

University of Windsor

Scholarship at UWindor

Electronic Theses and Dissertations

Theses, Dissertations, and Major Papers

1977

COMPACTNESS AND ORDER PROPERTIES IN ORDERED BANACH SPACES.

MUHAMMAD NASIR. CHAUDHARY

University of Windsor

Follow this and additional works at: <https://scholar.uwindsor.ca/etd>

Recommended Citation

CHAUDHARY, MUHAMMAD NASIR., "COMPACTNESS AND ORDER PROPERTIES IN ORDERED BANACH SPACES." (1977). *Electronic Theses and Dissertations*. 1484.

<https://scholar.uwindsor.ca/etd/1484>

This online database contains the full-text of PhD dissertations and Masters' theses of University of Windsor students from 1954 forward. These documents are made available for personal study and research purposes only, in accordance with the Canadian Copyright Act and the Creative Commons license—CC BY-NC-ND (Attribution, Non-Commercial, No Derivative Works). Under this license, works must always be attributed to the copyright holder (original author), cannot be used for any commercial purposes, and may not be altered. Any other use would require the permission of the copyright holder. Students may inquire about withdrawing their dissertation and/or thesis from this database. For additional inquiries, please contact the repository administrator via email (scholarship@uwindsor.ca) or by telephone at 519-253-3000ext. 3208.



National Library of Canada

Cataloguing Branch
Canadian Theses Division

Ottawa, Canada
K1A 0N4

Bibliothèque nationale du Canada

Direction du catalogage
Division des thèses canadiennes

NOTICE

The quality of this microfiche is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us a poor photocopy.

Previously copyrighted materials (journal articles, published tests, etc.) are not filmed.

Reproduction in full or in part of this film is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30. Please read the authorization forms which accompany this thesis.

**THIS DISSERTATION
HAS BEEN MICROFILMED
EXACTLY AS RECEIVED**

AVIS

La qualité de cette microfiche dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de mauvaise qualité.

Les documents qui font déjà l'objet d'un droit d'auteur (articles de revue, examens publiés, etc.) ne sont pas microfilmés.

La reproduction, même partielle, de ce microfilm est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30. Veuillez prendre connaissance des formules d'autorisation qui accompagnent cette thèse.

**LA THÈSE A ÉTÉ
MICROFILMÉE TELLE QUE
NOUS L'AVONS REÇUE**

COMPACTNESS AND ORDER PROPERTIES
IN ORDERED BANACH SPACES

by

Muhammad Nasir Chaudhary

A Dissertation
submitted to the Faculty of Graduate Studies
through the Department of
Mathematics in Partial Fulfillment
of the requirements for the Degree
of Doctor of Philosophy at
The University of Windsor.

Windsor, Ontario, Canada

1976

© M. N. Chaudhary

000140

Dedicated to my mother

ABSTRACT

(a) We give some conditions under which a positive linear map between two ordered Banach spaces is compact or weakly compact. First we show that an order bounded linear map from an order unit normed space into an ordered Banach space with compact (w-compact) order intervals is compact (w-compact), and then we use it to derive other results. Necessary conditions are also considered under which a continuous or compact map is order bounded. Further we look at maps majorized by compact maps, and we give some conditions which make $K(X,Y)$ an order ideal in $L(X,Y)$. Next we discuss a convergence theorem. It is proved that the pointwise limit of an increasing net of positive, compact maps from an AM-space into an AL-space is positive and compact.

(b) Necessary and sufficient conditions are given which ensure that $L(X,Y)$ and $K(X,Y)$ have certain order properties, such as being regular, α -directed or α -additive. Generally these are extensions of Ng's results for dual spaces, and supplement the earlier results of Ellis and Wickstead. We also discuss the existence of weak order units and quasi interior points in $K(X,Y)$.

(c) Finally we give a characterization of ordered Banach spaces with compact order intervals. We prove that in regular ordered Banach spaces, order-intervals are compact iff the order convex cover of a compact set is compact. This generalizes a previous result of Wickstead who considered ordered Banach spaces with R.D.P.

ACKNOWLEDGEMENTS

The author wishes to express his deep gratitude for Dr. H. R. Atkinson, whose capable guidance, helpful suggestions, tolerance and consideration enabled the author to complete this dissertation.

Thanks are also due to Dr. C. S. Wong, Dr. T. Traynor, Dr. S. K. Tsui, and Dr. F. W. Lemire for their concern and advice.

The author expresses his gratitude for all the members of his family whose encouragement and confidence helped him to complete his work.

Mrs. Doreen Truant deserves thanks who accepted the challenge of typing the manuscript.

TABLE OF CONTENTS

	Page
Abstract	ii
Acknowledgements	iv
DEFINITIONS AND NOTATIONS	1
CHAPTER ONE: ORDER, COMPACTNESS AND CONTINUITY	6
Section 1. Order and continuity	
Section 2. Duality	
Section 3. The space of continuous linear maps	
Section 4. Compact maps	
CHAPTER TWO: COMPACTNESS OF POSITIVE MAPS BETWEEN ORDERED BANACH SPACES	19
Section 1. Compactness of positive maps	
Section 2. Majorized maps	
Section 3. A convergence theorem	
CHAPTER THREE: ORDER PROPERTIES OF $L(X, Y)$	35
CHAPTER FOUR: ORDER PROPERTIES OF $K(X, Y)$	47
Section 1. Sufficient conditions	
Section 2. Necessary conditions	
Section 3. Weak order units and quasi interior points	
CHAPTER FIVE: COMPACTNESS IN REGULAR ORDERED BANACH SPACES	67
REFERENCES	72
VITA AUCTORIS	76

DEFINITIONS AND NOTATIONS

(For elementary properties of the ordered vector spaces, the reader is referred to [15] and [29]).

1. A subset P of a real vector space is called a Wedge if $P + P \subseteq P$ and $\alpha P \subseteq P$, $\alpha \geq 0$. P is a Cone if $P \cap -P = \{0\}$. A real vector space X along with a wedge is called an ordered vector space, the wedge being denoted by X_+ . The dual wedge in X^* is denoted by X_+^* . This wedge will be a cone in X^* if X_+ is generating ($X = X_+ - X_+$).

A set $C \subseteq X$ is order convex iff whenever $x, y \in C$ and $x \leq z \leq y$ we have that $z \in C$. The simplest order convex set is the order interval $[x, y] = \{z \in X: x \leq z \leq y\}$. The order convex cover of a set B is the set

$[B] = (B + X_+) \cap (B - X_+)$, and is the smallest order convex set containing B . An order-convex subspace of X is called an order-ideal. X is order complete if every majorized subset of X has a least upper bound. X is directed upwards if for $x, y \in X$, there is $z \geq x, y$.

2. A normed vector space X ordered by a cone X_+ which is closed with respect to the strong topology, will be called an ordered normed vector space. If further X is norm complete then it is an ordered Banach space.

Let $\alpha \geq 1$. We say that X_+ is α -normal if $[U] \subseteq \alpha \cdot U$ where U is the closed unit ball of X . X_+ is α -generating iff for each $x \in X$, there are $u, v \in X_+$ with $x = u - v$ and $\|u\| + \|v\| \leq \alpha \|x\|$.

We say that X is regular iff it satisfies:

(R_1): For each $x, y \in X$ such that $-x \leq y \leq x$, we have

$\|y\| \leq \|x\|$; and

(R_2): For each $x \in X$ and $\epsilon > 0$, there is a $y \in X_+$ such that $y \geq x, -x$ and $\|y\| \leq \|x\| + \epsilon$.

X is said to be (α, n) -additive if for any n positive elements x_1, \dots, x_n we have $\sum_{i=1}^n \|x_i\| \leq \alpha \|\sum_{i=1}^n x_i\|$.

X is (α, n) -directed iff for any n vectors x_1, \dots, x_n in U , there is a $y \in \alpha U$ such that $y \geq x_i$ ($i = 1, \dots, n$). X is α -additive (α -directed) iff it is (α, n) -additive (resp.

(α, n) -directed for every $n \in \mathbb{N}$. X is approximately (α, n) -directed iff it is $(\alpha + \epsilon, n)$ -directed for all $\epsilon > 0$.

X is said to have a monotone norm if $x, y \in X$ with $0 \leq y \leq x$ implies that $\|y\| \leq \|x\|$.

X is called an order unit normed space if there is $e \in X_+$ such that, for $x \in X$ there exists a positive integer n with $-n e \leq x \leq n e$, and Minkowski functional of the order interval $[-e, e]$ defines the norm on X . X is said to be a base normed space if there is a convex subset B of X_+ such that, for $x \in X_+, x \neq 0$ there is a unique positive number

$f(x)$ with $x/f(x) \in B$, and the Minkowski functional of $CO(B \cup -B)$ defines the norm on X .

An approximate order unit in X is an upward directed set $\{e_\lambda : \lambda \in \Lambda\}$ in X such that for each $x \in X$, there exists $\delta \in \Lambda$ and $\alpha > 0$ with $-\alpha e_\delta \leq x \leq \alpha e_\delta$. If the Minkowski functional of $\{x : \text{there exists } \lambda \in \Lambda \text{ with } -e_\lambda \leq x \leq e_\lambda\}$ is a norm, then X with this norm is called approximate order unit normed.

3. An ordered vector space X has the Riesz decomposition property (R.D.P.) iff for every $x, y \geq 0$ and every $z \in [0, x + y]$, there are $x_1, y_1 \geq 0$ such that $x_1 \in [0, x]$, $y_1 \in [0, y]$ and $z = x_1 + y_1$. Every vector lattice has the R.D.P. but the converse is not true.

X is said to be a simplex space iff X_+ is 1-normal, X has the R.D.P. and the open unit ball of X is directed upwards.

If X is a vector lattice we denote the lattice operations by \vee and \wedge . For any x , $|x| = x \vee -x$, $x^+ = x \vee 0$, $x^- = -x \vee 0$. If X is norm complete, it is a Banach lattice iff $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. A subset A of X is said to be solid if $a \in A$ and $|x| \leq |a|$ implies that $x \in A$. The smallest solid set containing a subset B of X is called the solid hull of B .

Important examples of Banach lattices arise when the norm and order have further connections. A norm on a vector lattice X is an L-norm iff $\|x + y\| = \|x\| + \|y\|$ for x, y in X_+ . An L-normed Banach lattice is called an AL-space.

Dually a norm on a vector lattice is called an M-norm iff $\|x \vee y\| = \max \{\|x\|, \|y\|\}$ for x, y in X_+ . An AM-space is a Banach lattice with an M-norm.

4. A compact convex set K in a locally convex space X is called a simplex iff the space of real continuous affine functions on K has the R.D.P. The extreme boundary of a convex set K will be denoted by ∂K . A simplex K is a Bauer simplex if ∂K is closed.

We say that x is an extremal point of X_+ if each point of the order-interval $[0, x]$ is a positive scalar multiple of x . When x is an extremal point, the set $\{\alpha x: \alpha \geq 0\}$ is called an extreme ray.

When K is a simplex and X is a topological vector space, the set valued map $\phi: K \rightarrow 2^X$ is termed lower semi-continuous (l.s.c.) if, whenever $W \subseteq X$ is open, the set $\{k \in K: \phi(k) \cap W \neq \emptyset\}$ is open in K . ϕ is termed affine if $\phi(k)$ is a non-empty convex set and

$$\lambda \phi(k_1) + (1 - \lambda) \phi(k_2) \subseteq \phi(\lambda k_1 + (1 - \lambda) k_2)$$

whenever $k_1, k_2 \in K$ and $0 \leq \lambda \leq 1$.

