May 18th, 9:00 AM - May 21st, 5:00 PM

Exploring argumentation, objectivity, and bias: The case of mathematical infinity

Ami Mamolo
University of Ontario Institute of Technology

Exploring Argumentation, Objectivity, and Bias: A Look at Mathematical Infinity

AMI MARIA MAMOLO
Faculty of Education
University of Ontario Institute of Technology
11 Simcoe St. N., Oshawa, ON
Canada
ami.mamolo@uoit.ca

Abstract: In a quest for truth, infinity boggles. It eludes and escapes finite reason, baffles logical expectation, and draws to light bias in moments of supposed objectivity. This presentation explores historical and present-day ideas, arguments, and biases related to complexities and ambiguities of mathematical infinity. Or rather, of mathematical infinities. Of interest: novice and expert a priori and a posteriori knowledge as elicited by falsidical and dialethical paradoxes, and the contextually-dependent nature of mathematical correctness.

Keywords: ambiguity, bias, infinity, intuitive and formal reasoning, nonstandard analysis, objectivity, paradox, set theory

1. Introduction

This paper presents an overview of several years of my research into individuals' reasoning, argumentation, and bias when addressing problems, scenarios, and symbols related to mathematical infinity. There is a long history of debate around what constitutes "objective truth" in the realm of mathematical infinity, dating back to ancient Greece (e.g., Dubinsky, Weller, McDonald, & Brown, 2005). Modes of argumentation, hindrances, and intuitions have been largely consistent over the years and across levels of expertise (e.g., Brown, McDonald, & Weller, 2010; Fischbein, Tirosh, & Hess, 1979; Tsamir, 1999). This paper takes a playful approach to examining the interrelated complexities of notions of objectivity, bias, and argumentation as manifested in different presentations and normative interpretations or resolutions of a well-known paradox of infinity—the ping pong ball conundrum. This conundrum is a deviation of the Ross-Littlewood 'super-task'—a task which occurs within a finite interval of time, yet which involves infinitely many steps (Thompson, 1954). It requires a bit of imagination and something akin to a cognitive leap (Mamolo, 2010).

> *The ping pong ball conundrum: Imagine a very large barrel and infinitely many ping pong balls numbered 1, 2, 3, ... You will use these in a thought experiment that will last no more and no less than 60 seconds. In the first 30 seconds, the task is to place the first 10 balls (#1-10) into the barrel, and instantaneously remove ball #1. In half of the remaining time, the next 10 balls (#11-20) are placed into the barrel and ball #2 is removed. Again, in half the remaining time (and working more and more quickly), the next 10 balls (#21-30) are placed in the barrel, and ball #3 is removed, and so on. Continuing in this way until the experiment is over, at the end of the 60 seconds, how many ping pong balls remain in the barrel?*

Bondy, P., & Benacquista, L. (Eds.). *Argumentation, Objectivity, and Bias: Proceedings of the 11th International Conference of the Ontario Society for the Study of Argumentation (OSSA), 18-21 May 2016*. Windsor, ON: OSSA, pp. 1-11.

Paradoxes that have highlighted the inherent anomalies of the infinite have had such a profound impact on mathematics and mathematical thought that Bertrand Russell (1913) attributed to them "the foundation of a mathematical renaissance" (p. 347). Cantor's (1915) work establishing a theory of transfinite numbers offered a first means of rigorous and consistent resolution to paradoxes of infinity—including David Hilbert's *Grand Hotel*. Indeed, the profoundness of Cantor's work and ideas inspired Hilbert (1925) to praise them as providing mathematics "with the deepest insight into the nature of the infinite" procured by "a discipline which comes closer to a general philosophical way of thinking" (pp.138-9). Cantor's theory, though controversial at the time, added depth and rigour to emerging conceptions of infinity, which included Bolzano's (1950) progressive views that the infinite is more than "that *which has no end*" (p. 82).

