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Time-dependent pointer states, determination of the preferred basis of measurement, and decoherence of quantum systems

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This dissertation includes 3 original papers that have been previously submitted for publication in peer reviewed journals, as follows[1]:

Thesis Chapter	Publication title/full citation	Publication status
3	Submitted to *Annals of Physics*	Under Review
4	Submitted to *Journal of Physics* A	Under Review
5	Submitted to *Journal of Physics* A	Under Review

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Abstract

We present a general analytic method for evaluating the generally time-dependent pointer states of a subsystem, which are defined by their capability not to entangle with the states of another subsystem. We explore the conditions under which the pointer states of the system become independent of time; so that a preferred basis of measurement can be realized. We relate the mathematical conditions for having time-independent pointer states to some classes of possible symmetries in the Hamiltonian of the total composite system. Indeed, our theory would serve as a generalization of the existing theory for determination of the preferred basis of measurement. By exploiting this new theory we can obtain those regimes of the parameter space for a given total Hamiltonian defining our system-environment model for which a preferred basis of measurement can be realized. Moreover, we can predict the corresponding preferred basis of measurement for each regime. We can also obtain the time-dependent pointer states of the system and the environment in most of the other regimes where the pointer states of the system are time-dependent and a preferred basis of measurement cannot be realized at all. This ability to obtain time-dependent pointer states is specifically important in decoherence studies; as these pointer states, although they evolve with time and cannot represent the preferred basis of measurement, they correspond to those initial conditions for the state of the system and the environment for which we can have longer decoherence times.

In the next step, we consider a spin-boson Hamiltonian which is generalized such that

the Hamiltonians for the system ($\hat{H}_{\mathcal{S}}$) and the interaction with the environment ($\hat{H}_{\rm int}$) do not commute with each other. Considering a single-mode quantized field in exact resonance with the tunneling matrix element of the system, we calculate the time-dependent pointer states of the system and the environment for the case that the environment initially is prepared in the coherent state. We also obtain a closed form for the offdiagonal element of the reduced density matrix of the system and study the decoherence of the central system in our model. We will show that for the case that the system initially is prepared in one of its pointer states, the offdiagonal element of the reduced density matrix of the system will be a *sinusoidal function* with a slow decaying envelop which is characterized by a decay time proportional to $\bar{n}$.

I dedicate this dissertation to my dear mother.

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I would like to give special thanks to my supervisor, Dr. G.W.F. Drake, for his complete support and encouragement through all my PhD program. I also would like to thank my family and specially my mother for their love and unconditional support; and all my professors from whom I had the opportunity to learn; from the beginning of my Bachelor's degree all the way to the end of my PhD career. Finally, I would like to thank Dr. W.E. Baylis for supporting the graduate students in our department and also for his encouragements.

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Chapter 1

Introduction

In this dissertation we study four different but closely related problems, within the context of quantum foundation and quantum optics. In essence, the corresponding four chapters, i.e. chapters 3 to chapter 6, build up the main part of my research contribution throughout the six years of my PhD career. The content of chapters 3 to chapter 6 of this dissertation has been prepared in paper format and in terms of four distinct papers for publication at *Journal of Physics* A and *Annals of Physics*. Wherever we refer to the papers within this dissertation, papers I [1] and II [2] respectively refer to the contents of chapters 3 and 6; while papers III [3] and IV [4] respectively refer to the contents of chapters 4 and 5.

A part of the literature review and theoretical background is presented in chapter two. However, the main portion of the literature review and theoretical background for each of the four problems of this dissertation is presented at the beginning of the corresponding chapters; i.e. within chapters 3 to 6. Also each of the main chapters (chapters 2 to 6) has its own introduction section, where we introduce our motivation and the significance of the problem being studied within that chapter.

As we will discuss, the main question within the context of quantum measurement and the quantum-to-classical transition is about how in practice classical systems and properties around us emerge from the underlying quantum domain. In essence, the abovementioned

question mainly is composed of three distinct issues:

1. The problem of the preferred basis of measurement. What singles out the observable which will be measured through a specific system-apparatus interaction. For example, why a specific interaction with a two-level system would result in the measurement of the upper or lower levels of the system (the $|\uparrow\rangle$ and $|\downarrow\rangle$ states; i.e. the eigenstates of the σ_z operator) while another different interaction may result in the measurement of the eigenstates of say the σ_x operator. In other words, how can we know whether a specific interaction would result in realization of a specific basis of measurement or not? And how can we determine the observable which will be measured?

2. The problem of the nonobservability of interference effects. Why is it so difficult to observe quantum interference effects on macroscopic scales?

3. The problem of definite outcomes. Why do measurements have outcomes at all? And supposing that even we do know the observable which will be measured through a measurement interaction, what selects a particular outcome among the different possible outcomes of measurement? This problem usually is referred to as the collapse problem. However, whether such a "collapse" of wave function is objective or subjective still is a subject of debate.

Now we do know that from the abovementioned steps of measurement, the first two questions for sure can be described within the framework of the standard quantum mechanics. However, as we will emphasize within chapters 2 and 6, decoherence cannot solve the collapse problem and it is mainly responsible in describing the second question, i.e. the problem of the nonobservability of interference effects. In fact, this dissertation is also mainly about the first two questions and the question of how to identify the generally time-dependent pointer states of the system and the environment (for a given total Hamiltonian describing a system-environment model), which are characterized as the states which keep their individuality and do not entangle with the states of another subsystem (rather than the collapse problem; which is the very last step of quantum measurement).

Also in writing this dissertation we have not used any specific interpretation of quantum

mechanics; (neither the Copenhagen interpretation nor the many-words interpretation of quantum mechanics). The Copenhagen interpretation assumes that the apparatus and the measuring devices are ruled by the laws of classical physics and not by the rules of quantum physics [5]; however, no longer this interpretation is taken that serious anymore, [6] and now the orthodox view of quantum mechanics tries to evade big assumptions like this. We also have not used the many words interpretation of quantum mechanics. In other words, what we have in this dissertation is not based on any specific interpretation; and the author believes that the followers of all interpretations would agree on the results of this dissertation; since they are merely based on the main structure of the standard quantum mechanics, and no further assumptions. In fact, we believe that before having a solution for the problem of definite outcomes using (or proposing) any interpretation for quantum physics is unjustifiable. Indeed, we have deliberately evaded talking much on the interpretations of quantum mechanics; since we especially wanted the reader's attraction to be drawn more to the significance of this work with respect to the idea of entanglement, which is very important within the context of applied physics; rather than to make the reader think that this work is about interpretations of quantum physics and ideas which may not be that testable (like the many worlds interpretation and so on); or making him/her to think that this knowledge may not be important for applications. Therefore, here our main question is about entanglement and the states which may be immune to the entanglement with the environment; rather than how we should (or should not) interpret quantum physics. However, as we will see, the knowledge which we obtain through this quest for obtaining pointer states will also shed light on the questions which we have in the context of the problem of the preferred basis of measurement. We also will obtain some valuable knowledge about certain aspects of decoherence and decoherence of the models which we study in this research. Nonetheless, the problem of definite outcomes (the collapse problem) and whether it can be possible to describe this problem just within the framework of the standard quantum mechanics or not, still is a big question to be solved.

In the conclusion chapter we will describe more on the linkage between evolution of pointer states and the results of this work, and the ideas of quantum computation and

quantum control; which would suggest some ideas for future work. This dissertation is organized as follows:

After this introduction and in chapter 2 we review the concept of tracing over the environmental degrees of freedom and we discuss the main aspects of the phenomenon of decoherence.

In chapter 3 we discuss time-dependent pointer states and the problem of determination of the preferred basis of measurement. We will also discuss the limitations in the current theories for determination of the preferred basis of measurement. As we will show, pointer states of a system in contact with an environment, which are characterized by their ability not to entangle with the states of another subsystem, quite easily may become time-dependent (for example due to the existence of non-commutative contributions in the Hamiltonian of the total composite system); and hence, generally one must distinguish between pointer states of a subsystem, and the preferred basis of measurement, which consists of time-*independent* pointer states which can be realized *only* in certain regimes. We will present a general formulation for obtaining the generally time-dependent pointer states of the system and the environment and we will study the conditions under which the pointer states of the system may become time-independent; so that a preferred basis of measurement can be realized. The author believes that this chapter along with the fourth chapter are the most significant chapters of this dissertation, as well as his PhD research.

In chapter 4 we apply our formulation for obtaining time-dependent pointer states (discussed in chapter 3) in order to obtain the time-dependent pointer states of the system and the environment for a spin-boson model which is generalized such that the Hamiltonians for the system ($\hat{H}_{\mathcal{S}}$) and the interaction with the environment ($\hat{H}_{\rm int}$) do not commute with each other. We will obtain general expressions for the elements of the reduced density matrix of the system in our model; and we will study the decoherence of the state of the system in our model. As we will show, the offdiagonal elements of the reduced density matrix of the system in this model and for the case that the system initially is prepared in one of its pointer states exhibit a decayo-sinusoidal decoherence; a behavior that has not been observed in other simpler models of decoherence (which often simply show an exponential

decay in the evolution of coherences of the system).

In chapter 5 we do a similar calculation for the quantized atom-field model and in a nonresonance regime; i.e. we will obtain the time-dependent pointer states of the system and the environment for the quantized atom-field model and in a nonresonance regime. We will also obtain general expressions for the elements of the reduced density matrix of the system; and we will study the decoherence of the central system in our model.

In chapter 6 we will discuss some of the issues with Zurek's proof of the Born Rule; and we will present our own proof of the Born rule.

Finally, in chapter 7 we will add some more notes on the significance of our results, and will conclude.

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Chapter 2

On tracing over the environmental degrees of freedom and decoherence

2.1 Introduction

Tracing over the environmental degrees of freedom [1] is usually interpreted as *averaging* over the effect of the environmental degrees of freedom on the state of the system [2]. In the first part of this article, after exploring the exact meaning of tracing over the environmental degrees of freedom we will point out that the interpretation of tracing over the environment as "averaging" over the environment (as in [2]) in fact is not a good description for the process. Especially, it is not accurately true and it does not give us a clear insight about the process.

In the second part of this article, which is designed to serve as a short tutorial we will address some important questions about decoherence. Some questions to be addressed are: what is the exact meaning of the off-diagonal elements of a reduced density matrix? Does decoherence by itself mean that we no longer have the superposition of the states of the system, and is decoherence by itself a nonunitary process? Also, some examples from the literature of misleading or improper statements and interpretations will be discussed.

As we will discuss, the subtle difference between the meaning of the off-diagonal elements of a *reduced* density matrix and the meaning of the off-diagonal elements of a density matrix *before* tracing over the environmental degrees of freedom is often neglected. This often produces ambiguity in interpretation and understanding of the phenomenon of decoherence. As an example, it is often falsely supposed that when the off-diagonal elements of the reduced density matrix are zero, the system under measurement is no longer in a pure state and we no longer have a *superposition* of possible states of the system. Indeed, the duty of the other important step of measurement, ie. the determination of definite outcomes (whose mechanism is yet to be understood), has often been attributed to decoherence; while it is important to know exactly what cannot be described by decoherence. We encounter this mistake in claims like the very common claim that "we do not observe a real Schrodinger cat, because of *decoherence*". In this article we present a standard definition for the phenomenon of decoherence, as just *one* of the steps of the quantum-to-classical transition.

2.2 Tracing over the environmental degrees of freedom

There is a physical meaning behind tracing over the environmental degrees of freedom and in fact, it is not exactly *averaging* over the effect of the environment. As we will elaborately discuss in this section, tracing over the environmental degrees of freedom is in fact, the addition of different probabilities when there is a fine structure including the possible states of the system in addition to some degrees of freedom from another interacting system, when we are calculating the probabilities. [1] Also, tracing over the environmental degrees of freedom is the addition of different interference terms, when we are calculating the interference between different possible states of the system. Here, we will clarify the above statements using an example for tracing over the environmental degrees of freedom which is provided by the Jaynes-Cummings Model (JCM) of quantum optics [4].

Consider the state of the two-level atom (2LA) in the Jaynes-Cummings model, which involves a two-level atom with upper and lower levels that can respectively be represented

[1] As we will see in this section, this indeed follows from Born's rule [3] which defines the probabilities in quantum mechanics.

by $|a\rangle$ and $|b\rangle$, interacting with a single-mode quantized electromagnetic field inside an ideal cavity, represented by creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$.

Suppose that ψ_{an} (ψ_{bn}) is the probability amplitude for finding the 2LA in the upper (lower) state and having n photons in the field. In the double basis set formed by $\{|a,n\rangle\}$'s and $\{|b,n\rangle\}$'s the global state of the system and the environment (the field) generally can be represented by the following matrix

$$|\psi_{\rm tot}(t)\rangle = \begin{pmatrix} \psi_{a0} \\ \psi_{a1} \\ \vdots \\ \psi_{aN} \\ \psi_{b0} \\ \psi_{b1} \\ \vdots \\ \psi_{bN} \end{pmatrix} \tag{2.1}$$

where N is the maximum possible number of the field photons. Here note that although the state space for the state of the two-level atom (2LA) alone is a two dimensional space, the creation of correlations between the states of the 2LA and the field creates a fine structure which includes the states of the system in addition to possible states of the environment. The result can generally can be described in a $2N+2$ dimensional space.

Now, the diagonal element of the reduced density matrix of the system (the 2LA) is given by

$$\begin{aligned} \rho_{11}^{(\rm red)} &= \langle a| \sum_n \langle n|\psi_{\rm tot}\rangle\langle\psi_{\rm tot}|n\rangle \ |a\rangle \\ &= \sum_n \langle a,n|\psi_{\rm tot}\rangle\langle\psi_{\rm tot}|a,n\rangle \\ &= |\psi_{a0}|^2 + |\psi_{a1}|^2 + ... + |\psi_{aN}|^2. \end{aligned} \tag{2.2}$$

We note that since ψ_{an} is the probability amplitude for finding the 2LA in the upper state and having n photons in the field, the above expression for the diagonal element of the reduced density matrix can be considered as sum of all "fine probabilities" for finding the

system in the state $|a\rangle$ while having a certain number of photons in the field (we also note that the number states in the Hilbert space of the environment form a complete basis set). In this sense, tracing over the environmental degrees of freedom can be interpreted as the addition of different contributions that may make a possible state of the system happen.

In fact, the concept of reduced density matrices arises most naturally when we want to compute the expectation value of an observable $\hat{\mathcal{O}}$ in the Hilbert space of the system $\mathcal{S}$, which is entangled with an environment $\mathcal{E}$ (or more generally any other subsystem). Such an operator can be written as $\hat{\mathcal{O}} = \hat{\mathcal{O}}_{\mathcal{S}} \otimes \hat{I}_{\mathcal{E}}$, where $\hat{I}_{\mathcal{E}}$ is the identity operator in the Hilbert space of the environment. Its expectation value generally can be computed using the trace rule,

$$\langle \hat{\mathcal{O}} \rangle = Tr(\hat{\rho}\hat{\mathcal{O}}), \tag{2.3}$$

with $\hat{\rho}$ as the total density matrix for the global state of the system and the environment. As we will show here[2], the trace operation of equation (2.3) can be simplified to a great extent by analytically carrying out the tracing over the environmental degrees of freedom. Suppose that $\{|\psi_m\rangle\}$ and $\{|\phi_n\rangle\}$ are some orthonormal basis sets in the Hilbert spaces of the system and the environment respectively. Then

$$\begin{aligned} \langle \hat{\mathcal{O}} \rangle &= \mathrm{Tr}(\hat{\rho}\hat{\mathcal{O}}) \\ &= \sum_{mn} \langle \phi_n | \langle \psi_m | \hat{\rho}\ (\hat{\mathcal{O}}_{\mathcal{S}} \otimes \hat{I}_{\mathcal{E}}) | \psi_m \rangle | \phi_n \rangle \\ &= \sum_{m} \langle \psi_m |\ (\sum_{n} \langle \phi_n | \hat{\rho} | \phi_n \rangle)\ \hat{\mathcal{O}}_{\mathcal{S}}\ | \psi_m \rangle \\ &= \sum_{m} \langle \psi_m |\ (\mathrm{Tr}_{\mathcal{E}} \hat{\rho})\ \hat{\mathcal{O}}_{\mathcal{S}}\ | \psi_m \rangle \\ &= \sum_{m} \langle \psi_m |\ \hat{\rho}_{\mathcal{S}} \hat{\mathcal{O}}_{\mathcal{S}}\ | \psi_m \rangle\ = \mathrm{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}} \hat{\mathcal{O}}_{\mathcal{S}}), \end{aligned} \tag{2.4}$$

where $\hat{\rho}_{\mathcal{S}}$ is the reduced density matrix for the system. Thus, in order to obtain information about the result of measurements on the system $\mathcal{S}$, which is entangled with an environment $\mathcal{E}$, we can take advantage of the simpler mathematical object which is obtained by tracing over the environmental degrees of freedom.

[2]The discussion on this paragraph can be found in many textbooks, such as the book by Schlosshauer [2].

The above discussion is in fact the source of the traditional interpretation of tracing over the environmental degrees of freedom as an "averaging" over the degrees of freedom of the environment. From the definition of the reduced density matrix, $\hat{\rho}_{\mathcal{S}} = \mathrm{Tr}_{\mathcal{E}}\hat{\rho} = \sum_n \langle\phi_n|\hat{\rho}|\phi_n\rangle$, we can calculate its elements as

$$\rho_{\mathcal{S}}^{ij} = \sum_n \langle\psi_i|\langle\phi_n|\ \hat{\rho}\ |\phi_n\rangle|\psi_j\rangle. \tag{2.5}$$

Now, let us take a closer look to the above expression.

Let m_0 and n_0 be the the number of orthonormal basis states in $\{|\psi_m\rangle\}$ and $\{|\phi_n\rangle\}$ for the system and the environment respectively. The total state of the composite system generally can be represented by a column matrix of dimension $N = m_0 n_0$, whose elements in the double basis set formed by $\{|\psi_m\rangle\}$ and $\{|\phi_n\rangle\}$ are given by the scalar products $\langle\psi_m\phi_n|\psi_{\mathrm{tot}}\rangle$. Also let $\langle\psi_m\phi_n|\psi_{\mathrm{tot}}\rangle = \psi_{mn}$. Now, we can calculate the elements of the reduced density matrix of the system as follows

$$\begin{aligned} \rho_{\mathcal{S}}^{ij} &= \sum_n \langle\psi_i|\langle\phi_n|\ \hat{\rho}\ |\phi_n\rangle|\psi_j\rangle \\ &= \sum_n \langle\psi_i|\langle\phi_n|\psi_{\mathrm{tot}}\rangle\langle\psi_{\mathrm{tot}}|\phi_n\rangle|\psi_j\rangle \\ &= \sum_n \psi_{in}\psi_{jn}^*, \end{aligned} \tag{2.6}$$

where we have assumed that the global system starts in a pure state. For example, the diagonal elements read

$$\rho_{\mathcal{S}}^{ii} = \sum_n |\psi_{in}|^2. \tag{2.7}$$

What we are doing here in fact is just the generalization of our previous calculation for the Jaynes-Cummings model. In the above equation ψ_{in} is the probability amplitude for having the environment in the state $|\phi_n\rangle$ while the system is in the state $|\psi_i\rangle$. Hence, as we already mentioned, when we are calculating the diagonal elements of the reduced density matrix (i.e. the probabilities), tracing over the environmental degrees of freedom is nothing but the addition of different "fine-probabilities" that may contribute in the realization of a specific state of the system. We can also write the expectation value of an operator acting on the Hilbert space of the system in another useful form by choosing $\{|\psi_m\rangle\}$ (of equation

(2.4)) be the eigenvectors of the system observable $\hat{\mathcal{O}}_{\mathcal{S}}$ with eigenvalues o_m

$$\begin{aligned} \langle\hat{\mathcal{O}}\rangle &= \mathrm{Tr}(\hat{\rho}\hat{\mathcal{O}}) \\ &= \sum_m \langle\psi_m| \left(\sum_n \langle\phi_n|\hat{\rho}|\phi_n\rangle\right) \hat{\mathcal{O}}_{\mathcal{S}} \; |\psi_m\rangle \\ &= \sum_{mn} o_m |\psi_{mn}|^2. \end{aligned} \tag{2.8}$$

Here, the meaning of "averaging" is embedded in summation over the complete set of eigenvalues corresponding to the system observable $\hat{\mathcal{O}}_{\mathcal{S}}$, in addition to giving a weight to each eigenvalue by including the appropriate probabilities. However, note that generally we can not discriminate between averaging over the environment and averaging over the system since in general, the global state of the system and the environment is a highly entangled state. To clarify this point note that, *only if* the states of the system and the environment were not entangled, ie. if $|\psi_{\mathrm{tot}}(t)\rangle = |\psi_{\mathcal{S}}(t)\rangle \otimes |\psi_{\mathcal{E}}(t)\rangle$ (with $|\psi_{\mathcal{S}}(t)\rangle$ and $|\psi_{\mathcal{E}}(t)\rangle$ as some state vectors of the system and the environment respectively), we could rewrite equation (2.8) as

$$\langle\hat{\mathcal{O}}\rangle = \sum_{mn} o_m |\langle\psi_m|\psi_{\mathcal{S}}\rangle\langle\phi_n|\psi_{\mathcal{E}}\rangle|^2. \tag{2.9}$$

By letting

$$\langle\psi_m|\psi_{\mathcal{S}}\rangle = c_m \quad \text{and} \quad \langle\phi_n|\psi_{\mathcal{E}}\rangle = d_n, \tag{2.10}$$

this will read

$$\langle\hat{\mathcal{O}}\rangle = \sum_{mn} o_m |c_m|^2 |d_n|^2. \tag{2.11}$$

In this case also

$$\mathrm{Tr}_{\mathcal{E}}(\hat{\rho}) = \sum_n \langle\phi_n|\psi_{\mathrm{tot}}\rangle\langle\psi_{\mathrm{tot}}|\phi_n\rangle = \sum_n |d_n|^2 |\psi_{\mathcal{S}}\rangle\langle\psi_{\mathcal{S}}|. \tag{2.12}$$

In the above equations, by summation over n and weighting by $|d_n|^2$ (which is the probability of finding the environment in the state $|\phi_n\rangle$) we are averaging over the environment while by summation over m and weighting by $|c_m|^2$ (which in fact is the probability of finding the system in the state $|\psi_m\rangle$) we are averaging over the degrees of freedom of the system. Hence, only in this case can we conclude that the trace operation over the environmental degrees of freedom can be interpreted as averaging over the environment. In any other case,

ie. if the states of the system and the environment are entangled, we can not write the global state of the total composite system in a product form like $|\psi_{\rm tot}(t)\rangle = |\psi_{\mathcal{S}}(t)\rangle \otimes |\psi_{\mathcal{E}}(t)\rangle$ and hence one cannot write the probabilities $|\psi_{mn}|^2$ in equation (2.8) as the product of probabilities corresponding to the Hilbert space of the system and probabilities corresponding to the Hilbert space of the environment, as in equation (2.11). As a result, one cannot discriminate between averaging over the environment and averaging over the system if the global state of the total composite system is an entangled one and the interpretation of tracing over the environmental degrees of freedom as "averaging" over the environment is by no means accurate.

Next, let us come back to the general expression in equation (2.6) in order to study the off-diagonal elements of the reduced density matrix which are given by

$$\rho_{\mathcal{S}}^{ij} = \sum_n \psi_{in}\psi_{jn}^*; \quad \text{with} \quad i \neq j \ . \tag{2.13}$$

For a two-state system like the two-level atom in our previous example of the Jaynes-Cummings model with upper and lower levels represented by $|a\rangle$ and $|b\rangle$, this will read

$$\rho_{\mathcal{S}}^{(ab)} = \psi_{a0}\psi_{b0}^* + \psi_{a1}\psi_{b1}^* + ... + \psi_{aN}\psi_{bN}^*. \tag{2.14}$$

Note that if we denote the quantum state of the total composite system corresponding to the case that the two-level atom is in the upper or lower states respectively by $|\psi_a^{(\rm tot)}\rangle$ and $|\psi_b^{(\rm tot)}\rangle$, the expression in equation (2.14) can be recognized as the overlap $\langle\psi_b^{(\rm tot)}|\psi_a^{(\rm tot)}\rangle$ between these states (or equally the scalar product between the two state-vectors). This is similar to the example of a Young's double-slit experiment with light where the interference contribution to the total intensity on the screen is given by the scalar product of the two electric fields: $I_{\rm int} = \vec{E_1} \ . \ \vec{E_2} = \sum_i E_{1i} \ E_{2i}$. Here also we have summation over just one index and although one might call this a kind of averaging, but if we want to talk more precisely we should say that in fact, it is the addition of different interference terms. Therefore, we understand that when calculating the *off-diagonal elements* of reduced density matrices also, the tracing over the environmental degrees of freedom is not exactly an *averaging*. But in fact, it exactly calculates the *interference* between two possible states of the system by adding the contributions from the interference of different branches of

the state of the total composite system which can contribute in creating the interference (equation (2.14)).

2.3 Other notes on reduced density matrices and decoherence

We note that, not all of the possible states of a composite system are necessarily able to interfere. A good example in the classical limit is two wavelets of light that meet each other at a specific point on a distant screen and having perpendicular polarizations, which of course will not interfere. They just add like different orthogonal components of a vector without interfering; while both of them do exist at the same time and the (classical) superposition is still there. Similarly, when the off-diagonal elements of the reduced density matrix are zero, it just means that we have no interference contribution to the total intensity (like $I = I_1 + I_2 + I_{12}$ with $\mathrm{I}_{12} = 0$). It does not necessarily mean that there is no superposition of the states. In the quantum limit also, for example if we consider a pure state of the EPR type $|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2$, the reduced density matrix for any of the two subsystems has no offdiagonal elements if the $|\pm\rangle$ states make an orthogonal basis. However, as we will describe, the fact that the reduced density matrix has no offdiagonal elements does not necessarily mean that there is no superposition of the two states of the systems. It just means that the two branches of the total system are orthogonal and hence, they are not able to interfere.

We can understand the exact meaning of the suppression of the offdiagonal elements of density matrices, which is formally referred to as decoherence, more easily if we study the decoherence of the state of the system in the pointer states basis of measurement. Here, as we will describe, the pointer states of the system and the environment are defined as pairs of states which are characterized by their ability not to entangle with the states of another subsystem. In other words, the pointer states of a subsystem (although generally they can be time-dependent states but they) maintain their individuality, as well as their one-to-one correspondence with the pointer states of another subsystem during the time evolution of

the composite system . Any states other than the pointer states of the system are subject to entanglement with the states of the environment; so that they lose their individuality (this means that the system cannot be ascribed with a well-defined state, due to the entanglement with the environment) and one cannot consider a one-to-one correspondence between some well-defined states from the system and some states from the environment [2, 5, 6].

Now once again let us consider a two-state system $\mathcal{S}$ with a preferred set of basis states $|s_0\rangle$ and $|s_1\rangle$, which after premeasurement by the environment become coupled with two states of the environment $|\varepsilon_0\rangle$ and $|\varepsilon_1\rangle$ respectively[3]. Before premeasurement by the environment, which creates a one-to-one correspondence between the pointer states of the system and those of the environment, the global state of the system and the environment can be represented by

$$|\psi_{\mathcal{SE}}\rangle = \{\alpha|s_0\rangle + \beta|s_1\rangle\} \otimes |\varepsilon_{\mathrm{i}}\rangle, \tag{2.15}$$

where $|\varepsilon_{\mathrm{i}}\rangle$ is the initial state of the environment before any coupling between the states of the system and the environment. Also, after premeasurement is complete the global state of the system and the environment is given by

$$|\psi_{\mathcal{SE}}\rangle = \alpha|s_0\rangle|\varepsilon_0(t)\rangle + \beta|s_1\rangle|\varepsilon_1(t)\rangle. \tag{2.16}$$

The states appearing in the above equation are the pointer states of the system and the environment [2, 5, 6, 7]. Also, in writing the above equation it is assumed that the pointer states of the system are time-independent; so that they can represent the preferred basis of measurement. Now, the off-diagonal element of the reduced density matrix of the system is given by

$$\rho_{12}^{\mathcal{S}} = \alpha\beta^*.\langle\varepsilon_1(t)|\varepsilon_0(t)\rangle. \tag{2.17}$$

while initially it simply was $\rho_{12}^{(\mathrm{tot})} = \alpha\beta^*$. However, what really is a measure for the existence of superposition of the states is exactly the product $\alpha\beta^*$; for only when $\alpha\beta^* \neq 0$, can we assume that the system is in the superposition of its two possible states. On the other hand, the off-diagonal element of the *reduced* density matrix includes another factor. i.e. the time-dependent overlap between the two pointer states of the environment $\langle\varepsilon_1(t)|\varepsilon_0(t)\rangle$.

[3]We will discuss the premeasurement by the environment more elaborately in chapter 3.

Hence, we may have $\alpha\beta^* \neq 0$, while $\langle\varepsilon_1(t)|\varepsilon_0(t)\rangle = 0$ and therefore $\rho_{12}^{\mathcal{S}} = 0$. In this case, although both of the two basis states of the system $|s_0\rangle$ and $|s_1\rangle$ still can be measured in a successive measurement on the state of the system, and in this sense the system is still in a superposition of its two possible states, the off-diagonal element of the density matrix is zero and this is because of the fact that after the states of the system are coupled to the corresponding states of the environment, as the overlap between the pointer states of the environment decreases from the initial value of $\langle\varepsilon_{\rm i}|\varepsilon_{\rm i}\rangle = 1$, the off-diagonal element of the reduced density matrix of the system starts to vanish. This is exactly what happens during the phenomenon of decoherence by the environment.

In essence, this is the time evolution of the pointer states of the environment that causes the offdiagonal elements of the density matrix to vanish[4]. Here we should also mention that the study of several models of decoherence has revealed that the rate at which the overlap between the pointer states of the environment decreases (which also is a measure for the decoherence rate) often is increased as we increase the number of the environmental degrees of freedom [6, 7, 8, 9].

Hence, it is important to distinguish between the meaning of the off-diagonal elements of a density matrix *before* and *after* tracing over the environmental degrees of freedom. As we discussed, although the off-diagonal elements of a density matrix before tracing over the environmental degrees of freedom exactly refer to the existence of superpositions, we should note that after tracing over the environmental degrees of freedom, they just refer to interference. Furthermore, we should exactly clarify what we mean by "coherence" in the word "decoherence". Indeed, if by "coherence" we refer to the existence of superposition of pointer states, then "decoherence" of course cannot be a good description for the suppression of the off-diagonal elements of density matrices; since as we discussed, up to this stage of a quantum measurement we can only conclude that such "coherence" between different components of the pure state of a quantum system can only be delocalized into the larger

[4]However, note that if we choose not to work in the complete basis set which contains the pointer states of the environment, we can equivalently describe the decoherence by the environment, which results in the nonobservability of any further interference between the states of the system, through the dephasing in the summation represented by equation (2.14).

composite system as a result of quantum entanglement, and it is not yet disappeared (see eqs. (15), (16) and (17)). However, we can justify the use of the word "decoherence" if by "coherence" we refer to the phase coherence among different interference terms in the summation represented in equation (2.13), in which when there is phase coherence between different contributions, they will add constructively and we expect to have interference.

The above point actually is in contrast with what is often advocated by many authors regarding decoherence. In fact, the word "coherence" is often wrongly used by taking it to be synonymous with superposition. For example, in his paper entitled "the decoherence puzzle" [10], Stamp interchangeably uses the two words of "coherence" and "superposition" instead of each other; eg. when he uses the expression "coherence (i.e., superpositions)". Many other authors use this word as equivalent to superposition. However, it is important to correctly use this word by attributing it to the phase coherence rather than superpositions; since its usage as equivalent to quantum superposition can imply that decoherence is related to the loss of superpositions. As another example, Vedral in his book entitled *Introduction to quantum information science* [11] writes:

> *Dissipation implies loss of energy to the environment, while decoherence implies the loss of coherence, ie. superpositions, and may not involve any energy exchange.*

We should also note that *reduced* density matrices generally do not specify the state of the system; in the sense that by looking at a density matrix we cannot discern if the system, which it intends to describe, is in a *pure* or in a *mixed* state. This can also be understood by noting that the reduced density matrix corresponding to the pure state of a composite system with environmental pointer states that are orthogonal literally has no difference with a density matrix that can also represent a mixed state. For example, for a pure state of the EPR type

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2), \tag{2.18}$$

if the $|\pm\rangle$ states make an orthogonal basis, clearly the reduced density matrix for any of the two subsystems has no offdiagonal elements. e.g. for the first subsystem the reduced

density matrix reads

$$\hat{\rho}^{(1)} = \frac{1}{2}\{|+\rangle_1\langle+|_1 + |-\rangle_1\langle-|_1\}. \tag{2.19}$$

Here we emphasize that in the state of the total composite system, represented by equation (2.18), we do have a superposition of the states of each subsystem; as we have *coexistence* of the $|+\rangle_{1(2)}$ and $|-\rangle_{1(2)}$ states of each subsystem; *although* each of the $|+\rangle_{1(2)}$ and $|-\rangle_{1(2)}$ states is correlated with some corresponding states from the other subsystem, and we cannot ascribe a state vector to each of the subsystems alone[5].

Now, note that if we consider the density matrix corresponding to a mixed state of the total composite system given by

$$\rho^{\text{tot}} = \frac{1}{2}\{|+\rangle_1\langle+|_1|-\rangle_2\langle-|_2 + |+\rangle_2\langle+|_2|-\rangle_1\langle-|_1\}, \tag{2.20}$$

tracing over the degrees of freedom of the second subsystem obviously would result in a reduced density matrix for the first system as that of equation (2.19), which was the reduced density matrix which we obtained from the total density matrix of the *pure* state, given by equation (2.18).

Now if we consider the purity, $\text{Tr}(\rho^2)$, for the reduced density matrix given by equation (2.19), it would be equal to $\frac{1}{2}$; which based on that one might conclude that the system is in a mixed state. However, we do know that the reduced density matrix of equation (2.19) could equally be obtained from the pure state of equation (2.18), *or* the mixed state represented by the total density matrix of equation (2.20). Therefore, although our reduced density matrix will show the same observable properties no matter if it is obtained from the pure state of equation (2.18) or the mixed state of equation (2.20), by having such a reduced density matrix we cannot conclude whether we have the superposition of the states of the system, or the system is part of a composite system in a mixed state[6].

[5]It is true that here we cannot ascribe a well-defined state to the system alone. But, *none of the possible branches of the system yet are selected at this stage of a measurement.* So, they still coexist at this step; although they coexist while they are coupled with some corresponding pointer states from the environment, through a perfect one-to-one correspondence between the pointer states of the system and the environment.

[6]The essence of this argument has been established in another way through the so-called purification theorem [2, 12], which states that any arbitrary nonpure state can always be regarded as the reduced state of the pure state of a larger composite system.

As another explanation, by looking at equation (2.17) we see that if we have time-independent pointer states for the environment (i.e. system 2 in this example) which are orthogonal to each other, definitely the coherences of the reduced density matrix of the system (i.e. system 1 in this example) will be zero. However, as we discussed, this necessarily does not mean that we do not have the superposition of the states of the system. Regarding the reduced density matrix of equation (2.19) also, the fact that it does not have any offdiagonal elements simply is a result of having $\langle +|-\rangle_2$ as equal to zero (see equations (2.17) and (2.18)); and necessarily it does not mean that we do not have a superposition of the $|\pm\rangle_1$ states.

In essence, one should be careful that the so called "purity", defined by $\mathrm{Tr}(\rho^2)$, is not a measure for the purity of the state of the system when we are considering *reduced* density matrices. Moreover, decoherence is not responsible for conversion of superpositions into mixed states, as it only makes the pointer states of the environment orthogonal to each other; removing the possibility of any interference effects (see equation (2.17)). This point also is in contrast to claims which one often encounters, like the following claim by Zurek [13]

> *... For our purposes, the effect of the last term on quantum superpositions is of greatest interest. I shall show that it destroys quantum coherence, eliminating offdiagonal terms responsible for quantum correlations between spatially separated pieces of the wave packet. It is therefore responsible for the classical structure of the phase space, as it converts superpositions into mixtures of localized wave packets which, in the classical limit, turn into the familiar points in phase space.*

We finish this article by discussing the relationship between the unitarity of the evolution of the total system, which is imposed by evolution according to the Schrödinger equation, and the decoherence of the state of the system.

We do know that decoherence is not a nonunitary process by itself; and this can be understood by considering the fact that the total composite system is assumed to be a closed system whose evolution is governed by the Schrödinger equation; and hence we do

expect that the evolution of the global state of the system and the environment preserves the norm of the total composite system. Also, as each of the subsystems that make the total composite system cannot be regarded as a closed system, naturally we do not expect that the evolution of each of the subsystems follows a unitary procedure. Nonetheless, in what follows we carefully examine the unitarity of the evolution of the global state of the system and the environment and its consequences, with specific attention to the case that the pointer states of the system necessarily are *not* orthogonal. This can be useful with respect to certain experimental settings for the study of the interference effects in which the pointer states of the system necessarily are not orthogonal.

Again let us consider the global state of the system and the environment represented by equation (2.16), which is created after the interaction between the two-level system $\mathcal{S}$ (initially prepared in the state $|\psi_S\rangle = \alpha|s_0\rangle + \beta|s_1\rangle$) and the environment $\mathcal{E}$ determines the pointer states of the system and the environment as the pairs of states which maintain their individuality, as well as the one-to-one correspondence between them, during their evolution with time [2]. It is important to note that the pointer states of the system and those of the environment generally are time-dependent. This is in contrast to the Von Neumann scheme for *ideal* quantum measurement which assumes that the measurement interaction does not change the states of the system. In fact, a good example for the time dependence of pointer states during the system-environment interaction again is provided by studying the evolution of the two-level atom in the Jaynes-Cummings model of quantum optics [14]. In appendix A we describe that in this model, and in the limit of a large average number of photons in the field, the pointer states of the system and the environment (the electromagnetic field) are time-dependent. Moreover, as we will show, the pointer states of the system and those of the environment generally are not orthogonal within themselves; as their overlap evolves with time. For example, as we will see in appendix A, the pointer states of the environment initially are not orthogonal; However, they become orthogonal within a very short time of the order of $t_c = 1/\mathrm{g}$; where g is the atom-field coupling constant.

Here in order to emphasize that the pointer states of the system and the environment

are time-dependent, let us rewrite equation (2.16) as follows

$$|\psi_{S\mathcal{E}}\rangle = \alpha|s_0(t)\rangle|\varepsilon_0(t)\rangle + \beta|s_1(t)\rangle|\varepsilon_1(t)\rangle. \tag{2.21}$$

We emphasize that the states appearing in the above equation must be discriminated from the instantaneous Schmidt states which can be obtained by diagonalizing the density matrix at each instant of time and which generally are not unique. Mainly because of the fact that the pointer states of the system and the environment, appearing in the diagonal state of the total composite system in the above equation, are the states which are characterized by their ability not to entangle with the states of another subsystem,[7] and one can examine this property of candidate pointer states at least in principle by solving the Schrödinger equation[8]. However, Schmidt states generally are not unique, and in addition to that they necessarily do not exhibit the quasiclassical properties of pointer states and hence, generally are not expected to preserve a one-to-one (system-environment) correspondence amongst themselves when evolved according to the Schrödinger equation. Moreover, the states appearing in equation (2.21) are not necessarily orthogonal at all times; while by definition the Schmidt states must be orthogonal within themselves at all times.

Coming back to our discussion regarding the unitarity in the evolution of the state of the total system, now the norm of the state of the total composite system is related to

$$\begin{aligned}\langle\psi_{\rm tot}|\psi_{\rm tot}\rangle = |\alpha|^2\ \langle s_0(t)|s_0(t)\rangle\ \langle\varepsilon_0(t)|\varepsilon_0(t)\rangle \\ +|\beta|^2\ \langle s_1(t)|s_1(t)\rangle\ \langle\varepsilon_1(t)|\varepsilon_1(t)\rangle \\ +\alpha\beta^*\ \langle s_1(t)|s_0(t)\rangle\ \langle\varepsilon_1(t)|\varepsilon_0(t)\rangle + \text{c.c.}\end{aligned} \tag{2.22}$$

As we discussed, decoherence is related to the suppression of the factor $\langle\varepsilon_1(t)|\varepsilon_0(t)\rangle$ of the third term in the above equation. Now *if* the pointer states of the system are orthogonal,

[7] This is often referred to as the *stability criterion* for determination of pointer states.

[8] In other words, pointer states of a system emerge dynamically, as a result of the natural evolution of the global state of the system and the environment, as those states that are the most robust to the interaction with the environment. In the sense that they become least entangled with the environment in the course of their evolution with time and a one-to-one (system-environment) correspondence is preserved between pointer states from the system and those of the environment.

we note that the third term in equation (2.22) will be zero; no matter how big is the factor $\langle\varepsilon_1(t)|\varepsilon_0(t)\rangle$ and hence, no matter if decoherence occurs or not. However, nonorthogonal pointer states are not *a priori* forbidden; as they can arise in certain experimental settings where we can observe interference effects. As an example, the pointer states of the system in the JCM (as are discussed in appendix A) are not orthogonal at all times. Or as another example in the context of quantum optics, it has been shown that for a harmonic oscillator interacting with an environment in thermal equilibrium and in the weak coupling limit the interaction between the system and the environment will result in the superselection of coherent states [4, 15] as the pointer states of the system which are characterized by maximal stability [15]; while the coherent states are well known to be nonorthogonal amongst themselves.

In such cases like that of the Jaynes-Cummings model where the pointer states of the system are *not* orthogonal at all times, although none of the two factors $\langle s_1(t)|s_0(t)\rangle$ and $\langle\varepsilon_1(t)|\varepsilon_0(t)\rangle$ of equation (2.22) is uniformly equal to zero, still we expect the last two terms in equation (2.22) to be zero; since $\langle\psi_{\rm tot}|\psi_{\rm tot}\rangle$, given by equation (2.22), must always (i.e. for all possible α and β) be equal to the unity; including for the case that $\alpha = 0$ or $\beta = 0$. Therefore, we must have $\langle s_0(t)|s_0(t)\rangle \times \langle\varepsilon_0(t)|\varepsilon_0(t)\rangle = \langle s_1(t)|s_1(t)\rangle \times \langle\varepsilon_1(t)|\varepsilon_1(t)\rangle = 1$ and $\langle s_1(t)|s_0(t)\rangle \times \langle\varepsilon_1(t)|\varepsilon_0(t)\rangle = 0$. Only in this case the norm of the state of the total composite system will always be preserved, although decoherence is in progress all throughout the evolution of the state of the total system due to the decay of the overlap between the pointer states of the environment $\langle\varepsilon_1(t)|\varepsilon_0(t)\rangle$.

The last of the above conditions is possible only if at those probable times for which $\langle\varepsilon_1(t)|\varepsilon_0(t)\rangle \neq 0$ we have the zeros of $\langle s_1(t)|s_0(t)\rangle$ to take place; and whenever $\langle s_1(t)|s_0(t)\rangle \neq 0$ we have the zeros of $\langle\varepsilon_1(t)|\varepsilon_0(t)\rangle \neq 0$ to take place. In appendix A considering the approximation for obtaining the pointer states of the Jaynes-Cummings model, given by $t \ll \hbar\sqrt{\bar{n}}/\mathrm{g}$ (with $\bar{n}$ as the average number of photons in the environment) [8], we show how this condition exactly is satisfied for $t \ll \hbar\sqrt{\bar{n}}/\mathrm{g}$. We expect this condition to be satisfied even for arbitrary large times (as long as pointer states can be realized for the state of the system and the environment, so that we can represent the state of the total composite

system in a diagonal form in terms of pointer states); although for the Jaynes-Cummings model one may not be able to do the same kind of calculation in any analytical way for arbitrary large times.

In essence, although decoherence progresses and the overlap between the pointer states of the environment constantly decreases with time, the last two contributions in the norm of the state of the total composite system (equation (2.22)) are always zero, and this way it is guaranteed that the norm of the state of the total composite system is preserved. The generalization of the above discussions to the case that the system has more than two possible states can be an interesting problem to be explored.

2.4 Conclusion

We studied the exact meaning of tracing over the environmental degrees of freedom as a procedure that (by considering the fine structure which is created due to the formation of correlations between the states of the system and those of the environment) calculates the probabilities for finding the system in specific states, by adding all "fine-probabilities" for finding the system in specific states, when we are calculating the diagonal elements of the reduced density matrix. It also merely calculates the interference between two possible states of the system, by adding the contributions from the interference of corresponding branches of the global state of the system and the environment, when we are calculating the off-diagonal elements of the reduced density matrix (equation (2.13)).

We emphasized that the suppression of the off-diagonal elements of the reduced density matrix, which is formally referred to as decoherence, should not be taken as the suppression of the quantum superposition and this is simply because of the fact that the off-diagonal elements of a reduced density matrix exactly refer to interference rather than the existence of superposition. In fact, after the formation of correlations between the states of the system and those of the environment creates a one-to-one correspondence between the pointer states and up to the stage that decoherence occurs we can *only* conclude that the superposition initially confined to the system has been *delocalized* from the system into the larger composite system (eqs. (15) and (16)). Hence, decoherence is not responsible for the

conversion of superpositions into mixed states as it only makes the pointer states orthogonal to each other; removing the possibility of any interference effects. Also the so-called purity of a *reduced* density matrix is not actually a measure for the purity of the state of the system.

We note that the presented discussions are *not* necessarily in favor of an Everettian interpretation of quantum mechanics; if we keep in mind that decoherence is just *one* of the steps in the procedure of the quantum-to-classical transition, which only refers to the interference effects. Hence, although decoherence provides us with a description for the usual nonobservability of quantum interference effects in macroscopic scales, but the mechanism by which we observe definite outcomes of measurement still remains a fundamental question to be answered.

Finally, we emphasized that the pointer states of the system must be discriminated from the instantaneous Schmidt states; and we discussed that decoherence has no contradiction with the unitary evolution of the global state of the system and the environment; as the total composite system is assumed to be a closed system. Hence, of course decoherence must be considered as a unitary process. However, as we showed, the unitarity of the evolution of the total composite system requires that at those probable times for which the overlap between the pointer states of the system is not zero, we must have the zeros of the overlap between the pointer states of the environment; and whenever the overlap between the pointer states of the *environment* is not zero, we must have the zeros of the overlap between the pointer states of the *system* to take place.

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Chapter 3

Time-dependent pointer states and determination of the preferred basis of measurement

3.1 Introduction

This chapter is organized as follows:

In section 3.2 (which is our introductory review and discussion section to better identify what the problem is) after reviewing the orthodox theory for determination of the pointer states of measurement we discuss an important restriction of this theory regarding the so-called *commutativity criterion* for determination of the pointer states. In fact, we will show that the pointer states of the system and the environment, which appear in the diagonal state of the total composite system

$$|\psi_{S\mathcal{E}}\rangle = \sum_{k=1}^{N} \alpha_k |s_k\rangle|\varepsilon_k\rangle \tag{3.1}$$

after premeasurement by the environment [1], generally are time-dependent and the commu-

[1] In this dissertation whenever we talk about the environment it refers to all the subsystems which are

tativity criterion, first introduced by Zurek [1-3], although sufficient in order to assure the requirement of having a faithful measurement, is too restrictive and generally does not hold valid for the pointer states of measurement in all situations. In other words, it is not the *minimal* condition for having a faithful measurement. We will show that the commutativity criterion can be valid *only* in certain regimes and under the specific conditions in which the pointer states of the system [2] are independent of time.

In section 2 we will also very briefly review some of the other predictability criteria which have been exploited to obtain the pointer states of measurement. We will discuss some of the restrictions and difficulties which one should expect while using these methods.

In section 3 we present a method in order to calculate the (generally) time-dependent pointer states of the system and the environment for an arbitrary total Hamiltonian defining the system-environment model and in section 4, using this method, we will exactly discuss under which conditions we can have time-independent pointer states; and also how we can predict the preferred basis of measurement in each of the corresponding regimes. As we will see, only under specific conditions time-independent pointer states can be realized; therefore a preferred basis of measurement does not necessarily exist in an arbitrary regime.

In section 5 we will discuss the significance of our theory more elaborately and will conclude.

outside our system of interest and hence it can include the apparatus as well.

[2]The pointer states of a system are characterized by their ability not to entangle with the states of the environment (i.e. the requirement of faithful measurement) and appear in the diagonal state of the total composite system after premeasurement by the environment. As we elaborately describe in this chapter, generally we should distinguish between the set of pointer states in equation (3.1) and the preferred basis of measurement; mainly because of the fact that the pointer states of a subsystem generally are time-dependent and a preferred basis of measurement does not exist, unless under the specific conditions (discussed in section 4) in which the pointer states of measurement become time-independent. Moreover, the pointer states of a system are not necessarily orthonormal amongst themselves at all times. Therefore, they cannot necessarily form a basis for the Hilbert space of the system at all times.

3.2 Review and discussion: identifying the problem

3.2.1 The Schmidt decomposition

Before proceeding to the question of the pointer states of measurement we would talk about the old *Schmidt Decomposition Theorem* [4] which states that an arbitrary pure state $|\Psi\rangle$ of the composite system $\mathcal{AB}$, made up of two subsystems $\mathcal{A}$ and $\mathcal{B}$ endowed with Hilbert spaces $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$, can always be written in a diagonal form

$$|\Psi\rangle = \sum_i \lambda_i \ |a_i\rangle \ |b_i\rangle, \tag{3.2}$$

where the Schmidt states $|a_i\rangle$ and $|b_i\rangle$ are orthonormal amongst themselves and form the so-called *Schmidt bases* of $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$ respectively, and the expansion coefficients λ_i generally are some complex numbers fulfilling $\sum_i |\lambda_i|^2 = 1$. Moreover, it is shown that this decomposition is unique if and only if the expansion coefficients λ_i are all different from one another. Note that the above statement basically refers to the fact that in describing the total state of a composite system in a diagonal form with orthonormal basis states generally there is a *basis ambiguity*, as the diagonal decomposition is not always unique.

We also note that the reduced density matrices $\hat{\rho}_{\mathcal{A}}$ and $\hat{\rho}_{\mathcal{B}}$ for the two subsystems which are obtained by tracing operation will be diagonal in the Schmidt bases $\{|a_i\rangle\}$ and $\{|b_i\rangle\}$, as one can easily verify, because of the fact that these states are orthogonal amongst themselves. Therefore, the Schmidt bases correspond to the orthonormal basis states which diagonalize the reduced density matrices.

