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MEDHAT MOUNIR. GHOBRIAL

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LA THÈSE A ÉTÉ MICROFILMÉE TELLE QUE NOUS L'AVONS RÉCU
ANALYSIS AND STABILITY OF
ORTHOTROPIC CYLINDRICAL CANTILEVER SHELLS

by

Medhat Mounir Ghobrial

A Dissertation submitted to the Faculty of Graduate Studies through the Department of Civil Engineering in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy at The University of Windsor

Windsor, Ontario, Canada

1977
ABSTRACT

The main objective of this work is to present a thorough study of the behaviour of orthotropic cantilever cylindrical shells, specially those made of corrugated sheets, in both the prebuckling and buckling stages. The need for this study arises from the fact that although these shells are frequently built as grain bins, oil storages ... etc, such study is not available in the literature.

Therefore, this work is composed of:

1. Complete solution for the prebuckling elastic stage, where closed-form expressions are given for the displacement and internal force components using an 8th order and an approximate 4th order governing systems. For practical use, curves are given for the calculation of the force components at the critical locations of the shell.

2. Theoretical treatment of the stability problem, where Trefftz's variational theory is employed. The overall buckling load is obtained for shells with different properties.

3. Experimental programme to verify the theoretical results obtained. Experimentally obtained results show reasonable agreement with the theoretically obtained ones for both the prebuckling and buckling stages.
ACKNOWLEDGEMENTS

This present work has been carried out under the advice of Dr. G. Abdel-Sayed, Professor of Civil Engineering at the University of Windsor. Dr. Abdel-Sayed has always been the main source of inspiration and active guidance for almost the past three years. To him, the author wishes to express his deepest sense of gratitude and appreciation, thanks to the golden opportunity of working under his supervision.

The author is grateful to his doctoral committee for their suggestions and encouragement at the time of his comprehensive examination.

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TABLE OF CONTENTS

ABSTRACT ........................................................................ ii

ACKNOWLEDGEMENTS .................................................. iii

TABLE OF CONTENTS .................................................... iv

LIST OF TABLES ............................................................ v

LIST OF FIGURES ........................................................... vi

NOTATIONS ................................................................... xi

CHAPTER

I. INTRODUCTION .......................................................... 1

II. PREBUCKLING ANALYSIS .......................................... 5

III. BUCKLING ANALYSIS .............................................. 40

IV. EXPERIMENTAL INVESTIGATION .............................. 73

V. OBSERVATIONS AND CONCLUSIONS .......................... 85

REFERENCES ............................................................... 88

TABLES .................................................................. 90

FIGURES .................................................................. 94

APPENDIX I ................................................................. 179

VITA AUCTORIS ........................................................... 187
<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-1</td>
<td>Buckling Load for Isotropic Shells with Restrained Upper Edges</td>
</tr>
<tr>
<td>3-4</td>
<td>Buckling Load for Isotropic Shells with Flexible Upper Edges</td>
</tr>
<tr>
<td>3-5</td>
<td>Buckling Load for Orthotropic Shells with Flexible Upper Edges</td>
</tr>
</tbody>
</table>
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Fig.</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-1</td>
<td>The Non-dimensioned Cylindrical Coordinate System</td>
<td>95</td>
</tr>
<tr>
<td>2-2</td>
<td>The Shell Element</td>
<td>96</td>
</tr>
<tr>
<td>2-3</td>
<td>Geometry of Corrugation</td>
<td>95</td>
</tr>
<tr>
<td>2-4a</td>
<td>Wind Load Distribution</td>
<td>97</td>
</tr>
<tr>
<td>2-4b</td>
<td>Pressure due to Eccentric Discharge</td>
<td>97</td>
</tr>
<tr>
<td>2-5</td>
<td>Radial Displacement along the Totally to Flexible Upper Edge of Orthotropic</td>
<td>98</td>
</tr>
<tr>
<td>2-10</td>
<td>Shell under Wind Loading</td>
<td></td>
</tr>
<tr>
<td>2-11</td>
<td>Radial Displacement along the Totally to Flexible Upper Edge of Isotropic</td>
<td>104</td>
</tr>
<tr>
<td>2-16</td>
<td>Shell under Wind Loading</td>
<td></td>
</tr>
<tr>
<td>2-17</td>
<td>Comparison of Radial Displacement</td>
<td>110</td>
</tr>
<tr>
<td>2-18</td>
<td>with Reference (22)</td>
<td></td>
</tr>
<tr>
<td>2-19</td>
<td>Maximum Circumferencial Moment (at $x=L/R$, $\phi=0$) for Shells with Flexible Upper Edges; under Wind Load</td>
<td>112</td>
</tr>
<tr>
<td>2-20</td>
<td>Maximum Circumferencial Moment (along the generator $\phi=0$) for Shells with Restrained Upper Edges; under Wind Load</td>
<td>113</td>
</tr>
<tr>
<td>2-21</td>
<td>Typical Distribution of $M_{\phi}$ along the generator $\phi=0$ for Shells with Flexible Upper Edges; under Wind Load</td>
<td>114</td>
</tr>
<tr>
<td>2-22</td>
<td>Typical Distribution of $M_{\phi}$ along the generator $\phi=0$ for Shells with Restrained Upper Edges; under Wind Load</td>
<td>115</td>
</tr>
<tr>
<td>2-23</td>
<td>Deviation in results of $M_{\phi,\text{max}}$ between the two approaches, for Shells with Restrained Upper Edges; under Wind Load</td>
<td>116</td>
</tr>
</tbody>
</table>
Fig. 2-24 Typical Distribution of $M_x$ along the generator $\phi=0$ for Shells with Flexible Upper Edges; under Wind Load ............... 117

Fig. 2-25 Typical Distribution of $M_x$ along the generator $\phi=0$ for Shells with Restrained Upper Edges; under Wind Load .......... 119

Fig. 2-26 Ratio of Maximum $M_x$ to Maximum $M_\phi$ vs. $E_\phi/E_x$ ............................................... 120

Fig. 2-27 Maximum Longitudinal Bending Moment (at $x=\phi=0$) for Shells with Flexible Upper Edges; under Wind Load .......... 121

Fig. 2-28 Maximum Longitudinal-Bending Moment (at $x=\phi=0$) for Shells with Restrained Upper Edges; under Wind Load .......... 122

Fig. 2-29 Maximum Hoop Force (at $x=L/2R$, $\phi=0$) for Shells with Restrained Upper Edges; under Wind Load ....................... 123

Fig. 2-30 Maximum Hoop Force (at $x=L/2R$, $\phi=0$) for Shells with Flexible Upper Edges; under Wind Load ....................... 124

Fig. 2-31 Maximum Membrane Shear Force (at $x=0$, $\phi=\pi/2$) for Shells with Restrained Upper Edges; under Wind Load ............... 125

Fig. 2-32 Maximum membrane Shear Force (at $x=0$, $\phi=\pi/2$) for Shells with Flexible Upper Edges; under Wind Load ............... 126

Fig. 2-33 Maximum Longitudinal Force (at $x=\phi=0$) for Shells with Restrained Upper Edges; under Wind Load ....................... 127

Fig. 2-35 Maximum Longitudinal Force (at $x=\phi=0$) for Shells with Flexible Upper Edges; under Wind Load ....................... 130

Fig. 2-37 Deviation in results of $N_{x_{\text{max}}}$ between the two approaches, for shells with Restrained Upper Edges; under Wind Load .......... 131
<table>
<thead>
<tr>
<th>Fig.</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-38</td>
<td>Typical Distribution of $N_\phi$ along the generator $\phi=0$ for Shells with Flexible Upper Edges; under Wind Load</td>
<td>132</td>
</tr>
<tr>
<td>2-39</td>
<td>Typical Distribution of $N_\phi$ along the generator $\phi=0$ for Shells with Restrained Upper Edges; under Wind Load</td>
<td>133</td>
</tr>
<tr>
<td>2-40</td>
<td>Typical Distribution of $N_X$ along the generator $\phi=0$ for Shells with Flexible Upper Edges; under Wind Load</td>
<td>134</td>
</tr>
<tr>
<td>2-41</td>
<td>Typical Distribution of $N_X$ along the generator $\phi=\pi/2$ for Shells with Restrained Upper Edges; under Wind Load</td>
<td>135</td>
</tr>
<tr>
<td>2-42</td>
<td>Typical Distribution of $N_{Xe}$ along the generator $\phi=\pi/2$ for Shells with Flexible Upper Edges; under Wind Load</td>
<td>136</td>
</tr>
<tr>
<td>2-43</td>
<td>Typical Distribution of $N_{Xe}$ along the generator $\phi=\pi/2$ for Shells with Restrained Upper Edges; under Wind Load</td>
<td>137</td>
</tr>
<tr>
<td>3-1</td>
<td>A Deformed Shell Element</td>
<td>138</td>
</tr>
<tr>
<td>3-2</td>
<td>Comparison of Buckling Load to Isotropic Shells with References (6) and (7)</td>
<td>139</td>
</tr>
<tr>
<td>3-5</td>
<td>Buckling Load of Isotropic Shells vs. L/R for Shells with Restrained Upper Edges; under Wind Load</td>
<td>142</td>
</tr>
<tr>
<td>3-6</td>
<td>Buckling Load of Isotropic Shells vs. R/t for Shells with Restrained Upper Edges; under Wind Load</td>
<td>143</td>
</tr>
<tr>
<td>3-7</td>
<td>Buckling Load of Isotropic Shells vs. L/R for Shells with Free upper Edges; under Wind Load</td>
<td>144</td>
</tr>
<tr>
<td>3-8</td>
<td>Buckling Load of Isotropic Shells vs. R/t for Shells with Free Upper Edges; under Wind Load</td>
<td>145</td>
</tr>
<tr>
<td>Fig.</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>------</td>
<td>-------------</td>
<td>------</td>
</tr>
<tr>
<td>3-9</td>
<td>Increase in Buckling Load due to Restraining the Upper Edge</td>
<td>146</td>
</tr>
<tr>
<td>3-10</td>
<td>Buckling Load of Orthotropic Shells with Flexible Upper Edges; under Wind Load</td>
<td>147</td>
</tr>
<tr>
<td>3-11</td>
<td>Buckling Load for Shells with Restrained upper edges obtained by</td>
<td>148</td>
</tr>
<tr>
<td>3-13</td>
<td>Using the two prebuckling approaches</td>
<td></td>
</tr>
<tr>
<td>3-14</td>
<td>Deviation in Buckling Load between the two Prebuckling Approaches for Shells with Restrained Upper Edges</td>
<td>151</td>
</tr>
<tr>
<td>3-15</td>
<td>Buckling Configurations for Orthotropic Shells with Flexible Upper Edges; under Wind Load</td>
<td>152</td>
</tr>
<tr>
<td>4-1</td>
<td>Set up of Shell No.1</td>
<td>155</td>
</tr>
<tr>
<td>4-2</td>
<td>Floor System of Shells</td>
<td>156</td>
</tr>
<tr>
<td>4-3</td>
<td>Set up of Shell No.2</td>
<td>157</td>
</tr>
<tr>
<td>4-4</td>
<td>Loading System</td>
<td>157</td>
</tr>
<tr>
<td>4-5</td>
<td>Shell No.1 before Loading</td>
<td>158</td>
</tr>
<tr>
<td>4-6 to 4-8</td>
<td>Shell No.1 Profile during Loading</td>
<td>159</td>
</tr>
<tr>
<td>4-9</td>
<td>Shell No.1 Profile at buckling Load</td>
<td>162</td>
</tr>
<tr>
<td>4-10</td>
<td>Shell No.2 before Loading</td>
<td>162</td>
</tr>
<tr>
<td>4-11</td>
<td>Interior of Shell No.2</td>
<td>163</td>
</tr>
<tr>
<td>4-12</td>
<td>Buckling Configuration of Shell No.2</td>
<td>163</td>
</tr>
<tr>
<td>4-13</td>
<td>Vertical Displacement of the Upper Edge at the buckling Load</td>
<td>164</td>
</tr>
<tr>
<td>4-14</td>
<td>Load Distribution</td>
<td>164</td>
</tr>
<tr>
<td>Fig.</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>4-15</td>
<td>Load-Deflection Diagram for points along the edge $x = L/R$ for Shell No.1.</td>
<td>165</td>
</tr>
<tr>
<td>4-16</td>
<td>Load-Deflection Diagram for points along the generator $\phi = 0$ for Shell No.1.</td>
<td>166</td>
</tr>
<tr>
<td>4-17</td>
<td>Typical Distribution of Radial Displacement along the Upper Edge for Shell No.1</td>
<td>167</td>
</tr>
<tr>
<td>4-18</td>
<td>Typical Distribution of Radial Displacement along the generator $\phi = 0$ for Shell No.1.</td>
<td>168</td>
</tr>
<tr>
<td>4-19</td>
<td>Load-$N_\phi$ Diagram at the point $(x = L/R, \phi = 0)$ for Shell No.1.</td>
<td>169</td>
</tr>
<tr>
<td>4-20</td>
<td>Load-$N_\phi$ Diagram at the point $(x = L/2R, \phi = 0)$ for Shell No.1.</td>
<td>170</td>
</tr>
<tr>
<td>4-21</td>
<td>Load-$N_\phi$ Diagram at the point $(x = L/2R, \phi = 0)$ for Shell No.1.</td>
<td>171</td>
</tr>
<tr>
<td>4-22</td>
<td>Typical Distribution of the Radial Displacement along the generator $\phi = 0$ for Shell no.2.</td>
<td>172</td>
</tr>
<tr>
<td>4-23</td>
<td>Typical Distribution of the Radial Displacement along the Upper Edge for Shell No.2.</td>
<td>173</td>
</tr>
<tr>
<td>4-24</td>
<td>The relation between the Relative Rigidity $D_\phi/E_b I_b$ and $N_\phi$ at $x = L/R, \phi = 0$.</td>
<td>174</td>
</tr>
<tr>
<td>4-25</td>
<td>Load-Deflection Diagram for points along the generator $\phi = 0$ for Shell No.2.</td>
<td>175</td>
</tr>
<tr>
<td>4-26</td>
<td>Load-Deflection Diagram for points along the Upper Edge for Shell No.2.</td>
<td>176</td>
</tr>
<tr>
<td>4-27</td>
<td>Load-$N_\phi$ Diagram at the point $(x = L/2R, \phi = 0)$ for Shell No.2.</td>
<td>177</td>
</tr>
<tr>
<td>4-28</td>
<td>Load-$N_\phi$ Diagram at the point $(x = L/2R, \phi = 0)$ for Shell No.2.</td>
<td>178</td>
</tr>
</tbody>
</table>
NOTATIONS

\( A_{mn}, B_{mn}, C_{mn} \) = parameters governing the virtual displacement components in x-, φ-, z-directions, respectively;

\( a_n, b_m \) = constants governing the distribution of loading in the vertical and horizontal directions, respectively;

\( B_x, B_\phi \) = bending rigidity in xz- and oz-planes, respectively;

\( B_{x\phi} \) = torsional rigidity;

\( c \) = corrugation pitch;

\( D_x, D_\phi \) = axial rigidity in x- and φ-directions, respectively;

\( D_{x\phi} \) = shear rigidity in xφ-plane;

\( E \) = modulus of elasticity of the material of the corrugated sheets;

\( F \) = stability submatrix independent of load;

\( f \) = half depth of corrugation;

\( G \) = stability submatrix multiplied by the load parameter;

\( K_{xx}, K_{\phi\phi} \) = change of curvature of the middle surface in the x- and φ-directions, respectively;

\( L \) = height of the shell;

\( M_x, M_\phi \) = bending moment per unit length acting about the xz- and oz-planes, respectively;
\( M_{x\phi}, M_{\phi x} \) = torsional moment per unit length acting about the \( \phi \)- and \( x \)-axes respectively;

\( N_x, N_{\phi} \) = axial force per unit length acting in the \( x \)- and \( \phi \)-directions, respectively;

\( N_{x\phi} \) = shear force per unit length acting in the \( x-\phi \) plane;

\( P, Q \) = number of terms taken into consideration from the series describing the virtual displacements during buckling;

\( P_x, P_{\phi}, P_z \) = external applied load per unit area of the middle surface acting in the \( x \)-, \( \phi \)- and \( z \)-directions, respectively;

\( Q_x, Q_{\phi} \) = lateral shear force per unit length acting perpendicular to \( x \)- and \( \phi \)-axes, respectively;

\( Q_{x\text{eff}} \) = actual lateral shear force per unit length acting perpendicular to \( x \)-axis after employing Kirchhoff's condition;

\( R_{1,2,3} \) = functions in the stability matrix independent of load

\( R \) = radius of curvature of the shell;

\( S_{1,2,3} \) = functions in the stability matrix multiplied by the load parameter;

\( t \) = average thickness of the shell;

\( U \) = total potential energy of the shell;

\( V \) = strain energy of the shell;

\( W \) = workdone exerted by the applied load;
\( u, v, w \) = non-dimensional displacement components in the 
\( x-, \phi- \) and \( z- \) directions, respectively;

\( \alpha_1, \alpha_2 \) = real parts in the complex roots of the
characteristic equation;

\( \beta_1, \beta_2 \) = imaginary parts in the complex roots of the
characteristic equation;

\( \varepsilon_x, \varepsilon_\phi \) = axial strains at the middle surface in the
\( x- \) and the \( \phi- \) directions, respectively;

\( \lambda \) = eigenvalue;

\( \mu \) = Poisson's ratio;

\( \xi, \eta, \zeta \) = virtual displacement components during buckling
in \( x-, \phi- \) and \( z- \) directions, respectively;

\( \sigma_x, \sigma_\phi \) = axial stresses in the \( x- \) and \( \phi- \) directions,
respectively;

\( \tau_{x\phi} \) = shear stress in \( x\phi- \) plane;
CHAPTER I
INTRODUCTION

Cantilever cylindrical shells are often built for use as oil storages, grain bins, chimneys...etc. They are of a complete circular cross section with a lower edge connected to the floor system. The upper edge could be free or connected to a roof system, usually in the form of a conical shell.

These shells are usually built of cold-formed corrugated sheet metals. According to previous studies, the rigidity properties of the sheets can be obtained as functions of the average geometric proportions of the corrugation, therefore, the shell can be treated as if made of an orthotropic material (1).

The pre-buckling analysis of isotropic cantilever shells has been studied before by Hoff(2), Vlasov (3), Krajcinovic(4) and others, while an approximate prebuckling analysis has been presented (5) for the case of orthotropic shells. For buckling analysis, Johns etal(6) and Langhar(7) presented solutions for the case of isotropic shells only. Although buckling is often a cause of failure for the cantilever cylindrical orthotropic shells, a solution for their stability problem is not available in the literature.

Therefore, the present investigation deals with a thorough theoretical study for both the pre-buckling and buckling analysis of orthotropic cantilever shells supported
by experimental verification leading to concrete results both academically and practically.

The pre-buckling analysis is based on the governing system of differential equations given by (8). This system of equations is considered to be adequately describing the pre-buckling elastic behaviour of shells made of corrugated sheets. Also, the approximate system presented in (5) is adopted in this work. Both of the two systems are solved and considered as the basis for the pre-buckling and the buckling analysis.

A numerical programme for the pre-buckling analysis is carried out for a practical range of variables, loading conditions and boundary conditions. Typical distribution for various internal force components and displacement components in the longitudinal and circumferencial directions are established. Charts are prepared for calculating various internal force components for practical applications.

The deviation in results between the approximate approach and the more exact one is studied and limits of validity of the approximate approach for the pre-buckling behaviour are suggested.

