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Commentary on: Michel Dufour's "Argument and explanation in mathematics"

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1. ARGUMENTS, EXPLANATIONS, AND PROOFS

Michel Dufour rightly observes that mathematics "brings together three crucial notions of human rational interaction, namely proof, argument and explanation" (Dufour, 2013, p. 2). He adds that these concepts "can easily overlap, at least in their day-to-day meaning" (ibid.). What do we know about this overlap? We know that there are some arguments which are neither proofs nor explanations. Beyond this, there is a remarkable lack of consensus about all three questions. For each pair of concepts, the conventional wisdom might seem to be that the terms do not overlap (as shown in Fig. 1).
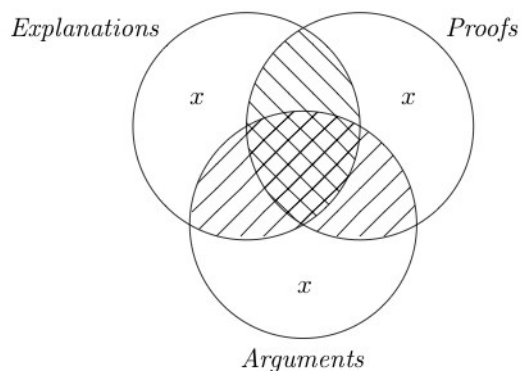

Figure 1: How explanations, proofs, and arguments are related according to conventional wisdom

Although few if any discussions address all three concepts, some people have argued that no proofs are arguments (for example, Perelman & Olbrechts-Tyteca, 1969; Johnson, 2000); some people have argued that no explanations are arguments (for example, Walton, 2004; Copi et al., 2007); and some people have argued that no proofs are explanations (for example, Resnik & Kushner, 1987). Each of these claims is either grounded in a wide practice, or at least is claimed to be so by its proponents. For example, Johnson states that his "requirement of a dialectical tier" for arguments, which he takes to disqualify proofs, "was to shift the paradigm ... [to] examples more illustrative of the best practices" (Johnson, 2002, p. 322); a long list

could be compiled of critical thinking texts and argumentation theory authors who deny that explanations are arguments (see McKeon, 2013, n. 1, for a wide selection); and Resnik and Kushner claim that in actual mathematical practice, as opposed to philosophical reflection on that practice, explanation is "barely acknowledged" (Resnik & Kushner, 1987, p. 151).

On the other hand, some people have argued that all proofs are arguments (for example, Van Bendegem, 2005; Aberdein, 2009) and some people have argued that all explanations are arguments (for example, Hempel & Oppenheim, 1948; McKeon, 2013). And while nobody seems to have claimed that *all* proofs are explanations, plenty of people have argued that *some* proofs are explanations (for example, Steiner, 1978; Kitcher, 1981b; Cellucci, 2008). Although each of these challenges to purported conventional wisdom was articulated independently, they share a common strategy of showing not just that the conventional wisdom was wrong, but that it was never conventional in the first place.
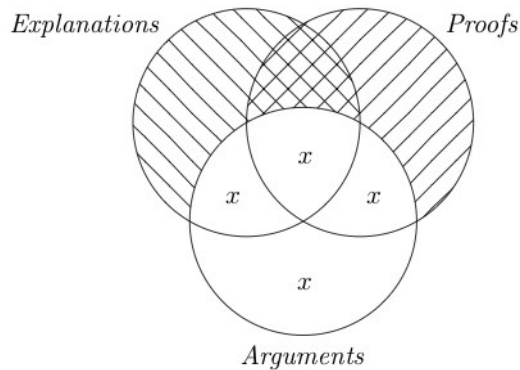

Figure 2: How explanations, proofs, and arguments are related according to the challengers

The profound claim in Dufour's paper is that the three challenges are interlinked. If that's right, we should expect them to stand or fall together. I take Dufour to be saying that they stand (as shown in Fig. 2). Clearly, not every combination of relationships between pairs of the three concepts will be logically possible. For example, if all proofs are arguments and no explanations are, or vice versa if all explanations are arguments and no proofs are, then no explanations can be proofs. But it would be logically possible, if less symmetrically pleasing, to adopt a hybrid position, such as all proofs are arguments but only some explanations are. Nonetheless, there is strong evidence for the individual components of the position shown in Fig. 2. Mathematicians do describe proofs as explanatory and talk about proofs as arguments (as Dufour has illustrated). And the textbook view of explanations as distinct from arguments is difficult to justify either pragmatically or structurally (as both Dufour and McKeon demonstrate).