3.2.2 Premeasurement by the environment

In order to define the pointer states of measurement and describe their exact distinction from the Schmidt bases and also from the preferred basis of measurement we need to describe what we mean by a faithful measurement. Faithful measurement in the usual sense concerns the requirement of a one-to-one correspondence between the states of the system and the apparatus. Just in this case, the state of the apparatus can be viewed as a reliable pointer (indicator) for the state of the system. This requirement can describe the

Von Neumann scheme [5] for quantum measurement which states that if the system starts out in a superposition of the basis states $|s_i\rangle$,

$$|\psi\rangle = \sum_i c_i |s_i\rangle \tag{3.3}$$

then, the system-apparatus combination will evolve according to

$$|\psi\rangle|a_r\rangle = (\sum_i c_i|s_i\rangle)|a_r\rangle \rightarrow |\psi\rangle = \sum_i c_i|s_i\rangle|a_i\rangle \tag{3.4}$$

where $|a_r\rangle$ is the initial "ready" state of the apparatus. We note that the Von Neumann scheme described by equation (3.4), which usually is also referred to as *premeasurement*, can be obtained just by assuming the requirement of the faithful measurement of the state of the system initially prepared in the state $|s_i\rangle$, i.e. the requirement that $|s_i\rangle|a_r\rangle \rightarrow |s_i\rangle|a_i\rangle$ and the linearity of Schrödinger's equation.

The states appearing in the righthand side of equation (3.4) are the pointer states of the two subsystems. The main characteristic of the pointer states of the system and the environment is their capability to maintain their individuality, as well as the one-to-one correspondence between themselves (the requirement of faithful measurement), during the interaction. This means that any states other than the pointer states of the system are subject to entanglement with the states of the environment so that they lose their individuality (meaning that one no longer can ascribe a well-defined state to the system alone, due to the entanglement with the environment) and one cannot consider a one-to-one correspondence between some well-defined states from the system and some states from the environment. However, the Von Neumann scheme assumes that the measurement interaction is ideal in the sense that it does not change the state of the system. In other words, it assumes a *quantum nondemolition procedure*. However, this assumption about premeasurement is not necessarily true. In fact, as we will show, the pointer states, which are characterized by their ability not to entangle with the states of another subsystem, generally are time-dependent. So, we must differentiate between the pointer state of a subsystem and the preferred basis of measurement. In this dissertation we refer to the preferred basis of measurement as time-independent pointer states which can be realized only in certain regimes.

3.2.3 Example: evolution of the two-level atom in the Jaynes-Cummings model

To clarify this better here we present a physical example, represented by the evolution of the two-level atom in the Jaynes-Cummings model (JCM) of quantum optics, in order to show how in practice a diagonal state with *time-dependent* pointer states can be created as a result of the natural evolution of the global state of the system and the environment.

Consider the state of the two-level atom in the Jaynes-Cummings model of quantum optics which involves a two-level atom, with upper and lower levels that can respectively be represented by $|a\rangle$ and $|b\rangle$, interacting with a single-mode quantized electromagnetic field inside an ideal cavity, represented by creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$. For exact resonance and in the rotating-wave approximation, the interaction Hamiltonian for the composite system can be written as

$$\hat{H}_{\rm int} = \hbar {\rm g}(\hat{a}^\dagger \sigma_- + \sigma_+ \hat{a}). \tag{3.5}$$

where ${\rm g} = -\varrho_{12}.\hat{\epsilon}\sqrt{\frac{w}{2\hbar\varepsilon_\circ V}}$ is the atom-field coupling constant, with $\varrho_{12} = e\langle a|\mathbf{r}|b\rangle$ as the atomic electric-dipole transition matrix element. ($\hat{\epsilon}$ is the field polarization vector, ω is the atomic transition frequency which is taken to be resonant with the frequency of the cavity eigenmode, and V is the cavity mode volume). Also σ_+ and σ_- are the atomic flipping operators given by

$$\sigma_+ = |a\rangle\langle b| \qquad \text{and} \qquad \sigma_- = |b\rangle\langle a|. \tag{3.6}$$

Consider the field to be initially in the coherent state $|\nu\rangle$

$$|\Phi_{\rm field}(t_0)\rangle = |\nu\rangle = \sum_{n=0}^{\infty} {\rm c}_n |n\rangle; \quad \text{with} \quad {\rm c}_n = \frac{{\rm e}^{-\frac{1}{2}|\nu|^2}\nu^n}{\sqrt{n!}}, \tag{3.7}$$

where $|\nu|^2 = \bar{n}$ is the average number of photons in the coherent state, and $\nu = |\nu|e^{-i\phi}$. In general, the exact solution for an initial atomic state $|\psi_{\rm atom}(t_0)\rangle = \alpha|a\rangle + \beta|b\rangle$ and a field state initially prepared in the coherent state, is a highly entangled state of the field and the atom [6]. However, Gea-Banacloche [7] has shown that for a large average number of photons, if we consider the evolution of the initial atomic states $|+\rangle$ and $|-\rangle$, defined by

$$|\pm\rangle = \frac{1}{\sqrt{2}}({\rm e}^{-{\rm i}\phi}|a\rangle \pm |b\rangle) \tag{3.8}$$

(here ϕ is the same as the phase of $\nu = |\nu|e^{-i\phi}$), the evolution of the global state of the system and the field (the environment) would be very interesting. Gea-Banacloche proved that when the initial atom-field state is $|\pm\rangle|\nu\rangle$, in the limit of $\bar{n} \to \infty$ the global state of the two-level atom (2LA) and the field will evolve as follows:

$$|\pm\rangle|\nu\rangle|_{t=0} \to \frac{1}{\sqrt{2}}(\mathrm{e}^{-\mathrm{i}\phi}\mathrm{e}^{\mp \mathrm{i}gt/(2\sqrt{\bar{n}})}|a\rangle \pm |b\rangle) \times |\Phi_{\pm}(t)\rangle, \tag{3.9}$$

where

$$|\Phi_{\pm}(t)\rangle = \mathrm{e}^{-\bar{n}/2}\sum_{n=0}^{\infty}\frac{\bar{n}^{n/2}}{\sqrt{n!}}\mathrm{e}^{-\mathrm{i}n\phi}\mathrm{e}^{\mp \mathrm{i}gt\sqrt{n}}|n\rangle, \tag{3.10}$$

gives us the time evolution of the state of the field. This result holds for any time, provided that t goes to infinity slowly enough to have $t/\bar{n} \to 0$. Since the time scale for the JCM revivals is $t_{\mathrm{R}} = 2\pi\sqrt{\bar{n}}/\mathrm{g}$ [6], $t_{\mathrm{R}}/\bar{n} \to 0$ as $\bar{n} \to \infty$ (a typical value for $\mathrm{g}/2\pi$ is 44 kHz in a micromaser experiment [7]). Hence, the approximate solution in equation (3.10) holds accurately over a large number of revivals, as long as $\bar{n}$ is large enough.

The states $|+\rangle$ and $|-\rangle$ form a basis set for the two-level atom (2LA); therefore, the evolution of any other initial atomic state with an initial coherent field can be expressed as a linear combination of the evolution of $|+\rangle|\nu\rangle$ and $|-\rangle|\nu\rangle$.

$$\begin{aligned}(\gamma|+\rangle + \delta|-\rangle)\ |\nu\rangle|_{t=0} \to \gamma\ |+(t)\rangle\ |\Phi_{+}(t)\rangle + \delta\ |-(t)\rangle\ |\Phi_{-}(t)\rangle \\ \text{with} \qquad |+(t)\rangle = \frac{\mathrm{e}^{-\mathrm{i}\phi}\mathrm{e}^{-\mathrm{i}gt/(2\sqrt{\bar{n}})}|a\rangle + |b\rangle}{\sqrt{2}} \\ \text{and} \qquad |-(t)\rangle = \frac{\mathrm{e}^{-\mathrm{i}\phi}\mathrm{e}^{+\mathrm{i}gt/(2\sqrt{\bar{n}})}|a\rangle - |b\rangle}{\sqrt{2}}.\end{aligned} \tag{3.11}$$

The time-dependent states $|\pm(t)\rangle$ and $|\Phi_{\pm}(t)\rangle$ appearing in the above equations are *the pointer states of the system (the 2LA) and the environment (the field) which are characterized by their ability not to entangle with each other.* As we observe, in the limit of large $\bar{n}$ in which equation (3.9) is valid, the global state of the 2LA and the field remains as a product state if the atom is initially prepared in one of the states $|\pm\rangle$. This means that, one can at all times assign a well-defined pure state to the atom initially prepared in one of the states $|\pm\rangle$ and clearly, no other initial atomic states have this characteristic, as is obvious from equation (3.11). In other words, for an initial coherent field and in the limit of large

$\bar{n}$, when the 2LA is initially prepared in one of the two states $|\pm\rangle$, the field and the atom never entangle; while equations (3.9) and (3.11) indicate that for any initial atomic state other than the $|\pm\rangle$ states, the states of the field and the atom will not remain separated and they entangle throughout the interaction. Also, equation (3.11) indicates that for an arbitrary initial atomic state and in the limit of large $\bar{n}$, there is always a one-to-one correspondence between a (preferred) set of pointer states from the system (the 2LA) and some corresponding states of a field which is initially prepared in a coherent state.

In essence, we observe that in the limit of large $\bar{n}$ and for an initial coherent field, in fact the coherent field does a Von Neumann premeasurement on the state of the 2LA; which makes the global state in a diagonal form and superselects a preferred set of pointer states of the system. However, here the premeasurement by the field definitely is not an ideal premeasurement, as the initial atomic states $|\pm\rangle$ evolve by acquiring a phase factor $\mathrm{e}^{\mp \mathrm{i}gt/(2\sqrt{\bar{n}})}$; except for $t \ll t_{\mathrm{R}}$, for which this change is negligible and the right hand side of equation (3.11) can be approximated by $\gamma\ |+\rangle|\Phi_+(t)\rangle + \delta\ |-\rangle|\Phi_-(t)\rangle$.

3.2.4 Schmidt states versus pointer states

It can be shown that in the limit of $\bar{n} \to \infty$, which corresponds to the classical limit for which equations (3.9) to (3.11) are valid, the field states $|\Phi_+(t)\rangle$ and $|\Phi_-(t)\rangle$ almost promptly become orthogonal [7]. However, as is obvious from equation (3.11), the pointer states of the system are not orthogonal at all times; and hence the diagonal state of the total composite system (represented by equation (3.11)), which is created after premeasurement, cannot represent a Schmidt decomposition at all times; as by definition the Schmidt states of the system and the environment must be orthogonal amongst themselves.

The Schmidt states obtained by diagonalizing the density matrix of the system at each instant of time necessarily are not the same as the pointer states at corresponding times; even in certain regimes and those times long enough so that the pointer states of the system and the environment can be considered as orthogonal amongst themselves. (For our example of the JCM the pointer states of the system are almost orthogonal provided that t goes to infinity slowly enough to have $t/\sqrt{\bar{n}} \to 0$, as can be seen from equation (3.11); also

the pointer states of the environment, given by equation (3.10), become orthogonal within a time of the order of 1/g due to decoherence, as was proved by Gea-Banacloche [7]). This is basically because of the fact that the Schmidt basis obtained this way will not necessarily exhibit the quasiclassical properties which are characteristic of the dynamical pointer states of measurement. In fact, as we described, the pointer states of the system emerge dynamically as those states that do not entangle with the environment; while the Schmidt states (which generally are not unique) necessarily are not robust against the entanglement with the environment. Hence, the pointer states of measurement generally cannot be obtained simply by diagonalizing the instantaneous density matrix of the system. (However, it is shown that only when the Schmidt states of the system are very nearly degenerate they can be significantly different from those environment-selected pointer states which are orthogonal amongst themselves (i.e. pointer states at certain regimes and sufficiently long times, so that they can be considered as orthogonal); while they are almost the same as the environment-selected pointer states whenever they are far from degeneracy and the pointer states of the system and the environment can be considered as orthogonal amongst themselves. The interested reader for example can refer to the interesting article by Albrecht [8]. This result in fact is just as we expect from the condition for the uniqueness of the Schmidt decomposition; since when the Schmidt states of the system are very nearly degenerate, this indicates that not all of the expansion coefficients in the diagonal state of the total composite system (equation (3.2)) are different and hence the Schmidt states are not unique. Therefore, in this case the Schmidt states which we obtain by some procedure necessarily will not be the same as the instantaneous pointer states of measurement which likewise diagonalize the state of the total composite system, but in addition to that do not entangle with the states of another subsystem at a subsequent time).

3.2.5 The commutativity criterion

In fact, the measurement cannot be considered faithful if the one-to-one correspondence between the states of the system and the apparatus is not preserved. In other words, if the further interaction with an outer environment does not maintain the one-to-one correspon-

dence between the states of two subsystems, no longer can we have a faithful measurement. For example if at an initial time t_0 the states of the system which appear in the diagonal state of the total composite system are the two states $|\psi_1(t_0)\rangle$ and $|\psi_2(t_0)\rangle$, then they can be considered as the instantaneous pointer states of the system if their further interaction with the environment preserves the one-to-one correspondence with the environment. In other words their evolution must be of the following form

$$|\psi_1(t_0)\rangle|E_0\rangle \rightarrow |\psi_1(t)\rangle|E_1(t)\rangle \quad \text{and} \quad |\psi_2(t_0)\rangle|E_0\rangle \rightarrow |\psi_2(t)\rangle|E_2(t)\rangle. \tag{3.12}$$

Now if we consider a superposition of these states at an initial time t_0 like

$$|\psi_{\pm}(t_0)\rangle = \alpha\ |\psi_1(t_0)\rangle \pm \beta\ |\psi_2(t_0)\rangle, \quad \text{with} \quad \alpha\beta \neq 0 \tag{3.13}$$

then, due to the interaction with the environment such a state will evolve according to

$$|\psi_{\pm}(t_0)\rangle|E_0\rangle \rightarrow \alpha\ |\psi_1(t)\rangle|E_1(t)\rangle \pm \beta\ |\psi_2(t)\rangle|E_2(t)\rangle. \tag{3.14}$$

This means that any superposition of $|\psi_1(t_0)\rangle$ and $|\psi_2(t_0)\rangle$ states (the states which appear in the diagonal state of the total composite system *and* do not entangle with the states of the environment) immediately entangles with the environment and hence it will lose its individuality and become unobservable. Indeed, the pointer states of the system emerge dynamically as those states that are the least sensitive, or the most robust, to the interaction with the environment; in the sense that they do not entangle with the environment. This is commonly referred to as the *stability criterion* for the selection of the pointer states [1, 2, 3]. In essence, some states are robust in spite of the environmental interaction, while other states rapidly entangle with the environment, lose their individuality and therefore become unobservable in practice. However, the information about only those states of the system that do not entangle with the environment can be passed all the way to the observer and these are the pointer states of measurement.

As we already mentioned by studying the evolution of the 2LA in the Jaynes-Cummings model of quantum optics, the pointer states characterized by their ability not to entangle with the environment, generally are *time-dependent* and hence, in general we should distinguish between the set of (time-dependent) pointer states and the preferred basis of

measurement. In section 3 we discuss why indeed the states which may be able to satisfy the requirement of faithful measurement generally are expected to be time-dependent. Also, in section 4 we will discuss the exact conditions under which the pointer states of measurement can be time-independent, so that a preferred set of basis states can exist as the basis of measurement.

With this introduction, now our ultimate goal is to find the preferred basis of measurement. However, we must first identify the pointer states of the system for an arbitrary total Hamiltonian defining the system-environment model. As we described, our general selection criterion is given by the *stability criterion*; i.e. the set of the pointer states of the system is given by those states of the system that do not entangle with the environment. In other words, they keep their individuality; so that they are able to hold a one-to-one correspondence with the states of the environment. In order to find these states, we should look for system states $|s_i(t)\rangle$ (like those of equation (3.11) for the JCM) such that the composite system-environment state, when starting from a product state $|s_i(t_0)\rangle|E_0\rangle$ at $t = 0$, remains in the product form $|s_i(t)\rangle|E_i(t)\rangle$ at all subsequent times $t > 0$ under the action of the total Hamiltonian. Now we show that the commutativity criterion for the determination of the pointer states of measurement, first introduced by Zurek [2, 3], although is sufficient in order to assure the requirement of having a faithful measurement, it is too restrictive and generally does not hold in all situations. In other words, it is not the minimal condition for having a faithful measurement.

Two regimes are often considered. In the *quantum measurement limit* the interaction between the system and the environment is so strong as to dominate the evolution of the system. Therefore, in this limit it is assumed that the intrinsic dynamics of the system and the environment is negligible in comparison with the evolution induced by the interaction. i.e.

$$\hat{H} \approx \hat{H}'; \tag{3.15}$$

and hence, the evolution of the composite system-environment state is approximately given by the evolution operator $\mathrm{e}^{-i\int_0^t \hat{H}'(t')\ dt'}$.

The other limit which also sometimes is considered, corresponds to the case that the

Hamiltonian for the system almost dominates the interaction between the system and the environment as well as the self-Hamiltonian of the environment. Hence, in this limit which frequently is called as the *quantum limit of decoherence*, the following approximation is assumed

$$\hat{H} \approx \hat{H}_{\mathcal{S}}. \tag{3.16}$$

As we show in section 4, the result of these approximations is that the pointer states of the system turn out to be independent of time (unlike those of the JCM in equation (3.11) which are obtained for the "exact-resonance" regime).

In his famous 1981 paper Zurek argued that in the quantum measurement limit the preferred set of pointer states for the system should be given by those states of the system that are eigenstates of the part of the interaction Hamiltonian $\hat{H}'$ pertaining to the Hilbert space of the system; since in this case we have

$$\mathrm{e}^{-\mathrm{i}\int_0^t \hat{H}'(t')\ dt'}|s_i\rangle|E_0\rangle = |s_i\rangle\ \mathrm{e}^{-\mathrm{i}\int_0^t(\lambda_i(t')\hat{E})\ dt'}|E_0\rangle \equiv |s_i\rangle|E_i(t)\rangle, \tag{3.17}$$

provided $\hat{H}' = \hat{S}\otimes\hat{E}$; with $\hat{S}$ and $\hat{E}$ denoting some operators in the Hilbert space of the system and the environment respectively; and $\hat{S}\ |s_i\rangle = \lambda_i|s_i\rangle$. As we see from equation (3.17), in this case the state of the system does not entangle with the state of the environment. (If we consider the more general form of the interaction Hamiltonian given by $\hat{H}' = \sum_\alpha \hat{S}_\alpha \otimes \hat{\mathcal{E}}_\alpha$, then a sufficient condition for $\{|s_i\rangle\}$ to form a set of pointer states of the system is that the $|s_i\rangle$ be simultaneous eigenstates of *all* the system operators $\hat{S}_\alpha$.)

Equivalently, if we define the pointer observable as the observable for the system whose eigenstates are these pointer states $|s_i\rangle$ of the system, i.e.

$$\hat{O}_{\mathcal{S}} = \sum_i o_i|s_i\rangle\langle s_i|, \tag{3.18}$$

since the $|s_i\rangle$ are eigenstates of $\hat{H}'$, it follows that $\hat{O}_{\mathcal{S}}$ must commute with $\hat{H}'$,

$$[\hat{O}_{\mathcal{S}}, \hat{H}'] = 0. \tag{3.19}$$

This condition is often referred to as the *commutativity criterion* and was first discussed by Zurek in his paper of 1981 [2]. However, this is not the only possible situation for

identifying the pointer states of a system (i.e. as we will discuss here, in many situations the pointer states of the system, which are characterized by their ability not to entangle with the environment, are time-dependent and do not satisfy the commutativity criterion); since we note that *if* the pointer states of the system are eigenstates of the Hamiltonian, as Zurek has proposed, for an interaction Hamiltonian which does not explicitly depend on time (like that of our example of the JCM at the zero-detuning regime) the most that can change about the pointer states of the system under the effect of the evolution operator is an overall phase factor and basically they would remain unaltered under the effect of the evolution operator. In other words, Zurek's pointer states are basically independent of time. However, the point is that in order to satisfy the requirement of faithful measurement the pointer states of the system do not have to be independent of time like Zurek's pointers. For example a one-to-one correspondence between some well-defined states of the system and some states from the environment is preserved in our example represented by the JCM (equation (3.11)); while as we saw, the pointer states which appear in the diagonal state of the total composite system in this example clearly depend on time.

Indeed, if we have pointer states which change by more than an overall phase factor (as those of the JCM), they cannot be eigenstates of the Hamiltonian at all times; and as one can easily verify, the pointer states of the system in the JCM, given by equation (3.11), also are not the eigenstates of the total Hamiltonian nor the interaction Hamiltonian at all times; although they satisfy the requirement of faithful measurement and do not entangle with the environment during the interaction. As a result, although the commutativity criterion fulfills the requirement of having a faithful measurement, we notice that it is not the minimal condition for determining the pointer states of measurement; in the sense that it is not always valid and we might have time-dependent pointer states like those of the JCM which do not satisfy any kind of commutation relation (including the total Hamiltonian of the composite system or the interaction Hamiltonian in the quantum measurement limit) at all times.

3.2.6 Bloch vector and determination of the preferred basis of measurement

The reduced density matrix of a two-level system $\hat{\rho}_{\mathcal{S}}(t)$ generally can be expressed in terms of the Bloch vector $\mathbf{R}(t) \equiv (R_x, R_y, R_z)$ [9] as follows

$$\hat{\rho}_{\mathcal{S}}(t) = \frac{1}{2}(\hat{I} + \mathbf{R}(t).\hat{\sigma}) = \frac{1}{2}(\hat{I} + R_x\sigma_x + R_y\sigma_y + R_z\sigma_z); \tag{3.20}$$

from which one can easily verify that the Bloch vector components must be defined by

$$R_x = \rho_{ab} + \rho_{ba} \qquad R_y = i(\rho_{ab} - \rho_{ba}) \quad \text{and} \quad R_z = \rho_{aa} - \rho_{bb}. \tag{3.21}$$

In the above equation we used the notation $\rho_{ab} = \langle a|\hat{\rho}_{\mathcal{S}}(t)|b\rangle$ (with $|a\rangle$ and $|b\rangle$ representing a complete set of basis states for the two-level system) and etc.

The Bloch vector here can be interpreted as the polarization of the state of the two-level system. This is because the direction of $\mathbf{R}$ tells us into what set of eigenstates the reduced density matrix of the system can be decomposed. For example, if $R_x = R_y = 0$ and $R_z \neq 0$ then

$$\hat{\rho}_{\mathcal{S}}(t) = \frac{1}{2}(\hat{I} + R_z\sigma_z) = \frac{1}{2}\begin{pmatrix} 1+R_z & 0 \\ 0 & 1-R_z \end{pmatrix}; \tag{3.22}$$

therefore, $\hat{\rho}_{\mathcal{S}}$ will commute with σ_z and can be decomposed in terms of the eigenstates of σ_z. As a result, generally speaking in a certain regime of the parameter space only if the components of the Bloch vector settle in some asymptotic values at $t \to \infty$, can we conclude that at sufficiently long times there can exist a preferred basis of measurement represented by the eigenstates of $\hat{\rho}_{\mathcal{S}}$. Otherwise, i.e. if the Bloch vector constantly changes its direction, a preferred basis of measurement cannot be realized in the corresponding regime.

As an example, Gea-Banacloche has studied the Bloch sphere evolution of the two-level atom in the Jaynes-Cummings model of quantum optics and for an initial coherent field [10]. As one can see from his studies, away from the specific features which he studied (such as collapses and revivals and state preparation in the evolution of the two-level atom), the Bloch vector does not have any asymptotic behavior in most of the regimes. In chapter 5 [11] we discuss some regimes of the parameter space for the quantized atom-field model for

which the Bloch vector has an asymptotic behavior for large times; so that a preferred basis of measurement can be realized.

As another example, consider the spin-spin model described by the total Hamiltonian

$$\hat{H} = \hat{H}_{\mathcal{S}} + \hat{H}' = -\frac{1}{2}\Delta_0\hat{\sigma}_x + \frac{1}{2}\hat{\sigma}_z \otimes \sum_{i=1}^{N} \mathrm{g}_i\hat{\sigma}_z^{(i)}, \tag{3.23}$$

where the first term (representing the self-Hamiltonian of the central system) accounts for the intrinsic dynamics of the central spin-half particle and Δ_0 is the so-called tunneling matrix element. Also, the second term in equation (3.23) corresponds to the linear interaction between the $\hat{\sigma}_z$ coordinate of the central spin and the $\hat{\sigma}_z$ coordinates of N environmental spin-half particles, with coupling strengths g_i.

The evolution of the Bloch vector has been studied for this model by Cucchietti et al. [12]. They considered a Gaussian spectral density for the initial state of the environmental spins and considered two main regimes. In the regime that the self-Hamiltonian of the system is negligible compared to the interaction between the system and the environment they found that $\mathbf{R}(t \to \infty) \to R_z$; while for the regime that the interaction between the system and the environment is negligible compared to the self-Hamiltonian of the system they found that $\mathbf{R}(t \to \infty) \to R_x$. This basically shows that in the first regime the preferred basis of measurement is determined as the eigenstates of the $\hat{\sigma}_z$ operator, while in the second regime they are determined as the eigenstates of the $\hat{\sigma}_x$ operator; just in agreement with the predictions of the commutativity criterion for the quantum measurement limit and the quantum limit of decoherence.

3.2.7 Other methods for determination of the pointer states of measurement

As we discussed, the pointer states of the system keep their individuality; as they do not entangle with the states of the environment and hold a one-to-one correspondence with the pointer states of the environment (see equation (3.12)). Therefore, if one initially prepares the system in one of its pointer states $|s_i\rangle$, on a further observation on the state of the environment he can expect only to observe the corresponding pointer state of the

environment $|e_i\rangle$ (and vice versa). In this sense pointer states are *predictable*; just as classical states are.

Also we discussed that the commutativity criterion for determination of the pointer states cannot be a reliable criterion in many of the realistic cases; because pointer states, characterized by their ability not to entangle with the states of another subsystem, generally are time-dependent (we will mathematical prove this latter point within the next section). Realizing the unreliability of the commutativity criterion for determination of the pointer states of measurement, some researchers have tried to introduce a better predictability criterion [13, 14, 15]. However, these other criteria for determination of the pointer states of measurement more or less suffer from the same limitations of the commutativity criterion; again because pointer states generally are time-dependent. Hence, we may have the problem of stability of solutions with respect to time. Moreover, the mathematical calculations required while using these methods often are quite difficult; even for some very simple models.

Among these methods, Zurek has introduced the so-called "predictability sieve" [13, 14, 15], which exploits the Von Neumann entropy to measure the loss of predictability caused by evolution. In this case pointer states correspond to the least entropy producing states and predictability is a function of time and a functional of the initial state of the system. So, pointer states are sought by maximizing the predictability functional over the initial state of the system. Also, the purity of a system $\mathrm{Tr}[\rho^2](\mathrm{t})$ (more exactly "purity loss time") has been exploited to measure the loss of predictability in the so-called "purity sieve" [13].

One main issue regarding the different predictability sieves which have been introduced so far is that there is no *a priori* reason to expect that all of these criteria lead to the same set of pointer states; although in the macroscopic limit the difference between various sieves is expected to be negligible [16]. However, the authors believe that the problem of the stability of solutions for pointer states (which arises from the fact that the pointer states of measurement generally are time-dependent) and the calculational difficulties involved in these methods are the main issues with these methods.

3.2.8 Another aspect of pointer states: redundant encoding of information in the environment

In some of the more recent publications the main focus is on the information encoded in the environment from the state of the system and the spread of this information through the environment; rather than the system environment interactions. In this approach the environment acquires the role of a communication channel and the pointer states of the system correspond to those states of the system whose imprints on the subsystems of the environment are *most redundantly* and *most robustly* encoded in the environment. In other words, the main focus of this kind of approach is the transformation of the information encoded in an ensemble of environmental "witness states" all the way to the observer. This process indeed is another step of the measurement process. However, we do not focus on this aspect of the measurement process in this research.

The research on this aspect of pointer states is carried out under the headings of "the environment as a witness" and "quantum Darwinism"; and it has been claimed by Ollivier [17, 18] and also by Blume-Kohout and Zurek [19, 20] that the environment-selected pointer states of the system not only are the states which are the least entangled with the states of the environment (i.e. the states which are the most robust against the environmental interactions) but also they are the states which can be imprinted most completely and most redundantly in many distinct subsets of the environment.

3.3 Identifying time-dependent Pointer States of measurement for an arbitrary Hamiltonian

In order to be able to obtain the pointer states of the system and the environment for an arbitrary total Hamiltonian defining the system-environment model we first need to find those probable initial states of the system which do not entangle with the states of the environment throughout their evolution with time; and then we should obtain their time evolution. Finally, we should obtain their corresponding states from the environment which

in fact, are the pointer states of the environment. As we will see in this section, existence of pointer states often requires having a sufficiently large environment which contains a large number of degrees of freedom. In other words, pointer states characterized by their ability not to entangle with the states of another subsystem, do not necessarily exist in any arbitrary regime.

After developing our method in this section we will exploit it in order to rederive the time-dependent pointer states of the two-level-atom and the field (initially prepared in the coherent state) in the JCM and for the exact resonance regime. As we will see, the previous results obtained by Gea-Banacloche (equations (3.8) to (3.11)) are easily obtained using our method. Also, in chapter 4 (paper III [21]) and chapter 5 (paper IV [11]) we will show how easily we can use this method in order to obtain the time-dependent pointer states of the system and the environment for the generalized spin-boson model (SBM) and also for the quantized atom-field model and in some nonresonance regimes.

Consider a two-state system $\mathcal{S}$ with two arbitrary basis states $|a\rangle$ and $|b\rangle$, initially prepared in the state

$$|\psi^{\mathcal{S}}(t_0)\rangle = \alpha|a\rangle + \beta|b\rangle \quad \text{with} \quad |\alpha|^2 + |\beta|^2 = 1, \tag{3.24}$$

and an environment initially prepared in the state

$$|\phi^{\mathcal{E}}(t_0)\rangle = \sum_{n=0}^{\infty} c_n|\varphi_n\rangle, \tag{3.25}$$

where $\{|\varphi_n\rangle\}$'s are a complete set of basis states for the environment. For the two-state system with the two basis states $|a\rangle$ and $|b\rangle$ we can take the set of any four linearly independent operators in the Hilbert space of the system as a complete set of basis operators, which can induce any change to the initial state of the two-state system given by equation (3.24). For example, we can take the Pauli operators in addition to the identity operator $\hat{I} = |a\rangle\langle a| + |b\rangle\langle b|$ as our complete set of basis operators; or equivalently we can take the four operators $|a\rangle\langle a|$, $|a\rangle\langle b|$, $|b\rangle\langle a|$ and $|b\rangle\langle b|$ as our complete set of basis operators. So, the time evolution operator for the global state of the system and the environment, which (for

a two-state system) generally is of the form

$$\hat{U}_{\rm tot}(t) = \sum_{\alpha=1}^{4} \hat{S}_\alpha \otimes \hat{\mathcal{E}}_\alpha \ , \tag{3.26}$$

can be considered as

$$\hat{U}_{\rm tot}(t) = \hat{\mathcal{E}}_1|a\rangle\langle a| + \hat{\mathcal{E}}_2|a\rangle\langle b| + \hat{\mathcal{E}}_3|b\rangle\langle a| + \hat{\mathcal{E}}_4|b\rangle\langle b|. \tag{3.27}$$

In the above equation $\hat{\mathcal{E}}_i$'s depend on the total Hamiltonian defining the system-environment model. For example, for the Jaynes-Cummings model of quantum optics and for exact resonance and in the rotating wave approximation (RWA), it can be shown [6] that the $\hat{\mathcal{E}}_i$'s are given by the following relations

$$\begin{aligned} \hat{\mathcal{E}}_1 = \cos(gt\sqrt{\hat{a}^\dagger\hat{a}+1}) \ , \quad \hat{\mathcal{E}}_2 = -i \ \frac{\sin(gt\sqrt{\hat{a}^\dagger\hat{a}+1}}{\sqrt{\hat{a}^\dagger\hat{a}+1}} \ \hat{a}) \\ \hat{\mathcal{E}}_3 = -i\hat{a}^\dagger \ \frac{\sin(gt\sqrt{\hat{a}^\dagger\hat{a}+1}}{\sqrt{\hat{a}^\dagger\hat{a}+1}}) \ , \quad \hat{\mathcal{E}}_4 = \cos(gt\sqrt{\hat{a}^\dagger\hat{a}}). \end{aligned} \tag{3.28}$$

Using equations (3.24) to (3.27) we can write the global state of the system and the environment as follows

$$\begin{aligned} |\psi^{\rm tot}(t)\rangle = \hat{U}_{\rm tot}(t).\ (\alpha|a\rangle + \beta|b\rangle) \otimes (\sum_{n=0}^{\infty} c_n|\varphi_n\rangle) \\ = \mathbf{A}(t)\ |a\rangle + \mathbf{B}(t)\ |b\rangle \quad \text{with} \quad \mathbf{A}(t) = \sum_{n=0}^{\infty} c_n\{\alpha\hat{\mathcal{E}}_1 + \beta\hat{\mathcal{E}}_2\}\ |\varphi_n\rangle \\ \text{and} \quad \mathbf{B}(t) = \sum_{n=0}^{\infty} c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle. \end{aligned} \tag{3.29}$$

In order to find those probable initial states of the system which do not entangle with the states of the environment we first define $\hat{G}(t)$ as the operator in the Hilbert space of the environment which relates the vectors $\mathbf{A}(t)$ and $\mathbf{B}(t)$ to each other

$$\mathbf{A}(t) = \hat{G}(t)\mathbf{B}(t) \ \ \text{or} \quad \sum_n c_n\{\alpha\hat{\mathcal{E}}_1 + \beta\hat{\mathcal{E}}_2\}\ |\varphi_n\rangle = \hat{G}(t)\sum_n c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle. \tag{3.30}$$

Now, for the global state of the system and the environment, which is given by

$$\begin{aligned} |\psi^{\rm tot}(t)\rangle = \mathbf{A}(t)\ |a\rangle + \mathbf{B}(t)\ |b\rangle = \hat{G}(t)\mathbf{B}(t)\ |a\rangle + \mathbf{B}(t)\ |b\rangle \\ = \{\hat{G}(t)|a\rangle + |b\rangle\} \times (\sum_{n=0}^{\infty} c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle), \end{aligned} \tag{3.31}$$

we observe that *if* for some initial states of the system and the environment $\hat{G}(t)$ turns out to become in the form

$$\hat{G}(t) = G(t) \times \hat{I}_{\mathcal{E}}, \tag{3.32}$$

with $G(t)$ as a scalar (rather than an operator) and $\hat{I}_{\mathcal{E}}$ representing the identity operator in the Hilbert space of the environment, then those initial states of the system and the environment will not entangle with each other, and hence they can represent the initial pointer states of the system and the environment. This result simply is because of the fact that if for some initial states of the system and the environment $\hat{G}(t)$ turns out to become a scalar in the form of equation (3.32), $G(t)$ will be independent of the indices of the environment (i.e. independent of n); as in this case all components of $\mathbf{B}(t)$ will be mapped into their corresponding components from $\mathbf{A}(t)$ through the *same* scalar function $G(t)$ (which will keep the two vectors $\mathbf{A}(t)$ and $\mathbf{B}(t)$ parallel to each other). Therefore, in this case $\hat{G}(t)$ will not enter the summation in the expression $\sum_n c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle$ of equation (3.31); and (as one can see from equation (3.31)) the states of the system and the environment respectively represented by $\{G(t)|a\rangle + |b\rangle\}$ and $\sum_n c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle$ will not entangle to each other. In other words, if for some initial states of the system and the environment the operator $\hat{G}(t)$ becomes proportional to the identity operator, the two vectors $\mathbf{A}(t)$ and $\mathbf{B}(t)$ will stay parallel with each other throughout their evolution with time, and the states of the system and the environment will not entangle with each other; therefore (as one can see from equation (3.31)), in this case pointer states can be realized for the system and the environment given by

$$\begin{aligned} |\pm(t)\rangle &= \mathcal{N}\ \{G(t)|a\rangle + |b\rangle\} \qquad \text{and} \\ |\Phi_{\pm}(t)\rangle &= \mathcal{N}^{-1}(\sum_{n=0}^{\infty} c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle). \end{aligned} \tag{3.33}$$

In the above equation we have represented the pointer states of the system by $|\pm(t)\rangle$ and those of the environment by $|\Phi_{\pm}(t)\rangle$. Also, $\mathcal{N}$ is the normalization factor for the pointer states of the system (clearly $\mathcal{N} = \frac{1}{\sqrt{2}}$ if $|G(t)| = 1$, as we will see for the example of JCM in exact resonance and the rotating wave approximation).

As we will see in this chapter, generally there is no guaranty for the condition (3.32) to be satisfied; and satisfaction of this condition often may require having a sufficiently large environment which contains a large number of degrees of freedom. However, *if* in some regime and for a given Hamiltonian defining a system-environment model we can find initial states for the system and the environment which satisfy this condition, we do know that pointer states can be realized for the system and the environment and these initial states would correspond to the initial pointer states of the system and the environment.

In order to find the pointer states of the system and the environment for a given total Hamiltonian defining the system-environment model, and for a given initial state of the environment, our main goal would be finding those possible initial states of the system for which $\hat{G}(t)$ (which is defined through equation (3.30)) is of the form of relation (3.32). Within the following paragraphs we will consider our previous example of the JCM with an initial coherent field and exploit this method in order to rederive the time-dependent pointer states of the two-level-atom and the field, which we already saw in equations (3.10) and (3.11). As we will see, this task is not as difficult as it might initially seem and finding initial states of the system which make the operator $\hat{G}(t)$ proportional to the identity operator in the Hilbert space of the environment often can be done quite easily when dealing with a sufficiently large environment which contains a large number of degrees of freedom. However, for an initial state of the environment which does not correspond to a sufficiently large environment (as we will see) we might not have any initial states which can satisfy our condition (3.32) for determining the pointer states of the system and the environment. This means that pointer states, which are characterized by their ability not to entangle with the states of another subsystem, do not necessarily exist in any arbitrary regime.

For our example of the JCM if we use the number states as the complete set of basis states for the environment, then equation (3.28) for the environmental operators ($\hat{\mathcal{E}}_i$'s) would suggest us that in this case we have

$$\hat{\mathcal{E}}_1|\varphi_n\rangle = f_1(n,t)|\varphi_n\rangle, \qquad \hat{\mathcal{E}}_2|\varphi_n\rangle = f_2(n,t)|\varphi_{n-1}\rangle$$
$$\hat{\mathcal{E}}_3|\varphi_n\rangle = f_3(n,t)|\varphi_{n+1}\rangle \quad \text{and} \quad \hat{\mathcal{E}}_4|\varphi_n\rangle = f_4(n,t)|\varphi_n\rangle; \tag{3.34}$$

with

$$f_1(n,t) = \cos(gt\sqrt{n+1}), \qquad f_2(n,t) = -i\sin(gt\sqrt{n})$$
$$f_3(n,t) = -i\sin(gt\sqrt{n+1}) \quad \text{and} \quad f_4(n,t) = \cos(gt\sqrt{n}). \tag{3.35}$$

Using equation (3.34) and our definition of the operator $\hat{G}(t)$ (equation (3.30)), we can write

$$\sum_n \{\alpha c_n f_1(n,t) + \beta c_{n+1} f_2(n+1,t)\}\ |\varphi_n\rangle = \hat{G}(t) \sum_n \{\alpha c_{n-1} f_3(n-1,t) + \beta c_n f_4(n,t)\}\ |\varphi_n\rangle. \tag{3.36}$$

The above relation is valid whenever equation (3.34) is valid for a total Hamiltonian defining the system-environment model (as for our example of the JCM).

Now, for the pointer states $\hat{G}(t)$ must satisfy the condition (3.32) for obtaining the pointer states of the system and the environment, i.e. $\hat{G}(t) = G(t) \times \hat{I}_{\mathcal{E}}$. Therefore, since $\{|\varphi_n\rangle\}$ is a complete set of basis states for the environment, for the initial pointer states we can open the summations in equation (3.36) and equalize terms from the two sides of this equation which correspond to the same basis state $|\varphi_n\rangle$ to obtain

$$G(t) = \frac{\{\alpha c_n f_1(n,t) + \beta c_{n+1} f_2(n+1,t)\}}{\{\alpha c_{n-1} f_3(n-1,t) + \beta c_n f_4(n,t)\}}; \quad \forall\ n > 0. \tag{3.37}$$

The above result for $G(t)$ which generally depends on n would contradict our initial assumption of $\hat{G}(t)$ being a scalar *unless* if we can find certain initial states for the system for which $G(t)$ turns out to become independent of n. [3] So now we should seek for those particular initial states of the system which can make $G(t)$ independent of the index n of the states of the environment. For this purpose we assume the field to be initially in the coherent state and use the $f_i(n,t)$ functions from equation (3.35) in order to simplify

[3] We would like to see if the condition can be satisfied for *any* initial state of the system and the environment with $G(t)$ becoming independent of the index n of the states of the environment. So, if finally we can find any specific set of initial states for the system and the environment which satisfies this condition with $G(t)$ independent of the indices of the environment, then we have reached our goal and our assumption has not been in vain.

equation (3.37) for our example of the JCM in exact resonance and the RWA:

$$G(t)=\frac{\{\alpha c_n\cos(gt\sqrt{n+1})-i\beta c_n e^{-i\varphi}\sqrt{\frac{\bar{n}}{n+1}}\sin(gt\sqrt{n+1})\}}{\{\beta c_n\cos(gt\sqrt{n})-i\alpha c_n e^{i\varphi}\sqrt{\frac{n}{\bar{n}}}\sin(gt\sqrt{n})\}};\tag{3.38}$$

as for the coherent field (represented by equation (3.7)) we have $c_{n+1}=c_n e^{-i\varphi}\sqrt{\frac{\bar{n}}{n+1}}$ and $c_{n-1}=c_n e^{i\varphi}\sqrt{\frac{n}{\bar{n}}}$. However, in the limit of a large average number of photons $\bar{n}\to\infty$ we can replace the factors $\sqrt{\frac{n}{\bar{n}}}$ and $\sqrt{\frac{\bar{n}}{n+1}}$ by unity, since the Poisson distribution of the coherent field is extremely sharp for $\bar{n}\to\infty$ and hence, (in the summations of equation (3.36)) for $\bar{n}\to\infty$ and $n\approx\bar{n}$ we have $\sqrt{\frac{n}{\bar{n}}}\approx 1$ and $\sqrt{\frac{\bar{n}}{n+1}}\approx 1$, while for n being far from $\bar{n}$ the corresponding c_n coefficients are negligible. As a result, equation (3.38) for $G(t)$ can be further simplified to

$$G(t)=\frac{\{\alpha c_n\cos(gt\sqrt{n+1})-i\beta c_n e^{-i\varphi}\sin(gt\sqrt{n+1})\}}{\{\beta c_n\cos(gt\sqrt{n})-i\alpha c_n e^{i\varphi}\sin(gt\sqrt{n})\}}.\tag{3.39}$$

In fact, as Gea-Banacloche has shown [7], the difference between such approximate expressions which are obtained by assuming $\sqrt{\frac{n}{\bar{n}}}\approx 1$ and $\sqrt{\frac{\bar{n}}{n+1}}\approx 1$ and the exact expressions (where we keep these factors) goes to zero as $\bar{n}\to\infty$. Moreover, in this limit we can use

$$\sqrt{n+1}-\sqrt{n}\approx\frac{1}{2\sqrt{\bar{n}}}\tag{3.40}$$

as one can easily verify for example by writing the Taylor expansion of $\sqrt{n}$ about $\bar{n}$.

Now, from equation (3.39) clearly we have $\alpha=\beta G(0)$; and hence by this substitution we find

$$G(t)=\frac{G(0)\cos(gt\sqrt{n+1})-i\ e^{-i\varphi}\sin(gt\sqrt{n+1})}{\cos(gt\sqrt{n})-i\ G(0)\ e^{i\varphi}\sin(gt\sqrt{n})}.\tag{3.41}$$

By looking at the above equation one would easily see that if $G(0)=\pm e^{-i\varphi}$ (i.e. if $\alpha=\pm\beta e^{-i\varphi}$), $G(t)$ will be independent of the index n of the states of the environment; since in this case we have

$$G(t)=\pm e^{-i\varphi}\ \frac{e^{\mp igt\sqrt{n+1}}}{e^{\mp igt\sqrt{n}}}=\pm e^{-i\varphi}\ e^{\mp igt(\sqrt{n+1}-\sqrt{n})}\ ;\tag{3.42}$$

however, at the limit of a large average number of photons we can use equation (3.40) to replace the factor $\sqrt{n+1}-\sqrt{n}$ by $\frac{1}{2\sqrt{\bar{n}}}$ and find

$$G(t)=\pm e^{-i(\varphi\pm gt/2\sqrt{\bar{n}})}\tag{3.43}$$

which clearly is independent of the index n of the states of the environment.

This result simply means that for $\alpha = \pm\beta e^{-i\varphi}$ which is equivalent to having

$$\alpha = \frac{e^{-i\varphi}}{\sqrt{2}} \quad \text{and} \quad \beta = \pm\frac{1}{\sqrt{2}} \tag{3.44}$$

(since we must have $|\alpha|^2 + |\beta|^2 = 1$) the states of the system and the environment will not entangle with each other. Moreover, using equation (3.33) which gives us the general time evolution of the pointer states of the system; and $G(t)$ of equation (3.43) (which is independent of the index n of the states of the environment) we find the time evolution of the pointer states of the system as follows

$$|\pm(t)\rangle = \frac{e^{-i\varphi}e^{\mp igt/(2\sqrt{\bar{n}})}|a\rangle \pm |b\rangle}{\sqrt{2}}. \tag{3.45}$$

Next, let us use equation (3.33) in order to obtain the corresponding pointer states of the environment; i.e. $|\phi_\pm(t)\rangle$. Here, substituting α and β from equation (3.44) and using equations (3.34) and (3.35) we have

$$|\phi_\pm(t)\rangle = \sqrt{2}\ \sum_{n=0}^{\infty} c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle$$

$$= \sum_{n=0}^{\infty} c_n\{\mp i\ e^{-i\varphi}\sin(\mathrm{g}t\sqrt{n+1})\ |n+1\rangle + \cos(\mathrm{g}t\sqrt{n})\ |n\rangle\}. \tag{3.46}$$

(In writing the above equation for mathematical convenience we made use of the fact that an overall phase is not important in determining the pointer states of the environment and only the relative phases are important). However, for the coherent field we had $c_{n+1} = c_n e^{-i\varphi}\sqrt{\frac{\bar{n}}{n+1}}$; also, as we already discussed, at the limit of $\bar{n} \to \infty$ we can neglect the factor $\sqrt{\frac{\bar{n}}{n+1}}$ and replace $c_n\mathrm{e}^{-\mathrm{i}\varphi}$ of the above equation by c_{n+1} to have

$$|\phi_\pm(t)\rangle = \sum_{n=0}^{\infty}\{\mp i\ c_{n+1}\ \sin(\mathrm{g}t\sqrt{n+1})\ |n+1\rangle + c_n\cos(\mathrm{g}t\sqrt{n})\ |n\rangle\}$$

$$= \sum_{n=0}^{\infty} c_n\{\mp i\ \sin(\mathrm{g}t\sqrt{n}) + \cos(\mathrm{g}t\sqrt{n})\}|n\rangle = \sum_{n=0}^{\infty} c_n e^{\mp igt\sqrt{n}}\ |n\rangle, \tag{3.47}$$

where in the second line of the above equation we used the substitution $n \longrightarrow n-1$ and made use of the fact that the first term in the first summation is equal to zero and hence we can keep the lower limit of the summation as $n = 0$.

This way we easily reproduced the previous results first introduced by Gea-Banacloche (given by equations (3.10) and (3.11)) through using our general method for finding the pointer states of the system and the environment. We will further demonstrate the generality and usefulness of this method in chapter 4 (paper III [21]) and chapter 5 (paper IV [11]), where we obtain the time-dependent pointer states of the system and the environment for the generalized spin-boson model [21] and also for the quantized atom-field model and in some nonresonance regimes [11] (these are new results not contained in previous works). However, the significance of this formulation is not only because of providing us with a method for obtaining the pointer states of the system and the environment for a given total Hamiltonian. In fact, as we discuss in more detail in the next section, this formulation specifically is useful because of the insight which it can bring us regarding the general properties of pointer states under different regimes and circumstances; and more importantly the insight which it brings us regarding the question of determination of the preferred basis of measurement.

3.4 Determination of the preferred basis of measurement

We already showed that *if* in some regime and for some specific values of α and β of the initial state of the system the function $\hat{G}(t)$, defined by equation (3.30), turns out independent of the states of the environment, then in that regime the corresponding values for α and β are related to the initial pointer states of the system, which will not entangle with the states of the environment. Moreover, the time evolution of the pointer states of the system and the environment can be obtained with the help of these initial values of α and β and by using equation (3.33). Nevertheless, by looking at equation (3.30) which defines $G(t)$, we notice that as the evolution operators $\hat{\mathcal{E}}_i$ (which correspond to the Hilbert space of the environment) generally are time-dependent, generally we expect the function $\hat{G}(t)$ and therefore the pointer states of the system (given by equation (3.33)) to be dependent on time.

The above consideration basically means that the states from the system which might be able not to entangle with the states of another subsystem necessarily are not independent of

time and hence we must search for the special conditions under which we might have time-independent pointer states, so that a preferred set of basis states can be identified as the basis of measurement. However, before discussing our criteria for identifying time-independent pointer states and their further consequences here we briefly discuss an instructive physical example where the pointer states of the system turn out to be time-independent.

Our example is represented by the simplified spin-boson model (SBM) which is composed of a central spin-half particle surrounded by an environment of N bosonic particles. For our simplified spin-boson model we consider a single-mode quantized field for the environment; and moreover, we disregard a possible contribution to the self-Hamiltonian of the system which can induce transitions between the upper and lower states of the central system (i.e. an intrinsic tunneling contribution proportional to $\hat{\sigma}_x$ Pauli matrix that would generate the intrinsic dynamics of the central spin). So, we consider the following total Hamiltonian for our model

$$\hat{H} = \frac{1}{2}\omega_0\hat{\sigma}_z + \omega\hat{a}^\dagger\hat{a} + \hat{\sigma}_z \otimes (\mathrm{g}\hat{a}^\dagger + \mathrm{g}^*\hat{a}); \tag{3.48}$$

where in the above equation ω_0 is the splitting between the states of the spin-half particle and ω is the frequency of the cavity eigenmode. The third term, with g as the spin-field coupling constant, represents the interaction between the central spin-half particle and a single-mode quantized field; which in fact is the quantized form of the famous $-\vec{\mu}.\mathbf{B}$ Hamiltonian due to the interaction between a particle of magnetic dipole-moment $\vec{\mu}$ and a magnetic field $\mathbf{B}$.