The theory adopted in the buckling analysis is Trefftz's variational theory which is believed to be more rigorous than the conventional energy theories as it has been stated by Langhaar(9). The theory was first developed by E. Trefftz in 1932 as a useful tool for solving the stability problem of elastic systems. However, since then
it has not been used very frequently because of the amount of computation involved especially during the pre-computer era. In the recent years, however, a few investigators began to use this theory and found that, if applied adequately, it could yield a very accurate solution for the problem investigated. Boresi (10) started using Trefftz's theory for the two-dimensional problem of thick ring under uniform radial pressure, then Almroth (11) used the same procedure for another two-dimensional problem of a ring under non-uniform radial load. In 1973 Johns and others (12) tried the first three-dimensional problem of the stability of isotropic cantilever shells under nonuniform load, supporting their results by extensive experimental programme.

An expression for the second variation of the total potential energy of the orthotropic shells is established, from which a stability matrix is constructed. Two computer programmes for computing the buckling load are prepared; one is based on the approximate pre-buckling analysis and the other on the more exact formulation.

The buckling load is calculated for the same cases considered before in the pre-buckling numerical scheme. The deviation in buckling load while employing the approximate and the more exact pre-buckling approaches is examined to study the validity of the approximate solution during the buckling stage.

It was necessary to verify the results of both the pre-buckling and buckling analysis. Unfortunately
information needed for such a verification does not exist in the literature, neither for the analysis nor for the stability, except for the special case of isotropic shells. So, experimental information was the only available support.

An experimental programme was carried out on shells made of light gauge corrugated sheets. The radial displacement, the circumferential moment, the hoop force, the buckling load and the buckling configuration were recorded during testing.

The experimental results show good agreement with the theoretical ones in both the pre-buckling and buckling behaviour.
CHAPTER II
PRE-BUCKLING ANALYSIS

2.1 Introduction

The present chapter deals with the pre-buckling elastic behaviour of cantilever orthotropic cylindrical shells. The behaviour is assumed to be linear and the formulation of the governing equations follows the main assumptions of the elastic linear theory. Closed form expressions are derived for the displacement components and internal force components. Numerical results are obtained and comparison is made between different solutions. Also expressions for the pre-buckling membrane strain components which are required for the buckling analysis, as it will be shown later, are obtained.

2.2 Governing Differential Equations

Considering the non-dimensional cylindrical coordinate system and the shell element shown in Figs. (2-1) and (2-2), three groups of equations can be obtained. First, six equilibrium equations which can be reduced to the following three equations in the absence of body forces and after eliminating the lateral shear forces $Q_x$ and $Q_\phi$ and assuming that $N_{x\phi} = N_{\phi x}$ and $M_{x\phi} = M_{\phi x}$:

\[ N_{x,x} + N_{x,\phi} + R P_{\phi x} = 0 \quad (2-1a) \]

\[ R N_{\phi,\phi} + R N_{x\phi,x} - M_{\phi,\phi} - M_{x\phi,x} + R^2 P_{\phi} = 0 \quad (2-1b) \]
\[ M_{\phi,\phi} + 2M_{x\phi,x\phi} + M_{x,xx} + RN_{\phi} + R^2P_z = 0 \] (2-1c)

in which,

the comma denotes a derivative with respect to \( x \) or \( \phi \);

\( N_x, N_\phi = \) the axial force per unit length acting at the middle surface in \( x \) and \( \phi \) directions, respectively;

\( N_{x\phi} = N_{\phi x} = \) shearing force per unit length acting at the middle surface in \( x-\phi \) plane;

\( M_x, M_\phi = \) bending moment per unit length acting in \( x-z \) and \( x-\phi \) planes, respectively;

\( M_{x\phi} = M_{\phi x} = \) torsional moment per unit length acting about \( \phi \) and \( x \) axis, respectively;

\( P_x, P_\phi, P_z = \) external loading per unit area of the middle surface acting in \( x-, \phi-, z- \) directions, respectively;

\( R = \) radius of shell.

Second, the geometric relationships, which can be written as follows, after assuming that Poisson's ratio\((\nu)\) is equal to zero.

\[ \varepsilon_x = u_x \] (2-2a)

\[ \varepsilon_\phi = v_\phi - w \] (2-2b)

\[ \gamma_{x\phi} = v_x + u_\phi \] (2-2c)

\[ K_{xx} = (w_{xx})/R \] (2-2d)

\[ K_{\phi\phi} = (w_{\phi\phi} + w)/R \] (2-2e)

\[ K_{x\phi} = (w_{x\phi})/R \] (2-2f)
in which,

\( u, v, w \) = non-dimensional displacement components at the middle surface in the \( x-, \phi-, z \)-directions, respectively, as shown in Fig.(2-1);

\( \varepsilon_x, \varepsilon_\phi \) = axial strain of the middle surface in the \( x \) and \( \phi \) directions, respectively;

\( \varepsilon_{x\phi} \) = shear strain of the middle surface in \( x-\phi \) plane;

\( K_{xx}, K_{\phi\phi} \) = change of curvature of the middle surface in the \( x \) and \( \phi \) directions, respectively.

Third, the elastic relationships of orthotropic material are recalled. Using these relationships and employing Eqs. (2-2) after integrating along the shell thickness, the internal force components, as given by (8), can be expressed in terms of the displacement components as follows:

\[
\begin{align*}
N_\phi &= D_\phi (v_\phi - w) - \frac{B_\phi}{R^2} (w + w_{\phi\phi}) \\
N_x &= D_x u_x \\
N_{x\phi} &= D_{x\phi} (v_x + u_\phi) \\
M_\phi &= \frac{B_\phi}{R} (w + w_{\phi\phi}) \\
M_x &= \int \frac{B_\phi}{R} w_{xx} \\
M_{x\phi} &= M_{\phi x} = -\frac{B_{x\phi}}{R} w_x \\
Q_{x\phi} & = Q_{x\phi}^{\text{eff}} = -\left( \frac{B_x}{R^2} w_{xxx} + 2 \frac{B_{x\phi}}{R^2} w_{x\phi\phi} \right)
\end{align*}
\]

in which,
\(D_x, D_\phi = \) axial rigidity in \(x-\) and \(\phi-\)direction, respectively;

\(D_{x\phi} = \) shear rigidity in \(x-\phi\) plane;

\(B_x, B_\phi = \) bending rigidity in \(xz-\) and \(\phi z-\)planes, respectively;

\(B_{x\phi} = \) torsional rigidity.

The rigidity properties of a shell made of corrugated sheets are given by (1) as functions of the geometric proportions of corrugation and expressed as follows:

\[
D_x = \frac{E}{6(1-\mu^2)} \left(\frac{t}{f}\right)^2 t \tag{2-4a}
\]

\[
D_\phi = \frac{E}{c} \frac{t}{E} \tag{2-4b}
\]

\[
D_{x\phi} = \frac{E}{2(1+\mu)} \cdot \frac{c}{E} \tag{2-4c}
\]

\[
B_x = \frac{c}{2} \frac{E t^3}{12(1-\mu^2)} \tag{2-4d}
\]

\[
B_\phi = 0.522 \frac{E f^2 t}{E} \tag{2-4e}
\]

\[
B_{x\phi} = \frac{c}{2} \frac{E t^3}{12(1+\mu)} \tag{2-4f}
\]

in which,

\(t, f, c\) and \(c\) = geometric proportions of corrugation as shown in Fig. (2-3)

\(E = \) modulus of elasticity of the material

\(\mu = \) Poisson's ratio

By substituting Eqs. (2-3) and (2-4) in the equilibrium equations, Eqs. (2-1), a governing system of differential equations in \(u, v\) and \(w\) can be obtained. After
neglecting the higher order derivatives of $u$ and $v$ with respect to $x$ and $\phi$ when they appear in the same term with derivatives of $w$, also neglecting the terms multiplied by the small quantity $(\frac{t}{r})^2$ in the expressions of the membrane internal force components, the governing system of differential equations is reduced to the following (8):

\[
D_x u_{xx} + D_x (u_{\phi \phi} + v_{x \phi}) = - RP_x \tag{2-5a}
\]

\[
D_\phi (v_{\phi \phi} - w_\phi) + D_x (u_{\phi x} + v_{xx}) = - RP_\phi \tag{2-5b}
\]

\[
B_\phi (w_{\phi \phi \phi \phi} + 2w_{\phi \phi} + w) + B_x w_{xxxx} + 2B_x w_{xxx \phi} - R^2 D_\phi (v_\phi - w) = R^3 P_z \tag{2-5c}
\]

This system of differential equations has been found to be suitable to describe the elastic behaviour of shells made of corrugated sheets (8).

However, the longitudinal bending rigidity $B_x$ and torsional rigidity $B_{x \phi}$ of grain bins are normally too small when compared to the bending rigidity $B_\phi$. This observation enhances the confidence to utilize Valsov's approximation (3) in which the longitudinal bending moment $M_x$, and the twisting moment $M_{x \phi}$, are neglected.

Applying this approximation to Eqs. (2-5) leads to the following system of differential equations (5):

\[
D_x u_{xx} + D_x (u_{\phi \phi} + v_{x \phi}) = - RP_x \tag{2-6a}
\]

\[
D_\phi (v_{\phi \phi} - w_\phi) + D_x (u_{\phi x} - v_{xx}) = - RP_\phi \tag{2-6b}
\]

\[
B_\phi (w_{\phi \phi \phi \phi} + 2w_{\phi \phi} + w) - R^2 D_\phi (v_\phi - w) = R^3 P_z \tag{2-6c}
\]
Both the approximate system, Eqs.(2-6) and the more exact system, Eqs.(2-5), are solved in the following pre-buckling analysis and considered later in the computation of the buckling loads.

2.3 Loading

In addition to the axi-symmetrical pressure exerted by the material stored in the shell, the shell can be subjected to nonsymmetric loading due to wind, and - in case of grain bins - due to eccentric discharge of the stored material. These two types of loading could be critical as far as the stability of the shell is concerned. In general, a load acting in the z-direction can be represented analytically in the following form of Fourier's series:

\[ P_z = P_0 \sum_{n=0,1,2}^{\alpha} \sum_{m=1,2,\ldots}^{\alpha} a_n b_m \cos n\phi \sin \frac{n_m R}{L} \]

(2-7)

In which,

- \( P_0 \) = the maximum positive intensity of load;
- \( b_m, a_n \) = constants governing the distribution of loading in the vertical and horizontal directions respectively;
- \( \frac{n_m R}{L} \) =

\( L \) = total height of shell
2.3.1 Wind Loading

According to the NPL measurements made by Cowdrey and O’Neill (13), for the pressure around a model cooling tower at high Reynold’s number of the order of $10^7$, and the studies made after that by Rish (14), $a_n$ and $b_m$ for this case of loading can be expressed as follows:

$$a_n = 0, 1, 2, \ldots = -0.387, 0.338, 0.533, 0.471, 0.166, -0.066, -0.055$$

with internal suction of $0.607 P_o$ (for open shells only) which gives $a_0 = +0.22$, and

$$b_m = 1, 3, \ldots = \frac{4}{m \pi}$$

This distribution is shown in Fig. (2-4a).

2.3.2 Eccentric Discharge

The load distribution in this case is shown in Fig. (2-4b). The distribution in the $\phi$-direction has been proposed by Jenike (15), while it is assumed to have the shown shape in the $x$-direction. Therefore, the constants governing the distribution can be given as:

$$a_n = 1, 3, 5, \ldots = \frac{4}{n \pi} \sin \frac{n \pi}{8}$$

$$b_m = 1, 3 = 0.667$$

The wind pressure is constant in the $x$-direction. Thus, it can be expressed in the following single series:
\[ P = P_0 \sum_{n=0,1,2} a_n \cos n\phi \]  

(2-8)

Eq. (2-8) is simpler than Eq. (2-7). Therefore it should be used in both the pre-buckling and buckling analysis in the cases where the wind load is the only nonsymmetric type of loading as in the case of oil storages and chimneys. However, solutions using both formulas; the double and the single series, are presented in this study.

2.4 Boundary Conditions

2.4.1 The Eighth Order System

The first system of governing differential equations, Eqs. (2-5), yields an 8th order characteristic equation. Therefore eight constants of integration are encountered in the solution and calculated by satisfying eight boundary conditions. These conditions are:

a) at \( x = 0 \)

The lower edge is considered totally clamped to the floor, i.e.

\[ w = 0 \]  

(2-9a)

\[ v = 0 \]  

(2-9b)

\[ u = 0 \]  

(2-9c)

\[ w_x = 0 \]  

(2-9d)
b) at \( x = L/R \)

\[
N_x = 0 \tag{2-9e}
\]

\[
M_x = 0 \tag{2-9f}
\]

The upper edge of the shell may be connected to an edge member or to a conical roof as in the case of bins. This roof system provides partial restraint to the upper edge of the shell. Therefore, two extreme cases are considered, one for totally flexible and the other for totally restrained upper edge.

- Totally restrained edge

In this case the edge remains circular without change in radius. Therefore the circumferential moment \( M_\phi \) as well as the circumferential strain \( \varepsilon_\phi \) vanish along the edge, i.e. at \( x = L/R \):

\[
M_\phi = 0 \tag{2-9g}
\]

\[
\varepsilon_\phi = 0 \tag{2-9h}
\]

- Totally flexible edge

In this case, no membrane or lateral shear forces are transmitted along the edge of the shell, i.e. at \( x = L/R \):

\[
N_x\phi = 0 \tag{2-9i}
\]

\[
Q_{\text{eff}} = Q_x + \frac{\partial M_{x\phi}}{\partial \phi} = 0 \tag{2-9j}
\]
2.4.2 The Approximate System

The approximate solution is based on assuming that \( M_x = M_{x\phi} = 0 \). Thus, conditions (2-9f) and (2-9j) are already satisfied. Also conditions (2-9a) and (2-9b) are reduced to one independent condition, and conditions (2-9g) and (2-9h) are reduced to another independent condition. The following are the independent boundary conditions to be satisfied:

a) at \( x = 0 \)
\[
\begin{align*}
    w &= v = 0 \\
    u &= 0
\end{align*}
\]  
(2-10a)
(2-10b)

b) at \( x = L/R \)
\[
\begin{align*}
    N_x &= 0 \\
    N_{x\phi} &= 0 \quad \text{(totally flexible edge)} \\
    M_{\phi} &= 0 \quad \text{(totally restrained edge)}
\end{align*}
\]  
(2-10c)
(2-10d)
(2-10e)

2.5 Solution

2.5.1 Eighth Order System (Eqs. 2-5)

2.5.1.1 Solution for load component with \( n = 0 \)

In this case all the derivatives with respect to \( \phi \) will drop out. Therefore, Eqs.(2-5) are reduced to the following:
\[
    u_{xx} = 0
\]  
(2-11a)
\[ v_{xx} = 0 \quad (2-11b) \]

\[ B_\phi w + B_x w_{xxxx} + R^2 D_\phi w = R^3 P_{zo} \quad (2-11c) \]

where \( P_{zo} = P_0 a_0 \sum_{m=1,3,...} b_m \sin n_m x \)

By integrating both Eqs. (2-11a) and (2-11b) twice and applying boundary conditions (2-9c) and (2-9e) for the first one and (2-9b) and (2-9h) for the second one, the following is obtained.

\[ u_0 = 0 \quad (2-12a) \]

\[ v_0 = 0 \quad (2-12b) \]

The third equation, Eq. (2-10c) can be rewritten as follows:

\[ w_{xxxx} + S^* w = T \quad (2-13) \]

in which,

\[ S^* = (B_\phi + R^2 D_\phi) / B_x \]

\[ T = \frac{R^3 P_{zo}}{B_x} \]

The particular solution for this equation is given as:

\[ w_{po} = \frac{T}{S^*} \quad (2-14a) \]

while the homogeneous solution is as follows:

\[ w_{ho} = (A_1 \cos ax + A_2 \sin ax) e^{ax} \]

\[ + (A_3 \cos ax + A_4 \sin ax) e^{-ax} \quad (2-14b) \]

in which, \( \alpha = S^*/\sqrt{2} \)
Adding the homogeneous and the particular solutions together, the following is the complete solution for \( w_0 \):

\[
        w_0 = (A_1 \cos \alpha x + A_2 \sin \alpha x) e^{\alpha x} + (A_3 \cos \alpha x + A_4 \sin \alpha x) e^{-\alpha x} + \frac{T}{S_0} \quad (2-14c)
\]

where \( A_1, A_2, A_3 \) and \( A_4 \) are four constants of integrations to be obtained by satisfying the corresponding four boundary conditions for this case which are given as:

\[
        w = 0 \quad , \quad w_x = 0 \quad \text{at} \quad x = 0 \quad (2-15)
\]

\[
        w_{xx} = 0 \quad , \quad w_{xxx} = 0 \quad \text{at} \quad x = L/R
\]

Eqs. (2-12) are the exact solution for load component with \( n = 0 \).

2.5.1.2 Solution for load component with \( n \geq 0 \)

1. Homogeneous Solution:

The solution for the homogeneous part of Eqs. (2-5) can be assumed as follows:

\[
        w_n = A e^{\gamma x} \cos n\phi \quad (2-16a)
\]

\[
        v_n = B e^{\gamma x} \sin n\phi \quad (2-16b)
\]

\[
        u_n = C e^{\gamma x} \cos n\phi \quad (2-16c)
\]

where \( n = 1, 2, 3, \ldots \) and \( A, B, C \) and \( \gamma \) can be complex numbers. By substituting this solution in the homogeneous part of Eqs. (2-5), three homogeneous equations in \( A, B \) and \( C \) can be obtained. A non-trivial solution for this equation requires that the determinant of
coefficients be zero. This condition leads to the following characteristic equation:

\[
\begin{align*}
\gamma^8 - \gamma^6 \left( n^2 \left( \frac{\phi}{x} \right) + 2 \frac{B_x \phi}{B_x} \right) + \\
\gamma^4 \left( n^2 (n^2 - 1) \frac{B_x^2}{B_x^2} \right) + \gamma^2 \left( n^2 (n^2 - 1)^2 \frac{B_x^2 \phi}{B_x^2} \right) +
\end{align*}
\]

\[\{ n^2 (n^2 - 1)^2 \left( \frac{B_x}{B_x} \frac{D_x \phi}{D_x} \right) \} = 0 \quad (2-17)\]

This eighth order algebraic equation in \( \gamma \) is the general characteristic equation for the system (2-5). It can be solved using the computer subroutine POLRT of the IBM 360/65 scientific subroutines library.