2. VARIETIES OF EXPLANATION

A diverse range of theories has been proposed to account for scientific explanation (Salmon, 1989). However, many of these theories are unsuitable as accounts of explanation in mathematics, since they make essential use of concepts which have no role in mathematical practice, such as causation or laws of nature. Dufour discusses the three best-known accounts of explanation in mathematics: those of Mark Steiner, Philip Kitcher and Bas van Fraassen. For Steiner, proofs are explanatory if they turn on a "characterizing property" of some entity, which he defines as "a property unique to a given entity or structure within a family or domain of such entities or structures" (Steiner, 1978, p. 143). For van Fraassen, explanations answer why-questions with reference to a contrast class—before we can answer, we need to know if "Why P?" is "Why P rather than Q?" or "Why P rather than R?" (Van Fraassen, 1977, p. 149). For Kitcher explanatory proofs unify, that is they derive a lot of material from a little, something he claims can be achieved by minimizing the number of "argument patterns" or distinctive forms of argument (Kitcher, 1981a, p. 520).

Dufour acknowledges that both Steiner's and van Fraassen's theories have been criticized as not adequate to account for mathematicians' explanation practices (Dufour, 2013, pp. 5 ff.). Several authors complain that Steiner's account of explanation is vulnerable to the charge that characterizing properties are often very difficult to find (Resnik & Kushner, 1987, p. 149; Hafner & Mancosu, 2005, p. 240; Cellucci, 2008, p. 206). David Sandborg argues that, since mathematical propositions are either necessarily true or necessarily false, Van Fraassen must assess all proofs as equally explanatory, thereby making the theory useless as an analysis of mathematicians' practice of distinguishing some proofs as more explanatory than others (Sandborg, 1998, p. 613). Even if this specific feature of Van Fraassen's approach could be overcome, Sandborg identifies a more general problem for why-question theories: if you don't understand a proof, how do you know which why-question to ask? Conversely, for many proofs, knowing which why-question would elicit an explanatory answer is tantamount to knowing that answer (Sandborg, 1998, p. 621).

However, Kitcher's account has faced similarly robust criticism. Johannes Hafner and Paolo Mancosu have a nice example of three proofs of the same theorem, drawn from a graduate textbook, whereby they show that just totting up the numbers of schematic argument patterns not only does not identify the proof the textbook authors described as most explanatory, it actually misidentifies as most explanatory the proof the authors found least explanatory (Hafner & Mancosu, 2008, p. 166). In general, what are sometimes called "nuclear flyswatter" proofs, in which a single, disproportionately powerful technique is repeatedly applied, will be rated highly by Kitcher, whereas elegant combinations of several distinct but simple techniques will not. To remedy this, Kitcher needs an account of *qualitative* differences of proof method, which he does not have.

Dufour alludes to an important distinction which I believe has previously been overlooked in the context of mathematical explanation, that between *trace*, *strategic* and *deep* explanation (Dufour, 2013, p. 6). This originates in the literature

on expert systems, whence it was retrieved by Douglas Walton (Walton, 2004, p. 73; cf. Clancey, 1983; Southwick, 1991; Ye, 1995):

1. *Reasoning trace explanations*: ... explain why a conclusion was reached, or a decision made, by describing the reasoning steps that led to the conclusion. ...
2. *Strategic explanations*: rather than explaining a result by listing rules used, these explanations describe the strategy employed by the problem solver. Strategic explanation give the user an insight into the problem-solving methodology.
3. *Deep explanations*: deep, or model-based explanations *justify* system results by linking them to a deep, causal[1] model. Thus, deep explanations attempt to give the underlying reasons for an action or state. (Southwick, 1991, p. 2)

Perhaps surprisingly, explaining a mathematical proof and explaining the output of an expert system are very similar activities. In both cases a result is obtained as the end state of protracted ratiocination. This is quite unlike the characteristic situation in natural science, where the item requiring explanation is a naturally occurring phenomenon, not the product of a reasoning process. Following that process (trace explanation) or even stating why specific steps were chosen over others (strategic explanation) cannot provide full understanding of why the result is correct (deep explanation). Dufour rightly emphasizes the importance of that understanding (Dufour, 2013, p. 7); in order to say more about how it is obtained we need to say what deep explanations comprise. The existing theories of mathematical explanation attempt this task but, as we have seen, with limited success. In the final section I will make a new proposal for the analysis of deep explanation in mathematics.