Paradoxes have been described as occasioning major epistemological reconstructions (e.g., Quine, 1966), and in what follows I highlight such occasions as they emerged for both novices and experts. As mentioned, the approach I take is a playful one: data from several years of my research is represented in a fictionalized retelling in the dialectic tradition well-known to philosophers and existent in mathematics education literature (e.g., Lakatos, 1976); it focuses on the actual thoughts, arguments, biases, and debates that emerged in my research and that have been discussed in prior works (e.g., Mamolo, 2009, 2010, 2014; Mamolo & Zazkis, 2008). Connections to historical conceptions, arguments, and biases are woven in with some creative liberty, and two formal resolutions are presented. The paper concludes with discussion of some of the implications of these resolutions with connection to current conceptualisations of objectivity (e.g., Daston, 1992). Of interest is the perception that one single objective truth about "actual" mathematical infinity exists—indeed, this is brought to question at an axiomatic level.

2. Thoughts, biases, and arguments

Imagine a very large barrel…

"You could never have a barrel that big," says Alpha "there isn't enough space on earth for such a large barrel."

"It's a thought experiment… let's imagine…" replies Beta.

Imagine a very large barrel… moving more and more quickly…

"That's impossible. No one could move that fast," protests Gamma. Beta, who has accepted the premise, rolls his eyes. But Beta's opinion isn't a popular one and there are more protests. Allis and Koetsier (1995) applied what they call the "abstract continuity principle" to handle a variant of the paradox in terms of sequences of actions and address some of the controversy (Van Bendegem, 1994) around tasks that require letting go of physical restraints such as speed and acceleration.)

"The experiment is impossible—it will never end," this time Delta has chimed in. "The sixty seconds will last for ever. So the barrel will always be full of ping pong balls."

"How is that?" asks Alpha. "If the experiment is impossible, then how can you say it will always be full of balls?"

"What I mean is, the process is impossible since the time interval is halved infinitely many times, so the sixty seconds never ends. Since you're putting balls in the barrel for eternity, it will always be full," explains Delta.

"*Eternity*," muses Epsilon, "didn't Aristotle describe infinity as *inexhaustible*?" (Moore, 1995). "Surely, Aristotle would claim the experiment must last forever, because the process 'would require the whole of time'" (Dubinsky, et al., 2005, p. 341).

"That's right!" chimes Gamma. "Even with one second left we can still divide this amount of time into infinitely small amounts of time (if physics does not apply). Therefore, the experiment will continue into eternity and the number of balls will be infinite in the barrel."

"I *partially* agree," says Beta, "no pun intended. The increments of time, ½, ¼, 1/8, 1/16, and so on, form a convergent series when summed. We know this because the sequence of its partial sums converges, and the limit as n tends to infinity of the set of partial sums has to exist as a real number. We know that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, so 'the end' of the experiment happens at one minute, or sixty seconds!" (Beta, who is quick with numbers, has shifted units from seconds to minutes, identifying 30 seconds with a ½ minute, 15 seconds with a ¼ minute, and so on.)

Alpha scratches his nose, "Ok, yes, I do remember that from Calculus. So, is this a calculus problem? And, are you saying that Aristotle was wrong?!"

Nervous laughter.

"Aristotle wasn't wrong." Quine, who had been walking by and overheard, couldn't help but intervene. "When resolving certain paradoxes of infinity, a fallacy can emerge—'the mistaken notion that an infinite succession of intervals of time has to add up to all eternity' (Quine, 1966, p. 5). Aristotle is thinking of one kind of infinity—*potential* infinity, but this paradox involves a different type—*actual* infinity." And with a wry grin, Quine disappeared down the hall, leaving the class to grapple with what he considered a falsidical paradox.

"*Actual* infinity—a completed entity which encompasses the potential… The 'infinite present at a moment in time' (Dubinsky et al., 2005, p. 341)," Epsilon read aloud. "Hm.. I need time to wrap my head around that! Like 2000 years maybe!"

"Let's go back to what Beta was saying. Thinking about this as limits was starting to make sense to me," Alpha starts writing on the board, "as $n \to \infty$, the limit equals one."

