otimes f_j)(e_k \otimes e_l) = f_i(e_k)f_j(e_l) = \delta_{i,k}\delta_{j,l} = F_{i\otimes j}(e_k \otimes e_l)$$

for $i, j, k, l = 1, 2, \cdots, n$.

Let W be another vector space over $\mathbb{C}$. Then every linear map of vector spaces $F : V \to W$ induces the adjoint map $F' : W' \to V'$ by composition; that is,

$$F'(f) = f \circ F \quad (f \in W').$$

Now, we have the following theorem about the duality of finite-dimensional Hopf algebras.

Theorem 4.1.1.

(i) *Let (A, Δ_A) be a coalgebra. Then the dual space A' is an algebra with respect to the multiplication*

$$m_{A'} : \ A' \otimes A' \hookrightarrow (A \otimes A)' \xrightarrow{(\Delta_A)'} A', \ (m_{A'}(f \otimes g))(a) = (f \otimes g)(\Delta(a)).$$

A' is unital if and only if (A, Δ_A) is counital; in this case, we have $1_{A'} = \epsilon_A$.

(ii) *Let A be a finite-dimensional algebra. Then the dual space A' is a coalgebra with the comultiplication*

$$\Delta_{A'} : \ A' \xrightarrow{(m_A)'} (A \otimes A)' \cong A' \otimes A', \ (\Delta_{A'}(f))(a \otimes b) = f(ab).$$

$(A', \Delta_{A'})$ is counital if and only if A is unital; in this case, we have $\epsilon_{A'} = \iota_A(1_A)$, where $\iota_A : A \to A''$ is the canonical embedding.

(iii) *Let (A, Δ_A) be a finite-dimensional bialgebra. Then A' is a bialgebra under the multiplication $m_{A'}$ and the comultiplication $\Delta_{A'}$ defined above. In addition, if (A, Δ_A) is a Hopf algebra, then so is $(A', \Delta_{A'})$ with the antipode $S_{A'} = (S_A)'$. The natural isomorphism $\iota_A : A \to A''$ is a bialgebra isomorphism and a Hopf algebra isomorphism, respectively.*

(iv) *Let (A, Δ_A) be a Hopf $*$-algebra. Then A' is a $*$-algebra with respect to the involution given by $f^*(a) = \overline{f(S_A(a)^*)}$ $(a \in A, \ f \in A')$. In addition, if (A, Δ_A) is finite-dimensional, then $(A', \Delta_{A'})$ is a Hopf $*$-algebra again, and the natural isomorphism $\iota_A : A \to A''$ is a Hopf $*$-algebra isomorphism.*

PROOF. (i)-(iii) All these statements follow easily from the symmetry of the commutative diagrams that express the axioms for the structure maps involved.

(iv) For $f \in A'$, let $f^\sharp(a) = \overline{f(a^*)}$ $(a \in A)$. Then $f^\sharp \in A'$ and $f^* = {S_A}'(f^\sharp)$. It is easy to see that

$$(\alpha f + \beta g)^\sharp = \bar{\alpha} f^\sharp + \bar{\beta} g^\sharp, \quad (f^\sharp)^\sharp = f, \quad (fg)^\sharp = f^\sharp g^\sharp, \quad \text{and} \quad {S_A}'(f)^\sharp = {S_A}'(f^\sharp)$$

for all $f, g \in A'$ and $\alpha, \beta \in \mathbb{C}$. It follows that

$$(\alpha f + \beta g)^* = {S_A}'((\alpha f + \beta g)^\sharp) = \bar{\alpha} {S_A}'(f^\sharp) + \bar{\beta} {S_A}'(g^\sharp) = \bar{\alpha} f^* + \bar{\beta} g^*;$$
$$(f^*)^* = {S_A}'((f^*)^\sharp) = {S_A}'({S_A}'(f^\sharp)^\sharp) = {S_A}'({S_A}'(f)) = f;$$
$$(fg)^* = {S_A}'((fg)^\sharp) = {S_A}'(f^\sharp g^\sharp) = {S_A}'(g^\sharp){S_A}'(f^\sharp) = g^* f^*.$$

Also, we have

$$\begin{aligned}\Delta_{A'}(f^*) &= m_A'({S_A}'(f^\sharp)) = {S_A}'(f^\sharp) \circ m_A = f^\sharp \circ S_A \circ m_A \\ &= f^\sharp \circ m_A \circ (S_A \otimes S_A) \circ \Sigma = ({S_A}' \otimes {S_A}')(f^\sharp \circ m_A \circ \Sigma) \\ &= ({S_A}' \otimes {S_A}')((f \circ m_A)^\sharp) = ({S_A}' \otimes {S_A}')(m_A'(f)^\sharp) = \Delta_{A'}(f)^*,\end{aligned}$$

where for $g \otimes h \in A' \otimes A'$, we denote

$$(g \otimes h)^\sharp = g^\sharp \otimes h^\sharp \quad \text{and} \quad (g \otimes h)^* = g^* \otimes h^*.$$

Thus the involution defined in (iv) turns A' into a $*$-algebra.

Now denote by $\chi : \mathbb{C} \to \mathbb{C}$ the complex conjugation, and assume that A has finite dimension. Let $f \in A'$. Then

$$\begin{aligned}\Delta_{A'}(f^*) &= f^* \circ m_A = \chi \circ f \circ * \circ S_A \circ m_A \\ &= \chi \circ f \circ m_A \circ ((* \circ S_A) \otimes (* \circ S_A)) \\ &= \chi \circ \Delta_{A'}(f) \circ ((* \circ S_A) \otimes (* \circ S_A)) \\ &= \chi \circ (\sum f_{(1)} \otimes f_{(2)}) \circ ((* \circ S_A) \otimes (* \circ S_A)) \\ &= \sum (\chi \circ f_{(1)} \circ * \circ S_A) \otimes (\chi \circ f_{(2)} \circ * \circ S_A) \\ &= \sum f_{(1)}^* \otimes f_{(2)}^*.\end{aligned}$$

Thus $(A', \Delta_{A'})$ is a $*$-coalgebra. By (iii), $(A', \Delta_{A'})$ is also a Hopf algebra, and hence a Hopf $*$-algebra.

Finally, let us show that the natural Hopf algebra isomorphism $\iota_A : A \to A''$ is $*$-linear. For all $a \in A$ and $f \in A'$, we have

$$\begin{aligned}\iota_A(a)^*(f) &= \overline{\iota_A(a)(S_{A'}(f)^*)} = \overline{S_{A'}(f)^*(a)} = S_{A'}(f)(S_A(a)^*)\\ &= (f \circ S_A \circ * \circ S_A)(a) = f(a^*) = \iota_A(a^*)(f).\end{aligned}$$

Thus $\iota_A(a)^* = \iota_A(a^*)$, and hence ι_A is a Hopf $*$-algebra isomorphism. □

EXAMPLE 4.1.2. Let G be a finite group. Consider the associated Hopf algebra $\mathbb{C}(G)$ defined in Example 3.2.2. Denote by $(\delta_x)_{x\in G}$ the canonical basis of $\mathbb{C}(G)$, and by $(\xi_x)_{x\in G}$ the dual basis of $\mathbb{C}(G)'$, determined by

$$\xi_x(\delta_y) = \delta_{x,y} \ (x, y \in G).$$

Let us compute the structure maps of the Hopf algebra $\mathbb{C}(G)'$.

- The product of the elements $\xi_x, \xi_y \in \mathbb{C}(G)'$ is determined by

$$(\xi_x\xi_y)(\delta_z) = (\xi_x \otimes \xi_y)(\Delta(\delta_z)) = \sum_{\substack{a,b\in G\\ ab=z}} \xi_x(\delta_a)\xi_y(\delta_b) = \delta_{xy,z}.$$

  So, $\xi_x\xi_y = \xi_{xy}$. If e denotes the unit of G, then ξ_e is the unit of $\mathbb{C}(G)'$.
- The coproduct of $\xi_x \in \mathbb{C}(G)'$ is determined by

$$(\Delta_{\mathbb{C}(G)'}(\xi_x))(\delta_y \otimes \delta_z) = \xi_x(\delta_y\delta_z) = \delta_{x,y}\delta_{x,z}.$$

  So, $\Delta_{\mathbb{C}(G)'}(\xi_x) = \xi_x \otimes \xi_x$.
- The counit of $\mathbb{C}(G)'$ acts by evaluation at the unit $1_{\mathbb{C}(G)} = \sum_y \delta_y$. So,

$$\epsilon_{\mathbb{C}(G)'}(\xi_x) = \xi_x(1_{\mathbb{C}(G)}) = \sum_y \xi_x(\delta_y) = 1.$$

- The antipode $S_{\mathbb{C}(G)'}$, applied to $\xi_x \in \mathbb{C}(G)'$, acts by

$$(S_{\mathbb{C}(G)'}(\xi_x))(\delta_y) = \xi_x(S_{\mathbb{C}(G)}(\delta_y)) = \xi_x(\delta_{y^{-1}}) = \delta_{x,y^{-1}},$$

  and hence $S_{\mathbb{C}(G)'}(\xi_x) = \xi_{x^{-1}}$.

4.2. Dual pairings of Hopf algebras

Let (A, Δ_A) be a finite-dimensional bialgebra. In Theorem 4.1.1, we saw that the dual space A' of A is a bialgebra again. The relations between the structure maps of (A, Δ_A) and $(A', \Delta_{A'})$ can be expressed in terms of the natural pairings

$$(\cdot|\cdot): \ A \times A' \to \mathbb{C}, \ (a|f) := f(a)$$

and

$$(\cdot|\cdot): \ (A \otimes A) \times (A' \otimes A') \to \mathbb{C}, \ (a_1 \otimes a_2 | f_1 \otimes f_2) = f_1(a_1) f_2(a_2)$$

as follows: for all $a, b \in A$ and $f, g \in A'$,

$$(a|fg) = (a|m_{A'}(f \otimes g)) = (\Delta_A(a)|f \otimes g) = \sum (a_{(1)}|f)(a_{(2)}|g)$$

and

$$(ab|f) = (m_A(a \otimes b)|f) = (a \otimes b|\Delta_{A'}(f)) = \sum (a|f_{(1)})(b|f_{(2)});$$

that is, $fg = f * g$ and $\iota_A(ab) = \iota_A(a) * \iota_A(b)$. Therefore, the multiplication (comultiplication) on A and the comultiplication (multiplication) on A' are dual of each other, and the multiplications m_A and $m_{A'}$ are both convolutions.

Furthermore, if (A, Δ_A) is a Hopf ($*$-)algebra, then so is $(A', \Delta_{A'})$. In this case, the unit, counit, antipode (and involution) of (A, Δ) and $(A', \Delta_{A'})$ are related by similar equations. These relations motivate the following definition.

Definition 4.2.1. A *dual pairing* between two Hopf algebras (A, Δ_A) and (B, Δ_B) is a bilinear map $(\cdot|\cdot) : A \times B \to \mathbb{C}, \ (a, b) \mapsto (a|b)$ that satisfies

$$(a|b_1 b_2) = \sum (a_{(1)}|b_1)(a_{(2)}|b_2), \qquad (a_1 a_2|b) = \sum (a_1|b_{(1)})(a_2|b_{(2)}),$$

$$(a|1_B) = \epsilon_A(a), \qquad (1_A|b) = \epsilon_B(b), \qquad \text{and} \qquad (S_A(a)|b) = (a|S_B(b))$$

for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$. In the case where (A, Δ_A) and (B, Δ_B) are both Hopf $*$-algebras, the map $(\cdot|\cdot)$ is also required to satisfy

$$(a|b^*) = \overline{(S_A(a)^*|b)}, \qquad \text{and} \qquad (a^*|b) = \overline{(a|S_B(b)^*)}.$$

The dual pairing is *perfect* if for each $a_0 \in A \backslash \{0\}$, $(a_0|A) \neq \{0\}$ and $(A|a_0) \neq \{0\}$.

REMARK 4.2.2. (i) Let $(\cdot|\cdot) : A \times B \to \mathbb{C}$ be a dual pairing of Hopf algebras. Then each $a \in A$ defines a linear map $(a|\cdot) : B \to \mathbb{C},\ b \mapsto (a|b)$, and the map $A \to B'$ given by $a \mapsto (a|\cdot)$ is a unital algebra homomorphism. Similarly, we obtain a unital homomorphism $B \to A',\ b \mapsto (\cdot|b)$. The pairing is perfect if and only if these two homomorphisms are injective.

(ii) For every finite-dimensional Hopf algebra (A, Δ_A), the canonical pairing between A and A' is a perfect dual pairing of Hopf algebras. If (B, Δ_B) is an Hopf algebra and $(\cdot|\cdot) : A \times B \to \mathbb{C}$ is a perfect dual pairing, then B has finite dimension. Moreover, the homomorphisms $A \to B',\ a \mapsto (a|\cdot)$ and $B \to A',\ b \mapsto (\cdot|b)$ are Hopf algebra isomorphisms.

(iii) Let (A, Δ_A) and (B, Δ_B) be Hopf $*$-algebras. Let $(\cdot|\cdot) : A \times B \to \mathbb{C}$ be a dual pairing of Hopf $*$-algebras. The last two conditions in Definition 4.2.1 for Hopf $*$-algebra case are actually equivalent. Indeed, if $(a|b^*) = \overline{(S_A(a)^*|b)}$ for all $a \in A$ and $b \in B$, then

$$(a^*|b) = (S_A(S_A(a)^*)|b) = (S_A(a)^*|S_B(b)) = \overline{(a|S_B(b)^*)}.$$

The reverse implication follows similarly.

4.3. The restricted dual of a Hopf algebra

For every unital algebra A, there exists a largest subspace A^0 of A', for which the map $\Delta_{A'} = m_A{}' : A' \to (A \otimes A)'$ defines a comultiplication on A^0. Elements of this subspace can be characterized as follows.

LEMMA 4.3.1. *Let A be a unital algebra and let $f \in A'$. Then the following statements are equivalent.*

(i) $\Delta_{A'}(f) \in A' \otimes A'$*;*

(ii) $\ker f$ *contains a left ideal of A that has finite codimension;*

(iii) $\ker f$ *contains a right ideal of A that has finite codimension;*

(iv) $\ker f$ *contains an ideal of A that has finite codimension.*

PROOF. (i) $\Rightarrow$ (ii). Write $\Delta_{A'}(f) = \sum_i g_i \otimes h_i$ with $g_i, h_i \in A'$, where the g_i are linearly independent and the h_i are non-zero. Then as shown below, $J := \cap_i \ker h_i$ is a left ideal of A with finite codimension and contained in $\ker f$.

- $AJ \subseteq J$: if $b \in A$ and $c \in J$, then for all $a \in A$, we have

$$0 = \sum_i g_i(ab)h_i(c) = f(abc) = \sum_i g_i(a)h_i(bc),$$

  and the linear independence of the g_i implies that $h_i(bc) = 0$ for all i.
- J has finite codimension in A, because $\ker h_i \subseteq A$ has codimension 1 for each i and $\cap_i \ker h_i$ is a finite intersection.
- $f(J) = 0$, since $f(a) = f(1_A \cdot a) = \sum_i g_i(1_A)h_i(a) = 0$ for each $a \in J$.

(i) $\Rightarrow$ (iii). It is similar to the proof of the implication (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iv). Let $J \subseteq A$ be a left ideal of finite codimension that is contained in $\ker f$. Then $Hom(A/J, A/J)$ is a finite-dimensional algebra with the composition multiplication, and the map

$$\pi : A \to Hom(A/J, A/J), \ \pi(a)(b + J) := ab + J$$

is an algebra homomorphism. Let $I = \ker \pi$. Then I is an ideal of A with finite codimension in A. Finally, the relation $\pi(I) = 0$ implies that $I \subseteq J \subseteq \ker f$.

(iii) $\Rightarrow$ (iv). It is similar to the proof of the implication (ii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (i). Let $I \subseteq A$ be an ideal of finite codimension that is contained in $\ker f$. Denote by $\pi : A \to A/I$ the quotient map and by $\pi' : (A/I)' \to A'$ its adjoint. We show that $\Delta_{A'}(f)$ belongs to the space $(\pi' \otimes \pi')((A/I)' \otimes (A/I)') \subseteq A' \otimes A'$. Since $\Delta_{A'}(f)$ vanishes on $A \otimes I + I \otimes A$, it defines an element

$$g \in ((A \otimes A)/(A \otimes I + I \otimes A))' \cong ((A/I) \otimes (A/I))'.$$

Since A/I has finite dimension, $((A/I) \otimes (A/I))' \cong (A/I)' \otimes (A/I)'$, and g can be considered as an element of $(A/I)' \otimes (A/I)'$. It is easy to see that $\Delta_{A'}(f) = (\pi' \otimes \pi')(g)$. $\square$

Definition 4.3.2. The *restricted dual* of a unital algebra A is the subspace A^0 of A' consisting of all $f \in A'$ satisfying the equivalent conditions (i)-(iv) in Lemma 4.3.1.

Clearly, if A is finite-dimensional, then $A^0 = A'$. In general, we have the following proposition.

PROPOSITION 4.3.3.

(i) *Let A be a unital algebra and $\Delta_{A^0} = \Delta_{A'}|_{A^0}$. Then $\Delta_{A^0}(A^0) \subseteq A^0 \otimes A^0$, and (A^0, Δ_{A^0}) is a counital coalgebra, where the counit ϵ_{A^0} is given by $f \mapsto f(1_A)$.*

(ii) *Let (A, Δ_A) be a unital bialgebra. Then A^0 is a subalgebra of A', and (A^0, Δ_{A^0}) is a counital bialgebra. If (A, Δ_A) is also counital, then (A^0, Δ_{A^0}) is a unital and counital bialgebra with $1_{A^0} = \epsilon_A$. Furthermore, if (A, Δ_A) is a Hopf algebra, then so is (A^0, Δ_{A^0}) with $S_{A^0} = (S_A)'|_{A^0}$.*

(iii) *For every Hopf $*$-algebra (A, Δ), the formula $f^*(a) := \overline{f(S_A(a)^*)}$ defines an involution on A^0 that turns (A^0, Δ_{A^0}) into a Hopf $*$-algebra.*

PROOF. (i) First, we show that $\Delta_{A'}(A^0) \subseteq A^0 \otimes A^0$. Let $f \in A^0$ and write $\Delta_{A'}(f) = \sum_{i=1}^n g_i \otimes h_i$ with $g_i, h_i \in A'$, where the h_i are linearly independent. By the assumption on h_i, for every j, we can choose an element $a_j \in A$ such that $h_i(a_j) = \delta_{i,j}$ for all i, j. Then for all j and $a, b \in A$, we have

$$g_j(ab) = \sum_i^n g_i(ab)h_i(a_j) = f(aba_j) = \sum_i^n g_i(a)h_i(ba_j),$$

which implies that $\Delta_{A'}(g_j) = \sum_i^n g_i \otimes h_i(\cdot a_j) \in A' \otimes A'$. So, each $g_j \in A^0$. Therefore, $\Delta_{A'}(f) \in A^0 \otimes A'$. Similar argument shows that $\Delta_{A'}(f) \in A' \otimes A^0$. We conclude that $\Delta_{A'}(f) \in A^0 \otimes A^0$ for all $f \in A^0$, so that $\Delta_{A^0} : A^0 \to A^0 \otimes A^0$. The map Δ_{A^0} is coassociative because the multiplication m_A is associative. The assertion concerning the counit is evident from definition.

(ii) Suppose that (A, Δ_A) is a unital bialgebra. Let $f, g \in A^0$. To show that A^0 is a subalgebra of A', it suffices to show that $fg \in A^0$. Note that $\Delta_{A'}(f)$ and $\Delta_{A'}(g)$ are in $A' \otimes A'$. Then for all $a, b \in A$, we have

$$\begin{aligned}\Delta_{A'}(fg)(a \otimes b) = (fg)(ab) = (f \otimes g)(\Delta(ab)) &= \sum f(a_{(1)}b_{(1)})g(a_{(2)}b_{(2)}) \\ &= \sum \Delta_{A'}(f)(a_{(1)} \otimes b_{(1)})\Delta_{A'}(g)(a_{(2)} \otimes b_{(2)}) \\ &= \sum(\Delta_{A'}(f) \otimes \Delta_{A'}(g))(a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)}) \\ &= (\Delta_{A'}(f)\Delta_{A'}(g))(a \otimes b).\end{aligned}$$

This implies that

$$\Delta_{A'}(fg) = \Delta_{A'}(f)\Delta_{A'}(g) \in (A' \otimes A') \cdot (A' \otimes A') \subseteq A' \otimes A'.$$

Then $fg \in A^0$. Hence, $A^0 \subseteq A'$ is a subalgebra, and $\Delta_{A^0} : A^0 \to A^0 \otimes A^0$ is an algebra homomorphism. From (i), we know that (A^0, Δ_{A^0}) is counital and

$$(\epsilon_{A^0} \circ m_{A^0})(f \otimes g) = \epsilon_{A^0}(fg) = (f \otimes g)(1_A \otimes 1_A) = f(1_A)g(1_A) = \epsilon_{A^0 \otimes A^0}(f \otimes g).$$

Therefore, (A^0, Δ_{A^0}) is a counital bialgebra.

Next, suppose that A is a unital and counital bialgebra. It is easy to see that

$$\epsilon_A = 1_{A'} \quad \text{and} \quad (\Delta_{A'}(\epsilon_A))(a \otimes b) = \epsilon_A(ab) = \epsilon_A(a)\epsilon(b) \quad (a, b \in A).$$

Hence, $\epsilon_A \in A^0$ and $\epsilon_A = 1_{A^0}$. We also have

$$(\Delta_{A^0} \circ \eta_{A^0})(\lambda) = \lambda \Delta_{A^0}(\epsilon_A) = \lambda(\epsilon_A \otimes \epsilon_A) = \lambda 1_{A^0 \otimes A^0} = \eta_{A^0 \otimes A^0}(\lambda)$$

for all $\lambda \in \mathbb{C}$. Therefore, (A^0, Δ_{A^0}) is a unital and counital bialgebra.