5. A sequence $\{x_n\}$ in a normed vector lattice X is order convergent to $x \in X$ iff there is a downward directed sequence $\{z_n\}$ with infimum zero such that $|x_m - x| \leq z_n$ for all $m \geq n$. X has order continuous norm if every order convergent sequence in X converges in norm. If X and Y are ordered vector spaces, then a linear map $T: X \rightarrow Y$ is order bounded if it maps order intervals into order intervals. T is

positive if $Tx \in T_+$ when $x \in X_+$. Obviously a positive map is order bounded. A linear map between two normed vector lattices is order continuous if for each sequence $\{x_n\}$ in the domain space with order limit x , $\{Tx_n\}$ order converges to Tx .

6. Let X and Y be two ordered topological vector spaces. $L(X, Y)$ will denote the space of continuous linear maps from X into Y , and $K(X, Y)$ will be the subspace of compact maps. If K' is a simplex then the space of continuous affine maps from K' to X is denoted by $A(K', X)$. If \tilde{K} is a simplex with $\{o\}$ as an extreme point then $A_o(\tilde{K}, X)$ will denote the space of continuous affine functions from \tilde{K} to X , that vanish on $\{o\}$.

CHAPTER ONE

ORDER, COMPACTNESS AND CONTINUITY

1. Order and Continuity

Since the advent of the study of ordered topological vector spaces, it has been a prime concern to relate order theoretic properties with topological properties. In this context efforts have been made to answer the questions:

- (a) When is an order bounded (or positive) linear map from one ordered topological vector space into another necessarily continuous?
- (b) When is a continuous linear map from one ordered topological vector space into another necessarily order bounded?

The first question has been dealt with quite thoroughly, but the second one poses more problems. First we consider the linear functionals. It may be noted that an order bounded functional on an ordered topological vector space is not always continuous. For example, if we consider the space F of all those sequences $\{x_n\}$ which have finite number of non-zero elements, then $f(\{x_n\}) = \sum x_n$ defines a linear functional on F , which is positive but is not continuous with respect to the supremum norm on F . However the answer to the first question is affirmative in several cases:

Proposition 1.1: [15: 3.1.14]

Let X be an ordered topological vector space and X_+ have an interior point, then every order bounded linear functional on X is continuous.

Proposition 1.2: [15: 3.5.8]

Let X be a Fréchet space with a closed generating wedge. If f is an order bounded linear functional on X , then f is continuous, and is the difference between two continuous, positive, linear functionals.

Proposition 1.3: [29: II.2.17]

If X is a bornological space ordered by a sequentially complete strict b -cone [29] then a positive linear functional on X is continuous. (For ordered normed vector spaces, a strict b -cone is the same as a boundedly generating cone.)

The case of a normed vector lattice is even simpler:

Proposition 1.4 [31: II 5.2, 5.3]

Let X be a normed vector lattice. Then

- (a) Every positive linear functional on X is continuous.
- (b) Every real valued, order continuous lattice homomorphism on X is continuous.

The following proposition gives an answer to the second question:

Proposition 1.5 [15: 3.5.10]

Let X be a Fréchet space with a closed wedge X_+ . Then X_+ is normal if and only if every continuous linear functional on X is order bounded.

Similarly there are several results concerning the

continuity of an order bounded linear map when the range space is an ordered topological vector space.

Proposition 1.6: [15: 3.2.11]

Let X, Y be ordered topological vector spaces such that X_+ has an interior point and Y_+ is normal. Then every ordered bounded linear mapping from X into Y is continuous.

Proposition 1.7: [15: 3.2.12] [29: II.2.16]

If X and Y are ordered topological vector spaces, Y is locally convex and Y_+ is normal, then each of the following conditions implies that every positive linear mapping of X into Y is continuous:

- (a) X is a Mackey space on which every positive linear functional is continuous.
- (b) X is a bornological space with a sequentially complete strict b -cone.
- (c) X is of second category, metrizable and has a complete generating cone.

For metrizable spaces we can go one step further:

Proposition 1.8: [15: 3.5.5]

Let X, Y be topological vector spaces, X being complete and metrizable. If X is ordered by a closed, generating cone and f is a linear mapping from X into Y that maps order intervals into bounded sets, then f is continuous.

As before the results are quite satisfactory for normed lattices.

Proposition 1.9: [31: II.5.3]

Let X, Y be normed vector lattices and suppose X to be norm complete.

- (a) Every positive linear map from X into Y is continuous.
 (b) Every absolutely majorized linear map from X into Y is continuous.

(A linear map $S: X \rightarrow Y$ is called absolutely majorized if there exists a positive linear map $T: X \rightarrow Y$ such that $|Sx| \leq T_x$ for all $x \in X_+$.)

Thus we note that if X and Y are ordered Banach spaces in which X_+ is generating and Y_+ is normal, then a positive linear map from X into Y is continuous.

2. Duality

Duality has been from the beginning a useful tool in functional analysis. The systematic study of duality in the theory of ordered Banach spaces is only quite recent beginning with the work of Edwards and Ellis in 1964. Certain isolated results, such as those of Grosberg and Krein, and Reisz were discovered some thirty years earlier, but a comprehensive theory only developed in the years 1964-1970.

Some of the well known results are presented below and most of these will be used later. X will denote an ordered Banach space.

Grosberg and Krein [14] proved the first part and Ellis [12] proved the second part of the following theorem:

Theorem 2.1:

X_+ is α -normal for some $\alpha > 0$ if and only if X_+^* is α -generating.

X_+ is $(\alpha + \epsilon)$ -generating for all $\epsilon > 0$ if and only if X_+^* is α -normal.

Concerning the order unit normed and base normed spaces, we have:

Theorem 2.2 [Edwards]

X is order-unit normed if and only if X^* is base normed and the base is w^* -compact.

Theorem 2.3 [Ellis]

X is base normed if and only if X^* is order unit normed.

Edward's result was improved by Ng [25] and he also gave a characterization of approximate order unit normed spaces:

Theorem 2.4 [Ng]

The following statements are equivalent:

- (a) X^* is base normed.
- (b) X is approximate order unit normed.
- (c) X_+ is 1-normal, and the open unit ball of X is directed upwards.

Davies introduced the concept of regularity in discussing preduals of Banach lattices [8]. The two conditions for regularity turn out to be dual notions [25]:

Theorem 2.5

X satisfies (R_1) if and only if X^* satisfies (R_2) , and X satisfies (R_2) if and only if X^* satisfies (R_1) . Hence X is regular iff X^* is regular.

Davies' main result is:

Theorem 2.6

X^* is a Banach lattice iff X is regular and has the R.D.P.

Wickstead [34] has shown that X^* is a Banach lattice when X has the R.D.P. and X_+ is normal and generating. Moreover if X^* is a Banach lattice then it is also order-complete.

The next two results are due to Ng [25].

Theorem 2.7

Let X be an ordered Banach space. The following are equivalent:

- (a) The norm is additive on X_+ .
- (b) The open unit ball in X^* is directed upwards.
- (c) The closed unit ball in X^* is directed upwards.

Theorem 2.8

Let X be an ordered Banach space. The norm is additive on X^* iff the open unit ball in X is directed upwards.

Asimow [3], Ng [25] and Wickstead [34] established the following:

Theorem 2.9

If X is an ordered Banach space then X^* is (α, n) -additive iff X is approximately (α, n) -directed. Conversely X is (α, n) -additive iff X^* is (α, n) -directed.

Concerning the monotonicity of the norm Ng [25] proved the following two results:

Proposition 2.10

Let X be an ordered Banach space. The norm on X is monotone iff the following condition is satisfied:

For each f in X^* , there exists g in X^* with $\|g\| \leq \|f\|$ and $0, f \leq g$.

Proposition 2.11

Let X be an ordered Banach space. The norm on X^* is monotone iff the following condition is satisfied:

For each x in X and each positive ε , there exists y in X_+ such that $\|y\| \leq \|x\| + \varepsilon$ and $0, x \leq y$.

3. The Space of Continuous Linear Maps

The study of order properties of the dual of an ordered Banach space was further carried to the study of the order properties of $L(X, Y)$ the space of continuous linear mappings from one ordered Banach space X into another Y by Ellis [13] and Wickstead [34]. The work of Ellis [12] and Ng [25] served as a basis for this purpose and most of the results obtained so far are extensions of their results for dual spaces. However the lattice structure of $L(X, Y)$, when X and Y are both topological vector lattices, was considered earlier by Kantorowich, Peressini, Sherbert, Krengel and Schaefer (c.f.: Note on page 182 in [29]).

The following two results were proved by Ellis [13]. X and Y will denote ordered Banach spaces.

Theorem 3.1

If X has a base norm and Y has an order unit norm, then the operator norm in $L(X, Y)$ is an order unit norm.

Theorem 3.2

(a) If the operator norm in $L(X, Y)$ is an order unit norm, then X has a base norm and Y has an order unit norm.

(b) If the operator norm is a base norm in $L(X, Y)$ then X has an order unit norm and Y has a base norm.

When X has an order unit norm and Y has a base norm then it is not easy to characterize the order structure of $L(X,Y)$ with regard to the operator norm. Ellis [13] has remarked that in this case if $L(X,Y)$ is positively generated, then the operator norm is equivalent to a base norm but is not necessarily the same as that base norm.

Regarding the normality of $L(X,Y)_+$, Wickstead [34] proved:

Proposition 3.3

$L(X,Y)_+$ is normal if and only if X_+ is positively generating and Y_+ is normal.

But the positive generation of $L(X,Y)$ is rather difficult to determine. Necessary conditions are given in [34]:

Proposition 3.4

If $L(X,Y)$ is positively generated then X_+ is normal and Y is positively generated.

Unfortunately these do not constitute sufficient conditions. In fact there are only a few known cases when $L(X,Y)$ is positively generated. (We list some more in the subsequent chapters.) Theorem 3.1 is one of these. If X is finite dimensional and Y is positively generated; or if Y is finite dimensional and positively generated and X_+ is normal, then $L(X,Y)$ is positively generated.

Wickstead [34] also showed that $L(X,Y)$ is positively generated when $Y = C(\Omega)$ with Ω a stonian space and X satisfies one of the following conditions:

- (a) X is α -additive.
- (b) X has (R_1) .
- (c) X has a monotone norm.

Necessary and sufficient conditions for the order completeness of $L(X,Y)$ were obtained in [34]:

Proposition 3.5:

Let X_+ , Y_+ be normal and generating. Then the following are equivalent:

- (a) $L(X,Y)$ is order complete.
- (b) X has the R.D.P. and Y is an order-complete vector lattice.

Actually for the implication (b) \Rightarrow (a), Y need not be assumed to be a lattice.

"To determine the conditions under which $L(X,Y)$ is a lattice" is a well discussed problem [29], [31]. Some of the important results are given below. It may be pointed out that if X and Y are both lattices and Y is order complete, then $L(X,Y)_+ - L(X,Y)_+$ is an order complete lattice.

Theorem 3.6 [29: IV.3.6]

Suppose X is a vector lattice and Y is order complete lattice containing an order-unit, then $L(X,Y)$ is an order complete vector lattice.

Theorem 3.7 [29: IV.3.11]

Let X be a nuclear space having a normal cone, and Y be an order complete Banach lattice, then $L(X,Y)$ is an order complete vector lattice.

Theorem 3.8 [31: IV.4.4]

Let X be an AL-space and Y be an order complete AM-space with an order unit, then $L(X,Y)$ is an order complete AM-space with an order unit.

Theorem 3.9 [35: Prop. 7]

Let X have a normal, generating cone which is also the closed convex hull of its extreme rays, and let X have the R.D.P. Suppose also that Y is an AM-space. Then $L(X,Y)$ is a lattice.

Some interesting cases regarding the lattice structure of $L(X,Y)$ are discussed in [22].

4. Compact Maps

A systematic study of the space of compact maps from X into Y ($K(X,Y)$) was initiated by Wickstead in [34]. He considered the range space to be a simplex space and came up with some significant results.