The roots of these equations are the following eight complex roots:

\[
\begin{align*}
\gamma &= \pm \alpha_1 \pm i \beta_1 \\
\gamma &= \pm \alpha_2 \pm i \beta_2 
\end{align*}
\quad (2-18)\]

This leads to the following expression for \( w \):

\[
\begin{align*}
w_h &= \left\{ (K_1 \cos \beta_1 x + K_2 \sin \beta_1 x) e^{\alpha_1 x} \\
&+ (K_3 \cos \beta_1 x + K_4 \sin \beta_1 x) e^{-\alpha_1 x} \\
&+ (K_5 \cos \beta_2 x + K_6 \sin \beta_2 x) e^{\alpha_2 x} \\
&+ (K_7 \cos \beta_2 x + K_8 \sin \beta_2 x) e^{-\alpha_2 x} \} \cos n\phi 
\end{align*}
\]

By substituting the above expression for \( w \), Eq.(2-19), in the governing system, Eqs.(2-5), expressions for the
other displacement and internal force components are given in the terms the same eight constants of integration:

\[ v_h = \left[ \{ K_1 (M_1 \cos \beta_1 x - M'_1 \sin \beta_1 x) \\
+ K_2 (M'_2 \sin \beta_1 x - M_2 \cos \beta_1 x) \} e^{a_1 x} \\
+ \{ K_3 (M_1 \cos \beta_1 x - M'_1 \sin \beta_1 x) \\
+ K_4 (M'_2 \sin \beta_1 x - M_2 \cos \beta_1 x) \} e^{-a_1 x} \\
+ \{ K_5 (M_1 \cos \beta_1 x - M'_1 \sin \beta_1 x) \\
+ K_6 (M'_2 \sin \beta_1 x - M_2 \cos \beta_1 x) \} e^{a_2 x} \\
+ K_7 (M_1 \cos \beta_1 x - M'_1 \sin \beta_1 x) \} e^{-a_2 x} \] \sin nt \]

(2-20)

\[ u_h = \left[ \{ K_1 (a_1 \cos \beta_1 x - b_1 \sin \beta_1 x) \\
+ K_2 (a_1 \sin \beta_1 x + b_1 \cos \beta_1 x) \} e^{a_1 x} \\
- \{ K_3 (a_1 \cos \beta_1 x + b_1 \sin \beta_1 x) \\
+ K_4 (a_1 \sin \beta_1 x - b_1 \cos \beta_1 x) \} e^{-a_1 x} \\
+ \{ K_5 (a_2 \cos \beta_2 x - b_2 \sin \beta_2 x) \\
+ K_6 (a_2 \sin \beta_2 x + b_2 \cos \beta_2 x) \} e^{a_2 x} \\
- \{ K_7 (a_2 \cos \beta_2 x + b_2 \sin \beta_2 x) \\
+ K_8 (a_2 \sin \beta_2 x - b_2 \cos \beta_2 x) \} e^{-a_2 x} \] \cos nt \]

(2-21)
\[ N_{\theta h} = D_\theta \left[ \sum \left\{ \begin{array}{l}
K_1(N_1 \cos \beta_1 x - M_1' \sin \beta_1 x) \\
+ K_2(N_1 \sin \beta_1 x + M_1' \cos \beta_1 x) \right\} e^{a_1 x} \\
+ \{ K_3(N_2 \cos \beta_2 x - M_2' \sin \beta_2 x) \\
+ K_4(N_2 \sin \beta_2 x + M_2' \cos \beta_2 x) \right\} e^{-a_1 x} \\
+ \{ K_5(N_2 \cos \beta_2 x + M_2' \sin \beta_2 x) \\
+ K_6(N_2 \sin \beta_2 x - M_2' \cos \beta_2 x) \right\} e^{a_2 x} \\
+ \{ K_7(N_2 \cos \beta_2 x - M_2' \sin \beta_2 x) \\
+ K_8(N_2 \sin \beta_2 x + M_2' \cos \beta_2 x) \right\} e^{-a_2 x} \right] \cos \phi \] (2-22)

\[ N_{X h} = D_\chi \left[ \begin{array}{l}
\{ K_1(a_1 (a_1 \cos \beta_1 x - b_1 \sin \beta_1 x) \\
- \beta_1 (a_1 \sin \beta_1 x + b_1 \cos \beta_1 x)) \\
+ K_2\{ a_1 (a_1 \sin \beta_1 x + b_1 \cos \beta_1 x) \\
+ \beta_1 (a_1 \cos \beta_1 x - b_1 \sin \beta_1 x) \} \} e^{a_1 x} \\
+ \{ K_3(a_1 (a_1 \cos \beta_1 x - b_1 \sin \beta_1 x) \\
+ \beta_1 (a_1 \sin \beta_1 x - b_1 \cos \beta_1 x)) \\
+ K_4(a_1 (a_1 \sin \beta_1 x - b_1 \cos \beta_1 x) \\
- \beta_1 (a_1 \cos \beta_1 x + b_1 \sin \beta_1 x)) \} e^{-a_1 x} \\
+ \{ K_5(a_2 (a_2 \cos \beta_2 x - b_2 \sin \beta_2 x) \\
- \beta_2 (a_2 \sin \beta_2 x + b_2 \cos \beta_2 x)) \\
+ K_6(a_2 (a_2 \sin \beta_2 x + b_2 \cos \beta_2 x) \\
+ \beta_2 (a_2 \cos \beta_2 x - b_2 \sin \beta_2 x) \} \} e^{a_2 x} \\
+ \{ K_7(a_2 (a_2 \cos \beta_2 x + b_2 \sin \beta_2 x) \\
+ \beta_2 (a_2 \sin \beta_2 x - b_2 \cos \beta_2 x)) \\
+ K_8(a_2 (a_2 \sin \beta_2 x - b_2 \cos \beta_2 x) \\
- \beta_2 (a_2 \cos \beta_2 x + b_2 \sin \beta_2 x)) \} e^{-a_2 x} \right] \cos \phi \] (2-23)
\[ N_{\phi h} = D_{\phi h} \left[ \begin{array}{c} K_1(\psi_1 \cos \beta_1 x - \psi_1' \sin \beta_1 x) \\ + K_2(\psi_1 \sin \beta_1 x + \psi_1' \cos \beta_1 x) \end{array} \right] e^{\alpha_1 x} \\
- \left[ \begin{array}{c} K_3(\psi_1 \cos \beta_1 x + \psi_1' \sin \beta_1 x) \\ + K_4(\psi_1 \sin \beta_1 x - \psi_1' \cos \beta_1 x) \end{array} \right] e^{-\alpha_1 x} \\
+ \left[ \begin{array}{c} K_5(\psi_2 \cos \beta_2 x - \psi_2' \sin \beta_2 x) \\ + K_6(\psi_2 \sin \beta_2 x + \psi_2' \cos \beta_2 x) \end{array} \right] e^{\alpha_2 x} \\
- \left[ \begin{array}{c} K_7(\psi_2 \cos \beta_2 x + \psi_2' \sin \beta_2 x) \\ + K_8(\psi_2 \sin \beta_2 x - \psi_2' \cos \beta_2 x) \end{array} \right] e^{-\alpha_2 x} \right] \sin n\phi \\
(2-24) \]

\[ M_{x h} = \frac{B_x}{R} \left[ \begin{array}{c} K_1(R_1 \cos \beta_1 x - R_1' \sin \beta_1 x) \\ + K_2(R_1 \sin \beta_1 x + R_1' \cos \beta_1 x) \end{array} \right] e^{\alpha_1 x} \\
+ \left[ \begin{array}{c} K_3(R_1 \cos \beta_1 x + R_1' \sin \beta_1 x) \\ + K_4(R_1 \sin \beta_1 x - R_1' \cos \beta_1 x) \end{array} \right] e^{-\alpha_1 x} \\
+ \left[ \begin{array}{c} K_5(R_2 \cos \beta_2 x - R_2' \sin \beta_2 x) \\ + K_6(R_2 \sin \beta_2 x + R_2' \cos \beta_2 x) \end{array} \right] e^{\alpha_2 x} \\
+ \left[ \begin{array}{c} K_7(R_2 \cos \beta_2 x + R_2' \sin \beta_2 x) \\ + K_8(R_2 \sin \beta_2 x - R_2' \cos \beta_2 x) \end{array} \right] e^{-\alpha_2 x} \right] \cos n\phi \\
(2-25) \]

\[ M_{\phi h} = \frac{B_{\phi}}{R} (n^2 - 1) \left[ \begin{array}{c} (K_1 \cos \beta_1 x + K_2 \sin \beta_1 x) e^{\alpha_1 x} \\ + (K_3 \cos \beta_1 x + K_4 \sin \beta_1 x) e^{-\alpha_1 x} \\ + (K_5 \cos \beta_2 x + K_6 \sin \beta_2 x) e^{\alpha_2 x} \\ + (K_7 \cos \beta_2 x + K_8 \sin \beta_2 x) e^{-\alpha_2 x} \end{array} \right] \cos n\phi \\
(2-26) \]
\[ M_{x\phi h} = \frac{B x_0}{R} \sum_n \left[ \begin{array}{l} K_1(a_1 \cos \beta_1 x - \beta_1 \sin \beta_1 x) \\
 + K_2(a_1 \sin \beta_1 x + \beta_1 \cos \beta_1 x) \right] e^{a_1 x} \\
 - \{ K_3(a_1 \cos \beta_1 x + \beta_1 \sin \beta_1 x) \\
 + K_4(a_1 \sin \beta_1 x - \beta_1 \cos \beta_1 x) \right] e^{-a_1 x} \\
 + \{ K_5(a_2 \cos \beta_2 x - \beta_2 \sin \beta_2 x) \\
 + K_6(a_2 \sin \beta_2 x + \beta_2 \cos \beta_2 x) \right] e^{a_2 x} \\
 - \{ K_7(a_2 \cos \beta_2 x + \beta_2 \sin \beta_2 x) \\
 + K_8(a_2 \sin \beta_2 x - \beta_2 \cos \beta_2 x) \right] e^{-a_2 x} \right] \sin n\phi \] (2-27)

\[ Q_{x_{\text{eff}} h} = D_{\phi} \left[ \begin{array}{l} K_1(F_1 \cos \beta_1 x - F_1' \sin \beta_1 x) \\
 + K_2(F_1 \sin \beta_1 x + F_1' \cos \beta_1 x) \right] e^{a_1 x} \\
 - \{ K_3(F_1 \cos \beta_1 x + F_1' \sin \beta_1 x) \\
 + K_4(F_1 \sin \beta_1 x - F_1' \cos \beta_1 x) \right] e^{-a_1 x} \\
 + \{ K_5(F_2 \cos \beta_2 x - F_2' \sin \beta_2 x) \\
 + K_6(F_2 \sin \beta_2 x + F_2' \cos \beta_2 x) \right] e^{a_2 x} \\
 - \{ K_7(F_2 \cos \beta_2 x + F_2' \sin \beta_2 x) \\
 + K_8(F_2 \sin \beta_2 x - F_2' \cos \beta_2 x) \right] e^{-a_2 x} \right] \cos n\phi \] (2-28)

In which,

\[ M_1 = \frac{1}{n} \left\{ \Omega_1 (n^2 - 1)^2 + 1 \right\} - 2\Omega_2 n(a_1^2 - \beta_1^2) \\
+ \frac{\Omega_3}{n} (a_1^4 - 6a_1^2 \beta_1^2 + \beta_1^4) \]

\[ M_1' = -4\Omega_2 n a_1 \beta_1 + \frac{\Omega_1}{n} (a_1^3 \beta_1 - a_1 \beta_1^3) \]

\[ M_2 = \frac{1}{n} \left\{ \Omega_1 (n^2 - 1)^2 + 1 \right\} - 2\Omega_2 n(a_2^2 - \beta_2^2) \\
+ \frac{\Omega_3}{n} (a_2^4 - 6a_2^2 \beta_2^2 + \beta_2^4) \]
\[ M_2' = -4\Omega_2 n\alpha_2 \beta_2 + \frac{\Omega_3}{n}(\alpha_2^3 \beta_2 - \alpha_2^1 \beta_2^1) \]

\[ \Omega_1 = \frac{B_x}{R^2 D_\phi}, \quad \Omega_2 = \frac{B_x \phi}{R^2 D_\phi}, \quad \Omega_3 = \frac{B_x}{R^2 D_\phi} \]

\[ a_1 = \frac{D_\phi}{D_x} n^2 \left( \frac{n M_1'(\beta_1^3 - 3\alpha_1^2 \beta_1^1) + (1 - n M_1)(\alpha_1^3 - 3\alpha_1^2 \beta_1^1)}{(\alpha_1^3 - 3\alpha_1^2 \beta_1^1)^2 + (\beta_1^3 - 3\alpha_1^2 \beta_1^1)^2} \right) \]

\[ a_2 = \frac{D_\phi}{D_x} n^2 \left( \frac{n M_2'(\beta_2^3 - 3\alpha_2^2 \beta_2^1) + (1 - n M_2)(\alpha_2^3 - 3\alpha_2^2 \beta_2^1)}{(\alpha_2^3 - 3\alpha_2^2 \beta_2^1)^2 + (\beta_2^3 - 3\alpha_2^2 \beta_2^1)^2} \right) \]

\[ b_1 = \left\{ \frac{a_1(\beta_1^3 - 3\alpha_1^2 \beta_1^1) - \frac{D_\phi}{D_x} n^3 M_1'}{(\alpha_1^3 - 3\alpha_1^2 \beta_1^1)} \right\} \]

\[ b_2 = \left\{ \frac{a_2(\beta_2^3 - 3\alpha_2^2 \beta_2^1) - \frac{D_\phi}{D_x} n^3 M_2'}{(\alpha_2^3 - 3\alpha_2^2 \beta_2^1)} \right\} \]

\[ N_1 = \{n M_1 - \Omega_1(n^2 - 1) - 1\} \]

\[ N_2 = \{n M_2 - \Omega_1(n^2 - 1) - 1\} \]

\[ \psi_1 = (\alpha_1 M_1 - \beta_1 M_1' - n a_1) \]

\[ \psi_1' = (\alpha_1 M_1' + \beta_1 M_1 - n b_1) \]

\[ \psi_2 = (\alpha_2 M_2 - \beta_2 M_2' - n a_2) \]

\[ \psi_2' = (\alpha_2 M_2' + \beta_2 M_2 - n b_2) \]

\[ R_1 = \alpha_1^2 - \beta_1^2 \quad R_1' = 2\alpha_1 \beta_1 \]

\[ R_2 = \alpha_2^2 - \beta_2^2 \quad R_2' = 2\alpha_2 \beta_2 \]

\[ F_1 = \{\Omega_3(\alpha_1 R_1' - \beta_1 R_1') - 2n^2 \Omega_2 a_1\} \]

\[ F_1' = \{\Omega_3(\alpha_1 R_1 + \beta_1 R_1') - 2n^2 \Omega_2 \beta_1\} \]
\[ F_2 = \{ \Omega_1 (\alpha_2 \frac{R_2}{R_2} - \beta_2 \frac{R_1'}{R_2}) - 2n^2 \Omega_2 \alpha_2 \} \]

\[ F_2' = \{ \Omega_1 (\alpha_2 \frac{R_2'}{R_2} + \beta_2 \frac{R_2}{R_2}) - 2n^2 \Omega_2 \beta_2 \} \]

and \( K_1, K_2, \ldots \) and \( K_8 \) are eight constants of integration.

11. Particular Solution:

Assuming the following particular solution for Eqs. (2-5)

\[ u_p = \sum_{m=1,3, \ldots}^{\infty} \sum_{n=1,2, \ldots}^{\infty} C_{mn}' \cos n_m x \cos n_\phi \]  

(2-29a)

\[ v_p = \sum_{m=1,3, \ldots}^{\infty} \sum_{n=1,2, \ldots}^{\infty} B_{mn}' \sin n_m x \sin n_\phi \]  

(2-29b)

\[ w_p = \sum_{m=1,3, \ldots}^{\infty} \sum_{n=1,2, \ldots}^{\infty} A_{mn}' \sin n_m x \cos n_\phi \]  

(2-29c)

by substituting this assumed solution in the governing differential equations, Eqs. (2-5), the following can be obtained:

\[ A_{mn}' = K_{mn} B_{mn}' \]  

(2-30a)

\[ C_{mn}' = \frac{n . n_m}{n^2 + n_m^2} \frac{D_x}{D_\phi} B_{mn}' \]  

(2-30b)

\[ B_{mn}' = \frac{a_n b_m}{R^2} \left[ \frac{1}{K_{mn} \{ \Omega_1 (n^2 - 1)^2 + 2\Omega_2 n^2 n_m + \Omega_3 n_m^4 + 1 \} - n} \right] \]  

(2-30c)
in which,  
\[ \lambda = \frac{P_o R^3}{D \phi t^2} \]  
(a non-dimensional quantity with the load appearing as a parameter)

\[ K_{\text{mn}} = \frac{1}{n} \left\{ n^2 + \frac{Dx}{D\phi} n_m^2 - \frac{Dx}{D\phi} \frac{(n n_m)^2}{n^2 + n_m^2 \frac{Dx}{D\phi}} \right\} \]

By adding the particular solution to the previous homogeneous solution, then applying the eight boundary conditions, eight equations in the eight unknowns \( K_1, K_2, \ldots, K_8 \) are obtained. They are solved for each term in the two loading series Eq.(2-7). The sum of the solutions for load components with \( n = 0 \) to \( n = 6 \) for case of wind loading or with \( n = 1 \) to \( n = \alpha \) (or practically sufficient number of terms) for case of pressure due to the eccentric discharge, gives a complete solution for the pre-buckling analysis of the problem.

2.5.2 Approximate Solution (Eqs. 2-6)

In this approximate solution, two expressions to describe the load are considered, the single series formula and the double series formula, Eq.(2-7) and Eq.(2-8), respectively.

2.5.2.1 Solution for load component with \( n = 0 \)

1. Single Series Formula

\[ u_o = v_o = 0 \]  
(2-31a)

\[ w_o = \frac{R}{D \phi} P_o a_o = \lambda \frac{t^2}{R^2} a_o \]  
(2-31b)
25

ii. Double Series Formula

\[ u_0 = v_0 = 0 \]  

\[ w_0 = \frac{1}{R^2} \sum_{m=1,3,5} a_m b_m \sin n_m x \]  

(2-32a)

(2-32b)

2.5.2.2 Solution for load component with \( n = 1 \)

The loading in this case is,

\[ P_{z1} = P_0 \sum_{n=1}^{\infty} b_m \sin n_m x \]  

(2-33)

The corresponding deflection takes the form of a single cosine wave in the \( \phi \)-direction. Therefore

\[ M_\phi = -\frac{B_\phi}{R} (w + w_{\phi\phi}) = 0 \]  

(2-34a)

\[ B_\phi (w_{\phi\phi\phi} + 2w_{\phi\phi} + w) = 0 \]  

(2-34b)

Since \( M_x \) and \( M_{x\phi} \) are neglected, the shell is subjected to a state of pure membrane forces. In this case, the governing equations are:

\[ N_{x,x} + N_{x\phi,\phi} = 0 \]  

(2-35a)

\[ N_{\phi,\phi} - N_{x\phi,x} = 0 \]  

(2-35b)

\[ N_\phi = -RP_{z1} \]  

(2-35c)

By successive integration with applying the proper boundary conditions (those describing displacements and membrane internal force components) the solution is obtained as the following:
1. Single Series Formula

\[ u_1 = \lambda \frac{t^2 D_\phi}{R^2 D_x} a_1 \left( \frac{x^3}{6} - \frac{x^2 L}{2} + \frac{xL^2}{2} \right) \cos \phi \]  \hspace{1cm} (2-36a)

\[ v_1 = \lambda \frac{t^2 D_\phi}{R^2 D_x} a_1 \left( Lx \frac{D_x}{D_\phi} - \frac{x^2 D_x}{2} \frac{D_\phi}{D_x} + \frac{x^4 D_\phi}{24 D_x} - \frac{x^3 L}{6} \frac{D_\phi}{D_x} \right) \sin \phi \]  \hspace{1cm} (2-36b)

\[ w_1 = \lambda \frac{t^2}{R^2} a_1 \left[ Lx \frac{D_\phi}{D_x} - \frac{x^2 D_\phi}{2} \frac{D_x}{D_\phi} + \frac{x^4 D_\phi}{24 D_x} - \frac{x^3 L}{6} \frac{D_\phi}{D_x} \right] \cos \phi \]  \hspace{1cm} (2-36c)

2. Double Series Formula

\[ u_1 = \lambda \frac{t^2}{R^2} \frac{D_\phi}{D_x} a_1 \left[ \sum_{m=1,3, \ldots}^\infty \frac{b_m}{n_m} \frac{1}{n_m} \left( \cos n_m x - 1 \right) + \right. \]

\[ \left. \frac{m-1}{(-1)^{m+1}} x \right] \cos \phi \]  \hspace{1cm} (2-37a)

\[ v_1 = \lambda \frac{t^2}{R^2} a_1 \left[ \sum_{m=1,3, \ldots}^\infty \frac{b_m}{n_m} \sin n_m x \frac{D_\phi}{D_x} \right. \]

\[ \frac{b_m}{n_m} \left\{ \frac{m-1}{(-1)^{m+1}} \frac{x^2}{2} + \frac{1}{n_m} \left( \frac{1}{\sin n_m x} - x \right) \right\} \frac{D_\phi}{D_x} \right\} \sin \phi \]  \hspace{1cm} (2-37b)

\[ w_1 = \lambda \frac{t^2}{R^2} a_1 \left[ \sum_{m=1,3, \ldots}^\infty \sin n_m x \left( \frac{D_\phi}{D_x} \right) \frac{b_m}{n_m} \frac{D_\phi}{D_x} \right. \]

\[ \left. + \frac{D_\phi}{D_x} \left\{ \frac{m-1}{(-1)^{m+1}} \frac{x^2}{2} - \frac{x}{n_m} \right\} \right\} \cos \phi \]  \hspace{1cm} (2-37c)
2.5.2.3 Solution For Load Component with $n>1$

1. Homogeneous Solution

The solution given below was first presented in (5). The following is a summary for this presentation.