Finally, suppose that (A, Δ_A) is a Hopf algebra. Let $S_{A^0} = (S_A)'|_{A^0} : A^0 \to A'$ be the restriction of the adjoint of S_A to A^0. Let $f \in A^0$ and $a, b \in A$. Then we have

$$(\Delta_{A'}(f \circ S_A))(a \otimes b) = f(S_A(b)S_A(a)) = (\Delta_{A'}(f))(S_A(b) \otimes S_A(a)),$$

and hence

$$\Delta_{A'}(S_{A^0}(f))\Delta_{A'}(f \circ S_A) = \sum (f_{(2)} \circ S_A) \otimes (f_{(1)} \otimes S_A) \in A' \otimes A',$$

which implies that $S_{A^0}(A^0) \subseteq A^0$. Moreover, we have

$$\begin{aligned}\sum (S_{A^0}(f_{(1)})f_{(2)})(a) &= \sum S_{A^0}(f_{(1)})(a_{(1)})f_{(2)}(a_{(2)}) = \sum f_{(1)}(S_A(a_{(1)}))f_{(2)}(a_{(2)}) \\ &= f(\sum S_A(a_{(1)})a_{(2)}) = f(1_A)\epsilon(a) = \eta_{A^0}(\epsilon_{A^0}(f))(a)\end{aligned}$$

and similarly

$$\sum (f_{(1)}S_{A^0}(f_{(2)}))(a) = \eta_{A^0}(\epsilon_{A^0}(f))(a).$$

Therefore, (A^0, Δ_{A^0}) is a Hopf algebra with the antipode S_{A^0}.

(iii) It suffices to show that for every $f \in A^0$, the functional $f^* \in A'$ given by $a \mapsto \overline{f(S_A(a)^*)}$ belongs to A^0. Let $f \in A^0$ and $f^\sharp \in A'$ be the same as defined in the proof of Theorem 4.1.1(iv). Then

$$\Delta_{A'}(f^\sharp) = f^\sharp \circ m_A = \Delta_{A'}(f)^\sharp \circ \Sigma.$$

So, $f^\sharp \in A^0$ and $f^* = S_A{}'(f^\sharp) = S_{A^0}(f^\sharp) \in A^0$ by (ii). $\square$

PROPOSITION 4.3.4.

(i) *For every Hopf algebra (A, Δ_A), the map $A \times A^0 \to \mathbb{C}$, $(a, f) \mapsto f(a)$ is a dual pairing of Hopf algebras.*

(ii) *If $(\cdot|\cdot) : A \times B \to \mathbb{C}$ is a (perfect) dual pairing of Hopf algebras, then the induced maps $A \to B'$, $a \mapsto (a|\cdot)$ and $B \to A'$, $b \mapsto (\cdot|b)$ induce (injective) Hopf algebra morphisms $A \to B^0$ and $B \to A^0$, respectively.*

REMARK 4.3.5. As seen in Lemma 4.3.1, for every unital algebra A, the space A^0 is the largest subspace of A', on which $\Delta_{A'}$ defines a comultiplication.

CHAPTER 5

Multiplier Hopf algebras

Recall that a Hopf algebra is a pair (A, Δ), where A is a unital algebra with a homomorphism $\Delta : A \to A \otimes A$ and other structures. If we drop the assumption that A is unital and allow Δ to have values in the so called multiplier algebra of $A \otimes A$, we will get a multiplier Hopf algebra, a natural extension of the notion of a Hopf algebra. The main references of this chapter are [**7**] and [**8**].

5.1. Multiplier algebras

The concept of a multiplier Hopf algebra is based on the notion of a multiplier of an algebra.

Definition 5.1.1. Let A be an algebra. A *left/right multiplier* of A is a linear map $T_l/T_r : A \to A$ that satisfies $T_l(ab) = T_l(a)b/T_r(ab) = aT_r(b)$ for all $a, b \in A$. A *multiplier* of A is a pair (T_l, T_r) consisting of a left multiplier T_l and a right multiplier T_r satisfying $aT_l(b) = T_r(a)b$ for all $a, b \in A$.

Remark 5.1.2. (i) We denote the sets of all left multipliers, right multipliers, and multipliers of A as $L(A)$, $R(A)$, and $M(A)$, respectively. They are all algebras with respect to the multiplications

$$T_lS_l = T_l \circ S_l, \quad T_rS_r = S_r \circ T_r, \quad \text{and} \quad (T_l, T_r)(S_l, S_r) = (T_l \circ S_l, S_r \circ T_r),$$

respectively. $M(A)$ is called the *multiplier algebra* of A.

(ii) The multiplier algebra $M(A)$ always has a unit. We denote this by 1. It is easy to see that $1 = (id_A, id_A)$.

(iii) For every commutative algebra A, a map $T : A \to A$ is a left multiplier if and only if it is a right multiplier.

When working with multipliers, it is convenient to restrict to algebras and homomorphisms that are non-degenerate in the following sense.

DEFINITION 5.1.3. An algebra A is *non-degenerate* if

(a) for every $a \in A\backslash\{0\}$, we have $Aa \neq \{0\}$ and $aA \neq \{0\}$, and
(b) the linear span of AA is equal to A.

Let A and B be non-degenerate algebras. A homomorphism $\phi : A \to M(B)$ is *non-degenerate* if the linear span of $\phi(A)B$ and the linear span of $B\phi(A)$ both are equal to B.

We shall frequently use the following properties of multipliers.

PROPOSITION 5.1.4. *Let A and B be non-degenerate algebras.*

(i) *For every $a \in A$, the maps $l_a : b \mapsto ab$ and $r_a : b \mapsto ba$ define a multiplier (l_a, r_a) of A, and the injective algebra homomorphism*

$$\pi : \; A \to M(A), \; a \mapsto (l_a, r_a)$$

embeds A as an ideal in $M(A)$.

From now on, we identify A with the ideal $\pi(A)$ in $M(A)$.

(ii) *For all $a \in A$, $aT = T_r(a)$, $Ta = T_l(a)$, and $T = (T_l, T_r) \in M(A)$.*
(iii) *If A is unital, then $T1_A = T = 1_AT \in A$ for each $T \in M(A)$, and $M(A) = A$.*
(iv) *Let A be a $*$-algebra. For each $T \in M(A)$, the formulas $T^*a := (a^*T)^*$ and $aT^* := (Ta^*)^*$ $(a \in A)$ define a multiplier $T^* \in M(A)$. The involution $* : T \mapsto T^*$ turns $M(A)$ into a $*$-algebra.*
(v) *The tensor product $A \otimes B$ is non-degenerate, and there exists a natural algebraic embedding $M(A) \otimes M(B) \hookrightarrow M(A \otimes B)$.*
(vi) *Every non-degenerate homomorphism $\phi : A \to M(B)$ extends uniquely to a homomorphism $\tilde{\phi} : M(A) \to M(B)$, which satisfies that $\tilde{\phi}(1_{M(A)}) = 1_{M(B)}$. If A and B are $*$-algebras and ϕ is a $*$-homomorphism, then the extension $\tilde{\phi}$ is a $*$-homomorphism again.*

PROOF. (i) Let $a \in A$. For all $x, y \in A$, we have

$$l_a(xy) = axy = l_a(x)y, \quad r_a(xy) = xya = xr_a(y), \quad \text{and} \quad xl_a(y) = xay = r_a(x)y.$$

Therefore, (l_a, r_a) is a multiplier. Clearly, $\pi : A \to M(A)$ is an injective algebra homomorphism.

Let $T = (T_l, T_r) \in M(A)$. Then for all $x \in A$, we have

$$(l_a \circ T_l)(x) = aT_l(x) = T_r(a)x = l_{T_r(a)}(x)$$

and

$$(T_r \circ r_a)(x) = T_r(xa) = xT_r(a) = r_{T_r(a)}(x).$$

Hence, $(l_a, r_a)(T_l, T_r) = (l_a \circ T_l, T_r \circ r_a) = (l_{T_r(a)}, r_{T_r(a)})$. Similarly, we have $(T_l, T_r)(l_a, r_a) = (l_{T_l(a)}, r_{T_l(a)})$. Therefore, π embeds A as a two-sided ideal in $M(A)$.

(ii) This follows from the proof of (i).

(iii) Suppose that A is unital and $T \in M(A)$. Then $\pi(1_A) = (id_A, id_A)$, and hence $T1_A = T = 1_AT$. By (ii), $T = 1_AT = T_r(1_A) \in A$. Therefore, $M(A) = A$.

(iv) Let $T \in M(A)$. Define

$$U: \ A \to A, \ a \mapsto T_r(a^*)^* \qquad \text{and} \qquad V: \ A \to A, \ a \mapsto T_l(a^*)^*.$$

Then $T^* = (U, V) \in M(A)$ satisfying

$$T^*a = (a^*T)^* \qquad \text{and} \qquad aT^* = (Ta^*)^* \qquad (a \in A).$$

For all $T, S \in M(A)$, $a \in A$, and $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned} T^{**}a &= (a^*T^*)^* = (Ta)^{**} = Ta; \\ (\alpha T + \beta S)^*a &= (a^*(\alpha T + \beta S))^* = (a^*(\alpha T))^* + (a^*(\beta S))^* \\ &= (\alpha(a^*T))^* + (\beta(a^*S))^* = \bar{\alpha}(a^*T)^* + \bar{\beta}(a^*S)^* \\ &= \bar{\alpha}T^*a + \bar{\beta}S^*a = (\bar{\alpha}T^* + \bar{\beta}S^*)a; \\ (TS)^*a &= (a^*TS)^* = ((T^*a)^*S)^* = S^*T^*a. \end{aligned}$$

Similarly, we have

$$aT^{**} = aT, \quad a(\alpha T + \beta S)^* = a(\overline{\alpha}T^* + \overline{\beta}S^*), \quad \text{and} \quad a(TS)^* = aS^*T^*.$$

Therefore, the involution $* : T \mapsto T^*$ turns $M(A)$ into a $*$-algebra.

(v) Consider an element $x \in A \otimes B$. Write $x = \sum_i a_i \otimes b_i$, where the b_i are linearly independent. Suppose that for all $c \in A$ and $d \in B$, we have $x(c \otimes d) = 0$. Then

$$(f \otimes id)(x(c \otimes d)) = \sum_i f(a_i c) b_i d = 0 \qquad (f \in A').$$

Thus we obtain that $\sum_i f(a_i c) b_i = 0$ $(f \in A',\ c \in A)$ using the fact that B is non-degenerate. Since the b_i are linearly independent, for each i, we must have $f(a_i c) = 0$ for all $f \in A'$ and $c \in A$. It follows that each $a_i = 0$, since A is non-degenerate. So, $x = 0$. A similar argument shows that $x = 0$ if $(c \otimes d)x = 0$ for all $c \in A$ and $d \in B$.

It is clear that the linear span of $(A \otimes B)(A \otimes B)$ is equal to $A \otimes B$. Finally, one has a natural algebraic embedding

$$M(A) \otimes M(B) \hookrightarrow M(A \otimes B),\ T \otimes S \mapsto (T_l \otimes S_l, T_r \otimes S_r).$$

(vi) Let $\phi : A \to M(B)$ be a non-degenerate homomorphism. Then each $x \in B$ can be written as $x = \sum_i \phi(a_i) b_i = \sum_j b_j \phi(a_j)$ with $a_i, a_j \in A$ and $b_i, b_j \in B$. For $T \in M(A)$, let

$$\widetilde{\phi}(T)x = \sum_i \phi(Ta_i) b_i \quad \text{and} \quad x\widetilde{\phi}(T) = \sum_j b_j \phi(a_j T).$$

Note that for all $a, b \in A$, we have

$$(b\phi(a))\tilde{\phi}(T)x = b\phi(a) \sum_i \phi(Ta_i) b_i = b \sum_i \phi(aT)\phi(a_i) b_i = b\phi(aT)x.$$

Since $\phi : A \to \phi(B)$ is a non-degenerate homomorphism, the above equation implies that $\widetilde{\phi}(T)x$ is a well-defined element of B. Similarly, $x\widetilde{\phi}(T) \in B$ is also well-defined.

It is clear that $\widetilde{\phi}(T) \in M(B)$ for all $T \in M(A)$, and $\widetilde{\phi}(T) = \phi(T)$ if $T \in A$; that is, $\widetilde{\phi} : M(A) \to M(B)$ is an extension of ϕ. Also, $\tilde{\phi}(1_{M(A)})x = \sum_i \phi(a_i) b_i = x$ holds for all $x \in B$, which shows that $\tilde{\phi}(1_{M(A)}) = 1_{M(B)}$. It is easy to see from the definition that $\widetilde{\phi} : M(A) \to M(B)$ is a homomorphism, and it is a $*$-homomorphism if A and B are both $*$-algebras and ϕ is a $*$-homomorphism.

We show now the uniqueness of the extension. Let $\psi : M(A) \to M(B)$ be any homomorphic extension of ϕ. Let $T \in M(A)$ and $x = \sum_i \phi(a_i)b_i \in B$. Then

$$\psi(T)x = \sum_i \psi(T)\psi(a_i)b_i = \sum_i \psi(Ta_i)b_i = \sum_i \phi(Ta_i)b_i = \widetilde{\phi}(T)x$$

and similarly $x\psi(T) = x\widetilde{\phi}(T)$. Therefore, $\psi = \widetilde{\phi}$. □

Remark 5.1.5. (i) Suppose that A is a non-degenerate algebra.

(a) If a linear map T on A is both a left multiplier and a right multiplier of A, then $(T, T) \in M(A)$, since

$$aT(b) = T(ab) = T(a)b \qquad \text{for all } a, b \in A.$$

In this case (e.g., when A is commutative), we will simply use T to denote this multiplier (T, T) of A.

(b) In fact, the above equality implies that T is both a left integral and a right multiplier of A. More generally, if two linear maps T_1 and T_2 on A satisfy

$$aT_1(b) = T_2(a)b \qquad \text{for all } a, b \in A,$$

then the pair (T_1, T_2) is already a multiplier. Indeed, for all $a, b, c \in A$, we have

$$T_2(ab)c = abT_1(c) = aT_2(b)c.$$

which implies that $T_2(ab) = aT_2(b)$. Similarly, $T_1(ab) = T_1(a)b$.

(ii) Due to Proposition 5.1.4(ii), if $T = (T_l, T_r) \in M(A)$ and $a \in A$, we will always write $Ta = T_l(a)$ and $aT = T_r(a)$. Therefore, to define a multiplier T of A, we only have to define the linear maps $a \mapsto Ta$ and $a \mapsto aT$ on A, and show that $(aT)b = a(Tb)$ is satisfied for all $a, b \in A$. In particular, $T \in M(A)$ is the unit in $M(A)$ if and only if $Ta = aT = a$ for all $a \in A$. This follows from Remark 5.1.2(ii) that $1_{M(A)} = (id_A, id_A)$.

(iii) Suppose that A is a non-degenerate $*$-algebra. Then Proposition 5.1.4(iv) shows that for every $T = (T_l, T_r) \in M(A)$ and $a \in A$, we have

$$T^*a = (a^*T)^* = T_r(a^*)^* \qquad \text{and} \qquad aT^* = (Ta^*)^* = T_l(a^*)^*.$$

Therefore, we obtain that $T^* = (* \circ T_r \circ *, * \circ T_l \circ *)$.

(iv) Suppose that (A, m_A) is a non-degenerate algebra. Consider its opposite algebra $A^{op} = (A, m_{A^{op}})$, where $m_{A^{op}} = m_A \circ \Sigma$. Let $T = (T_l, T_r) \in M(A^{op})$. Then for all $a, b \in A$, we have

$$T_l(m_{A^{op}}(a \otimes b)) = m_{A^{op}}(T_l(a) \otimes b);$$

$$T_r(m_{A^{op}}(a \otimes b)) = m_{A^{op}}(a \otimes T_r(b));$$

$$m_{A^{op}}(a \otimes T_l(b)) = m_{A^{op}}(T_r(a) \otimes b);$$

that is,

$$T_l(ba) = bT_l(a), \quad T_r(ba) = T_r(b)a, \quad \text{and} \quad T_l(b)a = bT_r(a).$$

It follows that $T^{op} := (T_r, T_l) \in M(A)$ and $T = (T_l, T_r) \in M(A)^{op}$. Note that $(A^{op})^{op} = A$. Thus $T = (T_l, T_r) \in M(A^{op})$ if and only if $T^{op} = (T_r, T_l) \in M(A)$ if and only if $T = (T_l, T_r) \in M(A)^{op}$. Therefore, the map $M(A^{op}) \to M(A)$, $T \mapsto T^{op}$ is an anti-algebra isomorphism, and we obtain the algebra isomorphism $M(A^{op}) \cong M(A)^{op}$ via $T \mapsto T^{op}$ and the identity anti-algebra isomorphism $M(A) \to M(A)^{op}$.

(v) Corresponding to Proposition 5.1.4(vi), as shown below, for given non-degenerate algebras A and B, every antihomomorphism $\psi : A \to M(B)$ can be extended uniquely to an antihomomorphism $\tilde{\psi} : M(A) \to M(B)$ as well.

Let $\tau : M(B) \to M(B^{op})$ be the antihomomorphism discussed in (iv). Then $\psi_1 := \tau \circ \psi : A \to M(B^{op})$ is a homomorphism, which is also non-degenerate by the definition of the map τ. By Proposition 5.1.4(vi), ψ_1 can be extended uniquely to a homomorphism $\tilde{\psi}_1 : M(A) \to M(B^{op})$. Let $\tilde{\psi} = \tau^{-1} \circ \tilde{\psi}_1$. Then $\tilde{\psi} : M(A) \to M(B)$ is an antihomomorphism extension of ψ, which is certainly unique due to the uniqueness of $\tilde{\psi}_1$.

(vi) Let A and B be non-degenerate algebras, and let $\phi : A \to M(B)$ be a non-degenerate homomorphism. It can be seen from Proposition 5.1.4(vi) that the following assertions hold for the unique homomorphism extension $\tilde{\phi} : M(A) \to M(B)$ of ϕ.

(a) $\tilde{\phi} : M(A) \to M(B)$ satisfies

$$\tilde{\phi}(x)\phi(a) = \phi(xa) \quad \text{and} \quad \phi(a)\tilde{\phi}(x) = \phi(ax) \quad (a \in A,\ x \in M(A)).$$

(b) $\tilde{\phi} : M(A) \to M(B)$ is injective if ϕ is injective.

(c) $\tilde{\phi} : M(A) \to M(B)$ is bijective if ϕ is bijective.

In this case, $\tilde{\phi}$ is injective by (b), and we only need to show that $\tilde{\phi} : M(A) \to M(B)$ is surjective. Consider $\phi^{-1} : B \to A$, which is also a non-degenerate homomorphism. Then it has a homomorphism extension $\widetilde{\phi^{-1}} : M(B) \to M(A)$. For any $y \in M(B)$, let $x = \widetilde{\phi^{-1}}(y) \in M(A)$. Then by (a), we have

$$\tilde{\phi}(x)b = \phi(x\phi^{-1}(b)) = \phi(\widetilde{\phi^{-1}}(y)\phi^{-1}(b)) = \phi(\phi^{-1}(yb)) = yb$$

for all $b \in A$, which implies that $\tilde{\phi}(x) = y$. Hence, $\tilde{\phi} : M(A) \to M(B)$ is surjective.

(vii) In the rest of this thesis, the extension $\tilde{\phi}$ of a homomorphism or an anti-homomorphism $\phi : A \to M(B)$ will be simply denoted by ϕ.

5.2. Multiplier bialgebras

Let A be a non-degenerate algebra. From Proposition 5.1.4(v), we know that the tensor product $A \otimes A$ is still a non-degenerate algebra. Then the algebra A or B in Proposition 5.1.4 can be replaced by $A \otimes A$. It is clear that we have natural embeddings

$$A \otimes A \hookrightarrow M(A) \otimes M(A) \hookrightarrow M(A \otimes A),$$

whose composition is just the canonical embedding of $A \otimes A$ in $M(A \otimes A)$. Now, we are ready to talk about multiplier bialgebras.

Definition 5.2.1. A *multiplier bialgebra* is a non-degenerate algebra A equipped with a non-degenerate homomorphism $\Delta : A \to M(A \otimes A)$ such that

(i) $\Delta(A)(1 \otimes A)$ and $(A \otimes 1)\Delta(A)$ are subsets of $A \otimes A$;

(ii) Δ is coassociative in the sense that

$$(a \otimes 1 \otimes 1)(\Delta \otimes id)(\Delta(b)(1 \otimes c)) = (id \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c)$$

for all $a, b, c \in A$; that is, the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & M(A \otimes A) \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle id\otimes\Delta} \\ M(A \otimes A) & \xrightarrow[\Delta\otimes id]{} & M(A \otimes A \otimes A). \end{array}$$

Here, Δ is called a *comultiplication* on A. In addition, if A is a $*$-algebra and Δ is a $*$-homomorphism, then (A, Δ) is called a *multiplier $*$-bialgebra.*

A *morphism of multiplier ($*$-)bialgebras* (A, Δ_A) and (B, Δ_B) is a non-degenerate ($*$-)homomorphism $F : A \to M(B)$ satisfying $\Delta_B \circ F = (F \otimes F) \circ \Delta_A$; that is, the following square commutes:

$$\begin{array}{ccc} A & \xrightarrow{F} & M(B) \\ {\scriptstyle\Delta_A}\downarrow & & \downarrow{\scriptstyle\Delta_B} \\ M(A \otimes A) & \xrightarrow[F\otimes F]{} & M(B \otimes B). \end{array}$$

REMARK 5.2.2. (i) In Definition 5.2.1, the symbol 1 denotes the unit in $M(A)$, and the sets $A\otimes 1$ and $1\otimes A$ are considered as subsets of $M(A)\otimes M(A) \subseteq M(A\otimes A)$.

(ii) In the diagrams above, the homomorphisms

$$\begin{aligned} \Delta \otimes id : \ & A \otimes A \to M(A \otimes A) \otimes A \hookrightarrow M(A \otimes A \otimes A), \\ id \otimes \Delta : \ & A \otimes A \to A \otimes M(A \otimes A) \hookrightarrow M(A \otimes A \otimes A), \\ F \otimes F : \ & A \otimes A \to M(B) \otimes M(B) \hookrightarrow M(B \otimes B), \\ \Delta_B : \ & B \to M(B \otimes B) \end{aligned}$$

have been extended to the corresponding multiplier algebras.

(iii) Given multiplier bialgebras (A, Δ_A) and (B, Δ_B), we can construct the multiplier bialgebras $(A, \Delta_A)^{op}, (A, \Delta_A)^{cop}$, and $(A, \Delta_A)^{op,cop}$, and equip $A \oplus B$ and $A \otimes B$ with the structure of multiplier bialgebras in a similar way as in the case of bialgebras (cf. Remark 3.1.7(iii)).

5.3. Multiplier Hopf algebras

The theory of multiplier Hopf algebras is very similar to the theory of usual Hopf algebras. It is a very natural extension to the case where the underlying algebra is not unital. First of all, let us consider a motivating example.

EXAMPLE 5.3.1. Suppose that G is a group. Let $\mathbb{C}(G)$ be the algebra of all complex functions on G with pointwise operations. Let A denote the subalgebra $\mathbb{C}_{fin}(G)$ of $\mathbb{C}(G)$ consisting of complex functions on G with finite support.