"No, no, that's not right," insists Delta. "The limit *approaches* one, it doesn't equal one." A heated argument ensues – a common conception of post-secondary students is the idea that a limit is unreachable (Williams, 1991). Someone has the idea to check their favourite Calculus textbook (Stewart, 1999), which seems to lay the controversy to rest. The text, acting as an authority for proof (Harel & Sowder, 1998), was enough, for the time being, to convince (if only grudgingly) the class of the statement $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

"I think this is what Professor Quine was talking about. As n approaches infinity, the limit approaches one, but when you take the sum you include all of the infinity of intervals—*actual* infinity—and that's why the sum equals one…" Epsilon trails off, still thinking.

"Here, I've found something else," states Alpha. "Zeno's paradox—one of them anyway. It has a room and a traveller who cuts each distance in half infinitely many times. *The Dichotomy Paradox*… This looks to be the same as our problem."

"Well, almost," says Beta. "We still have to think about the balls. We've only been talking about the time intervals so far."

Imagine a very large barrel… place the first 10 balls in, and remove ball #1… and so on

"Alright," says Gamma, "so if we assume the experiment eventually ends—even though I really don't think it could—then, there should still at least be an infinite number of balls left in the barrel."

"Even though we've removed infinitely many balls?" asks Theta.

"There are infinitely many time periods," says Psi, "therefore infinitely many times during which 10 balls are put in and one thrown out. So—there are an infinite number of balls in the basket as well as an infinite number thrown out. It doesn't make sense to have this result, but there it is."

"My head hurts," sighs Delta.

"Wait," says Alpha, "are you saying that $\infty - \infty = \infty$? How is that possible?"

"This is definitely outside the realm of possibility!" pipes Gamma.

"I might be able to explain this," replies Theta carefully. "Every time the remaining time is halved, the equivalent change (+10 – 1) = 9 balls are added. So there will be an infinite of balls in the basket. Some may say that an infinite amount of balls have been taken out of the basket, which is true, but it is not an equivalent infinity to what is put in… There will be 9 times as many in the basket as you took out."

This explanation seems to cause a bit of panic.

"Are you suggesting that there can be different infinities? What do you mean by "not equivalent"? probes Epsilon.

"Well," says Theta, "I'm not really sure… it's just that more balls seem to go into the barrel at any given time than come out of it. So, if more go in at each step in the experiment, then at the end of the experiment (assuming it ends), there should be more in the barrel than out."

Beta nods, "There is 9x more balls in the barrel than out of the barrel at all times. At the end of the 60 seconds there are 9∞ balls in and ∞ balls out."

At this point, chaos ensues and indistinguishable voices proclaim:

"Impossible! Infinity is infinity is infinity!"

"You can't have more than one infinity!"

"9 infinity is still just infinity!"

"How can you compare infinity? What is bigger than endless?"

Meanwhile, in the courtyard just outside a game of Parcheesi is underway between Galileo and Bolzano. Epsilon, who had been gazing out the window pondering, took notice and had an idea. "Beta- why don't you and I ask Galileo and Bolzano out there what they think? They'll be able to help us resolve this problem!" The two students leave to pursue Epsilon's suggestion as the rest of the class continues to debate—strong opinions fly for both a "bigger" infinity remaining in the barrel and the idea that there can be nothing bigger than infinity.

Epsilon and Beta approached the pair in the courtyard and, interrupting a pretty intense Parcheesi game, presented the paradox and nature of their problem. Galileo and Bolzano smiled, happy for the distraction. Bolzano was the first to reply:

"Certainly most of the paradoxical statements encountered in the mathematical domain … are propositions which either immediately contain the idea of the infinite, or at least in some way or other depend upon that idea for their attempted proof" (Bolzano, 1950, p. 75).

"Paradoxical indeed," replied Galileo "I have spent some time myself working on Zeno's paradoxes—of which this seems to be a sort of extension."

"First, we must consider that the infinite is more than 'that which has no end'" (Bolzano, 1950, p. 82). "We could compare the sets by coupling, yet I have a better idea, which I think will put an end to your struggles Epsilon and Beta."

A slight twitch of Galileo's right eyebrow.

"Let us consider a similar problem, and you can then apply the reasoning in resolving your ping pong problem. Consider the sets of rational numbers $A = [0, 5]$ and $B = [0, 12]$. If we construct a map $5y = 12x$ for x in A and y in B, then we may couple each element in A with exactly one element in B, and vice versa."

