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Commentary on Ami Mamolo on argumentation and infinity

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Commentary on Ami Mamolo, "Exploring Argumentation, Objectivity, and Bias: A Look at Mathematical Infinity"

DANIEL H. COHEN
Department of Philosophy
Colby College
Waterville, ME 04901
USA
dhcohen@colby.edu

Few of us have the aesthetic capacities and sensibilities of the great poet Edna St. Vincent Millay, but she was wrong when she wrote in 1922, "Euclid alone has looked on beauty bare." There was at least one other: the inestimable Georg Ferdinand Ludwig Phillip Cantor. Half a century earlier, Cantor had seen what David Hilbert would later describe as "Cantor's Paradise." The elegance of the diagonal argument, the awe-inspiring heights of the Alephs, and the breathtaking beauty of the generalized continuum hypothesis are enough to convert even the most compromising constructivists among us into hard-core Platonists with a zeal to rival Kurt Gödel's own. This is where evidence of Intelligent Design really can be found, and Cantor saw the possibility of liberation, if not salvation, in the uncountable empyrean realm. It was irresistible.

But what about the Löwenheim-Skolem Theorem? That theorem, also from 1922, states that every consistent theory has a model with a countable domain, even theories like Cantor's that purport to be about *uncountable* domains. It is a rather awkward result, but the fact that it is always possible to find countable models for countable axiomatizations of first-order logic with set theory does not mean that it is impossible to find uncountable models. On the contrary: it is always possible to find uncountable models, so the theorem does not mandate that we make the interpretive contortions necessary to make sense of the restricted quantification in the countable ones. We can choose to play in the other ones. We do not have to ascend into the nondenumerable empyrean realm, but we may, and so we do. I think the full context for Hilbert's comment sums up the response of the mathematical community: "*Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben können*" (Hilbert 1926). We can translate this fairly literally as, "No one must be allowed to expel us from the Paradise that Cantor has created for us," but I think its spirit might be captured more succinctly and poetically as simply, "This is Paradise; to Hell with that!"

There is a clear and distinct bias visible in my rendition of Hilbert's response: a bias for beauty. There is ovewhelming beauty to be found in Cantor's Paradise. We love it here; we're staying; *amor omnia vincit*.

Professor Mamolo has taken us on a romp into the mathematics of transfinite cardinals at the edge of this paradise. The challenge is to rein in the flights of poetic fancy (as I have not) and say something precise about the infinite. Precision is not to be confused with objectivity, but it is undeniably a great help in negotiating the rarefied terrains of abstract mathematics. The precise language of set theory is designed for the job; ordinary languages are not. That mismatch generates the ambiguities that give rise to conflicting intuitions, fallacious argumentation, and paradoxes both veridical and falsidical.

One sort of difficulty comes into clearer focus when we consider the use of 1-1 functions mapping one set onto another as a measure of cardinality. That provides a precise criterion for

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determining whether two sets are the same size, independent of the determining how large either set is. That is, sometimes we can answer the question, "Are they the same size?" without answering the question, "How large are they?" Furthermore, the possibility of putting a set into a 1-1 correspondence with one of its own proper subsets provides an equally precise criterion for set infinitude. However, the possibility of putting a set, S_1 into a 1-1 correspondence with a proper subset of S_2 does not mean that S_1 is smaller than S_2 because it does not mean that S_1 cannot also be put into a 1-1 correspondence with S_2. That needs to be kept in mind when dealing with infinite sets of ping-pong balls – and keeping it "in mind" is precisely right, because "*in mind*" is the only place one ever has to deal with infintely many ping-pong balls. The infinite does not confront us in the rec room. Listen carefully to what Mamolo said:

> The sets of in-going and out-going balls, being numbered as they are, both correspond to the set of natural numbers. This correspondence ensures that at the end of the experiment, as many balls were removed from the barrel as went in.

Pay close attention to the exact correspondence that has been established here because there are many ways to puts equipollent sets into a 1-1 correspondence. The key to the paradox, as presented, lies in the apparently innocuous clause about the balls "*being numbered as they are*." The numbering might have been done differently. Suppose that the balls were numbered as we put them in the barrel by the counting numbers starting with 1, but we renumber them as we take them out, again starting with 1 – but we take them out randomly, not according to their original numbering. We might, for example, always take out the *second* ball from each group of ten we put in the barrel. In that case, the ball originally numbered 1 would remain in the barrel until the very end. Infinitely many balls put in; infinitely balls taken out, but this time one remains. The infinite is funny that way. The mathematics is not in dispute: adding aleph-null many elements – the cardinality of the counting numbers – to a set that already has aleph-null elements always results in a set with aleph-null elements, but taking away aleph-null elements from a set with aleph-null elements can result in a set with a cardinality anywhere between zero and aleph-null itself. The mathematical operation of subtraction is undefined on Cantor's transfinite cardinals. The physical operation of "taking away" ping-pong balls can be defined in several ways, some of which depend on how the balls are numbered, and some of which – but *only* some – leave an empty barrel. In the original problem, all the noise about putting balls into the barrel is a distraction: there are infintely many balls labelled 1, 2, 3, etc. Ball 1 is removed from the barrel at the 30-second mark; ball 2 is removed at the 45-second mark, and so on. The circuitous route that higher-numbered balls follow is irrelevant since the path of every numbered ball can be non-problematically traced, and every one of those paths ends with it being removed.

One way to avoid the paradox of the ping-pong balls and all its confounding paradoxical kin about the infinite is simply deny that there is any such thing as infinty. Agree with Aristotle in countenancing potential infinities but not actual infinities. Better yet, follow Gersonides, a 14th century commentator on Aristotle, who argued that infinity is properly thought of as an adverb of process rather than as either a definite quantity or, worse, a substance (Rudavsky, 1998). There is a real insight here: after all, we can count infinitely – augmenting without end – but we *cannot* count to infinity. Zeno might divide time and space infinitely, be he cannot divide it to an infinitesimal. That solves the problem, but the cost is high: it casts us out of Paradise. The axiom of infinity, like the Continuum Hypothesis, is provably independent of the other axioms of

Zermelo-Fraenkel set theory, so we do not have to ascend even into the lower, countable regions of the infinite. But as with the higher realms, we do not have to, but we may, so we do.

How does this relate to objectivity, bias, and argumentation? Mathematics is not just about purely formal, completely objective proofs. There is argumentation in mathematics (Aberdein 2005). It has arguments; arguments have arguers; and arguers have biases. Among those biases is the preference for beautiful theories, like Cantor's set theory. Mamolo is correct to point out that following Cantor into the transfinite realm is not "more correct" in any objective sense, than adopting Robinson's arithmetic with its nonstandard analysis as a way of allowing infinitesimals in the door – or, for that matter, a modern-day Gersonides who prefers a finite mathematical world without an axiom of infinity. But is it more beautiful? I think so, but that may be due to bias. Is it objectively more beautiful? There can be no such proof, but, oh, can there be argument!

References

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