The homogeneous solution is independent of load. Thus, this solution is valid for any load formula. Substituting the same previous homogeneous solution, Eqs.(2-16) in the approximate system of governing differential equations, the following fourth order characteristic equation appears:

$$
\gamma^4 - \gamma^2 \frac{D\phi}{D\xi} \left[ e_p \frac{n^2(n^2 - 1)^2}{1 + e_p(n^2 - 1)^2} \right] + \frac{D\phi}{D\xi} \left[ e_p \frac{n^4(n^2 - 1)^2}{1 + e_p(n^2 - 1)^2} \right] = 0
$$

(2-38)

It is obvious that this equation cannot be applied for the case of $n=1$. This occurred due to the approximation made to the system namely by neglecting $N_x$ and $N_{xx}$. This explains the reason for having a separate solution for the case of load component with $n=1$.

The above equation, Eq.(2-38), has four complex roots:

$$
\gamma = \pm \alpha \pm i\beta
$$

(2-39)

in which,

$$
\alpha = \sqrt[4]{\frac{D\phi n^2 k^2}{4 D\xi}} \sqrt[4]{1 + \xi^2 + 1}
$$

$$
\beta = \sqrt[4]{\frac{D\phi n^2 k^2}{4 D\xi}} \sqrt[4]{1 + \xi^2 - 1}
$$
\[ \xi' = \sqrt{\frac{4 D_{x\phi}^2}{k^2 D_x D_\phi} - 1} \]

\[ k^2 = \frac{\Omega_1 (n^2 - 1)^2}{1 + \Omega_1 (n^2 - 1)^2} \]

This leads to the following expressions for the displacement components

\[ w_h = \{(K_1 \cos \beta x + K_2 \sin \beta x) e^{ax} + (K_3 \cos \beta x + K_4 \sin \beta x) e^{-ax}\} \cos n\phi \quad (2-40a) \]

\[ v_h = \frac{1}{n} \left(1 + \Omega_1 (n^2 - 1)^2 \right) \{(K_1 \cos \beta x + K_2 \sin \beta x) e^{ax} + (K_3 \cos \beta x + K_4 \sin \beta x) e^{-ax}\} \sin n\phi \quad (2-40b) \]

\[ u_h = \left\{ \{ K_1 (a \cos \beta x - b \sin \beta x) + K_2 (b \cos \beta x + a \sin \beta x) \} e^{ax} - \{ K_3 (a \cos \beta x + b \sin \beta x) - K_4 (b \cos \beta x - a \sin \beta x) \} e^{-ax}\right\} \cos n\phi \quad (2-40c) \]

in which,

\[ a = -\frac{D_\phi}{D_x} \left\{ \frac{n^2 \Omega_1 (n^2 - 1)^2 (\alpha^3 - 3\alpha^2 \beta^2)}{\left(\alpha^3 - 3\alpha^2 \beta^2\right)^2 + (\beta^3 - 3\alpha^2 \beta)^2} \right\} \]

\[ b = a \left\{ \frac{(\beta^3 - 3\alpha^2 \beta)}{(\alpha^3 - 3\alpha^2 \beta)} \right\} \]

Expressions for the internal force components similar to those of the eighth order solution can be obtained.
ii. Particular Solution

- **Load as single series**

\[ u_p = 0 \]  \hspace{1cm} (2-61a)

\[ v_p = \sum_{n=2}^{\infty} C_m \sin n\phi \]  \hspace{1cm} (2-61b)

\[ v_p = \sum_{n=2}^{\infty} \frac{a_m}{n} \cos n\phi \]  \hspace{1cm} (2-61c)

in which,

\[ C_m = \lambda \cdot \frac{12 a_m}{(n^2 - 1)^2} \]

\[ R_m = C_m / n \]

- **Load as double series**

The solution presented in Eqs. (2-23) can be applied again here. In this case, \( R_{mn} \) only becomes as follows:

\[ R_{mn} = \lambda \cdot \frac{t^2}{K_m} \left\{ \frac{a_m b_m}{K_{mn} (n^2 - 1)^2 + 1} \right\} \]  \hspace{1cm} (2-42)

Again, by adding the homogeneous solution to any of the two particular solutions and applying the four boundary conditions, Eqs.(2-10), four equations in the four unknowns \( K_1, K_2, K_3 \) and \( K_4 \) are obtained. These equations are solved for each term in the load series Eqs.(2-7), (2-8). The sum of solutions for \( (n=0) + (n=1) + (n=1) \) gives a complete approximate solution for the pre-buckling analysis.
2.6 Numerical Scheme

In a general computer programme for the pre-buckling analysis part, the characteristic equation is solved first, then the equations describing the boundary conditions are constructed and solved through the programme. Finally the pre-buckling displacement components and internal force components are calculated as well as the membrane strain components. A computer programme for the pre-buckling analysis is a part of a larger programme calculating the the buckling load of shell.

The parameters considered in this numerical scheme are:

- The geometric proportions of the shell, i.e. length to radius ratio (L/R) and radius to shell thickness ratio (R/t)

- Rigidity ratios of the orthotropic shell i.e.

\[ \frac{D_x}{D_{x\phi}}, \frac{D_x}{D_{\phi}}, \frac{B_x}{E_{\phi}}, \frac{B_x}{B_{x\phi}} \]

- The boundary conditions at the upper edge, i.e. completely restrained edge or completely flexible edge.
2.7 Observations on the Pre-buckling Analysis

The pre-buckling displacement components and the internal force components were calculated for shells considering the variables mentioned in the previous section 2.6. Both the approximate and the more exact approaches were used, and the following was observed.

2.7.1 Radial Displacement

The radial displacement was calculated along the upper edge at \( x = L/R \) for different values of \( \phi \) from \( \phi = 0 \) to \( \phi = \pi/2 \).

For shells with flexible upper edge, the maximum radial displacement lies along the upper edge. The radial displacement appreciably increases with the decrease of \( D_x \), while fixing the proportions of the shell and the values of \( D_\phi \) and \( D_{x\phi} \), Figs. (2-5) to (2-10).

The deviation in results between the two approaches of the pre-buckling analysis depends on all the variables considered. It decreases steadily with the decrease of \( D_x/D_\phi \). The maximum deviation found was 12.7% for a shell with the following properties: \( L/R = 1.0 \), \( R/t = 100 \), \( \frac{D_x}{D_\phi} = \frac{B_x}{B_\phi} = 1.0 \) and \( \frac{D_\phi}{D_{x\phi}} = 2 \). For shells with \( \frac{D_x}{D_\phi} = 0.2 \), the maximum deviation was found to be 4.8%. This means that for shells made of corrugated sheets, where the ratio \( D_x/D_\phi \) is considerably smaller than 0.2, the deviation in radial displacement is negligible.
The results obtained for the special case where

\[
\frac{D_X}{D_\phi} = \frac{B_X}{B_\phi} = \frac{B_{X\phi}}{B_{\phi}} = 1.0 \quad \text{and} \quad \frac{D_\phi}{D_{X\phi}} = 2.0 \quad \text{(isotropic shell with)} \quad \mu = 0\]

show that the radial displacement rapidly increases with the increase of both L/R and R/t, Figs. (2-11) to (2-16). These results are compared with those presented by Johns (22) where Donnell's governing equations were employed, Figs. (2-17) and (2-18). In this comparison, only the results obtained by using the eighth order system are shown.

For shells with restrained upper edge, the maximum deviation in the approximate analysis is slightly higher than that obtained in the previous case of flexible upper edge (\(= 14.7\%\)). This can be attributed to the fact that the overall vertical rigidity of the shell is higher for the case of a restrained upper edge than that for the case of a flexible one which increases the effect of the neglected longitudinal bending moment in the approximation.

2.7.2 Circumferential Bending Moment \(M_\phi\)

The circumferential bending moment \(M_\phi\) was calculated along the generator \(\phi = 0\).

For the case of a shell with flexible upper edge, \(M_\phi\) has a zero value at \(x = 0\), where \(w = 0\), and a maximum value at \(x = L/R\), Fig. (2-21). The circumferential moment increases rapidly with the decrease in \(B_X/B_\phi\), Figs. (2-19) and (2-20). However the \((M_\phi \text{ vs } \frac{B_X}{B_\phi})\) relation is again a
function of the shell geometric proportions whereas the rate of change of $M_\phi$ with $B_x/B_\phi$ decreases with the increase of $R/t$ and the decrease of $L/R$, Figs. (2-19) and (2-20).

The deviation in the maximum circumferential moment due to the approximate analysis is slightly less than that found for the radial displacement. The maximum deviation was found to be 10.2% for an isotropic shell with $L/R = 1$ and $R/t = 100$.

Restraining the upper edge changes the distribution of $M_\phi$ as shown in Fig. (2-22) with zero points at top and bottom and maximum value near the mid-height of the shell. The maximum moment decreases to a value ranging between 10% to 66% of the value with flexible edge, depending on the different parameters. Thus, the restraining of the upper edge could be used to improve the overall performance of the shell.

The deviation due to the approximation is generally higher than that for the case of flexible upper edge as it has been previously noticed with the radial displacement. Fig. (2-23) gives the deviation against $L/R$ for shells with different $R/t$ and $D_x/D_\phi$ ratios. According to the figure, higher deviation occurs at lower $R/t$, lower $L/R$ and higher $D_x/D_\phi$ values. The deviation becomes negligible for all shells when $D_x/D_\phi$ becomes less than 0.2 or when $R/t$ becomes more than 400.
Figs. (2-19) and (2-20) show the maximum circumferential moment plotted against L/R - for shells with different R/t and B_x/B_φ values - for flexible and restrained upper edges, respectively. These curves can be used for practical applications to design shells whose properties lie within the ranges considered.

2.7.3 Longitudinal Bending Moment M_x

For cantilever shells, the vertical moment distribution has an approximately constant pattern, regardless of the boundary conditions at the upper edge. A local zone at the lower fixed edge exhibits a relatively high negative vertical bending moment. However, away from this local zone, which extends to about one sixth of the shell height, the moment decreases rapidly to a very small value of about 1/30 of the maximum value at x = 0. It continues to decrease every time it changes sign until it reaches zero at the upper edge.

The vertical bending moment was calculated along the generator φ = 0 where it has the maximum magnitude, Figs. (2-24) and (2-25).

Restraining the upper edge leads to a reduction in M_x. As shown in the example presented in Figs. (2-24) and (2-25), M_x at x = φ = 0 has been reduced to 64% of its original value before restraining the upper edge. The reduction in M_φ for the same shell was to 37% of the
original value. This observation, that the reduction in 
\( M_\phi \) is higher than that in \( M_x \), holds for all the shells 
investigated. This explains the reason for the deviation 
due to the approximate analysis being higher for the case 
of a shell with restrained upper edge than that for the 
case of one with flexible edge.

To test, in another way, the validity of the approx-
imate approach, Fig.(2-26) shows the relation between 
\( \left( \frac{M_{x_{\text{max}}}}{M_{\phi_{\text{max}}}} \right) \) and \( \left( \log \frac{B_\phi}{B_x} \right) \) for shell with \( L/R = 1 \) and 
\( R/t = 100 \). The shell is chosen because these proportions 
are associated with the maximum deviation between the two 
approaches. The figure shows that \( \left( \frac{M_{x_{\text{max}}}}{M_{\phi_{\text{max}}}} \right) \) rapidly 
decreases with the increase of \( B_\phi/B_x \). The moment ratio 
becomes smaller than 1.0 as \( B_\phi/B_x \) exceeds 25 and almost 
reaches zero at \( B_\phi/B_x = 320 \). Since the approximate 
analysis is based on assuming that the longitudinal moment 
is negligible compared to the circumferential moment, and 
since the ratio \( B_\phi/B_x \) for shells made of corrugated sheets 
is always greater than 320, therefore it could be concluded 
that the approximate approach should yield reliable results 
for shells made of such material.

Figs.(2-27) and (2-28) give the relation between 
\( L/R \) and \( M_{x_{\text{max}}} \) for different \( R/t \) and \( B_x/B_\phi \) values for shells 
with flexible and restrained upper edges, respectively. 
These figures can be used for practical design.
2.7.4 Membrane Internal Force Components

The membrane internal force components $N_\phi$ and $N_x$ were calculated along the generator $\phi = 0$ while $N_{x\phi}$ was calculated along the generator $\phi = \pi/2$.

Figs. (2-29) and (2-30) show the relation between $N_{\phi_{\text{max}}}$ with $L/R$ for different values of $R/t$ and $D_x/D_\phi$, for shell with restrained and flexible edges respectively. The same relation for $N_{x\phi_{\text{max}}}$ is shown in Figs. (2-31) and (2-32), while the ($N_x$ vs. $L/R$) relation is shown in Figs. (2-33) to (2-35) for shells with restrained edges and in Fig. (2-36) for shells with flexible edges. These curves can be used in practice to determine the maximum values of membrane force components.

The deviation due to the approximate analysis in the results of the membrane force components is relatively small compared to that found in bending moment and radial displacement. The maximum deviation in $N_x$ was 4.8%, in $N_\phi$ was less than 1% and in $N_{x\phi}$ was 8.4%. In the case of $N_{x\phi}$, the maximum deviation occurred in an isotropic shell with $L/R = 5$ and $R/t = 100$. This shell properties do not agree with the general trend where the maximum deviation always occurs at a small value for $L/R$. However, this is the only case in which this deviation from the general trend takes place.

Fig. (2-37) shows the deviation due to the approximate analysis in the value of $N_{x_{\text{max}}}$ for shells with
restrained upper edge as a function of L/R at different values of R/t and Dx/DΦ. The general trend of deviation mentioned before is clear in this figure.

The following is a discussion of the distribution of each membrane force component along the height of the shell.

a) Hoop Force Component NΦ

Figs. (2-38) and (2-39) show a typical vertical distribution of the hoop force at Φ = 0 for shell with flexible and restrained upper edge, respectively.

The value of NΦ at x = L/R depends on the upper edge conditions. It reaches zero when the edge is restrained while it decreases to about one fourth of the maximum value when the edge is flexible. For both cases, 90% of the maximum value is reached at one sixth of the height and the force does not decrease to a value lower than this value before reaching a height equal to five sixths of the total height. Between these upper and lower sixths, the distribution can be considered constant. The particular solution with the load expressed as a single series does not satisfy the pre-described boundary conditions, that NΦ = 0 at x = 0. Therefore the distribution obtained using this solution is a constant linear distribution as it is shown in Figs. (2-38) and (2-39) together with the distributions obtained by the other solutions. The previous deviation from the boundary conditions is the only one exhibited by this solution.
In conclusion, \( N_\phi \) is generally constant along the height. It is always compression with a very small value. It is almost independent of \( D_x/D_\phi \), \( R/t \) and \( L/R \). These results are valid for both sets of boundary conditions.

b) Longitudinal Force \( N_x \)

A typical vertical distribution of \( N_x \) is shown in Figs. (2-40) and (2-41) for shells with flexible and restrained upper edges, respectively. The example case shows the distribution along the generator \( \phi = 0 \). When the upper edge is flexible, the distribution starts with a maximum tension at \( x = 0 \), then the tension decreases rapidly along the height until it reaches zero at the top.

Restraining the upper edge reduces the maximum tension at \( x = \phi = 0 \) to a value between 66% to 22% depending on the other parameters.

c) Shearing Force \( N_{x\phi} \)

Figs. (2-42) and (2-43) show typical distribution of \( N_{x\phi} \) along the generator \( \phi = \pi/2 \). They indicate that this distribution is almost linear.

For a shell with flexible upper edge, Fig. (2-42), the maximum positive value occurs at the point \((\phi = \pi/2, x = 0)\) then the shearing force decreases until it reaches zero at the top to fulfill the boundary conditions.

By restraining the upper edge of the shell, the distribution is shifted parallel to itself to create
a negative shear at the upper third of the height and decrease the maximum positive value at the bottom by about 50%.
CHAPTER III
BUCKLING ANALYSIS

3.1 Trefftz's Variational Theory

The buckling of cantilever shells under lateral loading is essentially a non-first-order stability problem. This means that a thorough study for the actual behavior requires that a complete load - deflection diagram for the various states of loading be established to determine the maximum load carrying capacity of the shell. However, the theory required for determining this load - deflection relationship is essentially non-linear and there are many difficulties involved in its analysis. Even for the simple case of simply supported shells under uniform pressure (16, 17), there are only few solutions available and they do not all quantitatively agree with each other as pointed out by Langhaar (9).

Therefore, as a practical expedient it is satisfactory enough to concentrate on the less complicated, well established, infinitesimal theories.

The theory adopted in the present analysis is Trefftz's variational theory (18) together with employing Ritz's approach (19) in solving the variational problem involved.

The advantages of Trefftz's theory over the conventional energy method for buckling are discussed by Langhaar (9) and Borel (10). The theory is believed to
be more rigorous than the energy method given by Timoshenko (20) and it is possible to free the buckling theory from arbitrary physical assumptions (9).

The theory is based on the well known concept that a stationary conservative mechanical system is in stable equilibrium if, and only if, its total potential energy \( U \) has a relative minimum. For an elastic system, the potential energy:

\[
U = V - W
\]  

(3-1)

where \( V \) is the strain energy and \( W \) is the energy exerted by the external loads. Consequently due to any infinitesimal (virtual) displacement in the vicinity of the stable equilibrium configuration, the change in the potential energy \( \Delta U \) can be given as:

\[
\Delta U = \Delta V - \Delta W
\]  

(3-2)

and for any system

\[
\Delta V = \delta V + \frac{1}{2!} \delta^2 V + \frac{1}{3!} \delta^3 V + \ldots
\]  

(3-3)

where \( \delta^n V \) is "the \( n \)th variation of \( V \) and is a volume integral of a homogeneous polynomial of \( n \)th degree in the components of the virtual displacement vector and its derivatives".