"So, you've created a bijection," said Beta. "Does this mean the sets are equinumerous?"

Both men shook their heads.

"Although every quantity in *A* or *B* allows of coupling with one and only one in *B* or *A*, yet the set of quantities in *B* is other and greater than in *A*, since the *distance* between the two quantities in *B* is other and greater than the *distance* between the corresponding quantities in *A*" (Bolzano, reprinted 1950, p. 100).

"Here a difficulty presents itself which appears to me insoluble. Since it is clear that we may have one line greater than another, each containing an infinite number of points, we are forced to admit that, within one and the same class, we may have something greater than infinity, because the infinity of points in the long line is greater than the infinity of points in the short line. This assigning to an infinite quantity a value greater than infinity is quite beyond my comprehension" (Galilei, 1914/1965, p. 32).

Bolzano insisted, "two sets can still stand in a relation of inequality, in the sense that the one is found to be a whole and the other a part of that whole" (Bolzano, 1950, p. 98)

Galileo countered, "the attributes 'equal', 'greater,' and 'less,' are not applicable to infinite, but only to finite, quantities" (1914/1965, pp. 32-3).

Epsilon and Beta were beginning to regret their decision.

Galileo continued, "let us consider a different example: the set of natural numbers and the set of perfect squares."

"Clearly, one is found to be a whole and the other a part of that whole" chimed Bolzano.

"And yet," replied Galileo, "there are as many [squares] as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square" (Galilei, 1914/1965, p.32).

"Ah, but my friend, one-to-one correspondence never justifies us," Bolzano reasoned, "in inferring the *equality of the two sets, in the event of their being infinite*, with respect to the multiplicity of their members – that is, when we abstract from all individual differences… two sets can still stand in a relation of inequality, in the sense that the one is found to be a whole and the other a part of that whole" (1950, p.9 8).

"So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former" (Galilei, 1914/1965, p. 32). At this point, Galileo flips the Parcheesi board and the two men succumb to a shouting match, leaving poor Epsilon and Beta no nearer to the resolution of their problem.

Back in the classroom, Theta asks: "so, how did that go?"

Epsilon sighs, and Beta replies "I think there was something useful in the idea of a one-to-one correspondence, although neither felt that was sufficient to solve our problem. How does it go with you?"

"I think we are mostly in agreement that there are infinitely many balls remaining in the barrel, because even though there is a 1-1 correspondence between the sets {1, 2, 3, 4, …}, {9, 18, 27, 36, …}, the rate at which you are putting in is more than you are taking out. So even if there are just as many numbers in each set, they will never even out, because the process continues infinitely and you continue to put more in than you take out," explains Theta.

"I am not fully convinced," replies Alpha, "allow me to play devil's advocate. Let us see… If there is, in fact, infinitely many balls remaining in the barrel, what are their numbers? That is, which balls remain in the barrel?"

"Well," muses Delta, "There is an infinite number of balls in the barrel, however it is impossible to name a specific ball. As soon as a number is chosen, it is possible to determine the exact time… that ball was removed… I can't name a numbered ball that remains but then I also couldn't tell you how many balls we began with because there were infinity. Since you are always adding more than you are taking out, you can move at lightning speed, and you have infinity time intervals, I believe the task never ends."

"Delta is still insisting the task never ends!" laughs Psi.

"But this point is important," interjects Beta, "for every numbered ball, we know the exact time interval that the ball was removed—here is Galileo's correspondence in effect!"

"So, are you saying the barrel must be empty?" asks Gamma.

"That seems to be the argument," replies Alpha.

"There are conflicting views and now I am not sure whether there is none or infinite balls in the basket. My gut feeling seems to want to say that there are an infinite number but there seems to be none as well," reflects Epsilon.

Theta: "I will not accept a logical argument that the basket is empty. Such an argument would be flawed."

Delta: "I'm sure it makes sense if you're comfortable with the concept of infinity."

Psi: "I can't agree with 0 balls remaining. You put in more number of balls than you take out. I still think my original answer is correct!"