If Y is a simplex space, then $K = \{f \in Y^* : f \geq 0, \|f\| \leq 1\}$ is a compact simplex when given the w^* -topology and Y is isometrically isomorphic to $A_0(K)$, the space of continuous affine functions on K which vanish at zero. Similarly the following result enables us to relate the study of the order properties of $K(X,Y)$ with the study of continuous affine functions, vanishing at an extreme point, from a simplex into an ordered Banach space.

Proposition 4.1 [34: Prop. 4.1]

Let T be a bounded linear map from X into $A_0(K)$. Then there is an affine map τ of K into X^* , vanishing at zero, and

continuous for the w^* -topology on X^* , such that

$$(a) \quad (Tx)(k) = (\tau k)(x) \quad (x \in X, k \in K),$$

$$(b) \quad ||T|| = \sup \{ ||\tau k|| : k \in K \}.$$

Conversely if such a map τ is given, (a) defines a bounded linear map from X to $A_0(K)$ with norm defined by (b). T is compact iff τ is continuous for the norm topology of X^* .

If X is ordered by a closed cone then $T \geq 0$ if and only if $\tau \geq 0$.

An analogous result holds when $Y = A(K)$, K a simplex. Thus $K(X, Y)$ can be identified with $A_0(K, X^*)$ when Y is a simplex space, and with $A(K, X^*)$ when $Y = A(K)$. This approach is then based on Lazar's selection technique:

Theorem 4.2 [20]

Let X be a Fréchet space and $\phi: K \rightarrow 2^X$ be an affine lower semi-continuous set valued map such that $\phi(k)$ is closed for every $k \in K$. Then there exists an affine continuous selection for ϕ , that is, a continuous affine map $\pi: K \rightarrow X$ such that $\pi(k) \in \phi(k)$ for all $k \in K$.

Wickstead used Prop. 4.1 and Theorem 4.2 to prove the following results. X will denote an ordered Banach space.

Theorem 4.3

Let Y be a simplex space, and $\alpha \geq 1$. The following are equivalent:

$$(a) \quad x, y \in X \text{ and } 0 \leq y \leq x \text{ implies } ||y|| \leq \alpha ||x||.$$

$$(b) \quad T \in K(X, Y) \text{ and } ||T|| < 1 \text{ implies that there exists } S \in K(X, Y) \text{ with } S \geq T, 0 \text{ and } ||S|| < \alpha.$$

Theorem 4.4

Let X_+ be normal and generating and y be a simplex

space. Then X has the R.D.P. iff $K(X,Y)$ has the R.D.P.

Further $K(X,Y)$ is a lattice iff X has the R.D.P. and Y is a lattice.

Thus we note that $K(X,Y)$ may be positively generated when $L(X,Y)$ is not.

He also showed that if Y is a simplex space then $K(X,Y)$ is approximate order unit normed iff X is base-normed. Later he improved it in [35].

Proposition 4.5

The following are equivalent:

- (a) X is base normed and y is a.o.u. normed.
- (b) $K(X,Y)$ is a.o.u. normed.

In [35] Wickstead considered those spaces as the range, in which one of the following conditions holds:

- (a) compact sets are order bounded.
- (b) Order-intervals are norm compact.
- (c) The notions of norm compactness and order boundedness are identical.

Some of his major results are the following:

Theorem 4.6

The following are equivalent:

- (a) Every relatively compact subset of X is order bounded.
- (b) If Y is β -additive for some β , then $K(Y,X)$ is positively generated.
- (c) If Y is base-normed then $K(Y,X)$ is positively generated.
- (d) For some α , X is α -directed.

Theorem 4.7

Let X_+ be a generating cone and X have the R.D.P. then

the following are equivalent:

- (a) Order intervals in X are norm compact.
- (b) X is a lattice and the solid hull of each norm compact subset of X is norm compact.
- (c) X_+ is normal and is the closed convex hull of its extreme rays.
- (d) X is a complete vector lattice; if $|x| \wedge z = 0$ whenever z is extremal then $x = 0$; if $S \subseteq X$ is bounded above then the set of finite suprema from S converges in norm to $\text{Sup}(S)$.
- (e) If Z has a generating cone and has the R.D.P., then $K(Z, X)$ is order complete.

Theorem 4.8

Let X have the R.D.P. The following are equivalent:

- (a) X is linearly order isomorphic and homeomorphic to a space $C_0(T)$ of continuous functions vanishing at infinity on a discrete space T .
- (b) The notions of relative norm compactness and order boundedness coincide in X .
- (c) X is a lattice and the solid hull of every norm compact non-empty subset of X is norm relatively compact.
- (d) If Z_+ is normal and generating then $K(Z, X)$ is positively generated. If further Z has the R.D.P. then $K(Z, X)$ is a complete vector lattice.

CHAPTER TWO
COMPACTNESS OF POSITIVE MAPS
BETWEEN ORDERED BANACH SPACES

As can be seen from chapter one, a great deal of work has been done to investigate the continuity of positive linear maps between ordered Banach spaces X and Y , but very little is known regarding the compactness of such maps. Given an order bounded linear map T from X into Y , what (order theoretic and/or topological) conditions on X and Y will ensure that T is compact or weakly compact? In this chapter a start is made to answer this question. We also consider those linear maps which are majorized by compact maps. Section three contains some convergence theorems concerning the pointwise limit of increasing nets of compact maps.

In this, and the following chapters, X and Y will denote ordered Banach spaces unless specified.

1. Compactness of Linear Maps

Theorem 4.7 of the first chapter establishes a characterization of those spaces which have compact order-intervals. A similar result is given in [31] which states some conditions under which order-intervals are w -compact:

Theorem 1.1

For any Banach lattice X , the following conditions are equivalent:

- (a) Order-intervals are weakly compact in X .
- (b) X is order complete and every $f \in X^*$ is order continuous.
- (c) Each directed majorized family in X converges weakly.
- (d) Each directed family in X with infimum zero, norm converges to zero.
- (e) X is countably order complete and each decreasing sequence in X with infimum zero norm converges to zero.

Order-intervals are also weakly compact in AL-spaces and in normed vector lattices with order continuous norm.

Now the following proposition is an easy consequence of the properties of the spaces involved:

Proposition 1.2

Let X have an order-unit and T be an order-bounded linear map from X into Y . If the order-intervals are norm compact (resp. w. compact) in Y , then T is compact (resp. w. compact).

Since the order-intervals are norm compact in the sequence spaces ℓ_p , $1 \leq p < \infty$ and c_0 [35], we have:

Proposition 1.3

Let X be an AM-space.

- (a) Every positive linear map from X into ℓ_1 is compact.
- (b) If X has an order-unit, then every positive linear map from X into c_0 is compact.
- (c) Every positive linear map from c_0 into an AL-space is

compact.

(d) If Y has a base-norm then every positive linear map from X into Y is w -compact.

Proof

(a) Let T be a positive linear map from X into ℓ_1 .

X^{**} is an AM-space with an order-unit, and T has a norm preserving positive linear extension T_0 from X^{**} into Y [35. II 8.9].

Then T_0 is compact by Prop. 1.2, and hence its restriction $T: X \rightarrow \ell_1$ is also compact.

(b) Obvious from 2.2, since order-intervals are compact in c_0 .

(c) $c_0^* = \ell_1$ and the dual space of an AL-space is an AM-space with an order-unit. If T is a positive linear map from c_0 into an AL-space, then the conjugate of T i.e. T^* is positive and compact by (a). Hence T is also compact.

(d) Y^* has an order unit and X^* is an AL-space.

Let T be a positive linear map from X into Y . Since order-intervals are w -compact in an AL-space [30], T^* is w -compact by Prop. 1.2.

Corollary 1.4

A positive linear map from an AM-space into an AL-space is w -compact.

Further it can be easily seen that if the order-intervals are w -compact in Y and it has an order-unit, then every continuous linear map from some ordered Banach space into Y is w -compact. If X has an order-unit and Y is

reflexive with monotonic norm, then an order-bounded linear map from X into Y is also w -compact. Similarly using I.Prop.4.1 it can be proved that a positive linear map from a reflexive Banach space into a simplex space is w -compact. We also note that if X has an order-unit and Z is a Dual Banach space with a normal cone, then an order-bounded linear map from X into Z is w^* -compact.

As a corollary to one of these remarks, we have:

Proposition 1.5

Let X be reflexive, positively generated and Y be base-normed. Then a positive linear map T from X into Y is w -compact.

Proof

Y^* has an order-unit and X^* is a reflexive space with a normal cone.

Hence $T^*: Y^* \rightarrow X^*$ is w -compact, which further implies that T is w -compact.

In Prop. 1.2 and the consequent propositions, the order-unit has played an important role. In such spaces norm-bounded sets are order-bounded. It is natural to ask if we can weaken this assumption. In particular, do similar results hold if we only assume that we have an approximate order-unit normed space? The following proposition essentially settles this question.

Proposition 1.6

Let Y be an approximate order-unit normed Banach space which is β -directed. Then

- (a) norm bounded sets are order-bounded in Y .

(b) Y is an order-unit normed space.

Proof

(a) Let $K = Y_+^* \cap U$ where U is the closed unit ball in Y^* . Since Y has an approximate order unit, Y^* has a base norm.

Let $H(K)$ be the space of real homogeneous functions on Y_+^* , which are continuous on K , with the pointwise ordering. Y may be identified with a linear subspace of $H(K)$, and the given order coincides with the relative order as a subspace of $H(K)$, since Y_+ is closed. Also $H(K)$ is an AM-space with the supremum norm on K and the norm on Y is greater than the norm as a subspace of $H(K)$ · [11].

Let A be a norm bounded set in Y . Then it is norm-bounded in $H(K)$ as well, i.e. there exists $\lambda > 0$, such that

$$||a|| \leq \lambda \text{ for all } a \text{ in } A, \text{ where } ||a|| = \sup_{k \in K} |a(k)|.$$

This implies that for all a in A , and k in K ,

$$-\lambda \leq a(k) \leq \lambda.$$

Let B be the base in Y^* . Then $B = \{k \in K: ||k|| = 1\}$.

Let us define $f \in H(K)$ such that for all $b \in B$, $f(b) = 1$ and it is extended to K by homogeneity. $y \in Y_+^*$ implies that $y = \alpha b$ for some $b \in B$ and $\alpha \geq 0$, and $-\lambda \leq a(b) \leq \lambda$. Thus $-\lambda f(b) \leq a(b) \leq f(b)$, i.e. $-\lambda \alpha f(b) \leq \alpha \cdot a(b) \leq \lambda \alpha f(b)$. Hence $-\lambda f(y) \leq a(y) \leq \lambda f(y)$.

In the pointwise ordering of $H(K)$ this implies that

$$A \subseteq [-\lambda f, \lambda f].$$

Since Y is cofinal in $H(K)$ [35: Th. 1], there exists

$y \in Y$ such that $\lambda f \leq y$, i.e. $A \subseteq [-y, y]$.

(b) Let V be the closed unit ball in Y . (a) implies that V is order bounded, i.e. there exists $e \in Y$ such that $V \subseteq [-e, e]$.

$$\begin{aligned} \text{Then } \{y \in Y : \|y\| \leq 1\} &\subseteq \{y \in Y : -e \leq y \leq e\} \\ &\subseteq \{y \in Y : \|y\| \leq \|e\|\} \end{aligned}$$

since Y is 1-normal.

Thus the original norm in Y and the norm induced by e are equivalent, and therefore Y has the order unit e .

Next we look at some necessary conditions for a linear map to be compact. In theorem I. 4.6 we saw that if X is α -directed then relatively compact sets of X are order-bounded. This is also true when X is cofinal in $H(K)$, K as in Prop. 1.6. [35].