Similarly for the change in \( W \)

\[
\Delta W = \delta W + \frac{1}{2!} \delta^2 W + \frac{1}{3!} \delta^3 W + \ldots
\]  

(3-4)

Hence from equation (3-2)
\[ \Delta U = \delta V - \delta W + \frac{1}{2!} \delta^2 V - \frac{1}{2!} \delta^2 W + \ldots \]  

(3-5)

From the principle of virtual work, the first variation \((\delta V - \delta W)\) of the potential energy must vanish for any equilibrium configuration. Accordingly, if the virtual displacement components are small, the sign of \(\Delta U\) is controlled by the sign of \((\delta^2 V - \delta^2 W)\). Therefore, for all virtual displacements, equilibrium is stable, only if \((\delta^2 V - \delta^2 W) > 0\). If for certain values of virtual displacements the quantity \((\delta^2 V - \delta^2 W)\) changes its sign from positive to negative, equilibrium becomes unstable. Hence, buckling occurs when the second variation of potential energy changes from positive character to negative character. Therefore, the "condition of stability" is

\[ \delta^2 V - \delta^2 W = 0 \]  

(3-6)

This is a sufficient condition which can be established from the principles of calculus of variation. This means that the stability quest is now reduced to a mathematical study of the quantity \(\delta^2 V - \delta^2 W\).

Since the second variation of the total potential energy is a quadratic form in the virtual increments of the coordinates with the load appearing as the parameter, the critical load can be determined from the requirements of the theory of quadratic forms that the Hessian determinant be zero (21).
3.2 Strain Expressions in Buckling Analysis

Recognizing the fundamental importance of non-linear terms in the geometric relationships, the following expressions for strains and change of curvatures are considered in the buckling analysis.

\[
\bar{\varepsilon}_x = \bar{u}_x + \frac{1}{2} \bar{w}_x^2 \quad (3-7a)
\]

\[
\bar{\varepsilon}_\phi = (\bar{v}_\phi - \bar{w}) + \frac{1}{2} (\bar{v} + \bar{w}_\phi)^2 \quad (3-7b)
\]

\[
\bar{\gamma}_{x\phi} = \bar{u}_\phi + \bar{v}_x + \bar{w}_x (\bar{v} + \bar{w}_\phi) \quad (3-7c)
\]

\[
\bar{K}_{xx} = (\bar{w}_{xx})/R \quad (3-7d)
\]

\[
\bar{K}_{\phi\phi} = (\bar{w}_{\phi\phi} + \bar{w})/R \quad (3-7e)
\]

\[
\bar{K}_{x\phi} = (\bar{w}_{x\phi})/R \quad (3-7f)
\]

Here the bars denote the total strains and change of curvature during buckling i.e. the prebuckling component plus the virtual component. Let \( u, v \) and \( w \) present the displacement components vector defining the equilibrium configuration before buckling, and \( \xi, \eta, \zeta \) define the respective components of the incremental virtual displacements vector during buckling. Neglecting prebuckling rotations of shell element \( w_x \) and \( v+w_\phi \) which are very small, the expressions for strains and change of curvatures are as follows:

\[
\bar{\varepsilon}_x = u_x + \xi_x + \frac{1}{2} \zeta_x^2 \quad (3-8a)
\]
\[ \bar{\varepsilon}_\phi = \nu_\phi - \nu + \eta_\phi - \zeta + \frac{1}{2} (\eta + \zeta_\phi)^2 \]  
(3-8b)

\[ \bar{\tau}_{x\phi} = \nu_\phi + \nu_x + \zeta_\phi + \eta_x + \zeta_x (\eta + \zeta_\phi) \]  
(3-8c)

\[ \bar{\tau}_{xx} = (\nu_{xx} + \zeta_{xx})/\nu \]  
(3-8d)

\[ \bar{\tau}_{\phi\phi} = (\nu_{\phi\phi} + \nu + \zeta_{\phi\phi} + \zeta)/\nu \]  
(3-8e)

\[ \bar{\tau}_{x\phi} = (\nu_{x\phi} + \zeta_{x\phi})/\nu \]  
(3-8f)

3.3 Total Potential Energy Expression for Orthotropic Shell

3.3.1 Strain Energy Expression

The strain energy \( V \) for orthotropic shell can be given as:

\[ V = V_m + V_b \]  
(3-9)

in which

\( V_b \) is the bending strain energy;

\( V_m \) is the membrane strain energy.

With \( \nu = 0 \) for case of shells made of corrugated sheets the above \( V_m \) and \( V_b \) are:

\[ V_m = \frac{L}{R} \int_0^{2\pi} \int_0^R \left( \nu_x \varepsilon_x + \nu_{x\phi} \varepsilon_{x\phi} + \nu_{x\phi} \varepsilon_{x\phi} \right) dx d\phi \]  
(3-10a)

\[ V_b = -\frac{L}{R} \int_0^{2\pi} \int_0^R \left( \frac{1}{2} (\phi_{x\phi} \phi_{x\phi} + \phi_{x} \phi_{xx} + \phi_{x\phi} \phi_{x\phi}) \right) dx d\phi \]  
(3-10b)

Recalling equations (2-3) for internal force
components, the above equations can be expressed as follows:

\[ V_m = \frac{E^2}{2} \int_0^L \int_0^{2\pi} \left\{ D_x \left( \ddot{u}_x (\ddot{e}_x) + \ddot{v}_x \right) + D_\phi (\ddot{v}_\phi - \ddot{w}) \right. \\
- \frac{B_\phi}{R^2} \left( \ddot{w} + \ddot{w}_\phi \right) \left( \ddot{e}_\phi \right) \\
+ \left\{ D_{x\phi} (\ddot{u}_\phi + \ddot{v}_x) \right\} (\ddot{e}_x) \right\} \, dx \, d\phi \]

(3-11a)

\[ V_t = \frac{E^2}{2} \int_0^L \int_0^{2\pi} \left\{ D_\phi (\ddot{w} + \ddot{w}_x \phi) \ddot{e}_x + D_x (\ddot{w}_x) \ddot{e}_x \\
+ B_{x\phi} (\ddot{w}_x \phi) \ddot{e}_x \right\} \, dx \, d\phi \]

(3-11b)

Substituting equations (3-10) in equations (3-11), the strain energy can be expressed in terms of pre-buckling displacement components and virtual displacement components as follows:

\[ V_m = \frac{E^2}{2} \int_0^L \int_0^{2\pi} \left\{ D_x \left( \dot{u}_x + \xi_x + \frac{1}{2} \zeta_x^2 \right)^2 \\
+ D_\phi \left( \dot{v}_\phi - \dot{w} + \eta_\phi - \zeta + \frac{1}{2} (n + \zeta_\phi)^2 \right)^2 \\
+ D_{x\phi} \left( \dot{u}_\phi + \dot{v}_x + \dot{\xi}_x + n_x + \xi_x (n+\zeta_\phi) \right)^2 \\
- \frac{B_\phi}{R^2} \left( \dot{w} + \dot{w}_x \phi + \zeta + \zeta_\phi + \dot{v}_\phi - \dot{w} \\
+ \eta_\phi - \zeta + \frac{1}{2} (n + \zeta_\phi)^2 \right) \right\} \, dx \, d\phi \]

(3-12a)
\[ V_b = \frac{R^2}{2} \int_0^L \int_0^{2\pi} \left\{ \frac{B \phi}{R^2} (w + w\phi + \xi + \zeta\phi)^2 + \frac{B_x}{R^2} (w_{xx} + \xi_{xx})^2 + \frac{B_{\phi}}{R^2} (w_{\phi\phi} + \xi_{\phi\phi})^2 \right\} \, dx \, d\phi \quad (3-12) \]

### 3.3.2. External Work

The work exerted by the external applied load in the pretwisting stage can be expressed as follows:

\[ W = R^2 \int_0^{L/R} \int_0^{2\pi} \{(P) (w^*)\} \, dx \, d\phi \quad (3-13) \]

where \( w^* \) = Dimensional displacement component in \( z \) direction;

\( P \) = the external load function given as equation (2-7) or equation (2-8).

However, for the buckling analysis, the redistribution of load component on the deformed surface must be taken into consideration.

Consider the rectangular element shown in Fig. (3-1) at the middle surface of the shell. As the shell deforms, this element is dislocated and its dimensions undergo changes. The external load applied to this infinitesimal surface element is referred to with the notation \( dP \). Its components in the cylindrical coordinates system are \( dP_x \),
\( \text{dP}_\phi \) and \( \text{dP}_z \).

Evidently, for the undeformed cylinder we have
\[
\text{dP}_x = \text{dP}_y = 0, \quad \text{dP}_z = P R^2 d\phi \, dx
\]

Only loading in the radial direction (wind load or pressure due to eccentric discharge) was considered in the pre-buckling analysis. Considering the geometry of the deformed surface \( a'b'c'd' \), the projection of the unit vector in the direction of \( z \) axis on the new planes of axes of the deformed surface \( z'-\phi', z'-x', \) and \( x'-\phi' \), respectively are as follows:

\[
\lambda_x = \bar{w}_x \quad (3-14a)
\]
\[
\lambda_\phi = \bar{w}_\phi + \bar{v} \quad (3-14b)
\]
\[
\lambda_z = 1 \quad (3-14c)
\]

The area of the deformed infinitesimal element can be given after neglecting the second degree terms as follows:
\[
A' = R^2 (1 + \bar{v}_\phi + \bar{u}_x - \bar{w}) \, dx \, d\phi \quad (3-15)
\]

Therefore, the load components acting on the deformed surface can be given as the following after neglecting the second degree terms:

\[
\text{dP}'_x = P R^2 \bar{w}_x \, dx \, d\phi \quad (3-16a)
\]
\[
\text{dP}'_\phi = P R^2 (\bar{w}_\phi + \bar{v}) \, dx \, d\phi \quad (3-16b)
\]
\[
\text{dP}'_z = P R^2 (1 + \bar{v}_\phi + \bar{u}_x - \bar{w}) \, dx \, d\phi \quad (3-16c)
\]
The work exerted by the external load can be obtained by integrating over surface as follows:

\[ W = \int_0^{L/R} \int_0^{2\pi} \left\{ dP_x(\bar{u}^*) + dP_\phi(\bar{v}^*) + dP_z(\bar{w}^*) \right\} \]  

(3-17)

in which \( \bar{u}^*, \bar{v}^* \) and \( \bar{w}^* \) = The total dimensioned displacement components

\[ = \bar{u}.R, \bar{v}.R \text{ and } \bar{w}.R, \text{ respectively.} \]

Substituting equations (3-16) in equation (3-17) the following expression for the external work can be obtained:

\[ W = \frac{R^3}{2} \int_0^{L/R} \int_0^{2\pi} P \left\{ -2 \bar{w} + (\bar{w}^2 + 2 \bar{v} \bar{w}_\phi + \bar{v}^2 \right. \]

\[ - \bar{u}_x \bar{w} + \bar{w}_x \bar{u}) \right\} \ dx \ d\phi \]  

(3-18)

Since \( \bar{u} = u + \xi, \bar{v} = v + n \) and \( \bar{w} = w + \zeta \), the above equation will be

\[ W = \frac{R^3}{2} \int_0^{L/R} \int_0^{2\pi} P \left\{ -2(w + \zeta) + (w + \zeta)^2 \right. \]

\[ + 2(v + n)(w_\phi + \zeta_\phi) + (v + n)^2 \]

\[ - (u_x + \xi_x)(w + \zeta) \]

\[ + (w_x + \zeta_x)(u + \zeta) \right\} \ dx \ d\phi \]  

(3-12c)
3.4 Second Variation of Total Potential Energy

Substituting equations (3-12a), (3-12b) and (3-12c) in the total potential energy expression, Eq. (3-1), the following is the general formula giving the total potential energy of a shell in terms of the prebuckling displacement components and the virtual displacement components during buckling.

\[
U = \frac{L}{2R} \int_0^{2\pi} \int_0^R \left\{ D_x \left[ u_x + \xi_x + \frac{i}{2} \zeta_x^2 \right]^2 
+ D\phi \left[ v_\phi - w + n_\phi - \zeta + \frac{i}{2} (n + \zeta_\phi) \right]^2 
+ D_{x\phi} \left[ u_\phi \right. + v_x + \xi_\phi + n_x + \zeta_x (n + \zeta_\phi) \left. \right]^2 
+ \frac{E}{R^2} \left\{ (w + w_\phi + \zeta + \zeta_\phi) - w_\phi 
- \zeta_\phi - v_\phi - n_\phi - \frac{i}{2} (n + \zeta_\phi)^2 \right\} 
+ \frac{B}{R^2} \left\{ (w_{xx} + \zeta_{xx})^2 \right\} 
+ \frac{B_{x\phi}}{R^2} \left\{ (w_{x\phi} + \zeta_{x\phi})^2 \right\} 
\right. 
+ \frac{P}{R} \left( 2(w + \zeta) - (w + \zeta)^2 - 2(v + n) 
. \left( w_\phi + \zeta_\phi \right) - (v + n)^2 + (u_x + \xi_x) \right. 
. \left( w + \zeta \right) - (w_x + \zeta_x) (u + \xi) \right\} \, dx \, d\phi
\]

\( (3-19) \)
By expanding the above equation and applying the definition of the second variation of the total potential energy (the volume integral of a second degree polynomial in the virtual displacement components and their derivatives), the terms of the second variation can be separated and the following expression for the second variation of total potential energy is obtained:

\[
\delta^2 U = \frac{R^2}{2} \int_0^{2\pi} \int_0^L \left\{ D_x \xi_x^2 + D_\phi (\eta_\phi - \zeta)^2 + D_{x\phi} (\xi_\phi + \eta_x)^2 \right. \\
+ \frac{B_\phi}{R^2} (\xi + \zeta_\phi) (2\xi + \zeta_\phi - \eta_\phi) \\
+ \frac{B_x}{R^2} (\zeta_{xx})^2 + \frac{B_{x\phi}}{R^2} (\zeta_{x\phi})^2 \\
+ D_x (u_x) (\xi_x^2) \\
+ D_\phi (v_\phi - \omega) (n + \xi_\phi)^2 \\
+ D_{x\phi} (v_x + u_\phi) \{ 2(n + \xi_\phi) \xi_x \} \\
+ \left. R P (\zeta^2 + 2n\zeta_\phi - \eta^2 - \xi_x\zeta + \xi_\phi \zeta_x) \right\} \, dx \, d\phi
\] (3-20)

The underlined quantities represent the prebuckling membrane strain components \( \varepsilon_x, \varepsilon_\phi \) and \( \gamma_{x\phi} \), respectively. Hence it is clear that the second variation of total potential is a function of the rigidity properties of the
shell, the geometric proportions, the pre-buckling membrane strain components, the virtual displacement components and their derivatives and the applied load.

Equation (3-20) is the general expression for the second variation of the total potential energy of orthotropic shells (with $\mu = 0$). However, for the approximate solution of the prebuckling analysis where $N_x$ and $N_x\phi$ are neglected, the corresponding expression for the second variation is the same as the above equation when neglecting the terms $\frac{N_x}{N^2}(\xi_{xx})^2$ and $\frac{N_x\phi}{N^2}(\xi_{x\phi})^2$. "P" in the above equation represents the load formula whether it is a double series or a single series formula.

3.5 Expressions of the Virtual Displacement Components

As it has been stated before, the stability condition is the vanishing of the left hand side of Eq. (3-19) for certain values of the virtual displacement components. From that condition, the buckling load can be obtained. So it is required now to represent all possible virtual displacements. The virtual displacements have to meet the geometric requirements of the problem and satisfy the boundary conditions during buckling. The following set of expressions are chosen:

$$\xi = \sum_{m=1}^{P} \sum_{n=2}^{Q} A_{mn} x^m \cos n\phi$$  \hspace{1cm} (3-21a)
\[ n = \sum_{m=1}^{P} \sum_{n=2}^{Q} B_{mn} x^m \sin n\phi \quad (3-21b) \]

\[ \tau = \sum_{m=1}^{P} \sum_{n=2}^{Q} C_{mn} x^{(m+1)} \cos n\phi \quad (3-21c) \]

The above functions are complete in the shell region and satisfy the geometric requirements where the polynomial series in \( x \)-direction is chosen to represent any possible geometric configuration of a cantilever shell during buckling.

The boundary conditions are satisfied at the lower edge where \( \xi = \eta = \xi = \tau_x = 0 \) at \( x = 0 \), while they are not constrained at the upper edge at \( x = L/R \).

The values of \( P \) and \( Q \) are theoretically infinity. However, for practical considerations the number of terms taken from each series has to be limited to a certain value leading both series to a satisfactory convergent state. This number governs the size of the stability matrix.

The values of \( n = 0 \) and \( 1 \) are omitted as they represent the buckling under axisymmetric compression and the column buckling with undistorted cross section, respectively.
3.6 Stability Matrix

The expressions of the virtual displacement components, Eqs.(3-21) are substituted in the second variation of potential energy expression, Eq.(3-20) and all integrals are carried out. The result is a quadratic form in the parameters $A_{mn}$, $B_{mn}$ and $C_{mn}$. From the stability condition, the left hand side should be equal to zero at the buckling load. Differentiating this quadratic form with respect to each of these parameters, the result is a set of homogeneous linear algebraic equations in the parameters. The number of these equations is $[3 \times P \times (Q - 1)]$ where $P$ and $(Q - 1)$ are the number of terms taken into consideration from the polynomial and harmonic series describing the virtual displacement components.

A typical equation of this set can be written as follows:

$$
\left\{(P_{X1} + \lambda S_{X1}) A_{mn} + (P_{X2} + \lambda S_{X2}) B_{mn} + (P_{X3} + \lambda S_{X3}) C_{mn}\right\} = 0
$$

(3-22)

in which,

$P$ and $S$ are functions in $i$, $j$, $m$, $n$, $L/P$, $R/t$ and the rigidity properties of shell;

\begin{align*}
&i = 1, \ldots, P; \\
&j = 2, \ldots, Q; \\
&m = 1, \ldots, P; \\
&n = 2, \ldots, Q; \\
&K = 1, 2, 3; \\
&\lambda = \frac{P}{t^2 D} R^3 \quad \text{(The eigenvalue)}
\end{align*}
For the consistency of these set of homogeneous linear algebraic equations, the determinant of the coefficients must vanish. This determinant is the Kronecker determinant of the previous quadratic form. The matrix forming this determinant is called the stability matrix. Its size is dependent on the number of terms considered from the matrix describing the virtual displacement components and equals to \((2F \times (N - 1)) \times (2F \times (N - 1))\).