Slowly, Beta concedes: "I can now entertain the idea that there are no balls in the basket, but I don't like it."

3. Two contradictory truths

In the previous section, the informal resolutions and conceptions of learners were discussed. This data came from research with undergraduate and graduate students, some of whom had a limited background in mathematics and others who had advanced backgrounds in the subject (e.g., Mamolo, 2010; Mamolo & Zazkis, 2008). Notably, the level of mathematics background had little impact on learners' approaches to resolving the paradox, and indeed similar trends have been noted in other research as well (e.g., Ely, 2011; Mamolo, 2014; Radu & Weber, 2011). The historical connections highlight what can be considered as epistemological obstacles (in the sense of Duroux as described in Brousseau, 1997) related to philosophical beliefs, intuitions, and arithmetic properties of actual infinity. Overcoming an epistemological obstacle "means that the student will have to rise above his convictions, to analyse from outside the means he had used to solve problems in order to formulate the hypotheses he had admitted tacitly so far, and become aware of the possible rival hypotheses" (Sierpinska, 1987, p. 374). In some instances, the only way to overcome an obstacle—to rise, as Sierpinska (1987) wrote, above convictions, prior experience, and intuition—is through a *cognitive leap*. The call for such a leap (Mamolo, 2010) is in resonance with Hahn (1956), who noted the importance of separating realistic and intuitive considerations from conventional mathematical ones in understanding properties of actual infinity. So, let us now look to convention...

The standard approach to resolving this paradox is a set theoretic one. In this approach, the resolution may come as a bit of a surprise: After the experiment is over, at the end of the 60 seconds, zero ping-pong balls remain in the barrel. (The resolution presented below appears in similar form in Mamolo, 2010, 2014; Mamolo & Bogart, 2011; and Mamolo & Zazkis, 2008.)

The question 'how many?' in this context is a question of cardinality. As such, the question 'how many balls remain in the barrel?' can be interpreted as 'what is the cardinality of the set of balls which are not removed from the barrel?'. To compare the cardinalities of infinite sets in a consistent way, we rely on the work of Cantor (1915), in particular the fact that two sets are considered to have the same cardinality if and only if they can be put in one-to-one correspondence. Unpacking the experiment, we see there are three infinite sets to consider: the in-going ping-pong balls, the out-going ping-pong balls, and the intervals of time. To answer the question 'how many' we look to the existence (or not) of correspondences between pairs of sets.

The sets of in-going and out-going balls, being numbered as they are, both correspond to the set of natural numbers. This correspondence ensures that at the end of the experiment, as many balls were removed from the barrel as went in. The set of out-going balls and the set of time intervals, which can be represented as B = {1, 2, 3, … }, and T = {½, ¼, ⅛, …}, respectively, can also be put into one-to-one correspondence by pairing any $x \in B$ with $(½)^x \in T$. This correspondence assures that when the 60 seconds runs out, so do the balls. These facts are necessary but not sufficient to resolve the paradox. An essential feature of this thought experiment is the ordering of the out-going balls. It is not enough that the amount of out-going balls corresponds to the amount of time intervals. In order for the barrel to be empty at the end of the experiment the ping-pong balls must be removed consecutively, beginning from ball #1. Consequently, there will be a specific time for which each of the in-going balls is removed. Thus, at the end of the experiment, the barrel will be empty.

In a set theoretic context, the ordering of the out-going balls is essential for a resolution. If we did not know which balls were removed at which time interval, we would be dealing with a problem of transfinite subtraction that is undefined. Consequently, if we were to change the ordering of balls removed from the barrel, we could end up with a completely different resolution even when we have changed nothing else about the experiment. So if we vary the paradox even slightly, we might find infinitely different resolutions. But what if, rather than varying the paradox, we vary its context?

