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Anjan Kanti. Bhowmick

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ASSESSMENT OF NUMERICAL MODELS
FOR MULTIAXIAL FATIGUE

BY

ANJAN KANTI BHOWMICK

A Thesis
Submitted to the Faculty of Graduate Studies and Research through
The Department of Civil and Environmental Engineering
in Partial Fulfillment of the Requirements for the
Degree of Master of Applied Science at the
University of Windsor

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2002
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ABSTRACT

Many engineering components and structures are subjected to multiaxial stresses and strains under service conditions. These multiaxial stresses and strains influence the fatigue resistances of materials and components. As uniaxial-based life predictions are non-conservative, the need for advanced models predicting the fatigue life under multiaxial conditions is recognized by the industry. Research in this field is intensified in the last decade and now-a-days, engineers have a wide range of methods to estimate the fatigue life under multiaxial conditions. Each of these predictive methods are generally based on experimental observations and therefore subjected to restrictions, related to the inherent simplified assumptions. The common feature of most existing models is that the complex multiaxial loading case is reduced to an equivalent uniaxial case.

The objective of this research work is to assess existing models for different loading conditions. The approach is based on the experimental research findings of the Society of Automotive Engineers (SAE), which is used as a benchmark. The models are classified into three categories: (i) Empirical formulas based on modification of the Coffin-Manson equation; (ii) Critical plane models which are based on physical observation that fatigue crack initiate and grow on certain planes; (iii) Damage mechanics models based on energetic thermodynamic approach. The first two categories of models are implemented in commercial software “FE-Fatigue”. The software uses the results of linear finite element analysis to determine the critical locations. The fatigue life
prediction is therefore evaluated based on the stress-strain distribution obtained from the finite element analysis. The continuum damage mechanics approach uses a fundamental formulation based on energetic thermodynamics. This technique integrates an elasto-plastic-damage locally coupled constitutive law in the critical location where plastic deformation is accumulated. Based on its nature the continuum damage mechanics approach is restricted to low cycle fatigue.

Most engineering components contain stress concentrations or notches. In this case, local stress-strain states developed at notches become the main parameter for fatigue life estimation. The conventional methods based on empirical relations for component with simple geometry are evaluated for pure bending and torsional loading cases and the results are compared with the finite element based approach.
TO MY WIFE
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NOTATION

\( a \) = crack length
\( a \) = biaxiality ratio
\( \bar{A} \) = effective area
\( A \) = amplitude ratio
\( A_d \) = damaged area
\( A_f \) = area at fracture
\( A_0 \) = Original area
\( a_{yl} \) = elastic moduli

\( b \) = fatigue strength exponent
\( c \) = fatigue ductility exponent
\( D \) = damage variable
\( D_c \) = critical damage at crack initiation
\( D_{tc} \) = critical damage in pure tension
\( E \) = Young’s modulus
\( E_{dy} \) = Young’s modulus of the damaged material
\( E_o \) = Young’s modulus of the undamaged material
\( F \) = function potential of dissipation
\( K \) = strength coefficient
\( K_f \) = fatigue notch factor
$K_I = \text{elastic stress concentration factor}$

$K' = \text{cyclic strength coefficient}$

$K_{Ic} = \text{fracture toughness}$

$K_e = \text{strain concentration factor}$

$K_\sigma = \text{stress concentration factor}$

$N = \text{number of cycles}$

$N_f = \text{number of cycles to failure}$

$n = \text{strain hardening exponent}$

$n' = \text{cyclic hardening exponent}$

$P = \text{applied load}$

$\dot{\epsilon} = \text{accumulated plastic strain rate}$

$p_D = \text{damage threshold at which micro-crack nucleation starts}$

$P_f = \text{load at fracture}$

$q = \text{notch sensitivity factor}$

$r = \text{radial coordinate in the polar coordinate system}$

$RA = \text{reduction in area}$

$r_p = \text{radius of plastic zone}$

$R_e = \text{triaxiality ratio}$

$S = \text{damage strength material parameter}$

$S = \text{nominal stress}$

$S = \text{sectional area}$
$S_{ij} =$ damaged sectional area

$S_e =$ fatigue limit

$S_u =$ ultimate strength

$SR =$ stress ratio

$S_y =$ local elastic stress vector

$w_e =$ elastic strain density

$Y =$ elastic strain energy density

$\psi =$ free energy, a continuous scalar function

$\delta_y =$ kronecker delta

$\varepsilon =$ local strain

$\varepsilon_1, \varepsilon_2, \varepsilon_3 =$ principal strain

$\varepsilon^e =$ elastic strain

$\varepsilon_q =$ equivalent strain

$\varepsilon^{\varepsilon} =$ hydrostatic elastic strain

$\varepsilon^p =$ plastic strain

$\varepsilon_y =$ strain tensor

$\varepsilon_y^e =$ elastic strain tensor

$\varepsilon_y^p =$ plastic strain tensor

$\Delta \varepsilon =$ strain range

$\gamma_{\text{max}} =$ maximum shear strain
\( \dot{\lambda} \) = plastic multiplier

\( \nu \) = Poisson’s ratio

\( \theta_p \) = angle of absolute maximum principle stress

\( \sigma \) = local stress

\( \sigma_1, \sigma_2, \sigma_3 \) = principal stress

\( \sigma_u \) = stress amplitude

\( \sigma^* \) = damage equivalent stress

\( \sigma_{eq} \) = von Mises equivalent stress

\( \sigma_f \) = fatigue limit

\( \sigma_h \) = hydrostatic stress

\( \sigma_p \) = stress to rupture

\( \sigma_s \) = plastic threshold

\( \sigma_y \) = yield strength

\( \sigma_p^0 \) = deviatoric stress tensor

\( \Delta \sigma \) = stress range

\( \tau_{\text{max}} \) = maximum shear stress
CHAPTER 1

INTRODUCTION

1.1 General
Components of machines, vehicles, and structures are frequently subjected to repeated loads, also called cyclic loads. The resulting cyclic stresses can lead to microscopic physical damage to the materials involved. Even at stresses well below a given material’s ultimate strength, this microscopic damage can accumulate with continued cycling until it develops into a crack or other microscopic damage that leads to failure of the component. This process of damage and failure due to cycling loading is called fatigue.

One of the most important physical observations is that the fatigue process can generally be broken into three distinct phases: (i) A crack initiation phase, followed by (ii) crack propagation phase and finally (iii) unstable crack growth leading to fracture completely. The initiation life encompasses the development and early growth of a small crack. In this phase, both the shear and normal stresses and strains are the main parameters, which control the rate of crack extension. Last two phases together can be taken as the propagation life, portion of the total life spent growing a crack to failure. It is often very difficult to define the transition from initiation to propagation. This distinction depends upon many variables, including component size, material, loading condition and the methods used for life calculations. Though in uniaxial fatigue, this is not so much of a problem as the controlling parameters in both initiation and propagation lives are directly related to the uniaxial stress or strain. In multiaxial fatigue this is no longer the case.

Fatigue failure process due to cyclic stressing or straining can be divided into two areas: (i) low-cycle fatigue (LCF), when cyclic loading during each cycle introduces considerable plastic strain and (ii) high cycle fatigue (HCF) for which strain range is within the elastic range. The Stress based (S-N) approach is commonly used for high
cycle fatigue. This approach is still widely used in design applications where the stress is primarily within the elastic range and does not work well where the applied strains have a significant plastic component. In the case of low cycle fatigue, local strain-life approach is used. This method has gained acceptance as a useful method of evaluating the fatigue life of a notched component. Material properties obtained from smooth specimen strain controlled laboratory fatigue data are used for strain-life method.

It is seen over the years that fatigue evaluation of components and structures has become an integral part of the design process. Most real design situations, including rotating shafts, connecting links, automotive and aircraft components and many others involve a multiaxial state of cyclic stress. This often means that, at any point, the directions of the principal stresses can vary during the loading cycle and, therefore, as a function of time. Furthermore, the magnitudes of the principal stresses themselves may no longer be proportional to each other. Both these effects complicate the analysis required for the prediction of fatigue behavior.

During the analysis of components subjected to multiaxial loading, the problem often is reduced to an “equivalent uniaxial fatigue” case without thought as to whether the simplifying assumptions are valid for the specific load sequence or component being considered which results a very poor life prediction in compare to experimental result. The problem with all these equivalent stress-strain approach is that they do not take into account the fact that fatigue is essentially a directional process, with damage and cracking, taking place on particular planes. In addition there are serious problems in applying any of these equivalent stress and strain based approaches to situations of non-proportional loading. This has lead to much greater research emphasis being placed on understanding the underlying mechanisms of fatigue damage accumulation under multiaxial loading and given rise to a somewhat different approach based on predicting the extent of damage in specific directions and planes the component. This methodology is referred to as the critical plane approach.
Laboratory testing for fatigue life prediction is very complicated and in some cases almost impossible for complex loading conditions. A general trend is to make finite element modeling and then perform an elastic plastic analysis and then use equivalent uniaxial stress or strain criterion. Now-a-days elasto-plastic finite element analysis with cyclic stress-strain histories is rarely adopted, as this is an inconvenient and expensive procedure. Again equivalent uniaxial stress-strain criteria can only deal with multiaxial proportional or biaxial loading conditions with some simplified assumptions and deliver very poor results for non-proportional loading cases. A linear static finite element analysis, which is very convenient, is used here for stress-strain results. Fatigue life is then predicted for in phase or $90^\circ$ out of phase loadings using commercial software “FE-Fatigue”.

1.2 Research Objectives

The main objectives of this research work are to:

1. Get insight into commonly used fatigue criteria for multiaxial proportional and non-proportional loadings.

2. Investigate existing fatigue models using finite element based commercial code, FE-Fatigue and compare between models that can predict the fatigue crack initiation for multiaxial loading conditions.

3. Use continuum damage mechanics to model low cycle fatigue failure for multiaxial loading conditions.

4. Investigate the conventional methods for pure bending and torsional loading.
1.3 Thesis Outline

A literature review of available models for multiaxial fatigue is presented in Chapter 2 of this thesis. It includes a brief description on the stress-based approach, local strain approach and fracture mechanics approach. This chapter also has a detailed discussion on multiaxial proportional and non-proportional fatigue modeling and the multiaxiality assessment procedure.

In Chapter 3, an overview of models for multiaxial fatigue problems is presented. Finite Element analysis of the Society of Automotive Engineerings (SAE) test shaft is carried out, followed by a multiaxiality assessment to know the loading type and the suitable model to be used thereby. Results of the finite element analysis are used with FE-Fatigue to predict life for different loading conditions. Furthermore, details on the finite element analysis and FE-Fatigue used in the analysis are also presented in this chapter. Non-proportional multiaxial loading is considered and fatigue life calculations for these non-proportional multiaxial loading conditions are also carried out in this chapter. Predicted fatigue lives are then compared to experimental fatigue lives published in the literature (Fash et al. 1985) and (Leese and Socie, 1989). This chapter finishes with comments and general conclusions of the different multiaxial fatigue models used.

Chapter 4, continuum damage mechanics is used for fatigue life predictions for different loading conditions. This chapter includes a general discussion of damage mechanics, adapted for fatigue problems. A locally coupled analysis for fatigue crack initiation life predictions for the SAE notched shaft is presented. The fatigue life predictions are again compared to experimental results and comments and general conclusions on the damage mechanics approach are presented in this chapter.

Chapter 5, presents two conventional methods which can be used for pure bending and torsional cases. These methods are based on some empirical formulas and experimental observations. Calculations are done using conventional methods and the results are
compared with finite element based results. Moreover for pure torsion case, results are compared with experimental observations. A conclusion about the conventional approaches is drawn at the end of this chapter.

In Chapter 6, summary, conclusions and recommendations for future research are presented.
CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

Fatigue of materials is the process of accumulating damage and then failure due to cyclic loading. Fatigue is a complicated metallurgical process, which is difficult to accurately describe and model on the microscopic level. Despite the complexities, fatigue damage assessment for design of components and structures must be performed.

Historically, two over-riding considerations have promoted the development of fatigue analysis methods. The first has been the need to provide designers and engineers methods that are practical, easily implemented, and cost effective. The second consideration has been the need to reconcile between analytical approaches and physical observations. It has been through continued effort by many researchers that accepted design or analysis practices have been developed (Bannantine et al., 1990).

At present, there are three major approaches to analyzing and designing against fatigue failures. The traditional approach is to use the nominal stresses in the region of the component being analyzed, called stress-life approach. This approach is used mainly for high cycle fatigue where stresses and strains are elastic. No distinction between initiation and propagation is made in this approach, but deals with total life, or the life to failure of a component. The strain-based approach is an alternative. It involves more precise analysis of the localized yielding that may occur at stress raisers during cyclic loading. This approach is used when the strain is no longer purely elastic, but has a plastic component. Short (low cycle) fatigue lives generally occur under these conditions. Finally, the fracture mechanics approach is used specifically to predict propagation life from an initial crack or defect. This method is based upon linear elastic fracture mechanics (LEFM) principles, which are adapted for cyclic loading. Fracture mechanics
approach along with strain-life approach is sometimes used to predict a total (initiation and propagation) life. Each of the above mentioned methods has its domain of application, which may have some degree of overlap.

2.2 Stress-life approach
The stress-life, S-N, method was the first approach used in an attempt to understand and quantify metal fatigue. The number of cycle is calculated from the stress acting in the system. The S-N approach is still widely used in design applications where the applied stress is primarily within the elastic range of the material and the number of cycles to failure is large. The stress-life method does not work well in low-cycle fatigue applications where the applied strains have a significant plastic component.

2.2.1 Fatigue loading cycles
Before looking in more detail at the nominal stress procedure it is worth considering the general or typical types of cyclic stresses, which contribute to the fatigue process, such as those shown below in Figure 2.1(a). Referring to Figure 2.1, one can conveniently define several useful terms and symbols;

(i) The mean stress \( \sigma_m = \left( \frac{\sigma_{\text{max}} + \sigma_{\text{min}}}{2} \right) \)

(ii) The alternating range of stress \( \sigma_a = \left( \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2} \right) \)

(iii) The stress range \( \Delta \sigma = (\sigma_{\text{max}} - \sigma_{\text{min}}) \)

(iv) The stress ratio \( R = \frac{\sigma_{\text{min}}}{\sigma_{\text{max}}} \)

(v) The amplitude ratio \( A = \frac{\sigma_a}{\sigma_{\text{min}}} = \frac{1 - R}{1 + R} \)
Figure 2.1 Several constant-amplitude loading cycles. (a) Completely reversed, $R = -1$.
(b) Non zero mean stress. (c) Zero compression, $R = 0$ (Collins, 1981)
2.2.2 The S-N curve

S-N curve is a plot between nominal or average stress and cycles to failure. Such curves can be obtained from a variety of test apparatus ranging from relatively simple rotating bending machines to sophisticated closed-loop servohydraulic equipment. The basic S-N curve is the one with zero mean stress and can be tested using the rotating bending. The stress level at the surface of the specimen, nominal stress or average stress, is calculated using the beam equation \((S = Mc/I)\) for bending load and \(\sigma = P/A\) for uniaxial loading case) even if the resulting value exceeds the yield strength of the material. S-N data is nearly always presented in the form of a log-log plot of alternating stress amplitude or stress range versus cycles to failure. Certain materials, such as steels display a fatigue limit, \(S_o\), which represents an alternating stress level below which the material has an infinite life (Dowling, 1999). For most engineering purposes, infinite life is taken to be one million cycles.

S-N data approximating a straight line on a log-log plot, the corresponding equation is

\[
\sigma_a = AN_f^b
\]  

(2.1)

This equation is often used in a slightly different form,

\[
\sigma_a = \sigma_f(2N_f)^b
\]  

(2.2)

The fitting constants for the above two forms are related by

\[
A = 2^b \sigma_f,\ B = b
\]  

(2.3)

\(A\) and \(B\) are fitting test data for unnotched axial specimens tested under completely reversed \((\sigma_m = 0)\) loading. \(\sigma_f\) and \(b\) are taken as material properties and calculated for the stated test conditions.

Through many years of experience, particularly with steels, empirical relationship between fatigue and tensile properties have been developed. When the S-N curves for a number of different steels of varying strengths are plotted as the ratio of endurance limit, i.e., the stress amplitude at \(10^6\) cycles, \(S_o\), to the ultimate tensile strength, \(S_u\), all the curves tend to fall onto a single curve that can be approximated analytically as follows.
\[ S_6 = S_c \sim 0.5 \, S_u \quad \text{for } S_u < 1400 \, \text{MPa} \]  \hspace{1cm} (2.4)

and \[ S_6 = S_c \sim 700\,\text{MPa} \quad \text{for } S_u > 1400 \, \text{MPa} \]  \hspace{1cm} (2.5)

In addition to this, the stress at \(10^3\) cycles, \(S_3\), can be approximated by \(0.9 \, S_u\) and so utilizing these approximations, a generalized S-N curve can be generated for wrought steels, shown in Figure 2.2.

![Generalized S-N curve](image)

**Figure 2.2** Generalized S-N curve (Bannantine et al., 1990)

### 2.3 Strain-life approach

The strain-life method is based on the observation that in many critical locations such as notches the material responses to cyclic loading is strain rather than stress controlled. This arises from the fact that whilst most components are designed to confine nominal loads to the elastic region, stress concentrations such as notches often cause plastic deformation to occur locally. The material surrounding the plastically deformed zone remains fully elastic and so the deformation at the notch root is considered to be strain-controlled (Bannantine et al., 1990).
The strain-life method assumes similitude between the material in a smooth specimen tested under strain-control and the material at the root of a notch, Figure 2.3. For a given loading sequence, the fatigue damage in the specimen and the notch root are considered to be similar and so their lives will be similar.