This model has been studied by many people (for an interesting review the reader can refer to the article by Leggett et al. [22] or Schlosshauer's book [1]) and as one can easily show, the effective evolution operator in the interaction-picture for this model can be represented by

$$\begin{aligned} \hat{V}(t) &= \exp\{\hat{\sigma}_z \otimes (\lambda(t)\ \hat{a}^\dagger - \lambda^*(t)\ \hat{a})\} \\ &\text{with} \quad \lambda(t) = \frac{\mathrm{g}}{\omega}(1 - e^{i\omega t}) \end{aligned} \tag{3.49}$$

(the different notations for the time evolution operator between this example and our general formulation is just because our effective evolution operator, given by equation (3.49), in

fact differs from the actual evolution operator by an overall phase; which of course is not physically important). The key point regarding this evolution operator is that it contains *only one* of the Pauli spin operators (here $\hat{\sigma}_z$; as a result of which $\hat{\sigma}_z$ becomes a constant of motion in this simplified model). As we will see in the following paragraphs, this will result in having *time-independent* pointer states given by the eigenstates of $\hat{\sigma}_z$.

Now if we consider the initial state of the total composite system as

$$|\Psi^{\text{tot}}(t_0)\rangle = (\alpha|a\rangle + \beta|b\rangle)\ |\Phi^{\mathcal{E}}\rangle, \tag{3.50}$$

with $|a\rangle$ and $|b\rangle$ representing the eigenstates of $\hat{\sigma}_z$ and $|\Phi^{\mathcal{E}}\rangle$ representing some arbitrary initial state of the environment, then using the evolution operator given by equation (3.49) we easily obtain

$$\begin{aligned} |\Psi^{\text{tot}}(t)\rangle = \hat{V}(t)\ |\Psi^{\text{tot}}(t_0)\rangle = \alpha|a\rangle|\Phi_+(t)\rangle + \beta|b\rangle|\Phi_-(t)\rangle \\ \text{with} \quad |\Phi_+(t)\rangle = \hat{D}(\lambda(t))\ |\Phi^{\mathcal{E}}\rangle \\ \text{and} \quad |\Phi_-(t)\rangle = \hat{D}(-\lambda(t))\ |\Phi^{\mathcal{E}}\rangle. \end{aligned} \tag{3.51}$$

In the above equation $\hat{D}(\lambda(t))$, which generates the evolution of the pointer states of the environment, is defined by

$$\hat{D}(\lambda(t)) = \exp[\lambda(t)\ \hat{a}^\dagger - \lambda^*(t)\ \hat{a}] \tag{3.52}$$

(which in fact is the same as the displacement operator in quantum optics). The concrete form of the environmental pointer states $|\Phi_\pm(t)\rangle$ clearly would depend on our initial state of the environment $|\Phi^{\mathcal{E}}\rangle$. However, no matter what is our initial state of the environment here we clearly observe that the interaction between the system and the environment would select the time-independent eigenstates of $\hat{\sigma}_z$ as pointer states of the system, which will be robust against the entanglement with the environment.

We could equivalently arrive at this result by using our method for obtaining the pointer states of the system and the environment. (The following discussion might seem excessive. However, we are going through it in order to get into the roots of having time-independent pointer states and then finally relate the insight which we obtain through these examples to

our formulation for calculating the pointer states of measurement; and find out the possible conditions under which the pointer states of the system turn out to become independent of time; so that they can represent the preferred basis of measurement). In fact, we can expand the exponential in equation (3.49) to write

$$\begin{aligned}\hat{V}(t) &= \exp\{\hat{\sigma}_z \otimes (\lambda(t)\ \hat{a}^\dagger - \lambda^*(t)\ \hat{a})\} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\ (\hat{\Lambda}(t))^n\ \hat{\sigma}_z^n, \end{aligned} \tag{3.53}$$

where in the second line of the above equation we defined $\hat{\Lambda}(t) = \lambda(t)\ \hat{a}^\dagger - \lambda^*(t)\ \hat{a}$. But we have $\hat{\sigma}_z^{2l} = \hat{I}$ and $\hat{\sigma}_z^{2l+1} = \hat{\sigma}_z$. So, now we can write

$$\begin{aligned}\hat{V}(t) &= \sum_{l=0}^{\infty} \frac{1}{(2l)!}\ (\hat{\Lambda}(t))^{2l}\ \hat{I} + \sum_{l=0}^{\infty} \frac{1}{(2l+1)!}\ (\hat{\Lambda}(t))^{(2l+1)}\ \hat{\sigma}_z \\ &= \cosh(\hat{\Lambda}(t))\ \hat{I} + \sinh(\hat{\Lambda}(t))\ \hat{\sigma}_z. \end{aligned} \tag{3.54}$$

However, the last expression can be simplified as

$$\begin{aligned}\hat{V}(t) &= \exp(\hat{\Lambda}(t))|a\rangle\langle a| + \exp(-\hat{\Lambda}(t))|b\rangle\langle b| \\ &= \hat{D}(\lambda(t))\ |a\rangle\langle a| + \hat{D}(-\lambda(t))\ |b\rangle\langle b| \end{aligned} \tag{3.55}$$

Comparing this result with our general form for the evolution operator, given by equation (3.27), we find

$$\hat{\mathcal{E}}_1 = \hat{D}(\lambda(t))\ , \quad \hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 = 0 \quad \text{and} \quad \hat{\mathcal{E}}_4 = \hat{D}(-\lambda(t)). \tag{3.56}$$

Here we show that having time-*independent* pointer states given by the states $|a\rangle$ and $|b\rangle$ (i.e. the eigenstates of the $\hat{\sigma}_z$ operator, which are the basis states that we used in order to write our evolution operator in the form of equation (3.27)) is the result of having $\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 = 0$ in this example; and this is a general condition. i.e.

Theorem 1: *Whenever* in some basis $\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 = 0$, then those basis states will be time-independent pointer states of the system.

Proof: If $\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 = 0$, operating the evolution operator $\hat{U}_{\rm tot}(t) = \hat{\mathcal{E}}_1|a\rangle\langle a| + \hat{\mathcal{E}}_4|b\rangle\langle b|$ on the initial state of the total composite system $|\psi^{\rm tot}(t_0)\rangle = (\alpha|a\rangle + \beta|b\rangle) \otimes |\Phi^{\mathcal{E}}(t_0)\rangle$ we obtain

$$|\psi^{\rm tot}(t)\rangle = \alpha|a\rangle\ \hat{\mathcal{E}}_1\ |\Phi^{\mathcal{E}}(t_0)\rangle + \beta|b\rangle\ \hat{\mathcal{E}}_4\ |\Phi^{\mathcal{E}}(t_0)\rangle. \tag{3.57}$$

The above relation basically means if the system initially is prepared in the $|a\rangle$ state (i.e. $\beta = 0$), the evolution of the total composite system must be given by $|\psi^{\rm tot}(t_0)\rangle = |a\rangle\otimes|\Phi^{\mathcal{E}}(t_0)\rangle \rightarrow |a\rangle\ \hat{\mathcal{E}}_1\ |\Phi^{\mathcal{E}}(t_0)\rangle$ and if the system initially is prepared in the $|b\rangle$ state (i.e. $\alpha = 0$), the evolution of the total composite system must be given by $|\psi^{\rm tot}(t_0)\rangle = |b\rangle\otimes|\Phi^{\mathcal{E}}(t_0)\rangle \rightarrow |b\rangle\ \hat{\mathcal{E}}_4\ |\Phi^{\mathcal{E}}(t_0)\rangle$. In other words, the basis states $|a\rangle$ and $|b\rangle$ are the time-independent pointer states of the system; as they do not entangle with the states of the environment.

However, in this case the corresponding pointer states of the environment necessarily are not time-independent and are given by

$$|\phi_a(t)\rangle = \hat{\mathcal{E}}_1\ |\Phi^{\mathcal{E}}(t_0)\rangle \quad \text{and} \quad |\phi_b(t)\rangle = \hat{\mathcal{E}}_4\ |\Phi^{\mathcal{E}}(t_0)\rangle. \tag{3.58}$$

QED.

Here we note that although requiring the condition $\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 = 0$ would guarantee having time-independent pointer states in the interaction picture given by our initial basis states, these states can represent the preferred basis of measurement only if either $\hat{H}_{\mathcal{S}} \approx 0$ or they turn out to be eigenstates of the self-Hamiltonian of the system as well; since in general an arbitrary state of a system $|\alpha\rangle$ in the interaction picture is related to that of the Schrödinger picture by $|\alpha; t\rangle_{\rm S} = e^{-i\hat{H}_0 t}\ |\alpha; t\rangle_{\rm I}$. For our example of the simplified spin-boson model this further condition is satisfied and hence the basis states $|a\rangle$ and $|b\rangle$ do represent the preferred basis of measurement in this case.

We continue through studying the conditions under which the pointer states of the system may become independent from time by presenting three more theorems. In each theorem we present a condition for having time-independent pointer states and predict the corresponding stationary pointer states. We will relate the mathematical conditions of theorem 1 and theorem 2 for having time-independent pointer states to the symmetries in the Hamiltonian of the total composite system through theorem 3 and theorem 4; which will be our main physical criteria for predicting the preferred basis of measurement. We will discuss the significance of these new results more elaborately in our final conclusion (section 5), where we better clarify how these results can serve as a generalization of the existing theory for determination of the preferred basis of measurement.

Theorem 2: Whenever in some basis $|a\rangle$ and $|b\rangle$ for the state of a two-level system we

have $\hat{\mathcal{E}}_1 = \hat{\mathcal{E}}_4$ and $\hat{\mathcal{E}}_2 = e^{-i\varphi}\hat{\mathcal{E}}_3$, then we will have a pair of time-independent pointer states for the system given by $|\pm\rangle = \frac{1}{\sqrt{2}}\{|a\rangle \pm e^{i\frac{\varphi}{2}}|b\rangle\}$. e.g. if $\varphi = 0$ ($\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3$) and the basis states $|a\rangle$ and $|b\rangle$ represent the eigenstates of the $\hat{\sigma}_z$ operator, then our time-independent pointer states in the interaction picture should be represented by the eigenstates of the $\hat{\sigma}_x$ operator.
Proof: If $\hat{\mathcal{E}}_1 = \hat{\mathcal{E}}_4$ and $\hat{\mathcal{E}}_2 = e^{-i\varphi}\hat{\mathcal{E}}_3$, then the condition for obtaining the pointer states of the system and the environment reads

$$\sum_n c_n\{\alpha\hat{\mathcal{E}}_1 + \beta\hat{\mathcal{E}}_2\}\ |\varphi_n\rangle = \hat{G}(t)\sum_n c_n\{\alpha\ e^{i\varphi}\hat{\mathcal{E}}_2 + \beta\hat{\mathcal{E}}_1\}\ |\varphi_n\rangle \quad \text{and}$$
$$\hat{G}(t) \textit{ be proportional to the unit matrix.} \tag{3.59}$$

The above condition can be satisfied for $\alpha = \pm e^{-i\frac{\varphi}{2}}\beta$; since it is obvious from equation (3.59) that for $\alpha = \pm e^{-i\frac{\varphi}{2}}\beta$ we have $\hat{G}(t) = \pm e^{-i\frac{\varphi}{2}} \times \hat{I}_{\mathcal{E}}$. So, from equation (3.33) we see that in this case the pointer states of the system must be given by

$$|\pm(t)\rangle = \mathcal{N}\ \{G(t)|a\rangle + |b\rangle\} = \frac{1}{\sqrt{2}}\{|a\rangle \pm e^{i\frac{\varphi}{2}}|b\rangle\} \tag{3.60}$$

QED.

In chapter 5 we will show that the above condition with $\varphi = 0$ (i.e. $\hat{\mathcal{E}}_1 = \hat{\mathcal{E}}_4$ and $\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3$) is satisfied while studying the short-time evolution of the two-level atom in the quantized atom-field model and in the regime that $\hat{H}_{\mathcal{S}} \ll \hat{H}_{\mathcal{E}} \ll \hat{H}'$. As a result, in this regime our theorem predicts that the preferred basis of measurement must be given by the eigenstates of the $\hat{\sigma}_x$ operator. Interestingly, for this example and in this regime $\hat{H}'$ turns out to be proportional to $\hat{\sigma}_x$ and hence the Zurek theorem for determination of the preferred basis of measurement *also* predicts the preferred basis of measurement to be given by the eigenstates of $\hat{\sigma}_x$ (which are the eigenstates of the total Hamiltonian in the quantum measurement limit $\hat{H} \approx \hat{H}'$). In fact, as we will show, the above theorem always covers the predictions of Zurek's theory for determination of the preferred basis of measurement at corresponding limits; although, as we will see, it is much more general compared to the former theory.

Now, a very interesting question can be to ask: "How can we predict whether any of the conditions given by the above two theorems can be satisfied for an arbitrary total

Hamiltonian defining a system-environment model?" More specifically, "Can we predict any of these conditions from the symmetries in the Hamiltonian of the total composite system"? Being able to answer the above question is specifically important, since it is not always easy to calculate the time evolution operator for a given total Hamiltonian defining a system-environment model. So, it can be quite useful if we can obtain some information about the preferred basis of measurement before calculating the evolution operator and knowing the $\hat{\mathcal{E}}_i$ operators. In fact, if we can answer the above question for an arbitrary total Hamiltonian defining a system-environment model, we will be able to predict those regimes of the parameter space (the parameter space for example determines how big is the contribution of each term in the Hamiltonian of the total composite system) for which the pointer states of the system can become independent from time so that a preferred basis of measurement can be realized. Moreover, we will be able to predict the corresponding preferred basis of measurement for each regime.

In what follows we relate the mathematical conditions discussed in our aforementioned theorems for having time-independent pointer states to the symmetries in the Hamiltonian of the total composite system. We do this through the following two theorems:

Theorem 3: In the total Hamiltonian of the global composite system $\hat{H}_{\rm tot} = \hat{H}_{\mathcal{S}} + \hat{H}_{\mathcal{E}} + \hat{H}'$ if the interaction Hamiltonian between the system and the environment commutes with the self-Hamiltonian of the system, i.e. if $[\hat{H}_{\mathcal{S}}, \hat{H}'] = 0$, we must have time-independent pointer states for the system given by the eigenstates of $\hat{H}_{\mathcal{S}}$ provided $\hat{H}_{\mathcal{S}} \neq 0$ and $\hat{H}_{\mathcal{S}} \neq \hat{I}$; or the eigenstates of $\hat{H}'$ if $\hat{H}_{\mathcal{S}} = 0$ or $\hat{H}_{\mathcal{S}} = \hat{I}$.

Proof: The Hamiltonian, in the interaction picture, is given by

$$\hat{H}_{\rm int} = e^{i\hat{H}_0 t}\hat{H}' e^{-i\hat{H}_0 t} \quad \text{with} \quad \hat{H}_0 = \hat{H}_{\mathcal{S}} + \hat{H}_{\mathcal{E}}. \tag{3.61}$$

So, using the Baker-Hausdorff Lemma i.e.

$$e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} = \hat{B} + \alpha[\hat{A}, \hat{B}] + \frac{\alpha^2}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + ... \tag{3.62}$$

we can clearly see that if $[\hat{H}_{\mathcal{S}}, \hat{H}'] = 0$, then we must have $[\hat{H}_{\rm int}, \hat{H}_{\mathcal{S}}] = 0$. Therefore, the evolution operator in the interaction picture, which generally is given by

$$\hat{U}(t) = \mathcal{T}_{\leftarrow}\ \mathrm{e}^{-\mathrm{i}\int_0^{\mathrm{t}} \hat{\mathrm{H}}_{\mathrm{int}}(\mathrm{t}')\ \mathrm{dt}'} \tag{3.63}$$

(with $\mathcal{T}_{\leftarrow}$ representing the time-ordering operator in the above equation), also must commute with $\hat{H}_{\mathcal{S}}$. i.e.

$$[\hat{U}(t), \hat{H}_{\mathcal{S}}] = 0. \tag{3.64}$$

The above result simply means that if $[\hat{H}_{\mathcal{S}}, \hat{H}^{'}] = 0$, the eigenstates of $\hat{H}_{\mathcal{S}}$ must be time-independent pointer states of the system (provided $\hat{H}_{\mathcal{S}} \neq \hat{I}$ and $\hat{H}_{\mathcal{S}} \neq 0$); as they will be eigenstates of the evolution operator $\hat{U}(t)$ as well and cannot be changed by more than an overall phase factor under its effect[4].

Also for the case that $\hat{H}_{\mathcal{S}} = \hat{I}$ or $\hat{H}_{\mathcal{S}} = 0$ from equation (3.61) we can see that the operator $\hat{H}_{\text{int}}$ cannot change the eigenstates of $\hat{H}^{'}$ in the Hilbert space of the system by more than an overall phase factor. Therefore, the evolution operator $\hat{U}(t)$, given by equation (3.63), also cannot change the eigenstates of $\hat{H}^{'}$ by more than an overall phase factor and the eigenstates of $\hat{H}^{'}$ in the Hilbert space of the system will be the time-independent pointer states of the system QED.

In what follows, we study the relationship between the condition $[\hat{H}_{\mathcal{S}}, \hat{H}^{'}] = 0$ and the mathematical conditions for having time-independent pointer states presented in our first two theorems. In other words we establish the connection between theorem 3 and the first two theorems.

We already showed that for $[\hat{H}_{\mathcal{S}}, \hat{H}^{'}] = 0$ we must have $[\hat{U}(t), \hat{H}_{\mathcal{S}}] = 0$. Now, let us study what are the implications of the latter commutation relation regarding the environmental operators $\hat{\mathcal{E}}_i$ which appear in the evolution operator of the global composite system represented by equation (3.27).

The self-Hamiltonian of the two-level system, which here commutes with $\hat{H}^{'}$ (the interaction between the system and the environment), generally can be represented as

$$\hat{H}_{\mathcal{S}} = s_1|a\rangle\langle a| + s_2|a\rangle\langle b| + s_3|b\rangle\langle a| + s_4|b\rangle\langle b|. \tag{3.65}$$

[4]For example if $\hat{H}_{\mathcal{S}}$ and $\hat{H}^{'}$ are proportional to one of the Pauli Matrices $\hat{\sigma}_i$ (as in our example of the simplified SBM), then in principle $\hat{U}(t)$ can be expanded in terms of powers of $\hat{\sigma}_i$ (see equations (3.63) and (3.49)) . However, ${\hat{\sigma}_i}^{2l}$ (with l representing an integer) is equal to $\hat{I}$ the identity operator and ${\hat{\sigma}_i}^{2l+1} = \hat{\sigma}_i$. This means that here in principle the evolution operator $\hat{U}(t)$ can be written as a summation with terms which either contain $\hat{\sigma}_i$ or the identity operator. Therefore, it must commute with $\hat{\sigma}_i$ and $\hat{H}_{\mathcal{S}}$; resulting in having the eigenstates of $\hat{H}_{\mathcal{S}}$ as time-independent pointer states of the system.

In the above equation like always $|a\rangle$ and $|b\rangle$ are some basis states for the two-level system and s_i's are some numbers (rather than operators). As an example, for the simplified spin boson model (SBM) $\hat{H}_{\mathcal{S}}$ and $\hat{H}^{'}$ both are taken to be proportional to the $\hat{\sigma}_z$ operator; so in this case $s_1 = -s_4 = 1$ and $s_2 = s_3 = 0$. Now, using equations (3.27) and (3.65) we can rewrite $[\hat{U}(t), \hat{H}_{\mathcal{S}}] = 0$ as

$$\begin{aligned}[\hat{U}(t), \hat{H}_{\mathcal{S}}] = [\hat{\mathcal{E}}_1 \frac{(1+\hat{\sigma}_z)}{2} + \hat{\mathcal{E}}_4 \frac{(1-\hat{\sigma}_z)}{2} + \hat{\mathcal{E}}_2 \hat{\sigma}_+ + \hat{\mathcal{E}}_3 \hat{\sigma}_- \ , \\ s_1 \frac{(1+\hat{\sigma}_z)}{2} + s_4 \frac{(1-\hat{\sigma}_z)}{2} + s_2 \hat{\sigma}_+ + s_3 \hat{\sigma}_-] = 0. \end{aligned} \tag{3.66}$$

Simplifying the above 2×2 matrix relation we find an equivalent set of three equations given by

$$\begin{aligned} s_3(\hat{\mathcal{E}}_4 - \hat{\mathcal{E}}_1) + \hat{\mathcal{E}}_3 \ (s_1 - s_4) = 0 \quad \text{and} \\ s_2(\hat{\mathcal{E}}_1 - \hat{\mathcal{E}}_4) - \hat{\mathcal{E}}_2 \ (s_1 - s_4) = 0 \quad \text{and} \\ s_3 \hat{\mathcal{E}}_2 = s_2 \hat{\mathcal{E}}_3. \end{aligned} \tag{3.67}$$

In calculating equation (3.67) from equation (3.66) we used the following commutation relations

$$[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z \quad \text{and} \quad [\hat{\sigma}_z, \hat{\sigma}_+] = 2\hat{\sigma}_+ \quad \text{and} \quad [\hat{\sigma}_z, \hat{\sigma}_-] = -2\hat{\sigma}_- \ . \tag{3.68}$$

However, note that for $\hat{H}_{\mathcal{S}}$ to be Hermitian we must have $s_2 = s_3^*$. So, if we represent the phase of s_2 by $\varphi/2$; i.e. if $s_2 = |s_2| e^{-i\varphi/2}$, then $s_3 \hat{\mathcal{E}}_2 = s_2 \hat{\mathcal{E}}_3$ would mean that

$$\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 e^{-i\varphi}. \tag{3.69}$$

So, equations (3.67) would require $\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 e^{-i\varphi}$ (provided $s_2 = s_3^* \neq 0$); as well as either of the following situations

$$\begin{aligned} \hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 = 0 \quad \text{and} \quad \hat{\mathcal{E}}_1 = \hat{\mathcal{E}}_4 \quad \text{or} \\ \hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 = 0 \quad \text{and} \quad s_2 = s_3 = 0 \quad \text{or} \\ \hat{\mathcal{E}}_1 = \hat{\mathcal{E}}_4 \quad \text{and} \quad s_1 = s_4 \quad \text{or} \quad \text{etc.} \end{aligned} \tag{3.70}$$

Now,

(a) For the first one of the above conditions ($\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 = 0$ and $\hat{\mathcal{E}}_1 = \hat{\mathcal{E}}_4$) we have $\hat{U}(t) = \hat{\mathcal{E}}_1(|a\rangle\langle a| + |b\rangle\langle b|) = \hat{\mathcal{E}}_1\hat{I}$. So, all possible states of the system including the eigenstates of $\hat{H}_{\mathcal{S}}$ (as well as the states $|a\rangle$ and $|b\rangle$ themselves, also due to theorem 1 as here $\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 = 0$), will be time-independent in the interaction picture. However, as we pointed out, only those of the states which also are eigenstates of the self-Hamiltonian of the system can represent the preferred basis of measurement; since these are the only states which will not have any time-evolution in the Schrödinger picture. This is just in agreement with what is stated by theorem 3, which predicts the preferred basis of measurement as the eigenstates of $\hat{H}_{\mathcal{S}}$ for the case that $[\hat{H}_{\mathcal{S}}, \hat{H}'] = 0$.

(b) The second possible condition resulting from $[\hat{H}_{\mathcal{S}}, \hat{H}'] = 0$, given in equation (3.70) by $\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 = 0$ and $s_2 = s_3 = 0$, is a special case of theorem 1 and hence it predicts having time-independent pointer states given by the eigenstates of $\hat{\sigma}_z$. However, here we have $s_2 = s_3 = 0$; so $\hat{H}_{\mathcal{S}}$ must be given by $\hat{H}_{\mathcal{S}} = s_1|a\rangle\langle a| + s_4|b\rangle\langle b|$; whose eigenstates are the basis states $|a\rangle$ and $|b\rangle$. So as we observe, the time-independent pointer states of the system predicted by theorem 1 for this case are the same as the eigenstates of the self-Hamiltonian of the system $\hat{H}_{\mathcal{S}}$; just in agreement with what is stated by theorem 3.

(c) The third condition ($\hat{\mathcal{E}}_1 = \hat{\mathcal{E}}_4$ and $s_1 = s_4$) together with $\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 e^{-i\varphi}$ of equation (3.69) (provided $s_2 = s_3^* \neq 0$) is a special case of theorem 2 and hence it predicts having time-independent pointer states for the system given by $|\pm\rangle = \frac{1}{\sqrt{2}}\{|a\rangle \pm e^{i\frac{\varphi}{2}}|b\rangle\}$. Moreover, $s_1 = s_4$; so from equation (3.65) it is clear that $\hat{H}_{\mathcal{S}}$ must be given by $\hat{H}_{\mathcal{S}} = s_1\hat{I} + s_2\hat{\sigma}_+ + s_3\hat{\sigma}_-$. Also, $s_2 = s_3^*$ (and $s_2 = |s_2|e^{-i\varphi/2}$); so we must have $s_3 = s_2 e^{i\varphi}$. Therefore,

$$\hat{H}_{\mathcal{S}} = s_1\hat{I} + s_2\hat{\sigma}_+ + s_2 e^{i\varphi}\hat{\sigma}_- = \begin{pmatrix} s_1 & s_2 \\ s_2 e^{i\varphi} & s_1 \end{pmatrix}. \tag{3.71}$$

It is easy to verify that the eigenstates of the above matrix are the same as $|\pm\rangle = \frac{1}{\sqrt{2}}\{|a\rangle \pm e^{i\frac{\varphi}{2}}|b\rangle\}$ which were predicted by theorem 2 to be the time-independent pointer states of the system for this case. So as we observe, for the third possible condition resulting from $[\hat{H}_{\mathcal{S}}, \hat{H}'] = 0$ also theorem 2 predicts having time-independent pointer states for the system given by the eigenstates of the self-Hamiltonian of the system; just in agreement with what is stated by theorem 3.

As an example of having the condition discussed in theorem 3 suppose that both contributions $\hat{H}'$ and $\hat{H}_{\mathcal{S}}$ are proportional to the $\hat{\sigma}_z$ operator (i.e. $s_1 = -s_4 = 1$ and $s_2 = s_3 = 0$). Then according to equation (3.67) we must have $\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 = 0$ and hence, as a result of theorem 1 we must have time-independent pointer states for the system given by the eigenstates of $\hat{\sigma}_z$. Similarly, if both Hamiltonians $\hat{H}'$ and $\hat{H}_{\mathcal{S}}$ are proportional to $\hat{\sigma}_{x(y)}$, we find $\hat{\mathcal{E}}_1 = \hat{\mathcal{E}}_4$ and $\hat{\mathcal{E}}_2 = \pm\hat{\mathcal{E}}_3$ and hence, as a result of theorem 2 we must have time-independent pointer states for the system given by the eigenstates of $\hat{\sigma}_{x(y)}$.

Next, we discuss another possible symmetry in the Hamiltonian of a given system-environment model which also would lead to having time-*independent* pointer states for the system.

Theorem 4: For the Hamiltonian in the interaction picture, which in some basis $|a\rangle$ and $|b\rangle$ of the two-level system can be represented by

$$\hat{H}_{\rm int} = \hat{h}_{11}|a\rangle\langle a| + \hat{h}_{12}|a\rangle\langle b| + \hat{h}_{21}|b\rangle\langle a| + \hat{h}_{22}|b\rangle\langle b|, \tag{3.72}$$

we must have $\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3$ and $\hat{\mathcal{E}}_1 = \hat{\mathcal{E}}_4$ and hence, (according to theorem 2) time-independent pointer states for the system given by $|\pm\rangle = \frac{1}{\sqrt{2}}\{|a\rangle \pm |b\rangle\}$ provided

$$\hat{h}_{11} = \hat{h}_{22} \quad \text{and} \quad \hat{h}_{12} = \hat{h}_{21}. \tag{3.73}$$

Proof: For the time evolution operator in the interaction picture, which satisfies the Schrödinger equation, we have

$$i\hbar\frac{\partial}{\partial t}\hat{u}(t) = \hat{H}_{\rm int}\hat{u}(t) \quad \text{i.e.}$$
$$i\hbar\begin{pmatrix} \dot{\hat{\mathcal{E}}}_1 & \dot{\hat{\mathcal{E}}}_2 \\ \dot{\hat{\mathcal{E}}}_3 & \dot{\hat{\mathcal{E}}}_4 \end{pmatrix} = \begin{pmatrix} \hat{h}_{11} & \hat{h}_{12} \\ \hat{h}_{21} & \hat{h}_{22} \end{pmatrix}\begin{pmatrix} \hat{\mathcal{E}}_1 & \hat{\mathcal{E}}_2 \\ \hat{\mathcal{E}}_3 & \hat{\mathcal{E}}_4 \end{pmatrix}. \tag{3.74}$$

So, if at some regime of the parameter space we have $\hat{h}_{11} = \hat{h}_{22}$ and $\hat{h}_{12} = \hat{h}_{21}$ we would have the following set of four equations

$$\begin{aligned} i\hbar\dot{\hat{\mathcal{E}}}_1 &= \hat{h}_{11}\hat{\mathcal{E}}_1 + \hat{h}_{12}\hat{\mathcal{E}}_3, \\ i\hbar\dot{\hat{\mathcal{E}}}_2 &= \hat{h}_{11}\hat{\mathcal{E}}_2 + \hat{h}_{12}\hat{\mathcal{E}}_4, \\ i\hbar\dot{\hat{\mathcal{E}}}_3 &= \hat{h}_{11}\hat{\mathcal{E}}_3 + \hat{h}_{12}\hat{\mathcal{E}}_1, \\ i\hbar\dot{\hat{\mathcal{E}}}_4 &= \hat{h}_{11}\hat{\mathcal{E}}_4 + \hat{h}_{12}\hat{\mathcal{E}}_2. \end{aligned} \tag{3.75}$$

Now, this set of four equations is invariant under the transformation $\hat{\mathcal{E}}_1 \leftrightarrow \hat{\mathcal{E}}_4$ and $\hat{\mathcal{E}}_3 \leftrightarrow \hat{\mathcal{E}}_2$. Moreover, $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_4$ satisfy the same initial conditions; just as $\hat{\mathcal{E}}_2$ and $\hat{\mathcal{E}}_3$ do (since we must have $\hat{u}(t_0) = \hat{I}$; i.e. $\hat{\mathcal{E}}_1(t_0) = \hat{\mathcal{E}}_4(t_0) = 1$ and $\hat{\mathcal{E}}_2(t_0) = \hat{\mathcal{E}}_3(t_0) = 0$). As a result, whatever are the solutions of the set of equations (3.75) we must have $\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3$ and $\hat{\mathcal{E}}_1 = \hat{\mathcal{E}}_4$. Therefore, (according to theorem 2) we predict having time-independent pointer states for the system given by $|\pm\rangle = \frac{1}{\sqrt{2}}\{|a\rangle \pm |b\rangle\}$ QED.

Here also, we note that although the condition represented by equation (3.73) would guarantee having time-independent pointer states in the interaction picture given by $|\pm\rangle = \frac{1}{\sqrt{2}}\{|a\rangle \pm |b\rangle\}$, these states can represent the preferred basis of measurement only if either $\hat{H}_{\mathcal{S}} \approx 0$ or they turn out to be eigenstates of the self-Hamiltonian of the system as well; since in general an arbitrary state of a system $|\alpha\rangle$ in the interaction pictures is related to that of the Schrödinger picture by $|\alpha; t\rangle_{\rm S} = e^{-i\hat{H}_0 t}\ |\alpha; t\rangle_{\rm I}$.

As we will show in another article, the condition in theorem 4 is satisfied for the quantized atom-field model and in the regime that $\hat{H}_{\mathcal{S}} \approx 0$. In fact, an exact calculation of the pointer states of the two-level system in this regime and by using our method, would result in finding the states $|\pm\rangle = \frac{1}{\sqrt{2}}\{|a\rangle \pm |b\rangle\}$ (with $|a\rangle$ and $|b\rangle$ representing the atomic upper and lower states respectively) as the time-independent pointer states of the system; just as is predicted by theorem 4.

We should also mention that the above theorem can be generalized as follows:

Generalization of Theorem 4: For the Hamiltonian in the interaction picture, which in some basis $|a\rangle$ and $|b\rangle$ of the two-level system can be represented by equation (3.72) we must have $\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 e^{-i\varphi}$ and $\hat{\mathcal{E}}_1 = \hat{\mathcal{E}}_4$ and hence, (according to theorem 2) time-independent pointer states for the system given by $|\pm\rangle = \frac{1}{\sqrt{2}}\{|a\rangle \pm e^{-i\varphi}|b\rangle\}$ provided

$$\hat{h}_{11} = \hat{h}_{22} \quad \text{and} \quad \hat{h}_{12} = \hat{h}_{21} e^{-i\varphi}. \tag{3.76}$$

Proof: Using equation (3.74) for the case that $\hat{h}_{11} = \hat{h}_{22}$ and $\hat{h}_{12} = \hat{h}_{21} e^{-i\varphi}$ and then taking the second derivative with respect to time of the operators $\hat{\mathcal{E}}_i$ one can easily verify the following set of four equations

$$i\hbar\ddot{\hat{\mathcal{E}}}_1 = \{\dot{\hat{h}}_{11} + \frac{\hat{h}_{11}^2}{i\hbar} + \frac{\hat{h}_{12}^2}{i\hbar} e^{i\varphi}\}\hat{\mathcal{E}}_1 + \{\dot{\hat{h}}_{12} + \frac{2\hat{h}_{11}\hat{h}_{12}}{i\hbar}\}\hat{\mathcal{E}}_3,$$

$$
\begin{aligned}
i\hbar\ddot{\hat{\mathcal{E}}}_2 &= \{\dot{\hat{h}}_{11} + \frac{\hat{h}_{11}^2}{i\hbar} + \frac{\hat{h}_{12}^2}{i\hbar}e^{i\varphi}\}\hat{\mathcal{E}}_2 + \{\dot{\hat{h}}_{12} + \frac{2\hat{h}_{11}\hat{h}_{12}}{i\hbar}\}\hat{\mathcal{E}}_4,\\
i\hbar\ddot{\hat{\mathcal{E}}}_3 &= \{\dot{\hat{h}}_{11} + \frac{\hat{h}_{11}^2}{i\hbar} + \frac{\hat{h}_{12}^2}{i\hbar}e^{i\varphi}\}\hat{\mathcal{E}}_3 + \{\dot{\hat{h}}_{12}e^{i\varphi} + \frac{2\hat{h}_{11}\hat{h}_{12}}{i\hbar}e^{i\varphi}\}\hat{\mathcal{E}}_1,\\
i\hbar\ddot{\hat{\mathcal{E}}}_4 &= \{\dot{\hat{h}}_{11} + \frac{\hat{h}_{11}^2}{i\hbar} + \frac{\hat{h}_{12}^2}{i\hbar}e^{i\varphi}\}\hat{\mathcal{E}}_4 + \{\dot{\hat{h}}_{12}e^{i\varphi} + \frac{2\hat{h}_{11}\hat{h}_{12}}{i\hbar}e^{i\varphi}\}\hat{\mathcal{E}}_2.
\end{aligned}
\tag{3.77}
$$

this set of four equations is invariant under the transformation $\hat{\mathcal{E}}_1 \leftrightarrow \hat{\mathcal{E}}_4$ and $\hat{\mathcal{E}}_3 \leftrightarrow \hat{\mathcal{E}}_2 e^{i\varphi}$. Moreover, $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_4$ satisfy the same initial conditions; just as $\hat{\mathcal{E}}_2$ and $\hat{\mathcal{E}}_3 e^{-i\varphi}$ do (since we must have $\hat{U}(t_0) = \hat{I}$; i.e. $\hat{\mathcal{E}}_1(t_0) = \hat{\mathcal{E}}_4(t_0) = 1$ and $\hat{\mathcal{E}}_2(t_0) = \hat{\mathcal{E}}_3(t_0) = 0$. Also, the initial time t_0 is the same for all $\hat{\mathcal{E}}_i$'s; so in solving the set of equations (3.77) we should not worry about the equality of the other constant of integration). As a result, whatever are the solutions of the set of equations (3.77) we must have $\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 e^{-i\varphi}$ and $\hat{\mathcal{E}}_1 = \hat{\mathcal{E}}_4$. Therefore, (according to theorem 2) we predict having time-independent pointer states for the system given by $|\pm\rangle = \frac{1}{\sqrt{2}}\{|a\rangle \pm e^{-i\varphi}|b\rangle\}$ QED.

We note that theorem 4 definitely is not contained in theorem 3 and generally these two theorems refer to different conditions for having time-independent pointer states. This is because the condition for having time-independent pointer states represented in equation (3.76) of theorem 4 requires having $\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 e^{-i\varphi}$ and $\hat{\mathcal{E}}_1 = \hat{\mathcal{E}}_4$ as we discussed. However, according to equation (3.67) satisfaction of this latter condition necessarily will not lead to satisfaction of the condition $[\hat{H}_{\mathcal{S}}, \hat{H}'] = 0$ of theorem 3; unless $\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 = 0$ or $s_1 = s_4$ as well. In other words if the condition of theorem 4 (equation (3.76)) is satisfied but none of the conditions $\hat{\mathcal{E}}_2 = \hat{\mathcal{E}}_3 = 0$ or $s_1 = s_4$ are satisfied, then the condition for commutativity of $\hat{H}_{\mathcal{S}}$ and $\hat{H}'$ will not be satisfied.

3.5 Conclusion

Defining the pointer states of a subsystem as those states which are characterized by their ability not to entangle with the states of another subsystem, we presented a general method for evaluating the pointer states of a subsystem. This way we showed how in practice the global state of the system and the environment may evolve into a *diagonal* state (i.e. the Von Neumann scheme of measurement may be realized) as a result of the natural evolution

of the total composite system. As we showed, evaluation of the pointer states of the system requires finding those specific initial states of the system and the environment for which the operator $\hat{G}(t)$ (defined through equation (3.30)) may become independent of the states of the environment. However, as we could see from our example represented by the evolution of the two-level atom in the Jaynes-Cummings model of quantum optics and in the exact-resonance-regime such initial conditions for the state of the system and the environment necessarily do not exist in an arbitrary regime. (For this example as we saw, unless we have a large average number of photons which can make a sharp distribution function for the state of the electromagnetic field, the states of the two-level atom and the field will remain highly entangled and the pointer states of measurement cannot be realized at all.) As a result, even time-dependent pointer states necessarily do not exist for any arbitrary regime. In other words, premeasurement by the environment (the Von Neumann scheme of measurement), as a result of which the state of the total composite system becomes in a diagonal form, necessarily cannot be realized in any arbitrary regime.

In this chapter we distinguished between pointer states of measurement and the preferred basis of measurement; as time-independent pointer states which can arise only in certain regimes. We exactly showed why indeed time-independent pointer states (which require the operator $\hat{G}(t)$ defined through equation (3.30) be independent of time) cannot be expected in most of the regimes. In other words, the pointer states of the system, which do not entangle with the states of the environment and appear as a result of premeasurement by the environment, necessarily are not time-independent and the assumption of having quantum *nondemolition premeasurement* in the Von Neumann scheme of measurement practically is not a good assumption; as the pointer states which appear on the diagonal state of the total composite system may change by more than just an overall phase factor. Moreover, we explored those conditions under which the pointer states of the system may be independent from time; so that they can represent the preferred basis of measurement. These are new aspects not contained in the existing theory for determination of the preferred basis of measurement.

As we saw the conditions for having time-independent pointer states include the so-called

quantum limit of decoherence ($\hat{H} \approx \hat{H}_{\mathcal{S}}$) as well as the so-called quantum measurement limit ($\hat{H} \approx \hat{H}'$). In fact, time-independent pointer states for the system are predicted for these two regimes by using theorem 3 and just as special cases of the more general symmetry condition represented by this theorem. Therefore, our theorems cover the predictions of Zurek's theory for determination of the preferred basis of measurement at corresponding limits. Nonetheless, they present some other conditions as well under which the pointer states of the system would become independent of time and hence can we have a preferred basis of measurement. For example as we saw, in order to have the preferred basis of measurement given by the eigenstates of the interaction Hamiltonian in the Hilbert space of the system, necessarily we do not require $\hat{H}_{\mathcal{S}}$ and $\hat{H}_{\mathcal{E}}$ to be negligible and this prediction holds valid whenever $\hat{H}_{\mathcal{S}}$ commutes with a nonzero $\hat{H}'$ and no matter how big are the contributions from the self-Hamiltonian of the system and the self-Hamiltonian of the environment. Therefore, our criteria for predicting the time-independence of pointer states go beyond the limits in which $\hat{H}_{\rm tot} \approx \hat{H}'$ or $\hat{H}_{\rm tot} \approx \hat{H}_{\mathcal{S}}$ and will include some other cases as well; where all contributions can be present at the same time. In this sense, our theory not only provides us with a general method for obtaining the time-dependent pointer states of the system, but also would serve as a generalization for the existing theory for determination of the preferred basis of measurement.

As an application of our theory, one can use it in order to obtain those regimes of the parameter space for a given total Hamiltonian defining our system-environment model for which a preferred basis of measurement can be realized. Moreover, we can predict the corresponding preferred basis of measurement for each regime. In addition to that now we have a method in order to obtain the time-dependent pointer states in non-measurement regimes; where a time-independent basis of measurement cannot be realized at all. This ability to obtain *time-dependent* pointer states, which arise in the majority of regimes, is particularly important in decoherence studies; as such pointer states although evolve with time and cannot represent the preferred basis of measurement, they correspond to the initial conditions for the state of the system and the environment for which we can have long decoherence times. We will present some very interesting results regarding this problem in

chapter 4, where we obtain the time-dependent pointer states of the generalized spin-boson model and study the decoherence of the central system in this model.

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Chapter 4

Time-dependent pointer states of the generalized spin-boson model and consequences regarding the decoherence of the central system

4.1 Introduction

4.1.1 Foreword

In the previous chapter (paper I [1]) we discussed the pointer states of measurement[1] and we presented a general method for obtaining the pointer states of the system and the environment for a given total Hamiltonian defining the system-environment model. We used our method in order to rederive the time-dependent pointer states of the system and the environment (initially prepared in the coherent state) in the Jaynes-Cummings model (JCM) of quantum optics and for the exact resonance regime; verifying the previous results for the JCM [2, 3]. We also briefly discussed another simple example of a spin-boson model (SBM)[2] represented by the following well known Hamiltonian

$$\hat{H} = \frac{1}{2}\omega_0\hat{\sigma}_z + \omega\hat{a}^\dagger\hat{a} + \hat{\sigma}_z \otimes (\mathrm{g}\hat{a}^\dagger + \mathrm{g}^*\hat{a}). \tag{4.1}$$

This model basically is composed of a central spin-half particle (or other two-level system) surrounded by an environment of N bosonic particles such as photons. For this simplified spin-boson model we considered a single-mode quantized field for the environment. Moreover, we disregarded a possible contribution to the self-Hamiltonian of the system which can induce transitions between the upper and lower states of the central system (i.e. an intrinsic tunneling contribution proportional to $\hat{\sigma}_x$ Pauli matrix that would generate the intrinsic dynamics of the central spin).

[1] The pointer states of a subsystem are characterized by their ability not to entangle with the states of another subsystem (i.e. the requirement of faithful measurement) and appear in the diagonal state of the total composite system after premeasurement by the environment. As we elaborately described in chapter 3, generally we should distinguish between the pointer states of a system and the preferred basis of measurement. We proved explicitly that the pointer states of a subsystem generally are time-dependent and a preferred basis of measurement does not exist, unless under some specific conditions (discussed there in chapter 3) that the pointer states of measurement become time-independent. Moreover, the pointer states of a system necessarily are not orthonormal amongst themselves at all times. Therefore, necessarily they cannot form a basis for the Hilbert space of the system at all times.

[2] For a serious review and analysis of the spin-boson model in different regimes the interested reader can refer to the seminal article by Leggett *et al.* [4] or the book by Weiss [5]. Also a brief while very useful review of the model can be found in chapter 5 of Schlosshauer's book [6].

In the Hamiltonian of this simplified spin-boson model (represented by equation (4.1)) ω_0 is the splitting between the states of the spin-half particle. The second term in equation (4.1) represents the self-Hamiltonian of the electromagnetic field; where we have considered a single-mode quantized field with the frequency of ω for the environment. Also the third term, with g as the spin-field coupling constant, represents the interaction between the central spin-half particle and a single-mode quantized field; which in fact is the quantized form of the famous $-\boldsymbol{\mu}.\mathbf{B}$ Hamiltonian due to the interaction between a particle of magnetic dipole-moment $\boldsymbol{\mu}$ and a magnetic field $\mathbf{B}$.

The simplified model of equation (4.1) has been studied by many people. In particular, in the context of quantum computation and quantum information this model has been frequently used to gain some first insights into the decoherence of a single qubit interacting with a bosonic environment [7, 8, 9]. For this simplified model it is easy to show that the pointer states of the system must be time-independent and given by the eigenstates of the $\hat{\sigma}_z$ operator [6, 5]. However, as we discussed in chapter 3, the time independence of the pointer states in this frequently cited model merely is because of the fact that in the Hamiltonian of equation (4.1), which represents this model, the self-Hamiltonian of the system and the interaction between the system and the environment commute with each other $[\hat{H}_{\mathcal{S}}, \hat{H}_{\rm int}] = 0$. Indeed, time independence of pointer states by no means should be taken for granted; since, as we discussed in chapter 3, time-independent pointer states can be realized only under some specific conditions.

In this chapter we study another more challenging spin-boson model which is defined through the following total Hamiltonian

$$\hat{H} = -\frac{1}{2}\Delta_0\hat{\sigma}_x + \omega\hat{a}^\dagger\hat{a} + \hat{\sigma}_z \otimes (\mathrm{g}\hat{a}^\dagger + \mathrm{g}^*\hat{a}). \tag{4.2}$$

For this more general spin-boson model (unlike the simplified model) we consider a contribution in the self-Hamiltonian of the system which can induce transitions between the upper and lower states of the central system (i.e. the intrinsic tunneling contribution proportional to the $\hat{\sigma}_x$ Pauli matrix which would generate the intrinsic dynamics of the central spin). Δ_0 is the so-called tunneling matrix element. Also, it is assumed that the asymmetry energy in the self-Hamiltonian of the central system is negligible. Therefore, unlike the simplified

model here we do not have a contribution proportional to the $\hat{\sigma}_z$ Pauli matrix in the self-Hamiltonian of the system. However, one can easily verify that the modifications due to this contribution can be done quite easily, as such a term commutes with the interaction between the system and the environment, represented by the third term in equation (4.2).

As we will see, the addition of the tunneling contribution to the Hamiltonian of the total composite system would make the dynamics of the central spin and the electromagnetic field much more complicated. In fact, the pointer states that we will obtain in section 4.3 and for this more general model are by no means trivial; and they turn out to be *time-dependent*.

In the paper by Leggett *et al.* [4] they considered a general form of the Hamiltonian given by equation (4.2), where the environment can be represented by a spectral density $\boldsymbol{J}(\omega)$ (rather than considering a *single-mode* quantized field). They used the "influence-functional" method of Feynman and Vernon [10] to obtain general expressions for $P(t) \equiv \langle \hat{\sigma}_z(t) \rangle$ in the form of a power series in Δ. However, the general expressions they obtained for $P(t)$ in terms of the spectral density function $\boldsymbol{J}(\omega)$ are exceedingly cumbersome to calculate in most regimes. So, they assumed that $\boldsymbol{J}(\omega)$ is a smooth function of ω and moreover it is of the form of ω^s (where s is a real nonnegative number) up to some cutoff frequency of ω_c , *which is large compared to* Δ; to be able to simplify the general expressions they obtained for $P(t)$. Hence, they managed to study the behavior of $P(t) \equiv \langle \hat{\sigma}_z(t) \rangle$ only by considering the above assumption with $\Delta/\omega_c \ll 1$; which clearly is very different from our resonance regime.

This model can also be studied in the framework of the Born-Markov approximation in order to obtain an approximate master equation for the evolution of the reduced density matrix of the system [5]. The master equations obtained in this way are valid only in certain regimes; and moreover, one often may need to resort to numerical computation in order to be able to solve them. However, the main purpose of this chapter is (1) to obtain the time-dependent pointer states of the system and the environment, as well as expressions for the evolution of the reduced density matrix of the system in the exact resonance regime and for an environment initially prepared in a coherent state; and (2) to obtain approximate expressions *in closed form* for the evolution of the off-diagonal elements of the reduced

density matrix (for the case that the environment initially is prepared in a coherent state with a large average number of photons) which can be used so that we can understand the decoherence of the central system *in an analytical way*.

This chapter is organized as follows:

After this foreword we review our method for obtaining the pointer states of the system and the environment; and in section 3 we exploit it in order to calculate the time-dependent pointer states of the spin-boson model represented by the Hamiltonian of equation (4.2). In fact, as we will see, the pointer states which we obtain in this way for our model are much more complicated than the pointer states of the system and the environment in the Jaynes-Cummings model of quantum optics[1, 2]; it is no wonder that they were not calculated in any previous research.

In order to be able to exploit our method and obtain the pointer states of the system and the environment in our model (section 3) we need to know the time-evolution operator of our model in the appropriate form; and this task is done in section 2. In section 4 we discuss an interesting property of the pointer states which we obtain in section 3; i.e. their coincidence at specific times. As we will discuss, the coincidence of the pointer states of the system at specific times basically means that regardless of the initial state of the system, at some specific times the states of the system and the environment are not entangled with each other, and the system can be represented by a specific well-defined state of its own. In other words, our model exhibits an example of state preparation at some specific times.

In section 5 we exploit the pointer states of the system and the environment (which we obtain in section 3) in order to study the decoherence of the central system in our model. Using these pointer states, we obtain a closed form for the evolution of coherences of the central system in our model, and we discuss some of the interesting properties they exhibit. Finally, in section 6 we further discuss the significance of our results and the conclusions.

To further demonstrate the generality and usefulness of our method of obtaining pointer states, in chapter 5 (paper IV [11]) we will also obtain the time-dependent pointer states of the system and the environment for the quantized atom-field model and in some nonresonance regimes.

4.1.2 Review of the method

In order to be able to obtain the pointer states of the system and the environment for an arbitrary total Hamiltonian defining the system-environment model, we first need to find those probable initial states of the system which do not entangle with the states of the environment throughout their evolution with time; and then we should obtain their time evolution. Finally, we should obtain their corresponding states from the environment which in fact, are the pointer states of the environment. As we will see, existence of pointer states may require having a sufficiently large environment which contains a large number of degrees of freedom. In other words, pointer states characterized by their ability not to entangle with the states of another subsystem, do not necessarily exist in every arbitrary regime.