For a complete discussion of this variant, please see Mamolo & Bogart (2011). In what follows, I attempt to relay broad strokes of the resolution, as many of the details lie outside the scope of this paper. This variation of context relies on the work of Abraham Robinson who gave a rigorous mathematical foundation for the use of infinitesimals and infinite numbers in calculus (Loeb, 2000). Robinson's (1974) introduction of nonstandard analysis allows for the existence of an ordered extension field of the real numbers such that it contains both infinitesimal and infinite quantities. This extension, known as the hyperreals and denoted as $\mathbb{R}^*$, includes elements ε which is positive but less than every positive real number, and $\frac{1}{\varepsilon}$ which is greater than every positive real number (i.e., it is infinite). Included in the field of hyperreal numbers is the set of hyperintegers that extends the set of integers. Any positive hyperinteger that is not already an integer is called an *infinite integer*. With these bits as a foundation, we can reframe the paradox such that the collection of ping pong balls are numbered with the hyperintegers: 1, 2, 3…, $10W$ (where W is greater than any integer in $\mathbb{Z}$). We add and remove balls similarly before, continuing the process up to the step W, where, for every hyperinteger n less than or equal to W, at time $½^n$ from the end of a minute, balls numbered $10n-9, 10n-8, \dots 10n-1, 10n$ are placed into the barrel and ball numbered n is removed. This new context allows us to talk about the paradox in terms of measuring properties of numbers—a conceptualization that is consistent with intuitive understandings of infinity (e.g., Mamolo, 2009; Tall, 2001).

In a context of nonstandard analysis, the resolution of this paradox has three main parts: a description of the sets of balls, a nonstandard interpretation of quantity, and an application of that form of quantification to the problem at hand. The set of all balls can be described by A = {1, 2, 3, …, 10W-1, 10W}, where each hyperinteger between 1 and 10W occurs exactly once (a consequence of the transfer principle, which guarantees that the extended sets, functions, and relations continue to behave in familiar ways); let the set of balls removed from the barrel be R = {1, 2, …, W-1, W}; thus the set of balls remaining in the barrel at the end of the experiment is:

$$B = A \setminus R = \{W + 1, W + 2, \ldots 2W, 2W+1, \ldots 10W-1, 10W\}.$$

If we are to resolve this paradox in terms of measurement rather than cardinality, then to interpret the question of "how many balls" we need to consider measures of intervals (rather than cardinalities of sets). This reasoning is not unlike that of Bolzano's (1950), though his formulation fell short of being complete. To understand the 'size' of sets A, R, and B in this context, we need to measure the span of each of the intervals [0, 10W], [0, W], and [W, 10W]. Applying the transfer principle, we deduce that the length (i.e., measure) of the interval [W, 10W] is 9W and the length of the interval [0, W] is W. Lengths (unlike cardinalities) are additive in the usual way, and as such we can think of the measure of the removed balls as "∞" and the measure of the remaining balls as "9∞". Mamolo and Bogart (2011) note:

> An interesting feature of the hyperreal ball problem is that although we perform more steps of adding and removing balls than in the original version (one step for each hyperinteger from 1 to W, rather than just one for each natural number), the last step ends 2^{-W} seconds before the end of a minute. So the process takes (infinitesimally) less time than in the original version, which requires exactly a minute. (p. 622)

4. Conclusion

The title of this section is a bit of a misnomer. My intentions have been to invite curiosity and conversation, more so than draw any conclusions. However, in some concluding remarks, I turn my attention towards the notion(s) of objectivity, engaging as a tourist in a well-traversed philosophical field. In Section 2 of this paper, novice and expert ideas about infinity came into play in attempted resolutions of the ping pong ball paradox. In Section 3, hoping to resolve some of the controversy associated with the tacit biases and beliefs that influenced historical and modern day conceptions, two formal resolutions were presented—one which relied on cardinal conceptualization of numbers and the other which relied on a measurement approach. In each separate context, a rigorous, self-contained, logically consistent and correct argument was made to "resolve" the problem. In one context, the barrel was empty at the end of the sixty seconds. In the other context, it was infinitely full. Oh my.