![Diagram of a notch in a specimen with critical zone and smooth specimen marked]

**Figure 2.3** Equally stressed volume of material (Bannantine et al., 1990)

Crack growth is not explicitly accounted for in the strain-life method. Rather, failure of the component is assumed to occur when the equally stressed volume of material fails. Because of this, strain-life methods are often considered as initiation life estimates. The local strain-life approach has gained acceptance as a useful method of evaluating the fatigue life of a notched component. Both the American Society for Testing and Materials (ASTM) and the Society of Automotive Engineers (SAE) have recommended procedures and practices for conducting strain-controlled tests and using these data to predict fatigue lives (Bannantine et al., 1990).
A strain versus life curve is a plot of strain amplitude versus cycles to failure. Such a curve is employed in the strain-based approach for making life estimates. Before going to strain-life tests, stress-strain constitutive law of the material should be known.

### 2.3.1 Stress-strain relationships

A monotonic tension test of a smooth specimen is usually used to determine the engineering stress-strain behavior of a material where

\[
\text{Engineering stress, } S = \frac{P}{A_o} \quad (2.6)
\]

\[
\text{Engineering strain, } e = \frac{l - l_o}{l_o} = \frac{\Delta l}{l_o} \quad (2.7)
\]

The following terms are used in Figure 2.4:

- \( P \) = applied load,
- \( l_o \) = original length,
- \( d_o \) = original diameter,
- \( A_o \) = original area,
- \( l \) = instantaneous length,
- \( d \) = instantaneous diameter,
- \( A \) = instantaneous area.

![Diagram of a test specimen](image)

**Figure 2.4** Original and deformed (instantaneous) configuration of test specimen

(Bannantine et al., 1990)
In tension the true stress is larger than the engineering stress, due to change in cross-sectional area during deformation.

\[ \sigma = \frac{P}{A} \]  
(2.8)

True or natural strain, based on an instantaneous gage length, \( l \), is defined as

\[ \varepsilon = \frac{l}{l} = \ln \frac{l}{l_0} \]  
(2.9)

The true strain in terms of engineering strain is

\[ \varepsilon = \ln(1 + e) \]  
(2.10)

Equation (2.10) is valid up to necking. At necking the strain is no longer uniform throughout the gage length.

Assuming that the volume of the material remains constant during straining,

\[ A_0 l_0 = Al = \text{constant} \]  
(2.11)

Which allows us to write

\[ \frac{A_0}{A} = \frac{l}{l_0} \]  
(2.12)

True strain can then be stated in terms of cross-sectional area

\[ \varepsilon = \ln \frac{l}{l_0} = \ln \frac{A_0}{A} \]  
(2.13)

True stress can be again stated in terms of engineering stress

\[ \sigma = S \frac{A_0}{A} \]  
(2.14)

Combining equation (2.10) and equation (2.13) (valid up to necking) gives us

\[ \varepsilon = \ln(1 + e) = \ln \frac{A_0}{A} \]  
(2.15)

or \[ \frac{A_0}{A} = 1 + e \]  
(2.16)

Therefore, true stress can be stated as a function of engineering stress and strain using equations (2.14) and (2.16).
\[ \sigma = S(1 + e) \]  

(2.17)

This relation is valid only up to necking.

**Figure 2.5** Comparison of engineering and true stress-strain
(Bannantine et al., 1990)

The total true strain \( \varepsilon_t \) in tension test can be separated into elastic and plastic components, stated in equation form:

\[ \varepsilon_t = \varepsilon_e + \varepsilon_p \]  

(2.18)

The elastic strain is defined as

\[ \varepsilon_e = \frac{\sigma}{E} \]  

(2.19)

For most metals a log–log plot of true stress versus true plastic strain is modeled as a straight line. Consequently, this curve can be expressed using a power function

\[ \sigma = K(\varepsilon_p)^p \]  

(2.20)
or \( \varepsilon_p = \left( \frac{\sigma}{K} \right)^n \)  

(2.21)

where \( K \) is the strength coefficient and \( n \) is the strain-hardening exponent.

At fracture, two important quantities can be defined. These quantities are the true fracture strength and true fracture ductility. True fracture strength, \( \sigma_f \), is the true stress at final fracture.

\[ \sigma_f = \frac{P_f}{A_f} \]  

(2.22)

where \( A_f \) is the area at fracture and \( P_f \) is the load at fracture.

True fracture ductility, \( \varepsilon_f \), is the true strain at final fracture. This value can be defined in terms of the initial cross-sectional area and the area at fracture.

\[ \varepsilon_f = \ln \frac{A_0}{A_f} = \ln \frac{1}{1 - RA} \]  

(2.23)

where \( RA \) is defined as, \( RA = \frac{A_0 - A_f}{A_0} \)  

(2.24)

The strength coefficient, \( K \), can be defined in terms of the true stress at fracture, \( \sigma_f \), and the true strain at fracture, \( \varepsilon_f \).

\[ \sigma_f = K \left( \varepsilon_f \right)^n \]  

(2.25)

Rearranging

\[ K = \frac{\sigma_f}{\varepsilon_f^n} \]  

(2.26)

Therefore the expression for total strain can be rewritten as

\[ \varepsilon_t = \frac{\sigma}{E} + \left( \frac{\sigma}{K} \right)^n \]  

(2.27)
2.3.2 Cyclic stress-strain behavior

Cyclic stress-strain curves are useful for assessing the durability of structures and components subjected to repeated loading. The response of material subjected to cyclic inelastic loading is in the form of a hysteresis loop, as shown in Figure 2.6.

The total width of the loop is $\Delta \varepsilon$ or the total strain range. The total height of the loop is $\Delta \sigma$ or the total stress range.

Figure 2.6 A complete stress-strain cycle, a hysteresis loop (nCode International, 2001)

These can be stated in terms of amplitude:

$$\varepsilon_a = \frac{\Delta \varepsilon}{2} \quad (2.28)$$

where $\varepsilon_a$ is the strain amplitude and

$$\sigma_a = \frac{\Delta \sigma}{2} \quad (2.29)$$

where $\sigma_a$ is the stress amplitude.

The total strain is the sum of the elastic and plastic strain ranges,

$$\Delta \varepsilon = \Delta \varepsilon_e + \Delta \varepsilon_p \quad (2.30)$$

or in terms of amplitudes,
\[
\frac{\Delta \varepsilon}{2} = \frac{\Delta \varepsilon_e}{2} + \frac{\Delta \varepsilon_p}{2}
\]  

(2.31)

Using Hooke’s law, the elastic term may be replaced by \(\frac{\Delta \sigma}{E}\).

\[
\frac{\Delta \varepsilon}{2} = \frac{\Delta \sigma}{2E} + \frac{\Delta \varepsilon_p}{2}
\]  

(2.32)

Under strain control and early in life, the stress-strain response of most materials changes significantly with applied cyclic strains. However, after a relatively small number of cycles, typically no more than about 10% of total life, the hysteresis loops tend to stabilize so that the stress amplitude remains reasonably constant over the remaining portion of fatigue life. If the stress, strain co-ordinates of the tips from a number of stable hysteresis loops, of differing strain amplitude, are plotted in stress-strain space, then the locus of these points defines the cyclic stress-strain curve (Bannantine et al., 1990).

This is shown in Figure 2.7.

![Cyclic stress-strain curve](image)

**Figure 2.7** Cyclic stress-strain curve obtained by connecting tips of stabilized hysteresis loops (Bannantine et al., 1990).

Analogous to the monotonic stress-strain curve, a log-log plot of the completely reversed stabilized cyclic true stress versus true plastic strain can be approximated by a straight line as shown in Figure 2.8.
Figure 2.8 Log-log plot of true cyclic stress versus true cyclic plastic strain (Bannantine et al., 1990).

Similar to the monotonic relationship, it is possible to develop a power law function

$$\sigma_a = K'(\varepsilon_p)^{n'}$$  \hspace{1cm} (2.33)

where $\sigma_a$: the cyclically stable stress amplitude

$\varepsilon_p$: the cyclically stable plastic strain amplitude

$K'$: the cyclic strength coefficient

$n'$: the cyclic strain-hardening exponent

Rearranging Eq. 2.33 gives

$$\varepsilon_p = \left(\frac{\sigma_a}{K'}\right)^{n'}$$  \hspace{1cm} (2.34)

So the cyclic stress-strain relation comes as

$$\varepsilon_a = \frac{\sigma_a}{E} + \left(\frac{\sigma_a}{K'}\right)^{n'}$$  \hspace{1cm} (2.35)
2.4 The Strain-Life curve

Stress-life data may be represented by a straight line relationship when plotted using log scales. The relationship given by Basquin (Basquin, 1910) as stated in equation (2.2) is restated here

\[ \sigma_a = \sigma_f' (2N_f)^b \]  \hspace{1cm} (2.36)

where:

- \( \sigma_a \) : is the true cyclic stress amplitude
- \( \sigma_f' \) : is the regression intercept called fatigue strength coefficient
- \( 2N_f \) : number of half cycles, reversals, to failure
- \( b \) : is the regression slope called the fatigue strength exponent.
- \( \sigma_f' \) : is approximately taken equal to the monotonic fracture stress, \( \sigma_f \)

and \( b \) varies between \(-0.05\) and \(-0.12\) (Basquin, 1910).

The Basquin (Basquin, 1910) equation may be re-written in terms of elastic strain amplitude:

\[ \varepsilon_a = \frac{\sigma_a}{E} = \frac{\sigma_f' (2N_f)^b}{E} \]  \hspace{1cm} (2.37)

\( \varepsilon_a \) is the elastic strain amplitude.

Coffin (Coffin, 1954) and Manson (Manson, 1953) independently proposed that the plastic strain component of a fatigue cycle may also be related to life by a simple power law:

\[ \varepsilon_p = \varepsilon_f' (2N_f)^c \]  \hspace{1cm} (2.38)

where

- \( \varepsilon_p \) : is the plastic strain amplitude
- \( \varepsilon_f' \) : is the regression intercept called fatigue ductility coefficient
- \( 2N_f \) : number of half cycles, reversals, to failure
- \( c \) : is the regression slope called the fatigue ductility exponent.
\( \varepsilon'_f \) and \( c \) are considered to be mathematical properties with the fatigue ductility coefficient being approximately equal to the monotonic fracture strain, \( \varepsilon_f \), and \( c \) varies between \(-0.5\) and \(-0.7\) (Morrow, 1965).

Recent work by Morrow and Socie (Morrow and Socie, 1981) have indicated that the total strain amplitude, that is the sum of the elastic and plastic components, may be better correlated to life. Figure 2.9 illustrates schematically the nature of total strain-life curve.

![Diagram](image)

**Figure 2.9** Total strain-life curve (Bannantine et al., 1990).

Mathematically, this curve can be described by summing together the elastic and plastic components

\[
\varepsilon_i = \varepsilon_e + \varepsilon_p
\]

(2.39)

\[
\varepsilon_i = \frac{\sigma_f'(2N_f)^c}{E} + \varepsilon'_f(2N_f)^c
\]

(2.40)
2.4.1 Determination of cyclic fatigue properties

The cyclic material properties required to define the cyclic stress-strain curve and the strain-life curve are usually determined by carrying out tests, under strain control, on a series of smooth highly polished hour glass specimens. Typically, about 15 tests need to be performed at differing strain amplitudes. The fatigue parameters can then be calculated by regression analysis on the following curves:

- \( k' \) and \( n' \) : from a log stress vs log plastic strain regression.
- \( \sigma'_f \) and \( b \) : from a log elastic strain vs log \( 2N_f \) regression.
- \( \varepsilon'_f \) and \( c \) : from a log plastic strain vs log \( 2N_f \) regression.

2.4.2 Estimation from tensile properties

It is often difficult to gain access to measured cyclic properties. For this reason, a lot of effort has been put into finding ways of relating monotonic properties to cyclic properties. The first method of approximating the strain life relationship from monotonic properties was proposed by Manson (Manson, 1953) and later modified by Muralidharan and Manson (Muralidharan and Manson, 1988). The procedure is usually referred to as the method of universal slopes and can be applied to any metal.

Table 2.1 Method of universal slopes (nCode International, 2001)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Universal Slopes (Manson)</th>
<th>Modified Universal Slopes (Muralidharan and Manson)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma'_f )</td>
<td>1.9( \sigma_u )</td>
<td>0.623( \sigma_u ) 0.823( E^{0.168} )</td>
</tr>
<tr>
<td>( b )</td>
<td>-0.12</td>
<td>-0.09</td>
</tr>
<tr>
<td>( \varepsilon'_f )</td>
<td>0.76( \varepsilon_f^{0.6} )</td>
<td>0.0196( \varepsilon_f^{0.155} (\sigma_u / E)^{-0.53} )</td>
</tr>
<tr>
<td>( c )</td>
<td>-0.6</td>
<td>-0.56</td>
</tr>
<tr>
<td>( k' )</td>
<td>( (\sigma'_f / \varepsilon'_f)^{0.2} )</td>
<td>( (\sigma'_f / \varepsilon'_f)^{0.2} )</td>
</tr>
<tr>
<td>( n' )</td>
<td>0.2</td>
<td>0.2</td>
</tr>
</tbody>
</table>
More recently, Baumel and Seeger (Baumel and Seeger, 1990) have compiled an alternative approach based on the results of more than 1500 fatigue tests. Currently the approach is limited to plain carbon and low to medium alloy steels, aluminium and titanium alloys.

**Table 2.2 Uniform material law of Baumel and Seeger (Baumel and Seeger, 1990)**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Universal Material law plain and low alloy steel</th>
<th>Uniform Material Law Aluminium and Titanium alloys</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_f )</td>
<td>1.5( \sigma_u )</td>
<td>1.67( \sigma_u )</td>
</tr>
<tr>
<td>( b )</td>
<td>-0.087</td>
<td>-0.095</td>
</tr>
<tr>
<td>( \varepsilon_f )</td>
<td>0.59( \alpha )</td>
<td>0.35</td>
</tr>
<tr>
<td>( c )</td>
<td>-0.58</td>
<td>-0.69</td>
</tr>
<tr>
<td>( k' )</td>
<td>1.65( \sigma_u )</td>
<td>1.61( \sigma_u )</td>
</tr>
<tr>
<td>( n' )</td>
<td>0.15</td>
<td>0.11</td>
</tr>
</tbody>
</table>

The ductility factor \( \alpha \) is calculated from:

\[
\alpha = 1.0 \quad \text{for values of} \quad \frac{\sigma_u}{E} < 3 \times 10^{-3}
\]

\[
\alpha = \left(1.375 - 125 \frac{\sigma_u}{E}\right) \quad \text{for values of} \quad \frac{\sigma_u}{E} > 3 \times 10^{-3}
\]

**2.5 Fracture mechanics approach**

Fatigue life of a component has two stages: initiation, where the damage accumulates locally in the materials and propagation stages. The size of the crack at the transition from initiation to propagation is usually unknown and often depends on the analyst point of view and the size of the component being analyzed. The distinction between the initiation life and propagation life is important. At low strain amplitude up to ninety
percent of the life may be taken up with initiation, while at high amplitudes the majority of the fatigue life may be spent propagating a crack, as there is no flow here. Fracture mechanics approaches are used to estimate the propagation life (Bannantine et al., 1990). Fracture mechanics approaches require that an initial crack size be known or assumed. For components with imperfections or defects an initial crack size may be known. Alternatively, for an estimate of the total fatigue life of a defect-free material, fracture mechanics approaches can be used to determine propagation. Strain-life approaches may then be used to determine initiation life, with the total life being the sum of these two estimates.

2.5.1 Purpose of fracture mechanics

In general, the possibility of crack like defects being present at the start of service life is to be recognized. Even if the component is not cracked to start with, the remnant life after initiation may be significant or possibly dominant with respect to the total life. Furthermore, for inspection related activities, the remnant life of a component discovered to be cracked during inspection needs to be determined so that decisions regarding serviceability can be supported by reliable predictions (nCode International, 2001).

Faced with design problem the designer must be able to calculate the following criterion to crack stability (Figure 2.10):

1. The residual strength as a function of crack size
2. The crack size that can be allowed at the expected service load (the critical size)
3. How long it takes the crack to grow from a certain size to a critical size
4. The size of initial flaws that can be tolerated in a new component
5. The interval between inspections of cracked component.

Fracture mechanics tries to provide tools with which to answer these questions. The subject includes the materials science studies of fracture processes on an atomic scale, the growth of cracks, the analysis of the crack tip stresses and the behavior of cracks in these
stress fields, the provision of materials properties by testing, and finally the engineering application of these techniques to the analysis of real structures (Draper, 1999)

Figure 2.10 Relationship between crack length and failure load (Draper, 1999).

2.5.2 Linear elastic fracture mechanics: an overview

Linear elastic fracture mechanics principles are used to relate the stress magnitude and distribution near the crack tip to:

1. Remote stresses applied to the cracked component
2. The crack size and shape
3. The material properties of the cracked component

Linear elastic fracture mechanics (LEFM) is based on the application of the theory of elasticity to bodies containing cracks or defects. The assumptions used in the theory of LEFM are small displacements and general linearity between stresses and strains. The local stresses near the crack tip are of the general form shown in Figure 2.11.
Figure 2.11 Location of local stresses near a crack tip in cylindrical coordinates (Bannantine et al., 1990).

\[
\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_\eta (\theta) + \cdots \tag{2.41}
\]

where \( r \) and \( \theta \) are cylindrical coordinates of a point with respect to the crack tip and \( K \) is the stress intensity factor. It is the general form of the LEFM equations. As seen, a singularity exists such that as \( r \), the distance from crack tip, tends toward zero, the stresses go to infinity. Since materials plastically deform as the yield stress is exceeded, a plastic zone will form near the crack tip. The basis of LEFM remains valid, though, if this region of plasticity remains small in relation to the overall dimensions of the crack and cracked body (Bannantine et al., 1990).

2.5.3 Loading modes at the crack tip
There are generally three modes of loading, which involve different crack surface displacements. The three modes are shown in Figure 2.12.
Mode I: opening or tensile mode (the crack faces are pulled apart);
Mode II: sliding or in-plane shear (the crack surfaces slide over each other);
Mode III: tearing or anti-plane shear (the crack surfaces move parallel to the leading edge of the crack and relative to each other).