Proposition 1.7

Let the norm be monotonic in X , and the relatively compact sets be order bounded in Y . Then a compact map T from X into Y is order-bounded.

Proof

Let $[a, b]$ be an order-interval in X . Since the norm is monotonic in X , $[a, b]$ is norm-bounded. Therefore $T([a, b])$ is a relatively compact subsets of Y , and hence is order bounded in Y .

In particular this result is true when Y is an AM-space, because then relatively compact sets have a supremum in Y .

Similarly we can show that if X_+ is normal and Y is order unit normed then a continuous linear map from X into Y is order bounded.

2. Majorized Maps

In this section we shall consider positive linear maps which are majorized by compact maps. If $0 \leq T \leq T_0$ with T_0 compact, we like to know when T is also compact. In this regard first we discuss a few cases where $K(X, Y)$ forms an order ideal in $L(X, Y)$.

Proposition 2.1

Let X_+ be generating, and order convex cover of a compact set be compact in Y . Then $K(X, Y)$ is an order ideal in $L(X, Y)$.

Proof

Let $0 \leq T \leq S$ where $T \in L(X, Y)$ and $S \in K(X, Y)$. If $x \in X_+$ and $\|x\| \leq 1$, then $0 \leq T(x) \leq S(x)$. Thus if U is the closed unit ball in X , we have:

$$T(U_+) \subseteq \{y \in Y: 0 \leq y \leq S(x): x \in U^+\} \\ \subseteq S(U^+).$$

S being compact, $S(U^+)$ and hence $[S(U^+)]$ the order convex cover of $S(U^+)$ is compact. Therefore $T(U^+)$ is relatively compact, which implies that $T \in K(X, Y)$.

Now let $E \leq F \leq G$; $E, G \in K(X, Y)$, $F \in L(X, Y)$. Then $0 \leq F - E \leq G - E$. Since $K(X, Y)$ is a subspace of $L(X, Y)$, $G - E$ is compact. From above $F - E$ is compact. But $F = E + (F - E)$ and hence $F \in K(X, Y)$. The result follows from the definition of an order ideal.

Proposition 2.2

Let X be positively generated, and the notions of norm compactness and order-boundedness be identical in Y .

Then $K(X, Y)$ is an order ideal in $L(X, Y)$.

Proof.

Let $0 \leq T \leq T_0$ where T_0 is compact and $T \in L(X, Y)$.

If U is the closed unit ball in X then $T_0(U_+)$ is compact in Y . By hypothesis it is order bounded, and there is $e \in Y$ so that $T_0(U_+) \subseteq [0, e]$. Let $x \in U_+$, then $T(x) \leq T_0(x) \leq e$ i.e. $T(U_+) \subseteq [0, e]$. But $[0, e]$ is compact which implies that $T(U_+)$ is relatively compact. X being positively generated, T is compact.

The rest of the proof is the same as in Prop. 2.1.

In particular this result holds when Y satisfies the equivalent assertions of Theorem I.4.

Further we note from 1.2 and 1.3 that if one of the following conditions is satisfied:

- (a) X is order unit normed and order intervals are compact in Y .
- (b) X is an AM-space and $Y = \ell_1$.
- (c) X is an AM-space with order unit and $Y = c_0$.
- (d) $X = c_0$ and Y is an AL-space.

Then $K(X, Y)$ is an order ideal in $L(X, Y)$.

The following lemma is used later to determine the compactness of majorized maps.

Lemma 2.3

Let T be a positive linear map from X into Y which maps X_+ onto Y_+ . Then $T(A)$ is order-convex in Y whenever A is order-convex in X .

Proof

Let A be an order convex subset of X and $u, w \in T(A)$.

Let $u \leq v \leq w$ where $v \in y$. Then there exist $x, z \in A$ such that $Tx = u$, $Tz = w$.

Now $0 \leq v - u \leq w - u$ and by hypothesis there exist $a, b \in X_+$ such that $Ta = v - u$ and $Tb = w - u$.

i.e. $0 \leq Ta \leq T(b)$, and then $u \leq Ta + u \leq Tb + u$

Thus $u \leq T(a + x) \leq w$, and $v = T(a + x)$.

But $x \leq a + x \leq z$ implies that $a + x \in A$, and therefore $v \in T(A)$.

Proposition 2.4

Let X be approximate order unit normed and T_0 be a linear map from X into Y such that T_0 is compact and maps order convex sets into order convex sets. If $0 \leq T \leq T_0$, then T is a compact map.

Proof

Since X is approximate order unit normed, X_+ is 1-normal and the open unit ball V in X is directed upwards. Thus $[V] \subseteq 1 \cdot V \subseteq [V]$ so that V is order convex.

Let $y \in T[V]$, $y = T(x)$ ($x \in V$). Then there is a $v \in V_+$ such that $v \geq x$, $-x$ and thus $T(-v) \leq T(x) \leq T(v)$.

Also $T_0(-v) \leq T(x) \leq T(v)$ which by the order convexity of $T_0(V)$ implies that $T(x) \in T_0[V]$ and so $T[V] \subseteq T_0[V]$.

The compactness of T_0 now implies the compactness of T .

Obviously Prop. 2.4 is valid when X is an order unit

normed space.

Proposition 2.5

Let X be a base normed Banach space, and T, T_0 be as in Prop. 2.4. Then T is compact.

Proof

Let B denote the base of X_+ , then the closed unit ball U in X , $= \text{co}(BU - B)$.

X_+ is 2-normal [25. lemma 1]; implies $[U] \subseteq 2 \cdot U$.

Then $M = T_0([U]) \subseteq 2 \cdot T_0(U)$ is relatively compact and order-convex, because of the hypothesis.

Let $b \in B$, then $T(b) \leq T_0(b)$ & $T(-b) \geq T_0(-b)$

i.e. $T_0(-b) \leq T(+b) \leq T_0(b)$ and $T_0(+b) \in M$, since $\pm B \subseteq U$.

M is order-convex; implies $T(+b) \in M$.

Now let $y \in T(U)$; $y = Tx$, $x \in U$ and $x = \lambda b - \lambda' b'$; $b, b' \in B$ and $0 \leq \lambda, \lambda' \leq 1$. Thus $-b' \leq x \leq b$ and $T(-b') \leq Tx \leq T(b)$. But $T(b), T(-b') \in M$, implies $y = Tx \in M$, i.e. $T(U) \subseteq M$, and is therefore relatively compact.

Proposition 2.6

Let X be a Banach lattice and T, T_0 be as in Prop. 2.4.

Then T is compact.

Proof

Let U denote the closed unit ball in X .

Since X is a Banach lattice, X_+ is 2-normal and $[U] \subseteq 2 \cdot U$ [15. pg. 153].

Let $M = T_0([U]) \subseteq 2 \cdot T_0 \cdot (U)$, then M is relatively compact and order-convex. If $z \in U^+$, then $0 \leq Tz \leq T_0 z$.

implies $Tz \in M$; and similarly $T(-z) \in M$.

Now let $y \in Tu$ and $y = Tx$, $x \in U$. Then $x = x^+ - x^-$, and $||x^+||, ||x^-|| \leq ||x^+ + x^-|| = ||x|| = ||x|| \leq 1$.
i.e. $x^+, x^- \in U^+$, and $-x^- \leq x \leq x^+$. Thus $T(-x^-) \leq T(x) \leq T(x^+)$.
But $T(-x^-), T(x^+) \in M$ implies that $y = Tx \in M$ since M is order-convex, and hence $T(U) \subseteq M$. Therefore $T(U)$ is relatively compact.

If T, T_0 are as in Prop. 2.5, we can show that T will be compact if X_+ is normal and either X_+ is 1-generating or X has a directed open unit ball. The proofs are similar to those of 2.4 and 2.6, and will be omitted.

3. A Convergence Theorem

Here we discuss a convergence theorem for compact maps which is an analogue of Dini's Theorem. First we have a few definitions.

Let X and Y be Banach lattices. Then $T \in L(X, Y)$ is called regular if $T = T_1 - T_2$ for some $T_1, T_2 \in L(X, Y)_+$. Further if Y is order-complete, then $|T| \in L(X, Y)$, and the set of regular maps denoted by $L^r(X, Y)$, is a Banach lattice itself with $||T||_r = |||T|||$. In this case $L^r(X, Y)$ is also the set of order-bounded linear maps from X into Y .

$T \in L(X, Y)$ is called cone absolutely summing (c.a.s.) if for every positive summable sequence $\{x_n\}$ in X , the sequence $\{Tx_n\}$ is absolutely summable in Y . The set of c.a.s. maps is denoted by $L^l(X, Y)$, which is a Banach space

with $\|T\|_\ell = \sup \{ \sum_i \|Tx_i\| : \{x_i\} \text{ finite } \subset X, \sum_i \|x_i\| = 1 \}$

$T \in L(X, Y)$ is called Majorizing if for every null sequence $\{x_n\}$ in X , $\{\|Tx_n\|\}$ is an upper bounded sequence in Y . The set of Majorizing maps is denoted by $L^m(X, Y)$; this is a Banach space with

$$\|T\|_m = \sup \{ \|\sup_i \|Tx_i\|\| : \{x_i\} \text{ finite}, \|x_i\| \leq 1 \}.$$

In general $\|T\| \leq \|T\|_r, \|T\|_\ell, \|T\|_m$. The details can be found in [31: IV].

Further if X is an AM-space and Y is an AL-space, then it follows from [31: Ex. 3, page 352] that the sets $L^\ell(X, Y)$, $L^m(X, Y)$, $L^r(X, Y)$, Order bounded linear maps from X into Y , are all the same. Then we have:

Theorem 3.1

Let X be an AM-space, Y be an AL-space, and T, T_n , $n = 1, 2, \dots$ be linear maps from X into Y which belong to one of the following sets:

- (a) $L^\ell(X, Y)$
- (b) $L^m(X, Y)$
- (c) $L^r(X, Y)$
- (d) Order bounded linear maps.

If $T_1 \leq T_2 \leq \dots \leq T$ and $T_n(x) \rightarrow T(x)$ for all x in X ;

then

$$(1) \|T - T_n\| \rightarrow 0,$$

hence (2) T is compact when each T_n is compact.

Proof

(1) Let $x \in X_+$. Then $T_1(x) \leq T_2(x) \leq \dots \leq T(x)$. Since

$T_n(x) \rightarrow T(x)$ and cone is closed in Y , we have

$$T(x) = \sup_n T_n(x) \quad [15: 3.1.14].$$

We prove the theorem for case (a).

Since $L^\ell(X, Y)$ is an AL-space [31: IV.4.5], the norm is monotonic in $L^\ell(X, Y)$, and hence the order-intervals are norm-bounded in it. The sequence $\{T_n\}$ is contained in the order-interval $[T_1, T]$, and is therefore norm bounded. Then $\{T_n\}$ has a supremum in $L^\ell(X, Y)$ [31: II.8.2].

Let $S = \sup_n T_n$. This implies $S \leq T$ in $L^\ell(X, Y)$. But for $x \in X_+$, $S(x) \geq T_n(x)$ and $T(x) = \sup_n T_n(x)$. Thus $T(x) \leq S(x)$ i.e. $T \leq S$, which implies $S = T$ in $L^\ell(X, Y)$ and $T = \sup_n T_n$ in $L^\ell(X, Y)$.

Then since $L^\ell(X, Y)$ is an AL-space $\|T - T_n\|_\ell \rightarrow 0$ [15: 3.8.8].

But $\|T - T_n\| \leq \|T - T_n\|_\ell$, and therefore $\|T - T_n\| \rightarrow 0$.

(2) Since $K(X, Y)$ is a closed subspace of $L(X, Y)$, and each T_n is compact, T must be compact.