Consider the map

$$\pi: \ \mathbb{C}(G) \to M(A), \ F \mapsto T_F,$$

where $T_F f = fT_F = Ff$ $(f \in A)$. Clearly, π is an injective algebra homomorphism. To get that $\pi : \mathbb{C}(G) \to M(A)$ is surjective, let $T \in M(A)$ and we show that $T = T_F$ for some $F \in \mathbb{C}(G)$. Define $F \in \mathbb{C}(G)$ by

$$F(x) = (T\delta_x)(x) \qquad (x \in G).$$

Then for all $x, y \in G$, we have

$$(F\delta_x)(y) = F(y)\delta_{x,y} = (T\delta_y)(y)\delta_{x,y} = (T\delta_x)(x)\delta_{x,y}$$

and

$$(T\delta_x)(y) = T(\delta_x\delta_x)(y) = ((T\delta_x)\delta_x)(y) = (T\delta_x)(y)\delta_{x,y} = (T\delta_x)(x)\delta_{x,y}.$$

It follows that $T\delta_x = F\delta_x$ for all $x \in G$, and hence $Tf = Ff = T_F f$ for all $f \in A$ (since $A = \text{span}\,\{\delta_x : x \in G\}$); that is, $T = T_F$, noticing that A is commutative. Therefore, π induces an algebra isomorphism $M(A) \cong \mathbb{C}(G)$.

Also, we have $A \otimes A = \mathbb{C}_{fin}(G) \otimes \mathbb{C}_{fin}(G) \cong \mathbb{C}_{fin}(G \times G)$, and hence $M(A \otimes A) \cong \mathbb{C}(G \times G)$. The multiplication on G gives rise to a comultiplication Δ on A defined by

$$\Delta(f)((x, y)) = f(xy) \qquad (f \in A, \ x, y \in G). \tag{5.3.1}$$

Remark that

$$(\Delta(f)(1 \otimes g))((x, y)) = f(xy)g(y) \quad \text{and} \quad ((f \otimes 1)\Delta(g))((x, y)) = f(x)g(xy)$$

when $f, g \in A$ and $x, y \in G$, and that indeed $\Delta(f)(1 \otimes g)$ and $(f \otimes 1)\Delta(g)$ are in $A \otimes A$. The coassociativity of Δ follows from the associativity of the multiplication on G. Therefore, (A, Δ) is a multiplier bialgebra. We will see that it is also a multiplier Hopf algebra.

Let (A, Δ) be a multiplier bialgebra. Define $T_1, T_2 : A \otimes A \to A \otimes A$ by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \qquad \text{and} \qquad T_2(a \otimes b) = (a \otimes 1)\Delta(b). \tag{5.3.2}$$

In the case of Example 5.3.1, these two linear maps are given by

$$(T_1 f)(x, y) = f(xy, y) \qquad \text{and} \qquad (T_2 f)(x, y) = f(x, xy)$$

when $f \in \mathbb{C}_{fin}(G \times G)$ and $x, y \in G$. In this case, T_1 and T_2 are bijective, because they are dual, respectively, to the maps

$$(x, y) \mapsto (xy, y) \qquad \text{and} \qquad (x, y) \mapsto (x, xy)$$

on $G \times G$, which are invertible. This motivates the following definition.

DEFINITION 5.3.2. A multiplier ($*$-)bialgebra (A, Δ) is called a *multiplier Hopf ($*$-)algebra* if the linear maps $T_1, T_2 : A \otimes A \to A \otimes A$ defined by (5.3.2) are bijective.

A *morphism of multiplier Hopf ($*$-)algebras* is simply a morphism of the underlying multiplier ($*$-) bialgebras.

We saw that $\mathbb{C}_{fin}(G)$ in Example 5.3.1, with its natural multiplication, is a multiplier Hopf algebra.

NOTATION 5.3.3. As we mentioned before, *id* will denote the identity map (in most cases from A to A), and m will denote the multiplication map (here considered as a map not only from $A \otimes A$ to A, but also extended to $M(A) \otimes A$ and $A \otimes M(A)$). We often extend homomorphisms and antihomomorphisms (like ϵ, Δ, and S) to the corresponding multiplier algebra (cf. Proposition 5.1.4(vi) and Remark 5.1.5(v)).

We have used the Sweedler notation when dealing with ordinary Hopf algebras. The use of the Sweedler notation can be justified also in the case of multiplier Hopf algebras (see discussions in [**3**, Appendix] and [**10**, Section 2]). But this must be done with some care.

In general, the multiplier $\Delta(a) \in M(A \otimes A)$ can not be written as a sum $\sum_i a_{1,i} \otimes a_{2,i}$, where $a_{1,i}, a_{2,i} \in M(A)$. Nevertheless, we write this multiplier as a formal sum

$$\Delta(a) =: \sum a_{(1)} \otimes a_{(2)}.$$

For convenience, let $\Delta^{(0)} = id$. Then $\Delta^{(1)} := \Delta = (\Delta^{(0)} \otimes id) \circ \Delta$. Since Δ is non-degenerate, for $n \in \mathbb{N}$, we can define the non-degenerate homomorphism $\Delta^{(n)} : A \to M(A^{\otimes n+1})$ by

$$\Delta^{(n)} := (\Delta^{(n-1)} \otimes id_A) \circ \Delta,$$

and we write the multiplier $\Delta^{(n)}(a) \in M(A^{\otimes n+1})$ as a formal sum

$$\Delta^{(n)}(a) =: \sum a_{(1)} \otimes \cdots \otimes a_{(n+1)}.$$

We think of the multiplier $\Delta^{(n)}(a)$ as having $n+1$ *legs*, which are represented by the symbols $a_{(1)}, \ldots, a_{(n+1)}$, and treat the above formal sums like ordinary sums of elementary tensors in a tensor product of algebras. Thus we write

$$\Delta(a)(1 \otimes b) = \sum a_{(1)} \otimes a_{(2)}b \quad \text{and} \quad (a \otimes 1)\Delta(b) = \sum ab_{(1)} \otimes b_{(2)}$$

for all $a, b \in A$. The definition of a multiplier bialgebra implies that these two expressions belong to $A \otimes A$; thus the formal sums above stand for finite sums of elementary tensors in $A \otimes A$.

5.4. Constructions of the counit and the antipode

Let (A, Δ) be a multiplier Hopf algebra. We will construct a homomorphism $\epsilon : A \to \mathbb{C}$ and an antihomomorphism $S : A \to M(A)$, which are just the counit and the antipode, respectively, when (A, Δ) is a Hopf algebra.

Let us start with the counit. For $a \in A$, define the linear map $E(a) : A \to A$ by

$$E(a)b = m(T_1^{-1}(a \otimes b)) \qquad (b \in A).$$

Since

$$T_1(a \otimes bc) = \Delta(a)(1 \otimes bc) = \Delta(a)(1 \otimes b)(1 \otimes c) = T_1(a \otimes b)(1 \otimes c)$$

for all $a, b, c \in A$, the map T_1^{-1} has the same property. Then

$$E(a)(bc) = m(T_1^{-1}(a \otimes bc)) = m(T_1^{-1}(a \otimes b)(1 \otimes c)) = (E(a)b)c.$$

So, each $E(a)$ is a left multiplier on A, and the map $E : A \to L(A)$ is obviously linear.

Lemma 5.4.1. *For all $a, b \in A$, we have*

$$(id \otimes E)((a \otimes 1)\Delta(b)) = ab \otimes 1.$$

Proof. Let $a, b, c \in A$. By the surjectivity of T_1, we can write $b \otimes c = \sum_i \Delta(b_i)(1 \otimes c_i)$ for some $b_i, c_i \in A$. By applying $\Delta \otimes id$ and then multiplying $a \otimes 1 \otimes 1$ on the left, we get

$$\begin{aligned}(a \otimes 1)\Delta(b) \otimes c &= \sum_i (a \otimes 1 \otimes 1)(\Delta \otimes id)(\Delta(b_i)(1 \otimes c_i)) \\ &= \sum_i (id \otimes \Delta)((a \otimes 1)\Delta(b_i))(1 \otimes 1 \otimes c_i).\end{aligned}$$

Let ϕ be any linear functional on A. Applying $\phi \otimes id \otimes id$ on the both sides of the above equation, we obtain

$$\begin{aligned}(\phi \otimes id)((a \otimes 1)\Delta(b)) \otimes c &= \sum_i (\phi \otimes \Delta)((a \otimes 1)\Delta(b_i))(1 \otimes c_i) \\ &= \sum_i \Delta((\phi \otimes id)((a \otimes 1)\Delta(b_i)))(1 \otimes c_i) \\ &= \sum_i T_1((\phi \otimes id)((a \otimes 1)\Delta(b_i)) \otimes c_i),\end{aligned}$$

or

$$T_1^{-1}((\phi \otimes id)((a \otimes 1)\Delta(b)) \otimes c) = \sum_i (\phi \otimes id)((a \otimes 1)\Delta(b_i)) \otimes c_i.$$

By the definition of E, we have

$$E((\phi \otimes id)((a \otimes 1)\Delta(b)))c = \sum_i (\phi \otimes id)((a \otimes 1)\Delta(b_i))c_i.$$

So,

$$\begin{aligned}(\phi \otimes id)((id \otimes E)((a \otimes 1)\Delta(b))(1 \otimes c)) &= (\phi \otimes id)((a \otimes 1)\sum_i \Delta(b_i)(1 \otimes c_i)) \\ &= (\phi \otimes id)((ab \otimes 1)(1 \otimes c)).\end{aligned}$$

The arbitrariness of ϕ implies that

$$(id \otimes E)((a \otimes 1)\Delta(b))(1 \otimes c) = (ab \otimes 1)(1 \otimes c).$$

Therefore, $(id \otimes E)((a \otimes 1)\Delta(b)) = ab \otimes 1$. □

From the above lemma and the surjectivity of T_2, it is easy to obtain that $E(A) \subseteq \mathbb{C} \cdot 1$. So, we can define the counit $\epsilon : A \to \mathbb{C}$ by

$$\epsilon(a)1 = E(a) \qquad (a \in A). \tag{5.4.1}$$

Then we have

$$(id \otimes \epsilon)((a \otimes 1)\Delta(b)) = ab \qquad \text{for all } a, b \in A. \tag{5.4.2}$$

Due to the surjectivity of T_2, the functional $\epsilon \in A'$ satisfying (5.4.2) is unique. We will show that ϵ has the usual properties of the counit in a Hopf algebra.

PROPOSITION 5.4.2. *Suppose that (A, Δ) is a multiplier Hopf algebra. Let ϵ be defined by* (5.4.1)*. Then $\epsilon : A \to \mathbb{C}$ is a non-degenerate algebra homomorphism.*

PROOF. Let $a, b, c \in A$. By (5.4.2), we have

$$(id \otimes \epsilon)((a \otimes 1)\Delta(bc)) = abc.$$

Then

$$(id \otimes \epsilon)((a \otimes 1)\Delta(b)\Delta(c)) = (ab)c = (id \otimes \epsilon)((a \otimes 1)\Delta(b))c.$$

By the surjectivity of T_2, we get

$$(id \otimes \epsilon)((a \otimes b)\Delta(c)) = (id \otimes \epsilon)(a \otimes b)c = \epsilon(b)ac = \epsilon(b)(id \otimes \epsilon)((a \otimes 1)\Delta(c)).$$

Again, by the surjectivity of T_2, we have

$$(id \otimes \epsilon)(a \otimes bc) = \epsilon(b)(id \otimes \epsilon)(a \otimes c);$$

that is, $a\epsilon(bc) = a\epsilon(b)\epsilon(c)$, which implies that $\epsilon(bc) = \epsilon(b)\epsilon(c)$ by the arbitrariness of a.

Finally, from the definition of $E : A \to L(A)$ and the bijectivity of T_1, we have

$$\text{span }\{E(a)b \mid a, b \in A\} = A,$$

which follows that E is non-degenerate, and thus ϵ is non-degenerate. □

For all $a, b \in A$, by the definitions of E and ϵ, we have

$$(\epsilon \otimes id)(a \otimes b) = E(a)b = m(T_1^{-1}(a \otimes b)),$$

and hence

$$(\epsilon \otimes id)(\Delta(a)(1 \otimes b)) = ab. \tag{5.4.3}$$

Therefore, (5.4.2) and (5.4.3) imply that

$$(id \otimes \epsilon) \circ \Delta = (\epsilon \otimes id) \circ \Delta = id,$$

where $id \otimes \epsilon$ and $\epsilon \otimes id$ are the unique extensions to $M(A \otimes A)$.

Furthermore, as shown by A. Van Daele in [**8**, Proposition 5.7], if (A, Δ) is a multiplier Hopf $*$-algebra, then ϵ is a $*$-homomorphism.

Next, we will move on to constructing the antipode on (A, Δ). For each $a \in A$, define $S(a) : A \to A$ by

$$S(a)b = (\epsilon \otimes id)(T_1^{-1}(a \otimes b)) \qquad (b \in A). \tag{5.4.4}$$

Similar to the discussion on $E(a)$, each $S(a)$ is also a left multiplier on A, and $S : A \to L(A)$ is linear.

Lemma 5.4.3. *For all $a, b, c \in A$, we have*

$$(id \otimes S)((a \otimes 1)\Delta(b))(1 \otimes c) = (a \otimes 1)T_1^{-1}(b \otimes c).$$

Proof. Let $a, b, c \in A$. Write $b \otimes c = \sum_i \Delta(b_i)(1 \otimes c_i)$ for some $b_i, c_i \in A$. From the proof in Lemma 5.4.1, we have

$$T_1^{-1}((\phi \otimes id)((a \otimes 1)\Delta(b)) \otimes c) = \sum_i (\phi \otimes id)((a \otimes 1)\Delta(b_i)) \otimes c_i$$

for all $\phi \in A'$. By (5.4.2) and (5.4.4), we have

$$\begin{aligned} S((\phi \otimes id)((a \otimes 1)\Delta(b)))c &= \sum_i (\epsilon \otimes id)((\phi \otimes id)((a \otimes 1)\Delta(b_i)) \otimes c_i) \\ &= (\phi \otimes id)(\sum_i (id \otimes \epsilon)((a \otimes 1)\Delta(b_i)) \otimes c_i) \\ &= (\phi \otimes id)(\sum_i ab_i \otimes c_i) \\ &= (\phi \otimes id)((a \otimes 1)\sum_i (b_i \otimes c_i)). \end{aligned}$$

This implies that

$$(\phi \otimes id)((id \otimes S)((a \otimes 1)\Delta(b))(1 \otimes c)) = (\phi \otimes id)((a \otimes 1)T_1^{-1}(b \otimes c)).$$

Since this is true for all $\phi \in A'$, we finish our proof. □

LEMMA 5.4.4. *For all $a, b, c \in A$, we have*

$$m((id \otimes S)((a \otimes 1)\Delta(b))(1 \otimes c)) = a\epsilon(b)c.$$

PROOF. This is immediate by applying m to the equation in Lemma 5.4.3. □

The following right-side version of Lemma 5.4.3 and Lemma 5.4.4 was shown by A. Van Daele in [**8**, Lemma 4.5].

LEMMA 5.4.5. *Each $S(a)$ is also a right multiplier on A if we define*

$$bS(a) = (id \otimes \epsilon)(T_2^{-1}(b \otimes a)) \qquad (b \in A). \tag{5.4.5}$$

Corresponding to Lemma 5.4.3 and Lemma 5.4.4, for all $a, b, c \in A$, we have

$$(a \otimes 1)(S \otimes id)(\Delta(b)(1 \otimes c)) = T_2^{-1}(a \otimes b)(1 \otimes c)$$

and

$$m((a \otimes 1)(S \otimes id)(\Delta(b)(1 \otimes c))) = a\epsilon(b)c.$$

Given $a, b, c \in A$, write $b \otimes a = \sum_i (b_i \otimes 1)\Delta(a_i)$ for some $a_i, b_i \in A$. On the one hand, from (5.4.5) we have

$$bS(a) = (id \otimes \epsilon)(T_2^{-1}(b \otimes a)) = \sum_i (id \otimes \epsilon)(b_i \otimes a_i) = \sum_i b_i\epsilon(a_i),$$

so that

$$(bS(a))c = \sum_i b_i \epsilon(a_i)c.$$

On the other hand, we have

$$b \otimes S(a)c = (id \otimes S)(b \otimes a)(1 \otimes c) = \sum_i (id \otimes S)((b_i \otimes 1)\Delta(a_i))(1 \otimes c).$$

By Lemma 5.4.4, we obtain that

$$b(S(a)c) = \sum_i m((id \otimes S)((b_i \otimes 1)\Delta(a_i))(1 \otimes c)) = \sum_i b_i \epsilon(a_i)c.$$

Hence, we have

$$(bS(a))c = b(S(a)c) \qquad \text{for all } a, b, c \in A. \tag{5.4.6}$$

Therefore, $S(a) \in M(A)$. Clearly, the map $S : A \to M(A)$ satisfying the equation in Lemma 5.4.4 and the last equation in Lemma 5.4.5 is unique. Moreover, $S : A \to M(A)$ has the following property.

PROPOSITION 5.4.6. *Suppose that (A, Δ) is a multiplier Hopf algebra. Then the map $S : A \to M(A)$ is an algebra antihomomorphism; that is,*

$$S(ab) = S(b)S(a) \qquad \textit{for all } a, b \in A.$$

PROOF. Let $a, b, c, d \in A$. By Lemma 5.4.4, we have

$$\begin{aligned} m((id \otimes S)((c \otimes 1)\Delta(a)\Delta(b))(1 \otimes d)) &= c\epsilon(ab)d = c\epsilon(a)d\epsilon(b) \\ &= m((id \otimes S)((c \otimes 1)\Delta(a))(1 \otimes d))\epsilon(b). \end{aligned}$$

By the surjectivity of T_2, we get

$$\begin{aligned} m((id \otimes S)((c \otimes a)\Delta(b))(1 \otimes d)) &= m((id \otimes S)(c \otimes a)(1 \otimes d))\epsilon(b) \\ &= cS(a)d\epsilon(b) = c\epsilon(b)S(a)d \\ &= m((id \otimes S)((c \otimes 1)\Delta(b)))(1 \otimes S(a)d). \end{aligned}$$

By the surjectivity of T_2 again, we get

$$m((id \otimes S)(c \otimes ab)(1 \otimes d)) = m((id \otimes S)(c \otimes b))(1 \otimes S(a)d);$$

that is, $cS(ab)d = cS(b)S(a)d$. Therefore, $S(ab) = S(b)S(a)$. $\square$

In summary, we state below the main result of this section.

THEOREM 5.4.7. *Let (A, Δ) be a multiplier Hopf algebra.*

(i) There exists a unique non-degenerate homomorphism $\epsilon : A \to \mathbb{C}$, called the counit *of (A, Δ), satisfying*

$$(\epsilon \otimes id)(\Delta(a)(1 \otimes b)) = ab = (id \otimes \epsilon)((a \otimes 1)\Delta(b)), \tag{5.4.7}$$

or equivalently,

$$\sum \epsilon(a_{(1)})a_{(2)}b = ab = \sum ab_{(1)}\epsilon(b_{(2)})$$

for all $a, b \in A$; that is, the following diagram commutes:

$$\begin{tikzcd}
M(A \otimes A) \arrow[d, "\epsilon\otimes id"'] & A \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, hook] & M(A \otimes A) \arrow[d, "id\otimes\epsilon"] \\
M(\mathbb{C} \otimes A) \arrow[r, "\cong"'] & M(A) & M(A \otimes \mathbb{C}). \arrow[l, "\cong"]
\end{tikzcd}$$

If (A, Δ) is a multiplier Hopf $*$-algebra, then ϵ is a $*$-homomorphism.

(ii) There exists a unique antihomomorphism $S : A \to M(A)$, called the antipode *of A, such that*

$$m((id \otimes S)((a \otimes 1)\Delta(b))(1 \otimes c)) = a\epsilon(b)c \tag{5.4.8}$$

and

$$m((a \otimes 1)(S \otimes id)(\Delta(b)(1 \otimes c))) = a\epsilon(b)c, \tag{5.4.9}$$

or equivalently,

$$ab_{(1)}S(b_{(2)})c = a\epsilon(b)c = aS(b_{(1)})b_{(2)}c$$

for all $a, b, c \in A$. Moreover, $S : (A, \Delta) \to (A, \Delta)^{op,cop}$ is a morphism of multiplier Hopf algebras.; that is, $\Delta \circ S = (S \otimes S) \circ \Delta \circ \Sigma$, or equivalently,

$$\sum S(a)_{(1)} \otimes S(a)_{(2)} = \sum S(a_{(2)}) \otimes S(a_{(1)}) \quad \textit{for all } a \in A.$$

(iii) ([**8**, Proposition 5.8]) *If (A, Δ) is a multiplier Hopf $*$-algebra, then we have $S(A) \subseteq A$ and $S \circ * \circ S \circ * = id_A$.*

(iv) ([**7**, Theorem 2.1.12]) *Let $F : A \to M(B)$ be a morphism of multiplier Hopf algebras. Then $S_B \circ F = F \circ S_A$.*

Remark 5.4.8. (i) By Theorem 3.5.4, we can see that every Hopf algebra is a multiplier Hopf algebra. Conversely, if (A, Δ) is a multiplier Hopf algebra with an identity, the formulas in Theorem 5.4.7 are just the usual properties for the counit and the antipode. Thus (A, Δ) is a Hopf algebra.

(ii) If (A, Δ) is a multiplier Hopf algebra, then so is $(A, \Delta)^{op,cop}$, and they have the same antipode (cf. [**7**, Remark 2.1.10(ii)]). Therefore, $(A, \Delta)^{op}$ is a multiplier Hopf algebra if and only if $(A, \Delta)^{cop}$ is a multiplier Hopf algebra; in this case, they also have the same antipode. In the next section, we will see that if (A, Δ) and $(A, \Delta)^{cop}$ are both multiplier Hopf algebras, then their antipodes are inverse to each other.

5.5. Regular multiplier Hopf algebras

In the final section of this chapter, we will consider a regular multiplier Hopf algebra (A, Δ), where the antipode is a map from A to A (instead of $M(A)$) and is invertible.

Definition 5.5.1. A multiplier Hopf algebra (A, Δ) is *regular* if the multiplier bialgebra $(A, \Delta)^{cop}$ (or equivalently, $(A, \Delta)^{op}$) is a multiplier Hopf algebra.

Obviously, if (A, Δ) is a regular multiplier Hopf algebra, then $(A, \Delta)^{cop}$ and $(A, \Delta)^{op}$ are also regular multiplier Hopf algebras.