Most problems can be addressed by superimposing one or more of these modes, but Mode I is by far the most important for practical analysis.

![Diagram of crack modes](image)

**Figure 2.12** The three crack opening modes (Draper, 1999).

### 2.5.4 Stress intensity factor

The stress intensity factor, $K$ defines the magnitude of the local stresses, around the crack tip. The factor depends on loading, crack size, crack shape, and geometric boundaries, with the general form given by

$$K = f(g)\sigma \sqrt{\pi a}$$  \hspace{1cm} (2.42)

where

- $\sigma =$ remote stress applied to component
- $a =$ crack length
- $f(g) =$ correction factor that depends on specimen and crack geometry.

The stress intensity factor solutions are found for a wide variety of problems within LEFM. Stress intensity factors for a single loading mode can be added algebraically.
Consequently, stress intensity factors for a complex loading condition of the same loading mode can be determined by superposition.

2.5.5 Plastic zone size

Materials develop plastic strains as the yield stress is exceeded in the region near the crack tip. The amount of plastic deformation is restricted by the surrounding material, which remains elastic. The size of this plastic zone is dependent on the stress conditions of the body.

(a) Monotonic plastic zone size

The plastic zone sizes under monotonic loading have been estimated to be (Bannantine et al., 1990).

\[ r_y = \begin{cases} 
\frac{1}{2\pi} \left( \frac{K}{\sigma_y} \right)^2 & \text{Plane stress} \\
\frac{1}{6\pi} \left( \frac{K}{\sigma_y} \right)^2 & \text{Plane strain}
\end{cases} \]

where \( r_y \) is defined as shown in Figure 2.13.

![Diagram of monotonic plastic zone size](image)

**Figure 2.13** Monotonic plastic zone size (Bannantine et al., 1990).
(b) Cyclic plastic zone size

The reversed or cyclic plastic zone size is four times smaller than the comparable monotonic value. As the nominal tensile load is reduced, the plastic region near the crack tip is put into compression by the surrounding elastic body. As shown in Figure 2.14, the change in stress at the crack tip due to the reversed loading is twice the value of the yield stress (Bannantine et al., 1990).

The cyclic plastic zone size is smaller than the monotonic. Thus LEFM concepts can often be used in the analysis of fatigue crack growth problems even in materials that exhibit considerable amounts of ductility. The basic assumption that the plastic zone size is small in relationship to the crack and the crack body usually remains valid.

\[
\begin{align*}
    r_y &= \left\{ \frac{1}{2\pi} \left( \frac{K}{2\sigma_y} \right)^2 = \frac{1}{8\pi} \left( \frac{K}{\sigma_y} \right)^2 \right. \\
    r_y &= \left\{ \frac{1}{6\pi} \left( \frac{K}{2\sigma_y} \right)^2 = \frac{1}{24\pi} \left( \frac{K}{\sigma_y} \right)^2 \right. \\
\end{align*}
\]

Plane stress

Plane strain

Figure 2.14 Cyclic plastic zone size (Bannantine et al., 1990).
2.5.6 Fracture toughness

As the stress intensity factor reaches a critical value, $K_c$, unstable fracture occurs. This critical value of the stress intensity factor is known as the fracture toughness of the material. The fracture toughness can be considered the limiting value of the stress intensity just as the yield stress might be considered the limiting value of applied stress. Fracture toughness, $K_c$, is dependent on specimen geometry and metallurgical factors. The fracture toughness depends on both temperature and the specimen thickness.

2.5.7 Fatigue crack growth

If the crack length, $a$, is plotted versus the corresponding number of cycles, $N$, at which the crack is measured, it can be shown that most of the life of the component is spent while the crack length is relatively small. In addition, the crack growth rate increases with increased applied stress. The crack growth rate, $\frac{da}{dN}$, is obtained by taking the derivative of the above crack length, $a$, versus cycles, $N$, curve. Values of $\log \frac{da}{dN}$ can then be plotted versus $\log \Delta K$, for a given crack length, using the equation

$$\Delta K = K_{\text{max}} - K_{\text{min}} = f(g)\Delta \sigma \sqrt{a}$$

(2.43)

A plot of $\log \frac{da}{dN}$ versus $\log \Delta K$, a sigmoidal curve is shown in Figure 2.15.

This curve may be divided into three regions. At low stress intensities, Region I, cracking behavior is associated with threshold, $\Delta K_{th}$, effects. In the mid-region, Region II, the curve is essentially linear. Many structures operate in this region. Finally, in region III, at high $\Delta K$ values, crack growth rates are extremely high and little fatigue life is involved.
**Figure 2.15** Three regions of crack growth rate curve (Bannantine et al.1990).

**Region II**

LEFM tries to estimate crack growth behavior within region II. In this region the slope of the $\log \frac{da}{dN}$ versus $\log \Delta K$ curve is approximately linear and lies roughly between $10^{-6}$ and $10^{-3}$ in/cycle. Many curve fits to this region have been suggested. The Paris equation is the most widely accepted. In this equation

$$\frac{da}{dN} = C(\Delta K)^m$$  \hspace{1cm} (2.44)

where $C$ and $m$ are material constants and $\Delta K$ is the stress intensity range $K_{\text{max}} - K_{\text{min}}$. The crack growth life, in terms of cycles to failure, may be calculated using the equation (2.35). If $a_i$ is the initial crack length and $a_f$ is the final critical crack length, then

$$N_f = \int_{a_i}^{a_f} \frac{da}{C(\Delta K)^m}$$  \hspace{1cm} (2.45)
Region I
Region I of the sigmoidal crack growth rate curve is associated with threshold effects. Below the value of the threshold stress intensity factor, $\Delta K_{th}$, fatigue crack growth does not occur or occurs at a rate too slow to be measured. The fatigue threshold decreases with increasing the stress ratio. The threshold also depends on frequency of loading and environment.

Region III
In region III, rapid, unstable crack growth occurs. In many practical engineering situations this region may be ignored because it does not significantly affect the total crack propagation life. The point of transition from region II to region III behavior is dependent on the yield strength of the material, stress intensity factor, and stress ratio.

2.6 Multiaxial fatigue
In engineering components, cyclic loadings that cause complex states of stress and strain are common. Complex stress states-stress states in which the three principal stresses are non-proportional or whose directions change during a loading cycle, very often occur at geometric discontinuities such as notches or joint connections. Failure under these conditions, termed multiaxial fatigue, is an important design consideration for reliable operation and optimization of many engineering components.

Early development of multiaxial fatigue theories were based on extension of uniaxial fatigue theories to fatigue under combined stresses. These multiaxial fatigue theories based on equivalent uniaxial fatigue criteria with some simplified assumptions can only deal with proportional loading. Moreover, stress-based theories are usually limited to the high cycle fatigue (HCF) regime. In low cycle fatigue (LCF), the stresses and strains are no longer linearly related and the strain-based criteria are used in this regime. Strain life based methods are used in predicting multiaxial proportional fatigue life. Critical plane multiaxial fatigue theories based on the premise that failures occurs due to damage
developed on a critical plane, are currently used for multiaxial proportional and non-proportional loading conditions. All of the above theories and models are reviewed in this section. Before presenting multiaxial fatigue models, the multiaxiality parameters are presented.

2.6.1 Multiaxiality assessment

There are two essential parameters to assess multiaxiality:

(I) Biaxiality ratio, $a$, which is defined as the ratio of the smaller absolute in-plane principal to the larger

(II) Angle of the absolute maximum principal stress vector with the x-axis of the planar co-ordinate system, $\theta_p$.

Table 2.3 Criteria for multiaxiality assessment (nCode International, 2001).

<table>
<thead>
<tr>
<th>State of the Problem</th>
<th>Angle of absolute maximum principle stress ($\theta_p$)</th>
<th>Biaxiality ratio ($a$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniaxial</td>
<td>$\theta_p$ is constant</td>
<td>$a = 0$</td>
</tr>
<tr>
<td>Proportional Multiaxial</td>
<td>$\theta_p$ is constant</td>
<td>$-1 &lt; a &lt; +1$</td>
</tr>
<tr>
<td>Non-Proportional Multiaxial</td>
<td>$\theta_p$ may vary</td>
<td>$a$ may vary</td>
</tr>
</tbody>
</table>

2.6.2 Stress-based static yield criteria models

As fatigue damage is controlled by plastic deformation, yield criteria that describe plastic deformation might be used for describing fatigue behavior. At present there is no theoretical way of deducing the relationship between the stress components to correlate yielding for a multiaxial stress state with yielding in the uniaxial tension test. Therefore yield criteria are empirical relationships. Yield criterion must be consistent with a number of experimental observations. The three most common yield criteria are as follows:
(a) Maximum normal stress theory

The maximum normal stress theory applied to cyclic loading may be expressed as:

$$\Delta \sigma_{eq} = \Delta \sigma_i$$  \hspace{1cm} (2.46)

It would predict yielding failure when the magnitude of the principal stress becomes equal to the yield stress $\sigma_y$. But for multiaxial loading, it is no longer possible to predict yielding correctly through a consideration of the maximum principal stress alone, because the other principal stresses will also have an influence.

Since the hydrostatic stress component of a complex stress state does not influence the stress at which a material yields, the yield criterion must be based on the stress deviator. Moreover, for an isotropic material the yield criteria must be independent of the choice of axis, i.e. it must be an invariant functions of the stress deviator.

(b) Distortion-Energy, von Mises yield criterion

von Mises proposed that yielding will occur when the second invariant of the stress deviator, $J_2$ exceeds a critical value i.e.

$$J_2 = k^2$$  \hspace{1cm} (2.47)

where $$J_2 = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$  \hspace{1cm} (2.48)

The constant $k^2$ can be evaluated and related to yielding in at tension test by realizing that $\sigma_1 = \sigma_0$, and $\sigma_2 = \sigma_3 = 0$ and by using substitution for $J_2$,

$$k^2 = \frac{\sigma_0^2}{3}$$

and so the von Mises prediction of yielding in terms of the principal stresses becomes,

$$\sigma_0 = \frac{1}{\sqrt{2}}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{0.5}$$  \hspace{1cm} (2.49)

According to this criterion, yielding failure will occur when $\sigma_0$ exceeds the monotonic yield stress, $\sigma_y$.
(c) Maximum shear Stress, Tresca yield criterion

The Tresca yield criterion suggests that yielding will occur when the maximum shear stress under multiaxial loading reaches the value of the shear stress under uniaxial tension test. \( \sigma_1 \) and \( \sigma_3 \) being the maximum and minimum principal stress respectively, \( \tau_{\text{max}} \) is given by:

\[
\tau_{\text{max}} = (\sigma_1 - \sigma_3) / 2
\]  (2.50)

For uniaxial tension, \( \sigma_1 = \sigma_0 \), and \( \sigma_2 = \sigma_3 = 0 \) and so

\[
\tau_{\text{max}} = (\sigma_1 - \sigma_3) / 2 = \sigma_0 / 2
\]  (2.51)

and

\[
\sigma_1 - \sigma_3 = \sigma_0
\]  (2.52)

This criterion predicts that yielding failure will take place when \( \sigma_0 \) exceeds the monotonic yield stress, \( \sigma_y \).

2.6.3 Fatigue life prediction by stress-based models

This is called effective stress approach. Stress is calculated by different approaches, Principal stress criterion, von Mises yield criterion and maximum shear stress criterion. The calculated stress in every case is called effective stress, \( \sigma_e \). This effective stress is then used for fatigue life prediction.

The equivalent completely reversed stress amplitude \( \sigma_{ar} \) can now be obtained from the following equation:

\[
\sigma_{ar} = \frac{\sigma_e}{\frac{\sigma_m}{\sigma_f}}
\]  (2.53)

For fully reversed (zero mean stress) condition \( \sigma_{ar} = \sigma_e \) and fatigue life will be then

\[
N_f = \left( \frac{\sigma_{ar}}{\sigma_f} \right)^{1/h}
\]  (2.54)
where $\sigma'$ and $b$ are material constants. This equation is the empirical form of S-N curve. For cases where the principal axes rotate (non proportional loading) during cyclic loading, the application of these stress-based approaches are questionable. This also applies if cyclic loads occur at more than one frequency, or if there is difference in phase (other than $180^0$) between them.

2.6.4 Strain based models

Strain-based models are applicable for low-cycle fatigue where significant plasticity may occur. Models reviewed in this chapter make use of either strain only or some product of stress and strain.

(a) Maximum principal strain criterion

The maximum principal strain theory may be expressed as

$$\Delta \varepsilon_{eq} = \Delta \varepsilon_1$$  (2.55)

This criterion states that fatigue cracks start on planes, which experience the maximum principal strain amplitude. It can be related to strain life equation to get fatigue life.

$$\frac{\Delta \varepsilon_1}{2} = \frac{\sigma'}{2} (2N_f)^b + \varepsilon' (2N_f)^c$$  (2.56)

(b) Maximum shear strain criterion

The maximum shear strain can be calculated from the principal strains using Mohr’s circle (Figure 2.16), where the maximum shear strain is given by

$$\frac{\gamma_{max}}{2} = \frac{\varepsilon_1 - \varepsilon_3}{2}$$  (2.57)
It proposes that cracks will initiate on planes, which will experience maximum shear strain amplitude. For uniaxial stress, with an axial strain $\varepsilon_1$, and the other principal strains $\varepsilon_2 = -\nu \varepsilon_1$, $\gamma_{\text{max}} = \varepsilon_1 - \varepsilon_3 = (1 + \nu)\varepsilon_1$. For elastic strains, Poisson’s ratio $\nu$ is approximately 0.3 for steel, so that $\gamma_{\text{max}} = 1.3\varepsilon_1$.

As the shear strain amplitude is 1.3 times the axial strain amplitude, the elastic part of the strain life equation would be

$$\frac{\Delta\gamma_{\text{max}}}{2} = \frac{\Delta\varepsilon_1}{2} = 1.3 \frac{\sigma'}{E} (2N_f)^b$$

(2.58)

For purely plastic strains, $\nu = 0.5$ and so $\gamma_{\text{max}} = 1.5\varepsilon_1$, so

the plastic part of the strain life equation would be

$$\frac{\Delta\gamma_{\text{max}}}{2} = 1.5\varepsilon' (2N_f)^c$$

(2.59)
Combining equation (2.58) and (2.59), the strain life equation to get fatigue life is therefore
\[ \frac{\Delta \gamma_{\text{max}}}{2} = 1.3 \frac{\sigma'_f}{E} (2N_f)^b + 1.5 \epsilon'_f (2N_f)^c \] (2.60)

(c) Octahedral Shear Strain Theory:
The most popular theory is the octahedral strain theory and can be written in terms of the principal strains as:
\[ \epsilon_{eq} = A \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_4)^2 \right]^{1/3} \] (2.61)
where \( A \) is defined by
\[ A = 0.707 \left( \frac{1}{1 + \nu^*} \right) \] (2.62)
and \( \nu^* \) is the elasto-plastic Poisson’s ratio defined by
\[ \nu^* = (\nu_e \epsilon_e + \nu_p \epsilon_p) / \epsilon_i \] (2.62)
where
\( \nu_e = \) elastic Poisson’s ratio, about 0.3
\( \nu_p = \) plastic Poisson’s ratio, about 0.5
\( \epsilon_i = \) total strain amplitude
\( \epsilon_e = \) elastic strain amplitude
\( \epsilon_p = \) plastic strain amplitude
The calculated values of the equivalent strain can be used directly with the uniaxial strain life equation to calculate the fatigue life
\[ \epsilon_{eq} = \frac{\sigma'_f}{2} (2N_f)^b + \epsilon'_f (2N_f)^c \] (2.63)
2.6.5 Consideration for Notches, strain-based models

The most widely used procedure for estimating the notch strains in the plastic region is Neuber’s rule or equivalent approaches based on strain energy. Generalizing this simple concept to multiaxial loading has proven difficult, particularly for non-proportional loading.

Neuber’s rule may be written in terms of equivalent quantities for multiaxial loading.

\[ e^{eq} \sigma^{eq} = \epsilon^{eq} \epsilon^{eq} \]  

(2.65)

Equivalent stresses can be computed from von Mises flow criterion. Here, \( e^{eq} \sigma^{eq} \) and \( e^{eq} \epsilon^{eq} \) are defined as the elastically calculated notch stress and strain, and \( \sigma^{eq} \) and \( \epsilon^{eq} \) are the elastic-plastic notch stress and strain.

The cyclic stress-strain curve of the material provides a relationship between \( \sigma^{eq} \) and \( \epsilon^{eq} \).

\[ \epsilon^{eq} = \frac{\sigma^{eq}}{E} + \left(\frac{\sigma^{eq}}{K'}\right)^\frac{1}{n'} \]  

(2.66)

In multiaxial stress states, the material constants, \( K' \) and \( n' \), can be used to relate the individual stress and strain components shown here for plane stress.

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \left[f(E, K', n')\right] \begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]  

(2.67)

Equations 2.65, 2.66 and 2.67 provide five equations to solve for the three unknown components of stress and three unknown components of strain. An additional equation relating two of the unknowns must be obtained to find a solution.