In the rest of this section, as in Thm. 3.1, X will always be an AM-space and Y will be an AL-space.

Corollary 3.2

Let $T \in L(X, Y)$ be the point-wise limit of a non-decreasing sequence of positive, compact maps $\{T_n\}$ contained

in $L(X;Y)$. Then T is also positive and compact.

Proof

Let $x \in X_+$. Since $T_n(x) \rightarrow T(x)$, the cone is closed in Y and $T_n(x) \in Y_+$, we have that $Tx \in Y_+$.

Thus T is a positive map.

Now every positive map is order-bounded and therefore we apply 3.1(d).

In Theorem 3.1 we would like to remove the condition that T is itself a c.a.s. map. We have the following result.

Theorem 3.3

Let $T \in L(X,Y)$, and $\{T_r\}$ be a norm-bounded sequence in $L^{\ell}(X,Y)$ such that $T_1 \leq T_2 \leq \dots \leq T$ and $T_r(x) \rightarrow T(x)$ for $x \in X$. Then

(a) $T \in L^{\ell}(X,Y)$

Hence (b) T is compact, when each T_r is compact.

Proof

(a) Let $x \in X_+$. As before $Tx = \sup T_r(x)$.

Since $\{T_r\}$ is a non-decreasing norm-bounded sequence in $L^{\ell}(X,Y)$, which is an AL-space, there exists $S \in L^{\ell}(X,Y)$ such that $S = \sup_r T_r$. Thus $S \leq T$ in $L(X,Y)$.

Again for $x \in X_+$, $S(x) \geq T_r(x)$, $n = 1, 2, \dots$

Thus $S(x) \geq T(x)$, i.e. $S \geq T$ in $L(X,Y)$. Therefore $S = T$, and $T \in L^{\ell}(X,Y)$.

(b) Apply 3.1.

Corollary 3.4

Let $T \in L(X, Y)$ be the pointwise limit of a non-decreasing sequence $\{T_r\} \subseteq L(X, Y)$, of maps of finite rank.

Then T is compact.

Proof

Every T_r is a c.a.s. map because it is of finite rank.

It is also compact. Hence we can apply 3.3.

Instead of a non-decreasing sequence of maps in 3.1 and 3.3, we can have an upward directed family of maps. In fact we have the following:

Theorem 3.5

Let $T \in L^{\ell}(X, Y)$, and $\{T_{\alpha}\}_{\alpha \in I}$ be an upward directed family in $L^{\ell}(X, Y)$ such that $T_{\alpha} \leq T$, $\alpha \in I$, and $T_{\alpha}(x) \rightarrow T(x)$ for $x \in X$.

Then:

- (1) $\|T - T_{\alpha}\| \rightarrow 0$, and
- (2) T is compact, if each T_{α} is compact.

Corollary 3.6

Let $T \in L(X, Y)$ be the point wise limit of a directed net $\{T_{\alpha}\}_{\alpha \in I} \subseteq L(X, Y)$, of positive compact maps. Then T is also positive compact.

Although 1.3(a) has been proved in the first section, yet we prove it here again to exhibit an application of Th. 3.1.

Proposition 3.7

Let X be an AM-space. Then any positive linear map T from X into ℓ_1 is compact.

Proof

ℓ_1 is a base normed space with base

$B = \{\text{set of non-negative sequences } \{\xi_n\} \text{ such that } \sum \xi_n = 1\}.$

If $y \in \ell_+^1$, then $y = \sum_{i=1}^{\infty} \xi_i e_i$; $\xi_i \geq 0$, all i and where e_i is the element $\in \ell_1$, with 1 in the i th place and 0 elsewhere.

Let $x \in X$ and $y = Tx$. Then we define the operators $T_r: X \rightarrow \ell_1$ such that $T_r(x) = P_r(Tx)$, and where P_r is the projection of ℓ_1 on the span of $\{e_1, \dots, e_r\}$.

Since ℓ_1 is an AL-space, and T is positive, therefore T is continuous. This implies that for all r , T_r is continuous linear.

Now let $u \in X_+$, then $v = Tu \in \ell_+^1$. So if $v = \sum_i \xi_i e_i$, then $\xi_i \geq 0$, for all i and $P_r(v) = \sum_{i=1}^r \xi_i e_i \in \ell_+^1$.

Thus $T_r(u) = P_r(Tu) = P_r(v) \in \ell_+^1$, and T_r is positive.

Again from $y = \sum_{i=1}^{\infty} \alpha_i e_i \in \ell_+^1$, $\alpha_i \geq 0$; $P_r(y) = \sum_{i=1}^r \alpha_i e_i$,

and therefore $P_1 \leq P_2 \leq \dots \leq I$.

i.e. $T_1 \leq T_2 \leq \dots \leq T$, and also by definition of P_r and T_r ,

$T_r(x) \rightarrow T(x)$ for $x \in X$.

Finally for all r , T_r has a finite rank and is therefore compact. Now we apply cor. 3.2.

CHAPTER THREE

ORDER PROPERTIES OF $L(X,Y)$

Let X and Y be ordered Banach spaces. A short survey of the study of order structure of $L(X,Y)$ was given in section 3 of Chapter One. In this chapter some conditions are given under which $L(X,Y)$ will have certain useful order properties. In particular, we consider conditions in X and Y which ensure that $L(X,Y)$ will be regular, have a monotone norm or have a directed open unit hull. Study is also made to determine when the norm on $L(X,Y)$ is α -directed or α -additive. These are extensions of results in [25] and supplement the earlier results of Ellis [13] and Wickstead [34].

First we state a lemma due to Bonsall [6].

Lemma 1.1

Let X be a real vector space, with X_+ a wedge in X . Suppose P is a sublinear map from X into a complete vector lattice Y , and Q a super-linear map from X_+ into Y such that $Q(x) \leq P(x)$ for all $x \in X_+$. Then there is a linear map T from X into Y such that

$$\begin{aligned} T(x) &\leq P(x) & (x \in X) \\ Q(x) &\leq T(x) & (x \in X_+). \end{aligned}$$

In particular this lemma holds for functionals on X , when we take $Y = R$.

Proposition 1.2

Let Y be order unit normed, order complete lattice, and X satisfy (R_1) . Then $L(X, Y)$ satisfies (R_2) .

Proof

Let $T \in L(X, Y)$, and e be the order unit in Y . We define

$$P(x) = ||T|| \cdot ||x|| \cdot e, \quad x \in X,$$

$$Q(x) = \text{Sup} \{Tz: -x \leq z \leq x\}, \quad x \in X_+.$$

If $-x \leq z \leq x$, then

$$\begin{aligned} T(z) &\leq ||Tz|| \cdot e \\ &\leq ||T|| \cdot ||z|| \cdot e \\ &\leq ||T|| \cdot ||x|| \cdot e, \text{ since } X \text{ satisfies } (R_1), \\ &= P(x). \end{aligned}$$

Hence $Q(x)$ is well defined and $Q \leq P$. Also Q is super-linear and P is sublinear. Therefore Bonsall's lemma 1.1 gives us a linear map $S: X \rightarrow Y$ such that

$$S(x) \leq P(x), \quad x \in X; \text{ and } Q(x) \leq S(x), \quad x \in X_+.$$

$$\text{Thus } S(x) \leq ||T|| \cdot ||x|| \cdot e \text{ for } x \in X \text{ and}$$

$$-S(x) = S(-x) \leq P(-x) = ||T|| \cdot ||x|| \cdot e$$

$$\text{i.e. } -||T|| \cdot ||x|| \cdot e \leq S(x) \leq ||T|| \cdot ||x|| \cdot e.$$

Since Y is 1-normal, we have

$$||S(x)|| \leq ||T|| \cdot ||x|| \cdot ||e|| \leq ||T|| \cdot ||x||;$$

$$\text{i.e. } ||S|| \leq ||T|| \quad \text{and} \quad S \in L(X, Y).$$

Moreover from the definition of $Q(x)$ we have

$$+T(x) = T(+x) \leq Q(x) \leq S(x), \quad x \in X_+; \quad \text{i.e. } +T \leq S.$$

Thus given $T \in L(X, Y)$ there exists $S \in L(X, Y)$ such that

$$||S|| \leq ||T|| \quad \text{and} \quad +T \leq S.$$

and therefore $L(X,Y)$ satisfies (R_2) .

It is easy to show that if X has (R_2) and Y has (R_1) , then $L(X,Y)$ has (R_1) [36]. Combining this with the above result we have:

Proposition 1.3

Let X be regular and Y be an order unit normed, order complete lattice. Then $L(X,Y)$ is regular.

Proposition 1.4

Let $L(X,Y)$ satisfy (R_1) ; then X satisfies (R_2) and Y satisfies (R_1) .

Proof

Let $y \in Y_+$ with $||y|| = 1$, and $f, g, \in X^*$ such that $-f \leq g \leq f$.

We define $F, G: X \rightarrow Y$ by:

$$F(x) = f(x) \cdot y \quad x \in X$$

$$G(x) = g(x) \cdot y \quad x \in X$$

then $F, G \in L(X,Y)$.

Now let $x \in X_+$, then $-f(x) \leq g(x) \leq f(x)$

i.e. $-f(x) y \leq g(x) y \leq f(x) y$, in Y .

Thus $-F(x) \leq G(x) \leq F(x)$, $x \in X_+$

i.e. $-F \leq G \leq F$.

But $L(X,Y)$ satisfies (R_1) , and therefore $||G|| \leq ||F||$.

We also have $||G|| = \sup \{ ||G(x)|| : ||x|| \leq 1 \}$

$$= \sup \{ |g(x)| \cdot ||y|| : ||x|| \leq 1 \}$$

$$= \sup \{ |g(x)| : ||x|| \leq 1 \} = ||g||,$$

and similarly $||F|| = ||f||$.

Hence $\|g\| \leq \|f\|$ and therefore X^* satisfies (R_1) , which implies that X satisfies (R_2) . [25. Th. 7]

Next we prove that Y satisfies (R_1) .

Let $x_0 \in X_+$ ($x_0 \neq 0$). Since the cone is closed in S , there exists a positive $f \in X^*$ such that

$$f(x_0) = 1 \text{ and } \|f\| = 1$$

Let $-a \leq b \leq a$; $a, b \in Y$, and define $A, B: X \rightarrow Y$ such that

$$A(x) = f(x) \cdot a, \quad x \in X$$

$$B(x) = f(x) \cdot b, \quad x \in X.$$

Then $A, B \in L(X, Y)$ and since $-f(x) \cdot a \leq f(x) \cdot b \leq f(x) \cdot a$, we have

$$-A \leq B \leq A.$$

and therefore $\|B\| \leq \|A\|$;

where $\|A\| = \sup \{\|A(x)\| : \|x\| \leq 1\}$

$$= \sup \{\|f(x)\| \cdot \|a\| : \|x\| \leq 1\}$$

$$= \|f\| \cdot \|a\|.$$

$$\text{Similarly } \|B\| = \|f\| \cdot \|b\|.$$

$$\text{Therefore } \|f\| \cdot \|b\| \leq \|f\| \cdot \|a\|.$$

But $\|f\| \neq 0$ and hence $\|b\| \leq \|a\|$, i.e. Y satisfies (R_1) .

Combining Prop. 4.4 with Wong's result quoted in Prop.

1.3, we get:

Corollary 1.5

$L(X, Y)$ satisfies (R_1) iff X satisfies (R_2) and Y satisfies (R_1) .

Proposition 1.6

If $L(X, Y)$ satisfies (R_2) , then X satisfies (R_1) and

Y satisfies (R_2) .

Proof

(a) Let $y_0 \in Y_+$ with $\|y_0\| = 1$, and f be a positive continuous linear functional on Y with $f(y_0) = 1$ and $\|f\| = 1$.