Remark 5.5.2. Let (A, Δ) be a regular multiplier Hopf algebra. Then we have

$$\Delta(a)(b \otimes 1), (1 \otimes a)\Delta(b) \in A \otimes A$$

for all $a, b \in A$. In this case, using the extended Sweedler notation (cf. Notation 5.3.3), we have four expressions in $A \otimes A$:

$$\Delta(a)(b \otimes 1) = \sum a_{(1)}b \otimes a_{(2)}; \quad (b \otimes 1)\Delta(a) = \sum ba_{(1)} \otimes a_{(2)};$$

$$\Delta(a)(1 \otimes b) = \sum a_{(1)} \otimes a_{(2)}b; \quad (1 \otimes b)\Delta(a) = \sum a_{(1)} \otimes ba_{(2)}.$$

We say that in the products in the first line above, the first leg $a_{(1)}$ of $\Delta(a)$ is covered by b, whereas in the products in the second line, the second leg $a_{(2)}$ of $\Delta(a)$ is covered by b.

More generally, for $n \in \mathbb{N}$, $a \in A$ and $x \in A^{\otimes(n+1)}$, we write $x\Delta^{(n)}(a)$ and $\Delta^{(n)}(a)x$ in the extended Sweedler notation. If at least n legs of $\Delta^{(n)}(a)$ are covered by elements of A, then $x\Delta^{(n)}(a)$ and $\Delta^{(n)}(a)x$ belong to $A^{\otimes(n+1)}$.

Let $\Delta^{cop} := \Sigma \circ \Delta$, ϵ^{cop}, and S^{cop} denote the comultiplication, the counit, and the antipode on $(A, \Delta)^{cop}$, respectively. We will discuss the relationships between ϵ and ϵ^{cop} and between S and S^{cop}.

Proposition 5.5.3. *If (A, Δ) is a regular multiplier Hopf algebra, then we have $\epsilon = \epsilon^{cop}$.*

Proof. Let $a, b, c \in A$. On the one hand, for $(A, \Delta)^{cop}$, we have

$$(\epsilon^{cop} \otimes id)((1 \otimes a)\Delta(b)) = (id \otimes \epsilon^{cop})((a \otimes 1)\Delta^{cop}(b)) = ab.$$

On the other hand,

$$(\epsilon \otimes id)((1 \otimes a)\Delta(b))c = a(\epsilon \otimes id)(\Delta(b)(1 \otimes c)) = abc$$

implies that

$$(\epsilon \otimes id)((1 \otimes a)\Delta(b)) = ab.$$

The surjectivity of the map $a \otimes b \mapsto (1 \otimes a)\Delta(b)$ shows that $\epsilon = \epsilon^{cop}$. $\square$

Corollary 5.5.4. *If (A, Δ) is a regular multiplier Hopf algebra, then for all $a, b \in A$, we have*

$$(\epsilon \otimes id)(\Delta(a)(1 \otimes b)) = (\epsilon \otimes id)((1 \otimes a)\Delta(b)) = ab$$

and

$$(id \otimes \epsilon)(\Delta(a)(b \otimes 1)) = (id \otimes \epsilon)((a \otimes 1)\Delta(b)) = ab.$$

Proposition 5.5.5. *If (A, Δ) is a regular multiplier Hopf algebra, then $S(A) = A$, $S^{cop}(A) = A$, and S and S^{cop} are inverses of each other.*

Proof. Let $a, b, c, d \in A$. Write $a \otimes b = \sum_i \Delta(a_i)(1 \otimes b_i)$ for some $a_i, b_i \in A$. Then

$$S(a)b = (\epsilon \otimes id)(T_1^{-1}(a \otimes b)) = \sum_i (\epsilon \otimes id)(a_i \otimes b_i) = \sum_i \epsilon(a_i)b_i.$$

Now we have

$$b \otimes ac = (\sum_i \Delta^{cop}(a_i)(b_i \otimes 1))(1 \otimes c) = \sum_i \Delta^{cop}(a_i)(b_i \otimes c).$$

By applying $S^{cop} \otimes id$ and then multiplying $d \otimes 1$ on the left, we get

$$\begin{aligned} dS^{cop}(b) \otimes ac &= \sum_i (d \otimes 1)(S^{cop} \otimes id)(\Delta^{cop}(a_i)(b_i \otimes c)) \\ &= \sum_i (dS^{cop}(b_i) \otimes 1)(S^{cop} \otimes id)(\Delta^{cop}(a_i)(1 \otimes c)). \end{aligned}$$

Note that $S^{cop}(b) \in M(A)$. Then from Lemma 5.4.5 or (5.4.9), we have

$$\begin{aligned} d(S^{cop}(b)a)c = (dS^{cop}(b))ac &= \sum_i dS^{cop}(b_i)\epsilon^{cop}(a_i)c \\ &= dS^{cop}(\sum_i \epsilon(a_i)b_i)c = dS^{cop}(S(a)b)c, \end{aligned}$$

which implies that

$$S^{cop}(S(a)b) = S^{cop}(b)a. \tag{5.5.1}$$

The definition of $S : A \to M(A)$ and the surjectivity of T_1 imply that

$$A = \text{span } S(A)A = \text{span } S^{cop}(A)A.$$

Then we obtain from (5.5.1) that $S^{cop}(A) = A$. Hence, $S(A) = A$, noticing that $(A, \Delta)^{cop}$ is also a regular multiplier Hopf algebra and $S = (S^{cop})^{cop}$.

Since S^{cop} is an antihomomorphism, equation (5.5.1) also shows that $S^{cop}(b)a = S^{cop}(b)S^{cop}(S(a))$. Since $S^{cop}(A) = A$, we have $S^{cop}(S(a)) = a$, or $S^{cop} \circ S = id_A$. Similarly, $S \circ S^{cop} = (S^{cop})^{cop} \circ S^{cop} = id_A$. Therefore, $S^{cop} = S^{-1}$. □

Actually, the other direction of Proposition 5.5.5 is also true.

Proposition 5.5.6. *If (A, Δ) is a multiplier Hopf algebra such that $S(A) \subseteq A$ and $S : A \to A$ is bijective, then (A, Δ) is regular.*

PROOF. Let $a, b, c \in A$. By Lemma 5.4.3, we have

$$(id \otimes S)((c \otimes b)\Delta(a)) = (id \otimes S)((c \otimes 1)\Delta(a))(1 \otimes S(b)) = (c \otimes 1)T_1^{-1}(a \otimes S(b)).$$

This implies that

$$(c \otimes b)\Delta(a) = (c \otimes 1)(id \otimes S^{-1})(T_1^{-1}(a \otimes S(b))),$$

and thus

$$(1 \otimes b)\Delta(a) = (id \otimes S^{-1})(T_1^{-1}(a \otimes S(b))).$$

The bijectivity of the maps S and T_1 imply that $(1 \otimes b)\Delta(a) \in A \otimes A$, and the map $A \otimes A \to A \otimes A,\ a \otimes b \mapsto (1 \otimes b)\Delta(a)$ is bijective. Similarly, $\Delta(a)(b \otimes 1) \in A \otimes A$, and the map $A \otimes A \to A \otimes A,\ a \otimes b \mapsto \Delta(a)(b \otimes 1)$ is also bijective. Therefore, $(A, \Delta)^{op}$ is a multiplier Hopf algebra; that is, (A, Δ) is regular. □

Combining Proposition 5.5.5 and Proposition 5.5.6, we have the following conclusion.

PROPOSITION 5.5.7. *A multiplier Hopf algebra (A, Δ) is regular if and only if the antipode S is a linear bijection on A.*

The following result characterizes when a multiplier bialgebra is a regular multiplier Hopf algebra.

THEOREM 5.5.8. *Let (A, Δ) be a multiplier bialgebra over $\mathbb{C}$. Assume that $\Delta(A)(A \otimes 1)$ and $(1 \otimes A)\Delta(A)$ are both subsets of $A \otimes A$. If there exist a non-degenerate homomorphism $\epsilon : A \to \mathbb{C}$ and a linear bijection $S : A \to A$ such that* (5.4.7), (5.4.8), *and* (5.4.9) *are satisfied, then (A, Δ) is a regular multiplier Hopf algebra.*

PROOF. To prove that (A, Δ) is a multiplier Hopf algebra, we just need to show that the maps T_1 and T_2 on $A \otimes A$ are bijective. After that, the regularity of (A, Δ) follows immediately from Proposition 5.5.7. Consider the maps

$$R_1 : \ A \otimes A \to A \otimes A,\ a \otimes b \mapsto (id \otimes S)((1 \otimes S^{-1}(b))\Delta(a))$$

and

$$R_2 : \ A \otimes A \to A \otimes A,\ a \otimes b \mapsto (S \otimes id)(\Delta(b)(S^{-1}(a) \otimes 1)).$$

Then R_1 and R_2 are well-defined, since $S : A \to A$ is invertible, and $\Delta(A)(A \otimes 1)$ and $(1 \otimes A)\Delta(A)$ are subsets of $A \otimes A$. Now, for all $a, b, c, d \in A$, we have

$$\begin{aligned}(c \otimes d)T_1(R_1(a \otimes b)) &= (c \otimes d)T_1((id \otimes S)((1 \otimes S^{-1}(b))\Delta(a))) \\ &= \sum (c \otimes d)T_1(a_{(1)} \otimes S(a_{(2)})b) \\ &= \sum (c \otimes d)\Delta(a_{(1)})(1 \otimes S(a_{(2)})b) \\ &= \sum ca_{(1)} \otimes da_{(2)}S(a_{(3)})b \\ &= \sum ca_{(1)}\epsilon(a_{(2)}) \otimes db \\ &= (c \otimes d)(a \otimes b)\end{aligned}$$

and

$$\begin{aligned}(c \otimes d)R_1(T_1(a \otimes b)) &= (c \otimes d)R_1(\Delta(a)(1 \otimes b)) \\ &= \sum (c \otimes d)R_1(a_{(1)} \otimes a_{(2)}b) \\ &= \sum (c \otimes d)(id \otimes S)((1 \otimes S^{-1}(a_{(2)}b))\Delta(a_{(1)})) \\ &= \sum ca_{(1)} \otimes dS(a_{(2)})a_{(3)}b \\ &= \sum ca_{(1)}\epsilon(a_{(2)}) \otimes db \\ &= (c \otimes d)(a \otimes b).\end{aligned}$$

These imply that $T_1(R_1(a \otimes b)) = R_1(T_1(a \otimes b)) = a \otimes b$ for all $a, b \in A$. Thus $R_1 = T_1^{-1}$. Similarly, we can show that $R_2 = T_2^{-1}$. □

The corollary below is immediate by Theorem 5.5.8 and its proof.

Corollary 5.5.9. *If (A, Δ) is a regular multiplier Hopf algebra, then the maps $a \otimes b \mapsto \Delta(a)(b \otimes 1)$ and $a \otimes b \mapsto (1 \otimes a)\Delta(b)$ are linear bijections on $A \otimes A$. More precisely, they are the linear bijections*

$$(id \otimes S^{-1}) \circ T_1^{-1} \circ (id \otimes S) \circ \Sigma \quad \textit{and} \quad (S^{-1} \otimes id) \circ T_2^{-1} \circ (S \otimes id) \circ \Sigma.$$

Remark 5.5.10. Note that there are three cases where the multiplier Hopf algebra (A, Δ) is automatically regular (cf. Proposition 3.4.4 and Proposition 3.6.5 for the Hopf algebra case).

(a) When A is cocommutative, $\Delta = \Sigma \circ \Delta$. Then $(A, \Delta)^{cop} = (A, \Delta)$, and hence (A, Δ) is regular. In this case, $S = S^{cop} = S^{-1}$, and thus $S^2 = id$.

(b) When A is commutative, $(A, \Delta)^{op} = (A, \Delta)$, and thus (A, Δ) is also regular (cf. Definition 5.5.1). Moreover, by Remark 5.4.8(ii) and Proposition 5.5.5, we have $S = S^{cop} = S^{-1}$, and hence $S^2 = id$.

(c) If (A, Δ) is a multiplier Hopf $*$-algebra, then we have that $S(A) \subseteq A$ and $S \circ * \circ S \circ * = id_A$ (cf. Theorem 5.4.7(iii)). Therefore, the antipode $S : A \to A$ is bijective, and hence (A, Δ) is regular by Proposition 5.5.7. In this case, we have $S^{cop} = S^{-1} = * \circ S \circ *$.

CHAPTER 6

Integrals and their modular properties

A remarkable feature of a class of multiplier Hopf algebras is that they admit a nice duality, which is based on the existence of non-zero left and right invariant linear functionals, called integrals. In this chapter, these integrals and their modular properties will be discussed. The main references of this chapter are [**7**] and [**9**].

6.1. The concept of an integral

To motivate the definition of integrals, let us reformulate the concept of a Haar measure in terms of Hopf algebras.

Let G be a locally compact group with a left Haar measure λ. We note that if G is a discrete group, then the counting measure is a (left) Haar measure of G. Let $A \subseteq \mathbb{C}(G)$ be some Hopf algebra of functions on G with the comultiplication, counit, and antipode as in equation (3.2.3). If $A \subseteq L^1(G, \lambda)$, then the left Haar integral defines a linear functional

$$\phi : A \to \mathbb{C},\ f \mapsto \int_G f(y)d\lambda(y). \tag{6.1.1}$$

Left invariance of λ amounts to the fact that for each $f \in A$, the function F on G defined by

$$F(\cdot) = \int_G f(\cdot y)d\lambda(y)$$

satisfies $F(x) = \phi(f)$ for all $x \in G$. We replace the multiplication of G by the comultiplication of A, using the relation

$$f(xy) = \Delta(f)((x, y)) = \sum f_{(1)}(x)f_{(2)}(y) \qquad (x, y \in G),$$

and thus obtain

$$F = \sum f_{(1)} \int_G f_{(2)}(y)d\lambda(y) = (id \otimes \phi)\Delta(f).$$

Therefore, the invariance condition "$F(x) = \phi(f)$ for all $x \in G$" takes the form

$$(id \otimes \phi)\Delta(f) = \phi(f)1_A.$$

Now let (A, Δ) be a regular multiplier Hopf algebra. Given $\phi \in A'$ and $a, b \in A$, we define

$$((id \otimes \phi)\Delta(a))b = (id \otimes \phi)(\Delta(a)(b \otimes 1)) = \sum a_{(1)} b \phi(a_{(2)});$$
$$b((id \otimes \phi)\Delta(a)) = (id \otimes \phi)((b \otimes 1)\Delta(a)) = \sum b a_{(1)} \phi(a_{(2)}).$$

It is easy to see that $(id \otimes \phi)\Delta(a) \in M(A)$.

Similarly, we define $(\phi \otimes id)\Delta(a) \in M(A)$ by

$$((\phi \otimes id)\Delta(a))b = (\phi \otimes id)(\Delta(a)(1 \otimes b)) = \sum \phi(a_{(1)}) a_{(2)} b;$$
$$b((\phi \otimes id)\Delta(a)) = (\phi \otimes id)((1 \otimes b)\Delta(a)) = \sum \phi(a_{(1)}) b a_{(2)}.$$

Using these notation, we can define left invariant functions and right invariant functionals as follows.

Definition 6.1.1. Let (A, Δ) be a regular multiplier Hopf algebra. A linear map $\phi : A \to \mathbb{C}$ is

- *left invariant* if $(id \otimes \phi)\Delta(a) = \phi(a)1_{M(A)}$ for all $a \in A$;
- *right invariant* if $(\phi \otimes id)\Delta(a) = \phi(a)1_{M(A)}$ for all $a \in A$.

If ϕ is non-zero and left/right invariant, we call ϕ a *left/right integral* on (A, Δ). A left integral that is also right invariant is briefly called an *integral*.

Example 6.1.2. (i) Let G be a finite group. Consider the Hopf algebra $(\mathbb{C}(G), \Delta)$ defined in Example 3.2.2. The equality $S(\delta_x) = \delta_{x^{-1}}$ implies that $S : \mathbb{C}(G) \to \mathbb{C}(G)$ is invertible, and hence $(\mathbb{C}(G), \Delta)$ is a regular multiplier Hopf algebra with $\mathbb{C}(G) = M(\mathbb{C}(G))$ and $1_{M(\mathbb{C}(G))} = 1_G$ (the constant 1 function on G). Now consider the map

$$\phi : \mathbb{C}(G) \to \mathbb{C}, \; f \mapsto \sum_{x \in G} f(x).$$

Obviously, ϕ is linear. Moreover, for all $x \in G$, we have

$$(id \otimes \phi)\Delta(\delta_x) = \sum_{\substack{y,z\in G\\ yz=x}} (id \otimes \phi)(\delta_y \otimes \delta_z) = \sum_{\substack{y,z\in G\\ yz=x}} \delta_y \phi(\delta_z)$$
$$= \sum_{y\in G} \delta_y = 1_G = \phi(\delta_x) 1_{M(\mathbb{C}(G))}$$

and similarly $(\phi \otimes id)\Delta(\delta_x) = 1_G = \phi(\delta_x)1_{M(\mathbb{C}(G))}$. Therefore, $\phi : \mathbb{C}(G) \to \mathbb{C}$ is an integral on $(\mathbb{C}(G), \Delta)$.

(ii) More generally, as in Example 5.3.1, let $A = \mathbb{C}_{fin}(G)$, the algebra of complex functionals with finite support on a (discrete) group G. Then $M(A) \cong \mathbb{C}(G)$ and $1_{M(A)} = 1_G$. The map is still defined by $\phi : A \to \mathbb{C}$ by

$$\phi(f) = \sum_{x\in G} f(x).$$

Let $f, g \in A$ and $x \in G$. Then

$$(f((id \otimes \phi)\Delta(g)))(x) = ((id \otimes \phi)((f \otimes 1)\Delta(g)))(x)$$
$$= \sum ((id \otimes \phi)(f g_{(1)} \otimes g_{(2)}))(x)$$
$$= \sum_{y\in G} f(x) g_{(1)}(x) g_{(2)}(y)$$
$$= \sum_{y\in G} f(x) g(xy) = \sum_{y\in G} f(x) g(y) = f(x)\phi(g),$$

which implies that $f((id\otimes\phi)\Delta(g)) = \phi(g)f$. Similarly, we have $((id\otimes\phi)\Delta(g))f = \phi(g)f$. So,

$$(id \otimes \phi)\Delta(g) = \phi(g)1_G \quad \text{for all } g \in A,$$

and thus ϕ is left invariant. In this case, ϕ is also right invariant, since $\sum_x f(xy) = \sum_x f(x)$ for all $f \in A$ and $y \in G$.

Here are some characterizations of left invariance and right invariance.

Remark 6.1.3. (i) Let (A, Δ) be a regular multiplier Hopf algebra. Then $\phi \in A'$ is left invariant if and only if for all $f \in A'$ and $a, b \in A$, we have

$$(f \otimes \phi)(\Delta(a)(b \otimes 1)) = f((id \otimes \phi)(\Delta(a)(b \otimes 1)))$$
$$= f(b((id \otimes \phi)\Delta(a))) = f(b\phi(a)) = \phi(a)f(b)$$

and

$$(f \otimes \phi)((b \otimes 1)\Delta(a)) = f(\phi(a)b) = \phi(a)f(b).$$

Likewise, $\phi \in A'$ is right invariant if and only if

$$(\phi \otimes f)(\Delta(a)(1 \otimes b)) = \phi(a)f(b) \quad \text{and} \quad (\phi \otimes f)((1 \otimes b)\Delta(a)) = \phi(a)f(b)$$

for all $f \in A'$ and $a, b \in A$.

(ii) For a regular Hopf algebra (A, Δ), the invariance of linear maps can be characterized in terms of the convolution product defined in Section 3.3. Let $\phi \in A'$. Then

$$\phi \text{ is left invariant} \iff f * \phi = f(1_A)\phi \text{ for all } f \in A';$$

$$\phi \text{ is right invariant} \iff \phi * f = f(1_A)\phi \text{ for all } f \in A'.$$

These follow from the relations

$$(f \otimes \phi)(\Delta(a)) = (f * \phi)(a) \quad \text{and} \quad (\phi \otimes f)(\Delta(a)) = (\phi * f)(a) \quad (a \in A).$$

Recall that a linear map $\phi : A \to \mathbb{C}$ on an algebra A is called

- *faithful* if $\phi(aA) \neq 0$ and $\phi(Aa) \neq 0$ for every non-zero $a \in A$;
- *positive* if A is a $*$-algebra and $\phi(a^*a) \geq 0$ for all $a \in A$;
- *normalized* if A is unital and $\phi(1_A) = 1$.

Lemma 6.1.4. *(Cauchy-Schwarz Inequality) Suppose that A is a $*$-algebra and $\phi : A \to \mathbb{C}$ is a positive linear functional on A. Then*

$$|\phi(a^*b)|^2 \leq \phi(a^*a)\phi(b^*b) \qquad \text{for all } a, b \in A. \tag{6.1.2}$$

Proof. For ϕ, define $\langle \cdot, \cdot \rangle_\phi : A \times A \to \mathbb{C}$ by

$$\langle a, b\rangle_\phi = \phi(a^*b) \qquad (a, b \in A).$$

Then for all $a, b, c \in A$ and $\alpha, \beta \in \mathbb{C}$, we have

$$\langle \alpha a + \beta b, c\rangle_\phi = \phi((\alpha a + \beta b)^*c) = \bar{\alpha}\phi(a^*c) + \bar{\beta}\phi(b^*c) = \bar{\alpha}\langle a, c\rangle_\phi + \bar{\beta}\langle b, c\rangle_\phi;$$

$$\langle a, \alpha b + \beta c\rangle_\phi = \phi(a^*(\alpha b + \beta c)) = \alpha\phi(a^*b) + \beta\phi(a^*c) = \alpha\langle a, b\rangle_\phi + \beta\langle a, c\rangle_\phi;$$

$$\langle a, a\rangle_\phi = \phi(a^*a) \geq 0;$$

$$\langle a, b\rangle_\phi = \phi(a^*b) = \phi((b^*a)^*) = \overline{\phi(b^*a)} = \overline{\langle b, a\rangle_\phi}.$$

So, $\langle\cdot,\cdot\rangle_\phi : A \times A \to \mathbb{C}$ is a semi-inner product on A. Then

$$|\langle a, b\rangle_\phi|^2 \leq \langle a, a\rangle_\phi \langle b, b\rangle_\phi \qquad \text{for all } a, b \in A;$$

that is, $|\phi(a^*b)|^2 \leq \phi(a^*a)\phi(b^*b)$ for all $a, b \in A$. □

REMARK 6.1.5. Let (A, Δ) be a multiplier Hopf $*$-algebra, which is regular automatically by Remark 5.5.10(c). Let ϕ be a non-zero positive linear functional on A. We claim and prove the following statements.