2.6.6 Fatigue life prediction by Hoffmann and Seeger method

Hoffmann and Seeger (Hoffmann and Seeger, 1985) investigated two approximations for the missing equation for proportional loading. In their formulation, Neuber’s rule (Neuber, 1946) is written as
\[
\frac{(K_{i}^{eq} S)^{2}}{E} = \sigma^{eq} e^{eq}
\] (2.68)

Here, the nominal stress \( S \) can be defined in any convenient way such that it relates the stress concentration factor to the principal stress.

\[
^{e} \sigma_1 = K_i S
\] (2.69)

The elastic equivalent notch stress, \( ^{e} \sigma^{eq} \), is computed from the principal strains, the von Mises yield function and the knowledge of the elastic stress ratios.

\[
^{e} \sigma^{eq} = \frac{^{e} \sigma_1}{\sqrt{2}} \sqrt{\left(1 - \frac{^{e} \sigma_2}{^{e} \sigma_1}\right)^2 + \left(1 - \frac{^{e} \sigma_3}{^{e} \sigma_1}\right)^2 + \left(\frac{^{e} \sigma_2}{^{e} \sigma_1} - \frac{^{e} \sigma_3}{^{e} \sigma_1}\right)^2}
\] (2.70)

An equivalent stress concentration factor, \( K_{i}^{eq} \), now can be obtained as

\[
K_{i}^{eq} = \frac{^{e} \sigma^{eq}}{S}
\] (2.71)

and

\[
K_{i}^{eq} = \frac{K_i}{\sqrt{2}} \sqrt{\left(1 - \frac{^{e} \sigma_2}{^{e} \sigma_1}\right)^2 + \left(1 - \frac{^{e} \sigma_3}{^{e} \sigma_1}\right)^2 + \left(\frac{^{e} \sigma_2}{^{e} \sigma_1} - \frac{^{e} \sigma_3}{^{e} \sigma_1}\right)^2}
\] (2.72)

Equation 2.68 can be solved for the unknown notch equivalent stresses and strains.

\[
\frac{(K_{i}^{eq} S)^{2}}{E} = \sigma^{eq} e^{eq} = \left(\sigma^{eq} \right)^2 + \sigma^{eq} \left(\frac{\sigma^{eq}}{K}\right)^{1
\] (2.73)

For cyclic loading, \( K \) and \( n \) in equation (2.73) should be replaced by \( K' \) and \( n' \).

Hencky’s rule may be used to compute the principal stresses and strains from the equivalent stresses and strains. Plastic strains, \( ^{p} \varepsilon_i \), are a function of the deviatoric stresses, \( S \).

\[
^{p} \varepsilon_i = \frac{3}{2} \frac{^{p} e^{eq}}{\sigma^{eq}} S_i
\] (2.74)

This leads to a generalized formulation of Hooke’s law. Noting that one of the principal stresses \( \sigma_3 = 0 \) on the free surface gives
\[ \varepsilon_1 = \frac{\varepsilon_{eq}}{\sigma_{eq}} (\sigma_1 - \overline{\nu} \sigma_2) \quad (2.75) \]

\[ \varepsilon_2 = \frac{\varepsilon_{eq}}{\sigma_{eq}} (\sigma_1 - \overline{\nu} \sigma_1) \quad (2.76) \]

\[ \varepsilon_3 = \frac{\varepsilon_{eq}}{\sigma_{eq}} (-\overline{\nu} (\sigma_1 + \sigma_2)) \quad (2.77) \]

where \[ \overline{\nu} = \frac{1}{2} - \left( \frac{1 - \nu}{2} \right) \frac{\sigma_{eq}}{E \varepsilon_{eq}} \quad (2.78) \]

Principal Stresses can be related to the equivalent stress with the von Mises yield criterion.

\[ \sigma_{eq} = \sqrt{\frac{1}{2} \left[ (\sigma_1 - \sigma_2)^2 + \sigma_1^2 + \sigma_2^2 \right]} \quad (2.79) \]

The four equations, equation (2.69), (2.70), (2.72) and (2.73) have five unknowns \((\sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2, \text{ and } \varepsilon_3)\), and an additional assumption must be made to obtain the principal stresses and strains. Hoffmann and Seeger (Hoffmann and Seeger, 1985) suggested following assumptions:

1. The principal stresses and strains are fixed in orientation.

2. The ratio of the in-plane principal strains is constant \(\frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_2}{\varepsilon_1}\)

Hencky’s flow rule can be combined with the constant strain assumption to give a simple set of equations for estimating the notch principal stresses and strains when the principal stress directions remain fixed.

\[ a = \frac{\sigma_2}{\sigma_1} = \frac{\varepsilon_2 + \overline{\nu}}{\varepsilon_1} \quad (2.80) \]

\[ \sigma_1 = \frac{1}{\sqrt{1 - a + a^2}} \sigma_{eq} \quad (2.81) \]
\[ \varepsilon_i = \frac{1 - 3a}{\sqrt{1 - a^2}} \varepsilon^e \]  

Equivalent stress and strain are obtained from equation (2.73).

**Fatigue life prediction by Hoffmann and Seeger**

Once the principal stress and the principal strain are found for the critical locations, uniaxial strain-life equation can be used to predict fatigue life.

\[ \varepsilon_i = \frac{\sigma_i'}{2} (2N_f)^b + \varepsilon_i' (2N_f)^c \]  

Hoffmann and Seeger approximation method is found quite well in predicting fatigue life cycles for multiaxial proportional loadings (Hoffmann and Seeger, 1985). Fatigue life prediction for non proportional loadings by Hoffmann and Seeger method is not suitable as during this loading all the simplified assumption taken by Hoffmann and Seeger are violated. Nowadays, it is a general practice to use critical plane approach both for multiaxial proportional and non proportional loading. This approach is described in section 2.8 in this chapter.

**2.7 Multiaxial notch correction procedure**

Transient elastic-plastic FE analyses may be quite cumbersome for realistic FE models and input loading histories. Multiaxial notch correction procedure, which utilizes results from simple static linear-elastic FE analyses and the stress-superposition principle, is necessary. Inputs to this procedure are stresses and strains from the linear-elastic computations. Elastic-plastic strains are estimated from these elastic strains using a Neuber-type model, i.e. the overall strain energy density equivalence between linear-elastic and elastic plastic stress-strain states, shown in Figure 2.17.

The most used cyclic plasticity models, necessary for the notch correction, are the Mroz (Mroz, 1967) and Garud (Garud, 1981) models. These models are confined to the free surface conditions.
Figure 2.17 Overall strain energy density equivalence (nCode International, 2001)

A powerful feature of the implemented notch correction procedure is based on the ratios of strain energy density increments contributed by each pair of corresponding stress strain increments.

\[
\begin{align*}
S_{22}^e \Delta e_{22}^e + e_{22}^e \Delta S_{22}^e &= S_{22}^a \Delta e_{22}^a + e_{22}^a \Delta S_{22}^a \\
S_{23}^e \Delta e_{23}^e + e_{23}^e \Delta S_{23}^e &= S_{23}^a \Delta e_{23}^a + e_{23}^a \Delta S_{23}^a \\
S_{33}^e \Delta e_{33}^e + e_{33}^e \Delta S_{33}^e &= S_{33}^a \Delta e_{33}^a + e_{33}^a \Delta S_{33}^a
\end{align*}
\]

(2.84)

This assumption reduces the dependency of the results on geometry and constraint conditions at the notch tip significantly. The predictions of linear elastic-plastic strains and related stresses from linear-elastic inputs agree very favorably with transient elasto-plastic FEA results for wide range of loads, both for proportional and non proportional.
Cyclic plasticity modeling

Mroz (Mroz, 1967) has proposed that the uniaxial stress-strain material curve can be represented by a set of plasticity surfaces in three dimensional stress space. In the case of two dimensional stress state, the plasticity surfaces reduce to ellipses on the plane of principal stresses described by:

\[ \sigma_{eq} = \sqrt{\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2} \]  

(2.85)

and is illustrated in Figure 2.18

![Figure 2.18 Mroz approach for cyclic plasticity modeling (nCode International, 2001)](image)

The load path dependent memory effects are modeled by prescribing a transition rule for the ellipses moving with respect to each other over distances given by the stress increments. It is also assumed that the ellipses move inside each other and do not intersect. If the ellipses come in contact with one another they move together as a rigid body.

2.8 Critical plane models

Critical plane approaches have evolved from experimental observations of the nucleation and growth of cracks during loading. Depending on the material, stress-state,
environment, and strain amplitude, fatigue life usually will be dominated by crack growth along either shear planes or tensile planes. In this approach, stresses and strains during cyclic loading are determined for various orientations (planes) in the material and the stresses and strains acting on the most severely loaded plane are used to predict fatigue failure.

2.8.1 Normal strain criterion
In this approach strain amplitude normal to the critical plane is calculated (nCode International, 2001). Fatigue life is then predicted by

$$\frac{\Delta \varepsilon_n}{2} = \frac{\sigma_f}{E} (2N_f)^b + \varepsilon'_f (2N_f)^c$$

(2.86)

where \(\frac{\Delta \varepsilon_n}{2}\) is the strain amplitude normal to the critical plane. Otherwise this is the usual Coffin-Manson-Basquin equation.

2.8.2 Shear strain criterion
In this model shear strain amplitude is calculated along the critical plane (nCode International, 2001). Fatigue life is then calculated with the following equation

$$\frac{\Delta \gamma}{2} = \frac{(1 + \nu_e)\sigma_f}{E} (2N_f)^b + (1 + \nu_p)\varepsilon'_f (2N_f)^c$$

(2.87)

where

\(\frac{\Delta \gamma}{2}\) = shear strain amplitude along the critical plane, otherwise this is the maximum shear strain criterion.

\(\nu_e\) = elastic poison’s ratio

\(\nu_p\) = plastic poison’s ratio

2.8.3 Fatemi and Socie model (Fatemi and Socie, 1988)
Cracks virtually always have irregular shapes due to growth through the gain structure of the material. Thus, growth due to a shear stress alone tends to be difficult due to a
mechanical interlocking and friction effects involving irregularities on the faces of cracks as shown in Figure 2.19

![Figure 2.19](image)

**Figure 2.19** Physical basis of the Fatemi and Socie model (Fatemi and Socie, 1988)

Stresses and strains normal to the crack plane may have a major effect on the behavior, accelerating the growth if they tend to open the crack. Fatemi and Socie proposed an approach to include the normal strain effect. The proposed model is:

$$\frac{\Delta \gamma}{2} \left(1 + n \frac{\sigma_{n,\text{max}}}{\sigma_y}\right) = \frac{\tau'_f}{G} (2N_f)^b + \gamma'_f (2N_f)^c$$  \hspace{1cm} (2.88)

where

- $\Delta \gamma$ is the largest amplitude of shear strain for the critical plane.
- $\sigma_{n,\text{max}}$ is the maximum tensile stress normal to the plane of $\frac{\Delta \gamma}{2}$.
- $\tau'_f, b, \gamma'_f, c$ are constants found from completely reversed tests in pure shear.
- $n$ is an empirical constant, $n = 0.6$ to 1.0, depending on the material.
2.8.4 Smith, Watson and Topper (SWT) model (Smith et al., 1970)

Fatemi-Socie critical plane model has been developed primarily using the materials for which the dominant failure mechanism is shear crack nucleation and growth. An alternative damage model is needed for materials that fail predominantly by crack growth on planes of maximum tensile strain or stress. This model includes both the cyclic strain range and the maximum stress. SWT parameter also can be used in the analysis of both proportionally and non-proportionally loaded components constructed from materials that fail primarily due to mode I tensile cracking. This approach for multiaxial loading is based on the strain amplitude normal to the critical plane, $\frac{\Delta \varepsilon_n}{2}$ and maximum normal stress on the critical plane, $\sigma_{\text{n, max}}$.

![Diagram of tensile crack growth in SWT model](image)

**Figure 2.20** Tensile crack growth in SWT model (Smith et al., 1970)

\[
\frac{\Delta \varepsilon_n}{2} \sigma_{\text{n, max}} = \frac{\sigma'_f}{E} (2N_f)^{2b} + \sigma'_f \cdot \varepsilon'_f (2N_f)^b + c
\]  

(2.89)
CHAPTER 3

MULTIAXIAL FATIGUE LIFE PREDICTIONS

3.1 Introduction

Several multiaxial fatigue theories have been presented in the literature review. A common characteristic of all of the approaches is that the required material properties can be determined from standard uniaxial fatigue test data. However, still, there exists a lack of agreement on which model is most appropriate.

The Society of Automotive Engineering (SAE) notched shaft is taken as an example in this work to evaluate the validity of the existing fatigue theories used for multiaxial loading. Both in phase (multiaxial proportional) and 90° out of phase (non-proportional) loadings are considered for analysis.

In this chapter, a brief discussion on finite element based approach is presented first. Multiaxiality assessment of the problem is carried out next to decide which criterion would be best to predict fatigue life. As from the literature review, equivalent strain life approaches can predict the fatigue life with in some accepted factor for multiaxial proportional loading. Hoffmann and Segger tried to improve these usual effective strain approaches with some assumptions to deal with notch effect and also for proportional loading. These criteria are investigated here, using finite element based software FE-Fatigue. At last critical plane approaches are overviewed and used for predict life predictions.

Fash (Fash et al. 1985) performed an experimental investigation for the SAE notched shaft for in phase loading. He also performed a finite element analysis to evaluate the three dimensional stress-strain fields. The obtained strains were used to predict the fatigue life of the SAE notched shaft using effective strain criteria. Maximum principal strain and maximum shear strain criterion were considered. Fash found that the
correlation of these two methods is within a factor of ten. The larger scatter of Fash’s results was caused by the finite element mesh where he only used one element at the notch due to the lack of computer resources.

SAE notched shaft has been reanalyzed in this thesis and existing models are investigated to see which model can best predict for multiaxial loading.

3.2 Geometry and material properties of the SAE notched shaft
Figure 3.1 presents the geometry of the notched shaft. The shaft has the notch with 5 mm radius.

![Diagram of SAE notch shaft geometry and loading](image)

Figure 3.1 SAE notch shaft geometry and loading (Fash et al., 1985)

The shaft is made of SAE 1045 steel in hot rolled and normalized condition. Table 3.1 summarizes monotonic and cyclic parameters of this material.
Table 3.1  SAE 1045 HR material properties (nCode International, 2001)

<table>
<thead>
<tr>
<th>Monotonic Properties</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield strength (0.2%)</td>
<td>380 MPa</td>
</tr>
<tr>
<td>Ultimate strength</td>
<td>621 MPa</td>
</tr>
<tr>
<td>Elastic poisson’s ratio, (v_e)</td>
<td>0.3</td>
</tr>
<tr>
<td>Plastic poisson’s ratio, (v_p)</td>
<td>0.5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Axial cyclic</th>
<th>Torsional cyclic</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K')</td>
<td>1258 MPa</td>
</tr>
<tr>
<td>(n')</td>
<td>0.208</td>
</tr>
<tr>
<td>(\sigma_f')</td>
<td>948 MPa</td>
</tr>
<tr>
<td>(b)</td>
<td>-0.092</td>
</tr>
<tr>
<td>(\varepsilon_f')</td>
<td>0.26</td>
</tr>
<tr>
<td>(c)</td>
<td>-0.445</td>
</tr>
<tr>
<td>(E)</td>
<td>202000 MPa</td>
</tr>
<tr>
<td>(K')</td>
<td>614 MPa</td>
</tr>
<tr>
<td>(n_o')</td>
<td>0.217</td>
</tr>
<tr>
<td>(\tau_f')</td>
<td>505 MPa</td>
</tr>
<tr>
<td>(b_o)</td>
<td>-0.907</td>
</tr>
<tr>
<td>(\gamma_f')</td>
<td>0.413</td>
</tr>
<tr>
<td>(c_o)</td>
<td>-0.445</td>
</tr>
<tr>
<td>(G)</td>
<td>79000 MPa</td>
</tr>
</tbody>
</table>

With all material parameters it is possible to get monotonic and cyclic stress strain curve for this material. Figure 3.2 compares monotonic stress strain curve to cyclic stress strain curve. From this comparison it can be observed that the material is cyclically harden.
Figure 3.2 Stress-strain curves for monotonic tension and axial cyclic

3.3 Finite element analysis of the SAE shaft

The geometry of the SAE notched shaft is modeled in HYPERMESH software (Altair, 2000). Three-dimensional solid elements are used to model the shaft. To enhance accuracy of stress-strain calculation on the model surface, the three-dimensional elements are wrapped with thin shell elements with the same material properties as the solid elements. Although the shell elements increase computational cost of the model, they substantially increase the accuracy of the stress-strain predictions in the surface of the model since fatigue is fundamentally a surface phenomenon. Shells are chosen thin enough (0.0001mm thickness) such that they do not influence the stress-strain response of the solid elements.

In this finite element model six node hexahedral C3D6 and eight node C3D8 elements are chosen as solid elements and two dimensional general purpose shell elements S4 are used
as shell elements. Loading and boundary conditions are shown in Figure 3.1. In the SAE shaft, bending force and torsional loads are applied 150 mm from the notch root. As the model is analyzed up to 50 mm from the notch root, there will be three forces, bending force, bending moment and torsional moment acting on the front face, 50 mm from the notch root. Finite element model created in HYPERMESH is shown in Figure 3.3.

Figure 3.3 Three-dimensional finite element model

Meshing quality is checked through the process of meshing generation within HYPERMESH. Finite element model created in HYPERMESH is then exported in ABAQUS for analysis. Hexahedral solid elements are used in the model as well as in analysis as these elements are generally preferred for most cases because they are usually the most cost-effective elements for this type of analysis. Some wedge elements are also used in the model. It should be noted that all the solid elements and shell elements are used in the model with linear (first order) interpolation.
A linear static finite analysis is carried out using ABAQUS to calculate the local stresses and strains of the notched shaft. In this case, the bending force, bending moment and torque are applied as three separate static load cases. The magnitude of the bending force, bending moment and torque are taken arbitrarily as 1000N, 1000N-mm and 1000N-mm respectively. The static load cases will be combined within FE-Fatigue with actual bending and torque loading histories using the principle of superposition to produce stress histories for all or selected nodes of the model. Figure 3.4 shows a schematic representation of the FE model with all the loads applied.