Daston (1992) notes that few philosophical studies, "even those most directly concerned with objectivity in the sciences or with the historical context in which objectivity allegedly emerged once and for all, seriously entertain the hypothesis that objectivity might have an ongoing history intimately linked to the history of scientific practices and ideals" (p. 598). Yet, the paradox discussed provides a compelling reason to do just that—by and large, the set theoretic approach is the one commonly accepted as the normative standard. Not because it is "more correct", but rather there is a seeming resistance by mathematicians to accept Robinson's nonstandard analysis as a viable alternative to the epsilon-delta approaches to analysis laid down by Leibnitz and Newton in their attempts to formalize a system without dealing explicitly with those tricky creatures known as infinitesimals. The history of mathematical practices and ideals related to infinity do seem to suggest that 'objective' views of infinity depended largely on perspective. Aristotle's perspective that actual infinity cannot exist because it would take all of eternity to enumerate such a set, Galileo's view of the impossibility of comparing transfinite cardinalities, Bolzano's notions of 'distances' between elements, Cantor's sets, and Robinson's hyperreals all have some credence, yet each relies on a value-laden (if tacit) choice in perspective. Mathematicians, writes Daston (1992), have been "indifferent to public opinion… because the certainty or near-certainty of their 'demonstrations' freed them from evaluations based only on 'a certain nicety of taste'" (p. 606). Yet, it was a world view that dismissed the aesthetics of Platonism that led Aristotle to his conclusions, religious belief that led Kronecker to deny and supress Cantor, his one-time protégé (e.g., Rucker, 1982)—a perspective felt so heavily by Cantor that it is said to have contributed to feelings of persecution and paranoia experienced later in life (Aczel, 2000). Indeed, Cantor's own religious beliefs about the relationship between infinity and God is said to have convinced him of the veracity of his work—transfinite numbers and their properties were real because "God had told me so" (Aczel, 2000, p. 143). And, it is a sign of current value systems that young undergraduates suffer through cumbersome epsilon-delta proofs, when a viable alternative exists in Robinson's work.

Certainly, I am not the first to play with these ideas within the realm of mathematics. Gödel's incompleteness theorems (e.g., Goldstein, 2005) lay a foundation, and the ping pong balls provide a striking example of two complete and consistent axiomatic systems that are nevertheless incompatible in certain contexts. The idea that objectivity might lie at the heart of mathematics is deceiving. Rather, it is ambiguity that lies therein, and "mathematically true" has meaning only relative to underlying axioms (e.g., Byers, 2007; Devlin in Suri & Singh Bal, 2007). As Bertrand Russell (1903/1996) wrote, "mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true" (p. 75).

References

Aczel, A. (2000). *The Mystery of the Aleph: Mathematics, the Kabbalah, and the Search for Infinity*. New York: Simon & Schuster.

Allis, V., & Koetsier, T. (1995). On some paradoxes of the infinite II. *The British Journal for the Philosophy of Science* 46, 235 – 247.

Bolzano, B. (1950). *Paradoxes of the Infinite*. (F. Prihonsky, Trans.). (Introduction by D. Steele). London: Routledge and Kegan Paul Ltd.

Brown, A., McDonald, M., & Weller, K. (2010). Step by step: Infinite iterative processes and actual infinity. In: F. Hitt, et al. (Eds.), *Research in Collegiate Mathematics Education* (Vol. VII, pp. 115-142). Providence: Amer. Math. Soc.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. N. Balacheff, M. Cooper, R. Sutherland & V. Warfield, (Eds. & Trans.). Dordrecht: Kluwer Academic.

Byers, W. (2007). *How Mathematicians Think: Using Ambiguity, Contradiction and Paradox to Create Mathematics*. Princeton, NJ: Princeton University Press

Cantor, G. (1955). *Contributions to the Founding of the Theory of Transfinite Numbers*. (P. Jourdain, Trans.). New York: Dover Publications Inc. (Original work published 1915)

Daston, L. (1992). Objectivity and the escape from perspective. *Social Studies of Science* 22 (4), 597-618.

Dubinsky, E., Weller, K., McDonald, M.A., & Brown, A. (2005). Some historical issues and paradoxes regarding the concept of infinity: an APOS-based analysis: Part 1. *Educational Studies in Mathematics* 58, 335–359.

Ely, R. (2011). Envisioning the infinite by projecting finite properties. *Journal of Mathematical Behavior* 30, 1–18.