(a) ϕ is $*$-linear.

(b) If A is unital, then there exists a real number $r > 0$ such that $r\phi$ is normalized.

(c) ϕ is faithful if and only if $\phi(a^*a) > 0$ for all non-zero $a \in A$.

PROOF. (a) Since A is non-degenerate, it suffices to show that $\phi(a^*b) = \overline{\phi(b^*a)}$ for all $a, b \in A$. Let $a, b \in A$. Then we have

$$\phi((a+b)^*(a+b)) = \phi(a^*a) + \phi(a^*b) + \phi(b^*a) + \phi(b^*b)$$

and

$$\phi((a+ib)^*(a+ib)) = \phi(a^*a) + i(\phi(a^*b) - \phi(b^*a)) + \phi(b^*b).$$

Since ϕ is positive, we have

$$\phi(a^*b) + \phi(b^*a) \in \mathbb{R} \quad \text{and} \quad \phi(a^*b) - \phi(b^*a) \in i\mathbb{R}.$$

Therefore, $\phi(a^*b) = \overline{\phi(b^*a)}$.

(b) The Cauchy-Schwarz inequality (6.1.2) implies that

$$0 < |\phi(a)|^2 = |\phi(1_A a)|^2 \leq \phi(1_A 1_A^*)\phi(a^*a) = \phi(1_A)\phi(a^*a),$$

where $a \in A$ and $\phi(a) \neq 0$. Then $\phi(1_A) > 0$. Let $r = \frac{1}{\phi(1_A)}$. Then $r\phi$ is normalized.

(c) Let $a \in A$. On the one hand, if $\phi(Aa) = 0$, then in particular, $\phi(a^*a) = 0$. On the other hand, if $\phi(a^*a) = 0$, from the Cauchy-Schwarz inequality (6.1.2), for all $b \in A$, we have

$$0 \leq |\phi(b^*a)|^2 \leq \phi(b^*b)\phi(a^*a) = 0,$$

and thus $\phi(b^*a) = 0$. Hence, $\phi(\cdot a)$ vanishes on A if and only if $\phi(a^*a) = 0$. Similarly, $\phi(a\cdot)$ vanishes on A if and only if $\phi(aa^*) = 0$. Therefore, the assertion holds. $\square$

Now we turn back to integrals. We first show that integrals are always faithful.

PROPOSITION 6.1.6. *Every left (right) integral on a regular multiplier Hopf algebra is faithful.*

PROOF. Let (A, Δ) be a regular multiplier Hopf algebra with a left integral ϕ. Suppose that $a, b, c \in A$ and $\phi(Aa) = 0$. Then

$$(id \otimes \phi)((c \otimes 1)\Delta(b)\Delta(a)) = c((id \otimes \phi)\Delta(ba)) = c\phi(ba) = 0$$

together with the surjectivity of T_2 implies that $(id \otimes \phi)((b \otimes c)\Delta(a)) = 0$. Applying Δ and then multiplying $d \otimes 1$ ($d \in A$), we get

$$(id \otimes id \otimes \phi)(((d \otimes 1)\Delta(b) \otimes c)(\Delta \otimes id)(\Delta(a))) = 0.$$

Again, due to the surjectivity of T_2, we have

$$\begin{aligned} 0 &= (id \otimes id \otimes \phi)((d \otimes b \otimes c)(\Delta \otimes id)(\Delta(a))) \\ &= (1 \otimes b)(id \otimes id \otimes \phi)((1 \otimes 1 \otimes c)(id \otimes \Delta)((d \otimes 1)\Delta(a))), \end{aligned}$$

and hence

$$(id \otimes id \otimes \phi)((1 \otimes 1 \otimes c)(id \otimes \Delta)((d \otimes 1)\Delta(a))) = 0.$$

Let $F \in A'$ and put $p = (F \otimes id)((d \otimes 1)\Delta(a))$. Then applying $F \otimes id$ to the above equality, we get $(id \otimes \phi)((1 \otimes c)\Delta(p)) = 0$. Pick any $q \in A$. Since c and p are arbitrary, we can replace $(1 \otimes c)\Delta(p)$ by $(1 \otimes c)\Delta(p)(1 \otimes q)$. Let $\Delta(p)(1 \otimes q) = \sum_i a_i \otimes b_i$, where a_i are linearly independent. Then

$$\sum_i \phi(cb_i)a_i = \sum_i (id \otimes \phi)((1 \otimes c)(a_i \otimes b_i)) = 0,$$

which implies that $\phi(cb_i) = 0$ for all i. For each i, replace the above c by $cS(a_i)$, and then take the sum to get

$$\sum_i \phi(cS(a_i)b_i) = \phi(c\sum_i S(a_i)b_i) = 0.$$

By (5.4.9),

$$c\sum_i S(a_i)b_i = m((c \otimes 1)(S \otimes id)(\Delta(p)(1 \otimes q))) = c\epsilon(p)q.$$

Then we get $\phi(cq)\epsilon(p) = 0$. Since A is non-degenerate and $\phi \neq 0$, we conclude that $\epsilon(p) = 0$. On the other hand,

$$\epsilon(p) = F((id \otimes \epsilon)((d \otimes 1)\Delta(a))) = F(da)$$

for all $d \in A$ and $F \in A'$. Thus $a = 0$. Similarly, $\phi(aA) = 0$ implies that $a = 0$. The assertion for right integrals can also be proven using the same arguments. □

Before we consider the uniqueness and existence of left integrals and right integrals, let us clarify the relation between these two notions.

Proposition 6.1.7.

(i) *Let (A, Δ) be a regular multiplier Hopf algebra with a left/right integral ϕ. Then $\phi \circ S/\phi \circ S^{-1}$ is a right/left integral on (A, Δ).*

Therefore, a regular multiplier Hopf algebra (A, Δ) has a left integral if and only if it has a right integral.

(ii) *On every regular Hopf algebra, there exists at most one normalized left/right integral ϕ, and this ϕ is simultaneously a right/left integral satisfying $\phi = \phi \circ S = \phi \circ S^{-1}$.*

Proof. (i) Let ϕ be a left integral on (A, Δ). Then for all $a, b, c \in A$,

$$\begin{aligned} S(((\phi \circ S) \otimes id)(\Delta(a)(1 \otimes b))) &= (\phi \otimes id)(S \otimes S)(\Delta(a)(1 \otimes b)) \\ &= (id \otimes \phi)((S(b) \otimes 1)((S \otimes S) \circ \Sigma \circ \Delta)(a)) \\ &= (id \otimes \phi)((S(b) \otimes 1)\Delta(S(a))) \\ &= \phi(S(a))S(b). \end{aligned}$$

An application of S^{-1} shows that $((\phi\circ S)\otimes id)\Delta(a) = \phi(S(a))1_{M(A)}$, and thus $\phi\circ S$ is right invariant. By Proposition 5.5.7, the antipode S is a linear isomorphism of A, whence $\phi \circ S \neq 0$. Therefore, $\phi \circ S$ is a right integral on (A, Δ).

Similarly, we can show that if ϕ is a right integral on (A, Δ), then $\phi \circ S^{-1}$ is a left integral on (A, Δ).

(ii) Let (A, Δ) be a regular Hopf algebra with a normalized left integral ϕ. By (i), $\psi := \phi \circ S$ is a normalized right integral. For every normalized left integral $\widetilde{\phi}$, we have $\widetilde{\phi} = \psi$, because

$$\widetilde{\phi}(a) = \psi(1_A)\widetilde{\phi}(a) = (\psi \otimes \widetilde{\phi})(\Delta(a)) = \psi(a)\widetilde{\phi}(1_A) = \psi(a) \quad \text{for all } a \in A.$$

The proof for the uniqueness of a normalized right integral is similar. $\square$

For multiplier Hopf $*$-algebras (which are automatically regular), it is natural to consider positivity of integrals. Unfortunately, the correspondence between left integrals and right integrals obtained in Proposition 6.1.7 need not preserve positivity. However, we still have the following proposition, which motivates the concept of algebraic quantum group.

Proposition 6.1.8. ([**4**, Theorem 9.9]) *A multiplier Hopf $*$-algebra has a positive left integral if and only if it has a positive right integral.*

Definition 6.1.9. An *algebraic quantum group* is a multiplier Hopf $*$-algebra with a positive left integral and a positive right integral.

6.2. Existence and uniqueness

It is know that left or right integrals do not always exist on a regular multiplier Hopf algebra (cf. [**2**, Section 5.3]). However, it is possible to formulate extra assumptions on the pair (A, Δ) to assure their existence. For example, they always exist in the finite-dimensional case ([**9**, Proposition 5.1]).

In contrast to existence, uniqueness of integrals already holds for all regular multiplier Hopf algebras. The first main step towards the proof of uniqueness is the following result.

PROPOSITION 6.2.1. *Let (A, Δ) be a regular multiplier Hopf algebra with a left integral ϕ and a right integral ψ. Then*

$$\{\psi(\cdot a)|a \in A\} = \{\phi(\cdot a)|a \in A\} \quad \textit{and} \quad \{\psi(a\cdot)|a \in A\} = \{\phi(a\cdot)|a \in A\}.$$

PROOF. We only show the first " $\supseteq''$; other inclusions can be shown in similar ways. Let $a \in A$. It suffices to show that there exists $c \in A$ such that $\phi(\cdot a) = \psi(\cdot c)$. Choose $b \in A$ with $\psi(b) = 1$. The left invariance of ϕ implies that

$$\phi(xa) = \psi(b)\phi(xa) = (\psi \otimes \phi)(\Delta(x)\Delta(a)(b \otimes 1)) \quad \text{for all } x \in A.$$

By the surjectivity of T_1 and the regularity of (A, Δ), we can write $\Delta(a)(b \otimes 1) = \sum_i \Delta(c_i)(1 \otimes d_i)$ with $c_i, d_i \in A$. Then for all $x \in A$, we have

$$\phi(xa) = \sum_i (\psi \otimes \phi)(\Delta(x)\Delta(c_i)(1 \otimes d_i)) = \sum_i \psi(xc_i)\phi(d_i) = \psi(xc),$$

where $c = \sum_i c_i\phi(d_i) \in A$. $\square$

THEOREM 6.2.2. *Let (A, Δ) be a regular multiplier Hopf algebra with integrals. Then A has a unique (up to a scalar) left integral and a unique (up to a scalar) right integral.*

PROOF. Suppose that ϕ_1 and ϕ_2 are two left integrals, and ψ is a right integral on (A, Δ). Let $a, b, x \in A$ with $\psi(ab) = 1$. We can choose $c_i, y_i \in A$ such that $\sum_i (c_i \otimes 1)\Delta(y_i) = (1 \otimes x)\Delta(a)$ due to the surjectivity of T_2 and the regularity of (A, Δ). Then

$$\begin{aligned} \phi_1(x) &= \psi(ab)\phi_1(x) = (\psi \otimes \phi_1)((1 \otimes x)\Delta(a)\Delta(b)) \\ &= (\psi \otimes \phi_1)(\sum_i (c_i \otimes 1)\Delta(y_i)\Delta(b)) = \sum_i \psi(c_i)\phi_1(y_i b). \end{aligned} \tag{6.2.1}$$

By Proposition 6.2.1, there exists $d \in A$ such that $\phi_1(\cdot b) = \phi_2(\cdot d)$. Following the argument given in (6.2.1), we have

$$\begin{aligned} \psi(ad)\phi_2(x) &= (\psi \otimes \phi_2)(\sum_i (c_i \otimes 1)\Delta(y_i)\Delta(d)) \\ &= \sum_i \psi(c_i)\phi_2(y_i d) = \sum_i \psi(c_i)\phi_1(y_i b) = \phi_1(x). \end{aligned}$$

Therefore, $\phi_1 = \psi(ad)\phi_2$; that is, A has a unique (up to a scalar) left integral.

From Proposition 6.1.7, we know that $\psi \circ S^{-1}$ is a left integral, and $\phi_2 \circ S$ is a right integral. Replacing the above ϕ be $\psi \circ S^{-1}$, we conclude that there exists $\lambda \in \mathbb{C}$ such that

$$\psi \circ S^{-1} = \lambda\phi_2, \qquad \text{or equivalently,} \qquad \psi = \lambda(\phi_2 \circ S).$$

Therefore, A has a unique (up to a scalar) right integral. $\square$

COROLLARY 6.2.3. *Let (A, Δ) be a multiplier Hopf $*$-algebra with a positive left integral ϕ. Then there exists a number $z \in \mathbb{C}$ with $|z| = 1$ such that the right integral $z\phi \circ S$ is positive.*

PROOF. From Proposition 6.1.8, there exists a right positive integral, say ψ, on (A, Δ). Meanwhile, $\phi \circ S$ is a right integral by Proposition 6.1.7(i). The uniqueness of right integrals (cf. Theorem 6.2.2) implies that there exists $0 \neq \lambda \in \mathbb{C}$ such that $\psi = \lambda\phi \circ S$. Put $z = \lambda/|\lambda|$. Then $|z| = 1$, and $z\phi \circ S = (1/|\lambda|)\psi$ is clearly a positive right integral. $\square$

6.3. The modular element

If (A, Δ) is a regular Hopf algebra with a left integral ϕ such that $\phi(1) = 1$, then by Proposition 6.1.7(ii) and Theorem 6.2.2, every left or right integral is a scalar multiple of ϕ. So, in this case, left and right integrals coincide (up to a scalar). In general situation, left and right integrals are related by means of a multiplier of A, called the modular element of (A, Δ).

In fact, most results on the modular element are similar to those on the modular function of a locally compact group, which relates left Haar measures and right Haar measures. So, let us take a quick review of the locally compact group case.

Let G be a locally compact group with a left Haar measure λ and denote by $i : G \to G$ the inversion $x \mapsto x^{-1}$. Then λ^{-1} is a right invariant measure, where $\lambda^{-1}(E) = \lambda(i(E))$. In this case, there exists a strictly positive continuous function δ_G on G, called the *modular function of* G, such that

$$\int_G f(y)d\lambda(y) = \int_G f(y^{-1})\delta_G(y^{-1})d\lambda(y) = \int_G f(z)\delta_G(z)d\lambda^{-1}(z) \qquad (6.3.1)$$

for all positive λ-measurable functions f on G. The function δ_G is a homomorphism from G to the multiplicative group of positive real numbers; that is, for all $x, y \in G$, we have

$$\delta_G(xy) = \delta_G(x)\delta_G(y), \quad \delta_G(e) = 1, \quad \text{and} \quad \delta_G(x^{-1}) = \delta_G(x)^{-1}, \tag{6.3.2}$$

where $e \in G$ denotes the unit of G.

We can translate these in the language of integrals on regular multiplier Hopf algebras. Suppose that $A \subseteq L^1(G)$ is a multiplier Hopf algebra with structure maps as given in (3.2.3), and that ϕ is the left integral (6.1.1) on (A, Δ) given by λ. Then equation (6.3.1) takes the form

$$\phi(f) = (\phi \circ S)(f\delta_G) \qquad (f \in A). \tag{6.3.3}$$

Here, we assume for the moment that δ_G belongs to $M(A)$. Then the relations in (6.3.2) can be written as

$$\Delta(\delta_G) = \delta_G \otimes \delta_G, \quad \epsilon(\delta_G) = 1, \quad \text{and} \quad S(\delta_G) = \delta_G^{-1}.$$

As shown below, every regular multiplier Hopf algebra with integrals has a modular element δ that plays a similar role as δ_G does. We need to notice a small difference here. The modular element δ for a regular multiplier Hopf algebra coresponds to δ_G^{-1} rather than δ_G (comparing (6.3.3) with (6.3.8)). First, we prove the existence of this multiplier.

PROPOSITION 6.3.1. *Let (A, Δ) be a regular multiplier Hopf algebra with a left integral ϕ. There exists a multiplier $\delta \in M(A)$ such that*

$$(\phi \otimes id)\Delta(a) = \phi(a)\delta \qquad \textit{for all } a \in A. \tag{6.3.4}$$

PROOF. To simplify the notation, we assume that (A, Δ) is a regular Hopf algebra. First we show that $(\phi \otimes id)\Delta(\ker \phi) = \{0\}$. Let $a \in \ker \phi$ and $\omega \in A'$. Then $\phi * \omega \in A'$ is also left invariant, since

$$\begin{aligned}(f * (\phi * \omega))(b) &= ((f * \phi) * \omega)(b) = \sum (f * \phi)(b_{(1)})\omega(b_{(2)}) \\ &= \sum f(1_A)\phi(b_{(1)})\omega(b_{(2)}) = f(1_A)(\phi * \omega)(b)\end{aligned}$$

for all $f \in A'$ and $b \in A$ (cf. Remark 6.1.3(ii)). By Theorem 6.2.2, $\phi * \omega = \lambda_\omega \phi$ for some $\lambda_\omega \in \mathbb{C}$, which is also true when $\phi * \omega = 0$. It follows that

$$\omega((\phi \otimes id)\Delta(a)) = (\phi * \omega)(a) = \lambda_\omega \phi(a) = 0 \quad \text{for all } \omega \in A'.$$

Therefore, $(\phi \otimes id)\Delta(a) = 0$.

Next, we pick $a_0 \in A$ with $\phi(a_0) = 1$. Clearly, $a - \phi(a_0)a \in \ker \phi$ for any $a \in A$. Then we have

$$(\phi \otimes id)\Delta(a) - \phi(a)(\phi \otimes id)\Delta(a_0) = (\phi \otimes id)\Delta(a - \phi(a)a_0) = 0 \quad (a \in A).$$

Finally, let $\delta = (\phi \otimes id)\Delta(a_0) \in M(A)$. Then $(\phi \otimes id)\Delta(a) = \phi(a)\delta$ for all $a \in A$. $\square$

It is seen from Theorem 6.2.2 that $\delta \in M(A)$ is independent of ϕ, we call δ the *modular element* of (A, Δ). Now, it is not hard to find the behaviors of Δ, ϵ, and S on δ.

PROPOSITION 6.3.2. *Let (A, Δ) be a regular multiplier Hopf algebra with integrals. Then $\delta \in M(A)$ is invertible and*

$$\Delta(\delta) = \delta \otimes \delta, \quad \epsilon(\delta) = 1, \quad \textit{and} \quad S(\delta) = \delta^{-1}. \tag{6.3.5}$$

Here, Δ, ϵ, and S have been extended to the multiplier algebra $M(A)$.

PROOF. Let ϕ be a left integral on (A, Δ). Applying Δ and ϵ to (6.3.4), respectively, we get

$$\begin{aligned}\Delta(\phi(a)\delta) &= \sum \phi(a_{(1)})a_{(2)} \otimes a_{(3)} = \sum \phi(a_{(1)})\delta \otimes a_{(2)} \\ &= \delta \otimes (\sum \phi(a_{(1)})a_{(2)}) = \delta \otimes ((\phi \otimes id)\Delta(a)) = \phi(a)(\delta \otimes \delta)\end{aligned}$$

and

$$\epsilon(\phi(a)\delta) = \sum \phi(a_{(1)})\epsilon(a_{(2)}) = \sum \phi(a_{(1)}\epsilon(a_{(2)})) = \phi(a)$$

for all $a \in A$. Consequently, $\Delta(\delta) = \delta \otimes \delta$ and $\epsilon(\delta) = 1$. Furthermore,

$$S(\delta)\delta = m((S \otimes id)(\delta \otimes \delta)) = m((S \otimes id)\Delta(\delta)) = \epsilon(\delta)1_{M(A)} = 1_{M(A)}$$

and similarly $\delta S(\delta) = 1_{M(A)}$; in particular, δ is invertible in $M(A)$ with $\delta^{-1} = S(\delta)$. $\square$

REMARK 6.3.3. Let (A, Δ), ϕ, and δ be the same as above. By (6.3.5), we get

$$\Delta(\delta^{-1}) = \Delta(\delta)^{-1} = \delta^{-1} \otimes \delta^{-1}$$

and

$$S(\delta^{-1})\delta^{-1} = S(\delta^{-1})S(\delta) = S(1_{M(A)}) = 1_{M(A)} = S(\delta)S(\delta^{-1}) = \delta^{-1}S(\delta^{-1}).$$

Therefore, we have

$$\Delta(\delta^{-1}) = \delta^{-1} \otimes \delta^{-1} \qquad \text{and} \qquad S(\delta^{-1}) = \delta. \tag{6.3.6}$$

Now we consider the right integral version of equality (6.3.4). From (6.3.4), we have

$$\phi(S(a))\delta = (\phi \otimes id)\Delta(S(a)) = (S \otimes (\phi \circ S))\Delta(a)$$

for any $a \in A$. Applying S^{-1} to the above equality, we get

$$(id \otimes (\phi \circ S))\Delta(a) = \phi(S(a))S^{-1}(\delta) = \phi(S(a))\delta^{-1},$$

noticing that $S^{-1}(\delta) = \delta^{-1}$ (cf. (6.3.6)). Note that $\phi \circ S$ is a right integral, which is unique up to a scalar. We conclude that if ψ is a right integral on (A, Δ), then

$$(id \otimes \psi)\Delta(a) = \psi(a)\delta^{-1} \qquad \text{for all } a \in A. \tag{6.3.7}$$

The following lemma is needed to obtain the main result (Proposition 6.3.5) of this section.

LEMMA 6.3.4. ([**8**, Lemma 5.5]) *Let (A, Δ) be a regular multiplier Hopf algebra. Suppose that a, b, a_i, and b_i are elements of A. Then the followings are equivalent.*

(i) $\Delta(a)(1 \otimes b) = \sum_i \Delta(a_i)(b_i \otimes 1)$;
(ii) $a \otimes S^{-1}(b) = \sum_i (a_i \otimes 1)\Delta(b_i)$;
(iii) $(1 \otimes a)\Delta(b) = \sum_i S(b_i) \otimes a_i$.