![Schematic representation of three-dimensional model](image)

**Figure 3.4** A Schematic representation of three-dimensional model  
(Hoffmann and Seeger, 1985)

### 3.4 Fatigue life calculations using FE-Fatigue

FE-Fatigue is a fatigue data analysis tools. It integrates with industry standard Finite Element tools and CAE environments to enable fatigue analysis as part of the design analysis process. FE-Fatigue takes stress or strain data from a finite element (FE) calculation as input, and combines this information together with details of the variation of load with time and cyclic materials data. From this combination, it estimates the
fatigue damage for each supplied node or element on the FE model. FE-fatigue supports stress-life (S-N) method, local strain approach (E-N) and also multiaxial strain-life methods for proportional and non-proportional loading.

3.4.1 Assessing multiaxiality of the problem in FE-Fatigue
FE-Fatigue provides tools to assess the multiaxiality of a given problem by calculating the biaxiality ratio and the angle of the absolute maximum principal stress vector with the X-axis of the planar co-ordinate system. Biaxiality ratio is defined as the ratio of the smaller absolute in-plane principal to the larger. The multiaxiality assessment is based on the examination of the time histories, plots of biaxiality ratio and angle vs. absolute principal stress. The dominant value and the degree of scatter in these plots clearly show whether the problem is uniaxial, proportional or multiaxial and non-proportional.

Case 1: Uniaxial stress state- there is one principal stress which is significantly larger than the second for the whole of the load history and whose angle does not change. No special algorithm corrections need to be applied to convert elastic stresses and strains to elastic-plastic. The Neuber correction alone is sufficient.

Case 2: Proportional Multiaxial stress state- the ratio of the two principal stresses is non-zero, but remains constant for the duration of loading. The angle remains constant also. Procedures should be used to take into account the fact that the loading is non-uniaxial.

Case 3: Non-proportional and Multiaxial stress state- Either the biaxiality ratio or the angle of the maximum principal changes significantly through the time history. Damage models have been developed which can account for non-proportional and multiaxial loading conditions.
3.4.2 Time histories for in-phase loadings

Normalized in phase loading histories are shown in Figure 3.5. The time history is called in-phase as for both bending moment and torque, they have the same maximum and minimum values. The time histories here represent one full cycle. They are scaled during stress superposition within FE-Fatigue according to the applied scale factor.

**Figure 3.5** Normalized in-phase loading histories
3.4.3 Multiaxiality assessment

Multiaxiality assessment for the in-phase loading problem is done as explained in the previous section. Figure 3.6 and 3.7 show for biaxiality ratio distribution and distribution of angle of maximum principal stress with absolute maximum principal stresses over the in-phase loading histories (time histories). The analysis is done with bending moment of 1250 Nmm and torque of 880 Nmm.

\[ \text{Time range: 0 secs to 3 secs} \]

\[ \text{Absolute maximum principal stress} \]

\[ \text{Biaxiality ratio (No units)} \]

\[ \text{Screen 1} \]

**Figure 3.6** Biaxiality distribution over the time history

It is observed that the biaxiality ratio and the angle distribution are constant over the time history. Therefore the problem is multiaxial proportional. In the Previous section it was mentioned that for proportional loading the strain-based life prediction models are applicable. This will be investigated in this research work.
Figure 3.7 Angle distribution over the time history

3.5 Life prediction of SAE shaft under multiaxial in-phase (proportional) loading using strain based approach

The validities of strain based criteria are investigated by predicting the fatigue lives of the SAE notch shaft under different proportional multiaxial loadings. Absolute maximum principal strain, maximum shear strain and von Mises strain criteria are considered in this work. The predicted fatigue lives by FE-Fatigue are compared to experimental fatigue lives. FE-Fatigue uses Neuber correction to correct the elastic stresses and strains to elastoplastic.

3.5.1 Life prediction using absolute maximum principal strain

Here, the maximum principal strain is extracted to predict the fatigue life by using the following equation.
\[
\frac{\Delta \varepsilon_1}{2} = \frac{\sigma_f'}{E} (2N)^b + \varepsilon_f' (2N)^c
\]  
(3.1)

Parameters \( \sigma_f', b, \varepsilon_f' \), and \( c \) are the usual material constants.

### 3.5.2 Life prediction using maximum shear strain

Maximum shear strain criterion is also used here for fatigue life calculations. The equation is

\[
\frac{\Delta \gamma_{\text{max}}}{2} = 1.3 \frac{\sigma_f'}{E} (2N)^b + 1.5 \varepsilon_f' (2N)^c
\]  
(3.2)

### 3.5.3 Life prediction using von Mises strain criterion

Octahedral shear strain or von Mises strain criteria needs to determine and equivalent strain which is then used to predict fatigue life. The equivalent strain in terms of principal stress is defined as follows:

\[
\varepsilon_{eq} = A \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{0.5}
\]  
(3.3)

where \( A \) is given by

\[
A = 0.707 (1/(1 + \nu^*))
\]

and \( \nu^* \) is the elastic-plastic Poisson’s ratio defined by

\[
\nu^* = (\nu_c \varepsilon_c + \nu_p \varepsilon_p) / \varepsilon_i
\]

where \( \varepsilon_i = \varepsilon_c + \varepsilon_p \)

The calculated values of the equivalent strain can be used directly with the uniaxial strain life equation to get fatigue life

\[
\varepsilon_{eq} = \frac{\sigma_f'}{2} (2N_f)^b + \varepsilon_f' (2N_f)^c
\]  
(3.4)

Calculated predicted fatigue lives are compared with experimental cycles for these three strain life models are shown in Table 3.2. Predicted cycles versus experimental cycles are depicted in Figure 3.8.
Table 3.2 Life predictions by strain-life criteria

<table>
<thead>
<tr>
<th>Moment (N.m) (B)</th>
<th>Torsion (N.m) (T)</th>
<th>Experimental cycles</th>
<th>Absolute maximum principal strain criterion</th>
<th>Maximum shear strain criterion</th>
<th>von Mises Strain criterion</th>
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Figure 3.8 Experimental fatigue life vs. predicted life cycles for strain-life criteria

(a) Absolute maximum principal strain criterion
(b) Maximum shear strain criterion
(c) von Mises strain criterion

3.5.4 General discussion on strain-life criteria
Using strain criteria, life predictions for different multiaxial proportional loadings are carried out by FE-Fatigue and then compared with the experimental reported results. Experimental fatigue lives versus numerical estimated fatigue lives of the SAE notched shaft, have been shown for strain life criteria. It is seen that there is not any fixed strain life criterion that can predict the fatigue life best for different combinations of torsion and bending loadings.

- Absolute principal strain life method works fine when the biaxiality ratio is positive and constant for the time history. Absolute principal strain criterion cannot predict well when torsion dominates over bending moment and so for pure torsion case this criterion gives very poor estimations.
- Maximum shear strain criterion can predict very well when shear strain dominate over normal strain such as for torsion-dominated cases. For pure torsion case maximum shear criterion is the best criterion. For all other loading cases, maximum shear strain criterion gives conservative results.

- von Mises strain criterion takes the effects of all strain components present in the analysis and so deliver results with better accuracy than the other two strain life criteria.

Figure 3.9 below shows a comparison of life predictions between all the three strain life criteria with experimental life cycles.

![Graph comparing life predictions between different strain life criteria](image)

**Figure 3.9** Comparison between strain life criteria
Strain life criteria use Neuber correction to correct the elastic stresses and strains to elastoplastic. It is seen that Neuber correction alone is insufficient as the cyclic stress-strain curve is based on uniaxial data. Hoffmann and Seeger suggested a method for extending use of the Neuber correction to multiaxial loading. This approach is investigated in the following paragraph.

3.5.5 Hoffmann and Seeger method

Hoffmann and Seeger suggested a method for extending the use of the Neuber correction to multiaxial loading, subjected to the following assumptions:

1. The orientations of principal stress and strain axes are fixed.
2. The ratio of the in-plane principal strains is constant.
3. The uniaxial stress-strain curve can be extended for use with suitable equivalent stress and strain parameters such as the von Mises parameters.

The following steps are used:

1. Elastic values of the signed Von mise’s stress and strain are calculated by Finite element analysis. An equivalent stress using the von Mises criterion is defined.

\[ \sigma^{eq} = \sqrt{\frac{1}{2}(\sigma_1 - \sigma_2)^2 + \sigma_1^2 + \sigma_2^2} \]

2. Calculate the biaxiality ratio

\[ a = \frac{\sigma_2}{\sigma_1} = \frac{\varepsilon_2 + \nu}{\varepsilon_1} \]

3. Calculate equivalent strain

\[ \varepsilon^{eq} = \frac{\sigma^{eq}}{E} \]

4. Equivalent Poisson’s ratio

\[ \nu = \frac{1}{2} \left( \frac{1}{2} - \nu \right) \frac{\sigma^{eq}}{E \varepsilon^{eq}} \]
(5) Principal stress and strain are then calculated as

\[ \sigma_1 = \frac{1}{\sqrt{1 - \alpha + \alpha^2}} \sigma^{eq} \quad \text{and} \quad \varepsilon_1 = \frac{1 - \nu \alpha}{\sqrt{1 - \alpha + \alpha^2}} \varepsilon^{eq} \]

Once principal stress and principal strain are found for the critical locations, uniaxial strain-life equation can be used to predict fatigue life.

\[ \varepsilon_1 = \frac{\sigma_1^f}{2} (2N_f)^b + \varepsilon_1^f (2N_f)^c \]

(3.5)

Calculated predicted fatigue lives, and experimental cycles are shown in Table 3.3

**Table 3.3** Life predictions by Hoffmann and Seeger method

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<thead>
<tr>
<th>Moment (N.m) (B)</th>
<th>Torsion (N.m) (T)</th>
<th>Predicted Cycles (FE-Fatigue)</th>
<th>Experimental Cycles</th>
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<td>780</td>
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**Pure Torsion**

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</table>

63
Figure 3.10 Comparison between predicted and experimental fatigue life cycles for Hoffmann and Seeger method.
General discussion on Hoffmann and Seeger method

Hoffmann and Seeger method can be applied for proportional loading. This method is an extension of strain life criteria with some simplified assumptions to deal with notch effect in the structural component. While maximum shear strain method is conservative when predicting for proportional loading and again absolute maximum principal strain criterion cannot predict fatigue life well when shear stress dominates, Hoffmann-Seeger version of strain life criterion is the best choice for fatigue life prediction for proportional loading.

If the loading is non-proportional then most of the assumptions implicit in the Hoffmann and Seeger method are violated. Critical plane models may be used in that case. This will also be investigated in this research work. As fatigue crack initiation is a directional process and so critical plane approaches will also be able to best predict for multiaxial proportional loadings.

3.6 Critical plane methods

Critical plane approach is necessary to use when the stress tensor is mobile, that means when the principal stress direction rotates and also the biaxiality ratio varies over the time histories of the loads applied in the model. Critical plane should not be confused with the crack plane. Cracks typically start in shear mode and after a transition period grow in opening mode (mode I). The period, crack initiation life may include growth in a number of planes. The plane with the largest accumulated damage is said to be the critical plane. The local strain approach does not model cracks explicitly but attempts to predict initiation on the basis of bulk stress and strain parameters. The critical plane is simply the plane on which the stress and strain parameters are calculated. This method is used to rotate the co-ordinate system of the stresses and strains by means of a tensor rotation such that the x-y plane of the new co-ordinate system lies in the critical plane.
(a) Critical plane with normal strain criterion

In this approach strain amplitude normal to the critical plane, \( \frac{\Delta \varepsilon_n}{2} \), is calculated. Fatigue life is then predicted using

\[
\frac{\Delta \varepsilon_n}{2} = \frac{\sigma'_f}{E} (2N_f)^b + \varepsilon'_f (2N_f)^c
\]  

(3.6)

Calculated predicted fatigue lives, and experimental cycles for normal strain criterion is shown in Table 3.4.

**Table 3.4 Predicted fatigue life cycles based on normal strain criterion**

<table>
<thead>
<tr>
<th>Moment (N.m) (B)</th>
<th>Torsion (N.m) (T)</th>
<th>Predicted Cycles (FE-Fatigue)</th>
<th>Experimental Cycles</th>
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Following Figure 3.11 shows the comparison between predicted life cycles and experimental life cycles.

**Figure 3.11** Comparison between predicted and experimental fatigue life cycles for critical plane with normal strain criterion.
(b) Critical plane with Shear strain criterion

In this model shear strain amplitude, \( \frac{\Delta \gamma}{2} \), is calculated to along the critical plane. Fatigue life is then calculated with the following equation

\[
\frac{\Delta \gamma}{2} = \frac{(1 + \nu_e)\sigma_f'}{E} (2N_f)^s + (1 + \nu_p)\varepsilon_f'(2N_f)^c
\]  

(3.7)

Calculated predicted fatigue lives, and experimental cycles for shear strain criterion is shown in Table 3.5 and Figure 3.12 show a comparison between predicted and experimental life cycles.

**Table 3.5** Predicted fatigue life cycles based on critical plane approach with shear strain criterion

<table>
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<tr>
<th>Moment (N.m) (B)</th>
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Figure 3.12 Comparison between predicted and experimental fatigue life cycles for critical plane with shear strain criterion.
(c) Critical plane (Fatemi and Socie model)

Fatemi and Socie proposed a model, which is
\[
\frac{\Delta \gamma}{2} \left(1 + n \frac{\sigma_{n,\text{max}}}{\sigma_y}\right) = \frac{\tau'}{G} (2N_f)^b + \gamma' (2N_f)^c
\]

where \( \frac{\Delta \gamma}{2} \) is the largest amplitude of shear strain for the critical plane.

Calculated predicted fatigue lives, and experimental cycles for shear strain criterion is shown in Table 3.6 and Figure 3.13 shows a comparison between predicted and experimental life cycles.

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<tr>
<th>Moment (N.m) (B)</th>
<th>Torsion (N.m) (T)</th>
<th>Predicted Cycles (FE-Fatigue)</th>
<th>Experimental Cycles</th>
</tr>
</thead>
<tbody>
<tr>
<td>1460</td>
<td>0</td>
<td>164400</td>
<td>382000</td>
</tr>
<tr>
<td>1875</td>
<td>0</td>
<td>40439</td>
<td>43540</td>
</tr>
<tr>
<td>2600</td>
<td>0</td>
<td>7950</td>
<td>5676</td>
</tr>
<tr>
<td>T/B=0.7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1250</td>
<td>880</td>
<td>241900</td>
<td>325000</td>
</tr>
<tr>
<td>1550</td>
<td>1090</td>
<td>68670</td>
<td>97500</td>
</tr>
<tr>
<td>T/B=1.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>990</td>
<td>1390</td>
<td>246200</td>
<td>716382</td>
</tr>
<tr>
<td>1220</td>
<td>1710</td>
<td>71843</td>
<td>72000</td>
</tr>
<tr>
<td>T/B=1.9</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1355</td>
<td>2550</td>
<td>15870</td>
<td>5500</td>
</tr>
<tr>
<td>T/B=2.8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>845</td>
<td>1800</td>
<td>125200</td>
<td>200000</td>
</tr>
<tr>
<td>780</td>
<td>2180</td>
<td>62800</td>
<td>70681</td>
</tr>
<tr>
<td>T/B=3.6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>570</td>
<td>2180</td>
<td>80293</td>
<td>80287</td>
</tr>
<tr>
<td>851</td>
<td>2700</td>
<td>22735</td>
<td>10000</td>
</tr>
<tr>
<td>Pure Torsion</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>2000</td>
<td>239500</td>
<td>700000</td>
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<td>75700</td>
</tr>
<tr>
<td>0</td>
<td>3000</td>
<td>26334</td>
<td>6329</td>
</tr>
</tbody>
</table>
Figure 3.13 Comparison between predicted and experimental fatigue life cycles for critical Plane (Fatemi and Socie model ).
(d) **Critical Plane (Smith, Watson and Topper (SWT) model)**

SWT approach for multiaxial loading is based on the strain amplitude normal to the critical plane, \( \frac{\Delta \varepsilon_n}{2} \) and maximum normal stress on the critical plane, \( \sigma_{n,\text{max}} \).

\[
\frac{\Delta \varepsilon_n}{2} \sigma_{n,\text{max}} = \frac{\sigma_f'\varepsilon_f'}{E} (2N_f)^{2b} + \sigma_f' \varepsilon_f' (2N_f)^{b+c}
\] (3.9)

Calculated predicted fatigue lives, and experimental cycles for shear strain criterion is shown in Table 3.7 and Figure 3.14 shows a comparison between predicted and experimental life cycles.

**Table 3.7** Predicted fatigue life cycles based on SWT model

<table>
<thead>
<tr>
<th>Moment (N.m) (B)</th>
<th>Torsion (N.m) (T)</th>
<th>Predicted Cycles (FE-Fatigue)</th>
<th>Experimental Cycles</th>
</tr>
</thead>
<tbody>
<tr>
<td>1460</td>
<td>0</td>
<td>181000</td>
<td>382000</td>
</tr>
<tr>
<td>1875</td>
<td>0</td>
<td>44359</td>
<td>43540</td>
</tr>
<tr>
<td>2600</td>
<td>0</td>
<td>8742</td>
<td>5676</td>
</tr>
<tr>
<td></td>
<td></td>
<td>T/B=0.7</td>
<td></td>
</tr>
<tr>
<td>1250</td>
<td>880</td>
<td>252900</td>
<td>325000</td>
</tr>
<tr>
<td>1550</td>
<td>1090</td>
<td>71755</td>
<td>97500</td>
</tr>
<tr>
<td></td>
<td></td>
<td>T/B=1.4</td>
<td></td>
</tr>
<tr>
<td>990</td>
<td>1390</td>
<td>321300</td>
<td>716382</td>
</tr>
<tr>
<td>1220</td>
<td>1710</td>
<td>91206</td>
<td>72000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>T/B=1.9</td>
<td></td>
</tr>
<tr>
<td>1355</td>
<td>2550</td>
<td>24138</td>
<td>5500</td>
</tr>
<tr>
<td></td>
<td></td>
<td>T/B=2.8</td>
<td></td>
</tr>
<tr>
<td>845</td>
<td>1800</td>
<td>241700</td>
<td>200000</td>
</tr>
<tr>
<td>780</td>
<td>2180</td>
<td>151300</td>
<td>70681</td>
</tr>
<tr>
<td></td>
<td></td>
<td>T/B=3.6</td>
<td></td>
</tr>
<tr>
<td>570</td>
<td>2180</td>
<td>260400</td>
<td>80287</td>
</tr>
<tr>
<td>851</td>
<td>2700</td>
<td>54442</td>
<td>10000</td>
</tr>
<tr>
<td>Pure Torsion</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>2000</td>
<td>3931000</td>
<td>700000</td>
</tr>
<tr>
<td>0</td>
<td>2400</td>
<td>964200</td>
<td>75700</td>
</tr>
<tr>
<td>0</td>
<td>3000</td>
<td>209900</td>
<td>6329</td>
</tr>
</tbody>
</table>

72
Figure 3.14 Comparison between predicted and experimental fatigue life for critical plane (SWT model).
To compare critical plane approaches, fatigue life cycles for multiaxial proportional loading, absolute difference of experimental and predicted fatigue life cycles are determined for each critical plane approaches and for all multiaxial loading conditions. Table 3.8 presents comparison for different approaches. All the critical plane approaches are compared in Figure 3.15, against the experimental life cycles for different loading conditions.