Let $g \in X^*$, and $G: X \rightarrow Y$ be defined as

$$G(x) = g(x) \cdot y_0.$$

Then $G \in L(X, Y)$ and since $L(X, Y)$ satisfies (R_2) , given $\epsilon > 0$, there exists $H \in L(X, Y)_+$, with

$$\|H\| \leq \|G\| + \epsilon \quad \text{and} \quad \pm G \leq H.$$

$$\text{Since } \|G\| = \|g\|, \quad \|H\| \leq \|g\| + \epsilon$$

Now we define a functional h on X , as

$$h(x) = f(H(x)).$$

Then h is linear, positive, and

$$|h(x)| \leq \|f\| \cdot \|H\| \cdot \|x\| = \|H\| \cdot \|x\|$$

$$\text{i.e. } \|h\| \leq \|H\| \leq \|g\| + \epsilon.$$

Thus X^* satisfies (R_2) and therefore X satisfies (R_1)

[25. Th. 6].

(b) Next we prove that Y satisfies (R_2) . let $x_0 \in X_+$ with

$$\|x_0\| = 1, \text{ and } f \in X^* \text{ with } \|f\| = 1 \text{ and } f(x_0) = 1.$$

Let $y \in Y$. We define $T: X \rightarrow Y$ such that

$$T(x) = f(x) \cdot y, \quad x \in X.$$

Then $T \in L(X, Y)$, and there exists $S \in L(X, Y)_+$ such that

$$\|S\| \leq \|T\| + \epsilon, \quad \pm T \leq S, \quad \epsilon > 0.$$

Let $z = S(x_0)$. Then $z \geq \pm y$, and

$$\|z\| = \|Sx_0\| \leq \|S\| \leq \|T\| + \epsilon.$$

$$\text{But } \|T\| = \|y\| \text{ and therefore } \|z\| \leq \|y\| + \epsilon.$$

We can combine propositions 1.4 and 1.6 to obtain:

Proposition 1.7

Let $L(X,Y)$ be a regular space, then both X and Y are regular.

Next we look at the conditions under which $L(X,Y)$ has a monotone norm. If X satisfies:

(R_0) : For $x \in X$ with $||x|| < 1$, there is $y \geq 0$, x with $||y|| < 1$;

then X is boundedly generated i.e. X_+ is α -generating for some $\alpha > 0$. If further Y_+ is 1-normal then $L(X,Y)$ is α -normal [34]. But actually we have a more precise result:

Proposition 1.8

Let X satisfy (R_0) and Y_+ be 1-normal. Then the norm is monotone on $L(X,Y)$.

Proof

Let $S, T \in L(X,Y)$ with $0 \leq S \leq T$. For $x \in X$ with $||x|| < 1$, there are $y, z \in X_+$ such that $||y||, ||z|| < 1$ and $-z \leq x \leq y$.

Thus $-S(z) \leq S(x) \leq S(y)$, and $||S(x)|| \leq \max \{ ||S(z)||, ||S(y)|| \}$, since Y_+ is 1-normal. Therefore

$$\begin{aligned} ||S|| &= \sup \{ ||S(y)|| : y \in X_+, ||y|| < 1 \} \\ &\leq \sup \{ ||Ty|| : y \in X_+, ||y|| < 1 \} \\ &= ||T||. \end{aligned}$$

It is known that for a base-normed space the positive cone is $(1 + \epsilon)$ -generating for all $\epsilon > 0$ [25]. We also know that X_+^* is 1-normal iff X is $(1 + \epsilon)$ -generating (I. 2.1).

Thus we have:

Proposition 1.9

Let X_+ be $(1 + \epsilon)$ -generating for all $\epsilon > 0$. If the norm is monotone on Y , then so is the norm on $L(X, Y)$.

The proof is similar to that of Prop. 3.1 in [34] and is omitted.

Proposition 1.10

Let the norm be monotone on $L(X, Y)$. Then X satisfies (R_0) and Y has a monotone norm.

Proof

By an argument similar to that of Prop. 1.4 we can show that the norms on X^* and Y are monotone. Then [25: Prop. 6] implies that X satisfies (R_0) .

The converse to 1.10 also holds:

Proposition 1.11

Suppose $L(X, Y)$ has (R_0) . Then the norm is monotone on X , and Y has R_0 .

Proof

We can show, using a similar argument as that of Prop. 1.6, that X and Y have (R_0) and then norm is monotone on X [25: Prop. 5].

Sufficient conditions for $L(X, Y)$ to have (R_0) are given below:

Proposition 1.12

Let Y be an order complete, order unit normed lattice. If the norm is monotone on X , then $L(X, Y)$ has (R_0) .

Proof

Let e be the order unit in Y , and $T \in L(X, Y)$.

We define

$$Q(x) = \sup \{Tz : 0 \leq z \leq x\}, x \in X_+$$

$$\text{and } P(x) = ||T|| \cdot ||x|| \cdot e, x \in X.$$

Then Q is super-linear. Now we can follow the proof of Prop. 1.2 to show that $L(X,Y)$ has (R_0) .

Next we consider the α -directedness of $L(X,Y)$. The following result improves Theorem 3.4 of [34].

Proposition 1.13

let Y be an order unit normed space. $L(X,Y)$ is α -directed iff X is α -additive.

Proof

let $T_i \in L(X,Y)$, $||T_i|| \leq 1$, $i = 1, 2, \dots, n$. If e is the order unit in Y and U is the unit ball in X , then $T_i(U) \subseteq [-e, e]$, $i = 1, 2, \dots, n$.

$$\begin{aligned} \text{For } x \in X, T_i(x) &\leq ||T_i(x)|| e \\ &\leq ||x|| e \quad i = 1, \dots, n. \end{aligned}$$

Thus we set

$$r_i(x) = \inf \{\lambda : T_i(x) \leq \lambda e\}, i = 1, 2, \dots, n.$$

$$\text{and } r(x) = \max \{r_i(x) : i = 1, 2, \dots, n\}.$$

Obviously $r_i(x) \leq ||x||$, $i = 1, \dots, n$; and therefore

$$r(x) \leq ||x||.$$

let

$$q(x) = \sup \left\{ \sum_{j=1}^m r(x_j) : \sum_{j=1}^m x_j = x, x_j \geq 0, m = 1, 2, \dots \right\} (x \in X_+)$$

and

$$p(x) = \alpha ||x||, x \in X_+.$$

If $x \in X_+$ and $x = \sum_{j=1}^m x_j$, $x_j \geq 0$, we have

$$\sum_1^m r(x_j) \leq \sum_1^m ||x_j|| e \leq \alpha ||\sum_1^m x_j|| \leq \alpha ||x||$$

and therefore q is well-defined. Also q is superlinear, p is sublinear and $q(x) \leq p(x)$ $x \in X_+$.

Thus Bonsall's lemma 1.1 gives us a linear functional t on X ,

$$\begin{aligned} \text{with } t(x) &\leq p(x) & x \in X \\ q(x) &\leq t(x) & x \in X_+; \end{aligned}$$

i.e. $|t(x)| \leq \alpha ||x||$ and this implies that $t \in X^*$ with $||t|| \leq \alpha$.

Now we define $T: X \rightarrow Y$ by $T(x) = t(x) \cdot e$.

Then $T \in L(X, Y)$ and $||T|| \leq \alpha$. Also for $x \in X_+$,

$$T_1(x) \leq r(x) e \leq q(x) e \leq t(x) e \leq T(x).$$

Thus $L(X, Y)$ is (α, n) -directed for all n .

The proof of the converse is similar to [34: Th.3.4].

X is 1-additive iff norm is additive on X_+ and X is 1-directed iff the closed unit ball in X is directed upwards. Therefore we have:

Corollary 1.14

If Y is order unit normed, then $L(X, Y)$ has a directed closed unit ball iff norm is additive on X_+ .

In fact Propositions 1.2, 1.12 and 1.13 give us sufficient conditions for $L(X, Y)$ to be positively generated. Props. 1.2 and 1.12 can be generalized. We consider the following conditions: let $\alpha \geq 1$;

(a) $x, y \in X$ and $0 \leq x \leq y$ implies $||x|| \leq \alpha ||y||$.

(b) $x \in X$ and $||x|| < 1$ implies there is $y \geq 0, x$ with $||y|| < \alpha$

- (a') $x, y \in X$ and $-y \leq x \leq y$ implies $||x|| \leq \alpha ||y||$
 (b') $x \in X$ and $||x|| < 1$ implies there is $y \in X_+$; $x, -x \leq y$
 and $||y|| < \alpha$.

If Y is as in Prop.1.2 then $L(X,Y)$ has (b) iff X has (a), and $L(X,Y)$ has (b') iff X has (a'). Further if $L(X,Y)$ has (b) or (b') then so does Y .

From the remarks after I.3.2 we note that so far sufficient conditions are not known under which the operator norm on $L(X,Y)$ will be a base norm. However we can give sufficient conditions in order that the norm on $L(X,Y)_+$ be additive. We also consider the α -additivity of norm on $L(X,Y)_+$.

Proposition 1.15

Let the closed unit ball be directed upwards in X , and Y satisfy (R_1) and the norm be additive on Y_+ . Then the norm is additive on $L(X,Y)_+$.

Proof

Let $T \in L(X,Y)_+$ and U be the closed unit ball in X . We claim that

$$||T|| = \text{Sup } \{ ||Tx|| : x \in U_+ \} \text{ where } U_+ = U \cap X_+.$$

If $x \in U$, then $-x \in U$, and there exists $z \in U_+$ with $x, -x \leq z$ i.e. $-z \leq x \leq z$. and $-Tz \leq Tx \leq Tz$.

Since Y satisfies (R_1) , $||Tx|| \leq ||Tz||$, and our assertion is proved.

Next we pick $x, y \in U_+$ and $S, T \in L(X,Y)$. There exists $z \in U_+$ such that $x, y \leq z$. Since the norm is additive on

Y_+ and $Sx, Ty \in Y_+$, we have $||Sx|| + ||Ty|| = ||Sx + Ty||$.

But $Sx \leq Sz$ and $Ty \leq Tz$ implies $0 \leq Sx + Ty \leq Sz + Tz = (S + T)(z)$.

Therefore, the norm being monotone in Y_+ , we get

$$\begin{aligned} ||Sx|| + ||Ty|| &\leq ||(S + T)z|| \\ &\leq ||S + T|| \quad ||z|| \\ &\leq ||S + T||. \end{aligned}$$

Taking supremum respectively on x and y ,

$$||S|| + ||T|| \leq ||S + T||. \quad \text{i.e. } ||S|| + ||T|| = ||S + T||.$$

Corollary 1.16

Let X be l -directed, Y satisfy (R_1) and be α -additive.

Then $L(X, Y)$ is α -additive.

Proof

Let $T_i \in L(X, Y)_+$, $x_i \in U_+$, $i = 1, \dots, r$. There exists $z \in U_+$ with $z \geq x_i$, $i = 1, \dots, r$.

$$\text{Then } ||T_1 x_1|| + \dots + ||T_r x_r||$$

$$\leq \alpha \quad ||\sum_{i=1}^r T_i x_i||$$

$$\leq \alpha \quad ||\sum_{i=1}^r T_i(z)||$$

$$\leq \alpha \quad ||\sum_{i=1}^r T_i|| \quad ||z||$$

$$\leq \alpha \quad ||\sum_{i=1}^r T_i||.$$

Taking supremum respectively on x_1, \dots, x_r , we have

$$\sum_{i=1}^r ||T_i|| \leq \alpha \quad ||\sum_{i=1}^r T_i||.$$

Now we look at the lattice properties of $L(X, Y)$.

Proposition 1.17

Let X be base normed, have the R.D.P. and X_+ be closed convex hull of its extreme rays. If Y is a lattice then so also is $L(X, Y)$.