PROPOSITION 6.3.5. *Let (A, Δ), ϕ, and δ be the same as in Proposition 6.3.1. Then*

$$\phi(S(a)) = \phi(a\delta) \qquad \text{for all } a \in A. \tag{6.3.8}$$

PROOF. If $a, b \in A$ and

$$\Delta(a)(1 \otimes b) = \sum_i \Delta(a_i)(b_i \otimes 1) \tag{6.3.9}$$

for some $a_i, b_i \in A$, then by Lemma 6.3.4, we have

$$(1 \otimes a)\Delta(b) = \sum_i S(b_i) \otimes a_i. \tag{6.3.10}$$

Applying $(\phi \circ S) \otimes \phi$ to (6.3.9), we obtain

$$\begin{aligned}\sum_i \phi(S(b_i))\phi(a_i) &= \sum_i ((\phi \circ S) \otimes \phi)(\Delta(a_i)(b_i \otimes 1)) \\ &= ((\phi \circ S) \otimes \phi)(\Delta(a)(1 \otimes b)) = \phi(S(a))\phi(b).\end{aligned}$$

We also get, by applying $\phi \otimes \phi$ to (6.3.10) and using (6.3.4), that

$$\begin{aligned}\sum_i \phi(S(b_i))\phi(a_i) &= (\phi \otimes \phi)((1 \otimes a)\Delta(b)) = \phi((\phi \otimes id)((1 \otimes a)\Delta(b))) \\ &= \phi(a((\phi \otimes id)\Delta(b))) = \phi(a\phi(b)\delta) = \phi(a\delta)\phi(b).\end{aligned}$$

Therefore, $\phi(S(a)) = \phi(a\delta)$ for all $a \in A$. □

The following corollary is immediate by Theorem 6.2.2 and Proposition 6.3.5, which shows that δ/δ^{-1} is the multiplier that expresses a right/left integral in terms of a left/right integral.

COROLLARY 6.3.6. *Let (A, Δ) be a regular multiplier Hopf algebra with integrals and let $\delta \in M(A)$ be the modular element. Then for any left integral ϕ on (A, Δ) and any right integral ψ on (A, Δ), there exist scalars α and β with $\alpha\beta = 1$ such that $\psi = \alpha\phi(\cdot\delta)$ and $\phi = \beta\psi(\cdot\delta^{-1})$.*

REMARK 6.3.7. For all $a \in A$, applying (6.3.6) and (6.3.8), we obtain

$$\phi(S^2(a)) = \phi(S(a)\delta) = \phi(S(a)S(\delta^{-1})) = \phi(S(\delta^{-1}a)) = \phi(\delta^{-1}a\delta).$$

Then $\phi \circ S^2$ is also a left integral on (A, Δ), since $\phi \circ S^2 = \phi(\delta^{-1} \cdot \delta) \neq 0$ and

$$(id \otimes (\phi \circ S^2))\Delta(a) = (id \otimes \phi(\delta^{-1} \cdot \delta))\Delta(a) = \delta((id \otimes \phi)\Delta(\delta^{-1} a \delta))\delta^{-1}$$
$$= \phi(\delta^{-1} a \delta)\delta 1_{M(A)} \delta^{-1} = (\phi \circ S^2)(a) 1_{M(A)}$$

for all $a \in A$ (cf. (6.3.6)). By uniqueness, there is a complex number τ so that $\phi \circ S^2 = \tau\phi$. Therefore,

$$\phi(\delta^{-1} a \delta) = \tau\phi(a) \qquad \text{for all } a \in A.$$

In general, we do not have $\tau = 1$ though it holds in the following two cases.

- If A is commutative, it is obvious that $\tau = 1$.
- If A is cocommutative, then we have

  $$\phi(a) 1_{M(A)} = (id \otimes \phi)\Delta(a) = (\phi \otimes id)\Delta(a) = \phi(a)\delta,$$

  which implies that $\delta = 1_{M(A)}$, and hence $\tau = 1$.

6.4. The modular automorphism

In general, an integral ϕ on a regular multiplier Hopf algebra (A, Δ) need not be tracial in the sense that $\phi(ab) = \phi(ba)$ for all $a, b \in A$. However, there exists an automorphism σ of A satisfying $\phi(ab) = \phi(b\sigma(a))$ for all $a, b \in A$.

The construction of this modular automorphism proceeds in several steps. We start with the following important lemma.

Lemma 6.4.1. *Let (A, Δ) be a regular multiplier Hopf algebra with a left integral ϕ and a right integral ψ. Then for all $a, b \in A$, we have*

$$(S \otimes \phi)(\Delta(a)(1 \otimes b)) = (id \otimes \phi)((1 \otimes a)\Delta(b))$$

and

$$(\psi \otimes S)((a \otimes 1)\Delta(b)) = (\psi \otimes id)(\Delta(a)(b \otimes 1)).$$

In extended Sweedler notation, these formulas can be rewritten as

$$\sum S(a_{(1)})\phi(a_{(2)}b) = \sum b_{(1)}\phi(ab_{(2)}) \quad \text{and} \quad \sum \psi(ab_{(1)})S(b_{(2)}) = \sum \psi(a_{(1)}b)a_{(2)}.$$

Proof. Let $a, b \in A$ with

$$\Delta(a)(1 \otimes b) = \sum_i \Delta(a_i)(b_i \otimes 1)$$

for some $a_i, b_i \in A$. Applying $S \otimes \phi$ to the above equation and recalling from Lemma 6.3.4 that

$$(1 \otimes a)\Delta(b) = \sum_i S(b_i) \otimes a_i,$$

we get

$$\begin{aligned}(S \otimes \phi)(\Delta(a)(1 \otimes b)) &= \sum_i S((id \otimes \phi)(\Delta(a_i)(b_i \otimes 1))) = \sum_i S(\phi(a_i)b_i) \\ &= \sum_i (id \otimes \phi)(S(b_i) \otimes a_i) = (id \otimes \phi)((1 \otimes a)\Delta(b)).\end{aligned}$$

Similarly, we have that $(\psi \otimes S)((a \otimes 1)\Delta(b)) = (\psi \otimes id)(\Delta(a)(b \otimes 1))$. □

Proposition 6.4.2. *Let ϕ be a left integral on a regular multiplier Hopf algebra (A, Δ). Then the subspaces $\{\phi(\cdot a) | a \in A\}$ and $\{\phi(a \cdot) | a \in A\}$ of A' coincide.*

Proof. Take $c \in A$ with $\phi(S(c)) = 1$. Given $a \in A$, write

$$a \otimes c = \sum_i \Delta(a_i)(1 \otimes S^{-1}(c_i))$$

for some $a_i, c_i \in A$. Applying $id \otimes (\phi \circ S)$, we get

$$a = \sum_i (id \otimes \phi)((1 \otimes c_i)(id \otimes S)\Delta(a_i)) = \sum_i \sum a_{i(1)} \phi(c_i S(a_{i(2)})).$$

Let ψ be a right integral on (A, Δ) and $x \in A$. By Lemma 6.4.1, we have

$$\begin{aligned}\psi(xa) &= \sum_i \sum \psi(x a_{i(1)}) \phi(c_i S(a_{i(2)})) = \sum_i \sum \phi(c_i \psi(x a_{i(1)}) S(a_{i(2)})) \\ &= \sum_i \sum \phi(c_i \psi(x_{(1)} a_i) x_{(2)}) = \sum_i \sum \psi(x_{(1)} a_i) \phi(c_i x_{(2)}) \\ &= \sum_i \sum \psi(x_{(1)} \phi(c_i x_{(2)}) a_i) = \sum_i \sum \psi(S(c_{i(1)}) \phi(c_{i(2)} x) a_i) \\ &= \sum_i \sum \psi(S(c_{i(1)}) a_i) \phi(c_{i(2)} x) = \phi(bx),\end{aligned}$$

where $b = \sum_i \sum \psi(S(c_{i(1)}) a_i) c_{i(2)}$. By Proposition 6.2.1, we have $\{\phi(\cdot a) | a \in A\} \subseteq \{\phi(a \cdot) | a \in A\}$. The reverse inclusion can be shown similarly. □

REMARK 6.4.3. The element b in the proof of Proposition 6.4.2 is unique by the faithfulness of ϕ (cf. Proposition 6.1.6). By Proposition 6.2.1, we have the right side version of Proposition 6.4.2 for any right integral ψ on (A, Δ); in fact, we have

$$\{\phi(\cdot a) | a \in A\} = \{\phi(a \cdot) | a \in A\} = \{\psi(\cdot a) | a \in A\} = \{\psi(a \cdot) | a \in A\}. \qquad (6.4.1)$$

In the next chapter, this set of functionals on A will be defined as the dual $\hat{A}$ of A.

Let us now elaborate a little more on the above result.

THEOREM 6.4.4. *Let (A, Δ) be a regular multiplier Hopf algebra with a left integral ϕ and a right integral ψ. Then there exists two automorphisms σ and σ' of A such that*

$$\phi(ax) = \phi(x\sigma(a)) \quad \textit{and} \quad \psi(ax) = \psi(x\sigma'(a)) \quad (a, x \in A) \qquad (6.4.2)$$

Furthermore, ϕ is σ invariant and ψ is σ' invariant; that is,

$$\phi \circ \sigma = \phi \qquad \textit{and} \qquad \psi \circ \sigma' = \psi.$$

PROOF. We only prove the left integral case; the right integral case can be proved similarly. By (6.4.1), for every $a \in A$, there exists a unique $\sigma(a) \in A$ such that $\phi(ax) = \phi(x\sigma(a))$ for all $x \in A$. Consider the map $\sigma : A \to A$, $a \mapsto \sigma(a)$. For all $a, b, x \in A$ and $k \in \mathbb{C}$, we have

$$\phi(x\sigma(ab)) = \phi(abx) = \phi(bx\sigma(a)) = \phi(x\sigma(a)\sigma(b))$$

and

$$\begin{aligned} \phi(x\sigma(ka+b)) &= \phi((ka+b)x) = \phi(kax) + \phi(bx) \\ &= \phi(xk\sigma(a)) + \phi(x\sigma(b)) = \phi(x(k\sigma(a) + \sigma(b))). \end{aligned}$$

Thus $\sigma : A \to A$ is injective, linear, and multiplicative, since ϕ is linear and faithful. Also, $\sigma : A \to A$ is surjective by Proposition 6.4.2. Therefore, σ is an automorphism of A.

Furthermore, for all $a, b \in A$, we have

$$\phi(ab) = \phi(b\sigma(a)) = \phi(\sigma(a)\sigma(b)) = \phi(\sigma(ab)).$$

Therefore, $\phi = \phi \circ \sigma$, since $A = \text{span } AA$. □

In the rest of this section, we always assume that (A, Δ), ϕ, ψ, σ, and σ' are the same as in Theorem 6.4.4. Due to the uniqueness of left integrals and right integrals (up to a scalar), it is seen that σ and σ' are independent of ϕ and ψ. We call the pair σ and σ' the *modular automorphisms* of (A, Δ). We have the following results on the interrelationships between σ, σ', S, and Δ.

PROPOSITION 6.4.5. *The modular automorphisms σ and σ' of (A, Δ) satisfy*

$$S \circ \sigma' = \sigma^{-1} \circ S.$$

PROOF. For all $a, b \in A$, we have

$$\begin{aligned} \phi(S(b)\sigma(S(\sigma'(a)))) &= \phi(S(\sigma'(a))S(b)) = (\phi \circ S)(b\sigma'(a)) \\ &= (\phi \circ S)(ab) = \phi(S(b)S(a)). \end{aligned}$$

This implies that $\sigma \circ S \circ \sigma' = S$, or equivalently, $S \circ \sigma' = \sigma^{-1} \circ S$. □

PROPOSITION 6.4.6. *For all $a \in A$, we have*

$$\Delta(\sigma(a)) = (S^2 \otimes \sigma)\Delta(a) \qquad \textit{and} \qquad \Delta(\sigma'(a)) = (\sigma' \otimes S^{-2})\Delta(a).$$

PROOF. We prove the first equality. Since functionals of the form $\phi(b\cdot)$ $(b \in A)$ separate elements of A, it is enough to show that for all such functions $\omega := \phi(b\cdot)$ and all $a \in A$,

$$(id \otimes \omega)\Delta(\sigma(a)) = (id \otimes \omega)((S^2 \otimes \sigma)\Delta(a));$$

that is,

$$(id \otimes \phi)((1 \otimes b)\Delta(\sigma(a))) = (id \otimes \phi)((1 \otimes b)(S^2 \otimes \sigma)\Delta(a))$$

for all $b \in A$. Applying Lemma 6.4.1 twice shows that

$$\begin{aligned} &(id \otimes \phi)((1 \otimes b)\Delta(\sigma(a))) = (S \otimes \phi)(\Delta(b)(1 \otimes \sigma(a))) \\ &= (S \otimes \phi)((1 \otimes a)\Delta(b)) = (S^2 \otimes \phi)(\Delta(a)(1 \otimes b)) = (id \otimes \phi)((1 \otimes b)(S^2 \otimes \sigma)\Delta(a)) \end{aligned}$$

for all $a, b \in A$. In a similar way, or using the relation $S \circ \sigma' = \sigma^{-1} \circ S$, we can get the second equality. $\square$

we know from Remark 6.3.3 that there is a unique complex number τ satisfying $\phi \circ S^2 = \tau\phi$ for all left integrals ϕ on (A, Δ). We will use this number τ below to consider the relations between the modular automorphisms σ and σ', the antipode S, and the modular element δ.

PROPOSITION 6.4.7. *Both σ and σ' commute with S^2.*

PROOF. For all $a, b \in A$, on the one hand, we have

$$\phi(S^2(a)b) = \phi(b\sigma(S^2(a))),$$

and on the other hand, we have also

$$\begin{aligned}\phi(S^2(a)b) &= \phi(S^2(aS^{-2}(b))) = \tau\phi(aS^{-2}(b)) = \tau\phi(S^{-2}(b)\sigma(a)) \\ &= \phi(S^2(S^{-2}(b)\sigma(a))) = \phi(bS^2(\sigma(a))).\end{aligned}$$

Since ϕ is faithful, we conclude that $\sigma(S^2(a)) = S^2(\sigma(a))$ for all $a \in A$; that is, $\sigma \circ S^2 = S^2 \circ \sigma$. Similarly, we can show that $\sigma' \circ S^2 = S^2 \circ \sigma'$. $\square$

By Proposition 5.1.4(vi), the modular automorphism $\sigma : A \to A$ ($\subseteq M(A)$) can be extended to an algebra homomorphism $\tilde{\sigma} : M(A) \to M(A)$. From the bijectivity of σ, it is seen that $\tilde{\sigma}$ is also an automorphism on $M(A)$ (cf. Remark 5.1.5(vi)). However, to simplify notation, we use σ to denote its extension to $M(A)$, and the extension of σ' to $M(A)$ is still denoted as σ'.

PROPOSITION 6.4.8. *We have*

$$\sigma(\delta) = \sigma'(\delta) = \tau^{-1}\delta \quad \textit{and} \quad \delta\sigma(x) = \sigma'(x)\delta \quad \textit{for all } x \in M(A).$$

PROOF. For all $a, b \in A$, we have

$$\begin{aligned}\tau\phi(ab) &= \phi(S^2(ab)) = \phi(\delta^{-1}ab\delta) = \phi(b\delta\sigma(\delta^{-1}a)) \\ &= \phi(b\delta\sigma(\delta^{-1})\sigma(a)) = \phi(ab\delta\sigma(\delta^{-1})).\end{aligned}$$

This shows that $b\delta\sigma(\delta^{-1}) = \tau b$ for all $b \in A$; that is, $\delta\sigma(\delta^{-1}) = \tau 1_{M(A)}$. Similarly, we have $\delta\sigma'(\delta^{-1}) = \tau 1_{M(A)}$. These imply that $\sigma(\delta) = \sigma'(\delta) = \tau^{-1}\delta$.

Meanwhile, for all $a, b \in A$, we have

$$\phi(b\delta\sigma(a)) = \phi(ab\delta) = \phi(S(ab)) = \phi(S(b\sigma'(a))) = \phi(b\sigma'(a)\delta).$$

Therefore, $\delta\sigma(a) = \sigma'(a)\delta$ for all $a \in A$. Due to Remark 5.1.5(vi), it follows that for all $x \in M(A)$ and $a, b \in A$, we have

$$\sigma'(a)(\delta\sigma(x))\sigma(b) = \sigma'(a)(\sigma'(x)\delta)\sigma(b),$$

and hence $\delta\sigma(x) = \sigma'(x)\delta$. $\square$

If we apply ϵ to the last relation in Proposition 6.4.8, it is easy to get that $\epsilon \circ \sigma = \epsilon \circ \sigma'$, since $\epsilon(\delta) = 1$. It is also interesting to see that

$$\sigma' = S^{-1} \circ \sigma^{-1} \circ S = \delta\sigma(\cdot)\delta^{-1}.$$

CHAPTER 7

The duality of multiplier Hopf algebras

Let (A, Δ) be a regular multiplier Hopf algebra with integrals. In this chapter, we show that one can associate to (A, Δ) a regular multiplier Hopf algebra $(\hat{A}, \hat{\Delta})$ with integrals, called the dual of (A, Δ), so that the bidual $(\hat{\hat{A}}, \hat{\hat{\Delta}})$ is naturally isomorphic to (A, Δ). The main reference of this chapter is [**9**].

7.1. The duality of regular multiplier Hopf algebras

Throughout this section, (A, Δ) will be a regular multiplier Hopf algebra with a left integral ϕ and a right integral ψ. Put

$$\hat{A} := \{\phi(\cdot a) | a \in A\} \subseteq A'.$$

Then from Remark 6.4.3, we know that

$$\hat{A} = \{\phi(a\cdot) | a \in A\} = \{\psi(a\cdot) | a \in A\} = \{\psi(\cdot a) | a \in A\},$$

and $\hat{A}$ does not depend on the choice of ϕ and ψ.

Obviously, if A is unital, then ϕ and ψ are both in $\hat{A}$. In fact, the other direction is also true. More precisely, we have the following proposition.

PROPOSITION 7.1.1. *Let (A, Δ) be a regular multiplier Hopf algebra algebra with a left integral ϕ and a right integral ψ. Then the following statements are equivalent.*

(i) *A is unital;*
(ii) *$\phi \in \hat{A}$;*
(iii) *$\psi \in \hat{A}$.*

PROOF. (i) $\Rightarrow$ (ii). If A has a unit 1_A, then $\phi(a) = \phi(a1_A)$ for all $a \in A$, which implies that $\phi = \phi(\cdot 1_A) \in \hat{A}$.

(ii) $\Rightarrow$ (i). If $\phi \in \hat{A}$, then there exists $e \in A$ such that $\phi(a) = \phi(ae)$ for all $a \in A$. Since A is non-degenerate and ϕ is faithful, we have $a = ae$ for all $a \in A$.

Similarly, we can find $e' \in A$ with $a = e'a$ for all $a \in A$. Therefore, e is the unit of A, noticing that $e = e'e = e'$.

(i) $\Leftrightarrow$ (iii). This can be shown similarly. $\square$

We will equip $\hat{A}$ with the multiplication and the comultiplication that are dual to the comultiplication and the multiplication of (A, Δ), respectively. Let us first make $\hat{A}$ into an associative algebra in the usual way.

Given elements $\omega_1 = \phi(\cdot a_1)$ and $\omega_2 = \phi(\cdot a_2)$ of $\hat{A}$, we define a linear map $\omega_1\omega_2 : A \to \mathbb{C}$ by the formula

$$(\omega_1\omega_2)(x) = (\omega_1 \otimes \omega_2)\Delta(x) = (\phi \otimes \phi)(\Delta(x)(a_1 \otimes a_2)) \quad (x \in A). \tag{7.1.1}$$

Proposition 7.1.2. *Equation* (7.1.1) *defines a non-degenerate multiplication*

$$\hat{m} : \; \hat{A} \otimes \hat{A} \to \hat{A}, \; \omega_1 \otimes \omega_2 \mapsto \omega_1\omega_2$$

on $\hat{A}$ that makes $\hat{A}$ into an associative algebra over $\mathbb{C}$.

Proof. Let $\omega_1 = \phi(\cdot a_1)$ and $\omega_2 = \phi(\cdot a_2)$ as above. First of all, we need to show that $\omega_1\omega_2 \in \hat{A}$. Since (A, Δ) is a regular multiplier Hopf algebra, we can write $a_1 \otimes a_2 = \sum_i \Delta(c_i)(d_i \otimes 1)$ with $c_i, d_i \in A$ (cf. Corollary 5.5.9). Then since ϕ is left invariant, for all $x \in A$, we get

$$(\omega_1\omega_2)(x) = \sum_i (\phi \otimes \phi)(\Delta(xc_i)(d_i \otimes 1)) = \sum_i \phi(xc_i)\phi(d_i) = \phi(xb),$$

where $b = \sum_i \phi(d_i)c_i$. Hence, $\omega_1\omega_2 = \phi(\cdot b) \in \hat{A}$. The associativity of the multiplication on $\hat{A}$ follows from the coassociativity of Δ.

Moreover, since $\phi \neq 0$, there exists $b \in A$ with $\phi(b) = 1$. Let $\omega = \phi(\cdot a) \in \hat{A}$. By the regularity of (A, Δ), we can write $\Delta(a)(b \otimes 1) = \sum_i a_i^1 \otimes a_i^2$ for some $a_i^1, a_i^2 \in A$. Then

$$\begin{aligned} \omega(x) = \phi(b)\phi(xa) &= (\phi \otimes \phi)(\Delta(x)\Delta(a)(b \otimes 1)) \\ &= \sum_i (\phi \otimes \phi)(\Delta(x)(a_i^1 \otimes a_i^2)) = \sum_i (\omega_i^1\omega_i^2)(x) \end{aligned}$$

for all $x \in A$, where $\omega_i^1 = \phi(\cdot a_i^1)$ and $\omega_i^2 = \phi(\cdot a_i^2)$. This implies that $\omega = \sum_i \omega_i^1\omega_i^2$. So, $\hat{A}\hat{A}$ spans $\hat{A}$.