Consider a two-state system $\mathcal{S}$ with two arbitrary basis states $|a\rangle$ and $|b\rangle$, initially prepared in the state

$$|\psi^{\mathcal{S}}(t_0)\rangle = \alpha|a\rangle + \beta|b\rangle \quad \text{with} \quad |\alpha|^2 + |\beta|^2 = 1; \tag{4.3}$$

and an environment initially prepared in the state

$$|\Phi^{\mathcal{E}}(t_0)\rangle = \sum_{n=0}^{\infty} c_n|\varphi_n\rangle, \tag{4.4}$$

where $\{|\varphi_n\rangle\}$'s are a complete set of basis states for the environment. For the two-state system with the two basis states $|a\rangle$ and $|b\rangle$ we can take the set of any four linearly independent operators in the Hilbert space of the system as a complete set of basis operators, which can induce any change to the initial state of the two-state system given by equation (4.3). For example, we can take the Pauli operators in addition to the identity operator $\hat{I} = |a\rangle\langle a| + |b\rangle\langle b|$ as our complete set of basis operators; or equivalently we can take the four operators $|a\rangle\langle a|$, $|a\rangle\langle b|$, $|b\rangle\langle a|$ and $|b\rangle\langle b|$ as our complete set of basis operators. So, the time evolution operator for the global state of the system and the environment, which (for a two-state system) generally is of the form

$$\hat{U}_{\text{tot}}(t) = \sum_{\alpha=1}^{4} \hat{S}_\alpha \otimes \hat{\mathcal{E}}_\alpha \ , \tag{4.5}$$

can be considered as

$$\hat{U}_{\rm tot}(t) = \hat{\mathcal{E}}_1|a\rangle\langle a| + \hat{\mathcal{E}}_2|a\rangle\langle b| + \hat{\mathcal{E}}_3|b\rangle\langle a| + \hat{\mathcal{E}}_4|b\rangle\langle b|. \tag{4.6}$$

In the above equation $\hat{\mathcal{E}}_i$'s depend on the total Hamiltonian defining the system-environment model.

Using equations (4.3) to (4.6) we can write the global state of the system and the environment as follows

$$\begin{aligned} |\psi^{\rm tot}(t)\rangle &= \hat{U}_{\rm tot}(t).\ (\alpha|a\rangle + \beta|b\rangle) \otimes (\sum_{n=0}^{\infty} c_n|\varphi_n\rangle) \\ = \mathbf{A}(t)\ |a\rangle + \mathbf{B}(t)\ |b\rangle \quad &\text{with} \quad \mathbf{A}(t) = \sum_{n=0}^{\infty} c_n\{\alpha\hat{\mathcal{E}}_1 + \beta\hat{\mathcal{E}}_2\}\ |\varphi_n\rangle \\ &\text{and} \quad \mathbf{B}(t) = \sum_{n=0}^{\infty} c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle. \end{aligned} \tag{4.7}$$

In order to find those probable initial states of the system which do not entangle with the states of the environment we first define $\hat{G}(t)$ as the operator in the Hilbert space of the environment which relates the vectors $\mathbf{A}(t)$ and $\mathbf{B}(t)$ to each other

$$\mathbf{A}(t) = \hat{G}(t)\mathbf{B}(t) \quad \text{or} \quad \sum_n c_n\{\alpha\hat{\mathcal{E}}_1 + \beta\hat{\mathcal{E}}_2\}\ |\varphi_n\rangle = \hat{G}(t)\sum_n c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle. \tag{4.8}$$

Now, for the global state of the system and the environment, which is given by

$$\begin{aligned} |\psi^{\rm tot}(t)\rangle &= \mathbf{A}(t)\ |a\rangle + \mathbf{B}(t)\ |b\rangle = \hat{G}(t)\mathbf{B}(t)\ |a\rangle + \mathbf{B}(t)\ |b\rangle \\ &= \{\hat{G}(t)|a\rangle + |b\rangle\} \times (\sum_{n=0}^{\infty} c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle), \end{aligned} \tag{4.9}$$

we observe that *if* for some initial states of the system and the environment $\hat{G}(t)$ turns out to become in the form

$$\hat{G}(t) = G(t) \times \hat{I}_{\mathcal{E}}, \tag{4.10}$$

with $G(t)$ as a scalar (rather than an operator) and $\hat{I}_{\mathcal{E}}$ representing the identity operator in the Hilbert space of the environment, then those initial states of the system and the environment will not entangle with each other, and hence they can represent the initial

pointer states of the system and the environment. This result simply is because of the fact that if for some initial states of the system and the environment $\hat{G}(t)$ turns out to become a scalar in the form of equation (4.10), $G(t)$ will be independent of the indices of the environment (i.e. independent of n); as in this case all components of $\mathbf{B}(t)$ will be mapped into their corresponding components from $\mathbf{A}(t)$ through the *same* scalar function $G(t)$ (which will keep the two vectors $\mathbf{A}(t)$ and $\mathbf{B}(t)$ parallel to each other). Therefore, in this case $\hat{G}(t)$ will not enter the summation in the expression $\sum_n c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle$ of equation (4.9); and (as one can see from equation (4.9)) the states of the system and the environment respectively represented by $\{G(t)|a\rangle + |b\rangle\}$ and $\sum_n c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle$ will not entangle to each other.

In other words, if for some initial states of the system and the environment $\mathbf{A}(t) = \hat{G}(t)\mathbf{B}(t)$ is equal to $G\mathbf{B}(t)$, it means that for those initial states of the system and the environment $\mathbf{B}(t)$ becomes an eigenstate of the operator $\hat{G}(t)$; and the two vectors $\mathbf{A}(t)$ and $\mathbf{B}(t)$ will stay parallel with each other throughout their evolution with time; and as we discussed, in this case the states of the system and the environment will not entangle with each other and (as one can see from equation (4.9)) pointer states can be realized for the system and the environment given by

$$|\pm(t)\rangle = \mathcal{N}_\pm\ \{G(t)|a\rangle + |b\rangle\} \qquad \text{and}$$
$$|\Phi_\pm(t)\rangle = \mathcal{N}_\pm^{-1}(\sum_{n=0}^{\infty} c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle). \tag{4.11}$$

In the above equation we have represented the pointer states of the system by $|\pm(t)\rangle$ and those of the environment by $|\Phi_\pm(t)\rangle$. Also, $\mathcal{N}_\pm$ is the normalization factor for the pointer states of the system (clearly $\mathcal{N}_\pm = \frac{1}{\sqrt{2}}$ if $|G(t)| = 1$, as for the example of the JCM).

As we will see in this chapter, generally there is no guaranty for the condition (4.10) to be satisfied; and satisfaction of this condition often may require having a sufficiently large environment which contains a large number of degrees of freedom. However, *if* in some regime and for a given Hamiltonian defining a system-environment model we can find initial states for the system and the environment which satisfy this condition, we do know that pointer states can be realized for the system and the environment and these initial

states would correspond to the initial pointer states of the system and the environment. In fact, by looking at equation (4.9) we notice that in practice we can expect some states of the system and the environment to keep their individuality and not to entangle with each other *even* if they can satisfy our condition (given by equation (4.10)) only in a fraction of the Hilbert space of the environment where the c_n coefficients are not negligible. This of course will involve assuming some approximations in obtaining the pointer states which correspond to a given total Hamiltonian defining a physical model. However, as we will show in this chapter, in the end we can define a measure for the degree of entanglement between the states of the system and the environment, which after its calculation for the pointer states which we obtain for our model we can know exactly in which regimes our pointer states are valid and will not entangle with the states of another system. For example, this way we will show that the pointer states which we will obtain for our spin-boson model for an environment initially prepared in the coherent state are valid (ie. will not entangle with the states of the other subsystem throughout their evolution with time) only up to times of the order $\hbar\bar{n}/g$; where $\bar{n}$ is the average number of photons in the coherent state of the environment.

In order to find the pointer states of the system and the environment for a given total Hamiltonian defining the system-environment model, and for a given initial state of the environment, our main goal would be finding those possible initial states of the system for which $\hat{G}(t)$ (which is defined through equation (4.8)) is of the form of relation (4.10). In section 3 we will consider the spin-boson model represented by the Hamiltonian of equation (4.2) with an initial coherent field, and exploit this method in order to calculate the time-dependent pointer states of the system and the environment. As we will see, this task can be done when dealing with a sufficiently large environment which contains a large number of degrees of freedom; of course provided we do have the time-evolution operator for our system-environment model in the form of equation (4.6) and the $\hat{\mathcal{E}}_i$ operators. Nonetheless, for an initial state of the environment which does not correspond to a sufficiently large environment (as we will see) we might not have *any* initial states which can satisfy the condition (4.10) for determining the pointer states of the system and the environment. This means

that pointer states, which are characterized by their ability not to entangle with the states of another subsystem, do not necessarily exist in any arbitrary regime.

4.2 Calculation of the time-evolution operator

One main difficulty in calculating the pointer states of the system and the environment for our model (which is represented by the Hamiltonian of equation (4.2)) is that the calculation of the time-evolution operator for this model by no means is as easy as a similar calculation for the Jaynes-Cummings model or for the aforementioned simplified spin-boson model. We will obtain the time-evolution operator pertaining to our model in this section.

In order to calculate the time-evolution operator in the interaction picture for the Hamiltonian given by equation (4.2), first of all we need to have the Hamiltonian in the interaction picture, which is defined through the following equation

$$\hat{H}_{\rm int}(t) = e^{i\hat{H}_0 t}\hat{H}' e^{-i\hat{H}_0 t}. \tag{4.12}$$

Here $\hat{H}_0 = -\frac{1}{2}\Delta_0\hat{\sigma}_x + \omega\hat{a}^\dagger\hat{a}$ is the sum of the self-Hamiltonians of the system and the environment; and $\hat{H}' = \hat{\sigma}_z \otimes (\mathrm{g}\hat{a}^\dagger + \mathrm{g}^*\hat{a})$ is the Hamiltonian for the interaction between the system and the environment.

So, now we must calculate

$$\hat{H}_{\rm int}(t) = e^{-i\Delta_0\hat{\sigma}_x t/2}\ \hat{\sigma}_z\ e^{i\Delta_0\hat{\sigma}_x t/2} \otimes \{e^{i\omega\hat{a}^\dagger\hat{a}t}\ \mathrm{g}\hat{a}^\dagger\ e^{-i\omega\hat{a}^\dagger\hat{a}t} + e^{i\omega\hat{a}^\dagger\hat{a}t}\ \mathrm{g}^*\hat{a}\ e^{-i\omega\hat{a}^\dagger\hat{a}t}\}. \tag{4.13}$$

Using the Baker-Hausdorff Lemma, i.e.

$$e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} = \hat{B} + \alpha[\hat{A},\hat{B}] + \frac{\alpha^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \dots\ , \tag{4.14}$$

we can verify the following relations

$$\begin{aligned} e^{i\omega\hat{a}^\dagger\hat{a}t}\ \hat{a}\ e^{-i\omega\hat{a}^\dagger\hat{a}t} &= \hat{a}\ e^{-i\omega t}, \qquad \text{and} \\ e^{-i\Delta_0\hat{\sigma}_x t/2}\ \hat{\sigma}_z\ e^{i\Delta_0\hat{\sigma}_x t/2} &= \hat{\sigma}_z\cos(\Delta_0 t) - \hat{\sigma}_y\sin(\Delta_0 t); \end{aligned} \tag{4.15}$$

which would simplify equation (4.13) for $\hat{H}_{\rm int}(t)$ as follows

$$\hat{H}_{\rm int}(t) = \{\hat{\sigma}_z \cos(\Delta_0 t) - \hat{\sigma}_y \sin(\Delta_0 t)\}\{\mathrm{g}\hat{a}^\dagger e^{i\omega t} + \mathrm{g}^* \hat{a} e^{-i\omega t}\}. \tag{4.16}$$

Here, the commutator of $\hat{H}_{\rm int}(t)$ and $\hat{H}_{\rm int}(t' \neq t)$, i.e. $[\hat{H}_{\rm int}(t), \hat{H}_{\rm int}(t' \neq t)]$ with $\hat{H}_{\rm int}(t)$ given by equation (4.16), is not a function of a constant number. This in fact can make the evaluation of the time-evolution operator quite difficult [13].

In parallel with chapter 3 we consider the general form given by equation (4.6) for the evolution operator of the global spin-field system. For such time-evolution operator in the interaction picture, which satisfies the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\hat{U}(t) = \hat{H}_{\rm int}\hat{U}(t), \tag{4.17}$$

we have

$$i\hbar \begin{pmatrix} \dot{\hat{\mathcal{E}}}_1 & \dot{\hat{\mathcal{E}}}_2 \\ \dot{\hat{\mathcal{E}}}_3 & \dot{\hat{\mathcal{E}}}_4 \end{pmatrix} = \hat{H}_{\rm int}(t) \begin{pmatrix} \hat{\mathcal{E}}_1 & \hat{\mathcal{E}}_2 \\ \hat{\mathcal{E}}_3 & \hat{\mathcal{E}}_4 \end{pmatrix} = \{\mathrm{g}\hat{a}^\dagger e^{i\omega t} + \mathrm{g}^* \hat{a} e^{-i\omega t}\} \tag{4.18}$$
$$\times \begin{pmatrix} \hat{\mathcal{E}}_1 \cos(\Delta_0 t) + i\hat{\mathcal{E}}_3 \sin(\Delta_0 t) & \hat{\mathcal{E}}_2 \cos(\Delta_0 t) + i\hat{\mathcal{E}}_4 \sin(\Delta_0 t) \\ -\hat{\mathcal{E}}_3 \cos(\Delta_0 t) - i\hat{\mathcal{E}}_1 \sin(\Delta_0 t) & -\hat{\mathcal{E}}_4 \cos(\Delta_0 t) - i\hat{\mathcal{E}}_2 \sin(\Delta_0 t) \end{pmatrix}.$$

Now, we assume the transition matrix element Δ_0 to be in resonance with the cavity eigenmode ω and we use the rotating-wave approximation (RWA) [12, 14] (just as is assumed in the conventional Jaynes-Cummings model of quantum optics [12]). So, by entering the resonance condition $\Delta_0 = \omega$ and using the rotating wave approximation (i.e. disregarding the higher-frequency terms which contain $e^{\pm i(\omega+\Delta_0)t}$) the above equation will simplify to the following set of four equations

$$\begin{aligned}
i\hbar\dot{\hat{\mathcal{E}}}_1 &= \frac{\mathrm{g}\hat{a}^\dagger}{2}(\hat{\mathcal{E}}_1 - \hat{\mathcal{E}}_3) + \frac{\mathrm{g}^*\hat{a}}{2}(\hat{\mathcal{E}}_1 + \hat{\mathcal{E}}_3), \\
i\hbar\dot{\hat{\mathcal{E}}}_2 &= \frac{\mathrm{g}\hat{a}^\dagger}{2}(\hat{\mathcal{E}}_2 - \hat{\mathcal{E}}_4) + \frac{\mathrm{g}^*\hat{a}}{2}(\hat{\mathcal{E}}_2 + \hat{\mathcal{E}}_4), \\
i\hbar\dot{\hat{\mathcal{E}}}_3 &= \frac{\mathrm{g}\hat{a}^\dagger}{2}(\hat{\mathcal{E}}_1 - \hat{\mathcal{E}}_3) - \frac{\mathrm{g}^*\hat{a}}{2}(\hat{\mathcal{E}}_1 + \hat{\mathcal{E}}_3), \\
i\hbar\dot{\hat{\mathcal{E}}}_4 &= \frac{\mathrm{g}\hat{a}^\dagger}{2}(\hat{\mathcal{E}}_2 - \hat{\mathcal{E}}_4) - \frac{\mathrm{g}^*\hat{a}}{2}(\hat{\mathcal{E}}_2 + \hat{\mathcal{E}}_4).
\end{aligned} \tag{4.19}$$

In order to solve the above set of coupled differential equations, we proceed as follows. First, we take derivative with respect to time of the first equation. By replacing $\dot{\hat{\mathcal{E}}}_1$ and $\dot{\hat{\mathcal{E}}}_3$ from the first and the third equations in the resulting equation we find

$$i\hbar\ddot{\hat{\mathcal{E}}}_1 = \frac{-i|g|^2}{2\hbar}\{(1+2\hat{N})\hat{\mathcal{E}}_1 - \hat{\mathcal{E}}_3\}, \tag{4.20}$$

where $\hat{N} = \hat{a}^\dagger\hat{a}$ is the number operator. Similarly, by doing the same procedure on the third equation for $\dot{\hat{\mathcal{E}}}_3$ we find

$$i\hbar\ddot{\hat{\mathcal{E}}}_3 = \frac{-i|g|^2}{2\hbar}\{(1+2\hat{N})\hat{\mathcal{E}}_3 - \hat{\mathcal{E}}_1\}. \tag{4.21}$$

Next, we define $\hat{\mathcal{E}}_{++}$ and $\hat{\mathcal{E}}_{+-}$ as follows

$$\hat{\mathcal{E}}_{++} = \hat{\mathcal{E}}_1 + \hat{\mathcal{E}}_3 \qquad \text{and} \qquad \hat{\mathcal{E}}_{+-} = \hat{\mathcal{E}}_1 - \hat{\mathcal{E}}_3. \tag{4.22}$$

By adding and subtracting equations (4.20) and (4.21) we find

$$\begin{aligned} \ddot{\hat{\mathcal{E}}}_{++} &= \frac{-|g|^2}{\hbar^2}\hat{N}\hat{\mathcal{E}}_{++} \qquad \text{and} \\ \ddot{\hat{\mathcal{E}}}_{+-} &= \frac{-|g|^2}{\hbar^2}(\hat{N}+1)\ \hat{\mathcal{E}}_{+-}\ . \end{aligned} \tag{4.23}$$

These equations for $\hat{\mathcal{E}}_{++}$ and $\hat{\mathcal{E}}_{+-}$ can simply be solved to find

$$\begin{aligned} \hat{\mathcal{E}}_{++} &= \sin(\frac{|g|\sqrt{\hat{N}}\ t}{\hbar})\hat{A} + \cos(\frac{|g|\sqrt{\hat{N}}\ t}{\hbar})\hat{B} \qquad \text{and} \\ \hat{\mathcal{E}}_{+-} &= \sin(\frac{|g|\sqrt{\hat{N}+1}\ t}{\hbar})\hat{A}' + \cos(\frac{|g|\sqrt{\hat{N}+1}\ t}{\hbar})\hat{B}', \end{aligned} \tag{4.24}$$

where $\hat{A}$, $\hat{A}'$, $\hat{B}$ and $\hat{B}'$ are some time-independent *operators* (rather than constant *numbers*), which will be found from our initial conditions in the following paragraphs. Here we note that since these coefficients generally are some time-independent operators rather than constant numbers, and they do not necessarily commute with the number operator $\hat{N}$, we *must* have them on the right-hand side of the *sin* and *cos* functions (rather than having them on the left-hand side). Only in this way the solutions in equation (4.24) will satisfy equation (4.23).

Now using equations (4.22) and (4.24) we can obtain the operators $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_3$ as follows:

$$\hat{\mathcal{E}}_1 = \frac{1}{2}\{\sin(\frac{|g|\sqrt{\hat{N}}\ t}{\hbar})\hat{A} + \cos(\frac{|g|\sqrt{\hat{N}}\ t}{\hbar})\hat{B} + \sin(\frac{|g|\sqrt{\hat{N}+1}\ t}{\hbar})\hat{A}' + \cos(\frac{|g|\sqrt{\hat{N}+1}\ t}{\hbar})\hat{B}'\} \qquad \text{and} \tag{4.25}$$

$$\hat{\mathcal{E}}_3 = \frac{1}{2}\{\sin(\frac{|g|\sqrt{\hat{N}}\ t}{\hbar})\hat{A} + \cos(\frac{|g|\sqrt{\hat{N}}\ t}{\hbar})\hat{B} - \sin(\frac{|g|\sqrt{\hat{N}+1}\ t}{\hbar})\hat{A}' - \cos(\frac{|g|\sqrt{\hat{N}+1}\ t}{\hbar})\hat{B}'\}. \tag{4.26}$$

In quite the same manner we can calculate $\hat{\mathcal{E}}_2$ and $\hat{\mathcal{E}}_4$ as follows

$$\hat{\mathcal{E}}_2 = \frac{1}{2}\{\sin(\frac{|g|\sqrt{\hat{N}}\ t}{\hbar})\hat{C} + \cos(\frac{|g|\sqrt{\hat{N}}\ t}{\hbar})\hat{D} + \sin(\frac{|g|\sqrt{\hat{N}+1}\ t}{\hbar})\hat{C}' + \cos(\frac{|g|\sqrt{\hat{N}+1}\ t}{\hbar})\hat{D}'\} \qquad \text{and} \tag{4.27}$$

$$\hat{\mathcal{E}}_4 = \frac{1}{2}\{\sin(\frac{|g|\sqrt{\hat{N}}\ t}{\hbar})\hat{C} + \cos(\frac{|g|\sqrt{\hat{N}}\ t}{\hbar})\hat{D} - \sin(\frac{|g|\sqrt{\hat{N}+1}\ t}{\hbar})\hat{C}' - \cos(\frac{|g|\sqrt{\hat{N}+1}\ t}{\hbar})\hat{D}'\}; \tag{4.28}$$

where $\hat{C}$, $\hat{C}'$, $\hat{D}$ and $\hat{D}'$ also, generally are some time-independent *operators*, which will be determined from our original set of equations (4.19) and the initial conditions on $\{\hat{\mathcal{E}}_i\}$'s.

In order to obtain the eight operator coefficients which must be determined in the expressions for $\{\hat{\mathcal{E}}_i\}$'s, first of all we note that the time-evolution operator, given by equation (4.6), must satisfy the initial condition $\hat{U}(t=0) = \hat{I}_{\mathcal{S}} \otimes \hat{I}_{\mathcal{E}}$; where $\hat{I}_{\mathcal{S}} = |a\rangle\langle a| + |b\rangle\langle b|$ represents the identity operator in the Hilbert space of the system and $\hat{I}_{\mathcal{E}}$ is the identity operator in the Hilbert space of the environment. This means that we must have

$$\hat{\mathcal{E}}_1(0) = \hat{\mathcal{E}}_4(0) = \hat{I}_{\mathcal{E}} \qquad \text{and} \qquad \hat{\mathcal{E}}_2(0) = \hat{\mathcal{E}}_3(0) = 0 \tag{4.29}$$

From the above initial conditions and equations (4.25) to (4.28) we easily find four of the coefficients as follows

$$\hat{B} = \hat{B}' = \hat{D} = \hat{I}_{\mathcal{E}} \qquad \text{and} \qquad \hat{D}' = -\hat{I}_{\mathcal{E}}. \tag{4.30}$$

In order to find $\hat{A}$ and $\hat{A}'$ we proceed as follows. First, we use equation (4.19-a) to obtain $\hat{\mathcal{E}}_3$ as follows

$$(g^*\hat{a} - g\hat{a}^\dagger)\hat{\mathcal{E}}_3 = 2i\hbar\dot{\hat{\mathcal{E}}}_1 - (g^*\hat{a} + g\hat{a}^\dagger)\hat{\mathcal{E}}_1\ . \tag{4.31}$$

Replacing $\hat{\mathcal{E}}_1$ and $\dot{\hat{\mathcal{E}}}_1$ from equation (4.25) into the above equation, it reads

$$\begin{aligned}(\mathrm{g}^*\hat{a}-\mathrm{g}\hat{a}^\dagger)\hat{\mathcal{E}}_3 &= i|\mathrm{g}|\sqrt{\hat{N}}\ \{\cos(\frac{|\mathrm{g}|\sqrt{\hat{N}}\ t}{\hbar})\hat{A}-\sin(\frac{|\mathrm{g}|\sqrt{\hat{N}}\ t}{\hbar})\}\\ &+i|\mathrm{g}|\sqrt{\hat{N}+1}\ \{\cos(\frac{|\mathrm{g}|\sqrt{\hat{N}+1}\ t}{\hbar})\hat{A}'-\sin(\frac{|\mathrm{g}|\sqrt{\hat{N}+1}\ t}{\hbar})\}\\ &-(\frac{\mathrm{g}^*\hat{a}+\mathrm{g}\hat{a}^\dagger}{2})\{\sin(\frac{|\mathrm{g}|\sqrt{\hat{N}}\ t}{\hbar})\hat{A}+\cos(\frac{|\mathrm{g}|\sqrt{\hat{N}}\ t}{\hbar})\\ &+\sin(\frac{|\mathrm{g}|\sqrt{\hat{N}+1}\ t}{\hbar})\hat{A}'+\cos(\frac{|\mathrm{g}|\sqrt{\hat{N}+1}\ t}{\hbar})\}.\end{aligned} \tag{4.32}$$

At $t=0$ the above equation reduces to

$$i|\mathrm{g}|\sqrt{\hat{N}}\ \hat{A}+i|\mathrm{g}|\sqrt{\hat{N}+1}\ \hat{A}'-(\mathrm{g}^*\hat{a}+\mathrm{g}\hat{a}^\dagger)=(\mathrm{g}^*\hat{a}-\mathrm{g}\hat{a}^\dagger)\ \hat{\mathcal{E}}_3(t=0)=0. \tag{4.33}$$

Operating this last equation on $|n\rangle$ we have

$$i|\mathrm{g}|\sqrt{\hat{N}}\ \hat{A}\ |n\rangle+i|\mathrm{g}|\sqrt{\hat{N}+1}\ \hat{A}'\ |n\rangle=\mathrm{g}^*\sqrt{n}\ |n-1\rangle+\mathrm{g}\sqrt{n+1}\ |n+1\rangle. \tag{4.34}$$

Next, we use equation (4.19-c) to obtain $\hat{\mathcal{E}}_1$ as follows

$$(\mathrm{g}\hat{a}^\dagger-\mathrm{g}^*\hat{a})\hat{\mathcal{E}}_1=2i\hbar\dot{\hat{\mathcal{E}}}_3+(\mathrm{g}^*\hat{a}+\mathrm{g}\hat{a}^\dagger)\hat{\mathcal{E}}_3\ . \tag{4.35}$$

Replacing $\hat{\mathcal{E}}_3$ and $\dot{\hat{\mathcal{E}}}_3$ from equation (4.26) into the above equation, it reads

$$\begin{aligned}(\mathrm{g}\hat{a}^\dagger-\mathrm{g}^*\hat{a})\hat{\mathcal{E}}_1 &= i|\mathrm{g}|\sqrt{\hat{N}}\ \{\cos(\frac{|\mathrm{g}|\sqrt{\hat{N}}\ t}{\hbar})\hat{A}-\sin(\frac{|\mathrm{g}|\sqrt{\hat{N}}\ t}{\hbar})\}\\ &+i|\mathrm{g}|\sqrt{\hat{N}+1}\ \{-\cos(\frac{|\mathrm{g}|\sqrt{\hat{N}+1}\ t}{\hbar})\hat{A}'+\sin(\frac{|\mathrm{g}|\sqrt{\hat{N}+1}\ t}{\hbar})\}\\ &+(\frac{\mathrm{g}^*\hat{a}+\mathrm{g}\hat{a}^\dagger}{2})\{\sin(\frac{|\mathrm{g}|\sqrt{\hat{N}}\ t}{\hbar})\hat{A}+\cos(\frac{|\mathrm{g}|\sqrt{\hat{N}}\ t}{\hbar})\\ &-\sin(\frac{|\mathrm{g}|\sqrt{\hat{N}+1}\ t}{\hbar})\hat{A}'-\cos(\frac{|\mathrm{g}|\sqrt{\hat{N}+1}\ t}{\hbar})\}.\end{aligned} \tag{4.36}$$

At $t=0$ the above equation reduces to

$$i|\mathrm{g}|\sqrt{\hat{N}}\ \hat{A}-i|\mathrm{g}|\sqrt{\hat{N}+1}\ \hat{A}'=(\mathrm{g}\hat{a}^\dagger-\mathrm{g}^*\hat{a})\ \hat{\mathcal{E}}_1(t=0)=(\mathrm{g}\hat{a}^\dagger-\mathrm{g}^*\hat{a}). \tag{4.37}$$

Operating this last equation on $|n\rangle$ we have

$$i|\mathrm{g}|\sqrt{\hat{N}}\ \hat{A}\ |n\rangle-i|\mathrm{g}|\sqrt{\hat{N}+1}\ \hat{A}'\ |n\rangle=\mathrm{g}\sqrt{n+1}\ |n+1\rangle-\mathrm{g}^*\sqrt{n}\ |n-1\rangle. \tag{4.38}$$

Finally, we use equations (4.34) and (4.38) to obtain the coefficients $\hat{A}$ and $\hat{A}'$. Assuming g to be real and then adding equations (4.34) and (4.38) we find

$$\hat{A}\ |n\rangle = -i\sqrt{\frac{n+1}{\hat{N}}}\ |n+1\rangle = -i|n+1\rangle. \tag{4.39}$$

Comparing the above equation to $\frac{-i}{\sqrt{\hat{N}}}\ \hat{a}^\dagger\ |n\rangle = -i|n+1\rangle$ we find $\hat{A}$ as

$$\hat{A} = \frac{-i}{\sqrt{\hat{N}}}\ \hat{a}^\dagger. \tag{4.40}$$

Similarly, by subtracting equation (4.38) from equation (4.34) to find

$$\hat{A}'\ |n\rangle = -i\sqrt{\frac{n}{\hat{N}+1}}\ |n-1\rangle = -i|n-1\rangle \tag{4.41}$$

and comparing the above equation to $\frac{-i}{\sqrt{\hat{N}+1}}\ \hat{a}\ |n\rangle = -i|n-1\rangle$ we find $\hat{A}'$ as

$$\hat{A}' = \frac{-i}{\sqrt{\hat{N}+1}}\ \hat{a}\ . \tag{4.42}$$

One should pay attention to the order that the operators appear in equations (4.40) and (4.42); as they are not commuting operators.

By doing exactly the same procedure on equations (4.19-b) and (4.19-d) we would find the operator coefficients $\hat{C}$ and $\hat{C}'$ as follows

$$\hat{C} = \frac{i}{\sqrt{\hat{N}}}\ \hat{a}^\dagger = -\hat{A} \qquad \text{and} \qquad \hat{C}' = \frac{-i}{\sqrt{\hat{N}+1}}\ \hat{a} = \hat{A}'\ . \tag{4.43}$$

Now that we found all the operator coefficients, we can replace them back in equations (4.25) to (4.28) and write the $\{\hat{\mathcal{E}}_i\}$ in their final form

$$\hat{\mathcal{E}}_1 = \frac{1}{2}\{-i\sin(\frac{|g|\sqrt{\hat{N}}\ t}{\hbar})\frac{1}{\sqrt{\hat{N}}}\ \hat{a}^\dagger + \cos(\frac{|g|\sqrt{\hat{N}}\ t}{\hbar})$$
$$-i\sin(\frac{|g|\sqrt{\hat{N}+1}\ t}{\hbar})\frac{1}{\sqrt{\hat{N}+1}}\ \hat{a} + \cos(\frac{|g|\sqrt{\hat{N}+1}\ t}{\hbar})\}, \tag{4.44}$$

$$\hat{\mathcal{E}}_3 = \frac{1}{2}\{-i\sin(\frac{|g|\sqrt{\hat{N}}\ t}{\hbar})\frac{1}{\sqrt{\hat{N}}}\ \hat{a}^\dagger + \cos(\frac{|g|\sqrt{\hat{N}}\ t}{\hbar})$$
$$+i\sin(\frac{|g|\sqrt{\hat{N}+1}\ t}{\hbar})\frac{1}{\sqrt{\hat{N}+1}}\ \hat{a} - \cos(\frac{|g|\sqrt{\hat{N}+1}\ t}{\hbar})\}, \tag{4.45}$$

$$\hat{\mathcal{E}}_2 = \frac{1}{2}\{i\sin(\frac{|\mathrm{g}|\sqrt{\hat{N}}\ t}{\hbar})\frac{1}{\sqrt{\hat{N}}}\ \hat{a}^\dagger + \cos(\frac{|\mathrm{g}|\sqrt{\hat{N}}\ t}{\hbar})$$
$$-i\sin(\frac{|\mathrm{g}|\sqrt{\hat{N}+1}\ t}{\hbar})\frac{1}{\sqrt{\hat{N}+1}}\ \hat{a} - \cos(\frac{|\mathrm{g}|\sqrt{\hat{N}+1}\ t}{\hbar})\} \qquad \text{and} \tag{4.46}$$
$$\hat{\mathcal{E}}_4 = \frac{1}{2}\{i\sin(\frac{|\mathrm{g}|\sqrt{\hat{N}}\ t}{\hbar})\frac{1}{\sqrt{\hat{N}}}\ \hat{a}^\dagger + \cos(\frac{|\mathrm{g}|\sqrt{\hat{N}}\ t}{\hbar})$$
$$+i\sin(\frac{|\mathrm{g}|\sqrt{\hat{N}+1}\ t}{\hbar})\frac{1}{\sqrt{\hat{N}+1}}\ \hat{a} + \cos(\frac{|\mathrm{g}|\sqrt{\hat{N}+1}\ t}{\hbar})\}. \tag{4.47}$$

One can verify that the above set of operators satisfy the unitarity of the time-evolution operator $\hat{U}^\dagger\hat{U} = \hat{U}\hat{U}^\dagger = \hat{I}$.

4.3 Calculation of the time-dependent pointer states of the system and the environment

Using the time-evolution operator which we already obtained for our model in the exact-resonance regime, now we want to obtain the corresponding pointer states for the system and the environment. We consider the field to be initially prepared in the coherent state $|\nu\rangle$

$$|\Phi_{\text{field}}(t_0)\rangle = |\nu\rangle = \sum_{n=0}^{\infty} \mathrm{c}_n|n\rangle; \quad \text{with} \quad \mathrm{c}_n = \frac{\mathrm{e}^{-\frac{1}{2}|\nu|^2}\nu^n}{\sqrt{n!}}, \tag{4.48}$$

where $|\nu|^2 = \bar{n}$ is the average number of photons in the coherent state, and $\nu = |\nu|e^{-i\varphi}$. In this section we will show that in the regime that we are considering (i.e. the exact-resonance regime) and for the environment initially prepared in the coherent state, in the limit of a large average number of photons we must have pointer states for the system (the central spin) given by

$$|+(t)\rangle = -i\cos(\frac{\varphi}{2} + \frac{\mathrm{g}t}{4\hbar\sqrt{\bar{n}}})|a\rangle + \sin(\frac{\varphi}{2} + \frac{\mathrm{g}t}{4\hbar\sqrt{\bar{n}}})|b\rangle \qquad \text{and}$$
$$|-(t)\rangle = i\sin(\frac{\varphi}{2} - \frac{\mathrm{g}t}{4\hbar\sqrt{\bar{n}}})|a\rangle + \cos(\frac{\varphi}{2} - \frac{\mathrm{g}t}{4\hbar\sqrt{\bar{n}}})|b\rangle, \tag{4.49}$$

where $|a\rangle$ and $|b\rangle$ are eigenstates of the $\hat{\sigma}_z$ Pauli matrix.

We make the usual assumption that there exists no correlations between the system and the environment at $t = 0$. So, we consider the following initial state for the total composite

system

$$|\psi^{\rm tot}(t_0)\rangle = (\alpha|a\rangle + \beta|b\rangle) \otimes \sum_{n=0}^{\infty} {\rm c}_n|n\rangle \qquad \text{with} \qquad |\alpha|^2 + |\beta|^2 = 1. \tag{4.50}$$

Now we use our condition for obtaining the pointer states of the system and the environment given by equations (4.8) and (4.10):

$$\sum_n c_n\{\alpha\hat{\mathcal{E}}_1 + \beta\hat{\mathcal{E}}_2\}\ |\varphi_n\rangle = \hat{G}(t) \times \sum_n c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle;$$
$$\textit{with } \hat{G}(t) \textit{ be proportional to the unit matrix.} \tag{4.51}$$

For the $\{\hat{\mathcal{E}}_i\}$, given by equations (4.45) through (4.48), we can write

$$\hat{\mathcal{E}}_i|n\rangle = f_{i1}(n)|n+1\rangle + f_{i2}(n)|n\rangle + f_{i3}(n)|n-1\rangle, \tag{4.52}$$

where f_{ij}'s (with $i = 1, 2, 3, 4$ and $j = 1, 2, 3$) are given by

$$\begin{aligned}
f_{11}(n) = f_{31}(n) = -f_{21}(n) = -f_{41}(n) = \frac{-i}{2}\sin(\frac{{\rm g}t\sqrt{n+1}}{\hbar}) \equiv f_1(n),\\
f_{12}(n) = f_{42}(n) = \frac{1}{2}(\cos(\frac{{\rm g}t\sqrt{n}}{\hbar}) + \cos(\frac{{\rm g}t\sqrt{n+1}}{\hbar})) \equiv f_2(n),\\
f_{22}(n) = f_{32}(n) = \frac{1}{2}(\cos(\frac{{\rm g}t\sqrt{n}}{\hbar}) - \cos(\frac{{\rm g}t\sqrt{n+1}}{\hbar})) \equiv f_3(n) \qquad \text{and}\\
f_{13}(n) = f_{23}(n) = -f_{33}(n) = -f_{43}(n) = \frac{-i}{2}\sin(\frac{{\rm g}t\sqrt{n}}{\hbar}) \equiv f_1'(n).
\end{aligned} \tag{4.53}$$

Using equation (4.52) our condition, given by equation (4.51), becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_n\{\alpha(f_{11}(n)|n+1\rangle + f_{12}(n)|n\rangle + f_{13}(n)|n-1\rangle)\\
&\qquad +\beta(f_{21}(n)|n+1\rangle + f_{22}(n)|n\rangle + f_{23}(n)|n-1\rangle)\}\\
&= \hat{G}(t) \times \sum_{n=0}^{\infty} c_n\{\alpha(f_{31}(n)|n+1\rangle + f_{32}(n)|n\rangle + f_{33}(n)|n-1\rangle)\\
&\qquad +\beta(f_{41}(n)|n+1\rangle + f_{42}(n)|n\rangle + f_{43}(n)|n-1\rangle)\};\\
&\qquad \textit{and } \hat{G}(t) \textit{ be proportional to the unit matrix.}
\end{aligned} \tag{4.54}$$

Now, for pointer states $\hat{G}(t)$ must satisfy the condition (4.10) for obtaining the pointer states of the system and the environment, i.e. $\hat{G}(t) = G(t) \times \hat{I}_{\mathcal{E}}$. Therefore, since the number states $\{|n\rangle\}$ are a complete set of basis states for the environment, for the initial pointer states we

can open the summations in equation (4.54) and equalize terms from the two sides of this equation which correspond to the same number state $|n\rangle$ to obtain

$$\frac{c_n(\alpha f_{11}(n)+\beta f_{21}(n))+c_{n+1}(\alpha f_{12}(n+1)+\beta f_{22}(n+1))+c_{n+2}(\alpha f_{13}(n+2)+\beta f_{23}(n+2))}{c_n(\alpha f_{31}(n)+\beta f_{41}(n))+c_{n+1}(\alpha f_{32}(n+1)+\beta f_{42}(n+1))+c_{n+2}(\alpha f_{33}(n+2)+\beta f_{43}(n+2))}; \tag{4.55}$$

$$= G(t) \; for \; all \; n; \; \text{and} \; \hat{G}(t) \; must \; be \; proportional \; to \; the \; unit \; matrix.$$

The above result for $G(t)$, which generally depends on n, would contradict our initial assumption of $\hat{G}(t)$ being proportional to the unit matrix *unless* if we can find certain initial states for the system for which $G(t)$ turns out to become independent of n; since as we discussed, for pointer states, all components of the vector $\mathbf{A}$ ($A_n{}'s$) must be related to their corresponding components from $\mathbf{B}$ ($B_n{}'s$) through the *same* scalar factor G (see equations (4.8) and (4.10)).[3] So now we should seek those particular initial states of the system which can make $G(t)$ independent of the index n of the states of the environment.

Using equation (4.53), equation (4.55) would simplify as

$$G(t)=\frac{(\alpha-\beta)c_n f_1(n)+\alpha c_{n+1}f_2(n+1)+\beta c_{n+1}f_3(n+1)+(\alpha+\beta)c_{n+2}f_1'(n+2)}{(\alpha-\beta)c_n f_1(n)+\alpha c_{n+1}f_3(n+1)+\beta c_{n+1}f_2(n+1)-(\alpha+\beta)c_{n+2}f_1'(n+2)}. \tag{4.56}$$

However, for an initial coherent field (equation (4.48)) we have:
$c_{n+1}=c_n e^{-i\varphi}\sqrt{\frac{\bar{n}}{n+1}}$ and $c_{n+2}=c_n e^{-2i\varphi}\frac{\bar{n}}{\sqrt{(n+1)(n+2)}}$. Moreover, in the limit of a large average number of photons $\bar{n}\to\infty$ we can replace the factors $\sqrt{\frac{\bar{n}}{n+1}}$ and $\sqrt{\frac{\bar{n}}{n+2}}$ by unity[4]; since for $\bar{n}\to\infty$ the Poisson distribution of the coherent field is extremely sharp (with $\bar{n}$ at the center) and hence, for $\bar{n}\to\infty$ and $n\approx\bar{n}$ we have $\sqrt{\frac{\bar{n}}{n+1}}\approx 1$ and $\sqrt{\frac{\bar{n}}{n+2}}\approx 1$, while for

[3] We would like to see if the condition can be satisfied for *any* initial state of the system and the environment with $G(t)$ becoming independent of the index n of the states of the environment. So, if finally we can find any specific set of initial states for the system and the environment which satisfies this condition with $G(t)$ independent of the indices of the environment, then we have reached our goal and our assumption has not been in vain.

[4] This approximation has been used by Gea-Banacloche and other people [2, 3] in the study of the Jaynes-Cummings model of quantum optics. In fact, for the Jaynes-Cummings model it has been shown that an average number of photons only as large as twenty is enough to make this assumption a good approximation [2].

n being far from $\bar{n}$ the c_n coefficient is negligible. So, the corresponding terms (of n being far from $\bar{n}$) do not have any contribution in the summations of equation (4.54). As a result, equation (4.56) for $G(t)$ can be further simplified to

$$G(t) = \frac{(\alpha-\beta)f_1(n) + \alpha e^{-i\varphi} f_2(n+1) + \beta e^{-i\varphi} f_3(n+1) + (\alpha+\beta)e^{-2i\varphi} f_1'(n+2)}{(\alpha-\beta)f_1(n) + \alpha e^{-i\varphi} f_3(n+1) + \beta e^{-i\varphi} f_2(n+1) - (\alpha+\beta)e^{-2i\varphi} f_1'(n+2)}. \quad (4.57)$$

Replacing the f_i functions from equation (4.53), the above equation reads

$$\begin{aligned} G(t) = \{&(\alpha+\beta)e^{-i\varphi}\cos(\frac{gt\sqrt{n+1}}{\hbar}) - i(\alpha-\beta)\sin(\frac{gt\sqrt{n+1}}{\hbar}) \\ &+(\alpha-\beta)e^{-i\varphi}\cos(\frac{gt\sqrt{n+2}}{\hbar}) - i(\alpha+\beta)e^{-2i\varphi}\sin(\frac{gt\sqrt{n+2}}{\hbar})\} \\ \div\quad &\{(\alpha+\beta)e^{-i\varphi}\cos(\frac{gt\sqrt{n+1}}{\hbar}) - i(\alpha-\beta)\sin(\frac{gt\sqrt{n+1}}{\hbar}) \\ &+(\beta-\alpha)e^{-i\varphi}\cos(\frac{gt\sqrt{n+2}}{\hbar}) + i(\alpha+\beta)e^{-2i\varphi}\sin(\frac{gt\sqrt{n+2}}{\hbar})\}. \end{aligned} \quad (4.58)$$

In order to obtain the pointer states of the system, we should look for the probable initial states of the system (represented by the coefficients α and β in equation (4.50)) which can make the expression (in the above equation) for $G(t)$ *independent* of the index n of the states of the environment. On the other hand, by looking at equation (4.58) we realize that if $\alpha-\beta = \pm(\alpha+\beta)e^{-i\varphi}$ the expression for $G(t)$ will be considerably simplified. In what follows we show that for $\alpha-\beta = \pm(\alpha+\beta)e^{-i\varphi}$, which is equivalent to the initial conditions for the state of the system given by

$$\begin{aligned} &\alpha_+ = -i\cos(\varphi/2) \quad \text{and} \quad \beta_+ = \sin(\varphi/2) \quad \text{(for the plus sign)} \quad \text{or} \\ &\alpha_- = i\sin(\varphi/2) \quad \text{and} \quad \beta_- = \cos(\varphi/2) \quad \text{(for the minus sign)} \end{aligned} \quad (4.59)$$

(since $|\alpha|^2 + |\beta|^2 = 1$), $G(t)$ of equation (4.58) will be independent of the states of the environment; provided we have a large average number of photons in the field $\bar{n} \to \infty$. Therefore, the initial conditions of equation (4.59) correspond to the initial states of the system which do not entangle with the states of the environment. After that, we will obtain the time evolution of these initial pointer states and followed by that we obtain the corresponding pointer states of the environment.

For $\alpha-\beta = \pm(\alpha+\beta)e^{-i\varphi}$ the expression in equation (4.58) for $G(t)$ simplifies to

$$G(t) = \frac{e^{\mp\frac{igt\sqrt{n+1}}{\hbar}} \pm e^{-i\varphi}\ e^{\mp\frac{igt\sqrt{n+2}}{\hbar}}}{e^{\mp\frac{igt\sqrt{n+1}}{\hbar}} \mp e^{-i\varphi}\ e^{\mp\frac{igt\sqrt{n+2}}{\hbar}}}. \quad (4.60)$$

The above expression can be written as

$$G(t) = \frac{\{e^{-i\varphi/2}\ e^{\mp\frac{igt}{2\hbar}(\sqrt{n+1}+\sqrt{n+2})}\}\{e^{i\varphi/2}\ e^{\pm\frac{igt}{2\hbar}(\sqrt{n+2}-\sqrt{n+1})} \pm e^{-i\varphi/2}\ e^{\pm\frac{igt}{2\hbar}(\sqrt{n+1}-\sqrt{n+2})}\}}{\{e^{-i\varphi/2}\ e^{\mp\frac{igt}{2\hbar}(\sqrt{n+1}+\sqrt{n+2})}\}\{e^{i\varphi/2}\ e^{\pm\frac{igt}{2\hbar}(\sqrt{n+2}-\sqrt{n+1})} \mp e^{-i\varphi/2}\ e^{\pm\frac{igt}{2\hbar}(\sqrt{n+1}-\sqrt{n+2})}\}}$$
$$= \frac{\{e^{i\varphi/2}\ e^{\pm\frac{igt}{2\hbar}(\sqrt{n+2}-\sqrt{n+1})} \pm e^{-i\varphi/2}\ e^{\pm\frac{igt}{2\hbar}(\sqrt{n+1}-\sqrt{n+2})}\}}{\{e^{i\varphi/2}\ e^{\pm\frac{igt}{2\hbar}(\sqrt{n+2}-\sqrt{n+1})} \mp e^{-i\varphi/2}\ e^{\pm\frac{igt}{2\hbar}(\sqrt{n+1}-\sqrt{n+2})}\}} \quad (4.61)$$

From the Taylor series expansion of $\sqrt{n+2}-\sqrt{n+1}$ about $\bar{n}$ (the average number of photons in the environment), given by

$$\sqrt{n+2}-\sqrt{n+1} = \frac{1}{2\sqrt{\bar{n}}} - \frac{(n+1-\bar{n})}{4\bar{n}^{3/2}} + \ldots\ , \quad (4.62)$$

we notice that in the limit of a very large average number of photons we can replace $\sqrt{n+2}-\sqrt{n+1}$ by $\frac{1}{2\sqrt{\bar{n}}}$ [2]; since for very large $\bar{n}$ all terms containing the index n, which appear after the first term, are negligible; and *the series is convergent*. So, in the classical limit of $\bar{n}\to\infty$ we can rewrite equation (4.61) for $G(t)$ as

$$G(t) = \frac{\{e^{i\varphi/2}\ e^{\pm\frac{igt}{4\hbar\sqrt{\bar{n}}}} \pm e^{-i\varphi/2}\ e^{\mp\frac{igt}{4\hbar\sqrt{\bar{n}}}}\}}{\{e^{i\varphi/2}\ e^{\pm\frac{igt}{4\hbar\sqrt{\bar{n}}}} \mp e^{-i\varphi/2}\ e^{\mp\frac{igt}{4\hbar\sqrt{\bar{n}}}}\}}$$
$$= -i\cot(\frac{\varphi}{2}+\frac{gt}{4\hbar\sqrt{\bar{n}}}) \qquad \text{for the first sign}$$
$$\text{or} \qquad = i\tan(\frac{\varphi}{2}-\frac{gt}{4\hbar\sqrt{\bar{n}}}) \qquad \text{for the second sign;} \quad (4.63)$$

which clearly is independent of the index n of the states of the environment.

In appendix B by calculating the degree of entanglement between the states of the system and the environment for the pointer states which will be obtained from the above result, we will show that the this result is valid over a length of time which is proportional to $\bar{n}$, the average number of photons in the field. In fact, as we will see, the limitation which is imposed on the generality of our result is mainly caused from our focusing on that part of the Hilbert space of the environment which corresponds to nonnegligible c_n coefficients.

The result of equation (4.63)) simply means that for the initial states of the system given by

$$|+(t_0)\rangle = -i\cos(\varphi/2)|a\rangle + \sin(\varphi/2)|b\rangle \quad \text{and} \quad |-(t_0)\rangle = i\sin(\varphi/2)|a\rangle + \cos(\varphi/2)|b\rangle \quad (4.64)$$

(see equation (4.59)) the states of the system and the environment will not entangle with each other. Moreover, using equation (4.11) which gives us the general time evolution of the pointer states of the system; and $G(t)$ of equation (4.63) (which is independent of the index n of the states of the environment) we can find the time evolution of the pointer states of the system as follows

$$|+(t)\rangle = \mathcal{N}_+ \{-i\cot(\frac{\varphi}{2}+\frac{\mathrm{g}t}{4\hbar\sqrt{\bar{n}}})|a\rangle+|b\rangle\} \quad \text{and} \quad |-(t)\rangle = \mathcal{N}_- \{i\tan(\frac{\varphi}{2}-\frac{\mathrm{g}t}{4\hbar\sqrt{\bar{n}}})|a\rangle+|b\rangle\}; \tag{4.65}$$

where $\mathcal{N}_+$ and $\mathcal{N}_-$ are the normalization factors for the $|+(t)\rangle$ and $|-(t)\rangle$ states respectively. It is easy to verify that

$$\mathcal{N}_+ = \sin(\theta_+(t)) \quad \text{and} \quad \mathcal{N}_- = \cos(\theta_-(t)) \quad \text{with} \quad \theta_\pm(t) = \frac{\varphi}{2} \pm \frac{\mathrm{g}t}{4\hbar\sqrt{\bar{n}}} \tag{4.66}$$

So, we can rewrite equation (4.64) as

$$\begin{aligned} |+(t)\rangle &= -i\cos(\frac{\varphi}{2}+\frac{\mathrm{g}t}{4\hbar\sqrt{\bar{n}}})|a\rangle + \sin(\frac{\varphi}{2}+\frac{\mathrm{g}t}{4\hbar\sqrt{\bar{n}}})|b\rangle \qquad \text{and} \\ |-(t)\rangle &= i\sin(\frac{\varphi}{2}-\frac{\mathrm{g}t}{4\hbar\sqrt{\bar{n}}})|a\rangle + \cos(\frac{\varphi}{2}-\frac{\mathrm{g}t}{4\hbar\sqrt{\bar{n}}})|b\rangle, \end{aligned} \tag{4.67}$$

which is the same as equation (4.49); Q.E.D.