Proof

Let $F \in L(X, Y)$. We shall show that F^+ exists in $L(X, Y)$.

We define $G(p) = (F(p))^+$, $p \in \partial B$ where B is the base in X .

G can be extended to B by convexity and then to X by linearity. Obviously G is linear. Since the lattice operations are continuous in Y , G is continuous on ∂B and hence on X .

If $b \in B$, then

$$b = \sum_{i=1}^n \alpha_i p_i, \quad p_i \in \partial B \text{ and } \alpha_i > 0.$$

$$\text{Thus } G(b) = \sum_{i=1}^n \alpha_i G(p_i) = \sum_{i=1}^n \alpha_i F(p_i)$$

$$\geq \sum_{i=1}^n \alpha_i F(p_i) = F\left(\sum_{i=1}^n \alpha_i p_i\right) = F(b).$$

i.e. $G \geq F, 0$. Further if $E \geq F, 0$ then we can show that

$G \leq E$. Hence $G = F^+$.

CHAPTER FOUR
ORDER PROPERTIES OF $K(X,Y)$

The order properties of $L(X,Y)$, where X and Y are ordered Banach spaces, have been discussed in quite some detail but the order structure of $K(X,Y)$ needs more exploration. Section 4 in Chapter One contains a summary of Wickstead's results in this direction. His work was mostly concentrated on the positive generation and order completeness of $K(X,Y)$. In this chapter an effort is made to give an extensive study of order properties of $K(X,Y)$. Sufficient conditions are given in Section 1, and Section 2 covers the necessary conditions. The third section deals with the existence of weak order units and quasi interior points in $K(X,Y)$.

1. Sufficient Conditions

It was mentioned in section 4 of chapter one that if Y is a simplex space then $K(X,Y)$ can be identified with $A_0(C, X^*)$ where $C = \{f \in Y^* : f \geq 0, ||f|| \leq 1\}$.

In particular if $Y = A(\tilde{C})$, \tilde{C} a simplex then $K(X,Y)$ is identical with $A(\tilde{C}, X^*)$. Thus using duality we can derive a number of results from [4], [34] and [36].

Proposition 1.1

Let \tilde{C} be a simplex and $Y = A(\tilde{C})$.

- (1) If X_+ is normal then $K(X, Y)_+$ is generating.
- (2) If X is regular then $K(X, Y)$ is regular.
- (3) If the norm is additive on X_+ , then the closed unit ball is directed in $K(X, Y)$.
- (4) If X is α -additive, then $K(X, Y)$ is α -directed.
- (5) If X_+ is normal, positively generating and X has the R.D.P. then $K(X, Y)$ has these properties.

If, further, \tilde{C} is a Bauer simplex then $K(X, Y)$ is a Banach lattice; and if X is an AL-space then $K(X, Y)$ is an AM-space with an order unit.

Proof

We show that $A(\tilde{C}, X^*)$ has the required properties.

- (1) X^* is positively generated and theorem 2.3 of [4] implies that $A(\tilde{C}, X^*)$ is positively generated.
- (2) X^* is regular (I.2.5) and we can apply [36: II.3.8].
- (3) The closed unit ball of X^* is directed (I.2.7) and then so is that of $A(\tilde{C}, X^*)$ [36: II.3.4].
- (4) X^* is α -directed (I.2.9) and the result follows from [36: II.3.5].
- (5) X^* is a Banach lattice [34: Th.2.8] and using [4: 2.5] we see that $A(\tilde{C}, X^*)$ has the stated properties.

If \tilde{C} is a Bauer simplex, $A(\tilde{C}, X^*)$ is a Banach lattice [36: II.3.13]. When X is an AL-space, X^* is an AM-space with order unit. Now the result follows from [36.II.3.13] and [I.3.1].

When the range space is a simplex space, we have:

Proposition 1.2

Let Y be a simplex space.

- (1) If X_+ is normal, generating and X has the R.D.P. then $K(X,Y)_+$ is normal, generating and has the R.D.P.
- (2) If X is base normed and has the R.D.P. then $K(X,Y)$ is a simplex space.
- (3) If X is base normed with R.D.P.; and Y has a lattice order, then $K(X,Y)$ is an AM-space.

Proof

As before we will show that $A_0(C, X^*)$ has the required properties where $C = \{f \in Y_+^* : ||f|| \leq 1\}$.

- (1) X_+^* is normal, generating and has R.D.P. Since $\{0\}$ is a closed face of C , we can modify proof of [4: Th.2.4] to prove that $A_0(C, X^*)$ has these properties.
- (2) From (1) $K(X,Y)$ has the R.D.P. Theorem 4.8 of [34] implies that $K(X,Y)$ is approximate order unit normed. Hence $K(X,Y)$ is a simplex space.
- (3) X^* is an AM-space with order unit. Cor. 4.7 of [34] implies that $K(X,Y)$ is a lattice. Then from [25. Cor.2.Th.7] and (2) we see that it is in fact an AM-space.

Following the selection theorem technique we obtain,

Proposition 1.3

Let Y be a simplex space. Then $K(X,Y)$ is α -directed if and only if X is α -additive.

Proof

- (a) Let X be α -additive. Then X^* is α -directed and we show that $A_0(C, X^*)$ is also α -directed. We shall be using

Theorem I. 4.2.

Let $\sigma_i \in A_0(C, X^*)$, $i = 1, \dots, n$ with $||\sigma_i|| \leq 1$.

Define a map $\phi: K \rightarrow 2^{X^*}$ by

$$\phi(k) = \{y \in X^*: y \geq \sigma_i(k), i = 1, \dots, n; ||y|| < \alpha + \overline{\epsilon}, k \neq 0$$

$$\phi(0) = 0$$

$\phi(k)$ is closed and non-empty since X^* is α -directed. It is easy to show that it is convex and affine. To prove that it is l.s.c.; we take an open set $V \subseteq X^*$, $k_0 \in K$ such that $\phi(k_0) \cap V \neq \emptyset$. let $P = \{k \in K: \phi(k) \cap \overline{V} \neq \emptyset\}$. Then $k_0 \in P$. We want to find an open set $W \subseteq P$, which contains k_0 . This would imply that P is open and hence ϕ is l.s.c.

There exists $v_0 \in \phi(k_0) \cap V$ with $||v_0|| < \alpha + \epsilon$.

let $||v_0|| = (\alpha + \epsilon) - \delta$, $\delta > 0$ and suppose

$$\{v: ||v - v_0|| < \gamma\} \cap V \neq \emptyset \text{ for some } \gamma > 0.$$

Let W be the open set

$$\{k \in K: ||\delta_i(k) - \delta_i(k_0)|| < \frac{1}{2} \min\{\gamma, \delta\} / \alpha, i = 1, \dots, n. \}$$

If $\frac{1}{2} \min\{\gamma, \delta\} / \alpha = \beta$ (say), then for $k \in W$,

$$||(\sigma_i(k) - \sigma_i(k_0)) / \beta|| < 1, \quad i = 1, \dots, n \text{ and because}$$

X^* is α -directed, there exists $q \geq (\sigma_i(k) - \sigma_i(k_0)) / \beta$

with $||q|| \leq \alpha$.

Let $p = \beta \cdot q$ then

$$p \geq \sigma_i(k) - \sigma_i(k_0) \quad i = 1, \dots, n$$

and $||p|| \leq \alpha \cdot \beta \leq \frac{1}{2} \min\{\gamma, \delta\}$.

If we put $v = v_0 + p$ then $v \geq \sigma_i(k)$ $i = 1, \dots, n$; and

$$||v|| \leq ||v_0|| + ||p|| \leq (\alpha + \varepsilon) - \delta + \frac{1}{2}\delta < \alpha + \varepsilon.$$

Thus $v \in \phi(k)$. $(k \in W)$.

Also $||v - v_0|| = ||p|| \leq \frac{1}{2} \gamma < \gamma^*$, which implies that $v \in V$

i.e. $v \in \phi(k)$ V and therefore $k \in P$. Hence $k_0 \in W \cap P$

and P is open.

Now we apply Th. I.4.2 to obtain a continuous affine selection ϕ of ϕ such that

$$\phi(k) \in \phi(k) \quad k \in K.$$

$$\phi \geq \sigma_i \quad i = 1, \dots, n$$

$$||\phi|| \leq \alpha \quad \text{and} \quad \phi \in A_0(C, X^*).$$

Therefore $A_0(C, X^*)$, and hence $K(X, Y)$ is α -directed.

(b) Conversely let $K(X, Y)$ be α -directed, & $y_0 \in Y_+$ with

$$||y_0|| = 1. \quad \text{There exists } f \in Y_+^* \text{ such that } f(y_0) = 1 = ||y_0||. \quad \triangle$$

$$\text{Let } g_i \in X^* \text{ with } ||g_i|| \leq 1, \quad i = 1, \dots, n.$$

We define $G_i: X \rightarrow Y \quad i = 1, \dots, n$

$$\text{as } G_i(x) = g_i(x) \cdot y_0 \quad x \in X.$$

Then $G_i \in K(X, Y)$ and $||G_i|| \leq 1$.

Therefore there exists $G \geq G_i$ with $||G|| \leq \alpha$, $G \in K(X, Y)_+$.

$$\text{Let } g(x) = f(G(x))$$

If $||x|| \leq 1$, then

$$\begin{aligned} |g(x)| &= |f(G(x))| \leq ||f|| \cdot ||G(x)|| \\ &\leq ||G|| \end{aligned}$$

therefore $||g|| \leq ||G|| \leq \alpha$, and $g \in X^*$.

If $x \in X_+$, $f(G(x)) \geq f(G_i(x)) \quad i = 1, \dots, n.$

$$= g_i(x) \cdot f(y_0)$$

$$= g_i(x)$$

Thus $g \geq g_i \quad i = 1, \dots, n.$

This implies that X^* is α -directed and then by [I.2.9] X is α -additive.

Corollary 1.4

Let Y be a simplex space. The closed unit ball of $K(X, Y)$ is directed if and only if the norm is additive on X_+ .

Several of the proofs in Chapter Three are also valid for the space of compact maps. In particular Propositions 1.4, 1.6, 1.8, 1.9, 1.10, 1.13, 1.14, 1.15, 1.16 hold true for $K(X, Y)$.

Next we introduce a concept which is useful for the later work. let X be order complete, then X will be said to have property (S) if it satisfies the following condition:

"If $P \subseteq X$ is bounded above then the set of finite suprema from P converges in norm to $\text{Sup}(P)$."

Proposition 1.5

Suppose Y is order complete, 1-directed, and has the properties (R_1) and (S). If X is α -additive then $K(X, Y)$ is α -directed.

Proof

Let $T_i \in K(X, Y)$ with $\|T_i\| \leq 1, \quad i = 1, 2, \dots, n;$
and U be the unit ball in X . Then $P = \bigcup_{i=1}^n (T_i(U))$ is relatively

compact and hence order bounded (Th. I.4.6).

Let $e = \text{Sup}(P)$ and $\{y_j\}_{j=1}^m$ be a finite subset of P .

Since $\|y_j\| \leq 1$ and Y is 1-directed, there exists $y \in Y_+$

with $\|y\| \leq 1$ and $y \geq y_j$, $j = 1, \dots, m$.

Let $u = \text{Sup} \{y_j\}$, then $y_j \leq u \leq y$.

Similarly there is $z \in Y_+$ with $\|z\| \leq 1$ and $-y_j \leq z$, $j = 1, \dots, m$.

Again there exists $v \geq y, z$ and $\|v\| \leq 1$. Then

$$-v \leq y_j \leq u \leq y \leq v \text{ which implies that } \|u\| \leq 1$$

since Y has (R_1) .

Now let $\{z_\delta\}_{\delta \in I}$ be the family of finite suprema from P .