**Table 3.8** Relative differences in life cycles from experimental fatigue life for critical plane approaches.

| Pure Bending |  |  |  |
|--------------|--------------|-----------------|-----------------|-----------------|
| Moment (N.m) (B) | Torsion (N.m) (T) | Experimental cycles | Absolute difference in life cycles |
| | | | Fatemi and Socie | Smith, Watson and Topper | Normal Strain | Shear Strain |
| 1460 | 0 | 382000 | 217600 | 201000 | 146500 | 211500 |
| 1875 | 0 | 43540 | 3101 | 819 | 12718 | 1546 |
| 2600 | 0 | 5676 | 2274 | 3066 | 5148 | 2594 |
| T/B=0.7 | | | | | |
| 1250 | 880 | 325000 | 83100 | 72100 | 33500 | 78900 |
| 1550 | 1090 | 97500 | 28830 | 25745 | 15901 | 27541 |
| T/B=1.4 | | | | | |
| 990 | 1390 | 716382 | 470182 | 395082 | 426582 | 477082 |
| 1220 | 1710 | 72000 | 157 | 19206 | 10835 | 1241 |
| T/B=1.9 | | | | | |
| 1355 | 2550 | 5500 | 10370 | 18638 | 13852 | 9421 |
| T/B=2.8 | | | | | |
| 845 | 1800 | 200000 | 74800 | 41700 | 26000 | 92600 |
| 780 | 2180 | 70681 | 7881 | 80619 | 21873 | 22993 |
| T/B=3.6 | | | | | |
| 570 | 2180 | 80287 | 6 | 180113 | 51413 | 22391 |
| 851 | 2700 | 10000 | 12735 | 44442 | 22194 | 7337 |

Pure Torsion

| | | | | | |
|--------------|--------------|-----------------|-----------------|
| 0 | 2000 | 700000 | 460500 | 3231000 | 2200 | 584600 |
| 0 | 2400 | 75700 | 8801 | 888500 | 116900 | 32958 |
| 0 | 3000 | 6329 | 20005 | 203571 | 42213 | 7536 |
Figure 3.15 Comparison between critical plane approaches

3.7 Life Prediction of SAE shaft under multiaxial non-proportional loading

3.7.1 Time histories for 90° out of phase (Non-proportional) loading

The stresses results from the static load sub-cases are divided by the values of static loads applied to the FEA model (1000 N for the bending force, 1000 N.mm for bending moment and also 1000 N.mm for the torque). Normalized 90° out of phase loading histories are shown in Figure 3.16. They are scaled during stress superposition within FE-Fatigue.
Figure 3.16 Normalized 90° out of phase loading histories

3.7.2 Multiaxiality assessment for non-proportional loading

Multiaxiality assessment for 90° out of phase loading (non-proportional loading) problem is done in FE-Fatigue. Figure 3.17 shows biaxiality ratio distribution and distribution of angle of maximum principal stress with absolute maximum principal stresses over 90° out phase loading histories (time histories). The analysis is done with bending moment of 1850 N.mm and torque of 2100 N.mm applied in the model.
Figure 3.17 Multiaxiality assessment for non-proportional loading

It is seen that biaxiality ratio and angle distribution are mobile over the time history and so the problem is multiaxial non-proportional. From literature, it was concluded that only critical plane approaches can deal with non-proportional loading.
3.7.3 Critical plane approaches for non-proportional loading

There are some components where a combination of loads and geometric effects generates local loadings, which are not proportional. In plane biaxiality ratio varies over the time history of the cyclic loading and also does the angle of maximum principal stress. So, uniaxial strain-life methods and also Hoffmann and Seeger method with simplified approximations are no longer applicable to for these non-proportional multiaxial loading. From physical observation it is seen that fatigue crack initiation is a directional process and therefore critical plane approaches are there which should predict fatigue life better for non-proportional multiaxial loadings over all other existing strain-life methods.

In the following table, fatigue life predictions for different multiaxial non-proportional loading conditions are carried out for different critical plane approaches. The predicted fatigue lives by FE-Fatigue are compared to experimental fatigue lives that are reported in the literature (Leese and Socie, 1989).

**Table 3.9 Life prediction for multiaxial non-proportional loading conditions.**

<table>
<thead>
<tr>
<th>Moment (N-m)</th>
<th>Torsion (N-m)</th>
<th>Experimental</th>
<th>Fatemi and Socie</th>
<th>Normal Strain</th>
<th>Shear Strain</th>
<th>Smith, Watson and Topper</th>
</tr>
</thead>
<tbody>
<tr>
<td>1150</td>
<td>2700</td>
<td>13110</td>
<td>16711</td>
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<td>24078</td>
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<td>1850</td>
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<td>67695</td>
<td>51249</td>
<td>42463</td>
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<td>1698</td>
<td>2242</td>
<td>10840</td>
<td>33502</td>
<td>1.097E5</td>
<td>69117</td>
<td>66029</td>
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<td>1325</td>
<td>23980</td>
<td>14732</td>
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<td>16047</td>
</tr>
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<td>770</td>
<td>2180</td>
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<td>3.62E5</td>
<td>73507</td>
<td>1.013E6</td>
</tr>
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<td>1295</td>
<td>1710</td>
<td>45580</td>
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<tr>
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<td>1710</td>
<td>213800</td>
<td>1.606E5</td>
<td>9.053E5</td>
<td>3.301E5</td>
<td>4.63E5</td>
</tr>
</tbody>
</table>
A quantitative comparison among all the critical plane approaches for $90^0$ out of phase loading is shown in the following figures.

**Figure 3.18** Experimental vs. predicted life cycles for $90^0$ out of phase loading (Fatemi and Socie model)

**Figure 3.19** Experimental vs. predicted life cycles for $90^0$ out of phase loading (Normal strain criterion)
Figure 3.20 Experimental vs. predicted life cycles for 90° out of phase loading (Shear strain criterion)

Figure 3.21 Experimental vs. predicted life cycles for 90° out of phase loading (SWT model).
To compare critical plane approaches for multiaxial non-proportional loading, relative
difference of experimental and predicted fatigue life cycles are determined for all critical
plane approaches, Table 3.10 and then all the critical plane approaches are compared in
Figure 3.22 with respect to the experimental life cycles for different loading conditions.

**Table 3.10** Relative difference in life cycles from experimental fatigue life for
critical plane approaches for 90° out of phase loadings.

<table>
<thead>
<tr>
<th>Moment (N-m)</th>
<th>Torsion (N-m)</th>
<th>Experimental</th>
<th>Relative difference in life cycles</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Fatemi and Socie</td>
</tr>
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</tr>
<tr>
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<td>27470</td>
<td>15049</td>
</tr>
<tr>
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</tr>
<tr>
<td>1220</td>
<td>1710</td>
<td>213800</td>
<td>53200</td>
</tr>
</tbody>
</table>
**Figure 3.22** Comparison among critical plane approaches for $90^\circ$ out of phase (non-proportional) loadings.

### 3.7.4 Conclusion

The present study shows that critical plane models perform better than other approaches for both proportional and non-proportional loading.

Like maximum shear strain criterion in strain-life method, critical plane approach based on shear strain criterion has good performance in low-cycle fatigue region. Among all the critical plane models, shear strain criterion is found conservative in predicting the fatigue lives. This is especially noticeable in pure torsion tests.

Critical plane approach based normal strain criterion deals with the normal strain amplitude normal to the critical plane. Again fatigue cracks usually start in a shear mode. So life prediction with this criterion cannot deliver good results when applied for torsion tests. For pure bending or when the normal stress-strain is a major part of the critical
stress-strain developed in the notched component due to multiaxial loading, normal strain criterion works very well.

From the comparison among all the critical plane approaches, it is seen that Fatemi and Socie model is the best one in predicting for both proportional and non-proportional loadings. This is because this model takes the shear strain along the critical plane along with the maximum tensile stress normal to the plane of largest shear strain taken.

SWT model predicts life well when the normal stress and strain are more than the shear stress-strain in the critical region of the structural component. This model cannot predict well for pure torsion tests as the model is developed by assuming that the fail is predominantly by crack growth on planes of maximum tensile stress or strain. This approach overestimates the fatigue lives in low-cycle fatigue drastically.

Critical plane approaches are in better position in predicting fatigue life for proportional loading than for non-proportional and also it is clear from data analyses and graphical comparisons that Fatemi and Socie model delivers good results in compare to other critical plane criteria for non-proportional loading.
CHAPTER 4

DAMAGE MECHANICS APPROACH FOR CRACK INITIATION FOR LOW CYCLE FATIGUE

4.1 Introduction

Continuum damage mechanics is the branch in mechanics that deals with physical process involving deterioration of a component. This theory was originated by Kachanov (1958) to describe creep induced damage. The mechanism of deterioration is modeled using internal variables. In the simplest case, the internal variable, denoted as $D$ is scalar which varies between 0 for a virgin state and 1 for ultimate state where the material loose any capability of transmitting stresses. During the last two decades, damage mechanics has gained popularity and become a systematic theory based on a general framework of thermodynamics and continuum medium mechanics. It has been applied to model various phenomena starting from creep, ductile plastic damage, and brittle damage. The application of damage mechanics for fatigue problems is still an active area of research since 1974.

This chapter deals with the study of a damage mechanics model for the fatigue life assessment of the SAE shaft. The applicability of this model for multiaxial low cycle fatigue is discussed. The first part of this chapter summarizes the basic equation of this model, emphasizing on the coupling between plasticity and damage. The identification of the model parameters is discussed in the second part of this chapter with the application to Steel SAE 1045 HR. In the last part, the numerical procedure used to implement the model is summarized and the simulation predictions are compared with the experimental results.
4.2 Constitutive equation of the elasto-plastic-damage coupled model

4.2.1 Thermodynamics of elasto-plastic-damage coupled models

It is well known that low cycle fatigue of metals is governed by its elasto-plastic properties. The idea to include damage variable within the elasto-plastic constitutive equations aims to generalize this efficient model to take into account for the deterioration accumulated by the fatigue process.

The framework of the thermodynamics of irreversible process is particularly suitable for developing a complex model such as the one coupling between elasticity, plasticity and damage. This approach is based on the definition of local state variables. Each variable is associated to a particular phenomenon. The state potential, written as a function of the state variables, defines the state laws.

Definition of internal variables

1. Elastic strains, $\varepsilon^e$ is the strain tensor of component $\varepsilon^e_{ij}$ associated with the stress tensor $\sigma$.

2. $\varepsilon^p$ the plastic strain tensor of component $\varepsilon^p_{ij}$ defined from the partitioning of the total strain.

$$\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij} \quad (4.1)$$

3. Damage variable

   In the classical sense the damage variable, $D$ is defined as follows

$$D = \frac{A - \tilde{A}}{A} = \frac{A_D}{A} \quad (4.2)$$

where $A$ is the overall cross-sectional area and $\tilde{A}$ is the effective cross-sectional area which efficiently bears the load. It follows from this definition that $A_{ni}$ is the area of defects accumulated during the damage process.
where $A$ is the overall cross-sectional area and $\tilde{A}$ is the effective cross-sectional area which efficiently bears the load. It follows from this definition that $A_D$ is the area of defects accumulated during the damage process. One can conclude that

$$D = 0; \text{ corresponds to the undamaged state.}$$

$$D = D_c \leq 1; \text{ corresponds to the ultimate failure state.}$$

This definition depends on the orientation of the surface area, $A$ in the space. Within the context of isotropic damage theory it is assumed that this definition of damage is dependent of the orientation. The simplification leads to the theory of the isotropic damage, by considering that $D$ is independent of the orientation.

It appears natural to introduce the concept of effective stress $\tilde{\sigma}$ as the stress that is effectively transmitted through the section $\tilde{A}$ as

$$\tilde{\sigma} = \frac{P}{\tilde{A}} = \frac{P}{A(1-D)} = \frac{\sigma}{1-D} \quad (4.3)$$

Lemaitre has defined the principle of strain equivalence, which states that (Lemaitre, 1992)
Any strain constitutive equation for a damaged material may be derived in the same way as for a virgin material except that the usual stress is replaced by the effective stress.

Then if we consider Hooke’s law in one-dimensional case, one can write

$$
\varepsilon^e = \frac{\sigma}{E_o} = \frac{\sigma}{E_o(1-D)} = \frac{\sigma}{E_D}
$$

(4.4)

where \(\varepsilon^e\) is the elastic strain and \(E_o\) is the Young’s modulus of the undamaged material. If the denominator \(E_o(1-D)\) is replaced by \(E_D\), which represents the Young’s modulus of the damaged material, it follows that

$$
D = 1 - \frac{E_D}{E_o}
$$

(4.5)

where \(E_o\) is the effective Young’s modulus corresponding to \(D = 0\) (Lemaitre, 1992)

### 4.2.2 Kinetic constitutive equations

Within the framework of the thermodynamics of irreversible process, it is postulated that energy potentials exist from which one can derive state laws and the kinetic constitutive laws. For a model including elasticity, plasticity and damage, the free energy \(\psi\) is a continuous scalar function, convex with the state variables. In this study, we consider the case of ideal plasticity where no hardening is considered. In the case of isothermal case the free energy is

$$
\psi = \psi(\varepsilon, \varepsilon^e, \varepsilon^p, D)
$$

(4.6)

For elasto-plasticity, the partitioning of strains allows simplifying the free energy to include only \(\varepsilon^e = \varepsilon - \varepsilon^p\) and can be expressed simply as follows:

$$
\psi = \psi(\varepsilon^e, D)
$$

(4.7)

By extending the expression of free energy for purely elastic material, one can write

$$
\psi = \frac{1}{\rho} \left[ \frac{1}{2} a_{ijkl} \varepsilon_{ijkl} (1-D) \right]
$$

(4.8)
where $a_{ijkl}$ is the fourth tensor order elasticity operator. $\rho$ is the density of the material. Then, the elasticity law coupled with damage is obtained by the second thermodynamic principle

$$\sigma_{ij} = \rho \frac{\partial \psi}{\partial \varepsilon_{ij}^e} = a_{ijkl} \varepsilon_{kl}^e (1 - D)$$  \hspace{1cm} (4.9)

By inversion, and considering a homogeneous elasto-plastic-damageable material, we obtain

$$\varepsilon_{ij}^e = \frac{1 + \nu}{E} \frac{\sigma_{ij}}{1 - D} - \frac{\nu}{E} \frac{\sigma_{kk}}{1 - D} \delta_{ij}$$  \hspace{1cm} (4.10)

where $\nu$ is the Poisson’s ratio for undamaged state and $\delta_{ij}$ is the kronecker delta.

Since $D$ appears in the free energy form, one can define an associated variable, $Y$, defined as

$$Y = -\bar{Y} = \rho \frac{\partial \psi}{\partial D} = \frac{1}{2} a_{ijkl} \varepsilon_{ij}^e \varepsilon_{kl}^e$$  \hspace{1cm} (4.11)

In order to relate $Y$ with the elastic strain energy density $w_e$, let’s recall that

$$dw_e = \sigma_{ij} d\varepsilon_{ij}^e$$  \hspace{1cm} (4.12)

Integrating with the law of elasticity, and assuming no variation of damage, that is, $D =$ constant, yields

$$w_e = \int a_{ijkl} \varepsilon_{ij}^e (1 - D) d\varepsilon_{ij}^e = Y(1 - D)$$  \hspace{1cm} (4.13)

It follows that

$$Y = \frac{w_e}{1 - D}$$  \hspace{1cm} (4.14)

$Y$ is called the strain energy density release rate. This is the energy released by loss of stiffness in the material.

In the case of metal plasticity, the stress and strain can be splitted into deviatoric and hydrostatic parts.

$$\sigma_{ij} = \sigma_{ij}^D + \sigma_{ij}^{H} \delta_{ij}$$  \hspace{1cm} (4.15)
\[ \varepsilon^e_y = \varepsilon^{\text{el}}_y + \varepsilon^d_H \delta_y \]  

(4.16)

\( \sigma_H \) and \( \varepsilon^d_H \) are the hydrostatic stress and strains: \( \sigma_H = \frac{1}{3} \sigma_{kk}, \varepsilon^d_H = \frac{1}{3} \varepsilon^d_{kk} \).