Next, we obtain the corresponding pointer states of the environment. Using equations (4.11) and (4.52) we have

$$\begin{aligned} |\Phi_\pm(t)\rangle &= \mathcal{N}_\pm^{-1}\sum_{n=0}^{\infty} c_n\{\alpha_\pm\hat{\mathcal{E}}_3+\beta_\pm\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle \\ &= \mathcal{N}_\pm^{-1}\sum_{n=0}^{\infty} c_n\{\alpha_\pm\ [f_{31}(n)|n+1\rangle + f_{32}(n)|n\rangle + f_{33}(n)|n-1\rangle] \\ &\qquad +\beta_\pm\ [f_{41}(n)|n+1\rangle + f_{42}(n)|n\rangle + f_{43}(n)|n-1\rangle]\}, \end{aligned} \tag{4.68}$$

where in the above equation $\alpha_\pm$ and $\beta_\pm$ are those of the initial pointer states of the system given by equation (4.59). Let us first obtain $|\Phi_+(t)\rangle$; i.e. the pointer state of the environment corresponding to the $|+(t)\rangle$ state. Replacing the f_{ij} functions from equation (4.53), α_+ and β_+ from equation (4.59) and $\mathcal{N}_+$ from equation (4.66) we have

$$|\Phi_+(t)\rangle = \sum_{n=0}^{\infty}\frac{c_n}{2\sin(\theta_+(t))}\{-\sin(\frac{\mathrm{g}t\sqrt{n+1}}{\hbar})\ e^{-i\varphi/2}\ |n+1\rangle$$

$$+[-i\cos(\frac{gt\sqrt{n}}{\hbar})\ e^{i\varphi/2}+i\cos(\frac{gt\sqrt{n+1}}{\hbar})\ e^{-i\varphi/2}\ ]\ |n\rangle$$
$$+\sin(\frac{gt\sqrt{n}}{\hbar})\ e^{i\varphi/2}\ |n-1\rangle\}. \tag{4.69}$$

Using $c_{n\pm1}\approx c_n e^{\mp i\varphi}$ for the coherent field and in the limit of $\bar{n}\to\infty$, the above relation can be written as

$$|\Phi_+(t)\rangle=\sum_{n=0}^{\infty}\frac{c_n}{2i\sin(\theta_+(t))}\{-i\sin(\frac{gt\sqrt{n}}{\hbar})\ e^{i\varphi/2}$$
$$+[\cos(\frac{gt\sqrt{n}}{\hbar})\ e^{i\varphi/2}-\cos(\frac{gt\sqrt{n+1}}{\hbar})\ e^{-i\varphi/2}\ ]$$
$$+i\sin(\frac{gt\sqrt{n+1}}{\hbar})\ e^{-i\varphi/2}\ \}\ |n\rangle$$
$$=\sum_{n=0}^{\infty}\frac{c_n}{2i\sin(\theta_+(t))}\ \{e^{i(\varphi/2-\frac{gt\sqrt{n}}{\hbar})}-e^{-i(\varphi/2+\frac{gt\sqrt{n+1}}{\hbar})}\}\ |n\rangle. \tag{4.70}$$

Inserting

$2i\sin(\theta_+(t))=2i\sin(\frac{\varphi}{2}+\frac{gt}{4\hbar\sqrt{\bar{n}}})\equiv e^{i[\varphi/2+\frac{gt}{2\hbar}(\sqrt{n+1}-\sqrt{n})]}-e^{-i[\varphi/2+\frac{gt}{2\hbar}(\sqrt{n+1}-\sqrt{n})]}$

into the above equation, after simplifying we find the following final form for $|\Phi_+(t)\rangle$

$$|\Phi_+(t)\rangle=\sum_{n=0}^{\infty}c_n\ e^{\frac{-igt}{2\hbar}(\sqrt{n+1}+\sqrt{n})}\ |n\rangle. \tag{4.71}$$

Following exactly the same procedure one can also find $|\Phi_-(t)\rangle$ as follows

$$|\Phi_-(t)\rangle=\sum_{n=0}^{\infty}c_n\ e^{\frac{igt}{2\hbar}(\sqrt{n+1}+\sqrt{n})}\ |n\rangle. \tag{4.72}$$

For an initial coherent field and in the limit of a large average number of photons, which we are considering, the factor $\sqrt{n+1}+\sqrt{n}$ in equations (4.71) and (4.72) can also be replaced by

$$\sqrt{n+1}+\sqrt{n}\approx\frac{1}{2\sqrt{\bar{n}}}+2\sqrt{\bar{n}}\ . \tag{4.73}$$

However, in what follows we only use the expressions in the form of equations (4.71) and (4.72) for $|\Phi_\pm(t)\rangle$.

We should also mention that the pointer states of the system at $t=t_0$ are orthonormal (see equation (4.64)) and hence, they form a complete basis set for the state of the system. Therefore, the evolution of *any* initial state $\psi_{\mathcal{S}}(t_0)=\alpha'\ |+(t_0)\rangle+\beta'\ |-(t_0)\rangle$ with an initial

coherent field $|\nu\rangle$ can be expressed as a linear combination of the evolution of $|+(t_0)\rangle|\nu\rangle$ and $|-(t_0)\rangle|\nu\rangle$

$$(\alpha' \,|+(t_0)\rangle + \beta' \,|-(t_0)\rangle)\,|\nu\rangle \rightarrow \alpha' \,|+(t)\rangle\,|\Phi_+(t)\rangle + \beta' \,|-(t)\rangle\,|\Phi_-(t)\rangle, \tag{4.74}$$

where in the above equation the evolution of the pointer states of the system $|\pm(t)\rangle$ is given by equation (4.67) and the evolution of the pointer states of the environment $|\Phi_\pm(t)\rangle$ is given by equations (4.71) and (4.72).

4.4 State preparation at specific times

One interesting feature of the pointer states of the system, given by equation (4.67), is that at specific times they coincide with each other. In fact, by looking at equation (4.65) we notice that at those times for which $\tan(\frac{\varphi}{2} - \frac{gt}{4\hbar\sqrt{\bar{n}}}) = -\cot(\frac{\varphi}{2} + \frac{gt}{4\hbar\sqrt{\bar{n}}})$ the $|\pm(t)\rangle$ states are equal to each other. One can easily verify that at $t_1 = (4n+1)\pi\hbar\sqrt{\bar{n}}/g$ (with $n = 0, 1, 2, ...$) both of the $|\pm(t)\rangle$ states will be in the common state given by

$$|\pm(t_1)\rangle = i\sin(\frac{\varphi}{2} - \frac{\pi}{4})|a\rangle + \cos(\frac{\varphi}{2} - \frac{\pi}{4})|b\rangle; \tag{4.75}$$

while at $t_2 = (4n-1)\pi\hbar\sqrt{\bar{n}}/g$ (with $n = 1, 2, ...$) the $|\pm(t)\rangle$ states will be in the common state given by

$$|\pm(t_2)\rangle = i\sin(\frac{\varphi}{2} + \frac{\pi}{4})|a\rangle + \cos(\frac{\varphi}{2} + \frac{\pi}{4})|b\rangle. \tag{4.76}$$

The state preparation at these specific times basically means that whatever is the initial state of the system, at these specific times the states of the system and the environment are not entangled to each other and the system can be represented by a well-defined state of its own (see equation (4.74)). Moreover, as we see, these specific states clearly depend on the phase φ of the initial state of the coherent field. The same kind of phenomenon was also discovered in the simpler Jaynes-Cummings model of quantum optics by Gea-Banacloche [2] in 1991.

4.5 Consequences regarding the decoherence of the central spin

In this section we will use the pointer states of the system and the environment (which we already obtained in section 3) in order to study the decoherence of the central system in our model. However, prior to that we will use the time-evolution operator, already obtained in section 2, to obtain the general time evolution of the total composite system (i.e. $|\psi_{\rm tot}(t)\rangle$) in our model and by assuming an environment initially prepared in the coherent state. After that, we will calculate the offdiagonal element of the reduced density matrix of the system (i.e. $\rho_{12}^{(\mathcal{S})}(t)$) by tracing over the environmental degrees of freedom. Then, we will also obtain the coherences of the reduced density matrix of the system in another way by using the pointer states of the system and the environment obtained in section 3. In fact, this way we will obtain a *closed* form for the coherences of the reduced density matrix of the system; unlike the result of the first method which contains summations running on an infinite number of states of the environment. We will compare the two results by plotting $\rho_{12}^{(\mathcal{S})}(t)$ which we obtain from the two different methods. As we will see, the two results are in very good agreement with each other especially for short times; just as long as we are considering an average number of photons as large as twenty or more. Finally, we will discuss some very interesting features which can be observed in our study of the decoherence of the central system.

4.5.1 General expressions for the evolution of the state of the total composite system and the reduced density matrix of the system

Using equations (4.7) and (4.52) to obtain $|\psi_{\rm tot}(t)\rangle$, we can write

$$\begin{aligned}|\psi_{\rm tot}(t)\rangle = \sum_{n=0}^{\infty} c_n\{&(\alpha f_{11}(n)+\beta f_{21}(n))|a,n+1\rangle + (\alpha f_{12}(n)+\beta f_{22}(n))|a,n\rangle \\ &+(\alpha f_{13}(n)+\beta f_{23}(n))|a,n-1\rangle + (\alpha f_{31}(n)+\beta f_{41}(n))|b,n+1\rangle \\ &+(\alpha f_{32}(n)+\beta f_{42}(n))|b,n\rangle + (\alpha f_{33}(n)+\beta f_{43}(n))|b,n-1\rangle\},\end{aligned} \tag{4.77}$$

where f_{ij}'s are given by equation (4.53). Replacing these functions from equation (4.53) the above relation becomes

$$\begin{aligned}|\psi_{\rm tot}(t)\rangle = \sum_{n=0}^{\infty} c_n\{&\frac{-i}{2}(\alpha-\beta)\sin(\frac{{\rm g}t\sqrt{n+1}}{\hbar})|a,n+1\rangle + [(\frac{\alpha+\beta}{2})\cos(\frac{{\rm g}t\sqrt{n}}{\hbar})\\ &+(\frac{\alpha-\beta}{2})\cos(\frac{{\rm g}t\sqrt{n+1}}{\hbar})]|a,n\rangle - \frac{i}{2}(\alpha+\beta)\sin(\frac{{\rm g}t\sqrt{n}}{\hbar})|a,n-1\rangle\\ &-\frac{i}{2}(\alpha-\beta)\sin(\frac{{\rm g}t\sqrt{n+1}}{\hbar})|b,n+1\rangle + [(\frac{\alpha+\beta}{2})\cos(\frac{{\rm g}t\sqrt{n}}{\hbar}) -\\ &(\frac{\alpha-\beta}{2})\cos(\frac{{\rm g}t\sqrt{n+1}}{\hbar})]|b,n\rangle + \frac{i}{2}(\alpha+\beta)\sin(\frac{{\rm g}t\sqrt{n}}{\hbar})|b,n-1\rangle\}.\end{aligned} \tag{4.78}$$

The above equation can be simplified into the following final from

$$\begin{aligned}|\psi_{\rm tot}(t)\rangle &= \sum_{n=0}^{\infty}(c_{a,n}(t)|a,n\rangle + c_{b,n}(t)|b,n\rangle), \qquad \text{with}\\ c_{a,n}(t) &= \frac{-i}{2}(\alpha-\beta)c_{n-1}\sin(\frac{{\rm g}t\sqrt{n}}{\hbar}) + c_n[(\frac{\alpha+\beta}{2})\cos(\frac{{\rm g}t\sqrt{n}}{\hbar})\\ &+(\frac{\alpha-\beta}{2})\cos(\frac{{\rm g}t\sqrt{n+1}}{\hbar})] - \frac{i}{2}(\alpha+\beta)c_{n+1}\sin(\frac{{\rm g}t\sqrt{n+1}}{\hbar})\\ c_{b,n}(t) &= \frac{-i}{2}(\alpha-\beta)c_{n-1}\sin(\frac{{\rm g}t\sqrt{n}}{\hbar}) + c_n[(\frac{\alpha+\beta}{2})\cos(\frac{{\rm g}t\sqrt{n}}{\hbar})\\ &-(\frac{\alpha-\beta}{2})\cos(\frac{{\rm g}t\sqrt{n+1}}{\hbar})] + \frac{i}{2}(\alpha+\beta)c_{n+1}\sin(\frac{{\rm g}t\sqrt{n+1}}{\hbar}).\end{aligned} \tag{4.79}$$

For the state of the total composite system in our model, which is given by equation (4.79), we can do the trace operation over the basis states of the environment (i.e. the $\{|n\rangle\}$ which make a complete basis for the state of the environment) to obtain the reduced density matrix of the system $\mathcal{S}$

$$\begin{aligned}\rho^{\mathcal{S}}(t) &= \sum_{n=0}^{\infty}\langle n|\rho^{\rm tot}(t)|n\rangle = \sum_{n=0}^{\infty}\langle n|\psi_{\rm tot}(t)\rangle\langle\psi_{\rm tot}(t)|n\rangle\\ &= \sum_{n=0}^{\infty}\ (\ |c_{a,n}(t)|^2\ |a\rangle\langle a| + |c_{b,n}(t)|^2\ |b\rangle\langle b| + c_{a,n}(t)c^*_{b,n}(t)\ |a\rangle\langle b| + c.c.\ ).\end{aligned} \tag{4.80}$$

So, in the basis of the eigenstates of σ_z (i.e. in the basis of the $|a\rangle$ and $|b\rangle$ states) the elements of the reduced density matrix of the system must be given by

$$\rho^{\mathcal{S}}_{12}(t) = \sum_{n=0}^{\infty} c_{a,n}(t)\ c^*_{b,n}(t) = c_{a,0}\ c^*_{b,0} + c_{a,1}\ c^*_{b,1} + c_{a,2}\ c^*_{b,2} + ... \qquad \text{and}$$

$$\rho^{\mathcal{S}}_{11}(t) = 1 - \rho^{\mathcal{S}}_{22}(t) = \sum_{n=0}^{\infty}|c_{a,n}(t)|^2. \tag{4.81}$$

Replacing $c_{a,n}(t)$ and $c_{b,n}(t)$ from equation (4.79) in the above equation, after some algebra one finds

$$\rho_{12}^{\mathcal{S}}(t) = \gamma f_0(t) + \delta f_1(t) + \lambda f_2(t) + \lambda^* f_3(t) + (\lambda - \lambda^*) f_4(t) \qquad \text{and}$$
$$\rho_{11}^{\mathcal{S}}(t) = \gamma g_0(t) + \delta g_1(t) + \lambda g_2(t) + \lambda^* g_2^*(t); \tag{4.82}$$

where in the above equations the coefficients γ, δ and λ are given by

$$\gamma = \frac{1}{4}|\alpha - \beta|^2 \qquad \text{and} \qquad \delta = \frac{1}{4}|\alpha + \beta|^2 \qquad \text{and}$$
$$\lambda = \frac{1}{4}(|\alpha|^2 - |\beta|^2 + \alpha\beta^* - \beta\alpha^*). \tag{4.83}$$

Also the $f_i(t)$ and $g_i(t)$ functions are given by

$$f_0(t) = \sum_{n=0}^{\infty}(\ |c_{n-1}|^2 \sin^2(\frac{gt\sqrt{n}}{\hbar}) + i[c_{n-1}c_n^* + c_{n-1}^* c_n]$$
$$\times \sin(\frac{gt\sqrt{n}}{\hbar}) \cos(\frac{gt\sqrt{n+1}}{\hbar}) - |c_n|^2 \cos^2(\frac{gt\sqrt{n+1}}{\hbar}) \ ),$$
$$f_1(t) = \sum_{n=0}^{\infty}(\ |c_n|^2 \cos^2(\frac{gt\sqrt{n}}{\hbar}) - i[c_n c_{n+1}^* + c_n^* c_{n+1}]$$
$$\times \cos(\frac{gt\sqrt{n}}{\hbar}) \sin(\frac{gt\sqrt{n+1}}{\hbar}) - |c_{n+1}|^2 \sin^2(\frac{gt\sqrt{n+1}}{\hbar}) \ ),$$
$$f_2(t) = \sum_{n=0}^{\infty}(\ -ic_{n-1}c_n^* \sin(\frac{gt\sqrt{n}}{\hbar}) \cos(\frac{gt\sqrt{n}}{\hbar}) - c_{n-1}c_{n+1}^*$$
$$\times \sin(\frac{gt\sqrt{n}}{\hbar}) \sin(\frac{gt\sqrt{n+1}}{\hbar}) - ic_n c_{n+1}^* \sin(\frac{gt\sqrt{n+1}}{\hbar}) \cos(\frac{gt\sqrt{n+1}}{\hbar})),$$
$$f_3(t) = \sum_{n=0}^{\infty}(\ ic_n c_{n-1}^* \sin(\frac{gt\sqrt{n}}{\hbar}) \cos(\frac{gt\sqrt{n}}{\hbar}) + c_{n+1}c_{n-1}^*$$
$$\times \sin(\frac{gt\sqrt{n}}{\hbar}) \sin(\frac{gt\sqrt{n+1}}{\hbar}) + ic_{n+1}c_n^* \sin(\frac{gt\sqrt{n+1}}{\hbar}) \cos(\frac{gt\sqrt{n+1}}{\hbar})),$$
$$f_4(t) = \sum_{n=0}^{\infty}(\ |c_n|^2 \cos(\frac{gt\sqrt{n}}{\hbar}) \cos(\frac{gt\sqrt{n+1}}{\hbar})), \tag{4.84}$$
$$g_0(t) = \sum_{n=0}^{\infty}(\ |c_{n-1}|^2 \sin^2(\frac{gt\sqrt{n}}{\hbar}) + i[c_n c_{n-1}^*$$
$$-c_n^* c_{n-1}] \sin(\frac{gt\sqrt{n}}{\hbar}) \cos(\frac{gt\sqrt{n+1}}{\hbar}) + |c_n|^2 \cos^2(\frac{gt\sqrt{n+1}}{\hbar}) \ ),$$
$$g_1(t) = \sum_{n=0}^{\infty}(\ |c_n|^2 \cos^2(\frac{gt\sqrt{n}}{\hbar}) + i[c_n c_{n+1}^* - c_n^* c_{n+1}] \times \cos(\frac{gt\sqrt{n}}{\hbar}) \sin(\frac{gt\sqrt{n+1}}{\hbar})$$

$$+ |c_{n+1}|^2 \sin^2(\frac{gt\sqrt{n+1}}{\hbar})) \quad \text{and}$$

$$g_2(t) = \sum_{n=0}^{\infty}(-ic_{n-1}c_n^* \sin(\frac{gt\sqrt{n}}{\hbar}) \cos(\frac{gt\sqrt{n}}{\hbar}) + c_{n-1}c_{n+1}^*$$
$$\times \sin(\frac{gt\sqrt{n}}{\hbar}) \sin(\frac{gt\sqrt{n+1}}{\hbar}) + |c_n|^2 \cos(\frac{gt\sqrt{n}}{\hbar}) \cos(\frac{gt\sqrt{n+1}}{\hbar})$$
$$+ic_nc_{n+1}^* \sin(\frac{gt\sqrt{n+1}}{\hbar}) \cos(\frac{gt\sqrt{n+1}}{\hbar}))$$

Now, let us obtain the coherences of the reduced density matrix of the system in another way by using the pointer states of the system and the environment which we obtained in section 3. As we will see, in this way not only can we obtain a closed form for the coherences of the reduced density matrix of the system, but also we can acquire a better understanding regarding the characteristics of decoherence of the central system in our model.

As we saw, for a two-state system $\mathcal{S}$ in contact with an environment $\mathcal{E}$ after determination of the pointer states of the system and the environment, the state of the total composite system generally can be represented by equation (4.74). i.e. $|\psi_{\rm tot}(t)\rangle = \alpha' \mid +(t)\rangle \mid \Phi_+(t)\rangle + \beta' \mid -(t)\rangle \mid \Phi_-(t)\rangle$. For $|\psi_{\rm tot}(t)\rangle$ given by equation (4.74) the reduced density matrix of the system $\hat{\rho}_{\mathcal{S}}(t)$ can be calculated by tracing over the environmental degrees of freedom to obtain

$$\hat{\rho}_{\mathcal{S}}(t) = |\alpha'|^2 \times \mid +(t)\rangle\langle +(t)\mid + |\beta'|^2 \times \mid -(t)\rangle\langle -(t)\mid + \alpha'\beta'^*$$
$$\times \mid +(t)\rangle\langle -(t)\mid \times \langle \Phi_-(t)|\Phi_+(t)\rangle + \beta'\alpha'^* \times \mid -(t)\rangle\langle +(t)\mid \times \langle \Phi_+(t)|\Phi_-(t)\rangle. \quad (4.85)$$

So, in an arbitrary basis $|a\rangle$ and $|b\rangle$ for the state of the two-level system generally we have

$$\rho_{11}^{\mathcal{S}}(t) = 1 - \rho_{22}^{\mathcal{S}}(t) = |\alpha'|^2 \times \langle a| +(t)\rangle\langle +(t)|a\rangle + |\beta'|^2 \times \langle a| -(t)\rangle\langle -(t)|a\rangle + \alpha'\beta'^*$$
$$\times \langle a| +(t)\rangle\langle -(t)|a\rangle \times \langle \Phi_-(t)|\Phi_+(t)\rangle + \beta'\alpha'^* \times \langle a| -(t)\rangle\langle +(t)|a\rangle \times \langle \Phi_+(t)|\Phi_-(t)\rangle \quad (4.86)$$
$$\text{and} \qquad \rho_{12}^{\mathcal{S}}(t) = |\alpha'|^2 \times \langle a| +(t)\rangle\langle +(t)|b\rangle + |\beta'|^2 \times \langle a| -(t)\rangle\langle -(t)|b\rangle + \alpha'\beta'^*$$
$$\times \langle a| +(t)\rangle\langle -(t)|b\rangle \times \langle \Phi_-(t)|\Phi_+(t)\rangle + \beta'\alpha'^* \times \langle a| -(t)\rangle\langle +(t)|b\rangle \times \langle \Phi_+(t)|\Phi_-(t)\rangle.$$

For the system initially prepared in one of the pointer states $\mid \pm(t_0)\rangle$ (i.e. for $\alpha' = 0$ or $\beta' = 0$) the above expressions can be simplified. For example, for the system initially

prepared in the $|+(t_0)\rangle$ state (i.e. for $\alpha' = 1$ and $\beta' = 0$) generally we have

$$\rho_{11}^{\mathcal{S}}(t) = |\langle a|+(t)\rangle|^2 \qquad \text{and} \qquad \rho_{12}^{\mathcal{S}}(t) = \langle a|+(t)\rangle.\langle b|+(t)\rangle^* \; ; \tag{4.87}$$

and for the system initially prepared in the $|-(t_0)\rangle$ state (i.e. for $\beta = 1$ and $\alpha = 0$) generally we have

$$\rho_{11}^{\mathcal{S}}(t) = |\langle a|-(t)\rangle|^2 \qquad \text{and} \qquad \rho_{12}^{\mathcal{S}}(t) = \langle a|-(t)\rangle.\langle b|-(t)\rangle^*. \tag{4.88}$$

For our spin-boson model in the exact-resonance regime we can use the pointer states of the system, presented in equation (4.67), to calculate the above expressions for the elements of the reduced density matrix of the system. So, using equation (4.67) we have

$$\begin{aligned} &\rho_{11}^{\mathcal{S}}(t) = \cos^2(\frac{\varphi}{2} + \frac{\mathrm{g}t}{4\hbar\sqrt{\bar{n}}}) \text{ and } \rho_{12}^{\mathcal{S}}(t) = -\frac{i}{2}\sin(\varphi + \frac{\mathrm{g}t}{2\hbar\sqrt{\bar{n}}}) \quad \text{for} \quad |\psi_{\mathcal{S}}(t_0)\rangle = |+(t_0)\rangle \\ &\rho_{11}^{\mathcal{S}}(t) = \sin^2(\frac{\varphi}{2} - \frac{\mathrm{g}t}{4\hbar\sqrt{\bar{n}}}) \text{ and } \rho_{12}^{\mathcal{S}}(t) = \frac{i}{2}\sin(\varphi - \frac{\mathrm{g}t}{2\hbar\sqrt{\bar{n}}}) \quad \text{for} \quad |\psi_{\mathcal{S}}(t_0)\rangle = |-(t_0)\rangle. \end{aligned} \tag{4.89}$$

The above expressions for $\rho_{12}^{\mathcal{S}}(t)$ basically mean that for the system initially prepared in one of the pointer states, the offdiagonal element of the reduced density matrix of the system should be a sinusoidal function with frequency $\frac{\mathrm{g}}{2\hbar\sqrt{\bar{n}}}$. Also, it must have successive zeros which are apart from each other by $\Delta t = 2h\sqrt{\bar{n}}/\mathrm{g}$.

An examination of $\rho_{12}^{\mathcal{S}}(t)$ by plotting its more exact expression, given by equation (4.82), shows good agreement with the above result *only* as long as we have a very large average number of photons in the field. However, for a smaller average number of photons although we observe the oscillating behavior with the same frequency of $\frac{\mathrm{g}}{2\hbar\sqrt{\bar{n}}}$ when the system initially is prepared in one of the pointer states, we can clearly observe a decaying envelope which would destroy the offdiagonal element of the reduced density matrix at large times (causing decoherence of the state of the system *even* when the system initially is prepared in one of the pointer states). Also, we observe that this decay is specifically more significant for a smaller average number of photons. Hence, we can guess that the difference between the prediction of equation (4.89) and what we expect from the more exact expression of equation (4.82), for the case that we have a smaller average number of photons, must be due to the fact that in calculating the pointer states of the system we assumed having a large average number of photons in the environment (so that we have a sharp distribution

for the coherent state of the field and can assume $\sqrt{n+1}-\sqrt{n} \approx \frac{1}{2\sqrt{\bar{n}}}$). In other words, we guess that *the decoherence of the state of the central system when we start from one of the pointer states of the system must be due to having a limited number of photons in the field.*

In what follows our first goal is to make the appropriate corrections in equation (4.89) so that we can theoretically justify the decoherence of the state of the system when starting from one of the pointer states. Followed by that, we make corrections to the other elements of equation (4.86) (i.e. the expressions for $\langle a| + (t)\rangle\langle -(t)|a\rangle$, $\langle a| - (t)\rangle\langle +(t)|a\rangle$, $\langle a| + (t)\rangle\langle -(t)|b\rangle$ and $\langle a| - (t)\rangle\langle +(t)|b\rangle$). Finally, we will use equation (4.86), together with the corrections which we make for having a limited average number of photons, in order to obtain a closed form for $\rho_{12}^{\mathcal{S}}(t)$. As we will see, after these corrections our closed form for the offdiagonal element of the reduced density matrix of the system will be in good agreement with the more exact but cumbersome expression of equation (4.82) which we obtained in this section for $\rho_{12}^{\mathcal{S}}(t)$.

4.5.2 First order corrections due to having a finite average number of photons in the environment

In section 3, while obtaining our pointer states of the system and the environment, we assumed having a large average number of photons in the environment; so that we can substitute $\sqrt{n+1}-\sqrt{n}$ by $\frac{1}{2\sqrt{\bar{n}}}$ in our expressions. Now we consider the next order in the Taylor expansion of $\sqrt{n+1}-\sqrt{n}$ about $\bar{n}$, i.e. in

$$\sum_{n=0}^{\infty} |c_n|^2 \ (\sqrt{n+1}-\sqrt{n}) \approx \sum_{n=0}^{\infty} |c_n|^2 \ (\frac{1}{2\sqrt{\bar{n}}} - \frac{(n-\bar{n})}{4\bar{n}^{3/2}} + ...), \tag{4.90}$$

and make the appropriate corrections (due to having a finite average number of photons) in equation (4.89). In fact, by looking at equations (4.61) and (4.90) we notice that *only* at the limit of a large average number of photons, where $\sum_{n=0}^{\infty} |c_n|^2 \ (\sqrt{n+1}-\sqrt{n}) \approx \frac{1}{2\sqrt{\bar{n}}}$ is a good approximation and there is no need to consider the next terms in our expansion for $\sqrt{n+1}-\sqrt{n}$, the function $G(t)$ (defined by equation (4.8)) will be independent of the states of the environment and pointer states can be realized for the system and the environment, which do not entangle with each other.

We make corrections on $\rho_{12}^{\mathcal{S}}(t)$ of equation (4.89) by using the following substitution in our expressions

$$e^{\mp it'/2\sqrt{\bar{n}}} \rightarrow \sum_{n=0}^{\infty} |c_n|^2 \; e^{\mp it'(\sqrt{n+1}-\sqrt{n})} \qquad \text{where} \qquad t' = \frac{gt}{\hbar}. \tag{4.91}$$

Using equations (4.48) and (4.90) we have

$$\begin{aligned} \sum_{n=0}^{\infty} |c_n|^2 \; e^{-it'(\sqrt{n+1}-\sqrt{n})} &\approx \sum_{n=0}^{\infty} \frac{e^{-\bar{n}} \; \bar{n}^n}{n!} \; e^{-it'(\frac{1}{2\sqrt{\bar{n}}} - \frac{(n-\bar{n})}{4\bar{n}^{3/2}})} \\ = e^{-\bar{n}} \; e^{-\frac{3it'}{4\sqrt{\bar{n}}}} \times \sum_{n=0}^{\infty} \frac{(\bar{n} \; e^{\frac{it'}{4\bar{n}^{3/2}}})^n}{n!} &= e^{-\bar{n}} e^{-\frac{3it'}{4\sqrt{\bar{n}}}} \times \exp(\bar{n} \; e^{\frac{it'}{4\bar{n}^{3/2}}}) \\ = \exp(\bar{n} \; [e^{\frac{it'}{4\bar{n}^{3/2}}} - 1]) \; e^{-\frac{3it'}{4\sqrt{\bar{n}}}} &= \exp(\bar{n} \; e^{\frac{it'}{8\bar{n}^{3/2}}} [e^{\frac{it'}{8\bar{n}^{3/2}}} - e^{\frac{-it'}{8\bar{n}^{3/2}}}]) \; e^{-\frac{3it'}{4\sqrt{\bar{n}}}}; \end{aligned} \tag{4.92}$$

which simplifies as

$$\sum_{n=0}^{\infty} |c_n|^2 \; e^{-it'(\sqrt{n+1}-\sqrt{n})} \approx \exp(2i \; \bar{n} \; e^{\frac{it'}{8\bar{n}^{3/2}}} \sin(\frac{t'}{8\bar{n}^{3/2}})) \times e^{-\frac{3it'}{4\sqrt{\bar{n}}}}. \tag{4.93}$$

For an average number of photons large enough and times short enough for which $\frac{t'}{\bar{n}^{3/2}} \ll 1$ we can approximate $\sin(\frac{t'}{8\bar{n}^{3/2}})$ by $\frac{t'}{8\bar{n}^{3/2}}$ and $e^{\frac{it'}{8\bar{n}^{3/2}}}$ by $1 + \frac{it'}{8\bar{n}^{3/2}}$ in the above equation. In other words, provided t goes to infinity slowly enough to have $\frac{t'}{\bar{n}^{3/2}} \ll 1$ we can write

$$\sum_{n=0}^{\infty} |c_n|^2 \; e^{-it'(\sqrt{n+1}-\sqrt{n})} \approx \exp(2i \; \bar{n} \; (1 + \frac{it'}{8\bar{n}3/2}) \times (\frac{t'}{8\bar{n}3/2})) \times e^{-\frac{3it'}{4\sqrt{\bar{n}}}} \qquad \text{or}$$

$$\sum_{n=0}^{\infty} |c_n|^2 \; e^{-it'(\sqrt{n+1}-\sqrt{n})} \approx e^{-\frac{it'}{2\sqrt{\bar{n}}}} \; e^{-t'^2/32\bar{n}^2}. \tag{4.94}$$

So, to make the appropriate corrections in our expressions we should use the following substitution

$$e^{-it'/2\sqrt{\bar{n}}} \rightarrow e^{-it'/2\sqrt{\bar{n}}} \; e^{-t'^2/32\bar{n}^2}. \tag{4.95}$$

Making the above substitution in the expressions of equation (4.89) for $\rho_{12}^{\mathcal{S}}(t)$ we find

$$\begin{aligned} \rho_{12}^{\mathcal{S}}(t) = -\frac{i}{2} \sin(\varphi + \frac{t'}{2\sqrt{\bar{n}}}) &\rightarrow -\frac{i}{2} \; [\frac{e^{i(\frac{t'}{2\sqrt{\bar{n}}}+\varphi)} e^{-t'^2/32\bar{n}^2} - e^{-i(\frac{t'}{2\sqrt{\bar{n}}}+\varphi)} e^{-t'^2/32\bar{n}^2}}{2i}] \\ &= -\frac{i}{2} \sin(\varphi + \frac{t'}{2\sqrt{\bar{n}}}) \; e^{-t'^2/32\bar{n}^2} \quad \text{for} \quad |\psi_{\mathcal{S}}(t_0)\rangle = |+(t_0)\rangle \quad \text{and} \end{aligned}$$

$$\rho_{12}^{\mathcal{S}}(t) = \frac{i}{2}\sin(\varphi - \frac{t'}{2\sqrt{\bar{n}}}) \rightarrow \frac{i}{2}\,[\frac{e^{i(\varphi-\frac{t'}{2\sqrt{\bar{n}}})}e^{-t'^2/32\bar{n}^2} - e^{-i(\varphi-\frac{t'}{2\sqrt{\bar{n}}})}e^{-t'^2/32\bar{n}^2}}{2i}] \qquad (4.96)$$
$$= \frac{i}{2}\sin(\varphi - \frac{t'}{2\sqrt{\bar{n}}})\; e^{-t'^2/32\bar{n}^2} \quad \text{for} \quad |\psi_{\mathcal{S}}(t_0)\rangle = |-(t_0)\rangle.$$

One interesting aspect of the evolution of coherences given by the above equation is that for $\varphi = 0$ and $\varphi = \pi$ no matter whether the system initially is prepared in the $|+(t_0)\rangle$ state or the $|-(t_0)\rangle$ state, the evolution of $\rho_{12}^{\mathcal{S}}(t)$ is given by $\rho_{12}^{\mathcal{S}}(t) = \mp\frac{i}{2}\sin(\frac{t'}{2\sqrt{\bar{n}}})\; e^{-t'^2/32\bar{n}^2}$ (with the minus sign for $\varphi = 0$ and the plus sign for $\varphi = \pi$). Also, if $\varphi = \frac{\pi}{2}$ or $\varphi = \frac{3\pi}{2}$, for both initial pointer states the evolution of $|\rho_{12}^{\mathcal{S}}(t)|$ is given by $|\rho_{12}^{\mathcal{S}}(t)| = \frac{1}{2}\cos(\frac{t'}{2\sqrt{\bar{n}}})\; e^{-t'^2/32\bar{n}^2}$. In general, for $\varphi = n\pi/2$ the evolution of $|\rho_{12}^{\mathcal{S}}(t)|$ will be the same for both initial pointer states. However, as we will see in the coming paragraphs, this does not mean that for $\varphi = n\pi/2$ the evolution of $|\rho_{12}^{\mathcal{S}}(t)|$ becomes independent of the initial state of the system.

In Figure 1 we have used equation (4.96) to plot the evolution of $|\rho_{12}^{\mathcal{S}}(t)|$ for the case that the system initially is prepared in the $|+(t_0)\rangle$ state. We also used the more exact expression, given by equation (4.82), to plot the same function. As we see, the correction that we made on our initial expression for $\rho_{12}^{\mathcal{S}}(t)$ (due to having a finite average number of photons in the environment), for the case that the system initially is prepared in one of the pointer states, nicely describes the decaying envelope in the evolution of coherences of the reduced system $\mathcal{S}$, which is given by the factor $e^{-t'^2/32\bar{n}^2}$.

Coming back to equation (4.86) for the general evolution of coherences of the reduced system, we also need to calculate the expressions for $\langle a|+(t)\rangle\langle -(t)|b\rangle$ and $\langle a|-(t)\rangle\langle +(t)|b\rangle$, as well as the overlap between the pointer states of the environment $\langle\Phi_-(t)|\Phi_+(t)\rangle$, for our generalized spin boson model. Using equation (4.67) for the evolution of pointer states of the system we can evaluate $\langle a|+(t)\rangle\langle -(t)|b\rangle$ and $\langle a|-(t)\rangle\langle +(t)|b\rangle$ as follows

$$\langle a|+(t)\rangle\langle -(t)|b\rangle = -i\cos(\frac{\varphi}{2} + \frac{t'}{4\sqrt{\bar{n}}})\cos(\frac{\varphi}{2} - \frac{t'}{4\sqrt{\bar{n}}}) \qquad \text{and}$$
$$\langle a|-(t)\rangle\langle +(t)|b\rangle = i\sin(\frac{\varphi}{2} + \frac{t'}{4\sqrt{\bar{n}}})\sin(\frac{\varphi}{2} - \frac{t'}{4\sqrt{\bar{n}}}). \qquad (4.97)$$

However, here also we should make the appropriate correction due to having a finite average number of photons in the environment. Such correction can be made again by using equation

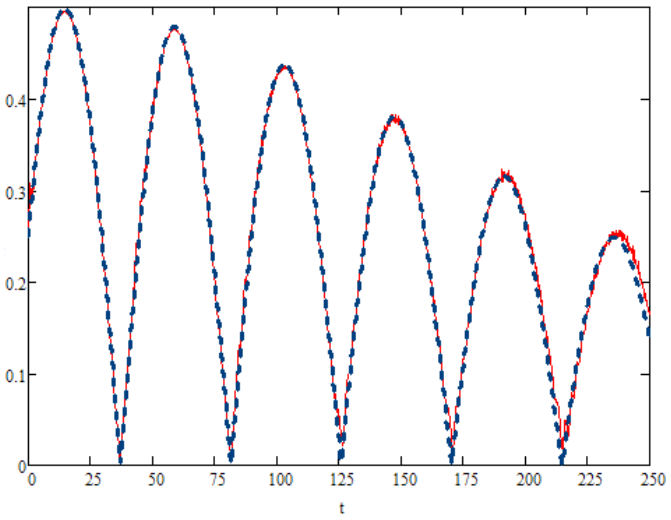

Figure 4.1: Evolution of $|\rho_{12}^{\mathcal{S}}(t')|$ (where $t' = gt/\hbar$) for the case that the system initially is prepared in the $|+(t_0)\rangle$ state. Here we chose $\varphi = \pi/6$ and $\bar{n} = 50$. The curve represented by dashed lines is plotted by using the approximate expression which we obtained from our pointer states, given by equation (4.96). The other curve with solid lines is obtained from the more exact expression of equation (4.82).

(4.95) to obtain

$$\langle a|+(t)\rangle\langle -(t)|b\rangle = -i\cos(\frac{\varphi}{2}+\frac{t'}{4\sqrt{\bar{n}}})\cos(\frac{\varphi}{2}-\frac{t'}{4\sqrt{\bar{n}}})\ e^{-t'^2/64\bar{n}^2} \qquad \text{and}$$
$$\langle a|-(t)\rangle\langle +(t)|b\rangle = i\sin(\frac{\varphi}{2}+\frac{t'}{4\sqrt{\bar{n}}})\sin(\frac{\varphi}{2}-\frac{t'}{4\sqrt{\bar{n}}})\ e^{-t'^2/64\bar{n}^2}. \qquad (4.98)$$

Finally, we use the expressions for $|\Phi_\pm(t)\rangle$, given by equations (4.71) and (4.72), in order to calculate the overlap between the pointer states of the environment $\langle\Phi_-(t)|\Phi_+(t)\rangle$. We have

$$|\Phi_\pm(t)\rangle = \sum_{n=0}^{\infty} c_n\ e^{\mp\frac{igt}{2\hbar}(\sqrt{n+1}+\sqrt{n})}\ |n\rangle = \sum_{n=0}^{\infty}\frac{e^{-\bar{n}/2}\ \bar{n}^{n/2}\ e^{-in\varphi}}{\sqrt{n!}}\ e^{\mp\frac{igt}{2\hbar}(\sqrt{n+1}+\sqrt{n})}\ |n\rangle. \qquad (4.99)$$

As we discussed, for a coherent field with a large average number of photons we can use

$$\sqrt{n} \approx \sqrt{\bar{n}} + \frac{(n-\bar{n})}{2\sqrt{\bar{n}}} - \frac{(n-\bar{n})^2}{8\bar{n}^{3/2}}; \qquad (4.100)$$

So, using the above relation and $t' = gt/\hbar$, equation (4.99) becomes

$$|\Phi_{\pm}(t)\rangle \approx \sum_{n=0}^{\infty} \frac{e^{-\bar{n}/2}\ \bar{n}^{n/2}\ e^{-in\varphi}}{\sqrt{n!}}\ e^{\mp\frac{it'}{2}(\sqrt{\bar{n}}+\frac{(n+1-\bar{n})}{2\sqrt{\bar{n}}}-\frac{(n+1-\bar{n})^2}{8\bar{n}^{3/2}}+\sqrt{\bar{n}}+\frac{(n-\bar{n})}{2\sqrt{\bar{n}}}-\frac{(n-\bar{n})^2}{8\bar{n}^{3/2}})}\ |n\rangle$$

$$= e^{-\bar{n}/2}\exp(\mp\frac{it'}{2}[\frac{3}{4}\sqrt{\bar{n}}+\frac{3}{4\sqrt{\bar{n}}}-\frac{1}{8\bar{n}^{3/2}}]) \times \sum_{n=0}^{\infty}\frac{\bar{n}^{n/2}\ e^{-in\varphi}}{\sqrt{n!}} \quad (4.101)$$

$$\times \exp(\mp\frac{it'}{2}\{(\frac{n}{\sqrt{\bar{n}}}) \times [\frac{3}{2}-\frac{n}{4\bar{n}}]-\frac{n}{4\bar{n}^{3/2}}\})\ |n\rangle.$$

For the coherent field and in the approximation that we are using, as we discussed, $\sum_n |c_n|^2\ (n/\bar{n}) \approx \sum_n |c_n|^2 = 1$. In other words, for large $\bar{n}$ effectively we would have $n \equiv \bar{n}$. So, in the above equation we replace the expression $[\frac{3}{2}-\frac{n}{4\bar{n}}]$ by $\frac{5}{4}$. Therefore, using equation (4.48) we can simplify equation (4.101) to

$$|\Phi_{\pm}(t)\rangle \approx \exp(\mp\frac{it'\sqrt{\bar{n}}}{2}[\frac{3}{4}+\frac{3}{4\bar{n}}-\frac{1}{8\bar{n}^2}]) \times |\ \nu\ \exp(\mp\frac{it'}{2\sqrt{\bar{n}}}[\frac{5}{4}-\frac{1}{4\bar{n}}])\ \rangle. \quad (4.102)$$

So now

$$\langle\Phi_{-}(t)|\Phi_{+}(t)\rangle \approx \exp(-it'\sqrt{\bar{n}}\ [\frac{3}{4}+\frac{3}{4\bar{n}}-\frac{1}{8\bar{n}^2}]) \times \langle\ \nu\ e^{\frac{it'}{2\sqrt{\bar{n}}}[\frac{5}{4}-\frac{1}{4\bar{n}}]}|\ \nu\ e^{-\frac{it'}{2\sqrt{\bar{n}}}[\frac{5}{4}-\frac{1}{4\bar{n}}]}\ \rangle. \quad (4.103)$$

Using the following formula for the scalar product of coherent states from quantum optics [12]

$$\langle\nu\ '|\nu\rangle = \exp[-(|\nu\ '|^2+|\nu|^2)/2+\nu\ '^*\ \nu]. \quad (4.104)$$

equation (4.103) becomes

$$\langle\Phi_{-}(t)|\Phi_{+}(t)\rangle \approx \exp(-it'\sqrt{\bar{n}}\ [\frac{3}{4}+\frac{3}{4\bar{n}}-\frac{1}{8\bar{n}^2}]) \times \exp(\ \bar{n}\ \{e^{\frac{-it'}{\sqrt{\bar{n}}}(\frac{5}{4}-\frac{1}{4\bar{n}})}-1\}). \quad (4.105)$$

So

$$|\langle\Phi_{-}(t)|\Phi_{+}(t)\rangle|^2 \approx \exp(-2\bar{n}\ \{\ 1-\ \cos([\frac{t'}{\sqrt{\bar{n}}}][\frac{5}{4}-\frac{1}{4\bar{n}}])\ \})$$

$$= \exp(-4\bar{n}\sin^2([\frac{t'}{2\sqrt{\bar{n}}}][\frac{5}{4}-\frac{1}{4\bar{n}}])). \quad (4.106)$$

For an average number of photons large enough and times short enough for which $\frac{t'}{\sqrt{\bar{n}}} \ll 1$ the above expression for the overlap between the pointer states of the environment reduces to

$$|\langle\Phi_{-}(t)|\Phi_{+}(t)\rangle|^2 \approx \exp(-t'^2\ [\frac{5}{4}-\frac{1}{4\bar{n}}]^2) \approx e^{-\frac{25}{16}t'^2}. \quad (4.107)$$

Now, using equations (4.86), (4.96), (4.98) and (4.105) we can calculate $\rho_{12}^{\mathcal{S}}(t)$ for our model and for $t' \ll \bar{n}^{3/2}$; when the total system initially is prepared in the state $|\psi^{\rm tot}(t_0)\rangle = (\alpha' \mid +(t_0)\rangle + \beta' \mid -(t_0)\rangle) \otimes |\nu\rangle$; with $\mid \pm(t_0)\rangle$ representing the pointer states of the system at t_0

$$\begin{aligned}\rho_{12}^{\mathcal{S}}(t) = |\alpha'|^2\{\frac{-i}{2}\sin(\varphi + \frac{t'}{2\sqrt{\bar{n}}})\ e^{-t'^2/32\bar{n}^2}\} + |\beta'|^2\{\frac{i}{2}\sin(\varphi - \frac{t'}{2\sqrt{\bar{n}}})\ e^{-t'^2/32\bar{n}^2}\} \\ +\alpha'\beta'^*\{-i\cos(\frac{\varphi}{2} + \frac{t'}{4\sqrt{\bar{n}}})\cos(\frac{\varphi}{2} - \frac{t'}{4\sqrt{\bar{n}}})\ e^{-t'^2/64\bar{n}^2}\} \times \\ \exp(-it'\sqrt{\bar{n}}\ [\frac{3}{4} + \frac{3}{4\bar{n}} - \frac{1}{8\bar{n}^2}]) \times \exp(\ \bar{n}\ \{e^{\frac{-it'}{\sqrt{\bar{n}}}(\frac{5}{4} - \frac{1}{4\bar{n}})} - 1\}) \\ +\beta'\alpha'^*\{i\sin(\frac{\varphi}{2} + \frac{t'}{4\sqrt{\bar{n}}})\sin(\frac{\varphi}{2} - \frac{t'}{4\sqrt{\bar{n}}})\ e^{-t'^2/64\bar{n}^2}\} \times \\ \exp(it'\sqrt{\bar{n}}\ [\frac{3}{4} + \frac{3}{4\bar{n}} - \frac{1}{8\bar{n}^2}]) \times \exp(\ \bar{n}\ \{e^{\frac{it'}{\sqrt{\bar{n}}}(\frac{5}{4} - \frac{1}{4\bar{n}})} - 1\}).\end{aligned} \quad (4.108)$$

In Figure 2 we used equation (4.108) to plot the short time evolution of $|\rho_{12}^{\mathcal{S}}(t)|$ for a case that the system initially is *not* prepared in one of its pointer states. We also used the more exact expression, given by equation (4.82), to plot the same function. As we see from this figure, equation (4.108) serves as a good approximation in closed form for the more exact relation, as long as we are not considering long times[5].

As we can see from equations (4.107) and (4.108), at sufficiently short times $t' \ll \sqrt{\bar{n}}$ the decay of the first two terms is characterized by the decaying factor $e^{-t'^2/32\bar{n}^2}$, while the decay of the other two terms is characterized by the much faster-decaying term due to the overlap between the pointer states of the environment $\langle\Phi_-(t)|\Phi_+(t)\rangle$ which is proportional to the factor $e^{-\frac{25}{32}t'^2}$. This fact clearly shows why indeed we should generally expect a much slower decoherence of the state of the system when the system initially is prepared in one of its pointer states, compared to the case that the system initially is *not* in any of its pointer states. Also, equation (4.108) shows that for $\varphi = n\pi/2$ and if the system initially is *not* prepared in one of its pointer states, the evolution of coherences of the central system would not be independent of the initial state of the system (just unlike the case that the

[5]For longer times, the approximate equation (4.108) shows internal oscillations in the evolution of $|\rho_{12}^{\mathcal{S}}(t)|$ which are misplaced compared to those of the plot which we obtain from the more exact expression of equation (4.82). However, the envelopes still do coincide with each other with great precession.

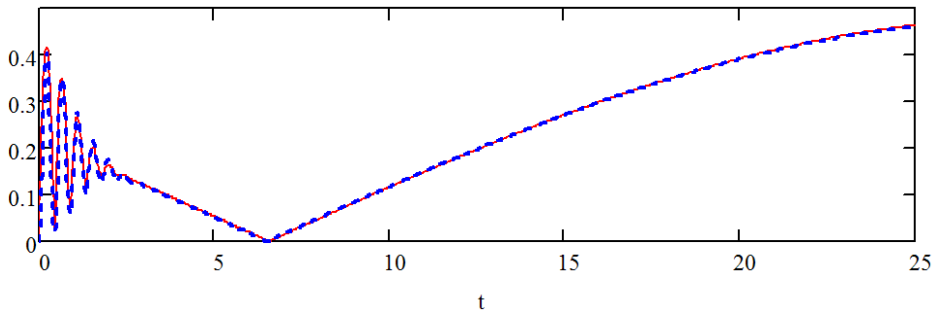

Figure 4.2: Short time evolution of $|\rho_{12}^{\mathcal{S}}(t')|$ for the case that the system initially is prepared in the $|b\rangle$ state. Here we chose $\varphi = \pi/6$ and $\bar{n} = 50$. The curve represented by dashed lines is plotted by using the approximate expression which we obtained from our pointer states, given by equation (4.108). The other curve with solid lines is obtained from the more exact expression of equation (4.82).

system initially is prepared in one of its pointer states). However, since the last two terms of equation (4.108) vanish much faster than the first two terms, at larger times and for $\varphi = n\pi/2$ we expect the evolution of coherences of the central system to be independent of the initial state of the system.