To complete the proof, we only need to show that the multiplication on $\hat{A}$ is non-degenerate. Suppose that $\omega_1\omega_2 = 0$ for all $\omega_2 = \phi(\cdot a_2) \in \hat{A}$. Then

$$(\omega_1\omega_2)(b) = (\phi \otimes \phi)(\Delta(b)(a_1 \otimes a_2)) = 0 \qquad \text{for all } a_2, b \in A.$$

Since T_1 is surjective, we can replace $\Delta(b)(1 \otimes a_2)$ by arbitrary $c \otimes d \in A \otimes A$. Then we obtain that $\phi(ca_1)\phi(d) = 0$ for all $c, d \in A$. This implies that $\omega_1 = \phi(\cdot a_1) = 0$. Similarly, we have $\omega_1 = 0$ if $\omega_2\omega_1 = 0$ for all $\omega_2 \in \hat{A}$. □

REMARK 7.1.3. Given $\omega_1, \omega_2 \in \hat{A}$ in the form $\omega_1 = \psi(\cdot b_1)$ and $\omega_2 = \psi(\cdot b_1)$, corresponding to (7.1.1), $\omega_1\omega_2 \in \hat{A}$ can also be defined by

$$(\omega_1\omega_2)(x) = (\psi \otimes \psi)(\Delta(x)(b_1 \otimes b_2)) \quad (x \in A). \tag{7.1.2}$$

Indeed, if $\omega_1 = \psi(\cdot b_1) = \phi(\cdot a_1)$ and $\omega_2 = \psi(\cdot b_1) = \phi(\cdot a_2)$, we have

$$(\psi \otimes \psi)(\Delta(x)(b_1 \otimes b_2)) = (\phi \otimes \phi)(\Delta(x)(a_1 \otimes a_2)) \quad \text{for all } x \in A.$$

That is, (7.1.1) and (7.1.2) define the same element of $\hat{A}$. To see this, let $p \in A$ with $\psi(p) = 1$ and write

$$\Delta(a_1)(p \otimes 1) = \sum_i \Delta(c_i)(1 \otimes d_i) \quad \text{and} \quad \Delta(a_2)(p \otimes 1) = \sum_j \Delta(c_j)(1 \otimes d_j)$$

for some $c_i, c_j, d_i, d_j \in A$. Then by the proof of Proposition 6.2.1, we have

$$b_1 = \sum_i c_i\phi(d_i) \qquad \text{and} \qquad b_2 = \sum_j c_j\phi(d_j).$$

Then for all $x \in A$, we have

$$\begin{aligned}
(\psi \otimes \psi)(\Delta(x)(b_1 \otimes b_2)) &= \sum_{i,j} (\psi \otimes \psi)(\Delta(x)(c_i\phi(d_i) \otimes c_j\phi(d_j))) \\
&= \sum_{i,j} \sum \phi(d_i)\psi(x_{(1)}c_i)\phi(d_j)\psi(x_{(2)}c_j) \\
&= \sum_{i,j} \sum (\psi \otimes \phi)(\Delta(x_{(1)}c_i)(1 \otimes d_i))(\psi \otimes \phi)(\Delta(x_{(2)}c_j)(1 \otimes d_j)) \\
&= \sum (\psi \otimes \phi)(\Delta(x_{(1)})\Delta(a_1)(p \otimes 1))(\psi \otimes \phi)(\Delta(x_{(2)})\Delta(a_2)(p \otimes 1)) \\
&= \sum \phi(x_{(1)}a_1)\psi(p)\phi(x_{(2)}a_2)\psi(p) = (\phi \otimes \phi)(\Delta(x)(a_1 \otimes a_2)).
\end{aligned}$$

Similarly, if $\omega_1 = \phi(c_1\cdot) = \psi(d_1\cdot)$ and $\omega_2 = \phi(c_2\cdot) = \psi(d_2\cdot)$, we have

$$(\phi \otimes \phi)((c_1 \otimes c_2)\Delta(\cdot)) = (\psi \otimes \psi)((d_1 \otimes d_2)\Delta(\cdot)).$$

Furthermore, it is also true that

$$(\psi \otimes \psi)(\Delta(\cdot)(b_1 \otimes b_2)) = (\phi \otimes \phi)((c_1 \otimes c_2)\Delta(\cdot)).$$

To show this, let $q \in A$ with $\phi(S(q)) = 1$, and write

$$b_1 \otimes q = \sum_i \Delta(a_i)(1 \otimes S^{-1}(d_i)) \qquad \text{and} \qquad b_2 \otimes q = \sum_j \Delta(a_j)(1 \otimes S^{-1}(d_j))$$

for some $a_i, a_j, d_i, d_j \in A$. Then by the proof of Proposition 6.4.2, we have

$$b_1 = \sum_i \sum a_{i(1)}\phi(d_i S(a_{i(2)})), \qquad b_2 = \sum_j \sum a_{j(1)}\phi(d_j S(a_{j(2)})),$$

$$c_1 = \sum_i \sum \psi(S(d_{i(1)})a_i)d_{i(2)}, \quad \text{and} \quad c_2 = \sum_j \sum \psi(S(d_{j(1)})a_j)d_{j(2)}.$$

Then for all $x \in A$, using Lemma 6.4.1, we have

$$\begin{aligned}
&(\phi \otimes \phi)((c_1 \otimes c_2)\Delta(x)) \\
&= \sum_{i,j} \sum (\phi \otimes \phi)(\psi(S(d_{i(1)})a_i)d_{i(2)}x_{(1)} \otimes \psi(S(d_{j(1)})a_j)d_{j(2)}x_{(2)}) \\
&= \sum_{i,j} \sum (\psi \otimes \psi)((S(d_{i(1)})\phi(d_{i(2)}x_{(1)})a_i \otimes S(d_{j(1)})\phi(d_{j(2)}x_{(2)})a_j)) \\
&= \sum_{i,j} \sum (\psi \otimes \psi)(x_{(1)}\phi(d_i x_{(2)})a_i \otimes x_{(3)}\phi(d_j x_{(4)})a_j) \\
&= \sum_{i,j} \sum (\phi \otimes \phi)(d_i\psi(x_{(1)}a_i)x_{(2)} \otimes d_j\psi(x_{(3)}a_j)x_{(4)}) \\
&= \sum_{i,j} \sum (\phi \otimes \phi)(d_i\psi(x_{(1)}a_{i(1)})S(a_{i(2)}) \otimes d_j\psi(x_{(2)}a_{j(1)})S(a_{j(2)})) \\
&= \sum_{i,j} \sum (\psi \otimes \psi)(x_{(1)}a_{i(1)}\phi(d_i S(a_{i(2)})) \otimes x_{(2)}a_{j(1)}\phi(d_j S(a_{j(2)}))) \\
&= (\psi \otimes \psi)(\Delta(x)(b_1 \otimes b_2)).
\end{aligned}$$

Therefore, if $\omega_1, \omega_2 \in \hat{A}$ have the forms $\omega_1 = \phi(\cdot a_1) = \psi(\cdot b_1) = \phi(c_1 \cdot) = \psi(d_1 \cdot)$ and $\omega_2 = \phi(\cdot a_2) = \psi(\cdot b_2) = \phi(c_2 \cdot) = \psi(d_2 \cdot)$, then

$$\begin{aligned}\omega_1\omega_2 &= (\phi \otimes \phi)(\Delta(\cdot)(a_1 \otimes a_2)) = (\psi \otimes \psi)(\Delta(\cdot)(b_1 \otimes b_2)) \\ &= (\phi \otimes \phi)((c_1 \otimes c_2)\Delta(\cdot)) = (\psi \otimes \psi)((d_1 \otimes d_2)\Delta(\cdot)). \end{aligned} \tag{7.1.3}$$

Next, we turn to the comultiplication $\hat{\Delta} : \hat{A} \to M(\hat{A} \otimes \hat{A})$. The obvious formula $\hat{\Delta}(\omega)(x \otimes y) = \omega(xy)$ $(x, y \in A,\ \omega \in \hat{A})$ only defines a map $\hat{\Delta} : \hat{A} \to (A \otimes A)'$, and it is not immediately clear that the image of this map can be identified with a subspace of $M(\hat{A} \otimes \hat{A})$. Therefore, we have to take a different approach to $\hat{\Delta}$. We will introduce $\hat{\Delta}$ by first defining the linear functionals $\hat{\Delta}(\omega_1)(1 \otimes \omega_2)$ and $(\omega_1 \otimes 1)\hat{\Delta}(\omega_2)$ on $A \otimes A$ for $\omega_1, \omega_2 \in \hat{A}$ by

$$(\hat{\Delta}(\omega_1)(1 \otimes \omega_2))(x \otimes y) = (\omega_1 \otimes \omega_2)((x \otimes 1)\Delta(y)) \quad (x, y \in A) \tag{7.1.4}$$

and

$$((\omega_1 \otimes 1)\hat{\Delta}(\omega_2))(x \otimes y) = (\omega_1 \otimes \omega_2)(\Delta(x)(1 \otimes y)) \quad (x, y \in A). \tag{7.1.5}$$

To justify (7.1.4) and (7.1.5), let us suppose that $\epsilon \in \hat{A}$ and $(\hat{A}, \hat{\Delta})$ is a multiplier bialgebra. In this case, ϵ is the unit of $\hat{A}$, and hence for any $\omega = \phi(\cdot a) \in \hat{A}$ and $x, y \in A$, we have

$$\begin{aligned}(\hat{\Delta}(\omega)(1 \otimes \epsilon))(x \otimes y) &= (\omega \otimes \epsilon)((x \otimes 1)\Delta(y)) = \sum (\omega \otimes \epsilon)(xy_{(1)} \otimes y_{(2)}) \\ &= \sum \phi(xy_{(1)}a)\epsilon(y_{(2)}) = \phi(xya) = \omega(xy)\end{aligned}$$

and similarly

$$((\epsilon \otimes 1)\hat{\Delta}(\omega))(x \otimes y) = \omega(xy).$$

We will show that these definitions are indeed the expected formulas. Let us first argue that the functionals defined in (7.1.4) and (7.1.5) are actually in $\hat{A} \otimes \hat{A}$.

Lemma 7.1.4. *If $\omega, \omega_1, \omega_2 \in \hat{A}$, then $\hat{\Delta}(\omega)(1 \otimes \omega_2)$ and $(\omega_1 \otimes 1)\hat{\Delta}(\omega)$ are in $\hat{A} \otimes \hat{A}$, and we have*

$$((\omega_1 \otimes 1)\hat{\Delta}(\omega))(1 \otimes \omega_2) = (\omega_1 \otimes 1)(\hat{\Delta}(\omega)(1 \otimes \omega_2)).$$

Proof. Let $\omega = \phi(\cdot a)$ and $\omega_2 = \phi(\cdot a_2)$. Due to the regularity of A, we can write

$$a \otimes a_2 = \sum_i \Delta(b_i)(c_i \otimes 1)$$

for some $b_i, c_i \in A$. Then for all $x, y \in A$,

$$\begin{aligned}(\hat{\Delta}(\omega)(1 \otimes \omega_2))(x \otimes y) &= (\omega \otimes \omega_2)((x \otimes 1)\Delta(y)) \\ &= (\phi \otimes \phi)((x \otimes 1)\Delta(y)(a \otimes a_2)) \\ &= \sum_i (\phi \otimes \phi)((x \otimes 1)\Delta(yb_i)(c_i \otimes 1)) \\ &= \sum_i \phi(xc_i)\phi(yb_i) = (\sum_i \phi(\cdot c_i) \otimes \phi(\cdot b_i))(x \otimes y).\end{aligned}$$

So,

$$\hat{\Delta}(\omega)(1 \otimes \omega_2) = \sum_i \phi(\cdot c_i) \otimes \phi(\cdot b_i) \tag{7.1.6}$$

is a well defined element of $\hat{A}\otimes\hat{A}$. Similarly, we can show that $(\omega_1 \otimes 1)\hat{\Delta}(\omega) \in \hat{A}\otimes\hat{A}$; however, in this case, we use $\psi(a_1\cdot)$ and $\psi(a\cdot)$ for ω_1 and ω, respectively, and write $a_1 \otimes a = \sum_i (1 \otimes b_i)\Delta(c_i)$ to obtain that

$$(\omega_1 \otimes 1)\hat{\Delta}(\omega) = \sum_i \psi(c_i\cdot) \otimes \psi(b_i\cdot). \tag{7.1.7}$$

Finally, for all $x, y \in A$, we have

$$\begin{aligned}(((\omega_1 \otimes 1)\hat{\Delta}(\omega))(1 \otimes \omega_2))(x \otimes y) &= \sum ((\omega_1 \otimes 1)\hat{\Delta}(\omega))(x \otimes y_{(1)})\omega_2(y_{(2)}) \\ &= \sum (\omega_1 \otimes \omega)(\Delta(x)(1 \otimes y_{(1)}))\omega_2(y_{(2)}) \\ &= \sum \omega_1(x_{(1)})\omega(x_{(2)}y_{(1)})\omega_2(y_{(2)})\end{aligned}$$

and

$$\begin{aligned}((\omega_1 \otimes 1)(\hat{\Delta}(\omega)(1 \otimes \omega_2)))(x \otimes y) &= \sum \omega_1(x_{(1)})(\hat{\Delta}(\omega)(1 \otimes \omega_2))(x_{(2)} \otimes y) \\ &= \sum \omega_1(x_{(1)})(\omega \otimes \omega_2)((x_{(2)} \otimes 1)\Delta(y)) \\ &= \sum \omega_1(x_{(1)})\omega(x_{(2)}y_{(1)})\omega_2(y_{(2)}).\end{aligned}$$

Therefore, $((\omega_1 \otimes 1)\hat{\Delta}(\omega))(1 \otimes \omega_2) = (\omega_1 \otimes 1)(\hat{\Delta}(\omega)(1 \otimes \omega_2))$. □

Let $\omega \in \hat{A}$. We define the linear maps

$$\hat{\Delta}(\omega)_l: \ \hat{A}\otimes\hat{A} \to \hat{A}\otimes\hat{A}, \ \omega_1\otimes\omega_2 \mapsto (\hat{\Delta}(\omega)(1\otimes\omega_2))(\omega_1\otimes 1) \tag{7.1.8}$$

and

$$\hat{\Delta}(\omega)_r: \ \hat{A}\otimes\hat{A} \to \hat{A}\otimes\hat{A}, \ \omega_1\otimes\omega_2 \mapsto (1\otimes\omega_2)((\omega_1\otimes 1)\hat{\Delta}(\omega)). \tag{7.1.9}$$

Then the corollary below is immediate by Lemma 7.1.4 and Remark 5.1.5(i.b).

COROLLARY 7.1.5. *Let $\omega \in \hat{A}$. Then $\hat{\Delta}(\omega) := (\hat{\Delta}(\omega)_l, \hat{\Delta}(\omega_2)_r) \in M(\hat{A}\otimes\hat{A})$*

Quite similar to the linear maps T_1 and T_2 on $A \otimes A$ given in Section 5.3, we can define the linear maps $\hat{T}_1, \hat{T}_2 : \hat{A}\otimes\hat{A} \to \hat{A}\otimes\hat{A}$ by

$$\hat{T}_1: \ \hat{A}\otimes\hat{A} \to \hat{A}\otimes\hat{A}, \ \omega_1\otimes\omega_2 \mapsto \hat{\Delta}(\omega_1)(1\otimes\omega_2) \tag{7.1.10}$$

and

$$\hat{T}_2: \ \hat{A}\otimes\hat{A} \to \hat{A}\otimes\hat{A}, \ \omega_1\otimes\omega_2 \mapsto (\omega_1\otimes 1)\hat{\Delta}(\omega_2). \tag{7.1.11}$$

Then equations (7.1.4) and (7.1.5) can be written respectively as

$$\hat{T}_1(\omega_1\otimes\omega_2)(x\otimes y) = (\omega_1\otimes\omega_2)(T_2(x\otimes y))$$

and

$$\hat{T}_2(\omega_1\otimes\omega_2)(x\otimes y) = (\omega_1\otimes\omega_2)(T_1(x\otimes y)).$$

Therefore, we have

$$\hat{T}_1 = (T_2)'|_{\hat{A}\otimes\hat{A}} : \hat{A}\otimes\hat{A} \to \hat{A}\otimes\hat{A} \quad \text{and} \quad \hat{T}_2 = (T_1)'|_{\hat{A}\otimes\hat{A}} : \hat{A}\otimes\hat{A} \to \hat{A}\otimes\hat{A},$$

and hence $\hat{T}_1$ and $\hat{T}_2$ are injective. The proof of Lemma 7.1.4 shows that they are also surjective. Therefore, we conclude that

$$\hat{T}_1, \hat{T}_2: \ \hat{A}\otimes\hat{A} \to \hat{A}\otimes\hat{A} \ \text{ are bijective.} \tag{7.1.12}$$

We need the following lemma to show that $\hat{\Delta}$ is multiplicative.

LEMMA 7.1.6. *For all $\omega, \omega_1, \omega_2 \in \hat{A}$ and $x, y \in A$, we have*

$$(\hat{\Delta}(\omega)(\omega_1\otimes\omega_2))(x\otimes y) = (\omega_1\otimes\hat{\Delta}(\omega)(1\otimes\omega_2))(\Delta^{cop}(x)\otimes y)$$

and

$$((\omega_1 \otimes \omega_2)\hat{\Delta}(\omega))(x \otimes y) = ((\omega_1 \otimes 1)\hat{\Delta}(\omega) \otimes \omega_2)(x \otimes \Delta^{cop}(y)).$$

PROOF. Let $\omega, \omega_1, \omega_2 \in \hat{A}$ and $x, y \in A$. Then

$$\begin{aligned}(\hat{\Delta}(\omega)(\omega_1 \otimes \omega_2))(x \otimes y) &= ((\hat{\Delta}(\omega)(1 \otimes \omega_2))(\omega_1 \otimes 1))(x \otimes y)\\ &= \sum \omega_1(x_{(2)})(\hat{\Delta}(\omega)(1 \otimes \omega_2))(x_{(1)} \otimes y)\\ &= (\omega_1 \otimes \hat{\Delta}(\omega)(1 \otimes \omega_2))(\Delta^{cop}(x) \otimes y).\end{aligned}$$

The second equality can be proved in a similar way. □

PROPOSITION 7.1.7. *The map $\hat{\Delta} : \hat{A} \to M(\hat{A} \otimes \hat{A})$ is a non-degenerate algebra homomorphism, which is coassociative in the sense that*

$$(\omega_1 \otimes 1 \otimes 1)(\hat{\Delta} \otimes id)(\hat{\Delta}(\omega)(1 \otimes \omega_2)) = (id \otimes \hat{\Delta})((\omega_1 \otimes 1)\hat{\Delta}(\omega))(1 \otimes 1 \otimes \omega_2)$$

for all $\omega, \omega_1, \omega_2 \in \hat{A}$.

PROOF. Clearly, $\hat{\Delta} : \hat{A} \to M(\hat{A} \otimes \hat{A})$ is linear. Let $\omega_1, \omega_2, \omega_3 \in \hat{A}$ and $x, y \in A$. On the one hand, write

$$\hat{\Delta}(\omega_2)(1 \otimes \omega_3) = \sum_i \omega_i' \otimes \omega_i''$$

for some $\omega_i', \omega_i'' \in \hat{A}$. On the one hand, from Lemma 7.1.6, we have

$$\begin{aligned}(\hat{\Delta}(\omega_1)\hat{\Delta}(\omega_2)(1 \otimes \omega_3))(x \otimes y) &= \sum_i (\hat{\Delta}(\omega_1)(\omega_i' \otimes \omega_i''))(x \otimes y)\\ &= \sum \sum_i (\omega_i' \otimes \hat{\Delta}(\omega_1)(1 \otimes \omega_i''))(x_{(2)} \otimes x_{(1)} \otimes y)\\ &= \sum \sum_i \omega_i'(x_{(2)})(\omega_1 \otimes \omega_i'')((x_{(1)} \otimes 1)\Delta(y))\\ &= \sum \sum_i \omega_1(x_{(1)}y_{(1)})\omega_i'(x_{(2)})\omega_i''(y_{(2)})\\ &= \sum \omega_1(x_{(1)}y_{(1)})(\hat{\Delta}(\omega_2)(1 \otimes \omega_3))(x_{(2)} \otimes y_{(2)})\\ &= \sum \omega_1(x_{(1)}y_{(1)})(\omega_2 \otimes \omega_3)((x_{(2)} \otimes 1)\Delta(y_{(2)}))\\ &= \sum \omega_1(x_{(1)}y_{(1)})\omega_2(x_{(2)}y_{(2)})\omega_3(y_{(3)}).\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}(\hat{\Delta}(\omega_1\omega_2)(1\otimes\omega_3))(x\otimes y) &= (\omega_1\omega_2\otimes\omega_3)((x\otimes 1)\Delta(y))\\ &= \sum(\omega_1\omega_2)(xy_{(1)})\omega_3(y_{(2)})\\ &= \sum\omega_1(x_{(1)}y_{(1)})\omega_2(x_{(2)}y_{(2)})\omega_3(y_{(3)}).\end{aligned}$$

So, $\hat{\Delta}(\omega_1\omega_2) = \hat{\Delta}(\omega_1)\hat{\Delta}(\omega_2)$.

Since $\hat{T}_1 : \hat{A}\otimes\hat{A}\to\hat{A}\otimes\hat{A}$ is bijective (cf. (7.1.12)) and $\hat{A}$ is non-degenerate, we have

$$\omega_1\otimes\omega_2 = \sum_i \hat{\Delta}(\omega_i^1\omega_i^2)(1\otimes\omega_i^3) = \sum_i \hat{\Delta}(\omega_i^1)(\hat{\Delta}(\omega_i^2)(1\otimes\omega_i^3))$$

for some $\omega_i^1, \omega_i^2, \omega_i^3 \in \hat{A}$. This implies that $\hat{A}\otimes\hat{A}\subseteq \operatorname{span}\{\hat{\Delta}(\hat{A})(\hat{A}\otimes\hat{A})\}$; that is,

$$\operatorname{span}\{\hat{\Delta}(\hat{A})(\hat{A}\otimes\hat{A})\} = \hat{A}\otimes\hat{A}.$$

Similarly, we have $\operatorname{span}\{(\hat{A}\otimes\hat{A})\hat{\Delta}(\hat{A})\} = \hat{A}\otimes\hat{A}$. Therefore, $\hat{\Delta} : \hat{A}\to M(\hat{A}\otimes\hat{A})$ is a non-degenerate algebra homomorphism.