For linear isotropic elasticity, coupled with damage,

\[ \varepsilon^{\text{el}}_y = \frac{1 + \nu \sigma^D_y}{E} \frac{\sigma^D_y}{1 - D}, \varepsilon^d_H = \frac{1 - 2\nu}{E} \frac{\sigma^D_H}{1 - D} \]  

(4.17)

Taking \( \delta_y^d \delta_y^d = 3 \)

\[ w_e = \frac{1}{2} \left( \frac{1 + \nu \sigma^D_y}{E} \frac{\sigma^D_y}{1 - D} + 3 \frac{1 - 2\nu}{E} \frac{\sigma^D_H}{1 - D} \right) \]  

(4.18)

Introducing von Mises equivalent stress defined as

\[ \sigma_{eq} = \left( \frac{3}{2} \sigma^D_y \sigma^D_y \right)^{1/2} \]  

(4.19)

Which leads \( Y \) to be equal to

\[ Y = \frac{w_e}{1 - D} = \frac{\sigma^2_{eq}}{2E(1 - D)^2} \left[ \frac{2}{3} (1 + \nu) + 3(1 - 2\nu) \left( \frac{\sigma_H}{\sigma_{eq}} \right)^2 \right] \]  

(4.20)

\[ = \frac{\sigma^2_{eq}}{2E(1 - D)^2} R_v \]

The factor \( R_v \) introduced is called the “triaxiality ratio.”

For identification purpose, it is interesting to define the one dimensional stress \( \sigma^* \) which, that for the same value of the damage, yields the same value of the elastic strain energy density as that of a three dimensional state. It can be shown that (Lemaitre 1992)

\[ \sigma^* = \sigma_{eq} R_v^{1/2} \]  

(4.21)

4.3 Kinetic law of damage evolution

To complete the formulation of the constitutive model, the kinematic laws modeling plasticity and damage evolution is to be defined. The laws of plasticity coupled with
damage are derived from a potential of dissipation $F$. In order to define the potential, the following hypotheses are made ((Lemaitre and Dufailly, 1994).

- Small deformation
- Uncoupled partitioning of the total strain in elastic and plastic strain

\[ \varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p \]  \hspace{1cm} (4.22)

- Linear isotropic elasticity
- Perfect plasticity (no hardening).

The later hypothesis requires justification. In most cases and especially for LCF problems damage develops when accumulated plastic strain is large enough, to consider the strain hardening saturated. The plasticity limits are defined as

\[ \sigma_f \leq \sigma_S \leq \sigma_u \]  \hspace{1cm} (4.23)

where

- $\sigma_f$ = fatigue limit; $\sigma_S$ = plastic threshold; and $\sigma_u$ = ultimate strength.

The plastic constitutive equations are derived in the classical way. However the yield function is defined in terms of the effective stress

\[ f = \tilde{\sigma}_{eq} - \sigma_S = 0 \]  \hspace{1cm} (4.24)

\[ \tilde{\sigma}_{eq} = \frac{\sigma_{eq}}{1 - D} \]  \hspace{1cm} (4.25)

The plastic strain rate is derived using the normality rule

\[ \dot{\varepsilon}_{ij}^p = \frac{\partial F}{\partial \sigma} = \frac{3}{2} \frac{\sigma_{ij}^D}{\sigma_{eq}} \frac{\dot{\lambda}}{1 - D} \]  \hspace{1cm} (4.26)

where $\dot{\lambda}$ is the plastic multiplier.

The accumulated plastic strain is defined as

\[ \dot{p} = \left( \frac{2}{3} \dot{\varepsilon}_{ij}^p \dot{\varepsilon}_{ij}^p \right)^{1/2} = \frac{\dot{\lambda}}{1 - D} \]  \hspace{1cm} (4.27)

To derive the kinetic law of damage evolution, a dissipation term related to damage is added to obtain the following expression of the problem of the potential of dissipation $F$.  

90
\[ F = f + \frac{Y^2}{2S(1-D)} \]  
(4.28)

where \( S \) is defined as the damage strength.

The damage evolution law is given by

\[ \dot{D} = \frac{\partial F}{\partial Y} \dot{\lambda} = \frac{Y}{S} \frac{\dot{\lambda}}{1-D} = \frac{Y}{S} \dot{p}, \text{ if } p \geq p_D \]  
(4.29)

\( p_D \) is a damage threshold at which micro-crack nucleation is expected to start.

The system state is known if the accumulated plastic strain is obtained. The numerical analysis of this problem is therefore reduced to a classical integration of the elasto-plastic constitutive law. To complete the model, the critical value of damage at the mesoscale \( D_c \) has to be evaluated. Stating that the energy dissipated by damage is constant, one can write

\[ \int_{0}^{D_c} \! Y dD = \text{constant at failure} \]

Using the expression for the strain energy density release rate, \( Y = \frac{\sigma^2_{S} R_c}{2E} D_c \) and considering proportional loading for which \( R_c = \text{constant} \), we obtain following expression.

\[ \int_{0}^{D_c} \! Y dD = \int_{0}^{D_c} \frac{\sigma^2_{S} R_c}{2E} dD = \frac{\sigma^2_{S} R_c}{2E} D_c \]  
(4.30)

By taking the pure tension test as a reference which gives the critical value of the damage at the mesocrack initiation, \( D_{1c} \), related to the stress to rupture \( \sigma_R \)

\[ \frac{\sigma^2_{S} R_c}{2E} D_c = \frac{\sigma^2_u}{2E} D_{1c} \]  
(4.31)

that gives

\[ D_c = D_{1c} \frac{\sigma^2_u}{\sigma^2_{S} R_c} \]  
(4.32)
4.4 Fatigue crack initiation for multiaxial loading
It is quite often that the damage is so localized and so the volume of the damage material is small in comparison to the macroscale of the structural component and even to the RVE. This allows to perform an uncoupled analysis at the macroscale and to consider the coupling between strain and damage only on the RVE of the critical point as shown schematically in Figure 4.2. This is the case of small-scale damage.

Figure 4.2 Locally coupled analysis of crack initiation (Lemaitre, 1992)

4.4.1 Localization of damage
Damage localization from stress concentration, of course, but also occurs, because some weakness always exists at the microscale. In General, the mechanical model is a two-scale volume element, elastic or elasto-plastic at the mesoscale and elastoplastic and damageable at the microscale.

Consider an element as shown in Figure 4.3 exhibiting elastic behavior everywhere except a small microvolume \( \mu \) representing a weak defect subjected to elasticity, plasticity and damage.
The matrix is elastic with a yield stress $\sigma_y$ and a fatigue limit $\sigma_f$. The inclusion has the same properties as the matrix except that it is perfectly plastic with a plastic threshold $\sigma_S^{\mu}$ and a fatigue limit $\sigma_f^{\mu}$. Its weakness comes from the value of the plastic threshold, which is assumed to be equal to the fatigue limit $\sigma_f$ of the material as it is the lowest stress giving rise to possible damage.

$$\sigma_S^{\mu} = \sigma_y^{\mu} = \sigma_f$$ (4.33)

Furthermore, the weakness also comes from the fatigue limit $\sigma_f^{\mu}$, is assumed to be reduced in the same proportion as the plastic threshold.

$$\sigma_f^{\mu} = \sigma_f \frac{\sigma_S^{\mu}}{\sigma_y^{\mu}} = \frac{\sigma_f^2}{\sigma_y^{\mu}}$$ (4.34)

As the fatigue limit is smaller than the yield stress, it allows for plasticity which induces damage at the microscale, whereas the matrix remains elastic or elastoplastic and undamaged for the same loading.

The second assumption, which simplifies calculations is the Lin-Taylor strain compatibility hypothesis, which states that the state of strain at the microscale is equal to the state of strain at the mesoscale as derived from the classical structural calculation

$$\varepsilon^{\mu} = \varepsilon$$ (4.35)
The inclusion being perfectly plastic, then from the yield criterion,

\[
\frac{\sigma_{eq}^\mu}{1-D} = \sigma_S^\mu
\]  
(4.36)

The triaxiality factor may also be expressed as a function of the stress at the mesoscale

\[
R^\mu_v = \frac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\sigma_H^\mu}{\sigma_{eq}^\mu}\right)^2
\]  
(4.37)

\[
\sigma_{eq}^\mu = (1-D)\sigma_S^\mu
\]  
(4.38)

\[
\sigma_H^\mu = \frac{E(1-D)}{1-2\nu} \epsilon_H^\mu
\] from the damaged elasticity at the microscale,

but \( \epsilon_H^\mu = \epsilon_H \) at the mesoscale since \( \epsilon^\mu + \epsilon^{\beta\mu} = \epsilon \) and \( t_c(\epsilon^{\beta\mu}) = 0; \)

\[
\epsilon_H = \frac{1-2\nu}{E} \sigma_H
\] from pure elasticity at the mesoscale;

then \( \sigma_H^\mu = (1-D)\sigma_H \) and

\[
\frac{\sigma_H^\mu}{\sigma_{eq}^\mu} = \frac{\sigma_H}{\sigma_S^\mu}
\]

The damage threshold \( p_D = \epsilon_{pd} \frac{\sigma_u - \sigma_f}{\sigma_{eq} - \sigma_f} \) becomes \( p_D = \epsilon_{pd} \frac{\sigma_u - \sigma_f}{\sigma_S^\mu - \sigma_f^\mu} \) because for the inclusion, \( \sigma_{eq} = \sigma_S^\mu \) and \( \sigma_f = \sigma_f^\mu \).

Therefore,

\[
\dot{D} = \frac{\sigma_S^\mu}{2ES} \left[ \frac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\sigma_H}{\sigma_S^\mu}\right)^2 \right] \dot{p} \quad \text{if} \quad p \geq p_D; \quad \text{if not} \quad \dot{D} = 0
\]  
(4.39)

The integration of this damage rate equation is performed and crack will initiate when

\[
D = D_c = D_{tc} \frac{\sigma_S^2}{\sigma_S^\mu^2 R_v}
\]  
(4.40)

4.4.2 Numerical procedure for calculation of fatigue crack initiation

Recalling the equations to be solved

\[
\epsilon_y = \epsilon_y^* + \epsilon_y^\mu
\]  
(4.41)
\[ \varepsilon_{ij}^e = \frac{1 + \nu}{E} \frac{\sigma_{ij}}{1 - D} \frac{\nu}{E} \frac{\sigma_{kk}}{1 - D} \delta_{ij} \] (4.42)

\[ \dot{\varepsilon}_{ij}^p = \frac{3}{2} \frac{\sigma_{ij}}{\sigma_S} \dot{p} \] if \( f = 0 \) and \( f = \frac{\sigma_{eq}}{1 - D} - \sigma_S \); if not \( \dot{\varepsilon}_{ij}^p = 0 \) (4.43)

\[ \dot{\sigma} = \frac{\sigma_S}{2ES} R_e \dot{p} \] if \( p \geq p_D \); if not \( \dot{\sigma} = 0 \) (4.44)

The numerical analyses used to integrate the coupled equations are standard strain-driven methods. At the beginning of time step, the total time strain increment is given. The stress and other material variables are given at the time step \( t_n \). To update these variables at time \( t_{n+1} \), the Newton-Raphson algorithm is used.

It is first assumed that the increment is entirely elastic. So we have

\[ \tilde{\sigma}_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu (\varepsilon_{ij} - \varepsilon_{ij}^p) \] (4.45)

with \( \lambda = \frac{ED}{(1-2\nu)(1+\nu)} \) and \( \mu = \frac{E}{2(1+\nu)} \) (4.46)

\( \varepsilon_{ij}^p \) is the plastic strain at the beginning of the increment.

Then if this elastic predictor satisfies the yield condition \( f \leq 0 \), the assumption is valid and the computation for this time increment ends.

If \( f > 0 \), the elastic state has to be corrected. The rate relations presented above are discretized in an incremental form and introduced to a fully implicit integration scheme. The solution at time \( t_{n+1} \) has to satisfy the following equations

\[ f = \tilde{\sigma}_{eq} - \sigma_S = 0 \] (4.47)

\[ \tilde{\sigma} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu (\varepsilon - \varepsilon_{ij}^p - \Delta \varepsilon^p) \] (4.48)

\[ \Delta \varepsilon^p = N \Delta p \] (4.49)

\[ \Delta D = \frac{Y}{S} \Delta p \] (4.50)
where \( N = \frac{3 \tilde{\sigma}^D}{2 \sigma_{eq}} \)  

(4.51)

By replacing \( \Delta \varepsilon^n \) in the second equation, the problem is reduced to the first equation with unknowns \( \sigma \) and \( p \).

\[
f = \tilde{\sigma}_{eq} - \sigma_S^\mu = 0
\]

(4.52)

\[
h_y = \tilde{\sigma}_y - \lambda \varepsilon_\mu \sigma_y - 2\mu(\varepsilon_y - \varepsilon_{y_n}) - 2\mu N \Delta p = 0
\]

(4.53)

This non-linear system is solved by Newton's method. The details of the implementation are given in (Lemaitre and Doghri, 1994).

4.5 Identification of materials properties

The determination of the coefficients \( S, \varepsilon_{p0}, \) and \( D_{ic} \) that characterize the damage, together with \( \sigma_f \) and \( \sigma_u \) related to plasticity, must be worked out for each material and temperature, from experiments performed on those cases in which the damage is easiest to measure. If a tensile test has been performed with measurement of the damage during unloading by elasticity change (Figure 4.4)

![Figure 4.4 Identification of damage coefficients (Lemaitre, 1992)](image)
\( \varepsilon_{pD} \) is the plastic strain below which there is no appreciable damage. \( \sigma_f \) is the fatigue limit can be determined from S-N fatigue test. The last parameter, \( S \), is determined from the slope of the curve: damage \( D \) versus the plastic strain \( \varepsilon_p \).

\[
\dot{D} = \frac{\sigma^2}{2E(1-D)^2} \dot{\varepsilon}_p
\]

\[
dD = \frac{\sigma^2}{2ES(1-D)^2}
\]

At each point of the curve, \( D \) is known, \( \sigma \) is known from the stress-strain curve, \( dD/d\varepsilon_p \) is estimated by

\[
S = \frac{\sigma^2}{2E(1-D)^2} \frac{dD}{d\varepsilon_p}
\]

**Calculation of material properties**

**(a) Calculation of tensile properties**

Yield strength (2% offset) \( \sigma_y = 380 \) MPa

Ultimate Strength \( \sigma_u = 621 \) MPa

Modulus of Elasticity, \( E = 202000 \) MPa

Poisson’s ratio = 0.3

Reduction in area \( RA = \frac{A_0 - A_f}{A_0} = 51\% \)

True strain at fracture, \( \varepsilon_f = \ln \frac{A_0}{A_f} = \ln \frac{1}{1 - RA} = \ln \frac{1}{1 - 0.51} = 0.7133 \)

True stress at fracture, \( \sigma_f = K(\varepsilon_f)^n \)

Where for SAE 1045 HR,

Strength coefficient, \( K = 1185 \)
Strain hardening exponent = 0.23
So true fracture stress as $\sigma_f = 1096$ MPa

Engineering stress to rupture, $\sigma_R = \frac{P_f}{A_0}$

True fracture stress, $\sigma_f = \frac{P_f}{A_f} = 1096$ MPa

$$\frac{\sigma_R}{\sigma_f} = \frac{A_f}{A_0}$$

$\sigma_R = 1096 \times (1 - 0.51) = 537.04$ MPa

(b) Determination of fatigue limit
One can determine the fatigue limit $\sigma_f$ from the S-N curve for the material SAE 1045.
The fatigue limit or endurance limit is the stress amplitude below which the material has

![Figure 4.5 S-N curve](nCode international, 2001)
an infinite life. From the S-N curve it is observed that $\sigma_f = 160$ MPa.

(c) Determination of damage properties

Critical damage $D_{tc}$ applied to the pure monotonic tension test,

$$D_{tc} = 1 - \frac{\sigma_R}{\sigma_u} = 1 - \frac{537.04}{621} = 0.135$$

Effective elastic modulus of elasticity

$$\tilde{E} = E \times (1 - D_{tc})$$

$$= 202000 \times (1 - 0.135) = 174689 \text{ MPa}$$

Engineering strain at rupture $= 0.61$ (assume)

Engineering plastic strain at the point of rupture $= 0.61 - \frac{537.04}{174689} = 0.607$

Engineering strain, $\varepsilon$ and true strain, $\varepsilon_t$ has a relation $\varepsilon = \ln(1 + \varepsilon)$

Engineering stress, $S$ and true strain, $\sigma$ has a relation $\sigma = S(1 + \varepsilon)$

True strain, $\varepsilon_t = \frac{\sigma}{\tilde{E}} + \left(\frac{\sigma}{K}\right)^{1/n}$

With some iteration, for ultimate strength of 621 MPa true stress is 685 MPa

Engineering strain at ultimate strength $= 0.1004$

Engineering plastic strain at the ultimate strength, $\varepsilon_{pD} = 0.1004 - \frac{621}{202000} = 0.0973$

This is the strain below which there is no appreciable damage.

The last parameter $S$ is determined from the slope of the curve: damage $D$ versus plastic strain $\varepsilon_p$ (Figure 4.4)

$$\frac{D - 0.135}{0.135} = \frac{\varepsilon_p - 0.61}{0.61 - 0.0973}$$

$$\frac{dD}{d\varepsilon_p} = 0.2647$$
Once \( \frac{dD}{d\varepsilon_p} \) is known and also \( \sigma \) is known from the stress strain curve, \( S \) can be calculated from the following equation

\[
S = \frac{\sigma^2}{2E(1-D)^2 \frac{dD}{d\varepsilon_p}}
\]

\[
= \frac{537.04^2}{2 \times 174689 \times (1 - 0.135)^2 \times 0.2647} = 4.5 \text{ MPa}
\]

4.6 Multiaxial fatigue life prediction using damage mechanics

Using the damage mechanics model, fatigue life predictions are carried out for different multiaxial loading conditions. The predicted life cycles by damage mechanics are compared to experimental fatigue lives.