The elements of this matrix are as follows:

(a) Like independent terms (R terms):

\[
R_{11} = 2 \frac{Dx \phi}{D\phi} \delta_{m,j} \frac{p}{m + \frac{1}{2} + \frac{1}{\bar{R}}} (\bar{R})^{m+1-1} \\
+ 2 \frac{Dx \phi}{D\phi} \delta_{m,j} \frac{p}{m + \frac{1}{2} + \frac{1}{\bar{R}}} (\bar{R})^{m+1-1}
\]

\[
R_{12} = 2 \frac{Dx \phi}{D\phi} \delta_{m,j} \frac{p}{m + \frac{1}{2} + \frac{1}{\bar{R}}} (\bar{R})^{m+1-1}
\]

\[
R_{13} = 0
\]

\[
R_{22} = 2 \frac{Dx \phi}{D\phi} \delta_{m,j} \frac{p}{m + \frac{1}{2} + \frac{1}{\bar{R}}} (\bar{R})^{m+1-1} \\
+ 2 \frac{Dx \phi}{D\phi} \delta_{m,j} \frac{p}{m + \frac{1}{2} + \frac{1}{\bar{R}}} (\bar{R})^{m+1-1}
\]

...
\[
R_{23} = -2\pi \delta n_j \frac{n}{m + 1 + 2} \left(\frac{L}{R}\right)^{m+1+2} \\
\quad - \frac{B_0}{R^2 D_\phi} \pi \delta n_j \frac{n(i^2 - 1)}{m + 1 + \frac{1}{2}} \left(\frac{L}{R}\right)^{m+1+2} \\
R_{33} = 2\pi \delta n_j \frac{1}{m + 1 + \frac{1}{3}} \left(\frac{L}{R}\right)^{m+1+3} \\
\quad + 2\pi \delta n_j \frac{B_0}{R^2 D_\phi} \frac{\nu(1-n^2)(1-i^2) + 1}{m + 1 + \frac{1}{3}} \left(\frac{L}{R}\right)^{m+1+3} \\
\quad + 2\pi \delta n_j \frac{B_0}{R^2 D_\phi} \frac{m(i+1)(i+1)}{m + 1 + \frac{1}{3}} \left(\frac{L}{R}\right)^{m+1+1} \\
\quad + 2\pi \delta n_j \frac{B_{00}}{R^2 D_\phi} \frac{n(i+1)(i+1)}{m + 1 + \frac{1}{3}} \left(\frac{L}{R}\right)^{m+1+1}
\]

(b) Load dependent terms (S terms)

1. Load is expressed as double series, 8th order pre-buckling solution

\[
S_{11} = 0 \\
S_{12} = 0
\]

\[
S_{13} = \left(\frac{t}{R}\right)^2 \frac{i + 1 - m}{m + 1 + \frac{1}{1}} \left(\frac{L}{R}\right)^{m+1+1} I_2(m_n,j)
\]

\[
S_{22} = 2I_4(m+1,n,j) + 2\left(\frac{t}{R}\right)^2 I_1(m_n,j) \\
\quad \left(\frac{L}{R}\right)^{m+1+1} \frac{m+1+1}{m + 1 + \frac{1}{1}} - 2\pi \delta n_j I_0(m+1+1)
\]
\[ S_{2,3} = -2J I_s(m+1, n, j) + 2 \frac{Dx}{D\phi} I_s(m+1, n, j) \]

\[ - 2(\frac{t}{R})^2 J I_1(m, n, j) (\frac{L/R}{m+1})^{m+2} \]

\[ + 2 \pi \delta_{n j} J I_o(m+1) \]

\[ S_{3,3} = 2 \frac{Dx}{D\phi} \]

\[ (m+1) (1+1) I_s(m+1, n, j) \]

\[ + 2nJ I_s(m+1, n, j) - 2 \frac{Dx}{D\phi} \]

\[ (1+1) n I_s(m+1, n, j) - 2 \frac{Dx}{D\phi} \]

\[ (1+m) j I_s(m+1, n, j) + 2 (\frac{t}{R})^2 \]

\[ \int \frac{(L/R)^{m+1}}{m+1} I_2(m, n, j) - 2\pi \delta_{n j} nJ I_o(m+1) \]

in which,

\[ I_s(m) = \sum_{s=1,3} b_s a_o \int_0^{L/R} w_o x^{m-1} dx \]

\[ I_1(m, n, j) = \sum_{r=0}^{6} \sum_{s=1,3} a_r b_s \int_0^{L/R} x^{m-1} \sin n_m x \]

\[ 2\pi \int_0^{2\pi} \cos r \theta \sin n \theta \sin j \phi d\phi \]
\[ I_2(m,n,j) = \sum_{r=0}^{6} \sum_{s=1,3}^{\infty} a_r b_s \int_0^{L/R} x^{m-1} \sin n_m x \, dx \]

\[ \int_0^{2\pi} \cos r \phi \cos n \phi \cos j \phi \, d\phi \]

\[ I_3(m,n,j) = \sum_{r=1}^{6} \sum_{s=1,3}^{\infty} \int_0^{L/R} x_1(x) x^{m-1} \, dx \]

\[ \int_0^{2\pi} \cos r \phi \cos n \phi \cos j \phi \, d\phi \]

\[ I_4(m,n,j) = \sum_{r=1}^{6} \sum_{s=1,3}^{\infty} \int_0^{L/R} x_2(x) x^{m-1} \, dx \]

\[ \int_0^{2\pi} \cos r \phi \sin n \phi \sin j \phi \, d\phi \]

\[ I_5(m,n,j) = \sum_{r=1}^{6} \sum_{s=1,3}^{\infty} \int_0^{L/R} x_3(x) x^{m-1} \, dx \]

\[ \int_0^{2\pi} \sin r \phi \cos n \phi \sin j \phi \, d\phi \]

\[ I_5(m,j,n) = \sum_{r=1}^{6} \sum_{s=1,3}^{\infty} \int_0^{L/R} x_4(x) x^{m-1} \, dx \]

\[ \int_0^{2\pi} \sin r \phi \cos j \phi \sin n \phi \, d\phi \]
\[ \epsilon_x = u_x = \sum_{r=1}^{6} \sum_{s=1,3}^{6} x_1(x) \cos r\phi \]

\[ \epsilon_\phi = v_\phi - w = \sum_{r=1}^{6} \sum_{s=1,3}^{6} x_2(x) \cos r\phi \]

\[ \gamma_{x\phi} = v_x + v_\phi = \sum_{r=1}^{6} \sum_{s=1,3}^{6} x_3(x) \sin r\phi \]

\( w_0 \) is the general prebuckling solution for load component with \( r=0 \).

ii. Load expressed as double series. Approximate Prebuckling Solution

\[ S_{11} = 0, \quad S_{12} = 0 \]

\[ S_{13} = \left( \frac{t}{R} \right)^2 \frac{1 - m + 1}{m + 1 + 1} I_2(m+1, n, j) \]

\[ S_{22} = 2 I_1(m+1, n, j) + 2 \left( \frac{t}{R} \right)^2 I_1(m+1, n, j) \]

\[ - 2 \pi \delta_{nj} \left( \frac{t}{R} \right)^2 I_0(m+1) \]

\[ - 2 \left( \frac{t}{R} \right)^2 \pi RT_2(m+1, n, j) \]

\[ S_{23} = -2j I_4(m+1+2, n, j) + 2 \frac{D\phi}{D\phi} (i+1) I_5(m+1, n, j) \]

\[ - 2 \left( \frac{t}{R} \right)^2 j I_4(m+1+2, n, j) + \]
\[ S_{3,3} = 2 \frac{Dx}{d\phi} (m+1) (i+1) I_3(m+i+1,n,j) \]

\[ + 2 n_j I_4(m+i+3,n,j) \]

\[ - 2 \frac{Dx}{d\phi} (l+1) n I_5(m+i+2,n,j) \]

\[ - 2 \frac{Dx}{d\phi} (l+m) j I_6(m+i+2,j,n) \]

\[ + 2 \left( \frac{t}{R} \right)^2 I_7(m+i+3,n,j) \]

\[ - 2 \left( \frac{t}{R} \right)^2 n_j RT2(m+i+3,n,j) \]

\[ - 2 \left( \frac{t}{R} \right)^2 (l+m) (l+1) RT1(m+i+1,n,j) \]

\[ - 2 \left( \frac{t}{R} \right)^2 (m+1) j RT3(m+i+2,n,j) \]

\[ - 2 \left( \frac{t}{R} \right)^2 (l+1) n RT4(m+i+2,j,n) \]

\[ - 2 \pi \delta_{nj} \left( \frac{t}{R} \right)^2 n_j I_6(m+1+3) \]

\[ + 2 \left( \frac{t}{R} \right)^2 \frac{a_1 b_2}{n_m^2} (1+m)(1+1) \frac{(L/R)^{m+i+1}}{m + 1} RSL(n,j) \]
in which,

\[ I_0(m) = \sum_{s=1,3} b_s a_0 \int_0^{L/R} x^{m-1} \sin n_m x \, dx \]

\[ R_{S1}(n,j) = \sum_{s=1,3} \int_0^{2\pi} (-1)^{(s-1)/2} \cos \phi \cos n\phi \cos j\phi \, d\phi \]

\[ R_{T1}(m,n,j) = \sum_{s=1,3} \frac{a_1 b_s}{n_m^2} \int_0^{L/R} x^{m-1} \sin n_m x \, dx \]

\[ \cdot \int_0^{2\pi} \cos \phi \cos n\phi \cos j\phi \, d\phi \]

\[ R_{T2}(m,n,j) = \sum_{s=1,3} a_1 b_s \int_0^{L/R} x^{m-1} \sin n_m x \, dx \]

\[ \cdot \int_0^{2\pi} \cos \phi \sin n\phi \sin j\phi \, d\phi \]

\[ R_{T3}(m,n,j) = \sum_{s=1,3} \frac{a_1 b_s}{n_m} \int_0^{L/R} x^{m-1} \cos n_m x \, dx \]

\[ \cdot \int_0^{2\pi} \sin \phi \cos n\phi \sin j\phi \, d\phi \]
\[ RT_4(m,n,j) = \sum_{s=1,3}^{a} a_1 \sum_{n_m}^{b_s} \int_{0}^{L/R} x^{n-1} \cos n_m x \, dx \]

\[ \cdot \int_{0}^{2\pi} \sin \phi \sin n \phi \cos j \phi \, d\phi \]

\[ I_3(m,n,j) = \sum_{r=2}^{6} \sum_{s=1,3}^{a} \int_{0}^{L/R} x_1(x) x^{m-1} \, dx \]

\[ \cdot \int_{0}^{2\pi} \cos r \phi \cos n \phi \cos j \phi \, d\phi \]

\[ I_4(m,n,j) = \sum_{r=2}^{6} \sum_{s=1,3}^{a} \int_{0}^{L/R} x_2(x) x^{m-1} \, dx \]

\[ \cdot \int_{0}^{2\pi} \cos r \phi \sin n \phi \sin j \phi \, d\phi \]

\[ I_5(m,n,j) = \sum_{r=2}^{6} \sum_{s=1,3}^{a} \int_{0}^{L/R} x_3(x) x^{m-1} \, dx \]

\[ \cdot \int_{0}^{2\pi} \sin r \phi \cos n \phi \sin j \phi \, d\phi \]

\[ I_6(m,j,n) = \sum_{r=2}^{6} \sum_{s=1,3}^{a} \int_{0}^{L/R} x_3(x) x^{m-1} \, dx \]

\[ \cdot \int_{0}^{2\pi} \sin r \phi \cos j \phi \sin n \phi \, d\phi \]
\[ \varepsilon_x = u_x = \sum_{r=2}^{6} \sum_{s=1,3}^{\infty} x_1(x) \cos r\phi \]

\[ \varepsilon_\phi = v_\phi - \kappa = \sum_{r=2}^{6} \sum_{s=1,3}^{\infty} x_2(x) \cos r\phi \]

\[ y_{x\phi} = v_x + u_\phi = \sum_{r=2}^{6} \sum_{s=1,3}^{\infty} x_3(x) \sin r\phi \]

iii. Load is expressed as single series.

Approximate prebuckling solution

\[ S_{11} = 0 \quad , \quad S_{12} = 0 \]

\[ S_{13} = \left( \frac{t}{R} \right)^2 \frac{1 - m + 1}{m + 1 + 1} \left( \frac{L}{R} \right)^{m+1} \frac{m+1}{m+1+1} I_2(n,j) \]

\[ S_{22} = 2 I_1(m+1, j) + 2 \left( \frac{t}{R} \right)^2 \frac{(L/R)^{m+1+1}}{m+1+1} I_1(n,j) \]

\[ - 2 \pi \delta_{n,j} a_0 \left( \frac{t}{R} \right)^2 \frac{(L/R)^{m+1+1}}{m+1+1} \]

\[ - 2 \left( \frac{t}{R} \right)^2 \frac{(L/R)^{m+1}}{m+1+1} I_1(n,j) \]

\[ S_{23} = -2J I_4(m+1, j) + \frac{D_\phi}{D_x} (i+1) I_5(m+i+1, n, j) \]

\[ - 2 \left( \frac{t}{R} \right)^2 J \frac{(L/R)^{m+1+2}}{m+1+2} I_1(n,j) + \]
\[ 2 \pi \delta_{nj} \frac{(\frac{L}{R})^2}{R} a_0 \frac{(L/R)^{m+1+2}}{m+1+2} \]

\[ + 2 \left( \frac{L}{R} \right)^2 (i+1) \frac{(L/R)^{m+1+2}}{(m+1+1)(m+1+2)} \text{RS2}(n,j) \]

\[ + 2 \left( \frac{L}{R} \right)^2 j \frac{(L/R)^{m+1+2}}{m+1+2} \text{IS}(n,j) \]

\[ S_{3,3} = 2 \frac{Dx}{D\phi} (m+1)(i+1) I_1(m+1+1,n,j) \]

\[ + 2nj I_3(m+1+3,n,j) \]

\[ - 2 \frac{Dx \phi}{D\phi} (i+1)n I_5(m+1+2,n,j) \]

\[ - 2 \frac{Dx \phi}{D\phi} (m+1)j I_5(m+1+2,j,n) \]

\[ + 2 \left( \frac{L}{R} \right)^2 \frac{(L/R)^{m+1+3}}{m+1+3} I_2(n,j) \]

\[ + 2 \left( \frac{L}{R} \right)^2 (1+m)(i+1) \frac{(L/R)^{m+1+3}}{(m+1+1)(m+1+2)(m+1+3)} \text{RS1}(n,j) \]

\[ - 2 \pi \delta_{nj} nj a_0 \left( \frac{L}{R} \right)^2 \frac{(L/R)^{m+1+3}}{m+1+3} \]

\[ - 2 \left( \frac{L}{R} \right)^2 (m+1)j \frac{(L/R)^{m+1+3}}{(m+1+2)(m+1+3)} \text{RS2}(n,j) \]

\[ - 2 \left( \frac{L}{R} \right)^2 (i+1)n \frac{(L/R)^{m+1+3}}{(m+1+2)(m+1+3)} \text{RS2}(j,n) \]

\[ - 2 \left( \frac{L}{R} \right)^2 nj \frac{(L/R)^{m+1+3}}{m+1+3} \text{IS}(n,j) \]
in which,

\[ R S 1(n,j) = a_1 \int_0^{2\pi} \cos \phi \cos n\phi \cos j\phi \, d\phi \]

\[ R S 2(n,j) = a_1 \int_0^{2\pi} \sin \phi \sin n\phi \cos j\phi \, d\phi \]

\[ I S (n,j) = a_1 \int_0^{2\pi} \cos \phi \sin n\phi \sin j\phi \, d\phi \]

\[ I_1(n,j) = \sum_{r=0}^{6} a_r \int_0^{2\pi} \cos r\phi \sin n\phi \sin j\phi \, d\phi \]

\[ I_2(n,j) = \sum_{r=0}^{6} a_r \int_0^{2\pi} \cos r\phi \cos n\phi \cos j\phi \, d\phi \]

\[ I_3(m,n,j) = \sum_{r=2}^{L/R} \int_0 x_1(x) x^{m-1} \, dx \int_0^{2\pi} \cos r\phi \cos n\phi \cos j\phi \, d\phi \]

\[ I_4(m,n,j) = \sum_{r=2}^{L/R} \int_0 x_2(x) x^{m-1} \, dx \int_0^{2\pi} \cos r\phi \sin n\phi \sin j\phi \, d\phi \]

\[ I_5(m,n,j) = \sum_{r=2}^{L/R} \int_0 x_3(x) x^{m-1} \, dx \int_0^{2\pi} \sin r\phi \cos n\phi \sin j\phi \, d\phi \]

\[ I_6(m,n,j) = \sum_{r=2}^{L/R} \int_0 x_4(x) x^{m-1} \, dx \int_0^{2\pi} \sin r\phi \cos j\phi \sin n\phi \, d\phi \]
\[ \varepsilon_x = u_x = - \sum_{r=2}^{6} x_r(x) \cos r\phi \]

\[ \varepsilon_\phi = v_\phi - \nu = \sum_{r=2}^{6} x_r(x) \cos r\phi \]

\[ \gamma_\phi = v_x + u_\phi = \sum_{r=2}^{6} x_r(x) \sin r\phi \]

and

\[ \delta_{nj} = \text{Kronecker delta} \]

\( b_s, a_p = b_m, b_n \) of the pre-buckling analysis respectively.

All the above terms are for the case of wind load only. Similar expressions can be written for the case of eccentric discharge, only by using the corresponding values of \( b_s, a_p \) and the corresponding terms in the series describing the load.

### 3.7 The Buckling Load

In a general computer programme on IBM 360/65, the pre-buckling analysis is carried out to calculate the pre-buckling membrane strain components. Thereafter the integrals are carried out numerically and the different elements of the stability matrix are calculated and arranged in their proper places in the matrix.
The stability matrix can be divided into two submatrices as follow:

\[ SM = R + \lambda S \]  \hspace{1cm} (3-23)

in which,

- \( R \) = the matrix containing load independent terms
- \( S \) = the matrix containing load dependent terms
- \( \lambda \) = the parameter containing the load value
  \[ \lambda = \frac{P_0 R^3}{D \phi t^2} \]  \hspace{1cm} "the eigenvalue"

The stability condition is:

\[ |R| + \lambda |S| = 0 \]  \hspace{1cm} (3-24)

Using the subroutine NROOT of the IBM scientific subroutines library, the eigenvalues can be obtained. The number of these eigenvalues must be \((3P \times (Q - 1))\). If \( P \) and \( Q \) are taken to be 5 and 6 respectively, the matrix \( R \) and the matrix \( S \) each will be of the size 75, which is also the number of eigenvalues. Each eigenvalue is associated with a certain mode of buckling which can be obtained.

From the requirements of subroutine NROOT, the matrix multiplied by \( \lambda \) should have a positive definite character which is the case with the matrix \( R \) but not with the matrix \( S \). Therefore Eq. (3-24) is rewritten as follows:

\[ |S| + \lambda' |R| = 0 \]  \hspace{1cm} (3-25)

where

\[ \lambda' = \frac{\lambda}{2} \]
The buckling load, hence, is given as the reciprocal of the largest eigenvalue fulfilling equation (3-25).

3.8 Buckling Configuration

For any eigenvalue $\lambda$, the homogeneous set of algebraic equations in $A_{mn}$, $B_{mn}$ and $C_{mn}$ can be written in the following matrix form:

\[
\begin{bmatrix}
R_{11} + \lambda S_{11} & R_{12} + \lambda S_{12} & R_{13} + \lambda S_{13} \\
R_{21} + \lambda S_{21} & R_{22} + \lambda S_{22} & R_{23} + \lambda S_{23} \\
R_{31} + \lambda S_{31} & R_{32} + \lambda S_{32} & R_{33} + \lambda S_{33}
\end{bmatrix} \begin{bmatrix}
A_{mn} \\
B_{mn} \\
C_{mn}
\end{bmatrix} = 0
\]

By assuming an arbitrary value for one of the parameters (say $A_{11}$) the above matrix form can be solved for the rest of the parameters using Gauss elimination technique. Hence, the relative values of the parameters $A_{mn}$, $B_{mn}$ and $C_{mn}$ associated with a certain eigenvalue $\lambda$ are obtained. These values are then substituted in the series describing the virtual displacement components during buckling, Eqs. (3-21). The summation of each series for the same order of terms considered before in establishing the stability matrix, gives the relative values of the virtual displacement components, $\xi$, $\eta$ and $\zeta$ at a certain
point \((x,t)\) at the middle surface of the shell, corresponding to a certain eigenvalue \(\lambda\).

Repeating the summation process at sufficient number of points at the middle surface of the shell, the relative values of the virtual displacement components can be obtained, hence a complete geometric configuration of the shell during buckling can be established.

3.1. Observations on the Buckling Analysis

The computer programmes were prepared to calculate the buckling load. One is based on the eighth-order reforming system of the prebuckling analysis. The other was on the approximate system. The variables taken into consideration were the geometric proportions of shell, the boundary conditions at the upper edge and the rigidity ratios of the shell.