3. EXPLANATION AND EXPLICATION

There is something reflexive to the project of writing about explanation, since ultimately one seeks to explain it. One strategy for diminishing this appearance of circularity originates with the work of Rudolf Carnap. He distinguishes between *explanation* and *explication*. Explanation is a relationship between a fact or natural phenomenon and its ostensible explanation. Explication is a process of clarification or conceptual analysis. Carnap proposes four requirements which good explications should meet:

1. The explicatum [the thing which explicates] is to be *similar to the explicandum* [the thing requiring explication] in such a way that, in most cases in which the explicandum has so far been used, the explicatum can be used; however, close similarity is not required, and considerable differences are permitted.

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[1] Although this version of the distinction references a *causal* model, the reference is inessential: the expert systems at issue happened to involve activities (auditing, diagnosing) naturally explained causally, so an appropriate deep explanation would be causal. In non-causal domains deep explanation would be non-causal. Not every presentation uses 'causal'—Walton, for example, omits it—and the expert systems themselves are inferential not causal.

2. The characterization of the explicatum, that is, the rules of its use (for instance, in the form of a definition), is to be given in an *exact* form, so as to introduce the explicatum into a well-connected system of scientific concepts.
3. The explicatum is to be a *fruitful* concept, that is, useful for the formulation of many universal statements (empirical laws in the case of a nonlogical concept, logical theorems in the case of a logical concept).
4. The explicatum should be as *simple* as possible; this means as simple as the more important requirements (1), (2), (3) permit (Carnap, 1950, p. 7).

So the task of the philosopher studying explanation is not to explain it but to explicate it: to construct a simple, fruitful and exact model which is demonstrably similar to the actual practice of explanation. Success in this task would yield more than a description, it should also be normative: practice that departed from the model could be criticized for so doing. Each of the accounts of mathematical explanation claims to have such a model, but none of them is a good fit with all four criteria. Perhaps the mistake has been to look too far afield; could explication itself act as deep mathematical explanation?

As yet no philosopher of mathematics appears to have argued that mathematical explanation just is explication, but some have defended a view of mathematics as explicatory. For example, "I would like to claim that the basic similarity between philosophy and mathematics is the focus on the explication of informal concepts" (Kuipers, 2005, p. 170) or "getting a clearly articulated grasp of the concepts is not merely prerequisite for mathematical knowledge: it is the whole story" (Eagle, 2008, p. 69). Conversely, some people have made the same point in the opposite direction, for example by arguing that philosophical explication resembles mathematical reasoning: "conceptual analysis, as exemplified by the famous Gettier programme in the analysis of knowledge, has the heuristic form of proofs and refutations that [the philosopher of mathematics] Lakatos identifies" (Harper, 2012, p. 236). So an explicative account of mathematical explanation would be continuous with a plausible position in the philosophy of mathematics.

Some new empirical research provides stronger support for this account. I recently collaborated in the design of a study in which 255 mathematicians were asked to characterise a proof of their choice using 80 different adjectives that have often been used to describe mathematical proofs (Inglis & Aberdein, 2013). A five-point Likert scale was provided for each adjective and the responses were subjected to a principal components analysis. The adjectives lined up on four major dimensions, which we characterized as aesthetics, intricacy, precision, and utility. The relevance of this study for present purposes concerns the position of 'explanatory' with respect to these four dimensions. The study suggests that explanatoriness is a multi-dimensional concept: it was positively correlated with precision and (more weakly) utility, negatively correlated with intricacy, and not correlated with aesthetics. That's to say, the mathematicians who completed the study expected explanatory proofs to be precise, useful and not intricate. Or, in other words, simple, fruitful and exact. The deep mathematical explanation to be found in mathematicians' descriptions of proofs is, in Carnap's sense, not explanation at all: it's explication.

REFERENCES

Aberdein, A. (2009). Mathematics and argumentation. *Foundations of Science*, *14*, 1–8.

Carnap, R. (1950). *Logical Foundations of Probability*. London: Routledge & Kegan Paul.