To show that $\hat{\Delta}$ is coassociative, we let $\omega, \omega_1, \omega_2 \in \hat{A}$ and write

$$\hat{\Delta}(\omega)(1\otimes\omega_2) = \sum_i \omega_i'\otimes\omega_i'' \quad\text{and}\quad (\omega_1\otimes 1)\hat{\Delta}(\omega) = \sum_j \omega_j'\otimes\omega_j''$$

with $\omega_i', \omega_i'', \omega_j', \omega_j'' \in \hat{A}$. Then for all $x, y, z \in A$, by (7.1.4) and (7.1.5), we have

$$\begin{aligned}&((\omega_1\otimes 1\otimes 1)(\hat{\Delta}\otimes id)(\hat{\Delta}(\omega)(1\otimes\omega_2)))(x\otimes y\otimes z)\\ &= \sum_i((\omega_1\otimes 1\otimes 1)(\hat{\Delta}\otimes id)(\omega_i'\otimes\omega_i''))(x\otimes y\otimes z)\\ &= \sum_i((\omega_1\otimes 1)\hat{\Delta}(\omega_i')\otimes\omega_i'')(x\otimes y\otimes z)\\ &= \sum_i(\omega_1\otimes\omega_i'\otimes\omega_i'')(\Delta(x)(1\otimes y)\otimes z)\\ &= \sum(\omega_1\otimes\hat{\Delta}(\omega)(1\otimes\omega_2))(x_{(1)}\otimes x_{(2)}y\otimes z)\\ &= \sum(\omega_1\otimes\omega\otimes\omega_2)(x_{(1)}\otimes(x_{(2)}y\otimes 1)\Delta(z))\\ &= \sum\omega_1(x_{(1)})\omega(x_{(2)}yz_{(1)})\omega_2(z_{(2)})\end{aligned}$$

and

$$
\begin{aligned}
&((id \otimes \hat{\Delta})((\omega_1 \otimes 1)\hat{\Delta}(\omega))(1 \otimes 1 \otimes \omega_2))(x \otimes y \otimes z) \\
&= \sum_j ((id \otimes \hat{\Delta})(\omega_j' \otimes \omega_j'')(1 \otimes 1 \otimes \omega_2))(x \otimes y \otimes z) \\
&= \sum_j (\omega_j' \otimes \hat{\Delta}(\omega_j'')(1 \otimes \omega_2))(x \otimes y \otimes z) \\
&= \sum_j (\omega_j' \otimes \omega_j'' \otimes \omega_2)(x \otimes (y \otimes 1)\Delta(z)) \\
&= \sum ((\omega_1 \otimes 1)\hat{\Delta}(\omega) \otimes \omega_2)(x \otimes yz_{(1)} \otimes z_{(2)}) \\
&= \sum (\omega_1 \otimes \omega \otimes \omega_2)(\Delta(x)(1 \otimes yz_{(1)}) \otimes z_{(2)}) \\
&= \sum \omega_1(x_{(1)})\omega(x_{(2)}yz_{(1)})\omega_2(z_{(2)}).
\end{aligned}
$$

Therefore, the coassociativity holds. □

Combining Lemma 7.1.4, Proposition 7.1.7, and (7.1.12), we see that $(\hat{A}, \hat{\Delta})$ is a multiplier Hopf algebra. Actually, $(\hat{A}, \hat{\Delta})$ is also regular.

Proposition 7.1.8. *The multiplier Hopf algebra $(\hat{A}, \hat{\Delta})$ is regular, where the antipode $\hat{S}$ is dual to S and the counit $\hat{\epsilon}$ is given by evaluation at 1. That is, for all $a \in A$, we have*

$$\hat{S}(\omega)(a) = \omega(S(a)),$$

and

$$\hat{\epsilon}(\phi(\cdot a)) = \phi(a) = \hat{\epsilon}(\phi(a\cdot)) \quad \textit{and} \quad \hat{\epsilon}(\psi(\cdot a)) = \psi(a) = \hat{\epsilon}(\psi(a\cdot)).$$

Proof. Let $\omega, \omega_1, \omega_2 \in \hat{A}$. First, analogue to (7.1.4) and (7.1.5), we define the linear functionals $\hat{\Delta}_0(\omega)(\omega_1 \otimes 1)$ and $(1 \otimes \omega_2)\hat{\Delta}_0(\omega)$ on $A \otimes A$ by

$$(\hat{\Delta}_0(\omega)(\omega_1 \otimes 1))(x \otimes y) = (\omega \otimes \omega_1)(\Delta(x)(y \otimes 1)) \quad (x, y \in A)$$

and

$$((1 \otimes \omega_2)\hat{\Delta}_0(\omega))(x \otimes y) = (\omega_2 \otimes \omega)((1 \otimes x)\Delta(y)) \quad (x, y \in A).$$

Following the proof of Lemma 7.1.4, we can show that both $\hat{\Delta}_0(\omega)(\omega_1 \otimes 1)$ and $(1 \otimes \omega_2)\hat{\Delta}_0(\omega)$ are actually in $\hat{A} \otimes \hat{A}$. Similar to the case $\hat{\Delta}(\omega)$, we can also define

$\hat{\Delta}_0(\omega) \in M(\hat{A} \otimes \hat{A})$. Then for all $x, y \in A$, by Lemma 7.1.6, we have

$$\begin{aligned}
(\hat{\Delta}_0(\omega)(\omega_1 \otimes \omega_2))(x \otimes y) &= ((\hat{\Delta}_0(\omega)(\omega_1 \otimes 1))(1 \otimes \omega_2))(x \otimes y) \\
&= \sum (\hat{\Delta}_0(\omega)(\omega_1 \otimes 1))(x \otimes y_{(1)})\omega_2(y_{(2)}) \\
&= \sum (\omega \otimes \omega_1)(\Delta(x)(y_{(1)} \otimes 1))\omega_2(y_{(2)}) \\
&= \sum \omega(x_{(1)}y_{(1)})\omega_1(x_{(2)})\omega_2(y_{(2)}) \\
&= \sum \omega_1(x_{(2)})(\omega \otimes \omega_2)((x_{(1)} \otimes 1)\Delta(y)) \\
&= \sum \omega_1(x_{(2)})(\hat{\Delta}(\omega)(1 \otimes \omega_2))(x_{(1)} \otimes y) \\
&= (\omega_1 \otimes \hat{\Delta}(\omega)(1 \otimes \omega_2))(\Delta^{cop}(x) \otimes y) \\
&= (\hat{\Delta}(\omega)(\omega_1 \otimes \omega_2))(x \otimes y).
\end{aligned}$$

So, $\hat{\Delta}_0(\omega)(\omega_1 \otimes \omega_2) = \hat{\Delta}(\omega)(\omega_1 \otimes \omega_2)$. Similarly, we have $(\omega_1 \otimes \omega_2)\hat{\Delta}_0(\omega) = (\omega_1 \otimes \omega_2)\hat{\Delta}(\omega)$. Therefore, we obtain that $\hat{\Delta}_0 = \hat{\Delta}$. In particular, we get that $(1 \otimes \hat{A})\hat{\Delta}(\hat{A}) \subseteq \hat{A} \otimes \hat{A}$ and $\hat{\Delta}(\hat{A})(\hat{A} \otimes 1) \subseteq \hat{A} \otimes \hat{A}$; that is, the first condition in Theorem 5.5.8 is satisfied.

Next, we consider the counit of $(\hat{A}, \hat{\Delta})$. Let $\omega = \phi(a\cdot) = \phi(\cdot a') = \psi(b\cdot) = \psi(\cdot b')$. To show that $\hat{\epsilon} : A \to \mathbb{C}$ is well-defined, it is suffices to show that $\phi(a) = \phi(a') = \psi(b) = \psi(b')$. Since $\phi(ax) = \phi(x\sigma(a))$ for all $x \in A$ and $\phi = \phi \circ \sigma$ (cf. Theorem 6.4.4), it is easy to see that $a' = \sigma(a)$ and $\phi(a) = \phi(\sigma(a)) = \phi(a')$. Similarly, $\psi(b) = \psi(b')$. Now we consider the pair $\phi(a')$ and $\psi(b)$. Uniqueness of right integrals implies that $\psi = k\phi \circ S$ for some scaler $k \in \mathbb{C}$. By Proposition 6.3.1 and Theorem 6.4.4, for all $x \in A$, we have

$$\omega(x) = \phi(xa') = \psi(bx) = k\phi(S(bx)) = k\phi(bx\delta) = k\phi(x\delta\sigma(b)),$$

which implies that $a' = k\delta\sigma(b)$. Then

$$\phi(a') = k\phi(\delta\sigma(b)) = k\phi(b\delta) = k\phi(S(b)) = \psi(b).$$

Therefore, $\phi(a) = \phi(a') = \psi(b) = \psi(b')$.

Clearly, $\hat{\epsilon}$ is linear. The proof of Proposition 7.1.2 shows that $\hat{\epsilon}$ is multiplicative. Hence, $\hat{\epsilon} : \hat{A} \to \mathbb{C}$ is a non-degenerate algebra homomorphism. Let $\omega_1 = \phi(\cdot a_1) \in \hat{A}$

and $\omega_2 = \phi(\cdot a_2) \in \hat{A}$ with

$$a_1 \otimes a_2 = \sum_i \Delta(b_i)(c_i \otimes 1)$$

for some $b_i, c_i \in A$. Let $x \in A$. From the proof of Proposition 7.1.2, we have

$$\omega_1\omega_2 = \sum_i \phi(c_i)\phi(\cdot b_i).$$

Notice that $\hat{\epsilon}(\omega) = \phi(a)$ for any $\omega = \phi(\cdot a) \in \hat{A}$. So, we have

$$(\hat{\epsilon} \otimes id)(\hat{\Delta}(\omega_1)(1 \otimes \omega_2)) = \omega_1\omega_2$$

by (7.1.6). Similarly, we have $(id \otimes \hat{\epsilon})((\omega_1 \otimes 1)\Delta(\omega_2)) = \omega_1\omega_2$; however, in this case, we write $\omega_1 = \psi(a_1\cdot)$ and $\omega_2 = \psi(a_2\cdot)$, and use equations (7.1.7) and (7.1.3). Therefore, the non-degenerate algebra homomorphism $\hat{\epsilon} : \hat{A} \to \mathbb{C}$, $\phi(\cdot a) \mapsto \phi(a)$ is the counit of $(\hat{A}, \hat{\Delta})$ (that is, (5.4.7) is satisfied).

Finally, we consider the antipode of $(\hat{A}, \hat{\Delta})$. Since $S : A \to A$ is bijective, so is the adjoint $S' : A' \to A'$. For any $\omega = \phi(a\cdot) \in A'$ and all $x \in A$, by Proposition 6.3.5, it is easy to check that

$$S'(\omega)(x) = \omega(S(x)) = \phi(aS(x)) = \phi(S(xS^{-1}(a))) = \phi(xS^{-1}(a)\delta),$$

which implies that $\hat{S}(\omega) = \phi(\cdot S^{-1}(a)\delta) \in \hat{A}$. Therefore, $\hat{S} = S'|_{\hat{A}} : \hat{A} \to \hat{A}$ is a linear bijection. Furthermore, for all $x \in A$, we have

$$\begin{aligned}
\hat{m}((\hat{S} \otimes id)(\hat{\Delta}(\omega_1)(1 \otimes \omega_2)))(x) &= (\hat{\Delta}(\omega_1)(1 \otimes \omega_2))((S \otimes id)\Delta(x)) \\
&= \sum (\hat{\Delta}(\omega_1)(1 \otimes \omega_2))(S(x_{(1)}) \otimes x_{(2)}) \\
&= \sum (\omega_1 \otimes \omega_2)((S(x_{(1)}) \otimes 1)\Delta(x_{(2)})) \\
&= \sum (\omega_1 \otimes \omega_2)(S(x_{(1)})x_{(2)} \otimes x_{(3)}) \\
&= \sum (\omega_1 \otimes \omega_2)(1 \otimes \epsilon(x_{(1)})x_{(2)}) \\
&= \omega_1(1)\omega_2(x) = (\hat{\epsilon}(\omega_1)\omega_2)(x).
\end{aligned}$$

Similarly, we get $\hat{m}((id \otimes \hat{S})((\omega_1 \otimes 1)\hat{\Delta}(\omega_2))) = \omega_1\hat{\epsilon}(\omega_2)$. So, the linear bijection $\hat{S} : \hat{A} \to \hat{A}$ satisfies (5.4.8) and (5.4.9). By Theorem 5.5.8, we conclude that $(\hat{A}, \hat{\Delta})$ is a regular multiplier Hopf algebra with counit $\hat{\epsilon}$ and antipode $\hat{S}$. □

REMARK 7.1.9. If (A, Δ) is a Hopf algebra, then the counit $\hat{\epsilon}$ on the dual multiplier Hopf algebra $(\hat{A}, \hat{\Delta})$ is given by $\hat{\epsilon}(\omega) = \omega(1_A)$ for all $\omega \in \hat{A}$. This follows from Proposition 7.1.8

Next, we describe integrals on the regular multiplier Hopf algebra $(\hat{A}, \hat{\Delta})$.

PROPOSITION 7.1.10.

(i) *For* $\omega = \phi(\cdot a) \in \hat{A}$*, let* $\hat{\psi}(\omega) = \epsilon(a)$*. Then* $\hat{\psi} : \hat{A} \to \mathbb{C}$ *is a right integral on* $(\hat{A}, \hat{\Delta})$*.*

(ii) *For* $\omega = \psi(a\cdot) \in \hat{A}$*, let* $\hat{\phi}(\omega) = \epsilon(a)$*. Then* $\hat{\phi} : \hat{A} \to \mathbb{C}$ *is a left integral on* $(\hat{A}, \hat{\Delta})$*.*

PROOF. We prove (i) only; (ii) can be shown similarly. Let $\omega_1 = \phi(\cdot a_1)$ and $\omega_2 = \phi(\cdot a_2)$. Then from the proof of Lemma 7.1.4, we have

$$\hat{\Delta}(\omega_1)(1 \otimes \omega_2) = \sum_i \phi(\cdot c_i) \otimes \phi(\cdot b_i),$$

where $a_1 \otimes a_2 = \sum_i \Delta(b_i)(c_i \otimes 1)$ for some $b_i, c_i \in A$. Then

$$(\hat{\psi} \otimes id)(\hat{\Delta}(\omega_1)(1 \otimes \omega_2)) = \sum_i \epsilon(c_i)\phi(\cdot b_i).$$

Inserting the relation

$$\sum_i \epsilon(c_i)b_i = \sum_i (\epsilon \otimes id)(\Delta(b_i)(c_i \otimes 1)) = (\epsilon \otimes id)(a_1 \otimes a_2) = \epsilon(a_1)a_2,$$

we get

$$(\hat{\psi} \otimes id)(\hat{\Delta}(\omega_1)(1 \otimes \omega_2)) = \epsilon(a_1)\phi(\cdot a_2) = \hat{\psi}(\omega_1)\omega_2.$$

Therefore, $\hat{\psi} \neq 0$ is right invariant and hence a right integral on $(\hat{A}, \hat{\Delta})$. □

Summarizing the above, we present the following duality result for regular multiplier Hopf algebras with integrals.

THEOREM 7.1.11. *(Duality for regular multiplier Hopf algebras) If* (A, Δ) *is a regular multiplier Hopf algebra with integrals, then the dual* $(\hat{A}, \hat{\Delta})$ *is a also a regular multiplier Hopf algebra with integrals.*

The bidual $(\hat{\hat{A}}, \hat{\hat{\Delta}})$ of a regular multiplier Hopf algebra (A, Δ) with integrals is naturally isomorphic to (A, Δ). The main step towards the proof of this assertion is the following lemma.

LEMMA 7.1.12. *Let* $\omega = \phi(\cdot a)$. *Then*

$$\hat{\psi}(\omega_1 \omega) = \omega_1(S^{-1}(a)) \qquad \textit{for all } \omega_1 \in \hat{A}.$$

PROOF. Let $\omega_1 = \phi(\cdot b)$ and write

$$b \otimes a = \sum_i \Delta(p_i)(q_i \otimes 1)$$

for some $p_i, q_i \in A$. Then we have

$$(\omega_1\omega)(x) = (\phi \otimes \phi)(\Delta(x)(b \otimes a)) = \sum_i (\phi \otimes \phi)(\Delta(xp_i)(q_i \otimes 1)) = \sum_i \phi(xp_i)\phi(q_i).$$

So, we get that $\hat{\psi}(\omega_1\omega) = \sum_i \epsilon(p_i)\phi(q_i)$ by Proposition 7.1.10. Recall the $\epsilon^{cop} = \epsilon$ and $S^{cop} = S^{-1}$ (cf. Proposition 5.5.3 and Proposition 5.5.5). Then we have

$$\begin{aligned} \sum \epsilon(p_i)q_i &= \sum_i m((S^{-1} \otimes id)(\Delta^{cop}(p_i)(1 \otimes q_i))) \\ &= \sum_i m_{A^{op}}((id \otimes S^{-1})(\Delta(p_i)(q_i \otimes 1))) \\ &= m_{A^{op}}((id \otimes S^{-1})(b \otimes a)) = S^{-1}(a)b. \end{aligned}$$

Therefore, $\hat{\psi}(\omega_1\omega) = \phi(\sum_i \epsilon(p_i)q_i) = \phi(S^{-1}(a)b) = \omega_1(S^{-1}(a))$. □

The above lemma shows that we can essentially identify the dual of $(\hat{A}, \hat{\Delta})$ with (A, Δ). This fact will be precisely stated in the following theorem.

THEOREM 7.1.13. *(Biduality for regular multiplier Hopf algebras) Let* (A, Δ) *be a regular multiplier Hopf algebra with integrals. Then the map* $\iota : A \to (\hat{A})'$ *given by*

$$\iota(a)(\omega) = \omega(a) \qquad (a \in A, \ \omega \in \hat{A})$$

takes values in $\hat{\hat{A}}$ *and defines an isomorphism of multiplier Hopf algebras between* (A, Δ) *and* $(\hat{\hat{A}}, \hat{\hat{\Delta}})$.

PROOF. Obviously, the map $\iota : A \to (\hat{A})'$ is linear, which is also injective since ϕ is faithful.

Note that every element of $\hat{\hat{A}}$ is of the form $\hat{\psi}(\cdot\omega)$ with $\omega \in \hat{A}$. Furthermore, since the antipode $S : A \to A$ is bijective, every $\omega \in \hat{A}$ can be written in the form $\omega = \phi(\cdot S(a))$ for some $a \in A$; conversely, every $a \in A$ defines $\omega = \phi(\cdot S(a)) \in \hat{A}$. By Lemma 7.1.12, we have

$$\hat{\psi}(\omega_1\omega) = \omega_1(S^{-1}(S(a))) = \omega_1(a) = \iota(a)(\omega_1)$$

for all $\omega_1 \in \hat{A}$, which implies that $\hat{\psi}(\cdot\omega) = \iota(a)$, and hence $\iota(A) = \hat{\hat{A}}$. Therefore, $\iota : A \to \hat{\hat{A}}$ is a linear isomorphism. Straightforward but lengthy calculations show that this isomorphism is compatible with the multiplication and comultiplication on A and $\hat{\hat{A}}$. $\square$

7.2. The duality of algebraic quantum groups

Recall from definition 6.1.9 that an algebraic quantum group is a multiplier Hopf $*$-algebra with a positive left integral and a positive right integral. By Remark 5.5.10(c), algebraic quantum groups are regular automatically. As shown below, the duality of regular multiplier Hopf algebras with integrals extends to a duality of algebraic quantum groups.

PROPOSITION 7.2.1. *Let (A, Δ) be a multiplier Hopf $*$-algebra with integrals. Then the formula*

$$\omega^*(x) := \overline{\omega(S(x)^*)} \qquad (x \in A, \omega \in \hat{A})$$

defines an involution on $\hat{A}$ that turns $(\hat{A}, \hat{\Delta})$ into a multiplier Hopf $*$-algebra.

PROOF. Let ϕ be a left integral on (A, Δ) and $\omega = \phi(\cdot a) \in \hat{A}$ for some $a \in A$. Since $* \circ S \circ * \circ S = id$, we have

$$\begin{aligned} \omega^*(x) &= \overline{\omega(S(x)^*)} = \overline{\phi(S(x)^* a)} = \overline{\phi(S(x)^* S(S(a)^*)^*)} \\ &= \overline{\phi((S(S(a)^*)S(x))^*)} = \overline{\phi(S(xS(a)^*)^*)} = \psi(xS(a)^*) \end{aligned}$$

for all $x \in A$, where $\psi(y) = \overline{\phi(S(y)^*)}$ $(y \in A)$. Following the proof of Proposition 6.1.7(i) and noticing that $1^*_{M(A)} = 1_{M(A)}$ and $\Delta(a^*) = \Delta(a)^*$ $(a \in A)$, we obtain that ψ is a right integral on (A, Δ). So, $\omega^* \in \hat{A}$.

It is immediately clear that $\omega^{**} = \omega$. The property $(\omega_1\omega_2)^* = \omega_2^*\omega_1^*$ follows naturally from the property that Δ is a $*$-homomorphism and that $(S \otimes S) \circ \Delta = \Sigma \circ \Delta \circ S$. □

THEOREM 7.2.2. *Let (A, Δ) be a multiplier Hopf $*$-algebra with integrals. If ϕ is a positive left integral on (A, Δ) and $\hat{\psi}$ is the right integral on $(\hat{A}, \hat{\Delta})$ determined by ϕ as in Proposition 7.1.10, then*

$$\hat{\psi}(\phi(\cdot a_1)^*\phi(\cdot a_2)) = \phi(a_1^*a_2) \qquad \text{for all } a_1, a_2 \in A.$$

In particular, $\hat{\psi}$ is positive. Likewise, if ψ is a positive right integral on (A, Δ), then the corresponding left integral $\hat{\phi}$ on $(\hat{A}, \hat{\Delta})$ is also positive.

PROOF. Given $a_1, a_2 \in A$, let $\omega_1 = \phi(\cdot a_1)$ and $\omega_2 = \phi(\cdot a_2)$. Applying Lemma 7.1.12 to ω_1^* and ω_2, and noticing that ϕ is $*$-linear (cf. Remark 6.4.3(i)), we get

$$\begin{aligned}\hat{\psi}(\phi(\cdot a_1)^*\phi(\cdot a_2)) &= \hat{\psi}(\omega_1^*\omega_2) = \omega_1^*(S^{-1}(a_2)) = \overline{\omega_1(S(S^{-1}(a_2))^*)} \\ &= \overline{\omega_1(a_2^*)} = \overline{\phi(a_2^*a_1)} = \overline{\phi((a_1^*a_2)^*)} = \overline{\overline{\phi(a_1^*a_2)}} = \phi(a_1^*a_2).\end{aligned}$$

The $\psi - \hat{\phi}$ version of Lemma 7.1.12 yields the second assertion. □

Summarizing, we conclude this section with the following duality theorem for algebraic quantum groups.

THEOREM 7.2.3. *(Duality and biduality for algebraic quantum groups) Let (A, Δ) be an algebraic quantum group. Then*

(i) *the dual $(\hat{A}, \hat{\Delta})$ is an algebraic quantum group;*

(ii) *the map $\iota : A \to (\hat{A})'$ given by*

$$\iota(a)(\omega) = \omega(a) \qquad (a \in A,\ \omega \in \hat{A})$$

takes values in $\hat{\hat{A}}$ and defines an isomorphism of algebraic quantum groups between (A, Δ) and $(\hat{\hat{A}}, \hat{\hat{\Delta}})$.

PROOF. (i) This follows directly from Theorem 7.1.11, Proposition 7.2.1, and Theorem 7.2.2.

(ii) By Theorem 7.1.13, it is clear that the map ι is an isomorphism of multiplier Hopf algebras. We just need to show that ι is $*$-linear. Indeed, for all $a \in A$ and $\omega \in \hat{A}$, we have

$$\begin{aligned}\iota(a)^*(\omega) &= \overline{\iota(a)(\hat{S}(\omega)^*)} = \overline{\hat{S}(\omega)^*(a)} = \hat{S}(\omega)(S(a)^*) \\ &= \omega(S(S(a)^*)) = \omega(a^*) = \iota(a^*)(\omega).\end{aligned}$$

$\square$

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