Fischbein, E., Tirosh, D., & Hess, P. (1979). The intuition of infinity. *Educational Studies in Mathematics* 10, 3–40.

Galilei, G. (1965). *Dialogues Concerning Two New Sciences*. (H. Crew & A. de Salvio, Trans.). New York, USA: Dover Publications, Inc. Retrieved from http://www.knovel.com.proxy.lib.sfu.ca/knovel2/Toc.jsp?BookID=449 (Original work published 1914)

Goldstein, R. (2005). *Incompleteness: The Proof and Paradox of Kurt Gödel*. New York: W. W. Norton & Company, Ltd.

Hahn, H. (1956). Infinity. In: J. Newman (Ed.), *The World of Mathematics* (Vol. 3, pp. 1593 – 1613). New York: Dover Publications Inc.

Harel, G., & Sowder, L. (1998). Students' proof schemes. In: E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research on Collegiate Mathematics Education* (Vol. III, pp. 234-283). AMS.

Hilbert, D. (1925). On the infinite. In: P. Benacerraf & H. Putnam (Eds.), *Philosophy of Mathematics: Selected Readings* (pp. 134 – 151). New Jersey: Prentice-Hall, Inc.

Lakatos, I. (1976). *Proofs and Refutations: The Logic of Mathematical Discovery*. Cambridge: Cambridge University Press.

Loeb, P. A. (2000). An introduction to non-standard analysis. In: P. A. Loeb & M. Wolff (Eds.), *Nonstandard Analysis for the Working Mathematician* (pp. 1–34). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Mamolo, A. (2009). Intuitions of ‘infinite numbers’: Infinite magnitude vs. infinite representation. *The Montana Mathematics Enthusiast* 6 (3), 305-330.

Mamolo, A. (2010). *Glimpses of Infinity: Intuition, Paradoxes, and Cognitive Leaps*. Saarbrücken, Germany: VDM Verlag.

Mamolo, A. (2014). How to act? A question of encapsulating infinity. *Canadian Journal of Science, Mathematics, and Technology Education* 14 (1), 1-22.

Mamolo, A., & Bogart, T. (2011). Riffs on the infinite ping-pong ball conundrum. *International Journal of Mathematical Education in Science and Technology* 42 (5), 615 – 623.

Mamolo, A., & Zazkis, R. (2008). Paradoxes as a window to infinity. *Research in Mathematics Education* 10, 167 – 182.

Quine, W. (1966). *Ways of Paradox and other Essays*. Kingsport: Random House, Inc.

Radu, I., & Weber, K. (2011). Refinements in mathematics undergraduate students’ reasoning on completed infinite iterative processes. *Educational Studies in Mathematics* 78, 165–180.

Robinson, A. (1974). *Non-Standard Analysis* (Rev. ed.). Amsterdam: North-Holland Publishing Company.

Rucker, R. (1982). *Infinity and the Mind: The Science and Philosophy of the Infinite.* Boston, USA: Birhauser.

Russel, B. (1996). *The Principles of Mathematics*. Norton. Retrieved from http://fair-use.org/bertrand-russell/the-principles-of-mathematics/ (Original work published 1903)

Sierpinska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics* 18, 371–397.

Stewart, J. (1999). *Single Variable Calculus* (4th ed.). Pacific Grove, USA: Brooks/Cole Publishing Company.

Suri, G., & Singh Bal, H. (2007). *A Certain Ambiguity: A Mathematical Novel*. New Jersey, USA: Princeton University Press

Tall, D. (2001). Natural and formal infinities. *Educational Studies in Mathematics* 48, 199–238.

Thomson, J. (1954). Tasks and super-tasks. *Analysis* XV, 1-13.

Tsamir, P. (1999). The transition from the comparison of finite sets to the comparison of infinite sets: Teaching prospective teachers. *Educational Studies in Mathematics* 38, 209-234.

Van Bendegem, J. P. (1994). Ross' Paradox is an impossible super-task. *British Journal for the Philosophy of Science* 45, 743-748.

Williams, S. (1991). Models of limits held by college calculus students. *Journal for Research in Mathematics Education* 22 (3), 219–236.