In this section we calculated the offdiagonal element of the reduced density matrix of the system in the basis of eigenstates of the $\hat{\sigma}_z$ operator. However, we can also calculate $\rho_{12}^{\mathcal{S}}(t)$ in the basis of the $|\pm(t_0)\rangle$ states, which are given by $|+(t_0)\rangle = -i\cos(\frac{\varphi}{2})|a\rangle + \sin(\frac{\varphi}{2})|b\rangle$ and $|-(t_0)\rangle = i\sin(\frac{\varphi}{2})|a\rangle + \cos(\frac{\varphi}{2})|b\rangle$. So, let us also study the decoherence of the state of the central system in the basis of the $|\pm(t_0)\rangle$ states and for short times $t' \ll \sqrt{\bar{n}}$.

From equation (4.67) it is clear that for $t' \ll \sqrt{\bar{n}}$ the pointer states of the system almost are time-independent and they can be approximated by $|\pm(t_0)\rangle$. So, in this limit the evolution of an arbitrary initial state of the central system $(\alpha'\,|+(t_0)\rangle + \beta'\,|-(t_0)\rangle)$ in contact with an initial coherent field from the environment approximately is given by $|\psi^{\rm tot}(t)\rangle = \alpha'\,|+(t_0)\rangle|\Phi_+(t)\rangle + \beta'\,|-(t_0)\rangle|\Phi_-(t)\rangle$. Therefore, we would have

$$\hat{\rho}_{\mathcal{S}}(t) = |\alpha'|^2\,|+(t_0)\rangle\langle+(t_0)| + |\beta'|^2\,|-(t_0)\rangle\langle-(t_0)|$$
$$+\alpha'\beta'^*\,|+(t_0)\rangle\langle-(t_0)| \times \langle\Phi_-(t)|\Phi_+(t)\rangle + \beta'\alpha'^*\,|-(t_0)\rangle\langle+(t_0)| \times \langle\Phi_+(t)|\Phi_-(t)\rangle \quad (4.109)$$

However, for this short range of times the evolution of the pointer states of the environment can be approximated by equation (4.102). So, in the $|\pm(t_0)\rangle$ basis and for $t' \ll \sqrt{\bar{n}}$ we must have

$$\rho_{12}^{\mathcal{S}}(t) = \alpha'\beta'^{*}\langle\Phi_{-}(t)|\Phi_{+}(t)\rangle$$
$$\approx \alpha'\beta'^{*}\exp(-it'\sqrt{\bar{n}}\,[\frac{3}{4}+\frac{3}{4\bar{n}}-\frac{1}{8\bar{n}^2}])\times\exp(\,\bar{n}\,\{e^{\frac{-it'}{\sqrt{\bar{n}}}(\frac{5}{4}-\frac{1}{4\bar{n}})}-1\}) \qquad (4.110)$$

Finally, using equation (4.107) we find that for $t' \ll \sqrt{\bar{n}}$

$$|\rho_{12}^{\mathcal{S}}(t)|^2 \approx |\alpha'\beta'|^2\exp(-t'^2\,[\frac{5}{4}-\frac{1}{4\bar{n}}]^2) \approx |\alpha'\beta'|^2\,e^{-\frac{25}{16}t'^2}. \qquad (4.111)$$

Hence, in the basis of the $|\pm(t_0)\rangle$ states the short-time decoherence of the state of the central system is characterized by the fast-decaying factor $e^{-\frac{25}{32}t'^2}$ when the system initially is *not* prepared in one of its pointer states; while in this basis the pointer states of the system almost do not decohere within short times.

4.6 Summary and conclusions

Considering a single-mode quantized field in exact resonance with the tunneling matrix element of the system, we obtained the time-evolution operator (equations (4.44) to (4.47)) for our model. Using this time-evolution operator then we calculated the pointer states of the system and the environment, which are characterized by their ability not to entangle with each other; for the case that the environment initially is prepared in a coherent state with a large average number of photons. Most importantly, we observed that for our spin-boson model represented by the Hamiltonian of equation (4.2) the pointer states of the system turn out to become time-dependent, as opposed to the pointer states of the simplified spin-boson model (for which $[\hat{H}_{\mathcal{S}}, \hat{H}_{\rm int}] = 0$). As we already mentioned, the simplified model has often been used in the context of quantum information and quantum computation to gain some insights regarding the decoherence of a single qubit [7, 9, 8]. However, in most of the practical situations different noncommutable perturbations may exist in the total Hamiltonian of a realistic system-environment model which would result

in having time-dependent pointer states for the system [1]. Indeed, the authors believe that the fact that the pointer states of a system generally are time-dependent and may evolve with time has not been seriously acknowledged in the context of quantum computation and quantum information. In specific, in the context of quantum error correction [15, 16] it is often assumed that the premeasurement by the environment does not change the initial pointer states of the system. In other words, quantum "nondemolition" premeasurement by the environment is often assumed [15, 16]; as is also assumed in Von Neumann scheme of measurement [17, 6]. Also, in the context of Decoherence-Free-Subspaces (DFS) theory the models which often are studied either contain self-Hamiltonian for the system which commutes with the interaction between the system and the environment, or it is assumed that we are in the *quantum measurement limit* [6] or in the *quantum limit of decoherence* [7, 18, 19, 20]. However, all of these assumptions are in fact a big simplification of the problem; since, as we discussed in chapter 3, they completely exclude the possibility of having pointer states for the system which may depend on time [1].

Another interesting point in obtaining the pointer states of the system and the environment for our model was the realization of the fact that *only* in the limit of a large average number of photons can we have a set of (time-dependent) pointer states for the system. In other words, unless we have a sufficiently large average number of photons which can make a sharp distribution function for the state of the electromagnetic field, there is always some degree of entanglement between the states of the system and the environment (see equations (4.61) and (4.90)) and the pointer states of measurement cannot be realized at all.

We also showed that at $t = (2n+1)\pi\hbar\sqrt{\bar{n}}/\mathrm{g}$ (with $n = 0, 1, 2, ...$) the $|\pm(t)\rangle$ pointer states of the system coincide with each other and hence, whatever is the initial state of the system, at these specific times the states of the system and the environment are not entangled with each other and the system can be represented by a well-defined state of its

[6] In the *quantum measurement limit* the interaction between the system and the environment is so strong as to dominate the evolution of the system $\hat{H} \approx \hat{H}_{\mathrm{int}}$. Also in the *quantum limit of decoherence* the Hamiltonian for the system almost dominates the interaction between the system and the environment as well as the self-Hamiltonian of the environment $\hat{H} \approx \hat{H}_{\mathcal{S}}$.

own. Using the time-evolution operator obtained in section 2, we also obtained a closed form (figure 2) for the offdiagonal element of the reduced density matrix of the system and studied the decoherence of the central system in our model. We showed that for the case that the system initially is prepared in one of its pointer states, the offdiagonal element of the reduced density matrix of the system will be a *sinusoidal function* with a slow decaying envelope which is characterized by a decay time proportional to $\bar{n}$ (through a decoherence factor calculated as $e^{-(gt/\hbar)^2/32\bar{n}^2}$); while for the case that the system initially is not prepared in one of its initial pointer states, it will experience a fast decoherence within a time of order $\hbar/g$. The "decayo-sinusoidal" evolution of coherences (figure 1) which we observe in our model and for the case that the system initially is prepared in one of its pointer states is a new form of decoherence which cannot be observed in the somewhat similar Jaynes-Cummings model of quantum optics [2].

It will be interesting to generalize this study to the case that the environment is not merely represented by a single-mode bosonic field; and consider some classes of spectral densities for the environment. Also, for the spin-boson model represented by the Hamiltonian of equation (4.2) at least in principle one should be able to obtain the pointer states of the system and the environment in some nonresonance regimes and for the single-mode quantized field.

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Chapter 5

Time-dependent pointer states of the quantized atom-field model in a nonresonance regime and consequences regarding the decoherence of the central system

5.1 Introduction

5.1.1 Foreword

In chapter 3 (paper I [1]) we discussed the pointer states of measurement[1] and we presented a general method for obtaining the pointer states of a two-level system and its environment, for a given total-Hamiltonian defining the system-environment model. As we discussed in chapter 3, time-independence of pointer states by no means should be taken for granted; since time-independent pointer states can be realized only under some very specific conditions (discussed there in chapter 3). We used our method in order to rederive the time-dependent pointer states of the system and the environment (initially prepared in the coherent state) in the Jaynes-Cummings model (JCM) of quantum optics and for the exact resonance regime; verifying the previous results for the JCM [2, 3]. Also, to further demonstrate the generality and usefulness of our method of obtaining pointer states, in chapter 4 we obtained the time-dependent pointer states of the system and the environment for the generalized spin-boson model and in the exact resonance regime.

In this chapter we study the quantized atom-field model *without* the assumption of resonance between the splitting of the states of the two-level atom ω_0 and the cavity eigenmode frequency ω (unlike the Jaynes-Cummings model). Our model basically consists of a two-level atom, with upper and lower levels that can respectively be represented by $|a\rangle$ and $|b\rangle$, interacting with a single-mode quantized bosonic field (such as photons) inside an ideal cavity, represented by creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$. The Hamiltonian for

[1] The pointer states of a subsystem are characterized by their ability not to entangle with the states of another subsystem and appear in the diagonal state of the total composite system after premeasurement by the environment. As we elaborately described in chapter 3, generally we should distinguish between the pointer states of a system and the preferred basis of measurement. We proved that the pointer states of a subsystem generally are time-dependent and a preferred basis of measurement does not exist, unless under some specific conditions (discussed there in chapter 3) for which the pointer states of measurement become time-independent. Moreover, the pointer states of a system are not necessarily orthonormal amongst themselves at all times. Therefore, they cannot necessarily form a basis for the Hilbert space of the system at all times.

the total composite system can be written as [5]

$$\hat{H} = \frac{1}{2}\omega_0\hat{\sigma}_z + \omega\hat{a}^\dagger\hat{a} + \mathrm{g}\chi\hat{\sigma}_x\hat{x}, \tag{5.1}$$

where $\mathrm{g} = -\boldsymbol{\varrho}_{12}.\boldsymbol{\epsilon}\sqrt{\frac{\omega_0}{2\hbar\varepsilon_\circ V}}$ is the atom-field coupling constant, with $\boldsymbol{\varrho}_{12} = e\langle a|\mathbf{r}|b\rangle$ as the atomic electric-dipole transition matrix element ($\boldsymbol{\epsilon}$ is the field polarization vector and V is the cavity mode volume). Also $\chi = \sqrt{2m\omega}$; so that $\chi\hat{x} = \hat{a} + \hat{a}^\dagger$.[2]

The main purpose of this chapter is to obtain the time-dependent pointer states of the system and the environment, as well as expressions for the evolution of the reduced density matrix of the system in the regime that $\hat{H}_\mathcal{E} \ll \hat{H}_\mathcal{S} \ll \hat{H}'$, but $\hat{H}_\mathcal{S} \neq 0$ and $\hat{H}_\mathcal{E} \neq 0$. In other words, to demonstrate how our formulation for obtaining time-dependent pointer states can be used in practice, here we consider a very specific regime of the parameter space and will obtain the corresponding pointer states of the system and the environment within that specific regime; as pointer states (*if* they exist in certain regimes of a system-environment model) generally depend on the specific regime of the parameter space which we are considering and generally acquire different forms in different regimes of the parameter space, *even* for a specifically given system-environment Hamiltonian.

For the Hamiltonian of equation (5.1), as we will show here, the special regime of $\hat{H}_\mathcal{E} \ll \hat{H}_\mathcal{S} \ll \hat{H}'$ is valid only and only if we have

$$1 \ll \sqrt{\frac{\omega_0}{\omega}} \ll |\boldsymbol{\varrho}_{12}.\boldsymbol{\epsilon}| \times \sqrt{\frac{m}{\hbar\varepsilon_\circ V}}\ , \quad \omega \neq 0 \quad \text{and} \quad \omega_0 \neq 0. \tag{5.2}$$

To show this, note that the condition $\hat{H}_\mathcal{S} \ll \hat{H}'$ requires that $\sqrt{\frac{\omega_0}{\omega}} \ll |\boldsymbol{\varrho}_{12}.\boldsymbol{\epsilon}| \times \sqrt{\frac{m}{\hbar\varepsilon_\circ V}}$; while the condition $\hat{H}_\mathcal{E} \ll \hat{H}_\mathcal{S}$ requires that $1 \ll \sqrt{\frac{\omega_0}{\omega}}$. Also, we should emphasize that we *must* have $\omega \neq 0$ ($\hat{H}_\mathcal{E} \neq 0$) and $\omega_0 \neq 0$; as otherwise we would have zero coupling $\mathrm{g}\chi$, and we cannot have $\hat{H}_\mathcal{S} \ll \hat{H}'$ (since we have $\mathrm{g}\chi = -\boldsymbol{\varrho}_{12}.\boldsymbol{\epsilon}\sqrt{\frac{m\omega\omega_0}{\hbar\varepsilon_\circ V}}$). Therefore, as we see, the regime that we are considering and the results of this article are valid *only* in the specific part of the parameter space where the inequalities of equation (5.2) are valid.

This chapter is organized as follows:

After this foreword we review our method for obtaining the pointer states of the system and the environment; and in section 3 we exploit it in order to calculate the time-dependent

[2]Here in this chapter we use the atomic units wherein $\hbar = 1$.

pointer states of the quantized atom-field model represented by the Hamiltonian of equation (5.1).

In order to be able to exploit our method and obtain the pointer states of the system and the environment in our model, we need to know the time-evolution operator of our model in the regime that we are considering. This task is done in section 2.

In section 4 we exploit the pointer states of the system and the environment (which we obtain in section 3) in order to study the decoherence of the central system in our model. Finally, in section 5 we further discuss the significance of our results and the conclusions.

5.1.2 Review of the method

In order to be able to obtain the pointer states of the system and the environment for an arbitrary total Hamiltonian defining our system-environment model we first need to find those probable initial states of the system which do not entangle with the states of the environment throughout their evolution with time; and then we should obtain their time evolution. Finally, we should obtain their corresponding states from the environment which in fact, are the pointer states of the environment. As we saw in chapter 3, existence of pointer states may require having a sufficiently large environment which contains a large number of degrees of freedom. In other words, pointer states characterized by their ability not to entangle with the states of another subsystem, do not necessarily exist in any arbitrary regime.

Consider a two-state system $\mathcal{S}$ with two arbitrary basis states $|a\rangle$ and $|b\rangle$, initially prepared in the state

$$|\psi^{\mathcal{S}}(t_0)\rangle = \alpha|a\rangle + \beta|b\rangle \quad \text{with} \quad |\alpha|^2 + |\beta|^2 = 1; \tag{5.3}$$

and an environment initially prepared in the state

$$|\Phi^{\mathcal{E}}(t_0)\rangle = \sum_{n=0}^{\infty} c_n|\varphi_n\rangle, \tag{5.4}$$

where $\{|\varphi_n\rangle\}$'s are a complete set of basis states for the environment. For the two-state system with the two basis states $|a\rangle$ and $|b\rangle$ we can take the set of any four linearly independent operators in the Hilbert space of the system as a complete set of basis operators,

which can induce any change to the initial state of the two-state system given by equation (5.3). For example, we can take the Pauli operators in addition to the identity operator $\hat{I} = |a\rangle\langle a| + |b\rangle\langle b|$ as our complete set of basis operators; or equivalently we can take the four operators $|a\rangle\langle a|$, $|a\rangle\langle b|$, $|b\rangle\langle a|$ and $|b\rangle\langle b|$ as our complete set of basis operators. So, the time evolution operator for the global state of the system and the environment, which (for a two-state system) generally is of the form

$$\hat{U}_{\rm tot}(t) = \sum_{\alpha=1}^{4} \hat{S}_\alpha \otimes \hat{\mathcal{E}}_\alpha \ , \tag{5.5}$$

can be considered as

$$\hat{U}_{\rm tot}(t) = \hat{\mathcal{E}}_1|a\rangle\langle a| + \hat{\mathcal{E}}_2|a\rangle\langle b| + \hat{\mathcal{E}}_3|b\rangle\langle a| + \hat{\mathcal{E}}_4|b\rangle\langle b|. \tag{5.6}$$

In the above equation $\hat{\mathcal{E}}_i$'s are operators acting on the Hilbert space of the environment, and depend on the total Hamiltonian defining the system-environment model. For example, for the Jaynes-Cummings model and for exact resonance and in the rotating wave approximation (RWA), it can be shown [5] that the $\hat{\mathcal{E}}_i$'s are given by the following relations

$$\begin{aligned} \hat{\mathcal{E}}_1 = \cos(\mathrm{g}t\sqrt{\hat{a}^\dagger\hat{a}+1}) \ , \quad \hat{\mathcal{E}}_2 = -i\ \frac{\sin(\mathrm{g}t\sqrt{\hat{a}^\dagger\hat{a}+1}}{\sqrt{\hat{a}^\dagger\hat{a}+1}}\ \hat{a}) \\ \hat{\mathcal{E}}_3 = -i\hat{a}^\dagger\ \frac{\sin(\mathrm{g}t\sqrt{\hat{a}^\dagger\hat{a}+1}}{\sqrt{\hat{a}^\dagger\hat{a}+1}}) \ , \quad \hat{\mathcal{E}}_4 = \cos(\mathrm{g}t\sqrt{\hat{a}^\dagger\hat{a}}). \end{aligned} \tag{5.7}$$

Using equations (5.3) to (5.6) we can write the global state of the system and the environment as follows

$$\begin{aligned} |\Psi^{\rm tot}(t)\rangle = \hat{U}_{\rm tot}(t).\ (\alpha|a\rangle + \beta|b\rangle) \otimes (\sum_{n=0}^{\infty} c_n|\varphi_n\rangle) \\ = \mathbf{A}(t)\ |a\rangle + \mathbf{B}(t)\ |b\rangle \quad \text{with} \quad \mathbf{A}(t) = \sum_{n=0}^{\infty} c_n\{\alpha\hat{\mathcal{E}}_1 + \beta\hat{\mathcal{E}}_2\}\ |\varphi_n\rangle \\ \text{and} \quad \mathbf{B}(t) = \sum_{n=0}^{\infty} c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle. \end{aligned} \tag{5.8}$$

In order to find those probable initial states of the system which do not entangle with the states of the environment we first define $\hat{G}(t)$ as the operator in the Hilbert space of the environment which relates the vectors $\mathbf{A}(t)$ and $\mathbf{B}(t)$ to each other

$$\mathbf{A}(t) = \hat{G}(t)\mathbf{B}(t) \ \ \text{or} \quad \sum_n c_n\{\alpha\hat{\mathcal{E}}_1 + \beta\hat{\mathcal{E}}_2\}\ |\varphi_n\rangle = \hat{G}(t)\sum_n c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle. \tag{5.9}$$

Now, for the global state of the system and the environment, which is given by

$$|\Psi^{\rm tot}(t)\rangle = \mathbf{A}(t)\ |a\rangle + \mathbf{B}(t)\ |b\rangle = \hat{G}(t)\mathbf{B}(t)\ |a\rangle + \mathbf{B}(t)\ |b\rangle$$
$$= \{\hat{G}(t)|a\rangle + |b\rangle\} \times (\sum_{n=0}^{\infty} c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle), \tag{5.10}$$

we observe that if for some initial states of the system and the environment $\hat{G}(t)$ turns out to become in the form

$$\hat{G}(t) = G(t) \times \hat{I}_{\mathcal{E}}, \tag{5.11}$$

with $G(t)$ as a scalar (rather than an operator) and $\hat{I}_{\mathcal{E}}$ representing the identity operator in the Hilbert space of the environment, then those initial states of the system and the environment will not entangle with each other, and hence they can represent the initial pointer states of the system and the environment. This result simply is because of the fact that if for some initial states of the system and the environment $\hat{G}(t)$ turns out to become a scalar in the form of equation (5.11), $G(t)$ will be independent of the indices of the environment (i.e. independent of n); as in this case all components of $\mathbf{B}(t)$ will be mapped into their corresponding components from $\mathbf{A}(t)$ through the *same* scalar function $G(t)$ (which will keep the two vectors $\mathbf{A}(t)$ and $\mathbf{B}(t)$ parallel to each other). Therefore, in this case $\hat{G}(t)$ will not enter the summation in the expression $\sum_n c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle$ of equation (5.10); and (as one can see from equation (5.10)) the states of the system and the environment respectively represented by $\{G(t)|a\rangle + |b\rangle\}$ and $\sum_n c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle$ will not entangle to each other.

In another word, if for some initial states of the system and the environment $\mathbf{A}(t) = \hat{G}(t)\mathbf{B}(t)$ is equal to $G\mathbf{B}(t)$, it means that for those initial states of the system and the environment, $\mathbf{B}(t)$ becomes an eigenstate of the operator $\hat{G}(t)$; and the two vectors $\mathbf{A}(t)$ and $\mathbf{B}(t)$ will stay parallel with each other throughout their evolution with time; and as we discussed, in this case the states of the system and the environment will not entangle with each other and (as one can see from equation (5.10)) pointer states can be realized for the system and the environment given by

$$|\pm(t)\rangle = \mathcal{N}_{\pm}\ \{G(t)|a\rangle + |b\rangle\} \qquad \text{and}$$

$$|\Phi_{\pm}(t)\rangle = \mathcal{N}_{\pm}^{-1}(\sum_{n=0}^{\infty} c_n\{\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4\}\ |\varphi_n\rangle). \tag{5.12}$$

In the above equation we have represented the pointer states of the system by $|\pm(t)\rangle$ and those of the environment by $|\Phi_{\pm}(t)\rangle$. Also, $\mathcal{N}_{\pm}$ is the normalization factor for the pointer states of the system (clearly $\mathcal{N}_{\pm} = \frac{1}{\sqrt{2}}$ if $|G(t)| = 1$, as for the example of the JCM in the exact-resonance regime).

The condition represented by equation (5.11) in fact is a *necessary* condition for obtaining pointer states; since unless $\hat{G}(t)$ turns out to be a scalar, the two vectors $\mathbf{A}(t) = \hat{G}(t) \times \mathbf{B}(t)$ and $\mathbf{B}(t)$ will not be parallel at all times and the operator $\hat{G}(t)$ will enter the summation over the environmental degrees of freedom (i.e. the summation over n) in equation (5.10), in which case the states of the system and the environment no longer will be separable in a tensor product form; and pointer states cannot be realized for the states of the system and the environment. Also, as we saw in chapter 4, generally there is no guaranty for the condition (5.11) to be satisfied; and satisfaction of this condition often may require having a sufficiently large environment which contains a large number of degrees of freedom. However, *if* in some regime and for a given Hamiltonian defining a system-environment model we can find initial states for the system and the environment which satisfy this condition, we do know that pointer states can be realized for the system and the environment and those initial states would correspond to the initial pointer states of the system and the environment.

In essence, in order to find the pointer states of the system and the environment for a given total Hamiltonian defining our system-environment model, and for a given initial state of the environment, our main goal would be finding those possible initial states of the system for which $\hat{G}(t)$ (which is defined through equation (5.9)) is of the form of relation (5.11). In section 3 considering the quantized atom-field model represented by the Hamiltonian of equation (5.1) and for the regime that $\hat{H}_{\mathcal{E}} \ll \hat{H}_{\mathcal{S}} \ll \hat{H}'$ (but $\hat{H}_{\mathcal{E}} \neq 0$ and $\hat{H}_{\mathcal{S}} \neq 0$), we exploit this method to obtain the time-dependent pointer states of the system and the environment; by assuming an initial state of the environment in the form of a Gaussian package in position space. As we will see, once we have the time-evolution operator for our

system-environment model in the form of equation (5.6) and the $\hat{\mathcal{E}}_i$ operators, this task can be done quite easily for our model.

5.2 Calculation of the time-evolution operator

In order to calculate the time-evolution operator in the interaction picture for the Hamiltonian of equation (5.1), first we need to have the Hamiltonian in the interaction picture, which is defined through the following equation

$$\hat{H}_{\rm int}(t) = e^{i\hat{H}_0 t}\hat{H}^{'}e^{-i\hat{H}_0 t}. \tag{5.13}$$

Here $\hat{H}_0 = \frac{1}{2}\omega_0\hat{\sigma}_z + \omega\hat{a}^\dagger\hat{a}$ is the sum of the self Hamiltonians of the system and the environment; and $\hat{H}^{'} = {\rm g}\chi\hat{\sigma}_x\hat{x}$ is the Hamiltonian for the interaction between the system and the environment. So, now we must calculate

$$\hat{H}_{\rm int}(t) = {\rm g}\ (e^{i\omega_0\hat{\sigma}_z t/2}\ \hat{\sigma}_x\ e^{-i\omega_0\hat{\sigma}_z t/2}) \otimes (e^{i\omega\hat{a}^\dagger\hat{a}t}\ \chi\hat{x}\ e^{-i\omega\hat{a}^\dagger\hat{a}t}), \tag{5.14}$$

where $\chi\hat{x} = \hat{a} + \hat{a}^\dagger$. However, $\hat{\sigma}_x = \hat{\sigma}_+ + \hat{\sigma}_-$; and $e^{i\omega_0\hat{\sigma}_z t/2}\ \hat{\sigma}_\pm\ e^{-i\omega_0\hat{\sigma}_z t/2} = \hat{\sigma}_\pm\ e^{\pm i\omega_0 t}$. Also $\hat{a}(t) = \hat{a}e^{-i\omega t}$. So

$$\begin{aligned}\hat{H}_{\rm int}(t) &= {\rm g}(\hat{\sigma}_+\ e^{i\omega_0 t} + \hat{\sigma}_-\ e^{-i\omega_0 t}) \otimes (\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}) \\ &= {\rm g}\{\hat{\sigma}_+(\hat{a}\ e^{i\Delta t} + \hat{a}^\dagger\ e^{i(\omega+\omega_0)t}) + c.c.\}, \qquad \text{with} \qquad \Delta = \omega_0 - \omega. \end{aligned} \tag{5.15}$$

Now in parallel with chapter 3, for the evolution operator of the global composite system we consider the general form given by equation (5.6). For such a time-evolution operator in the interaction picture, which satisfies the Schrödinger equation

$$i\frac{\partial}{\partial t}\hat{U}(t) = \hat{H}_{\rm int}\hat{U}(t), \tag{5.16}$$

we have

$$i\begin{pmatrix} \dot{\hat{\mathcal{E}}}_1 & \dot{\hat{\mathcal{E}}}_2 \\ \dot{\hat{\mathcal{E}}}_3 & \dot{\hat{\mathcal{E}}}_4 \end{pmatrix} = \hat{H}_{\rm int}(t)\begin{pmatrix} \hat{\mathcal{E}}_1 & \hat{\mathcal{E}}_2 \\ \hat{\mathcal{E}}_3 & \hat{\mathcal{E}}_4 \end{pmatrix}$$

$$
\begin{aligned}
&= \mathrm{g}\begin{pmatrix} 0 & \hat{a}\ e^{i\Delta t} + \hat{a}^\dagger\ e^{i(\omega+\omega_0)t} \\ \hat{a}^\dagger\ e^{-i\Delta t} + \hat{a}\ e^{-i(\omega+\omega_0)t} & 0 \end{pmatrix}\begin{pmatrix} \hat{\mathcal{E}}_1 & \hat{\mathcal{E}}_2 \\ \hat{\mathcal{E}}_3 & \hat{\mathcal{E}}_4 \end{pmatrix} \\
&= \mathrm{g}\begin{pmatrix} (\hat{a}\ e^{i\Delta t} + \hat{a}^\dagger\ e^{i(\omega+\omega_0)t})\ \hat{\mathcal{E}}_3 & (\hat{a}\ e^{i\Delta t} + \hat{a}^\dagger\ e^{i(\omega+\omega_0)t})\ \hat{\mathcal{E}}_4 \\ (\hat{a}^\dagger\ e^{-i\Delta t} + \hat{a}\ e^{-i(\omega+\omega_0)t})\ \hat{\mathcal{E}}_1 & (\hat{a}^\dagger\ e^{-i\Delta t} + \hat{a}\ e^{-i(\omega+\omega_0)t})\ \hat{\mathcal{E}}_2 \end{pmatrix}.
\end{aligned} \tag{5.17}
$$

Now, we assume $\omega \ll \omega_0$; so that $\Delta \approx \omega_0$ and $\omega + \omega_0 \approx \omega_0$. In other words, in the Hamiltonian of our total composite system, given by equation (5.1), we assume that the self-Hamiltonian of the system dominates the self-Hamiltonian of the environment. Therefore, equation (5.17) for the evolution of the time-evolution operator can be simplified to the following set of four equations

$$
\begin{aligned}
i\dot{\hat{\mathcal{E}}}_1 &= \mathrm{g}\chi\hat{x}\ e^{i\omega_0 t}\ \hat{\mathcal{E}}_3, \\
i\dot{\hat{\mathcal{E}}}_2 &= \mathrm{g}\chi\hat{x}\ e^{i\omega_0 t}\ \hat{\mathcal{E}}_4, \\
i\dot{\hat{\mathcal{E}}}_3 &= \mathrm{g}\chi\hat{x}\ e^{-i\omega_0 t}\ \hat{\mathcal{E}}_1, \\
i\dot{\hat{\mathcal{E}}}_4 &= \mathrm{g}\chi\hat{x}\ e^{-i\omega_0 t}\ \hat{\mathcal{E}}_2.
\end{aligned} \tag{5.18}
$$

In order to solve the above set of coupled differential equations, we proceed as follows. First, we take derivative with respect to time of the first equation. By replacing $\dot{\hat{\mathcal{E}}}_3$ from the third equation in the resulting equation we find

$$
\ddot{\hat{\mathcal{E}}}_1 = -(\mathrm{g}\chi\hat{x})^2\ \hat{\mathcal{E}}_1 + (\mathrm{g}\chi\hat{x}\omega_0\ e^{i\omega_0 t})\ \hat{\mathcal{E}}_3. \tag{5.19}
$$

Similarly, by doing the same procedure on the third equation for $\dot{\hat{\mathcal{E}}}_3$ we find

$$
\ddot{\hat{\mathcal{E}}}_3 = -(\mathrm{g}\chi\hat{x})^2\ \hat{\mathcal{E}}_3 - (\mathrm{g}\chi\hat{x}\omega_0\ e^{-i\omega_0 t})\ \hat{\mathcal{E}}_1. \tag{5.20}
$$

One can easily verify that *if* $\omega_0^2 \ll (\mathrm{g}\chi)^2$ (i.e. *if* $\hat{H}_{\mathcal{S}} \ll \hat{H}'$), so that $(\mathrm{g}\chi\hat{x})^2 + \omega_0^2/4 \approx (\mathrm{g}\chi\hat{x})^2$, the following solutions will satisfy the differential equations given by equations (5.19) and (5.20) for $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_3$:

$$
\hat{\mathcal{E}}_1 = \cos(\mathrm{g}\chi\hat{x}t)\ e^{i\omega_0 t/2} \qquad \text{and} \qquad \hat{\mathcal{E}}_3 = -i\sin(\mathrm{g}\chi\hat{x}t)\ e^{-i\omega_0 t/2}. \tag{5.21}
$$

In quite the same manner we can calculate $\hat{\mathcal{E}}_2$ and $\hat{\mathcal{E}}_4$ as follows

$$
\hat{\mathcal{E}}_2 = -i\sin(\mathrm{g}\chi\hat{x}t)\ e^{i\omega_0 t/2} \qquad \text{and} \qquad \hat{\mathcal{E}}_4 = \cos(\mathrm{g}\chi\hat{x}t)\ e^{-i\omega_0 t/2}. \tag{5.22}
$$

The above operators together with equation (5.6) make the time-evolution operator of our quantized atom-field model and for the regime that $\hat{H}_{\mathcal{E}} \ll \hat{H}_{\mathcal{S}} \ll \hat{H}'$, but $\hat{H}_{\mathcal{S}} \neq 0$ and $\hat{H}_{\mathcal{E}} \neq 0$. One can easily verify that the above set of operators satisfies the unitarity of the time-evolution operator given by $\hat{U}^{\dagger}\hat{U} = \hat{U}\hat{U}^{\dagger} = \hat{I}$ (with $\hat{I}$ representing the identity operator). Moreover, $\hat{\mathcal{E}}_1(0) = \hat{\mathcal{E}}_4(0) = 1$ and $\hat{\mathcal{E}}_2(0) = \hat{\mathcal{E}}_3(0) = 0$. So, these operators do satisfy the initial condition for the time-evolution operator given by $\hat{U}_{\rm tot}(t_0) = \hat{I}$.

5.3 Calculation of the time-dependent pointer states of the system and the environment

Using the time-evolution operator which we already obtained for our model and for the regime that $\hat{H}_{\mathcal{E}} \ll \hat{H}_{\mathcal{S}} \ll \hat{H}'$ (but $\hat{H}_{\mathcal{S}} \neq 0$ and $\hat{H}_{\mathcal{E}} \neq 0$), now we can obtain the corresponding pointer states of the system and the environment in this regime. For this purpose we assume that the system initially is prepared in the state $|\psi^{\mathcal{S}}(t_0)\rangle = \alpha|a\rangle + \beta|b\rangle$. Moreover, let us assume that the initial state of the environment can be represented by a Gaussian package in the position space

$$|\Phi^{\mathcal{E}}(t_0)\rangle = \mathcal{N}_0 \int_{-\infty}^{\infty} dx\ e^{-\alpha_\circ x^2}|x\rangle, \tag{5.23}$$

where $\mathcal{N}_0 = (2\alpha_\circ/\pi)^{1/4}$ is the normalization factor for this state. Now, the condition for determining the pointer states of the system and the environment, given by equations (5.9) and (5.11), reads

$$\begin{aligned}(\alpha\hat{\mathcal{E}}_1 + \beta\hat{\mathcal{E}}_2)\ |\Phi^{\mathcal{E}}(t_0)\rangle = \hat{G}(t) \times (\alpha\hat{\mathcal{E}}_3 + \beta\hat{\mathcal{E}}_4)\ |\Phi^{\mathcal{E}}(t_0)\rangle;\\ \textit{with } \hat{G}(t) \textit{ being proportional to the unit matrix.}\end{aligned} \tag{5.24}$$

(In other words, for an initial state of the system corresponding to one of its pointer states at $t = t_0$, the operator $\hat{G}(t)$ must be independent of the indices of the environment. i.e. x). Inserting the $\hat{\mathcal{E}}_i$'s from equations (5.21) and (5.22) into the above condition it reads

$$\int_{-\infty}^{\infty} dx\ [\alpha\cos(\mathrm{g}\chi x t) - i\beta\sin(\mathrm{g}\chi x t)]\ e^{-\alpha_\circ x^2 + i\omega_0 t/2}|x\rangle$$

$$= \hat{G}(t) \times \int_{-\infty}^{\infty} dx \; [-i\alpha \sin(\mathrm{g}\chi x t) + \beta \cos(\mathrm{g}\chi x t)] \; e^{-\alpha_\circ x^2 - i\omega_0 t/2} |x\rangle \tag{5.25}$$

and $\hat{G}(t)$ *be proportional to the unit matrix.*

For pointer states $\hat{G}(t)$ must satisfy the condition (5.11) for obtaining the pointer states of the system and the environment, i.e. $\hat{G}(t) = G(t) \times \hat{I}_{\mathcal{E}}$. Therefore, since the set $\{|x\rangle\}$ is a complete set of basis states for the environment, for initial pointer states we can simply equalize those terms from the two sides of equation (5.25) which correspond to the same $|x\rangle$ state and obtain

$$G(t) = \frac{\alpha \cos(\mathrm{g}\chi x t) - i\beta \sin(\mathrm{g}\chi x t)}{-i\alpha \sin(\mathrm{g}\chi x t) + \beta \cos(\mathrm{g}\chi x t)} \; e^{i\omega_0 t}. \tag{5.26}$$

The above result for $G(t)$, which generally depends on x, would contradict our initial assumption of $\hat{G}(t)$ being proportional to the unit matrix *unless* we can find certain initial states for the system for which $G(t)$ turns out to become independent of x; since, as we discussed, for pointer states, all components of the vector $\mathbf{A}$ ($A_x{}'s$) must be related to their corresponding components from $\mathbf{B}$ ($B_x{}'s$) through the *same* scalar factor G (see equations (5.9) and (5.11)).[3] So now we should seek those particular initial states of the system which can make $G(t)$ independent of the variable x of the states of the environment.

From equation (5.26) we easily see that for $\alpha = \pm\beta$, $G(t)$ turns out to become

$$G(t) = \pm e^{i\omega_0 t} \tag{5.27}$$

which clearly is independent of the variable x of the states of the environment.

The above result simply means that for the initial states of the system obtained from

$$\alpha_+ = \beta_+ = \frac{1}{\sqrt{2}} \quad \text{and} \quad \alpha_- = -\beta_- = \frac{1}{\sqrt{2}} \; , \quad \text{or} \quad |\pm(t_0)\rangle = \frac{1}{\sqrt{2}}(|a\rangle \pm |b\rangle), \tag{5.28}$$

(which correspond to the initial conditions for the state of the system given by $\alpha = \pm\beta$) the states of the system and the environment will not entangle with each other. Moreover,

[3] We would like to see if the condition can be satisfied for *any* initial state of the system and the environment with $G(t)$ becoming independent of the variable x of the states of the environment. So, if finally we can find any specific set of initial states for the system and the environment which satisfies this condition with $G(t)$ independent of the indices of the environment, then we have reached our goal.

using equation (5.12), which gives us the general time evolution of the pointer states of the system, and $G(t)$ of equation (5.27) (which is independent of the variable x of the states of the environment) we can find the time evolution of the pointer states of the system as follows

$$|\pm(t)\rangle = \mathcal{N}\ \{G(t)|a\rangle + |b\rangle\} = \frac{1}{\sqrt{2}}(e^{i\omega_0 t}\ |a\rangle \pm |b\rangle). \tag{5.29}$$

As we observe, in the regime that we are considering ($\hat{H}_{\mathcal{E}} \ll \hat{H}_{\mathcal{S}} \ll \hat{H}'$, with $\hat{H}_{\mathcal{E}} \neq 0$ and $\hat{H}_{\mathcal{S}} \neq 0$), $G(t)$ and the time evolution of the pointer states of the system are characterized by ω_0 of the self-Hamiltonian of the system; unlike the exact-resonance with the rotating wave approximation regime where the evolution of the pointer states of the system is characterized by the atom-field coupling constant g and the average number of photons $\bar{n}$, through the factor $\mathrm{g}/\sqrt{\bar{n}}$.

Next, we obtain the corresponding pointer states of the environment. Using equations (5.12), (5.23) and (5.28) we have

$$\begin{aligned} |\Phi_{\pm}(t)\rangle &= \mathcal{N}^{-1}(\alpha_{\pm}\hat{\mathcal{E}}_3 + \beta_{\pm}\hat{\mathcal{E}}_4)\ |\Phi^{\mathcal{E}}(t_0)\rangle \\ &= \mathcal{N}_0(\hat{\mathcal{E}}_3 \pm \hat{\mathcal{E}}_4)\ \int_{-\infty}^{\infty} dx\ e^{-\alpha_\circ x^2}|x\rangle; \end{aligned} \tag{5.30}$$

since $\mathcal{N}^{-1}\alpha_{\pm} = 1$ and $\mathcal{N}^{-1}\beta_{\pm} = \pm 1$. Therefore,

$$|\Phi_{\pm}(t)\rangle = (\frac{2\alpha_\circ}{\pi})^{\frac{1}{4}}\ \int_{-\infty}^{\infty} dx\ e^{-\alpha_\circ x^2 \mp i(\mathrm{g}\chi x \pm \omega_0/2)t}\ |x\rangle. \tag{5.31}$$

Also, the overlap between the pointer states of the environment can be calculated as

$$\langle\Phi_-(t)|\Phi_+(t)\rangle = e^{-(\mathrm{g}\chi t)^2/2\alpha_\circ}. \tag{5.32}$$

We should also mention that the pointer states of the system at $t = t_0$ (see equation (5.28)) are orthonormal and hence, they form a complete basis set for the state of the system. Therefore, the evolution of any initial pure state of the two-level system $|\psi_{\mathcal{S}}(t_0)\rangle = \alpha'\ |+(t_0)\rangle + \beta'\ |-(t_0)\rangle$ with an initial field $|\Phi_{\mathcal{E}}(t_0)\rangle$, in the form of equation (5.23), can be expressed as a linear combination of the evolution of $|+(t_0)\rangle|\Phi_{\mathcal{E}}(t_0)\rangle$ and $|-(t_0)\rangle|\Phi_{\mathcal{E}}(t_0)\rangle$

$$(\alpha'\ |+(t_0)\rangle + \beta'\ |-(t_0)\rangle)\ |\Phi_{\mathcal{E}}(t_0)\rangle \rightarrow \alpha'\ |+(t)\rangle\ |\Phi_+(t)\rangle + \beta'\ |-(t)\rangle\ |\Phi_-(t)\rangle, \tag{5.33}$$

where in the above equation the evolution of the pointer states of the system $|\pm(t)\rangle$ is given by equation (5.29) and the evolution of the pointer states of the environment $|\Phi_{\pm}(t)\rangle$ is given by equation (5.31).

5.4 Consequences regarding the decoherence of the central system

In this section first we use the time-evolution operator, already obtained in section 2, to obtain the general time evolution of the total composite system for our model and for an initial state of the environment in the form of a Gaussian package in position space, such as that of equation (5.23). After that, we will calculate the offdiagonal element of the reduced density matrix of the system (i.e. $\rho_{12}^{(\mathcal{S})}(t)$) by tracing over the environmental degrees of freedom. Then, we will also obtain the coherences of the reduced density matrix of the system in another way by using the pointer states of the system and the environment obtained in section 3. As we will see, the two results will be in perfect agreement with each other. Finally, we will discuss some interesting features which can be observed in our study of the decoherence of the central system.

Using equations (5.8), (5.21) and (5.22) to obtain $|\Psi_{\rm tot}(t)\rangle$, we can write

$$|\Psi_{\rm tot}(t)\rangle = \mathbf{A}(t)\ |a\rangle + \mathbf{B}(t)\ |b\rangle = (\alpha\cos(\mathrm{g}\chi\hat{x}t)\ e^{i\omega_0 t/2} - i\beta\sin(\mathrm{g}\chi\hat{x}t)\ e^{i\omega_0 t/2})\ |\Phi^{\mathcal{E}}(t_0)\rangle|a\rangle \tag{5.34}$$
$$+(-i\alpha\sin(\mathrm{g}\chi\hat{x}t)\ e^{-i\omega_0 t/2} + \beta\cos(\mathrm{g}\chi\hat{x}t)\ e^{-i\omega_0 t/2})\ |\Phi^{\mathcal{E}}(t_0)\rangle|b\rangle.$$

In the above equation $|\Phi^{\mathcal{E}}(t_0)\rangle$ is the initial state of the environment, represented by the Gaussian package of equation (5.23).

For the state of the total composite system in our model, which is given by equation (5.34), we can do the trace operation over the basis states of the environment (i.e. the $\{|x\rangle\}$ which make a complete basis for the state of the environment) to obtain the reduced density matrix of the system $\mathcal{S}$

$$\hat{\rho}_{\mathcal{S}}(t) = \int_{-\infty}^{\infty} dx\ \langle x|\hat{\rho}^{\rm tot}(t)|x\rangle = \int_{-\infty}^{\infty} dx\ \langle x|\Psi_{\rm tot}(t)\rangle\langle\Psi_{\rm tot}(t)|x\rangle$$

$$= \int_{-\infty}^{\infty} dx \ (\ |\psi_a(x,t)|^2 \ |a\rangle\langle a| + |\psi_b(x,t)|^2 \ |b\rangle\langle b| + \psi_a(x,t)\psi_b^*(x,t) \ |a\rangle\langle b| + c.c. \ ). \quad (5.35)$$

where

$$\psi_a(x,t) = (\frac{2\alpha_\circ}{\pi})^{\frac{1}{4}} \ [\alpha \cos(\mathrm{g}\chi x t) \ e^{i\omega_0 t/2} - i\beta \sin(\mathrm{g}\chi x t) \ e^{i\omega_0 t/2}] \ e^{-\alpha_\circ x^2} \quad \text{and}$$
$$\psi_b(x,t) = (\frac{2\alpha_\circ}{\pi})^{\frac{1}{4}} \ [-i\alpha \sin(\mathrm{g}\chi x t) \ e^{-i\omega_0 t/2} + \beta \cos(\mathrm{g}\chi x t) \ e^{-i\omega_0 t/2}] \ e^{-\alpha_\circ x^2}. \quad (5.36)$$

Using equations (5.35) and (5.36), after doing the integrations we easily find

$$\rho_{aa}^{\mathcal{S}}(t) = 1 - \rho_{bb}^{\mathcal{S}}(t) = \int_{-\infty}^{\infty} dx \ |\psi_a(x,t)|^2 = \frac{1}{2}[1 + (|\alpha|^2 - |\beta|^2) \ e^{-(\mathrm{g}\chi t)^2/2\alpha_\circ}] \quad \text{and}$$
$$\rho_{ab}^{\mathcal{S}}(t) = \int_{-\infty}^{\infty} dx \ \psi_a(x,t)\psi_b^*(x,t) = \frac{1}{2}[(\alpha\beta^* + \beta\alpha^*) + (\alpha\beta^* - \beta\alpha^*)e^{-(\mathrm{g}\chi t)^2/2\alpha_\circ}]e^{i\omega_0 t}. \quad (5.37)$$

(In the above equation we used the notation $\rho_{ab} = \langle a|\hat{\rho}_{\mathcal{S}}(t)|b\rangle$ and etc.) As we see from the above equations, for the initial pointer states of the system, for which $|\alpha| = |\beta|$ (see equation (5.28)), and also for very large times $t \to \infty$, the diagonal elements of the reduced density matrix of the system will be equal to the constant number of $\frac{1}{2}$. Also, for the initial pointer states of the system we have $\rho_{ab}^{\mathcal{S}}(t) = \frac{1}{2}(\alpha\beta^* + \beta\alpha^*)e^{i\omega_0 t}$. This means that for the initial pointer states of the system $|\rho_{ab}^{\mathcal{S}}(t)|$ will always be equal to the constant value of $\frac{1}{2}$; while for most of the other states (for which $\alpha\beta^* \neq \beta\alpha^*$) only at sufficiently large times $|\rho_{ab}^{\mathcal{S}}(t)|$ will converge to the constant value of $\frac{1}{2}(\alpha\beta^* + \beta\alpha^*)$, with a decoherence time given by

$$\tau_{\mathrm{dec}} = \frac{\hbar}{\mathrm{g}} \frac{\sqrt{2\alpha_\circ}}{\chi} = \frac{\hbar}{\mathrm{g}} \sqrt{\frac{\alpha_\circ}{m\omega}}. \quad (5.38)$$

The reduced density matrix of a two-level system $\hat{\rho}_{\mathcal{S}}(t)$ generally can be expressed in terms of the Bloch vector $\mathbf{R}(t) \equiv (R_x, R_y, R_z)$ [6] as follows

$$\hat{\rho}_{\mathcal{S}}(t) = \frac{1}{2}(\hat{I} + \mathbf{R}(t).\hat{\sigma}) = \frac{1}{2}(\hat{I} + R_x\sigma_x + R_y\sigma_y + R_z\sigma_z); \quad (5.39)$$

from which one can easily verify that the Bloch vector components must be defined by

$$R_x = \rho_{ab} + \rho_{ba} \qquad R_y = i(\rho_{ab} - \rho_{ba}) \quad \text{and} \quad R_z = \rho_{aa} - \rho_{bb}. \quad (5.40)$$

So now, using our expressions for the elements of the reduced density matrix of the system, given by equation (5.37), we can also calculate the components of the Bloch vector, which

are a measure for the polarization of the state of the two-level system [1, 7]. One would easily find

$$\begin{aligned} R_x(t) = \rho_{ab} + \rho^*_{ab} &= (\alpha\beta^* + \beta\alpha^*)\cos(\omega_0 t) + i(\alpha\beta^* - \beta\alpha^*)\sin(\omega_0 t)\ e^{-(\mathrm{g}\chi t)^2/2\alpha_\circ}, \\ R_y(t) = i(\rho_{ab} - \rho^*_{ab}) &= -(\alpha\beta^* + \beta\alpha^*)\sin(\omega_0 t) + i(\alpha\beta^* - \beta\alpha^*)\cos(\omega_0 t)\ e^{-(\mathrm{g}\chi t)^2/2\alpha_\circ}, \\ R_z(t) = \rho_{aa} - \rho_{bb} &= (|\alpha|^2 - |\beta|^2)\ e^{-(\mathrm{g}\chi t)^2/2\alpha_\circ}. \end{aligned} \tag{5.41}$$

For $t \to \infty$ and $\chi \neq 0$ we have

$$\begin{aligned} R_x(t) &\to (\alpha\beta^* + \beta\alpha^*)\cos(\omega_0 t), \\ R_y(t) &\to -(\alpha\beta^* + \beta\alpha^*)\sin(\omega_0 t) \quad \text{and} \\ R_z(t) &\to 0. \end{aligned} \tag{5.42}$$

The above result simply means that for $t \to \infty$ and if $\chi = \sqrt{2m\omega} \neq 0$ the pointer states of the system will evolve between the eigenstates of the $\hat{\sigma}_x$ and $\hat{\sigma}_y$ Pauli matrices; and therefore, a preferred basis of measurement is *not* determined in the regime that we are considering; although the eigenstates of $\hat{\sigma}_z$ are excluded from being realized for $t \to \infty$.