<table>
<thead>
<tr>
<th>Table 4.1 Life prediction using continuum damage mechanics.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pure Bending</td>
</tr>
<tr>
<td>Moment (N.m) (B)</td>
</tr>
<tr>
<td>1875</td>
</tr>
<tr>
<td>2600</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>1550</td>
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<td></td>
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<tr>
<td>1220</td>
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<td>1355</td>
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<td></td>
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<tr>
<td>780</td>
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<tr>
<td></td>
</tr>
<tr>
<td>851</td>
</tr>
<tr>
<td>Pure Torsion</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>
Figure 4.6 Comparison between life prediction using damage mechanics and experimental life cycles.
4.7 Conclusion

Multiaxial fatigue crack initiation lives are determined using a locally coupled analysis. Calculated lives are compared with experimental results. It is seen that damage mechanics approach usually gives lower fatigue life compare to experimental results. This is because damage is calculated in the microscale and also, stress-strain curve obtained from a classical tension-compression test at the meso-scale usually does not represent the true behavior for strain and damage because the space localization induces microplastic and damage zones much smaller than those of the specimen. Still, damage mechanics approach can be used for low cycle multiaxial fatigue life prediction. But for high cycle fatigue damage mechanics approach does not work well. Because here the damage is too localized to be represented by a field of damage variable defined in the mechanical framework of continuum mechanics.
CHAPTER 5

CONVENTIONAL METHODS FOR PURE TORSION AND BENDING LOADING

5.1 Introduction
Discontinuities, commonly known as notches act as stress raisers and therefore fatigue crack usually starts at such locations. At present, there are some conventional methods used to calculate fatigue lives for notched member. These methods are stress based and strain based. Stress-based approach employs elastic stress concentration factor to characterize the severity of a notch and determine the fatigue life using estimated S-N curves for the notched member. Strain-based approach treats notched members in a more detailed and rigorous manner than stress-based approach. This approach uses Neuber’s rule to estimate notch root stresses and strains, which are then used to calculate the fatigue life. This chapter investigates the use of these conventional methods. The Society of Automotive Engineers shaft is selected to carry out the investigation. Pure bending and torsion loading cases are only considered here. Results from conventional methods are compared with finite element based “FE-Fatigue” results. Moreover for pure torsion cases, the results are also compared with experimental results.

5.2 Stress-life approach
In this approach, the elastic stress concentration factor $K_r$ is employed to characterize the severity of the notch. The theoretical stress concentration factor is the ratio of the local notch stress to the nominal stress.

$$K_r = \frac{\sigma}{S} \quad (5.1)$$

Stress concentration factor is a function of geometry and mode of loading. Several theoretical stress concentration curves are available in literature (Peterson, 1974). For the present study only those for pure torsion loading are presented in Figure 5.1.
In the stress-life approach the effect of notches is accounted for by the fatigue notch factor, $K_f$. It is defined as the ratio of completely reversed stresses for the smooth member to that for the notched member. In the stress-life approach the effect of notches is accounted for by the fatigue notch factor, $K_f$. It is defined as the ratio of completely reversed stresses for the smooth member to that for the notched member.

$$K_f = \frac{S_e^{(\text{smooth})}}{S_e^{(\text{notched})}} \quad (5.2)$$

Whereas the theoretical stress concentration factor depends on the geometry and mode of loading, the fatigue notch factor is also dependent on material type. A useful concept to deal with notch effects is the notch sensitivity, $q$, which relates both $K_f$ and $K_t$. 

**Figure 5.1** Shaft with fillet torsion (Peterson, 1974)
\[
q = \frac{K_r - 1}{K_1 - 1}
\] (5.3)

**Figure 5.2** Shaft with fillet bending (Peterson, 1974)

The value of \( q \) between 0 to 1 is a measure of how severely a member is affected by a notch. The minimum value of \( q \) is zero where \( K_r = 1 \), that is, where the notch has no effect. Later on Peterson proposed the following empirical relationship for the determination of \( q \).

\[
q = \frac{1}{\left(1 + \frac{a}{r}\right)}
\] (5.4)

where \( r \) is the notch root radius and \( a \) is a material constant which depends on material strength and ductility. \( K_r \) can be written as a function of \( K_1, a, \) and \( r \).
\[ K_f = 1 + \frac{K_r - 1}{\frac{a}{r}} \]  

(5.5)

For ferrous-based wrought metals, \( a \) is approximately given by

\[ a = \left[ \frac{300}{S_u (ksi)} \right]^{1.8} \times 10^{-3} \text{in} \]  

(5.6)

\[ a = \left[ \frac{2070}{S_u (MPa)} \right]^{1.8} \times 0.025mm \]  

(5.7)

Although the value of \( K_f \) can be used to correct the entire S-N curve, a general trend is that the value of fatigue notch factor decreases with increasing stress level. The fatigue notch factor for stresses corresponding to lives of 1000 cycles has been defined as \( K_f' \). Figure 5.3 shows an empirical relationship between the correction for \( K_f' \) and ultimate strength of different materials. The notch effect at short lives is greatly reduced for low strength whereas the notch effect remains almost constant with life for high strength materials. The resulting corrected S-N curve for a notched member is shown in Figure 5.4.

![Figure 5.3](image)

**Figure 5.3** Relationship between \( K_f' \) and \( K_f \) with ultimate strength (Juvinall, 1967)
Figure 5.4 Estimation of S-N curve for notched material (Juvinall, 1967)

Sample calculations for torsion loading for SAE shaft

Ultimate strength, $S_u = 621$ MPa = 90.1 ksi

Notch root radius, $r = 5$ mm

\[
\frac{D}{d} = \frac{50}{40} = 1.25
\]

\[
\frac{r}{d} = \frac{5}{40} = 0.125
\]

With known $\frac{r}{d}$ and known $\frac{D}{d}$ from Figure 5.1, the value of elastic stress concentration factor can be determined and is found $K_f = 1.3$.

To determine the notch sensitivity factor, $q$

\[
q = \frac{1}{1 + \frac{a}{r}}
\]

Where $a$ has a relation with ultimate strength

\[
a = \left(\frac{2070}{S_u}\right)^{1.8} \times 0.025 \text{ mm} = 0.218 \text{ mm}
\]
With a known $a$ value of $q$ is found as 0.958.

Now, fatigue notch factor, $K_f$ is determined by

$$K_f = (K_r - 1)q + 1 = 1.287$$

For estimation of S-N curve for pure torsion loading,

Fatigue limit, $S_e = 0.5 \times S_u = 310.5$ MPa

Alternating stress at no of cycles of 1000 is, $S_{1000} = 0.9S_u = 558.9$ MPa

From Figure 5.3, the relation of $K_f$ and $K_r$ can be found 0.17 and from that $K_f = 1.049$

From the above information S-N curve can be estimated for pure torsion loading. Same calculations can be carried out for pure bending load to estimate the S-N curve for pure bending loading condition. Estimated S-N curves for both pure torsion and pure bending loading are shown in Figure 5.5 and Figure 5.6.

![Figure 5.5](image)

**Figure 5.5** Estimation of S-N curves for pure torsional loading
Figure 5.6 Estimation of S-N curves for pure bending loading.

For a torque of 2000 N.mm, nominal shear stress can be calculated as
\[
\tau_{\text{nom}} = \frac{Tc}{J} = \frac{16T}{\pi d^3} = 159.15 \text{ MPa}
\]

The alternating stress can then be found by
\[
\sigma = \sqrt{3} \tau_{\text{nom}} = 275.67
\]

From the estimated S-N curve for pure torsion loading, number of cycles to failure corresponding to this alternating stress is found as \( N_f = 435000 \) cycles.

In this way for all cases of pure bending and torsion loading cases, numbers of cycles are calculated using S-N based conventional approach.

5.3 Strain Life approach

The strain life approach may be applied in a fatigue analysis of a notched component. This method accounts for notch-root plasticity. If notch root strain history and smooth specimen strain-life data or fatigue properties are known, fatigue life evaluations may be
performed for notched members. The theoretical stress concentration factor, $K_i$, is often used to relate the nominal stresses, $S$, or strains, $e$, to the local values, $\sigma$ and $\varepsilon$. For increasing nominal stress, $S$, $K_i$ remains constant until yielding begins. Upon yielding the local stress, $\sigma$ and local strain, $\varepsilon$, are no longer linearly related and the local stress and strains are no longer related to the nominal values by $K_i$. Instead, the nominal and local values are related in terms of stress and strain concentration factors:

$$K_\sigma = \frac{\sigma}{S} \quad \text{or} \quad \sigma > \sigma_y$$

$$K_\varepsilon = \frac{\varepsilon}{e}$$

(5.8)

After yielding occurs, the local stress concentration, $K_\sigma$, decreases with respect to $K_i$, and $K_\varepsilon$ increases with respect to $K_i$. This is shown in Figure 5.7

![Figure 5.7 Effect of yielding on $K_\sigma$ and $K_\varepsilon$ (Bannatine et al., 1990)](image-url)
In other words, after yielding, the actual local stress is less than that predicted using \( K_t \), as shown in Figure 5.8

\[
\sigma_1 = K_t S \\
\sigma = K_\sigma S \\
S \\
\varepsilon = K_\varepsilon \varepsilon_1
\]

**Figure 5.8** Differences between local stress and strain predictions

(Bannatine et al., 1990)

Neuber (Neuber, 1961) analyzed a specific notch geometry and derived the following relationship

\[
K_t = \sqrt{K_\sigma K_\varepsilon} \tag{5.9}
\]

Elastic-plastic form of Neuber’s rule is given by

\[
K_t^2 S e = \sigma e \tag{5.10}
\]

The equation for the hysteresis curve is

\[
\frac{\Delta e}{2} = \frac{\Delta \sigma}{2E} + \left( \frac{\Delta \sigma}{2K'} \right)^{\frac{1}{\eta}} \tag{5.11}
\]

The nominal stress and strain relation can be obtained in the same way.

\[
\frac{\Delta e}{2} = \frac{\Delta S}{2E} + \left( \frac{\Delta S}{2K'} \right)^{\frac{1}{\eta}} \tag{5.12}
\]
Combining these three equations the following expression can be developed

\[ K' \frac{\Delta S}{2} \left[ \frac{\Delta S}{2E} + \left( \frac{\Delta S}{2K'} \right)^\frac{1}{n} \right] = \Delta \sigma \left[ \frac{\Delta \sigma}{2E} + \left( \frac{\Delta \sigma}{2K'} \right)^\frac{1}{n} \right] \]  

(5.13)

From this relation, equation (5.13) \( \Delta \sigma \) can be determined putting the value of \( \frac{\Delta S}{2} \).

Putting the value of \( \Delta \sigma \) in Equation (5.11) \( \frac{\Delta \varepsilon}{2} \) can be determined. Once \( \frac{\Delta \varepsilon}{2} \) is known, strain life equation can be used to determine the fatigue life.

\[ \frac{\Delta \varepsilon}{2} = \frac{\sigma'}{E} \left( 2N_f \right)^\frac{n}{b} + \varepsilon' \left( 2N_f \right)^\frac{c}{b} \]  

(5.14)

**Sample calculation for SAE shaft for pure torsion loading**

For the SAE shaft,

Elastic modulus, \( E = 202000 \) MPa, \( K' = 1258 \) MPa, \( n = 0.208 \), \( \sigma_f = 948 \) MPa,

\[ b = -0.09, \ c = -0.445, \ \text{and} \ \varepsilon_f = 0.26 \]

For pure torque of 2000 N.m, \( \frac{\Delta S}{2} = 275.67 \) MPa

Using \( \frac{\Delta S}{2} \) in equation (5.13), one can determine the value of \( \Delta \sigma \) as 640 MPa. Putting this \( \Delta \sigma \) in the equation of (5.11) \( \frac{\Delta \varepsilon}{2} \) can be determined as \( 2.9699 \times 10^{-3} \). Once \( \frac{\Delta \varepsilon}{2} \) is determined, using the strain life equation (5.14), fatigue life can be calculated and for this torsion loading case fatigue life comes, \( N_f = 70340 \) MPa.

In this way for all the cases of pure bending and torsion loading cases, number of cycles are calculated using strain based conventional approach.
5.4 Life prediction using conventional methods

Following table shows the fatigue life predictions for pure bending and torsion loading using conventional methods. Results are also compared with that from FE-Fatigue.

<table>
<thead>
<tr>
<th>Moment (N.m) (B)</th>
<th>Torsion (N.m) (T)</th>
<th>Life prediction for pure bending</th>
<th>From FE-Fatigue</th>
</tr>
</thead>
<tbody>
<tr>
<td>1400</td>
<td>0</td>
<td>475000</td>
<td>245400</td>
</tr>
<tr>
<td>1875</td>
<td>0</td>
<td>128000</td>
<td>58116</td>
</tr>
<tr>
<td>2600</td>
<td>0</td>
<td>9000</td>
<td>10685</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Moment (N.m) (B)</th>
<th>Torsion (N.m) (T)</th>
<th>Life Prediction for pure torsion</th>
<th>From FE-Fatigue</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Experimental cycles</td>
<td>Stress-based approach</td>
</tr>
<tr>
<td>0</td>
<td>2000</td>
<td>700000</td>
<td>435000</td>
</tr>
<tr>
<td>0</td>
<td>2400</td>
<td>75700</td>
<td>130000</td>
</tr>
<tr>
<td>0</td>
<td>3000</td>
<td>6329</td>
<td>17000</td>
</tr>
</tbody>
</table>

5.5 Conclusion

This study compares both conventional methods with finite element based FE-Fatigue results. Conventional methods are very easy to carry out. The good thing about these methods is that no finite element analysis is needed here. From comparison, it is seen that stress based conventional approach shows good correlation for pure bending loading cases. But for some pure torsion loading cases, this stress based method overestimates the fatigue life. Strain based conventional approach provides conservative results for most of the cases. For some loading cases, both stress based and strain based approaches cannot predict well. This is because both methods are based on some approximate relations.
CHAPTER 6

CONCLUSIONS

6.1 Summary
Elastic-plastic finite element analyses for cyclic loading is expensive and unfeasible sometimes. That is why linear-elastic three dimensional finite element analyses were conducted on the SAE notch shaft for pure bending, pure torsion and for different combinations of bending and torsion loadings. Calculated stresses and strains were used to predict the fatigue lives. Strain-life criteria and critical plane approaches were used for multiaxial proportional loading conditions. For non-proportional or 90° out of phase loading conditions, only critical plane approach is considered.

Strain based fatigue life predictions criteria are based on a uniaxial equation that relates the damage parameter to the number of cycles. These are called equivalent uniaxial fatigue criteria. Hoffmann and Seeger version of uniaxial strain life method is also used for fatigue life predictions for proportional loadings. The obtained fatigue lives, calculated using different strain-life criteria and critical plane approaches were compared to experimental fatigue lives in the published literature (Fash et al., 1985). The best criterion for multiaxial proportional loading was identified through the comparative studies. For non-proportional loading only critical plane approaches were considered, since other approaches are restricted to be used only when the biaxiality ratio and also the angle of maximum principle stress are constant throughout the time history. Obtained fatigue lives for 90° out of phase loading conditions were compared to experimental fatigue lives in the published literature (Leese and Socie, 1989) and a comparative study was done to select the optimum fatigue criterion for non-proportional multiaxial loading conditions.
A locally coupled damage mechanics approach is used for low cycle fatigue. The damage model is used for different combination of multiaxial loading cases for the SAE notched shaft. Obtained predicted lives are compared with the experimental results.

Conventional methods are also used for pure torsion and pure bending loading cases. Results from conventional methods are compared with the results from FE-Fatigue.

6.2 Conclusions
The following conclusions can be drawn from the analysis:

- Strain-based (Equivalent strain) approaches deliver good fatigue life predictions for proportional loadings. The absolute maximum principal strain criterion is found suitable for pure bending test. But this criterion cannot predict well in pure shear stress-strain cases. Therefore, absolute maximum principal strain criterion cannot be used solely to predict proportional multiaxial fatigue life. However, this criterion is recommended for fatigue crack initiation predictions in low-cycle regions, where normal strains control the deformation.

- The maximum shear strain criterion predicts the fatigue lives conservatively. Although, this criterion shows some good prediction in low-cycle fatigue regions, it cannot be considered as a reliable approach to predict high cycle fatigue crack initiations.

- Of all the strain based criteria, the von Mises (octahedral) criteria have been found to have the highest degree of acceptance with both conservative and non-conservative results reported.
• Equivalent strain approaches do not explain the observed nucleation and propagation of fatigue cracks on specific planes fixed with respect to the component being tested. Instead, they are a measure of average strain in a small volume of material. So for many Torsion/Bending load cases these approaches cannot account for the fatigue life. Again these methods are unsuitable for the case of non-proportional loading as well.

• Among all the strain based approaches Hoffmann and Seeger method is found to be more reliable. But this method is also based on the assumption that it does not work for non-proportional loading condition.

• Fatigue is a directional process. Critical plane approaches, which are based on some specific planes where damage accumulation is maximum, are found most effective for both multiaxial in phase and 90° out of phase loading cases.

• Among all the critical plane approaches, Fatemi and Socie model gives best fatigue life prediction for both multiaxial in phase and 90° out of phase loading cases. This is because this model takes the effect of both shear mode and tension mode of fatigue failure.

• SWT model cannot predict well when the significant part of stresses and strains are shear stress and shear strain. This is because this method is developed taking into consideration of tensile mode of fatigue failure.

• Damage mechanics approach gives the fatigue crack initiation life in microscale and so most results are found lower in comparison to experimental observations. Still damage mechanics can be applied for low cycle fatigue life prediction.
• Conventional methods are very easy to use but for all loading cases they do not give good correlation. This is because these methods are based on empirical relationship.

6.3 Recommendation for future research

• For non-proportional loading condition no model gives a very good fatigue life correlation. Future work can be done to improve the existing critical plane models for non-proportional loading.

• Damage mechanics approach does not work well for high cycle fatigue. Further research can be carried out in this field.

• In this work, damage mechanics model is used only for one gauss-point in the critical regions. Further research can be carried out considering an integration of more gauss points in the critical region.

• Further work can be done in developing stress concentration factors for different combinations of multiaxial loadings so that conventional methods can be carried out easily for complex loading conditions.

• Identical laboratory testing for fatigue failure problems differ widely and needs some statistical interpretation. Stochastic method can be introduced for better fatigue life correlation.
REFERENCES


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