Cellucci, C. (2008). The nature of mathematical explanation. *Studies in History and Philosophy of Science*, *39*, 202–210.

Clancey, W. J. (1983). The epistemology of a rule-based expert system: A framework for explanations. *Artificial Intelligence*, *20*, 215–251.

Copi, I. M., Cohen, C., & Flage, D. (2007). *Essentials of Logic*. Upper Saddle River, NJ: Prentice Hall/Pearson.

Dufour, M. (2013). Argument and explanation in mathematics. In D. Mohammed & M. Lewiński (Eds.), *Virtues of Argumentation: Proceedings of the 10th International Conference of the Ontario Society for the Study of Argumentation (OSSA), May 22–25, 2013*, (pp. 1–11). Windsor, ON: OSSA.

Eagle, A. (2008). Mathematics and conceptual analysis. *Synthese*, *161*, 67–88.

Hafner, J., & Mancosu, P. (2005). The varieties of mathematical explanation. In P. Mancosu, K.F. Jørgensen & S.A. Pedersen (Eds.), *Visualization, Explanation and Reasoning Styles in Mathematics*, (pp. 215–250). Dordrecht: Springer.

Hafner, J., & Mancosu, P. (2008). Beyond unification. In P. Mancosu (Ed.), *The Philosophy of Mathematical Practice*, (pp. 151–178). Oxford: Oxford University Press.

Harper, A. S. (2012). An oblique epistemic defence of conceptual analysis. *Metaphilosophy*, *43*, 235–256.

Hempel, C. G., & Oppenheim, P. (1948). Studies in the logic of explanation. *Philosophy of Science*, *15*, 135–175.

Inglis, M., & Aberdein, A. (2013). The personality of mathematical proofs. Draft.

Johnson, R. H. (2000). *Manifest Rationality: A Pragmatic Theory of Argument*. Mahwah, NJ: Lawrence Erlbaum Associates.

Johnson, R. H. (2002). Manifest rationality reconsidered: Reply to my fellow symposiasts. *Argumentation*, *16*, 311–331.

Kitcher, P. (1981a). Explanatory unification. *Philosophy of Science*, *48*, 507–531.

Kitcher, P. (1981b). Mathematical rigor—Who needs it? *Noûs*, *15*, 469–493.

Kuipers, T. A. (2005). Mathematics and explication: Reply to Jean Paul Van Bendegem. In R. Festa, A. Aliseda & J. Peijnenburg (Eds.), *Cognitive Structures in Scientific Inquiry: Essays in Debate with Theo Kuipers*, Vol. 2, (pp. 170–173). Amsterdam: Rodopi.

McKeon, M. W. (2013). On the rationale for distinguishing arguments from explanations. *Argumentation*, forthcoming.

Perelman, C., & Olbrechts-Tyteca, L. (1969). *The New Rhetoric: A Treatise on Argumentation*. Notre Dame, IN: University of Notre Dame Press.

Resnik, M., & Kushner, D. (1987). Explanation, independence, and realism in mathematics. *British Journal for the Philosophy of Science*, *38*, 141–158.

Salmon, W. C. (1989). Four decades of scientific explanation. *Minnesota Studies in the Philosophy of Science*, *13*, 3–219.

Sandborg, D. (1998). Mathematical explanation and the theory of why-questions. *British Journal for the Philosophy of Science*, *49*, 603–624.

Southwick, R. W. (1991). Explaining reasoning: An overview of explanation in knowledge-based systems. *The Knowledge Engineering Review*, *6*, 1–19.

Steiner, M. (1978). Mathematical explanation. *Philosophical Studies*, *34*, 135–151.

Van Bendegem, J.P. (2005). Proofs and arguments: The special case of mathematics. In R. Festa, A. Aliseda & J. Peijnenburg (Eds.), *Cognitive Structures in Scientific Inquiry: Essays in Debate with Theo Kuipers*, Vol. 2, (pp. 157–169). Amsterdam: Rodopi.

Van Fraassen, B. (1977). The pragmatics of explanation. *American Philosophical Quarterly*, *14*, 143–150.

Walton, D. N. (2004). A new dialectical theory of explanation. *Philosophical Explorations*, *7*, 71–89.

Ye, L. R. (1995). The value of explanation in expert systems for auditing: An experimental investigation. *Expert Systems with Applications*, *9*, 543–556.