The buckling loads of the shells investigated are shown in tables (3-1) to (3-7). All the shells were examined under wind pressure.

3.9.1 Comparison of Stability Results of Isotropic Shells under Wind Load with those obtained by (6) and (7).

A comparison is carried out between the results obtained for the isotropic shells and those given by Johns (6) and Langhaar (7) as shown in Figs. (3-2) to (3-4).
Both investigators studied shells with Poisson's ratio = 0.33. Johns' analysis is based on Donnell's prebuckling governing system. The results given by him are always greater than those obtained in the present work. This could be attributed to two factors, the effect of Poisson's ratio, and the error involved in the prebuckling analysis due to employing Donnell's approach.

The results given by Langhaar are very conservative, especially at high values for both L/R and R/t. The difference between the results given by the two investigators is rather high and sometimes reaches 100%.

The deviation in Langhaar results could be attributed to the approximation involved in his analysis where the membrane theory was considered as a basis for the pre-buckling analysis.

However, the experimental results obtained by Johns are also shown in the figures. These results - which do not closely conform to the theoretical results given by the same reference - are in good agreement with the theoretical values presented in this work as shown in Figs. (3-2) and (3-3).

3.9.2 Factors affecting the Buckling Load

The relation between the buckling load and both L/R and R/t is shown in Figs. (3-5) and (3-6) for shells with restrained upper edge and in Figs. (3-7) to (3-8) for
shells with free upper edge. For both edge conditions, the buckling load decreases with the decrease of L/R and/or h/t. However, the R/t ratio has more effect on the value of the buckling load than L/R.

Also, in the case of a shell with flexible upper edge, the effect of both L/R and h/t on the value of the buckling load is more than that in the case of shell with restrained upper edge.

Fig. (3-9) gives the ratio between the buckling load for a shell with flexible upper edge and that for a shell with restrained one, as a function of L/R for different values of h/t. As expected, restraining the upper edge increases the overall stiffness of the shell, thus it increases the value of the buckling load. The increase in the buckling load due to restraining the upper edge varies and depends on the shell proportions. For the cases investigated the ratio between flex. flexible and flex. restrained varied between 78% to 45%. The gain in buckling load increases with the increase of L/R and of h/t.

Fig. (3-10) shows the buckling load as a function of L/R for different values of D_x/D_φ and R/t for shells with flexible upper edge.

It is noticed from the figure that the buckling load decreases with the decrease of D_x/D_φ while fixing the value of D_φ.
3.9.3 The Approximate Prebuckling Analysis as a Basis for
The Stability of Cylindrical Cantilever Shells

The adequacy of the prebuckling approximate approach
to be employed as a basis for the buckling analysis is
examined for isotropic shells with restrained upper edges.
These shells have exhibited the highest deviation in the
prebuckling results due to the approximate approach.
Figs. (3-8) to (3-10) show the buckling load values
obtained with employing both prebuckling approaches.

The deviation in buckling load values between the
two approaches is presented in Fig. (3-11). This figure
gives the % deviation in the buckling load as a function
of L/R for different R/t values. The maximum deviation
was 15.74%. It occurred for the same isotropic shell that
yielded the maximum deviation in the prebuckling analysis
(L/R = 1, R/t = 100).

The relatively small deviation in the buckling loads
-especially for shells with R/t> 200)- can be explained by
examining the second variation of potential energy expres-
sion, Eq. (3-20). The contribution of the prebuckling
membrane strain components comprises a major part in the
second variation of potential energy. Consequently they
become an important factor in determining the buckling load
value. At the same time, the contribution of the change
of curvature components is relatively small. Therefore a
deviation in the buckling load is mainly a function of the
relatively small deviations in the membrane strain and internal force components.

The deviation in buckling load decreases with the increase of both L/R and R/t.

3.9.4. Buckling Configuration

The buckling configuration of orthotropic shells are calculated along the edge x = L/R and the generator φ = 0. The results are shown in Fig. (3-15) where the geometric configuration during buckling for shells having different L/R and E_φ/E_x values is shown. It is noticed that the configuration during buckling has the same pattern of a slightly deformed shell during the prebuckling elastic stage. This means that the buckling occurs without any sudden change of the configuration.
CHAPTER IV
EXPERIMENTAL INVESTIGATION

It was necessary to verify the analytical results obtained for the prebuckling and buckling stages. Unfortunately the information needed for such a verification is not available except for the special case of isotropic shells. Therefore, an experimental programme was carried out to test cantilever cylindrical shells made of light gauge cold-formed corrugated steel sheets.

4.1 Material

The material used was light gauge cold-formed corrugated steel sheets. The sheets were of semi-circular shape with an average radius $R = 37''$. The dimensions and proportions of the standard corrugation were as follows (Fig. 2-3):

$$t = 0.024''$$
$$f = 0.277''$$
$$l/c = 1.117$$

where

$t$ = average thickness of sheets
$f$ = average amplitude of corrugation wave
$l$ = curved length of one pitch of corrugation
$c$ = horizontal projection of the length of one pitch of corrugation
The mechanical properties were as follows:

Modulus of Elasticity \( (E) \) \( = \) \( 29.5 \times 10^6 \) (psi)

Poisson Ratio \( (\mu) \) \( = \) \( 0.33 \) \( (\mu E/\mu E) \)

Hence the rigidity properties of sheets are calculated as follows:

\[
D_x = \frac{E}{6(1 - \mu^2)} \left( \frac{L}{f} \right)^2 t = 1000.92 \text{ lb/in}
\]

\[
D_\phi = \frac{E}{c} t \frac{E}{f} = 804240.2 \text{ lb/in}
\]

\[
D_{x\phi} = \frac{Et}{2(1 + \mu)} \frac{c}{L} = 292324 \text{ lb/in}
\]

\[
B_x = \frac{c}{k} \frac{Et^3}{12(1 - \mu^2)} = 34.37 \text{ lb.in}
\]

\[
B_\phi = 0.522 \frac{E}{f^2} t = 28837.8 \text{ lb.in}
\]

\[
B_{x\phi} = \frac{\mu}{c} \frac{Et^3}{12(1 + \mu)} = 29.02 \text{ lb.in}
\]

4.2 Description of Shells

4.2.1 Shell No. 1 (Fig. 4-1)

Shell No. 1 was a cantilever shell with a completely free upper edge.

- Dimensions:
  
  Radius of shell = 37"
  
  Height of shell = 60"
Floor System:
The floor system was arranged in such a way to assume fixed end conditions at the lower edge. A circular steel angle 2" x 2" x 3/16" was bolted to the corrugated sheets at the lower edge and welded to the floor in the circumferential direction.
The lower edge was provided with two closed box section steel bracings 24" apart, Fig. (4-2). Each bracing was welded to the lower ring beam and bolted to the floor by two 2" diam. bolts.

4.2.2 Shell No.2 (Fig. 4-3)

Shell No.2 was of the same dimensions and having the same floor system as shell No.1. The shell was assumed to have a completely restrained upper edge. This edge condition was satisfied by providing the edge with a circular steel angle 2" x 2" x 3/8".

4.3 Loading System

An air bag was provided as a loading device to exert a uniform lateral radial pressure along the total height of the shell and covering an arc of 32° central angle. Unfortunately, the maximum pressure exerted by the bag was less than the buckling pressure of any of the two shells. Therefore, the air bag was used only for shell No.1 during
the prebuckling stage. Another system was used to load both shells up to the buckling pressure. This loading system consisted of four horizontal identical hydraulic jacks connected to Hydraulic Universal Testing Machine. The pressure was transmitted to the outer surface of the shell through eight rubber strips 3" height and 24" curved width, with a radius equal to the radius of the shell. The horizontal jacks were located at equal distance apart along the height of the shell in the vertical x-y plane.

By arranging the rubber strips also at equal distances along the total height of the shell, the loading was assumed to be a uniformly distributed load covering an arc of 30° central angle along the total height of the shell (Fig. 4-4).

4.4 Test Procedure

4.4.1 Loading

Each shell was loaded incrementally. The incremental load was chosen to be 1/20 of the predicted buckling load of each shell.

4.4.2 Measurement

Dial gauges of 0.001" accuracy were used to measure the radial deflection along the upper edge at $\phi = 0, \pi/6, \ldots, \pi$ and along the generator $\phi = 0$ at $x/(L/R) = 0.0, 0.17, 0.33, \ldots, 1.0$. 
The circumferencial strain was also measured and recorded at each incremental load at the points \((\phi = 0, x = L/R)\) and \((\phi = \pi/6, x = L/2R)\) on both surfaces of shell No.1. For shell No.2, \(\varepsilon_\phi\) was measured at the point \((\phi = 0, x = L/R)\). The strain gauges were 120Ω resistance strain gauges with 2.09 gauge factor on 1\(^\prime\) gauge length.

4.5 Test Results

4.5.1 Shell No.1

Tables (1-I), and (2-I), Appendix I, give the strain measurements recorded during test at the points \((\phi = 0, x = L/R)\) and \((\phi = \pi/6, x = L/2R)\) respectively, measured at both surfaces of the shell. The tables also give the observed circumferencial moment and force \((M_\phi, N_\phi)\) during test calculated according to the following formulae:

\[
M_\phi = \frac{(\varepsilon_1 - \varepsilon_2)}{2} \cdot \frac{B_\phi}{t/2}
\]

\[
= 0.522 t^2 E (\varepsilon_1 - \varepsilon_2) \quad (4-1)
\]

\[
N_\phi = \frac{(\varepsilon_1 + \varepsilon_2)}{2} \cdot D_\phi
\]

\[
= \frac{t}{2c} t E (\varepsilon_1 + \varepsilon_2) \quad (4-2)
\]

Tables (3-I) and (4-I), Appendix I, give the deflection recorded up to the buckling load, along the upper edge and the generator \(\phi = 0\), respectively.

The buckling load observed was 3.21 lb/in\(^2\). Fig. (4-5)
shows the shell before loading. Figs. (4-6) to (4-8) show the shell profile along the generator \( \phi = 0 \) during the different stages of loading, while Fig. (4-9) shows the shell at the final loading stage i.e. at the buckling load and the buckling configuration in the longitudinal direction along the generator \( \phi = 0 \) at this load.

4.5.2 Shell No. 2

Using the same expressions, Eqs. (4-1) and (4-2), to calculate \( M_\phi \) and \( N_\phi \), table (5-I), Appendix I, gives the observed surface strains and internal force components \( M_\phi \) and \( N_\phi \) at the point \( (\phi = 0, x = L/2R) \) at every incremental load. The deflection readings are given in tables (6-I) and (7-I), Appendix I, along the edge \( x = L/R \) and the generator \( \phi = 0 \), respectively. Fig. (4-10) shows the set-up of the shell and the loading system while Fig. (4-11) shows the shell interior and the location of the dial gauges. Fig. (4-12) illustrates the buckling configuration of the shell, while Fig. (4-13) shows the vertical displacement of the upper edge at the buckling load.

The critical load was reached at 3.85 lb/in\(^2\).

4.6 Discussion of Test Results

4.6.1 Load Formula

The distribution of the applied load in \( x- \) and \( \phi- \) directions is given in Fig. (4-14). This distribution can
be expressed in a Fourier series as the following:

\[
P_z = P_0 \sum_{n=1,2,\ldots}^{\infty} \sum_{m=1,3,\ldots}^{\infty} a_n \cos n\phi \cdot b_m \sin m \phi
\]

(4-3)

in which,

\[
a_n = \frac{2}{n \pi} \sin \left(\frac{n \pi}{6}\right)
\]

\[
b_m = 4 \left/ \pi m^2 m^2
\]

By providing the computer programmes for the prebuckling and buckling analysis with the above data describing the load, also with the rigidities and proportions information, the analytical results for the tested shells are obtained.

4.6.2 Shell No.1

The geometric proportions and rigidity ratios are as follows:

\[
\frac{L}{R} = 1.66
\]

\[
\frac{R}{t} = 1440
\]

\[
\Omega_1 = \frac{B_\phi}{R^2 D_\phi} = 2.76 \times 10^{-5}
\]

\[
\Omega_2 = \frac{B_{x\phi}}{R^2 D_\phi} = 2.63 \times 10^{-8}
\]

\[
\Omega_3 = \frac{B_x}{R^2 D_\phi} = 3.121 \times 10^{-8}
\]
\[ \frac{D}{D\phi} = 0.00124 \]

\[ \frac{D\phi}{Dx\phi} = 2.25 \]

4.6.2.1 Prebuckling Stage

Figs. (4-15) and (4-16) show the load-deflection relationship for points along the upper edge and the generator \( \phi = 0 \), respectively, both experimentally and analytically. Up to about three quarters of the buckling load, the observed and the predicted load-deflection relationship generally agree with each other with an average deviation of about 8.7%. However beyond that load, the experimental relationship starts to be nonlinear with an increasing rate and increasing deviation from the analytical one until the buckling load is reached which is defined as the load at which the deflection increases rapidly without any significant increase in the load, or the point at which the deflection becomes indifferent to the applied load.

Figs. (4-17) and (4-18) show typical radial displacement distribution along the upper edge and the generator \( \phi = 0 \) respectively, under unit applied pressure. The experimental curves represent only the linear stage of load-deflection diagram.

The theoretical buckling pressure for shell No.1 is 2.85 lb/in², i.e. 11.2% less than the observed buckling
pressure. It is noticed that the analytical results of shell No.1 obtained by employing any of the two prebuckling analysis approaches exhibit no significant difference from each other.

A comparison between the theoretical and experimental values of $M_\phi$ at the point $(x = L/R, \phi = 0)$ at which the maximum value of the circumferential moment occurs is shown in Fig. (4-19). Figs. (4-20) and (4-21) show the same relation for $N_\phi$ at $(x = L/R, \phi = 0)$ and $(x = L/2R, \phi = \pi/6)$ respectively. The three figures show a very good agreement between the predicted and the observed values.

4.6.2.2 Shell No.2

Figs. (4-22) and (4-23) show typical distribution of the radial displacement along the generator $\phi = 0$ and the edge $x = L/R$ respectively. The discrepancy between the predicted deflection assuming the upper edge to be completely restrained and the observed one is relatively high. The deviation increases along the height until it reaches its maximum value (828) at the top. This discrepancy can be attributed to the assumption that the upper ring beam is of infinite rigidity compared to the shell rigidity. The actual bending rigidity of the upper edge beam $E_bI_b$ is taken into consideration where $I_b$ is the local moment of inertia of the beam about its local y axis (see figure). The load transmitted to the circular beam.
is the lateral shear force $Q_{x\text{eff}}$ at $x = L/R$. Therefore the differential equation of the beam can be written, after neglecting the membrane shear force $N_{x\phi}$, as follows:

$$\frac{1}{R^3} \frac{\partial^4 w}{\partial \phi^4} = \frac{1}{E_b I_b} Q_{x\text{eff}}(x=L/R)$$

(4-4)

By integrating the above equation four times the radial deflection is obtained,

$$w(x=L/R) = \frac{R^3}{n^4 E_b I_b} Q_{x\text{eff}}(x=L/R) + C_1 \phi + C_2 \phi^2 + C_3 \phi + C_4$$

(4-5)

$w$ and $Q_{x\text{eff}}$ are expressed as infinite harmonic series in $\phi$, Eqs. (2-19) and (2-28). Since the above equation is valid for all values of $\phi$ along the edge, therefore each coefficient corresponding to a term in the series should vanish as well as the four constants of integration, i.e. $C_1 = C_2 = C_3 = C_4 = 0$. Hence the previous expression is reduced to

$$w(x=L/R) = \frac{R^3}{n^4 E_b I_b} Q_{x\text{eff}}(x=L/R)$$

(4-6)

The value $R^3 / n^4 E_b I_b$ can be considered as a spring coefficient relating the load to the deflection along the upper edge which can be considered as if
supported on elastic circular support. Accordingly, a new boundary condition is developed to replace the condition (2-12g) along the upper edge \( M_\phi = 0 \) at \( x = L/R \) and is given as follows:

at \( x = L/R \):

\[
w - \frac{R t^2}{n^4} \cdot \frac{D_\phi}{E_b I_b} Q_{x\text{eff}} = 0 \quad (4-7)
\]

The above equation is Eq. (4-6) after dividing it by the parameter \( \lambda \). The factor \( \frac{D_\phi}{E_b I_b} \) can be considered to describe the relative rigidity between the shell and the beam. Fig. (4-24) shows the relation between \( \frac{D_\phi}{E_b I_b} \) and \( M_\phi \) at \( (\phi = 0, x = L/R) \). From this figure the limit of considering the upper edges as a completely restrained edge is established (i.e. when \( M_\phi \) at \( x = L/R \) vanishes). By calculating the value of \( E_b I_b \) and using condition (4-6) for solving shell No.2, the theoretical radial displacement distribution is corrected as shown in Figs. (4-22) and (4-23) and the maximum deviation between the theoretical and experimental values reduces from 28% to 12%.

Figs. (4-25) and (4-26) give the experimental and theoretical load-deflection relationship at points along the generator \( \phi = 0 \) and the upper edge, respectively. The results obtained by considering both the ideally restrained and partially restrained edges are shown in the figure.

The relation between the internal forces \( M_\phi \) and \( N_\phi \)
and the load at point \((\phi = 0, x = L/2R)\) is shown in Figs. (4-27) and (4-28) respectively, where both theoretical and experimental results for \(M_\phi\) are in very good agreement. However, for \(N_\phi\) there is constant deviation which could be attributed to experimental error.

The theoretical buckling pressure for shell No. 2 is 3.41 lb/in\(^2\) with 11.53\% deviation from the observed buckling pressure. This theoretical value was based on assuming the ideal completely restrained upper edge conditions. By introducing the partially restrained upper edge conditions to the stability programme, the buckling pressure decreases to 3.26 lb/in\(^2\) and the deviation increases to 15.2\%.

In general, the experimental results for the pre-buckling analysis either for displacement or internal force components satisfactorily conform to the predicted results for both shells. As for the buckling loads although the results are still considered agreeable the relatively high discrepancy (11\% and 15.2\%) can be partially attributed to the accumulated effect of the conservative evaluation of shell rigidities in terms of the geometric proportion of the corrugation and to the assumption that the applied load exerted by the jacks is radially directed while it is actually parallel to the jacks. A part of the deviation, of course, is due to the difference between the actual behaviour of the shells and the mathematical model adapted, in addition to the random experimental errors encountered during testing.
CHAPTER IV
CONCLUSIONS

In this thesis a thorough study is carried out to determine the behaviour of cantilever orthotropic cylindrical shells subjected to lateral loads. A rigorous solution for the prebuckling analysis is introduced as well as an approximate one proposed for shells made of corrugated sheets. A theoretical treatment for the stability problem is presented using Trefftz's variational theory. The results are verified experimentally in both the buckling and the prebuckling stages. Curves are prepared to calculate the prebuckling internal force components as well as the buckling load. These curves can be used for practical use.

The following concluding remarks may be added here:

1. From the comparison between the experimental and the theoretical results, it is obvious that the linear analysis is satisfactorily accurate to describe the prebuckling behaviour of the shell. However, after about three quarters of the buckling load is reached, the deviation between the theory and the experimental results increases rapidly as the behaviour starts to be nonlinear with increasing rate. This deviation just before the buckling load, however, does not have significant effect on the outcome of the stability results.
2. Trefftz's variational theory is a successful tool in solving the stability problem of the three-dimensional structures. In spite of the complication involved in dealing with the stability of a three-dimensional orthotropic structure, the overall buckling load as well as the buckling configuration are obtained with relatively short computer time (6 min. and 9 min., using the approximate and the more exact approaches, respectively).