One can easily obtain the coherences of the reduced density matrix of the system in another way by using the pointer states of the system and the environment which we obtained in section 3. As we saw, for a two-state system $\mathcal{S}$ in contact with an environment $\mathcal{E}$ after determination of the pointer states of the system and the environment, the state of the total composite system generally can be represented by equation (5.33). i.e. $|\Psi_{\mathrm{tot}}(t)\rangle = \alpha'\ |+(t)\rangle\ |\Phi_+(t)\rangle + \beta'\ |-(t)\rangle\ |\Phi_-(t)\rangle$. For $|\Psi_{\mathrm{tot}}(t)\rangle$ given by equation (5.33) the reduced density matrix of the system $\hat{\rho}_{\mathcal{S}}(t)$ can be calculated by tracing over the environmental degrees of freedom to obtain

$$\begin{aligned} \hat{\rho}_{\mathcal{S}}(t) = |\alpha'|^2 \times |+(t)\rangle\langle+(t)| + |\beta'|^2 \times |-(t)\rangle\langle-(t)| + \alpha'\beta'^* \\ \times|+(t)\rangle\langle-(t)| \times \langle\Phi_-(t)|\Phi_+(t)\rangle + \beta'\alpha'^* \times |-(t)\rangle\langle+(t)| \times \langle\Phi_+(t)|\Phi_-(t)\rangle. \end{aligned} \tag{5.43}$$

So, in an arbitrary basis $|a\rangle$ and $|b\rangle$ of the state of the two-level system generally we have

$$\rho^{\mathcal{S}}_{aa}(t) = 1 - \rho^{\mathcal{S}}_{bb}(t) = |\alpha'|^2 \times \langle a|+(t)\rangle\langle+(t)|a\rangle + |\beta'|^2 \times \langle a|-(t)\rangle\langle-(t)|a\rangle + \alpha'\beta'^*$$

$$\times\langle a|+(t)\rangle\langle-(t)|a\rangle\times\langle\Phi_-(t)|\Phi_+(t)\rangle+\beta'\alpha'^*\times\langle a|-(t)\rangle\langle+(t)|a\rangle\times\langle\Phi_+(t)|\Phi_-(t)\rangle \quad (5.44)$$

$$\text{and}\qquad \rho^{\mathcal{S}}_{ab}(t)=|\alpha'|^2\times\langle a|+(t)\rangle\langle+(t)|b\rangle+|\beta'|^2\times\langle a|-(t)\rangle\langle-(t)|b\rangle+\alpha'\beta'^*$$

$$\times\langle a|+(t)\rangle\langle-(t)|b\rangle\times\langle\Phi_-(t)|\Phi_+(t)\rangle+\beta'\alpha'^*\times\langle a|-(t)\rangle\langle+(t)|b\rangle\times\langle\Phi_+(t)|\Phi_-(t)\rangle.$$

The expansion coefficients α' and β' for the state of the two-level system in the basis of the $|\pm(t_0)\rangle$ states are related to the corresponding coefficients in the $|a\rangle$ and $|b\rangle$ basis[4] through $\alpha'=\frac{1}{\sqrt{2}}(\alpha+\beta)$ and $\beta'=\frac{1}{\sqrt{2}}(\alpha-\beta)$. So now, for our quantized atom-field model and in the regime that we are considering one can use equations (5.29) and (5.32) to calculate the expressions in equation (5.44) for the elements of the reduced density matrix of the system; obtaining exactly the same results as those of equation (5.37).

One could similarly study the decoherence of the state of the system in the basis of the $|\pm(t_0)\rangle$ states. As one can see from equation (5.29), for $t\ll\omega_0^{-1}$ the pointer states of the system approximately can be represented by $|\pm(t_0)\rangle$. Therefore, in the basis of the $|\pm(t_0)\rangle$ states the short-time evolution of the off-diagonal element of the reduced density matrix of the system should be given by

$$\rho^{\mathcal{S}}_{12}(t)\approx\alpha'\beta'^*\langle\Phi_-(t)|\Phi_+(t)\rangle=\alpha'\beta'^*\ e^{-(g\chi t)^2/2\alpha_\circ} \quad (5.45)$$

Hence, in the basis of the $|\pm(t_0)\rangle$ states the short-time decoherence of the state of the system is characterized by the decaying factor $e^{-(g\chi t)^2/2\alpha_\circ}$, when the system initially is *not* prepared in one of its pointer states ($\alpha'\beta'^*\neq 0$); while in this basis the pointer states of the system almost do not decohere within short times; and $\rho^{\mathcal{S}}_{12}(t)\approx 0$ at all short times (i.e. for $t\ll\omega_0^{-1}$ for which $|\pm(t)\rangle\approx|\pm(t_0)\rangle$).

Finally, let us study whether the short-time decay of $\rho^{\mathcal{S}}_{12}(t)$, given by equation (5.45), might be reversible or not. As we will show here, the coherences of the reduced density matrix of the system, may revive at a later time. In such cases, of course we cannot have irreversible decoherence.

Using equation (5.44) for the offdiagonal element of the reduced density matrix of the system and equations (5.29) and (5.32), we can calculate the all-time evolution of $\rho^{\mathcal{S}}_{12}(t)$

[4]Now by $|a\rangle$ and $|b\rangle$ we mean the upper and lower levels of the two-level system; i.e. $|a\rangle$ and $|b\rangle$ no longer are some arbitrary basis states for the state of the two-level system.

for the regime that we are considering and in the basis of the initial pointer states of the system $|\pm(t_0)\rangle$ as follows

$$\rho_{12}^{\mathcal{S}}(t) = (|\beta'|^2 - |\alpha'|^2) \times [\frac{i}{2}\ \sin(\omega_0 t)] + \alpha'\beta'^{*} \times \cos^2(\omega_0 t/2) \times e^{-(g\chi t)^2/2\alpha_\circ} + \beta'\alpha'^{*} \times \sin^2(\omega_0 t/2) \times e^{-(g\chi t)^2/2\alpha_\circ}; \qquad (5.46)$$

which its short time evolution ($t \ll \omega_0^{-1}$) is the same as equation (5.45).

Now, clearly for $t \to \infty$ we have

$$\rho_{12}^{\mathcal{S}}(t) = (|\beta'|^2 - |\alpha'|^2) \times [\frac{i}{2}\ \sin(\omega_0 t)]. \qquad (5.47)$$

Therefore, except for $|\alpha'| = |\beta'|$, in the basis of the initial pointer states of the system and for $t \to \infty$ the offdiagonal element of the reduced density matrix of the system, $\hat{\rho}_{12}^{\mathcal{S}}$, will be oscillating with the frequency of ω_0. As a result, we should note that the short-time decay, represented by equation (5.45), can be reversible; as $\rho_{12}^{\mathcal{S}}(t)$ may revive at a later time.

5.5 Summary and conclusions

Considering the quantized atom-field model of quantum optics, we obtained the time-evolution operator for the regime that $\hat{H}_{\mathcal{E}} \ll \hat{H}_{\mathcal{S}} \ll \hat{H}'$ (but $\hat{H}_{\mathcal{S}} \neq 0$ and $\hat{H}_{\mathcal{E}} \neq 0$). Using this time-evolution operator then we calculated the corresponding pointer states of the system and the environment, which are characterized by their ability not to entangle with each other, by assuming an initial state of the environment in the form of a Gaussian package in position space. Most importantly, we observed that for our model represented by the Hamiltonian of equation (5.1) the pointer states of the system turn out to become *time-dependent*, as opposed to the pointer states of some simpler models, which often are cited in the context of quantum information and quantum computation [8-15]. However, in most of the practical situations different noncommutable perturbations may exist in the total Hamiltonian of a realistic system-environment model, which would result in having time-dependent pointer states for the system [1]. Indeed, the authors believe that the fact that the pointer states of a system generally are time-dependent and may evolve with time has not been seriously acknowledged in the context of quantum computation and quantum

information. Specifically, in the context of quantum error correction [11, 12] it is often assumed that the premeasurement by the environment does not change the initial pointer states of the system. In other words, quantum "nondemolition" premeasurement by the environment often is assumed [11, 12]; as is also assumed in the Von Neumann scheme of measurement [16, 7]. Also, in the context of Decoherence-Free-Subspaces (DFS) theory the models which often are studied either contain self-Hamiltonians for the system which commute with the interaction between the system and the environment, or it is assumed that we are in the *quantum measurement limit* [5] or in the *quantum limit of decoherence* [8, 13, 14, 15]. However, all of these assumptions are in fact a simplification of the problem; since, as we discussed in chapter 3, they completely exclude the possibility of having pointer states for the system which may depend on time.

Using the time-evolution operator obtained in section 2, we also obtained a closed form for the elements of the reduced density matrix of the system, and studied the decoherence of the central system in our model. We showed that for the case that the system initially is not prepared in one of its pointer states and in the basis of the initial pointer states of the system (i.e. the $|\pm(t_0)\rangle$ states), the short time ($t \ll \omega_0^{-1}$) evolution of the offdiagonal elements of the reduced density matrix of the system will demonstrate decoherence, with a decoherence factor given by $e^{-(g\chi t)^2/2\alpha_\circ}$; and a decoherence time which is inversely proportional to the square root of the mass of field particles.

It will be interesting to generalize this study to the case that the environment is not merely represented by a single-mode bosonic field; and consider some classes of spectral densities for the environment. Also, for the model represented by the Hamiltonian of equation (5.1) at least in principle one should be able to obtain the pointer states of the system and the environment in some other regimes of the parameter space.

[5] In the *quantum measurement limit* the interaction between the system and the environment is so strong as to dominate the evolution of the system $\hat{H} \approx \hat{H}_{\rm int}$. Also in the *quantum limit of decoherence* the Hamiltonian for the system almost dominates the interaction between the system and the environment as well as the self-Hamiltonian of the environment $\hat{H} \approx \hat{H}_{\mathcal{S}}$.

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Chapter 6

On Born's rule, $p_k = |\psi_k|^2$, for quantum probabilities

6.1 Introduction

Born's Rule [1] is one of the main postulates of quantum mechanics which provides the relation between deterministic quantum mechanics and the probabilistic nature of measurements. It will be valuable to understand the origin of (or possibly derive) this fundamental postulate of quantum mechanics which has never been violated by experiments. One can state it as follows [2]: suppose that we measure an observable represented by a Hermitian operator $\hat{O}$ with eigenstates $\{|o_i\rangle\}$ and corresponding eigenvalues $\{o_i\}$ (which are assumed to be discrete) on a system described by the state vector $|\psi\rangle$, which in the basis of the eigenstates $\{|o_i\rangle\}$ can be represented by $|\psi\rangle = \sum_i c_i|o_i\rangle$. Then, the probability for the measurement to yield the value o_i is given by $|c_i|^2$.

In 1957, Gleason published a paper [3] entitled "Measures on the closed subspaces of a Hilbert space", which could provide a motivation for Born's rule. However, as a purely mathematical theorem it could not provide much insight into the physics of Born's rule. Only many years later and following the recent insights gained in the context of the

quantum theory of measurement and decoherence, have there been attempts to provide a physical derivation of Born's rule during the last decade. To be mentioned, Deutsch [4] claimed a derivation from "the nonprobabilistic part of classical decision theory" and the "the nonprobabilistic axioms of quantum theory". Also, Zurek has proposed a derivation of Born's rule, based on an idea termed *environment-assisted invariance* or *envariance* [5, 6]. Zurek's derivation is based on a theorem which states that for an entangled global state of the system and the environment none of the measurable properties of the system, including probabilities of various outcomes, can depend on the phases of the Schmidt coefficients for the combined state of the system and the environment. Zurek has also argued that this theorem assures causality [5]. Zurek's approach was later analyzed by Schlosshauer and Fine [7]. Barnum [8] also analyzed and proposed a modified version of Zurek's derivation of Born's rule which is also based on his theorem of "envariance".

Recently, Brumer [9] showed that the quantum mechanical Born's rule has a well-defined purely classical limit and arises naturally from the Hilbert space formulation of classical mechanics. The recognition that Born's rule is not uniquely an element of quantum mechanics, as Brumer has mentioned, suggests that the origin of Born's rule should be independent of the subtle purely quantum details of quantum measurement process such as the origin of probabilities and the mechanism by which definite outcomes are realized.

This chapter is organized as follows: in section 2 after a brief review of Zurek's argument, we will show what is logically inconsistent in his proof of the "envariance" theorem. We will also show that one cannot assume the equality of probabilities in the finest structure of the global state of the system and the environment; as is assumed by Zurek in his proof of Born's rule. Also we will point out that a preferred basis of measurement does not necessarily exist in any arbitrary regime and specific conditions must be satisfied in order to be able to write the global state of the system and the environment in a diagonal form with time-independent pointer states for the system. In section 3 we discuss why indeed we would expect the phases of the expansion coefficients (*just* in the preferred basis of measurement) to be unimportant in determining the probabilities of finding the system in any of its possible quantized states. Finally, we present our proof of Born's rule in section

4 which is followed by a summary and conclusion in section 5.

6.2 Some Comments on Zurek's Proof

Considering the Von Neumann scheme of measurement [2], Zurek imagines a system S entangled with a dynamically decoupled environment $\mathcal{E}$ [1]. After premeasurement [2, 10] is complete we can present the global state of the system and the environment in a diagonal form given by

$$|\psi_{S\mathcal{E}}\rangle = \sum_{k=1}^{N} \alpha_k |s_k\rangle|\varepsilon_k\rangle. \tag{6.1}$$

The states appearing in the diagonal state of the total composite system in the above equation are the pointer states of the system and the environment and premeasurement by the environment just refers to the formation of a one-to-one correspondence between the pointer states of the system and those of the environment. So, premeasurement makes the state of the total composite system in a diagonal form; as is assumed in the Von Neumann scheme of measurement and as is required in order to have a "faithful measurement". However, here we just note that, as we will discuss, the pointer states which appear on the diagonal state of the total composite system in equation (6.1) generally are time-dependent. As a result, unlike what is assumed by Zurek, the preferred basis of measurement is not determined at this stage and the most that one can assume about the global state of the system and the environment at this stage is just a diagonal form with pointer states which generally are time-dependent. (Here, note that we differentiate between pointer states and the preferred basis of measurement. As we described in chapter 3, pointer state of a subsystem are characterized by their ability not to entangle with the states of another subsystem and generally are time-dependent; while the preferred basis of measurement is made up of time-*independent* pointer states which can be realized only in certain regimes [10].)

In this section we will present a physical example, represented by the evolution of the

[1] In this dissertation whenever we talk about the environment it refers to all the subsystems which are outside our system of interest and hence it may include the apparatus as well.

two-level atom in the Jaynes-Cummings model of quantum optics, in order to show how in practice a diagonal form with time-dependent pointer states can be created as a result of the natural evolution of the global state of the system and the environment. Generally, after this early step of measurement the pointer states of the system do not necessarily form a *basis* for the Hilbert space of the system (although decoherence by the environment may make the bipartite pointer states of the global system mutually orthogonal within an often fairly short amount of time); and as we described in detail in chapter 3, we should distinguish between the set of (time-dependent) pointer states in equation (6.1) and the preferred basis of measurement; mainly because of the fact that the pointer states which appear in equation (6.1) generally are time-dependent and the preferred basis of measurement is not yet determined at this stage. This in fact, raises another issue with Zurek's approach in his proof of Born's rule. However, for now we just discuss two other issues with his proof in sections 6.2.1 and 6.2.2; and in section 6.2.3 we will come back to the question of the preferred basis of measurement.

6.2.1 Envariance (environment-assisted invariance)

Let us first have a brief review of Zurek's argument about "envariance" [6]. Zurek defines an idea termed envariance (environment-assisted invariance) as follows:
When a global state $|\psi_{S\mathcal{E}}\rangle$ of the system and the environment can be transformed by $U_S = u_S \otimes \mathbf{1}_{\mathcal{E}}$ acting solely on S,

$$U_S|\psi_{S\mathcal{E}}\rangle = (u_S \otimes \mathbf{1}_{\mathcal{E}})|\psi_{S\mathcal{E}}\rangle = |\eta_{S\mathcal{E}}\rangle, \tag{6.2}$$

but the effect of U_S can be undone by acting solely on $\mathcal{E}$ with an appropriately chosen $U_{\mathcal{E}} = u_{\mathcal{E}} \otimes \mathbf{1}_S$:

$$U_{\mathcal{E}}|\eta_{S\mathcal{E}}\rangle = (u_{\mathcal{E}} \otimes \mathbf{1}_S)|\eta_{S\mathcal{E}}\rangle = |\psi_{S\mathcal{E}}\rangle; \tag{6.3}$$

then, $|\psi_{S\mathcal{E}}\rangle$ is called *envariant* under U_S. The following lemma is then presented [6].

Lemma. Any unitary transformation that is diagonal in the basis of the Schmidt states of the system (i.e. codiagonal with Schmidt eigenstates $\{|s_k\rangle\}$):

$$u_S = \sum_{k=1}^{N} \mathrm{e}^{-\mathrm{i}\phi_k}|s_k\rangle\langle s_k|, \tag{6.4}$$

is envariant.

Proof. It is easy to verify that the above transformation can be undone by the following "counter-transformation":

$$u_{\mathcal{E}} = \sum_{k=1}^{N} \mathrm{e}^{\mathrm{i}(\phi_k + 2\pi l_k)} |\varepsilon_k\rangle\langle\varepsilon_k|, \tag{6.5}$$

where l_k are arbitrary integers. Hence u_S is envariant. Q.E.D.

Note that by accepting the above proof, we have implicitly accepted that $u_{\mathcal{E}}$, given by (5), does not have any effect on the state of the system. As we will discuss, this is the source of the problem in Zurek's argument. He connects this implicit assumption to the following "fact":

Fact 1. Unitary transformations must act on the *system* to alter its state. (That is, when the evolution operator does not operate on the Hilbert space of the system; i.e., when it has a form $...\otimes\mathbf{1}_S\otimes...$ the state of the system remains the same).

Then, in order to prove that probabilities are independent of the phases of the Schmidt coefficients, two more facts are listed:

Fact 2. The state of the system S is all that is needed (and all that is available) to predict measurement outcomes, including their probabilities.

Fact 3. The state of a larger composite system that includes S as a subsystem is all that is needed (and all that is available) to determine the state of the system S.

With this preview the following theorem is presented by Zurek:

Theorem 1. For an entangled global state of the system and the environment no measurable properties of S, including probabilities of various outcomes, can depend on the phases of Schmidt coefficients.

Zurek's Proof. By definition of envariance and by the lemma that was already proved the effect of u_S (given by equation (6.4)) can be undone by a countertransformation of the form $\mathbf{1}_S\otimes u_{\mathcal{E}}$ (given by equation (6.5)) which (by fact 1) cannot alter the state of S. As $S\mathcal{E}$ is returned to the initial state, it follows from fact 3 that the state of S must have been restored. But (by fact 1) the system could not have been affected by the countertransformation. So the system must have been left unchanged by the envariant transformation u_S in the first place. It follows (from the above and fact 2) that measurable properties of S are unaffected

by envariant transformation u_S which can only change the phases of Schmidt coefficients. Hence, all measurable properties of S implied by its state must indeed be independent of the phases of Schmidt coefficients. Q.E.D.

Discussion: There is a delicate aspect of *Fact* 1 that must be noted. First, obviously by acting on the global state of the system and the environment (equation (6.1)), $u_{\mathcal{E}}$ (given by (5)) can change the phases of the Schmidt coefficients; although it has no effect on the basis states of the system $\{|s_k\rangle\}$'s. So, if the phases of the Schmidt coefficients have any effect on the final result of a measurement on the system, then we can conclude that $u_{\mathcal{E}}$ has affected the final (recorded) state of the system (Here we are using a *reductio ad absurdum*). Hence, we cannot assume that the operation of $u_{\mathcal{E}}$ on the global state of the system and the environment has no effect on the final (recorded) state of the system from the beginning; since this is equivalent to assuming that the phases of the Schmidt coefficients do not affect the result of a measurement on the state of the system; and we cannot have the result of a theorem as one of its assumptions. Again, we emphasize that in this case we can make this conclusion, although $u_{\mathcal{E}}$ has no effect on the basis states of the system $\{|s_k\rangle\}$'s. In other words, $u_{\mathcal{E}}$ has no direct effect on the system; however, it may have an *indirect* effect on the final state of the system, *if* the phases of the Schmidt coefficients can affect the final state of the system. This can be explained by arguing that $u_{\mathcal{E}}$ can affect the basis states of the environment; and the states of the environment may affect the states of the system, because of the perfect coupling that is created between the states of the system and the environment due to the premeasurement[2]. Therefore, we cannot exclude the possibility that $u_{\mathcal{E}}$ might be able to affect the states of the system from the very beginning.

[2]In fact, after the coupling of the states of the system with those of the environment, which leads to the diagonal form of equation (6.1), the subsystems S and $\mathcal{E}$ cannot be attributed quantum states of their own; instead they can only be described by the state of the global composite system. In other words, due to the entanglement, they lose their individuality. Similarly, the phases of the Schmidt coefficients cannot be attributed to the system or to the environment alone. In fact, the only thing that we can say about the phases of the Schmidt coefficients at this step is that they are a shared property between the states of the system and the environment. So, when $u_{\mathcal{E}}$ operates on the global state, given by equation (6.1), although it obviously has no effect on the basis states of the system $\{|s_k\rangle\}$'s, stating that it has no effect on the state of the system is meaningless at this step.

Zurek has also argued that this theorem assures causality [5]. Because, by causality, the state of the system must be independent of any operations carried out on a dynamically decoupled environment. Hence, if the phases of the Schmidt coefficients could be detected by a measurement on the system alone (while they can be transformed by an operation $u_{\mathcal{E}}$ on the Hilbert space of the environment), faster than light communication could be possible. In response to the causality argument also, we extend the following argument: Suppose that the phases of Schmidt coefficients for the combined state of the system and the environment do affect the probabilities of finding the system in any of its possible quantized states, then knowing that $u_{\mathcal{E}}$, given by (5), can change the phases of Schmidt coefficients and the phases of Schmidt coefficients do affect the state of the system, we can conclude that $u_{\mathcal{E}}$ not only is a transformation acting on the state of the environment, but also it does affect the state of the system. Therefore, it is not only a transformation acting on the state of the environment and there is no faster than light communication and no breach of causality if the effect of $u_{\mathcal{E}}$ (changing the phases of Schmidt coefficients) can be detected (or undone) by a measurement on the system. Indeed, it is definitely true that the state of the system must be independent of any operation carried out on a dynamically decoupled environment alone and this assures causality; however, as we discussed, we cannot assume that $u_{\mathcal{E}}$ operates *only* on the states of the environment before knowing whether the phases of the Schmidt coefficients have any effect on the final (recorded) state of the system or not.

As we will discuss in section 3, the phases of the expansion coefficients in the preferred basis of measurement indeed are unimportant in determining the probabilities of various outcomes. But, as we discussed, this is not because of the fact that the effect of a phase transformation on the Hilbert space of the system can be undone by a "counter-transformation" on the Hilbert space of the environment, as Zurek has proposed. Moreover, the Schmidt states are not necessarily the same as the preferred basis of measurement [10, 11] and as we will discuss, the unimportance of the phases of the expansion coefficients is just in the preferred basis of measurement. This means that the phases of the Schmidt coefficients are unimportant in determining the probabilities *only* in case that the Schmidt states coincide with the preferred basis of measurement. We will discuss the physical origins of the unim-

portance of the phases of the expansion coefficients in the preferred basis of measurement in section 3. For now, let us just move on to another question that will be raised after carefully examining Zurek's proof of Born's rule.

6.2.2 Zurek's "fine-graining"

The second part of Zurek's proof assumes that "under certain conditions probabilities of a subset of states of the system are equal" amongst themselves. In his paper [6], Zurek presents a proof for this assumption using the idea of "envariance under swaps" and also by assuming that the orthogonal states of the outer environment "are associated with subspaces of sufficient dimensionality so that a fine-graining" with coefficients which make the absolute value of all the Schmidt coefficients equal is possible [6] (this will be described in the next paragraphs). However, as we will show by an example, this assumption is not always true.

Consider the state of the two-level atom in the Jaynes-Cummings model of quantum optics which involves a two-level atom, with upper and lower levels that can respectively be represented by $|a\rangle$ and $|b\rangle$, interacting with a single-mode quantized electromagnetic field inside an ideal cavity, represented by creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$. For exact resonance and in the rotating-wave approximation, the interaction Hamiltonian for the composite system can be written as

$$H_I = \hbar g(\hat{a}^\dagger \sigma_- + \sigma_+ \hat{a}). \tag{6.6}$$

Where $g = -\varrho_{12}.\hat{\epsilon}\sqrt{\frac{w}{2\hbar\varepsilon_\circ V}}$ is the atom-field coupling constant, with $\varrho_{12} = e\langle a|\mathbf{r}|b\rangle$ as the atomic electric-dipole transition matrix element. ($\hat{\epsilon}$ is the field polarization vector, ω is the atomic transition frequency which is taken to be resonant with the frequency of the cavity eigenmode, and V is the cavity mode volume). Also σ_+ and σ_- are the atomic flipping operators given by

$$\sigma_+ = |a\rangle\langle b| \qquad \text{and} \qquad \sigma_- = |b\rangle\langle a|. \tag{6.7}$$

Consider the field to be initially in the coherent state $|\nu\rangle$

$$|\Phi_{\text{field}}(t_0)\rangle = |\nu\rangle = \sum_{n=0}^{\infty} c_n |n\rangle; \quad \text{with} \quad c_n = \frac{e^{-\frac{1}{2}|\nu|^2}\nu^n}{\sqrt{n!}}, \tag{6.8}$$

where $|\nu|^2 = \bar{n}$ is the average number of photons in the coherent state, and $\nu = |\nu|e^{-i\phi}$. In general, the exact solution for an initial atomic state $|\psi_{\rm atom}(t_0)\rangle = \alpha|a\rangle + \beta|b\rangle$ and a field state initially prepared in the coherent state, is a highly entangled state of the field and the atom [12]. However, Gea-Banacloche [13, 14] has shown that for a large average number of photons, if we consider the evolution of the initial atomic states $|+\rangle$ and $|-\rangle$, defined by

$$|\pm\rangle = \frac{1}{\sqrt{2}}(\mathrm{e}^{-\mathrm{i}\phi}|a\rangle \pm |b\rangle) \tag{6.9}$$

(here ϕ is the same as the phase of $\nu = |\nu|e^{-i\phi}$), the evolution of the global state of the system and the field (the environment) would be very interesting. Gea-Banacloche proved that when the initial atom-field state is $|\pm\rangle|\nu\rangle$, in the limit of a very large $\bar{n}$ the global state of the two-level atom (2LA) and the field will evolve as follows:

$$|\pm\rangle|\nu\rangle|_{t=0} \rightarrow \frac{1}{\sqrt{2}}(\mathrm{e}^{-\mathrm{i}\phi}\mathrm{e}^{\mp\mathrm{i}gt/(2\sqrt{\bar{n}})}|a\rangle \pm |b\rangle) \times |\Phi_\pm(t)\rangle, \tag{6.10}$$

where

$$|\Phi_\pm(t)\rangle = \mathrm{e}^{-\bar{n}/2}\sum_{n=0}^{\infty}\frac{\bar{n}^{n/2}}{\sqrt{n!}}\mathrm{e}^{-\mathrm{i}n\phi}\mathrm{e}^{\mp\mathrm{i}gt\sqrt{n}}|n\rangle, \tag{6.11}$$

gives us the time evolution of the state of the field. This result holds for any time, provided that t goes to infinity slowly enough to have $t/\bar{n} \rightarrow 0$. Since the time scale for the JCM revivals is $t_{\rm R} = 2\pi\sqrt{\bar{n}}/g$ [12], $t_{\rm R}/\bar{n} \rightarrow 0$ as $\bar{n} \rightarrow \infty$ (a typical value for $g/2\pi$ is 44 kHz in a micromaser experiment [13]). Hence, the approximate solution in equation (6.10) holds accurately over a large number of revivals, as long as $\bar{n}$ is large enough.

The states $|+\rangle$ and $|-\rangle$ form a basis set for the two-level atom (2LA); therefore, the evolution of any other initial atomic state with an initial coherent field can be expressed as a linear combination of the evolution of $|+\rangle|\nu\rangle$ and $|-\rangle|\nu\rangle$.

$$\begin{aligned}(\gamma|+\rangle + \delta|-\rangle)\ |\nu\rangle|_{t=0} \rightarrow \gamma\ |+(t)\rangle\ |\Phi_+(t)\rangle + \delta\ |-(t)\rangle\ |\Phi_-(t)\rangle \\ \text{with} \qquad |+(t)\rangle = \frac{\mathrm{e}^{-\mathrm{i}\phi}\mathrm{e}^{-\mathrm{i}gt/(2\sqrt{\bar{n}})}|a\rangle + |b\rangle}{\sqrt{2}} \\ \text{and} \qquad |-(t)\rangle = \frac{\mathrm{e}^{-\mathrm{i}\phi}\mathrm{e}^{+\mathrm{i}gt/(2\sqrt{\bar{n}})}|a\rangle - |b\rangle}{\sqrt{2}}.\end{aligned} \tag{6.12}$$

The time-dependent states $|\pm(t)\rangle$ and $|\Phi_\pm(t)\rangle$ appearing in the above equations are *the pointer states of the system (the 2LA) and the environment (the field) which are characterized by their ability not to entangle with each other.* As we observe, in the limit of large

$\bar{n}$ in which equation (6.10) is valid, the global state of the 2LA and the field remains as a product state if the atom is initially prepared in one of the $|\pm\rangle$ states. This means that, one can at all times assign a well-defined pure state to the atom initially prepared in one of the $|\pm\rangle$ states and clearly, no other initial atomic states have this characteristic, as is obvious from equation (6.12). In other words, for an initial coherent field and in the limit of large $\bar{n}$, when the 2LA is initially prepared in one of the two states $|\pm\rangle$, the field and the atom never entangle; while equations (6.10) and (6.12) indicate that for any initial atomic state other than the $|\pm\rangle$ states, the states of the field and the atom will not remain separated and they entangle throughout the interaction. Also, equation (6.12) indicates that for an arbitrary initial atomic state and in the limit of large $\bar{n}$, there is always a one-to-one correspondence between a (preferred) set of pointer states from the system (the 2LA) and some corresponding states of a field which is initially prepared in a coherent state. In appendix C we have shown that in the limit of $\bar{n} \to \infty$, which corresponds to the classical limit for which equations (6.10) to (6.12) are valid, the field states $|\Phi_+(t)\rangle$ and $|\Phi_-(t)\rangle$ almost promptly become orthogonal. However, as is obvious from equation (6.12), the pointer states of the system are not orthogonal at all times; and hence the diagonal state of the total composite system (represented by equation (6.12)), which is created after premeasurement, cannot represent a Schmidt decomposition at all times; as by definition the Schmidt states of the system and the environment must be orthogonal amongst themselves.

In essence, we observe that in the limit of large $\bar{n}$ and for an initial coherent field, in fact the coherent field does a Von Neumann premeasurement on the state of the 2LA (which makes the global state in a diagonal form) and moreover it superselects a preferred set of pointer states for the system, which are characterized by their ability not to entangle with the states of the environment. (However, note that the premeasurement by the field is not necessarily an ideal premeasurement, as the initial atomic states $|\pm\rangle$ slightly change by acquiring a phase factor $\mathrm{e}^{\mp \mathrm{i} g t/(2\sqrt{\bar{n}})}$; except for $t \ll t_{\mathrm{R}}$, for which this change is negligible and the right hand side of equation (6.12) can be approximated by $\gamma\,|+\rangle|\Phi_+(t)\rangle + \delta\,|-\rangle|\Phi_-(t)\rangle)$.

Coming back to Zurek's derivation of Born's rule [5, 6], let us follow the second part of his proof and consider the "fine-graining" which he suggests in order to convert the

entangled total state of the system and the environment, having the form of equation (6.12), with unequal coefficients into an entangled state with equal coefficients. For our physical example of the JCM with an initial coherent field and in the limit of fairly large average number of photons, let us use equation (6.11) for fine-graining. Hence, we can rewrite equation (6.12) as follows:

$$\begin{aligned}(\gamma|+\rangle + \delta|-\rangle)|\nu\rangle \rightarrow \gamma|+(t)\rangle|\Phi_+(t)\rangle + \delta|-(t)\rangle|\Phi_-(t)\rangle \\ \simeq \gamma \mathrm{e}^{-\bar{n}/2}\sum_{n=0}^{\infty}\frac{\bar{n}^{n/2}}{\sqrt{n!}}\mathrm{e}^{-\mathrm{i}n\phi}\mathrm{e}^{-\mathrm{i}gt\sqrt{n}}|n\rangle|+(t)\rangle \\ +\delta \mathrm{e}^{-\bar{n}/2}\sum_{n=0}^{\infty}\frac{\bar{n}^{n/2}}{\sqrt{n!}}\mathrm{e}^{-\mathrm{i}n\phi}\mathrm{e}^{\mathrm{i}gt\sqrt{n}}|n\rangle|-(t)\rangle. \qquad (6.13)\end{aligned}$$

Here, obviously the coefficients for each of the substates $|\pm(t), n\rangle$ depend on n and therefore, they are not equal; even disregarding the phases (and even if $|\gamma| = |\delta|$). However, here the main question is that whether it is 1) possible and 2) justified to write the environment states $|\Phi_\pm(t)\rangle$ in a basis set which makes all the expansion coefficients for the global state of the system and the environment equal. i.e. $|\Phi_+(t)\rangle = \sum_n \alpha_n|e_n\rangle$ and $|\Phi_-(t)\rangle = \sum_n \beta_n|e_n\rangle$, so that $\gamma\alpha_n = \delta\beta_n$ for all n.

From a mathematical point of view, if the orthogonal states of the environment are associated with subspaces of sufficient dimensionality, such a fine-graining is possible. However, there is a problem with such a resolution when thinking more exactly about the *physics* of this problem. First of all and most importantly, we note that the states of the environment can further couple with the states of an outer environment $\mathcal{C}$ which can be thought of as an apparatus, or simply another environment which surrounds both of our system of interest and its immediate environment $\mathcal{E}$. As a result, we can think of a "Von Neumann chain" of correlations [2] and instead of equation (6.13) we may write $(\gamma|+\rangle + \delta|-\rangle)|\nu\rangle|C_0\rangle \rightarrow (\gamma\ |+(t)\rangle|\Phi_+(t)\rangle + \delta\ |-(t)\rangle|\Phi_-(t)\rangle)|C_0\rangle$; with $|C_0\rangle$ as the initial state of $\mathcal{C}$. But, as we mentioned, the global state of the system and the environment can further interact with $\mathcal{C}$ in order to create the global state $|\psi_{S\mathcal{C}\mathcal{E}}\rangle = \gamma|+(t)\rangle\sum_n \alpha_n|e_n\rangle|c_n\rangle + \delta|-(t)\rangle\sum_n \beta_n|e_n\rangle|c_n\rangle$; where $\{|e_n\rangle\}$ make the pointer states of the environment which are determined through the interaction between the environment $\mathcal{E}$ and $\mathcal{C}$. As a result, we observe that because a *chain* of correlations is formed in practice, which determines the pointer states for the

environment, we are not free in choosing the basis set for the states of the environment $|\Phi_\pm(t)\rangle$ and a further resolution of the states of the environment is constrained with and determined by the nature of its interaction with an outer environment.

Also, note that in order to have expansion coefficients for the global state of the system and the environment which are all equal to each other, we must have $\gamma\alpha_n = \delta\beta_n$ or $\alpha_n/\beta_n = \delta/\gamma$ for all n. But this means that the expansion coefficients for the pointer states $|\Phi_\pm(t)\rangle$ of the environment, i.e. $\{\alpha_n\}$ and $\{\beta_n\}$, must depend on the initial state of the system via the ratio δ/γ. In other words, the fine-graining which creates equal expansion coefficients for the global state of the system and the environment must depend on the initial state of the system. But we do know that the (fine-structure) pointer states of the environment are determined by the nature of its interaction with the outer environment $\mathcal{C}$ and are independent from the initial state of the system. Hence, we can conclude that the mathematical fine-graining which can create equal expansion coefficients for the global state of the system and the environment generally, is *not* the same as the physical, real fine-graining that determines the pointer states of the environment through its interaction with the outer environment $\mathcal{C}$.

6.2.3 The pointer states of measurement

Obviously we cannot obtain the probability of measurement of a specific state if we cannot determine the preferred basis of measurement; and when we write the global state of the system and the environment in a diagonal form such as that of equation (6.1), in general the states of the system which appear on the diagonal can only represent the instantaneous pointer states of the system [3] and hence, generally we do not know if they can represent the preferred basis of measurement or not.

In chapter 3 we discussed the time-dependent pointer states of measurement and their

[3] Pointer states of a system are characterized by their ability not to entangle with the states of the environment (i.e. the requirement of faithful measurement) and appear in the diagonal state of the total composite system after premeasurement by the environment. In other words, the pointer states of the system emerge dynamically as those states that are the least sensitive, or the most robust, to the interaction with the environment; in the sense that they do not entangle with the environment.

exact distinction from the Schmidt states and also from the preferred basis of measurement [4]. There we proved that the pointer states, which appear in the diagonal state of the total composite system in equation (6.1), generally are time-dependent and thus specific conditions must be satisfied in order to be able to write the global state of the system and the environment in a diagonal form with time-*independent* pointer states for the system. We also studied the conditions under which the pointer states of measurement can be independent of time (so that a preferred set of basis states can exist as the basis of measurement) and predicted the preferred basis of measurement in each of the corresponding set of conditions. Here in this chapter we also presented the physical example of the Jaynes-Cummings model of quantum optics which basically shows how in practice a diagonal form such as that of equation (6.1), although with time-dependent pointer states, can be created as a result of the natural evolution of the global composite system.

In essence, a preferred basis of measurement does not exist in every arbitrary regime; as pointer states generally are time-dependent and Zurek's proof of Born's rule lacks an explicit assumption of being in certain regimes which would ensure us having time-independent pointer states, so that we can make sure that a preferred basis of measurement does exist and the states of the system in the diagonal state of the total composite system (equation (6.1)) do represent the preferred basis of measurement.

The above points must be seriously considered in any attempt to prove the Born rule for quantum probabilities and in fact they are some serious issues with Zurek's proof of Born's rule.

[4] As we discussed in chapter 3, generally we should distinguish between the set of pointer states in equation (6.1) and the preferred basis of measurement; mainly because of the fact that the pointer states of a subsystem generally are time-dependent and a preferred basis of measurement does not exist, unless under specific conditions which the pointer states of measurement become time-independent. Moreover, the pointer states of a system necessarily are not orthonormal amongst themselves at all times. Therefore, they cannot necessarily form a basis for the Hilbert space of the system at all times; and generally they cannot be obtained simply by diagonalizing the instantaneous density matrix of the system. In other words, by the end of premeasurement by the environment the global state of the system and the environment necessarily is not in a Schmidt decomposition form; as the time-dependent pointer states of the system and the environment generally are not orthonormal amongst themselves.

6.3 Unimportance of the Phases of the expansion Coefficients

In this section we discuss the conditions under which we can expect the phases of the expansion coefficients not to be important in determining the probabilities.

First of all we note that since the pointer states of the system and the environment in the state of the total composite system (equation (6.1)) generally can evolve with time, the phases of the expansion coefficients in equation (6.1) necessarily are not invariant; as they can evolve with the evolution of the pointer states. However, as we will discuss, under the specific conditions that the pointer states of the system become time-independent (so that they can represent the preferred basis of measurement) the phases of the expansion coefficients in equation (6.1) will be unimportant in determining the probabilities; regardless of the fact that they are not necessarily invariant.

Consider a typical Stern-Gerlach experiment with a magnetic field in the x-direction; so that the set up can measure the x-component of the spin of atoms. From our experience we know that there is a probability of $\frac{1}{2}$ for a filtered beam of atoms in the $|S_x;+\rangle$ state to be found in any of the $|\pm\rangle$ states (where $|\pm\rangle$ are the eigenstates of the $\hat{S}_z$ operator) in a succeeding experiment to measure the z-component of the spin of the atoms. However, this probability would be the same no matter if we use atoms in the $|S_x;+\rangle$ state or in the $|S_x;-\rangle$ state or in an eigenstate of *any* other spin operator corresponding to the plane which is perpendicular to the z-direction; for example the $|S_y;+\rangle$ and $|S_y;-\rangle$ states as well. Moreover, all these states can be expressed in terms of the complete basis set of the eigenstates of $\hat{S}_z$, represented by $|\pm\rangle$, as follows:

$$|S_\varphi;\pm\rangle = |\alpha_\pm(\varphi)|\ |+\rangle + |\beta_\pm(\varphi)|\ \mathrm{e}^{\mathrm{i}\theta_\pm(\varphi)}|-\rangle; \tag{6.14}$$

where in the above equation $|S_\varphi;\pm\rangle$ denote the pair of orthogonal eigenstates of an operator $\hat{S}_\varphi$ which can be assumed as the spinor states which are obtained by rotating the $|S_x;+\rangle$

state about the z-axis by an angle φ or $(\varphi - \pi)$ respectively[5]. In the above equation we have assumed that an overall phase is not important in determining the state of a spinor and hence we should only worry about the *relative* phase of the expansion coefficients. Also, one can easily verify that in order to satisfy the orthogonality of the pairs of states $|S_\varphi; \pm\rangle$ we must have

$$\theta_+(\varphi) - \theta_-(\varphi) = \pi \quad \text{and} \quad |\alpha_{+\varphi}\ \alpha_{-\varphi}| = |\beta_{+\varphi}\ \beta_{-\varphi}|. \tag{6.15}$$

Note that, although the phase difference $\theta_+(\varphi) - \theta_-(\varphi)$ is fixed by the requirement of orthogonality of the pair of states $|S_\varphi; \pm\rangle$, $\theta_\pm(\varphi)$ are important in order to uniquely determine the state of a spinor. For example, this phase determines whether we are dealing with the $|S_y; \pm\rangle$ states, or any other eigenstate of a spin operator corresponding to the plane which is perpendicular to the z-direction[6]. Moreover, as was previously mentioned, from our experience we know that for a succeeding spin-z measurement this phase is not important in determining the probability of finding the system in any of the $|\pm\rangle$ states. This can be understood by considering the symmetry which we have about the z-axis in this set up. In fact, for a succeeding spin-z measurement there is absolutely no reason to discriminate among any of the $|S_\varphi; \pm\rangle$ states and this is simply a result of the space symmetry. In this example phases are important in order to determine the state of the system; for example in order to determine whether the system is in the $|S_x; +\rangle$ state or in the $|S_x; -\rangle$ state; and hence they can be important for determining the probability of a spin measurement say in the x-direction. So, the phases of the expansion coefficients are not important in determining the probabilities *only* when the preferred basis of measurement is specified with certainty and these phases are the phases of the expansion coefficients in the preferred basis of measurement.

The same kind of discussion holds valid if the basis states of the system in equation (6.14) are coupled with some states from an outer environment or the measuring appara-

[5]From the Born rule we know that in equation (6.14) we must have $|\alpha_\pm(\varphi)| = |\beta_\pm(\varphi)| = \frac{1}{\sqrt{2}}$. However, here we do not make any use of this fact; as we are not allowed to assume the born rule in our assumptions. Moreover, we do not yet know if $|\alpha_\pm(\varphi)|$ and $|\beta_\pm(\varphi)|$ must be equal or not.

[6]However, first we should assign a value to $\theta_\pm(0)$ by some convention. Usually by convention $\theta_\pm(0)$ of the $|S_x; \pm\rangle$ states are set to be equal to 0 and π respectively.

tus; i.e. when we are dealing with bipartite states such as those of equation (6.1). (Since there will be a perfect coupling between the pointer states of the system and those of the environment after premeasurement by the environment; in the sense that the one-to-one correspondence among the states of the system and those of the environment is preserved for the pointer states of the two subsystems). Hence, it seems that no matter by what kind of mechanism and how the preferred basis of measurement is determined, as soon as the preferred basis of measurement is determined through the interaction between the system and the environment, a symmetry is introduced to the evolution of the state of the system, which causes the phases of the expansion coefficients to be unimportant. In our example of a SG-z experiment for example, for a magnetic field in the z-direction, which would filter the atoms based on their spin in the z-direction, all of the states of the system in equation (6.14), which can be obtained with different values of the phase φ, are practically identical; *as a rotation around the z-axis should not change anything about the system.* This symmetry can be represented by the commutativity of the phase transformation operator of the system (equation (6.4)) in the basis that we are using to write the global state of the total composite system[7] and the observable which represents the preferred basis of measurement.

Also, in our example of the Stern-Gerlach experiment, suppose that the preferred basis of measurement is known to be made up from the $|S_x;\pm\rangle$ states rather than the $|\pm\rangle$ states. As we mentioned before, in this case clearly the phases of the expansion coefficients in equation (6.14) are important for a SG-x experiment. However, this can also be represented by the non-commutativity of the phase transformation in the basis set of the $|\pm\rangle$ states (which basically can change the phase φ of the state of the system in equation (6.14)) and the operator $\hat{S}_x$ whose eigenstates represent the preferred basis of measurement in this case.

In fact, we just exploited the SG experiment as an *example*. However, regardless of the fact that whether *spin* is measured or any other observable of a two-state system, the above discussion can be carried on for any measurement on the state of a two-level system. The essence is that once a preferred direction in the Hilbert space of the system is determined, in the absence of a reason to break the symmetry of the problem around that

[7]For our example represented in equation (6.14) this basis was made up of the $|\pm\rangle$ states.

preferred direction, all directions perpendicular to the preferred direction of measurement are indifferent for a measurement on the pointer observable of the system; and this simply is a result of the space symmetry. Therefore, the phases of the expansion coefficients must be unimportant in determining the probabilities *just* in the preferred basis of measurement; as if one Schmidt decomposition for the global state of the system and the environment does not exactly coincide with the preferred basis of measurement, then the phases of the Schmidt coefficients can be important in determining the probabilities. This means that the phases of the *Schmidt coefficients* are not necessarily unimportant in determining the probabilities and one should emphasize that this unimportance of the phases of the expansion coefficients is just in the preferred basis of measurement. This is also in contrast to Zurek's so-called "envariance" theorem for unimportance of the phases of the expansion coefficients in the *Schmidt basis*; since, in those cases that the preferred basis of measurement necessarily does not coincide with the Schmidt decomposition of the global state of the system and the environment, Zurek's envariance argument which claims that the phases of the *Schmidt coefficients* must be unimportant in determining the probabilities can lead to the conclusion that the phases of the expansion coefficients in the *preferred basis* must be important in determining the probabilities[8]; a conclusion which doubtlessly is in contrast with our experiences.

6.4 Proof of Born's Rule

In this section we present our proof of Born's rule. Considering a two-level system S, entangled with an environment $\mathcal{E}$ we can rewrite equation (6.1) for the global state of the total composite system as follows:

$$|\psi_{S\mathcal{E}}\rangle = \gamma|s_0\rangle|\varepsilon_0\rangle + \delta|s_1\rangle|\varepsilon_1\rangle. \tag{6.16}$$

By presenting the example of the JCM of quantum optics in section (2) we already showed how in practice a diagonal form with time-dependent pointer states can be created as a

[8]Since a *phase* transformation in the Schmidt basis may change the *amplitudes* corresponding to decomposition of the same total state in another different basis.

result of the natural evolution of the total composite system. In fact, through the process of premeasurement by the environment a one-to-one correspondence is formed between some states from the system and some corresponding states from the environment (which are the pointer states of the system and the environment respectively) so that we can write the state of the total composite system in a diagonal form. (As we discussed in detail in chapter 3, such formation of a one-to-one correspondence between the pointer states of the system and the environment may require having a large environment.) Moreover, as we discussed these pointer states, which are characterized by their ability not to entangle with the states of the other subsystem, generally are time-dependent; and hence in general the states of the system, appearing in the diagonal state of the total composite system in the above equation, do not necessarily represent the preferred basis of measurement. However, as we proved in chapter 3, under specific conditions (which include the quantum measurement limit and the quantum limit of decoherence and some other specific conditions) the pointer states of measurement become time-independent and hence they can represent the preferred basis of measurement. Indeed, when representing the state of the total composite system by equation (6.16), here we explicitly assume that we are in one of the regimes which make the pointer states independent of time; so that we can assume that the states of the system in this equation represent the preferred basis of measurement.

Now, suppose that the environmental states $|\varepsilon_0\rangle$ and $|\varepsilon_1\rangle$ can be expressed in an orthonormal basis $\{|e_n\rangle\}$ as follows:

$$|\varepsilon_0\rangle = \sum_{n=0}^{\infty} c_n |e_n\rangle, \qquad \text{and} \qquad |\varepsilon_1\rangle = \sum_{n=0}^{\infty} d_n |e_n\rangle. \tag{6.17}$$

Hence,

$$|\psi_{S\mathcal{E}}\rangle = \gamma \sum_{n=0}^{\infty} c_n |s_0\rangle|e_n\rangle + \delta \sum_{n=0}^{\infty} d_n |s_1\rangle|e_n\rangle. \tag{6.18}$$

Here we prove that the probabilities for different states to occur are a general function of *only* the corresponding coefficients, denoted by $F(\text{coef})$; herein this function is to be determined. However, first we list the assumptions which will be used in this section in our proof of the Born rule:

1. In parallel with Zurek, we assume that "the state of a larger composite system that includes S as a subsystem is all that is needed to determine the state of the system".

2. We assume that the function F (which provides us with a rule for determination of probabilities from the states of the total composite system) is a universal function. In other words, we assume that probabilities can be obtained from a *unique* and well-defined function of the variables in the state of the total composite system.

3. We assume that sum of all probabilities is conserved: $\sum_i p_i = \text{constant}$.

4. We assume that the evolution of the state of the system and the environment is governed by Schrödinger's equation; and hence, the *norm* of the state of the total composite system is preserved. However, we do not require the evolution of each of the subsystems by themselves necessarily to be unitary. We will discuss this last assumption more elaborately in this section.

Theorem-I: For the global state of a system and the environment given by equation (6.16), $|\psi_{S\mathcal{E}}\rangle = \gamma|s_0\rangle|\varepsilon_0\rangle + \delta|s_1\rangle|\varepsilon_1\rangle$, the probability of finding the system in each of its possible states $|s_0\rangle$ and $|s_1\rangle$ is *only* a function of the absolute value of the corresponding coefficient. i.e.

$$P(|s_0\rangle) = F(|\gamma|) \qquad \text{and} \qquad P(|s_1\rangle) = F(|\delta|). \tag{6.19}$$

Proof. The first one of the above assumptions means that if we know the coefficients γ and δ with their phases, we must be able to predict the probability of finding the system in one of its pointer states. i.e.

$$P(|s_0\rangle) = F(\gamma, \delta). \tag{6.20}$$

Next, we note that because of the unitary evolution of states, given by Schrödinger's equation, we have

$$|\gamma|^2 + |\delta|^2 = 1 = \text{constant}. \tag{6.21}$$

Hence, if we know one of the coefficients we can find the magnitude of the other coefficient. In other words, we can know the other coefficient aside from the relative phase between the

two coefficients. Hence, Instead of equation (6.20) we can write

$$P(|s_0\rangle) = F(|\gamma|, \varphi), \tag{6.22}$$

where, φ is the relative phase between γ and δ. But we proved that the probabilities for various outcomes of the system are independent of the phases of the expansion coefficients in the preferred basis of measurement. Hence, the probability for finding the system in the state $|s_0\rangle$ is just a function of the coefficient γ

$$P(|s_0\rangle) = F(|\gamma|). \tag{6.23}$$

Q.E.D. The generalization of the above theorem for the case that the system has more than two states is also straightforward.