3. Restraining the upper edge of the shell decreases the value of the internal force components, especially the bending moments, hence it improves the performance of the shell. Also, a shell with a restrained upper edge always has a higher overall buckling load than a shell with a flexible edge having the same properties. Since these shells are usually thin and flexible, therefore their ultimate load carrying capacity is governed by the overall buckling. This means that providing the shell with a rigid upper edge appreciably increases its ultimate load carrying capacity.

4. The buckling configuration of the shells investigated has the same pattern of the slightly deformed shell during the prebuckling elastic stage. This means that the overall buckling of cantilever cylindrical shells under wind pressure occurs without any sudden change of the configuration or "bifurcation" in the circumferential and the longitudinal directions.
5. The approximate prebuckling approach can be used for cantilever shells made of corrugated sheets either for the analysis or for the stability. It can be employed in analysing other shells depending on the accuracy of the problem at hand and the amount of deviation that can be tolerated. However, considering 5% deviation in buckling load is acceptable, the approximate approach can be used for isotropic shells having $L/R > 3$ and/or $R/t > 300$. 
REFERENCES


5. Abdel-Sayed, G., "Analysis of Cylindrical Grain Storages made of Cold-Formed Steel", 3rd international specialty Conf. on Cold-Formed Steel Struc., pp. 927-950.


Table (3-1)

Buckling Load $P_{ocr}$ (lb/in$^2$) - $R/t = 100$
Isotropic Shells with a Restrained Upper Edge under Wind Load

<table>
<thead>
<tr>
<th>L/R</th>
<th>8th order system</th>
<th>Approx. system</th>
<th>Deviation %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>214.65</td>
<td>247.92</td>
<td>15.74</td>
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<tr>
<td>2</td>
<td>86.350</td>
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<tr>
<td>3</td>
<td>41.062</td>
<td>43.971</td>
<td>7.09</td>
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</table>

Table (3-2)

Buckling Load $P_{ocr}$ (lb/in$^2$) - $R/t = 200$
Isotropic Shells with a Restrained Upper Edge under Wind Load

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<tr>
<th>L/R</th>
<th>8th order system</th>
<th>Approx. system</th>
<th>Deviation %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>37.53</td>
<td>40.596</td>
<td>8.17</td>
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<tr>
<td>2</td>
<td>15.78</td>
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<tr>
<td>3</td>
<td>8.76</td>
<td>9.188</td>
<td>4.89</td>
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</table>
Table (3-3)

Buckling Load $P_{ocr}$ (lb/in$^2$)  $R/t = 300$

Isotropic Shells with a Restrained Upper Edge under Wind Load

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<tr>
<th>L/R</th>
<th>8th order system</th>
<th>Approx. system</th>
<th>Deviation %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>2</td>
<td>6.67</td>
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<td>2.88</td>
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<tr>
<td>3</td>
<td>3.84</td>
<td>3.917</td>
<td>2.03</td>
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Table (3-4)

Buckling Load $P_{ocr}$ (lb/in$^2$)

Isotropic Shells with a Flexible Upper Edge under Wind Load

<table>
<thead>
<tr>
<th>R/t</th>
<th>100</th>
<th>200</th>
<th>300</th>
</tr>
</thead>
<tbody>
<tr>
<td>L/R</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>147.54</td>
<td>23.62</td>
<td>8.66</td>
</tr>
<tr>
<td>2</td>
<td>67.95</td>
<td>9.21</td>
<td>3.84</td>
</tr>
<tr>
<td>3</td>
<td>38.60</td>
<td>5.36</td>
<td>2.22</td>
</tr>
</tbody>
</table>
Table (3-5)

Buckling Load $P_{ocr}$ (lb/in$^2$) $\times \frac{30 \times 10^6}{D_\phi / t}$ \hspace{1cm} $\frac{D_\phi}{D_x} = \frac{B_\phi}{B_x} = 4$

Flexible Upper Edge, under Wind Load

<table>
<thead>
<tr>
<th>R/t</th>
<th>L/R</th>
<th>100</th>
<th>200</th>
<th>300</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>2</td>
<td>1</td>
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<td>0.43</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>11.07</td>
<td>0.92</td>
<td>0.37</td>
</tr>
</tbody>
</table>

Table (3-6)

Buckling Load $P_{ocr}$ (lb/in$^2$) $\times \frac{30 \times 10^6}{D_\phi / t}$ \hspace{1cm} $\frac{D_\phi}{D_x} = \frac{B_\phi}{B_x} = 16$

Flexible Upper Edge, under Wind Load

<table>
<thead>
<tr>
<th>R/t</th>
<th>L/R</th>
<th>100</th>
<th>200</th>
<th>300</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>11.61</td>
<td>1.37</td>
<td>0.29</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>4.18</td>
<td>0.62</td>
<td>0.21</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3.17</td>
<td>0.39</td>
<td>0.16</td>
</tr>
</tbody>
</table>
Table (3-7)

Buckling Load $P_{cr}$ (lb/in$^2$) $\times \frac{30 \times 10^6}{D\phi / t}$ $\frac{D\phi}{Dx} = \frac{B\phi}{Bx} = 64$

Flexible Upper Edge, under Wind Load

<table>
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<tr>
<th>$L/R$</th>
<th>$R/t$</th>
<th>100</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.89</td>
<td>0.38</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.91</td>
<td>0.23</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.53</td>
<td>0.20</td>
<td></td>
</tr>
</tbody>
</table>
Figures

N.B. - All axes are nondimensional unless otherwise stated.
- The 8th order solution is referred to as the exact solution in curves.
Fig. (2-1) The Non-dimensional Cylindrical Coordinate System

\[ X^* = X/R \]
\[ Z^* = Z/R \]

Fig. (2-3) Geometry of Corrugation
The Shell Element
Fig. (2-4a) Wind Load Distribution

Fig. (2-4b) Pressure due to Eccentric Discharge
Fig. (2-5) Radial Displacement along the Totally Flexible Upper Edge of Orthotropic Shell under Wind Load
Fig. (2-6) Radial Displacement along the Totally Flexible Upper Edge of Orthotropic Shell under Wind Loading
Fig. (2-7) Radial Displacement along the Totally Flexible Upper Edge of Orthotropic Shell under Wind Loading
Fig. (2-8) Radial Displacement along the Totally Flexible Upper Edge of Orthotropic Shell under Wind Loading
Fig. (2-9) Radial Displacement along the Totally Flexible Upper Edge of Orthotropic Shell under Wind Loading
Fig. (2-10) Radial Displacement along the Totally Flexible Upper Edge of Orthotropic Shell under Wind Loading
Fig. (2-11) Radial Displacement along the Totally Flexible Upper Edge of Isotropic Shell under Wind Loading
Fig. (2-12) Radial Displacement along the Totally Flexible Upper Edge of Isotropic Shell under Wind Loading.
Fig. (2-13) Radial Displacement along the Totally Flexible Upper Edge of Isotropic Shell under Wind Loading
Fig. (2-14) Radial Displacement along the Totally Flexible Upper Edge of Isotropic Shell under Wind Loading
Fig. (2-15) Radial Displacement along the Totally Flexible Upper Edge of Isotropic Shell under Wind Loading
Fig. (2-16) Radial Displacement along the Totally Flexible Upper Edge of Isotropic Shell under Wind Loading.
Fig. (2-17) Comparison of Radial Displacement with Reference (22)
Fig. (2-18) Comparison of Radial Displacement with Reference (22)
Fig. (2-19) Maximum Circumferential Moment (at $x = L/R$, $\phi = 0$)
for Shells with Flexible Upper Edges; under Wind Load

$\max M_{\phi} = \left(10^4 \frac{P_0 R^2}{t} \right)$
Fig. (2-20) Maximum Circumferential Moment (along the generator $\phi = 0$) for Shells with Restrained Upper Edges; under Wind Load
Fig. (2-21) Typical Distribution of $M_\phi$ along the generator $\phi=0$ for Shells with Flexible Upper Edges; under Wind Load.
Fig. (2-22) Typical Distribution of $M_\phi$ along the generator $\phi = 0$ for Shells with Restrained Upper Edges; under Wind Load
Fig.(2-23) Deviation in results of $M_{\phi_{max}}$ between the two approaches, for shells with Restrained Upper Edge; under Wind Load.
Fig. (2-24a) Typical Distribution of $M_x$ along the generator $\phi=0$ for Shells with Flexible Upper Edges; under Wind Load
Fig. (2-24b) Typical Distribution of $M_x$ along the generator $\phi = 0$ for Shells with Flexible Upper Edges; under Wind Load
Fig. (2-25) Typical Distribution of $M_x$ along the generator $\phi=0$ for shells with restrained upper edges; under wind load.
Fig. (2-27) Maximum Longitudinal Bending Moment (at $x=\phi=0$) for Shells with Flexible Upper Edges; under Wind Load
Fig. (2-28) Maximum Longitudinal Bending Moment (at x=ϕ=0) for Shells with Restrained Upper Edges; under wind load
Fig. (2-29) Maximum Hoop Force (at $x=L/2R$, $\phi=0$) for Shells with Restrained Upper Edges; under Wind Load
Fig. (2-30) Maximum Hoop Force (at $x = L/2R$, $\phi = 0$) for shells with Flexible Upper Edges, under Wind Load
Fig. 2-31) Maximum Membrane Shear Force (at $x=0$, $\phi=\pi/2$) for Shells with Restrained Upper Edges; under Wind Load
Fig. (2-32) Maximum Membrane Shear Force (at $x=0$, $\phi=\pi/2$) for Shells with Flexible Upper Edges; under Wind Load.
Fig. (2-33) Maximum Longitudinal Force (at $x=\phi=0$) for Shells with Restrained Upper Edges; under Wind Load
Fig. (2-34) Maximum Longitudinal Force (at $x=\phi=0$) for Shells with Restrained Upper Edges; under Simd-Load.
Fig.(2-35) Maximum Longitudinal Force (at \( x=\phi=0 \))
for Shells with Restrained Upper Edges;
under Wind Load
Fig. (2-36) Maximum Longitudinal Force (at $x=\phi=0$) for Shells with Flexible Upper Edges; under Wind Load.
Fig. (2-37) Deviation in results of $N_{ymax}$ between the two approaches, for shells with Restrained Upper Edges; under Wind Load.
Fig. (2-38) Typical Distribution of $N_{\phi}$ along the generator
$\phi = 0$ for Shells with Flexible Upper Edges; under Wind Load
Fig. (2-39) Typical Distribution of $N_\phi$ along the generator $\phi = 0$ for Shells with Restrained Upper Edges under wind load.
Fig.(2-40) Typical Distribution of $N_x$ along the generator $\phi=0$ for Shells with Flexible Upper Edges under Wind Load.
Fig. (2-41) Typical Distribution of $N_x$ along the generator $\phi = 0$ for Shells with Restrained Upper Edges; under Wind Load

$N_x/(P_o R)$
Fig. (2-42) Typical Distribution of $N_{x\phi}$ along the generator $\phi=\pi/2$ for Shells with Flexible Upper Edges; under Wind Load $P/R$.
Typical Distribution of $N_x \phi$ along the generator $\phi=\pi/2$ for Shells with Restrained Upper Edges under Wind Load
Fig. (3-1) A Deformed Shell Element
Fig. (3-2) Comparison of Buckling Load of Isotropic Shells with References (6) and (7)
**Fig. (3-3)** Comparison of Buckling Load of Isotropic Shells with References (6) and (7)
Fig. (3-4) Comparison of Buckling Load of Isotropic Shells with References (6) and (7)
Fig. (3-5) Buckling Load of Isotropic Shells vs. L/R for Shells with Restrained Upper Edges; under Wind Load

\[ \frac{D_x}{D_\phi} = \frac{B_x}{B_\phi} = 1.0 \]
\[ \frac{D_\phi}{D_{x\phi}} = 2.0 \]
Fig. (3-6) Buckling Load of Isotropic Shells vs. R/t for Shells with Restrained Upper Edges; under Wind Load
Figure 3-7 Buckling Load of Isotropic Shells vs. L/R for Shells with Free Upper Edges; under Wind Load
Fig. (3-8) Buckling Load of Isotropic Shells vs. R/t for Shells with Free Upper Edges; under Wind Load
Fig. (3-9) Increase in Buckling Load due to Restraining the Upper Edge
Fig. (3-10) Buckling Load of Orthotropic Shells with Flexible Upper Edges; under Wind Load
Fig. (3-11) Buckling Load for Shells with Restrained Upper Edges obtained by using the two Prebuckling Approaches
Fig. (3-12) Buckling Load for Shells with Restrained Upper Edges obtained by using the two Prebuckling Approaches
Fig. (3-13) Buckling Load for Shells with Restrained Upper Edges obtained by using the two Prebuckling Approaches
Fig. (3-14) Deviation in Buckling Load between the two Prebuckling approaches for Shells with Restrained Upper Edges;
Fig. (3-15) Buckling Configurations for Orthotropic Shells with Flexible Upper Edges; under Wind Load
\( \frac{D_\phi}{D_x} = \frac{B_\phi}{B_x} = 4.0 \)

\( \frac{D_\phi}{D_x} = \frac{B_\phi}{B_x} = 16 \)

Fig. (3-15) (cont.)
\[L/R = 3.0\]

\[\frac{D\Phi}{D_x} = \frac{B\Phi}{B_x} = 4.0\]

\[L/R = 3.0\]

\[\frac{D\Phi}{D_x} = \frac{B\Phi}{B_x} = 16.0\]

Fig. (3-15) (cont.)
Fig. (4-1) Set up of Shell No. 1
Fig. (4-2) Floor System of Shells
Fig. (4-3) Set up of Shell No.2

Fig. (4-4) Loading System
Fig.(4-5) Shell No.1 before Loading
Fig.(4-5) Shell No.1 Profile during Loading
Fig. (4-7) Shell No.1 Profile during Loading
Fig.(4-8) Shell No.1 Profile during Loading
Fig. (4-9)
Shell No. 1 Profile at Buckling Load

Fig. (4-10) Shell No. 2 before Loading
Fig. (4-11) Interior of Shell No.2

Fig. (4-12)
Buckling Configuration of Shell No.2
Fig. (4-13)  Vertical Displacement of the Upper Edge at the Buckling Load

Fig. (4-14)  Load Distribution

$\phi$-Direction
Fig. (4-15). Load-Deflection Diagram for points along the edge $x=L/R$ for Shell No.1
Fig. (4-16) Load-Deflection Diagram for points along the generator $\phi=0$ for Shell No. 1
Fig.(4-18) Typical Distribution of Radial Displacement along the generator θ=0 for Shell No.1
Fig. (4-19) Load-\(M_\phi\) Diagram at the point \((x=1/R, \phi=0)\) for Shell No.1
Fig. 8-20 Load-N_0 Diagram at the point (x=L/2, \phi=0) for Shell No.
Fig. (4-21) Load-$N_\phi$ Diagram at the point ($x=L/2R$, $\phi=0$) for Shell No. 1

$P_0$ (lb/in$^2$)
Fig. (4-22) Typical Distribution of the Radial Displacement along the generator $\phi=0$ for Shell No. 2
Fig. (4-23) Typical Distribution of the Radial Displacement along the Upper Edge for Shell No. 2
Fig. (4-24) The Relation between the Relative Rigidity $\frac{D_{\phi}}{E_{b}I_{b}}$ and $M_{\phi}$ at $x=L/R, \phi=0$.
Fig.(4-25) Load-Deflection Diagram for points along the generator $\phi=0$ for Shell No.2
Fig. (4-27) Load-$N_\phi$ Diagram at the point $(x=L/2R, \phi=0)$ for Shell No. 2
Fig. (4-28) Load-\(N_\phi\) Diagram at the point \((x=L/2R, \phi=\alpha)\) for Shell No.2
Appendix I

Experimental Measurements
Table (1-I)

Strain Measurements and Observed Values of $N_\phi$ and $M_\phi$ at the point ($x = L/R, \phi = 0$) - Shell No. 1

<table>
<thead>
<tr>
<th>Pressure (lb/in²)</th>
<th>$(\epsilon_1+\epsilon_2)/2$ (µε)</th>
<th>$(\epsilon_1-\epsilon_2)/2$ (µε)</th>
<th>$N_\phi$ (lb/in)</th>
<th>$M_\phi$ (lb·in/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.27</td>
<td>26.5</td>
<td>5.75</td>
<td>21.3</td>
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Table (2-I)

Strain Measurement and Observed Values of \( N_\phi \) and \( M_\phi \) at the point \( (x = L/2R, \phi = \pi/6) \) - Shell No.1.

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Table (3-I)

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$(1 \times 10^{-3} \text{ in})$

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Table (4-I)

Radial Deflection at points along the generator $\phi = 0$ for
Shell No.1
$(1 \times 10^{-3}\text{in.})$

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Table (6-I)
Radial Deflection at points along the edge $x = \frac{L}{R}$ for Shell No.2
($1 \times 10^{-3}$ in)

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Table (7-I)
Radial Deflection at points along the Generator $\phi = 0$ for Shell No.2
($1 \times 10^{-3}$ in)

| $|L/R|$  | 1    | 0.8  | 0.6  | 0.4  | 0.2  |
|------|------|------|------|------|------|
| Pressure (lb/in²) | | | | | |
| 0.27 | 0    | 2    | 3    | 0    |
| 0.40 | 49   | 56   | 74   | 61   | 26   |
| 0.54 | 113  | 147  | 167  | 130  | 56   |
| 0.68 | 178  | 244  | 263  | 199  | 87   |
| 0.81 | 244  | 341  | 358  | 267  | 117  |
| 0.95 | 306  | 428  | 446  | 330  | 143  |
| 1.08 | 365  | 519  | 535  | 393  | 177  |
| 1.22 | 429  | 610  | 628  | 410  | 210  |
| 1.36 | 493  | 656  | 672  | 459  | 240  |
| 1.49 | 557  | 703  | 718  | 524  | 251  |
| 1.63 | 620  | 783  | 797  | 589  | 276  |
| 1.76 | 680  | 851  | 862  | 652  | 294  |
| 1.90 | 743  | 892  | 914  | 714  | 308  |
| 2.04 | 804  | 965  | 973  | 781  | 346  |
| 2.17 | 826  | 1053 | 1061 | 847  | 385  |
| 2.31 | 865  | 1139 | 1145 | 926  | 424  |
| 2.44 | 903  | 1243 | 1249 | 990  | 470  |
| 2.58 | 969  | 1322 | 1330 | 1067 | 510  |
| 2.72 | 1040 | 1418 | 1477 | 1142 | 555  |
| 2.85 | 1108 | 1509 | 1522 | 1223 | 600  |
| 3.00 | 1184 | 1605 | 1622 | 1301 | 652  |
| 3.12 | 1265 | 1700 | 1718 | 1390 | 700  |
| 3.26 | 1345 | 1801 | 1827 | 1480 | 755  |
| 3.40 | 1435 | 1906 | 1935 | 1579 | 820  |
| 3.53 | 1533 | 2017 | 2052 | 1680 | 883  |
| 3.67 | 1646 | 2267 | 2307 | 1806 | 1034 |
| 3.80 | 1855 | 2667 | 2639 | 2102 | 1231 |
VITA AUCTORIS

1948  Born, October 24 in Cairo, Egypt

1965  Entered the Faculty of Engineering, University of Cairo, Giza, Egypt

1970  Graduated with a Bachelor of Science Degree (Honours) in Civil Engineering (Structural Division)

Appointed as Teaching Assistant in the Structural Engineering Department, University of Cairo

1974  Graduated with a Master of Science Degree in Engineering (Structures) from University of Cairo

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Awarded a teaching and research assistantship for graduate study at the University of Windsor, Ontario, Canada