Hence, now for these probabilities we can write

$$F(|\gamma|) + F(|\delta|) = 1, \tag{6.24}$$

$$\sum_{n=0}^{\infty} F(|\gamma c_n|) + \sum_{n=0}^{\infty} F(|\delta d_n|) = 1; \tag{6.25}$$

where, for example, $F(|\gamma|)$ is the probability of finding the global system in the state $|s_0, \varepsilon_0\rangle$ and $F(|\gamma c_n|)$ is the probability of finding the global system in the substate $|s_0, e_n\rangle$.

The statement in theorem-I can also be applied for the possible states of the environment in equation (6.17). In other words: $P(|e_n\rangle \mid |\varepsilon_0\rangle) = F(|c_n|)$ and $P(|e_n\rangle \mid |\varepsilon_1\rangle) = F(|d_n|)$; where, $P(|e_n\rangle \mid |\varepsilon_{0(1)}\rangle)$ is the conditional probability for finding the environment in the state

$|e_n\rangle$, when it is already measured in the state $|\varepsilon_{0(1)}\rangle$ [9]. Hence, we also have

$$\sum_{n=0}^{\infty} F(|c_n|) = \sum_{n=0}^{\infty} F(|d_n|) = 1. \tag{6.26}$$

Now we can write

$$\begin{aligned} F(|\gamma c_n|) &= F(|\gamma|)F(|c_n|) \\ F(|\delta d_n|) &= F(|\delta|)F(|d_n|). \end{aligned} \tag{6.27}$$

The above equation means that the probability of finding the total system in the state $|s_{0(1)}, e_n\rangle$ is equal to the probability of finding it in the state $|s_{0(1)}, \varepsilon_{0(1)}\rangle$ times the probability of finding the environment, already measured in the state $|\varepsilon_{0(1)}\rangle$ in the substate $|e_n\rangle$. In the case of our previous example of the JCM, this situation is like first measuring the atomic energy operator $\hat{H}_S$ in order to find its energy eigenvalues and then doing a measurement on the number operator $\hat{N}$, on the Hilbert space of the field in Fock space. Clearly, such two operations are commutable and one measurement does not have any effect on the other measurement. Hence, this will justify the multiplicative resolution in equation (6.27).

So now equation (6.25) reads

$$\begin{aligned} &F(|\gamma|)\sum_{n=0}^{\infty} F(|c_n|) + (1 - F(|\gamma|))\sum_{n=0}^{\infty} F(|d_n|) \\ &= F(|\gamma|)\{\sum_{n=0}^{\infty} F(|c_n|) - \sum_{n=0}^{\infty} F(|d_n|)\} + \sum_{n=0}^{\infty} F(|d_n|) = 1. \end{aligned} \tag{6.28}$$

[9] For this to be justified we need to be sure that the phases of the expansion coefficients in equation (6.17) are unimportant in determining the probabilities of finding the environment in any of its substates ($\{|e_n\rangle\}$'s); However, this can be justified by noting that the states of the environment also, can further couple with the states of an outer environment $\mathcal{C}$, which can be thought of as an apparatus with corresponding pointer states $|C_n\rangle$ in order to form a global state in diagonal form such as: $|\psi_{\mathcal{EC}}\rangle = \sum_{n=0}^{\infty} c_n|e_n\rangle|C_n\rangle$ when the environment is already measured in the state $|\varepsilon_0\rangle$ or $|\psi_{\mathcal{EC}}\rangle = \sum_{n=0}^{\infty} d_n|e_n\rangle|C_n\rangle$ when the environment is already measured in the state $|\varepsilon_1\rangle$. So, the coupling of the substates of the environment with some states from an outer environment and the formation of a one-to-one correspondence between these states can justify the unimportance of the phases of the expansion coefficients $\{c_n\}$'s and $\{d_n\}$'s through a discussion exactly like the one that we made regarding the unimportance of the phases of the expansion coeffiecients for the global state of the $S\mathcal{E}$ composite system.

Now the unitarity of Schrödinger's equation which preserves the norm of every state implies that for the global state of the system and the environment given by equation (6.18) we must have

$$|\gamma|^2 \sum_{n=0}^{\infty} |c_n|^2 + |\delta|^2 \sum_{n=0}^{\infty} |d_n|^2 = 1 = \text{constant}. \tag{6.29}$$

Similarly, for equation (6.16) this implies that

$$|\gamma|^2 + |\delta|^2 = 1 = \text{constant}. \tag{6.30}$$

Hence, also we have

$$\sum_{n=0}^{\infty} |c_n|^2 = \sum_{n=0}^{\infty} |d_n|^2 = 1 = \text{constant}. \tag{6.31}$$

(We will come back to the issue of unitarity in the evolution of states. As we will discuss, the unitarity of Schrödinger's equation has nothing to do with the probability interpretation for the wave amplitudes. Hence, this will not bring any circularity to our proof). So, now equations (6.28) and (6.29) read

$$\begin{aligned} &F(|\gamma|)\{\sum_{n=0}^{\infty} F(|c_n|) - \sum_{n=0}^{\infty} F(|d_n|)\} + \sum_{n=0}^{\infty} F(|d_n|) \\ &= |\gamma|^2\{\sum_{n=0}^{\infty} |c_n|^2 - \sum_{n=0}^{\infty} |d_n|^2\} + \sum_{n=0}^{\infty} |d_n|^2 = 1. \end{aligned} \tag{6.32}$$

But due to equations (6.26) and (6.31) the first terms in the above equation are equal to zero. So the above equation simplifies as

$$\sum_{n=0}^{\infty} F(|d_n|) = \sum_{n=0}^{\infty} |d_n|^2 = 1. \tag{6.33}$$

We will prove, through the following theorem, that the above result is possible only if

$$F(|x|) = |x|^2; \tag{6.34}$$

which together with *Theorem-I* completes our proof of the Born rule for quantum probabilities.

Theorem-II: Suppose that we have a function F which satisfies the condition $\sum_{n=0}^{m} F(|x_n|) = 1$ for all integers m, where $m \geq 1$; while requiring $\sum_{n=0}^{m} |x_n|^2 = 1$. Then the function F must be uniquely determined as $F(|x|) = |x|^2$.

Proof: In the first step of our proof let us consider the special case that $m = 1$. This means that we have an equation like $F(|x|) + F(|y|) = 1$; while $|x|^2 + |y|^2 = 1$. In other words, we have $F(z) + F(1 - z) = 1$ where $z = |x|^2$; and we need to find the function F. Let

$$F(z) = \sum_{n=-\infty}^{\infty} a_n \; z^n = ... + a_{-2} \; z^{-2} + a_{-1} \; z^{-1} + a_0 + a_1 \; z + a_2 \; z^2 + ... \; ; \tag{6.35}$$

where we must determine the a_n's from our condition, given by $F(z) + F(1 - z) = 1$. Substituting F (given by equation (6.35)) in this condition, we find

$$\begin{aligned} F(z) + F(1-z) = ... + a_{-2} \; [z^{-2} + (1-z)^{-2}] + a_{-1} \; [z^{-1} + (1-z)^{-1}] \\ +2a_0 + a_1 + a_2 \; [z^2 + (1-z)^2] + a_3 \; [z^3 + (1-z)^3] + ... = 1. \end{aligned} \tag{6.36}$$

However, all the functions inside square brackets in the above equation are linearly independent functions, as one can easily see for example by verifying that the Wronskian of these functions is not zero. Therefore, we must have: $2a_0 + a_1 = 1$; and $a_{-1} = a_2 = a_{-2} = a_3 = a_{-3} = ... = 0$. Therefore,

$$F(z) = a_0 + a_1 z = a_0 + [1 - 2a_0]z. \quad \forall \; a_0. \tag{6.37}$$

For example, for $a_0 = 0$ we have $F(z) = z = |x|^2$; while for $a_0 = 1$ we have $F(z) = 1 - z = 1 - |x|^2$ and etc. Here, all these functions satisfy the equation $F(z) + F(1 - z) = 1$.

Next, in order to find the parameter a_0 we consider the next order, i.e. m=2, in our general equation $\sum_{n=0}^{m} F(|x_n|) = 1$. Now our equation reads: $F(|x|) + F(|y|) + F(|w|) = 1$; while requiring $|x|^2 + |y|^2 + |w|^2 = 1$. Calling $|x|^2 = z$ and $|y|^2 = z'$, it reads

$$F(z) + F(z') + F(1 - z - z') = 1. \tag{6.38}$$

Now using assumption 2, i.e. requiring the function F to be the same function in all orders of m, we can substitute F, which was obtained in the previous order of m as in equation (6.37), in the above equation to obtain

$$F(z) + F(z') + F(1 - z - z') = a_0 + [1 - 2a_0]z + a_0 + [1 - 2a_0]z'$$
$$+a_0 + [1 - 2a_0][1 - z - z'] = 1 + a_0 = 1. \quad \text{i.e.} \quad \mathrm{a_0 = 0} \quad \text{or} \quad \mathrm{F(z) = z}; \tag{6.39}$$

QED.

Here we should also comment that a key point about Schrödinger's equation is its unitarity which preserves the norm of the states. However, if we take the constant in equations (6.29) and (6.30) any other constant number like C; then still we will obtain equation (6.33), as one can easily verify.

We also note that the unitarity of Schrödinger's equation has nothing to do with the probability interpretation for wave amplitudes. Hence, this does not bring any circularity to our proof of Born's rule. In fact, the unitarity of Schrödinger's equation is simply because of the fact that for a potential that does not explicitly depend on time, the solutions of this equation are separable as:

$$\psi(x,t) = T(t)u(x); \quad \text{where} \quad T(t) = C\mathrm{e}^{-\mathrm{i}Et/\hbar}, \tag{6.40}$$

and C is a constant. Hence, this will imply that the norm of the states must be time-independent. In other words, the equation preserves the norm of the states. However, even for time-dependent potentials we do know that the Schrödinger equation will preserve the norm of states, since the time evolution operator in its most general form, whose evolution is governed by the Schrödinger equation, always is a unitary operator (of course provided the Hamiltonian of the system is a Hermitian operator). (Here also note that we do not need to use Gleason's theorem [3] (which is a purely mathematical theorem that defines the "measures" on the closed subspaces of a Hilbert state) in order to assume that a *measure* related to a state such as $|\psi\rangle = \sum_n a_n|n\rangle$ in the Hilbert space is given by $\langle\psi|\psi\rangle$ $(= \sum_n |a_n|^2)$ and then to exploit the unitarity of Schrödinger's equation in order to conclude equations (6.29) to (6.31). Indeed, equation (6.35) suggests that for a general state vector like $|\psi\rangle =$

$\sum_n a_n|n\rangle$, the mathematical quantity $\langle\psi|\psi\rangle = \sum_n |a_n|^2$ (whatever it is!) must be time-independent and hence a conserved quantity in time. In other words, the conservation of the mathematical quantity which usually is called the "norm" of a quantum state in the Hilbert space, does not depend on the Gleason theorem [3] which attributes the second power of this quantity to the "measures" on the Hilbert space).

Also, as is obvious from equation (6.35), the unitarity of the evolution of states indicates that the *norm* of a quantum state is not the only quantity which is conserved with time. However, if we equalize any quantity other than the magnitude of the norm of a quantum state to the power of *two* to the sum of all probabilities (which contains terms which each of them is a functions of only a single variable, as in equation (6.33)), due to the cross terms which are functions of more than one variable, the resulting equation (an equation such as $(|\gamma|^2 + |\delta|^2)^n = F(\gamma) + F(\delta)$, with $n \neq 1$ and arbitrary γ and δ) in general will not have any answers. As a result, any powers of the norm of a quantum state other than its second power cannot represent probabilities.

6.5 Summary and conclusions

We discussed Zurek's proof for the independence of quantum probabilities from the phases of the Schmidt coefficients, based on the idea of "envariance" and discussed what is logically inconsistent in his proof of the "envariance" theorem. Also, as we discussed in section 3, the phases of the Schmidt coefficients are not necessarily unimportant in determining the probabilities and one should emphasize that this unimportance of the phases of the expansion coefficients applies just in the preferred basis of measurement. Moreover, the unimportance of the phases of the expansion coefficients in the preferred basis of measurement is not because of the fact that the effect of a phase transformation on the Hilbert space of the system can be undone by a "counter-transformation" on the Hilbert space of the environment, as Zurek has proposed. In fact, it seems that this unimportance of phases originates from a simple symmetry in the Hilbert space of the system which even is *limited* after the preferred basis of measurement is determined through the interaction of the system with all its surrounding environments. For instance, in our example of the Stern-Gerlach experiment if we

do not know the operator which is going to be measured at all; say if the direction of the magnetic field is randomly chosen in our experiment, then all the eigenstates such as $|S_x;\pm\rangle$ and $|S_y;\pm\rangle$ states in addition to the eigenstates of the $\hat{S}_z$ operator, i.e. $|\pm\rangle$ states, can be assumed to be equally probable; and this is the result of a simple symmetry in the space before any specific direction is chosen as the preferred direction of measurement. However, if a specific direction for example the z-direction is chosen by creating a magnetic field in the z-direction (so that our measuring apparatus measures the z-component of the spin of atoms), then all states except the $|\pm\rangle$ states are excluded from the set of all possible results of the measurement. Moreover, the phases of the Schmidt coefficients will be unimportant in obtaining either of the $|\pm\rangle$ states only if our (bipartite) Schmidt states contain the $|\pm\rangle$ states of the system. However, this unimportance of the phases of the expansion coefficients in the preferred basis of measurement again is the result of a simple symmetry in the space which is created after the determination of the preferred basis of measurement has specified a specific direction as the preferred direction of measurement.

We also discussed why indeed a fine-graining, which can create equal expansion coefficients for the global state of the system and the environment, generally cannot be physically justified; although it is mathematically possible. As we discussed, such a fine-graining generally is *not* the same as the physical real fine-graining which is the result of the further interaction of the pointer states of the environment with an outer environment C; i.e. a further premeasurement by C on the pointer states of the environment.

In section 2 we also briefly discussed the pointer states of measurement. As we discussed, the pointer states of the system, which are characterized by their ability not to entangle with the environment (the stability criterion), generally are time-dependent and hence one must differentiate between the pointer states of the system and the preferred basis of measurement. In fact, as we pointed out, when we write the global state of the system and the environment in a diagonal form such as that of equation (6.1), generally the states of the system which appear on the diagonal can only represent the instantaneous pointer states of the system and in fact, a preferred basis of measurement necessarily is not determined in an arbitrary regime. Moreover, we cannot obtain the probabilities of realiza-

tion of specific states before the preferred basis of measurement is determined. However, by referring to chapter 3 we discussed that under specific conditions the pointer states of measurement become time-independent and hence they can represent the preferred basis of measurement. Indeed, by explicitly assuming that we are in one of the regimes which make the pointer states independent of time, we can justify our presentation of the state of the total composite system by equation (6.1), which is in a diagonal form with time-independent pointer states (i.e. the preferred basis of measurement) on its digonal.

Finally we presented our proof of Born's rule in section 4. In essence, we can say that what really is behind the quantum mechanical Born's rule seems to mainly originate from two issues. Firstly, the *unitarity* of the evolution of states, which is given by Schrödinger's equation and secondly, a symmetry which is introduced to the dynamics of the global state of the system and the environment by the environment-induced superselection of the preferred basis of measurement. As we discussed, this symmetry is created after the coupling between the pointer states of the system and those of the environment creates a one-to-one correspondence between these states; and after the preferred basis of measurement is realized (of course under some *specific* conditions) as the set of time-independent pointer states in the next step. The resulting symmetry, which is caused through the interaction with the environment, is an environment-induced symmetry, as Zurek had pointed out; and is responsible for the unimportance of the phases of the expansion coefficients in the preferred basis of measurement.

Also, in our introduction we referred to Brumer's work [9], which shows that the quantum mechanical Born's rule has a well defined purely classical limit. As Brumer has pointed out, this indicates that the origin of Born's rule should be independent of subtle purely quantum details of the quantum measurement process such as the origin of probabilities and the mechanism by which definite outcomes are realized. In fact, as the reader may have noticed, our proof of Born's rule also is along this line; because we did not make any assumption about the origin of the appearance of probabilities.

In summary, just as we expected, it appears that Born's rule is independent of the subtle details of the quantum measurement process such as the appearance of probabilities and the

mechanism by which definite outcomes are realized. Therefore, as we observe, the problem of definite outcomes and the origin of probabilities still remains a fundamental question to be answered.

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Chapter 7

Conclusion

1) Defining the pointer states of a subsystem as those states which are characterized by their ability not to entangle with the states of another subsystem, we presented a general method for evaluating the pointer states of a subsystem. This way we showed how in practice the global state of the system and the environment may evolve into a *diagonal* state (i.e. the Von Neumann scheme of measurement may be realized) as a result of the natural evolution of the total composite system. As we showed, evaluation of the pointer states of the system requires finding those specific initial states of the system and the environment for which the operator $\hat{G}(t)$ (defined through equation (3.30)) may become independent of the states of the environment. However, as we could see from our example represented by the evolution of the two-level atom in the Jaynes-Cummings model and in the exact-resonance-regime, such initial conditions for the states of the system and the environment necessarily do not exist in an arbitrary regime. (For this example as we saw, unless we have a large average number of photons which can make a sharp distribution function for the state of the electromagnetic field, the states of the two-level atom and the field will remain highly entangled and the pointer states of measurement cannot be realized at all.) As a result, even time-dependent pointer states necessarily do not exist for any arbitrary regime. In other words, premeasurement by the environment (the Von Neumann scheme of measurement),

as a result of which the state of the total composite system becomes in a diagonal form, necessarily cannot be realized in any arbitrary regime.

In this research we distinguished between pointer states of measurement and the preferred basis of measurement; as time-independent pointer states which can arise only in certain regimes. We exactly showed why indeed time-independent pointer states (which require the operator $\hat{G}(t)$ defined through equation (3.30) be independent of time) cannot be expected in most of the regimes. In other words, pointer states of a system, which do not entangle with the states of the environment and appear as a result of premeasurement by the environment, necessarily are not time-independent and the assumption of having quantum *nondemolition premeasurement* in the Von Neumann scheme of measurement practically is not a good assumption; as the pointer states which appear on the diagonal state of the total composite system may change by more than just an overall phase factor. Moreover, we explored those conditions under which the pointer states of the system may be independent from time; so that they can represent the preferred basis of measurement. These are new aspects not contained in the existing theory for determination of the preferred basis of measurement.

As we saw the conditions for having time-independent pointer states include the so-called quantum limit of decoherence ($\hat{H} \approx \hat{H}_{\mathcal{S}}$) as well as the so-called quantum measurement limit ($\hat{H} \approx \hat{H}_{\text{int}}$). In fact, time-independent pointer states for the system are predicted for these two regimes by using theorem 3 of chapter 3, and just as special cases of the more general symmetry condition represented by this theorem. Therefore, our theorems cover the predictions of Zurek's theory for determination of the preferred basis of measurement at corresponding limits. Nonetheless, they present some other conditions as well under which the pointer states of the system would become independent of time and hence can we have a preferred basis of measurement. For example as we saw, in order to have the preferred basis of measurement given by the eigenstates of the interaction Hamiltonian in the Hilbert space of the system, necessarily we do not require $\hat{H}_{\mathcal{S}}$ and $\hat{H}_{\mathcal{E}}$ to be negligible and this prediction holds valid whenever $\hat{H}_{\mathcal{S}}$ commutes with a nonzero $\hat{H}_{\text{int}}$ and no matter how big are the contributions from the self-Hamiltonian of the system and the self-Hamiltonian of the

environment. Therefore, our criteria for predicting the time-independence of pointer states go beyond the limits in which $\hat{H}_{\rm tot} \approx \hat{H}_{\rm int}$ or $\hat{H}_{\rm tot} \approx \hat{H}_{\mathcal{S}}$ and will include some other cases as well; where all contributions can be present at the same time. In this sense, our theory not only provides us with a general method for obtaining the generally time-dependent pointer states of the system and the environment, but also it can serve as a generalization for the existing theory for determination of the preferred basis of measurement.

As an application of our theory, one can use it in order to obtain those regimes of the parameter space (corresponding to the total Hamiltonian defining a given system-environment model) for which a preferred basis of measurement can be realized. Moreover, we can predict the corresponding preferred basis of measurement for each regime. In addition to that now we also have a method in order to obtain time-dependent pointer states in non-measurement regimes; where a time-independent basis of measurement cannot be realized at all. This ability to obtain *time-dependent* pointer states, which arise in the majority of regimes, is particularly important in decoherence studies; as such pointer states although evolve with time and cannot represent the preferred basis of measurement, they correspond to the initial conditions for the state of the system and the environment for which we can have long decoherence times. We presented some interesting results regarding this problem in chapter 4, where we obtained the time-dependent pointer states of the generalized spin-boson model and studied the decoherence of the central system in this model.

2) In chapter 4, considering a single-mode quantized field in exact resonance with the tunneling matrix element of the system, we obtained the time-evolution operator for our model. Using this time-evolution operator we then calculated the pointer states of the system and the environment, which are characterized by their ability not to entangle with each other; for the case that the environment initially is prepared in the coherent state. We showed that our solution for the pointer states of the system and the environment is valid over a length of time which is proportional to $\bar{n}$, the average number of bosons in the environment.

Most importantly, we observed that for our spin-boson model represented by the Hamil-

tonian of equation (4.2) the pointer states of the system turn out to become time-dependent, as opposed to the pointer states of the simplified spin-boson model (for which $[\hat{H}_{\mathcal{S}}, \hat{H}_{\rm int}] = 0$). As we already mentioned, the simplified model has often been used in the context of quantum information and quantum computation to gain some insights regarding the decoherence of a single qubit [1, 2, 3]. However, in practical situations often different noncommutable perturbations may exist in the total Hamiltonian of a realistic system-environment model which would result in having time-dependent pointer states for the system [4]. Indeed, the authors believe that the fact that the pointer states of a system generally are time-dependent and may evolve with time has not been seriously acknowledged in the context of quantum computation and quantum information. Specifically, in the context of quantum error correction [5, 6] it is often assumed that the premeasurement by the environment does not change the initial pointer states of the system. In other words, quantum "nondemolition" premeasurement by the environment is often assumed [5, 6]; as is also assumed in Von Neumann scheme of measurement [7, 8]. Also, in the context of Decoherence-Free-Subspaces (DFS) theory the models which often are studied either contain a self-Hamiltonian for the system which commutes with the interaction between the system and the environment, or it is assumed that we are in the *quantum measurement limit* [1] or in the *quantum limit of decoherence* [1, 9, 10, 11]. However, all of these assumptions are in fact a big simplification of the problem; since, as we discussed in chapter 3, they completely exclude the possibility of having pointer states for the system which may depend on time.

Another interesting point in obtaining the pointer states of the system and the environment for our spin-boson model was the realization of the fact that *only* in the limit of a large average number of photons can we have a set of (time-dependent) pointer states for the system. In other words, unless we have a sufficiently large average number of photons which can make a sharp distribution function for the state of the electromagnetic field, there is

[1] In the *quantum measurement limit* the interaction between the system and the environment is so strong as to dominate the evolution of the system $\hat{H} \approx \hat{H}_{\rm int}$. Also in the *quantum limit of decoherence* the Hamiltonian for the system almost dominates the interaction between the system and the environment as well as the self-Hamiltonian of the environment $\hat{H} \approx \hat{H}_{\mathcal{S}}$.

always some degree of entanglement between the states of the system and the environment (see equations (4.61) and (4.90)) and the pointer states of measurement cannot be realized at all.

We also showed that at $t = (2n+1)\pi\hbar\sqrt{\bar{n}}/\mathrm{g}$ (with n $= 0, 1, 2, ...$) the $|\pm(t)\rangle$ pointer states of the system coincide with each other and hence, whatever is the initial state of the system, at these specific times the states of the system and the environment are not entangled with each other and the system can be represented by a well-defined state of its own. Using the time-evolution operator obtained in section 2, we also obtained a closed form (figure 4.2) for the offdiagonal element of the reduced density matrix of the system and studied the decoherence of the central system in our model. We showed that for the case that the system initially is prepared in one of its pointer states, the offdiagonal element of the reduced density matrix of the system will be a sinusoidal function with a slow decaying envelope characterized by one over the square of the average number of photons in the environment; while for the case that the system initially is not prepared in one of its initial pointer states, it will experience a fast decoherence within a time of order $\hbar/\mathrm{g}$. The "decayo-sinusoidal" evolution of coherences (figure 4.1) which we observe in our model and for the case that the system initially is prepared in one of its pointer states is a new form of decoherence which cannot be observed in the somewhat similar Jaynes-Cummings model of quantum optics [12, 13].

It will be interesting to generalize this study to the case that the environment is not merely represented by a single-mode bosonic field; and consider some classes of spectral densities for the environment. Also, for the spin-boson model represented by the Hamiltonian of equation (4.2) at least in principle one should be able to obtain the pointer states of the system and the environment in some nonresonance regimes and for the single-mode quantized field.

3) In chapter 5, considering the quantized atom-field model, we obtained the time-evolution operator for the regime that $\hat{H}_{\mathcal{E}} \ll \hat{H}_{\mathcal{S}} \ll \hat{H}'$ (but $\hat{H}_{\mathcal{S}} \neq 0$ and $\hat{H}_{\mathcal{E}} \neq 0$). Using this time-evolution operator then we calculated the corresponding pointer states of

the system and the environment, which are characterized by their ability not to entangle with each other, by assuming an initial state of the environment in the form of a Gaussian package in position space. Again we observed that for our quantized atom-field model, represented by the Hamiltonian of equation (5.1), also the pointer states of the system turn out to become *time-dependent*, as opposed to the pointer states of some simpler models, which often are cited in the context of quantum information and quantum computation [1-3, 5,6, 9-11].

Using the time-evolution operator which we obtained in section 5.2, we also obtained a closed form for the elements of the reduced density matrix of the system, and studied the decoherence of the central system in our quantized atom-field model. We showed that for the case that the system initially is not prepared in one of its pointer states and in the basis of the initial pointer states of the system (i.e. the $|\pm(t_0)\rangle$ states), the short time ($t \ll \omega_0^{-1}$) evolution of the offdiagonal elements of the reduced density matrix of the system will demonstrate decoherence, with a decoherence time which is inversely proportional to the square root of the mass of the field particles.

Generally we can predict that only in *certain* regimes of the parameter space we may have (generally time-dependent) pointer states for the system and the environment; and those pointer states must depend on the specific regime of the parameter space which we are considering; however, outside those certain regimes of the parameter space the entanglement between the states of the system and those of the environment no longer can be negligible and hence, pointer states cannot be realized for the system and the environment at all. So far, we do know that in the exact-resonance with the rotating-wave-approximation regime (which was studied by Gea-Banacloche, as was referenced in our chapters 3 and 5) and also in the regime which we considered within chapter 5 of this dissertation, pointer states can be realized for the system and the environment. However, the question of identifying *all* those regimes of the parameter space where pointer states may exist, and also the corresponding pointer states in *each* of those regimes is an open question which should be investigated. So, for our model, represented by the Hamiltonian of equation (5.1), at least in principle one should be able to obtain the pointer states of the system and the environment

in some other regimes of the parameter space. Also, it will be interesting to generalize this study to the case that the environment is not merely represented by a single-mode bosonic field; and consider some classes of spectral densities for the environment.

4) Finally, in chapter 6 we discussed Zurek's proof for the independence of quantum probabilities from the phases of the Schmidt coefficients, based on the idea of "envariance" and discussed what is logically inconsistent in his proof of the "envariance" theorem. Also, as we discussed in section 6.3, the phases of the Schmidt coefficients are not necessarily unimportant in determining the probabilities and one should emphasize that this unimportance of the phases of the expansion coefficients applies just in the preferred basis of measurement. Moreover, the unimportance of the phases of the expansion coefficients in the preferred basis of measurement is not because of the fact that the effect of a phase transformation on the Hilbert space of the system can be undone by a "counter-transformation" on the Hilbert space of the environment, as Zurek has proposed. In fact, it seems that this unimportance of phases originates from a simple symmetry in the Hilbert space of the system which even is *limited* after the preferred basis of measurement is determined through the interaction of the system with all its surrounding environments. For instance, in our example of the Stern-Gerlach experiment if we do not know the operator which is going to be measured at all; say if the direction of the magnetic field is randomly chosen in our experiment, then all the eigenstates such as $|S_x;\pm\rangle$ and $|S_y;\pm\rangle$ states in addition to the eigenstates of the $\hat{S}_z$ operator, i.e. $|\pm\rangle$ states, can be assumed to be equally probable; and this is the result of a simple symmetry in the space before any specific direction is chosen as the preferred direction of measurement. However, if a specific direction for example the z-direction is chosen by creating a magnetic field in the z-direction (so that our measuring apparatus measures the z-component of the spin of atoms), then all states except the $|\pm\rangle$ states are excluded from the set of all possible results of the measurement. Moreover, the phases of the Schmidt coefficients will be unimportant in obtaining either of the $|\pm\rangle$ states if and only if our (bipartite) Schmidt states contain the $|\pm\rangle$ states of the system. However, this unimportance of the phases of the expansion coefficients in the preferred basis of mea-

surement again is the result of a simple symmetry in the space which is created after the determination of the preferred basis of measurement has specified a specific direction as the preferred direction of measurement.

We also discussed why indeed a fine-graining, which can create equal expansion coefficients for the global state of the system and the environment, generally cannot be physically justified; although it is mathematically possible. As we discussed, such a fine-graining necessarily is *not* the same as the physical real fine-graining which may happen as a result of the further interaction of the pointer states of the environment with an outer environment C; i.e. a further premeasurement by an outer apparatus or environment on the pointer states of the environment.

Also, as we discussed, when we write the global state of the system and the environment in a diagonal form such as that of equation (6.1), generally the states of the system which appear on the diagonal can only represent the instantaneous pointer states of the system. This is because of the fact that the pointer states of the system generally are time-dependent and one must differentiate between the pointer states of the system and the preferred basis of measurement; as the preferred basis of measurement necessarily cannot be determined in *any* arbitrary regime. Moreover, we cannot obtain the probabilities of realization of specific states before the preferred basis of measurement is determined. However, by referring to the discussions in chapter 3, we discussed that under specific conditions the pointer states of measurement become time-independent and hence they can represent the preferred basis of measurement. Indeed, only by explicitly assuming that we are in one of the regimes in which the pointer states are independent of time, can we justify our presentation of the state of the total composite system by equation (6.1), which is in a diagonal form with time-independent pointer states (i.e. the preferred basis of measurement) on its diagonal.

Finally we presented our proof of Born's rule in section 6.4. In essence, we can say that what really is behind the quantum mechanical Born's rule seems to mainly originate from two issues. Firstly, the *unitarity* of the evolution of states, which is given by Schrödinger's equation and secondly, a symmetry which is introduced to the dynamics of the global state of the system and the environment by the environment-induced superselection of the

preferred basis of measurement. As we discussed, this symmetry is created after the coupling between the pointer states of the system and those of the environment creates a one-to-one correspondence between these states; and after the preferred basis of measurement is realized (of course under some *specific* conditions) as the set of time-independent pointer states in the next step. The resulting symmetry, which is caused through the interaction with the environment, is an environment-induced symmetry, as Zurek had pointed out; and is responsible for the unimportance of the phases of the expansion coefficients in the preferred basis of measurement.

In our introduction to chapter 6 we referred to Brumer's work [14], which shows that the quantum mechanical Born's rule has a well defined purely classical limit. As Brumer had pointed out, this indicates that the origin of Born's rule should be independent of subtle purely quantum details of the quantum measurement process such as the origin of probabilities and the mechanism by which definite outcomes are realized. In fact, as the reader may have noticed, our proof of Born's rule also is along this line; because we did not make any assumption about the origin of the appearance of probabilities.

In summary, just as we expected, it appears that Born's rule is independent of the subtle details of the quantum measurement process such as the appearance of probabilities and the mechanism by which definite outcomes are realized. Therefore, as we observe, the problem of definite outcomes and the origin of probabilities still remains a fundamental question to be answered.

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Chapter 8

Appendices

Appendix A

The purpose of this appendix is to present the time-dependent pointer states of the system (the two-level atom) and the environment (the quantized electromagnetic field) in the Jaynes-Cummings model of quantum optics. Gea-Banacloche [13] has shown that if we consider the field to be initially prepared in the coherent state $|\nu\rangle$

$$|\nu\rangle = \sum_{n=0}^{\infty} \mathrm{c}_n |n\rangle; \quad \text{with} \quad \mathrm{c}_n = \frac{\mathrm{e}^{-\frac{1}{2}|\nu|^2}\nu^n}{\sqrt{n!}}, \tag{A-1}$$

(with $|\nu|^2 = \bar{n}$ as the average number of photons in the coherent state, and $\nu = |\nu|e^{-i\phi}$), and in the limit of a large average number of photons in the field, the interaction between the two-level-atom and the coherent field will determine the following time-dependent pointer states for the system

$$|+(t)\rangle = \frac{\mathrm{e}^{-\mathrm{i}\phi}\mathrm{e}^{-\mathrm{i}\mathrm{g}t/(2\sqrt{\bar{n}})}|a\rangle + |b\rangle}{\sqrt{2}} \tag{A-2}$$

$$\text{and} \quad |-(t)\rangle = \frac{\mathrm{e}^{-\mathrm{i}\phi}\mathrm{e}^{+\mathrm{i}\mathrm{g}t/(2\sqrt{\bar{n}})}|a\rangle - |b\rangle}{\sqrt{2}}.$$

In the above relations g is the atom-field coupling constant; while $|a\rangle$ and $|b\rangle$ are the atomic upper and lower states respectively. As it is obvious from equation (A-2), the pointer states

of the system are *not* always orthogonal. In fact, using equation (A-2) in order to calculate the overlap between the pointer states of the system, we can find

$$|\langle -(t)| + (t)\rangle|^2 = \sin^2(t'/2\sqrt{\bar{n}}); \quad \text{and} \quad t' = gt/\hbar. \tag{A-3}$$

The pointer states of the system, represented by equation (A-2), will maintain a one-to-one correspondence with the following states from the environment (the electromagnetic field) all throughout the interaction

$$|\Phi_{\pm}(t)\rangle = e^{-\bar{n}/2} \sum_{n=0}^{\infty} \frac{\bar{n}^{n/2}}{\sqrt{n!}} e^{-in\phi} e^{\mp igt\sqrt{n}} |n\rangle. \tag{A-4}$$

One can easily show that for short enough times, $t \ll t_R = 2\pi\sqrt{\bar{n}}/g$, the overlap between the pointer states of the environment would satisfy the following equation [13]

$$|\langle \Phi_-(t)|\Phi_+(t)\rangle|^2 = \exp\{-4\bar{n}\, \sin^2(t'/2\sqrt{\bar{n}})\}\ (\simeq e^{-g^2t^2}) \tag{A-5}$$

Therefore, although the field states $|\Phi_+(t)\rangle$ and $|\Phi_-(t)\rangle$ initially are not orthogonal, but they will become orthogonal within a very short time of the order of $t_c = 1/g$; which results in decoherence [13]. In Fig. 8.1 we plotted the two factors $|\langle -(t)| + (t)\rangle|^2$ and $|\langle \Phi_-(t)|\Phi_+(t)\rangle|^2$ of the pointer states of the system and the environment. As we can see from this picture at those times for which the overlap between the pointer states of the system is not zero, we have the zeros of the overlap between the pointer states of the environment; and whenever the overlap between the pointer states of the *environment* is not zero, we have the zeros of the overlap between the pointer states of the *system*. This way it is guaranteed that the factor $\langle s_1(t)|s_0(t)\rangle \times \langle \varepsilon_1(t)|\varepsilon_0(t)\rangle$ of equation (2.22) is constantly equal to zero.

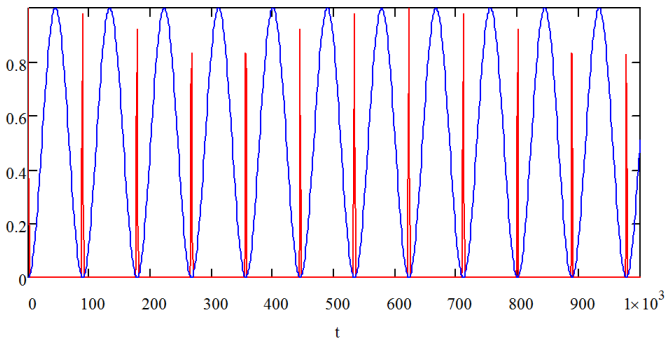

Figure 8.1: Time evolution of the square magnitude of the overlap between the pointer states of the system and the square magnitude of the overlap between the pointer states of the environment. Here we chose $\bar{n} = 200$.

Appendix B

In this section we introduce a measure for degree of entanglement between the states of the system and those of the environment, and by calculating this measure for the initial pointer states of our model (which are given by equation (4.9)) we show that our result for the pointer states of the system and the environment is valid over a length of time which is proportional to $\bar{n}$, the average number of photons in the field; since as we will see, only within up to this range of times our pointer states of the system and the environment can stay separated and will not considerably entangle with the states of another subsystem.

For the global state of the system and the environment, given by equation (4.9), we have

$$\begin{aligned} |\psi^{\text{tot}}(t)\rangle &= |\mathbf{A}(t)\rangle\ |a\rangle + |\mathbf{B}(t)\rangle\ |b\rangle \\ &= |\mathbf{A}'(t)\rangle\ [G(t)|a\rangle] + |\mathbf{B}(t)\rangle\ |b\rangle; \quad \text{where} \quad |\mathbf{A}'(t)\rangle = \frac{|\mathbf{A}(t)\rangle}{G(t)}. \end{aligned} \tag{B-1}$$

For our pointer states (given by $|\pm(t)\rangle = \mathcal{N}_{\pm}\ \{G_{\pm}(t)|a\rangle + |b\rangle\}$) we can define a degree of entanglement through the following relation

$$q(t) = \frac{\langle \mathbf{A}(t)|G_{\pm}(t)\mathbf{B}(t)\rangle}{|\mathbf{A}(t)|^2}; \tag{B-2}$$

which also is equal to

$$q(t) = \frac{\langle \mathbf{A}'(t)|\mathbf{B}(t)\rangle}{|\mathbf{A}'(t)|^2}. \tag{B-3}$$

The above function basically is the overlap between the vectors $|\mathbf{A}'(t)\rangle$ and $|\mathbf{B}(t)\rangle$ of equation (B-1); which is *normalized* to the unity, since for our pointer states of the system we have $|\mathbf{A}(t)\rangle = G_{\pm}(t)|\mathbf{B}(t)\rangle$. From equation (B-1) it is clear that for perfect pointer states, where there is no entanglement between the states of the system and the environment, $q(t)$ must always remain equal to the unity (i.e. $|\mathbf{A}'(t)\rangle$ and $|\mathbf{B}(t)\rangle$ must perfectly coincide with each other, in which case the states of the system and the environment in equation (B-1) will not entangle with each other; and we will have pointer states for the system which are given by equation (4.11)); all throughout the evolution of the system and the environment. Only in this case the states of the system and the environment in equation (B-1) will always stay separated and one can assign each of the two subsystems with well-defined states of their own.

Our goal is to calculate our measure of entanglement $q(t)$ for the pointer states of the system and the environment which we obtained for our model in chapter 4; and to study its evolution with time. For the global global state of the system and the environment, given by equation (B-1), one can calculate the density matrix of the total composite system as

$$\begin{aligned}\hat{\rho}^{\rm tot}(t) = |a\rangle\langle a| \otimes |\mathbf{A}(t)\rangle\langle\mathbf{A}(t)| + |b\rangle\langle b| \otimes |\mathbf{B}(t)\rangle\langle\mathbf{B}(t)| \\ +|a\rangle\langle b| \otimes |\mathbf{A}(t)\rangle\langle\mathbf{B}(t)| + |b\rangle\langle a| \otimes |\mathbf{B}(t)\rangle\langle\mathbf{A}(t)|. \end{aligned} \tag{B-4}$$

We note that the environmental states $|\mathbf{A}(t)\rangle$ and $|\mathbf{B}(t)\rangle$ necessarily are not orthogonal and do not necessarily make a complete set of basis states for the Hilbert space of the environment. Now, tracing over the environmental degrees of freedom (which implies writing the states $|\mathbf{A}(t)\rangle$ and $|\mathbf{B}(t)\rangle$ in terms of a complete set of basis states in the Hilbert space of the environment, before doing the trace operation), one can easily find

$$\hat{\rho}^{\mathcal{S}}(t) = |a\rangle\langle a|\langle\mathbf{A}(t)|\mathbf{A}(t)\rangle + |b\rangle\langle b|\langle\mathbf{B}(t)|\mathbf{B}(t)\rangle + |a\rangle\langle b|\langle\mathbf{B}(t)|\mathbf{A}(t)\rangle + |b\rangle\langle a|\langle\mathbf{A}(t)|\mathbf{B}(t)\rangle; \tag{B-5}$$

and therefore

$$\rho^{\mathcal{S}}_{12}(t) = \langle\mathbf{B}(t)|\mathbf{A}(t)\rangle; \tag{B-6}$$

where in the above equation $\rho^{\mathcal{S}}_{12}(t)$ is the offdiagonal element of the reduced density matrix of the two-level system in the $|a\rangle$ and $|b\rangle$ basis.

From equation (B-6) we can easily see that if the system initially is prepared in one of its initial pointer states; i.e. if $|\psi^{\mathcal{S}}(t_0)\rangle = |\pm(t_0)\rangle$, then we have:

$$\begin{aligned}\langle\mathbf{A}(t)|G_{\pm}(t)\mathbf{B}(t)\rangle &= G^*_{\pm}(t)\langle\frac{\mathbf{A}(t)}{G_{\pm}(t)}|G_{\pm}(t)\mathbf{B}(t)\rangle \\ &= G^*_{\pm}(t)\langle\mathbf{B}(t)|\mathbf{A}(t)\rangle = G^*_{\pm}(t)\rho^{\mathcal{S}}_{12}(t)\end{aligned} \tag{B-7}$$

Therefore,

$$q(t) = \frac{\langle\mathbf{A}(t)|G_{\pm}(t)\mathbf{B}(t)\rangle}{|\mathbf{A}(t)|^2} = \frac{G^*_{\pm}(t)\rho^{\mathcal{S}}_{12}(t)}{|\mathbf{A}(t)|^2}; \quad \text{or} \tag{B-8}$$

$$|q(t)| = \frac{|\rho^{\mathcal{S}}_{12}(t)|}{|\mathbf{A}(t)| \times |\mathbf{B}(t)|} = \frac{|\rho^{\mathcal{S}}_{12}(t)|}{\sqrt{\rho^{\mathcal{S}}_{11}(t)} \times \sqrt{1-\rho^{\mathcal{S}}_{11}(t)}}. \tag{B-9}$$

In calculating the above equations we must keep in mind that $\rho^{\mathcal{S}}_{11}(t)$ and $\rho^{\mathcal{S}}_{12}(t)$ must be calculated in the basis of the $|a\rangle$ and $|b\rangle$ basis states; and also they must be calculated for

the case that the system initially is prepared in one of its initial pointer states, given by equation (4.64).

For our spin-boson model and for example for the case that the system initially is prepared in the $|+(t_0)\rangle$ state, from equations (4.89), (4.95) and (4.96) we had

$$\rho_{11}^{\mathcal{S}}(t) = \cos^2(\frac{\varphi}{2} + \frac{t'}{4\sqrt{\bar{n}}})\ e^{-t'^2/32\bar{n}^2} \quad \text{and} \quad |\rho_{12}^{\mathcal{S}}(t)| = \frac{1}{2}|\sin(\varphi + \frac{t'}{2\sqrt{\bar{n}}})|\ e^{-t'^2/32\bar{n}^2}. \tag{B-10}$$

Therefore,

$$|q(t)| = \frac{|\rho_{12}^{\mathcal{S}}(t)|}{\sqrt{\rho_{11}^{\mathcal{S}}(t)} \times \sqrt{1-\rho_{11}^{\mathcal{S}}(t)}} \simeq 1 \quad \text{if and only if} \quad t' \ll \bar{n}. \tag{B-11}$$

One can easily verify that for the case that the system initially is prepared in the $|-(t_0)\rangle$ state also, we would have the same result for the degree of entanglement between the states of the system and the environment. These results basically indicate that our result for the pointer states of the system and the environment is valid over a length of time which is proportional to $\bar{n}$, the average number of photons in the field; since within times of the order of $\hbar\bar{n}/g$ the degree of entanglement, calculated for our pointer states, will stay close to the unity and our calculated pointer states will be immune to entanglement.

Appendix C

The purpose of this appendix is to show that in the limit of $\bar{n} \to \infty$, which corresponds to the classical limit for which equations (6.10) to (6.13) are valid, the field states $|\Phi_+(t)\rangle$ and $|\Phi_-(t)\rangle$ almost promptly become orthogonal and hence the expression in equation (6.12) approximately represents a Schmidt decomposition after a very short period of time.

One can easily calculate the short-time evolution of the JCM [12, 13], where by "short-time" we mean $t \ll t_R$. Here $t_{\rm R} = 2\pi\sqrt{\bar{n}}/{\rm g}$ [5] is the revival time for Rabi oscillations. One has

$$ {\rm g}t\sqrt{n} = gt\sqrt{\bar{n}} + \frac{\rm g}{2}\frac{n-\bar{n}}{\sqrt{\bar{n}}}t - \frac{\rm g}{8}\frac{(n-\bar{n})^2}{\bar{n}^{\,3/2}}t + \ldots \,. \tag{C-1} $$

Now, as long as $t \ll t_R = 2\pi\sqrt{\bar{n}}/{\rm g}$, the third term and all the higher order terms may be ignored [12]. Hence, we can write

$$ \begin{aligned} |\Phi_\pm(t)\rangle &= {\rm e}^{-\bar{n}/2}\sum_{n=0}^{\infty}\frac{\bar{n}^{\,n/2}}{\sqrt{n!}}{\rm e}^{-{\rm i}n\phi}{\rm e}^{\mp{\rm i}{\rm g}t\sqrt{n}}|n\rangle \\ &\simeq {\rm e}^{\mp{\rm i}{\rm g}t\sqrt{\bar{n}}/2}{\rm e}^{-\bar{n}/2}\sum_{n=0}^{\infty}\frac{\bar{n}^{\,n/2}}{\sqrt{n!}}{\rm e}^{-{\rm i}n(\phi\pm{\rm g}t/2\sqrt{\bar{n}})}|n\rangle \\ &= {\rm e}^{\mp{\rm i}{\rm g}t\sqrt{\bar{n}}/2}|\nu{\rm e}^{\,\mp{\rm i}{\rm g}t/2\sqrt{\bar{\rm n}}}\rangle, \end{aligned} \tag{C-2} $$

where $\nu = \sqrt{\bar{n}}{\rm e}^{-{\rm i}\phi}$, and the notation in the ket state means a coherent state with the amplitude $|\nu|$ and and the phase $\phi \pm {\rm g}t/2\sqrt{\bar{n}}$. Then, it follows that for the time considered and in the limit of large $\bar{n}$, the evolution of the system initially prepared in the state $(\gamma|+\rangle + \delta|-\rangle)|\nu\rangle$ (see equation (6.12)) is given by

$$ \begin{aligned} (\gamma|+\rangle + \delta|-\rangle)|\nu\rangle \to \gamma\ &{\rm e}^{-{\rm i}{\rm g}t\sqrt{\bar{\rm n}}/2}\ |+\rangle\ |\nu{\rm e}^{-{\rm i}{\rm g}t/2\sqrt{\bar{\rm n}}}\rangle \\ &+\delta\ {\rm e}^{+{\rm i}{\rm g}t\sqrt{\bar{n}}/2}\ |-\rangle\ |\nu{\rm e}^{{\rm i}{\rm g}t/2\sqrt{\bar{n}}}\rangle. \end{aligned} \tag{C-3} $$

Now, if we look closely at this equation we will notice that in fact it does represent a Schmidt decomposition after only a short period of time, because we can easily verify that the overlap between the two field states $|\nu{\rm e}^{{\rm i}{\rm g}t/2\sqrt{\bar{\rm n}}}\rangle$ and $|\nu{\rm e}^{-{\rm i}{\rm g}t/2\sqrt{\bar{\rm n}}}\rangle$ is given by

$$ \begin{aligned} &|\langle\nu{\rm e}^{-{\rm i}{\rm g}t/2\sqrt{\bar{\rm n}}}|\nu{\rm e}^{{\rm i}{\rm g}t/2\sqrt{\bar{\rm n}}}\rangle|^2 \\ &= \exp\{-4\bar{\rm n}\,\sin^2({\rm g}t/2\sqrt{\bar{\rm n}})\} \simeq {\rm e}^{-{\rm g}^2{\rm t}^2} \end{aligned} \tag{C-4} $$

where the approximation holds for the short times we are considering, namely, $t \ll t_{\mathrm{R}}$ and in the limit of large $\bar{n}$ (which corresponds to the classical limit for which equations (6.10) to (6.13) are valid). In obtaining the last result we have just used the formula for the scalar product of two coherent states [5] given by

$$\langle \nu\,'|\nu\rangle = \exp[-(|\nu\,'|^2 + |\nu|^2)/2 + \nu\,'^*\,\nu]. \tag{C-5}$$

Hence, in the limit of large $\bar{n}$ and for times $t \ll t_{\mathrm{R}}$, within a time of the order of $t_c = 1/\mathrm{g}$ (which is in fact the collapse time for Rabi oscillations) the field states $|\Phi_+(t)\rangle$ and $|\Phi_-(t)\rangle$ become orthogonal (so, this time can represent the decoherence time as well). We also note that in the limit of very large $\bar{n}$, $t_c \ll t_{\mathrm{R}} = 2\pi\sqrt{\bar{n}}/\mathrm{g}$. Therefore, in the limit of a large average number of photons the field states $|\Phi_+(t)\rangle$ and $|\Phi_-(t)\rangle$ almost promptly become orthogonal and hence the expression in equation (6.12) represents a Schmidt decomposition after a very short period of time. (We should also mention that if we use the more accurate expressions in equation (6.11) for the field states $|\Phi_+(t)\rangle$ and $|\Phi_-(t)\rangle$ and plot $|\langle\Phi_+(t)|\Phi_-(t)\rangle|^2$ at the limit of large $\bar{n}$, we will get the same result. i.e, again we will observe that the two states become orthogonal within a very short time of the order of $t_c = 1/\mathrm{g}$).

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