By hypothesis $\{z_\delta\}$ converges in norm to e , i.e. $\|e - z_\delta\| \rightarrow 0$.

Then given $\epsilon > 0$, there exists γ such that $\|e - z_\gamma\| < \epsilon$

and therefore $\|e\| < \|z_\gamma\| + \epsilon \leq 1 + \epsilon$.

Thus $\|e\| \leq 1$ and we can follow the proofs of Prop. III.1.2. and III.1.13 to obtain $T \in K(X, Y)$ such that $T \geq T_i$, $i = 1, \dots, n$ and $\|T\| \leq \alpha \|e\| \leq \alpha$.

Similarly one can show that if Y is order complete, α -directed, has Property (S), Y_+ is 1-normal, and X is 1-additive then $K(X, Y)$ is α -directed.

Corollary 1.6

Let Y be order complete with properties (R_1) and (S) and suppose the closed unit ball in Y is directed. If the norm is additive on X_+ , the closed unit ball of $K(X, Y)$ is directed.

Proposition 1.7

Let Y be an order complete lattice with compact order intervals and directed closed unit ball.

(a) If X satisfies (R_1) then $K(X, Y)$ satisfies (R_2) .

(b) If the norm is monotone on X , then $K(X, Y)$ satisfies (R_0) .

Proof

Let $T \in K(X, Y)$, with $||T|| \leq 1$, and U be the closed unit ball in X . Since Y satisfies the hypothesis of Prop. 1.5 [I.4.7] we can show as in 1.5, that there exists $e \in Y$ with $||e|| \leq 1$ and $T(U) \subseteq [-e, e]$.

(a) Let $P(x) = ||T|| \cdot ||x|| \cdot e$, $x \in X$;

③ $Q(x) = \text{Sup} \{Tz: -x \leq z \leq x\}$, $x \in X_+$.

A proof similar to that of Prop. III.1.2 gives us $S \in L(X, Y)$ with $S \leq P$, $S \geq Q$ and $||S|| \leq ||T||$.

Thus $S(x) \leq ||T|| \cdot ||x|| \cdot e$.

If $||x|| \leq 1$, then $S(x) \leq e$; i.e. $S(U) \subseteq [-e, e]$.

Since $[-e, e]$ is compact, $S \in K(X, Y)$.

(b) This follows from part (a) and III.1.12.

Prop. 1.7 can be generalized as follows: If Y satisfies the hypothesis of 1.7 and (a), (a'), (b), (b') are the four properties given after Prop. III.1.14 then

(i) If X has (a), $K(X, Y)$ has (b).

(ii) If X has (a'), $K(X, Y)$ has (b').

In the following propositions we consider the lattice structure of $K(X, Y)$.

Proposition 1.8

Let X be regular with the R.D.P. and Y satisfy the equivalent assertions of Thm. 1.4.7. If further Y is 1-

directed then $K(X, Y)$ is an order complete lattice.

Proof

Since X has R.D.P. and X_+ is generating $K(X, Y)$ is order complete by Thm. 1.4.7.

X has (R_1) and Y satisfies the hypothesis of Prop. 1.7(a), therefore $K(X, Y)$ is positively generated.

Both implications together give the required result.

Proposition 1.9

Let Y be reflexive. If $K(Y^*, X^*)$ is a lattice, then so also is $K(X, Y)$.

Proof

Let $E, F \in K(X, Y)$. Then $E^*, F^* \in K(Y^*, X^*)$ and $M = E^* \vee F^*$ exists,

Let $G = M^*|_X$. Then since $E = E^{**}|_X$ we have $G \geq E, F$.

If further $H \geq E, F$, then $H^* \geq E^*, F^*$ and therefore $M \leq H^*$.

Thus $G \leq H$ and $G = E \vee F$.

Corollary 1.10

Let X be base-normed, have the R.D.P., and Y be a reflexive lattice. Then $K(X, Y)$ is a lattice.

Proof

Y^*, X^* are Banach lattices and X^* has an order unit.

(By [I. 4.4] $K(Y^*, X^*)$ is a lattice.)

(Note: Props. 1.9, 1.10 are valid for $L(X, Y)$ as well. One can use [I. 3.6]).

Similarly one can show that if Y is reflexive and $K(Y^*, X^*)$ is order complete, then so also is $K(X, Y)$.

2. Necessary Conditions

In this section we look at necessary conditions in order that $K(X,Y)$ have certain order-properties.

Proposition 2.1:

Let $K(X,Y)$ be α -directed. Then X is α -additive and Y is α -directed.

Proof

(a) First we prove that Y is α -directed. Let $x_0 \in X_+$ with $||x_0|| = 1$. There exists $f \in X_+^*$ such that $f(x_0) = 1 = ||f||$.

Let $y_i \in Y$, $||y_i|| \leq 1$ $i = 1, \dots, n$.

We set $T_i(x) = f(x) \cdot y_i$ $x \in X$.

Then $T_i \in K(X,Y)$ and $||T_i|| \leq ||f|| \cdot ||y_i|| \leq 1$.

Therefore there exists $S \in K(X,Y)$ with $||S|| \leq \alpha$ and $S \geq T_i$ $i = 1, \dots, n$.

Let $z = Sx_0 \in Y$. Then $z \geq T_i(x_0) = f(x_0) y_i = y_i$

and $||z|| \leq ||S|| \cdot ||x_0|| \leq \alpha$.

(b) To show that X is α -additive, the proof is the same as in Prop. 1.3.

Corollary 2.2

If the closed unit ball is directed in $K(X,Y)$ then the norm is additive on X_+ and the closed unit ball in Y is directed.

In Prop. III.1.7 we showed that if $L(X,Y)$ is regular then both X and Y are regular. The same proof holds for $K(X,Y)$ as well. Thus we have:

Proposition 2.3

Let $K(X, Y)$ be regular. Then both X and Y are regular.

. Similarly from Prop. I.3.3 and Prop. I.3.4 we get:

Proposition 2.4

(a) If $K(X, Y)$ is positively generated then X_+ is normal and Y is positively generated.

(b) If $K(X, Y)_+$ is normal, then X is positively generated and Y_+ is normal.

Propositions III.1.10 and III.1.11 give us:

Proposition 2.5

If the norm is monotone on $K(X, Y)$ then the norm on Y is also monotone and X satisfies (R'_0) .

Further if $K(X, Y)$ has (R_0) , then the norm is monotone on X and Y has (R_0) .

Proposition 2.6

Let $K(X, Y)$ be α -additive. Then X is approximately α -directed and Y is α -additive.

Proof

(a) We choose $x_0, y_i, T_i, i = 1, \dots, n$, and f as in 2.1.

$$\begin{aligned} ||T_i|| &= \text{Sup}\{||f(x) \cdot y_i|| : ||x|| \leq 1\} \\ &= ||y_i|| \cdot ||f|| = ||y_i||. \end{aligned}$$

$$\text{Thus } \sum_{i=1}^n ||y_i|| = \sum_{i=1}^n ||T_i|| \leq \alpha \cdot ||\sum_{i=1}^n T_i|| = \alpha ||\sum_{i=1}^n y_i||;$$

$$\text{because } (\sum T_i)(x) = \sum T_i(x) = f(x) \cdot \sum y_i$$

$$\text{implies } ||\sum_{i=1}^n T_i|| = ||\sum_{i=1}^n y_i||. \quad \text{Therefore } Y \text{ is } \alpha\text{-additive.}$$

(b) Let $y_0 \in Y_+$ with $\|y_0\| = 1$, and

$f_i \in X_+^*$ with $\|f_i\| \leq 1$, $i = 1, \dots, n$.

There exists $g \in Y_+^*$ such that $g(y_0) = 1 = \|g\|$.

We set $F_i(x) = f_i(x) \cdot y_0$ so that $\|F_i\| = \|f_i\|$ and $F_i \in K(X, Y)$.

$(\sum_{i=1}^n F_i)(x) = y_0 \cdot \sum_{i=1}^n f_i(x)$ implies that $\|\sum_{i=1}^n F_i\| = \|\sum_{i=1}^n f_i\|$.

Therefore $\sum_{i=1}^n \|f_i\| = \sum_{i=1}^n \|F_i\| \leq \alpha \|\sum_{i=1}^n F_i\| = \alpha \|\sum_{i=1}^n f_i\|$,

and X^* is (α, n) -additive. By Thm. I.2.9. X is approximately α -directed.

Corollary 2.7

If the norm is additive on $K(X, Y)_+$ then the open unit ball is directed in X and the norm is additive on Y_+ .

Combining Prop. 2.6 with 2.4 we have:

Proposition 2.8

If $K(X, Y)$ is base normed then X is approximate order unit normed and Y is base normed.

Proof

From Prop. 2.6 we note that Y and X^* are 1-additive, since $K(X, Y)$ is so. Also the positive generation of $K(X, Y)$ implies the positive generation of Y and X^* (Prop. 2.4). Thus Y and X^* are both base normed. By I.2.4 X is approximate order unit normed.

Proposition 2.9

Let X_+ be normal and positively generating.

(a) If $K(X, Y)$ is a lattice, then X has the R.D.P. and Y

is a lattice.

(b) If $K(X, Y)$ has the R.D.P. then so do X and Y .

(c) If Y is positively generated and $K(X, Y)$ is order complete, then X has the R.D.P. and Y is a complete vector lattice.

Proof

(a) Let $y, z \in Y$ and $x_0 \in X$ with $\|x_0\| = 1$. There is $f \in X_+^*$ with $f(x_0) = 1$. We define $F(x) = f(x) \cdot y$ and $E(x) = f(x) \cdot z$, $x \in X$.

Since $E, F \in K(X, Y)$, there is $G = E \vee F \in K(X, Y)$. Let $v = G(x_0)$. Then $y = F(x_0) \leq G(x_0) = v$ and similarly $z \leq v$.

If $t \geq y, z$, we define $T(x) = f(x) \cdot t$, $x \in X$. Thus $t = T(x_0)$ and $T \geq E, F$ i.e. $T \geq G$ and $t \geq v$. Hence $v = \sup(y, z)$ and this implies that Y is a lattice.

The proof that X has the R.D.P. is similar to [34: Prop. 3.16].

(b) This can be proved in a similar way to part (a).

(c) The proof is the same as in [34: Prop. 3.16].

Corollary 2.10

Let X_+ be normal and positively generating.

(1) If $K(X, Y)$ is a simplex space, then X is base-normed with the R.D.P.; and Y is a simplex space.

(2) If $K(X, Y)$ is an AM-space, then X is base-normed with R.D.P. and Y is an AM-space.

Proof

(1) $K(X, Y)$ is approximate order unit normed and has R.D.P. Therefore X is base normed and Y has approximate order unit

[I.4.5], and further X, Y has R.D.P. by 2.9(b). Thus Y is a simplex space.

(2) $K(X, Y)$ is now a simplex space with a lattice order. Therefore by 2.9(a) Y is a lattice and hence by (1) is an AM-space.

Combining 2.9(a) with 2.8 we note that if $K(X, Y)$ is an AL-space then X is a simplex space and Y is an AL-space.

3. Weak Order Units and Quasi Interior Points

An element x_0 of X_+ is said to be a weak order unit of X if, for each non zero $x \in X_+$ there is a $z \in X_+$ such that $z \neq 0$ and $z \leq x_0$, $z \leq x$. An element x_0 of X_+ is a quasi interior point of X_+ if the linear hull of the order interval $(0, x_0)$ is dense in X . When X is a lattice then x_0 is a weak order unit in X_+ if $x \in X$ and $|x| \wedge x_0 = 0$ implies that $x = 0$.

In the following we discuss conditions on X and Y under which $K(X, Y)$ has a weak order unit or a quasi interior point. First we consider some necessary conditions.

Proposition 3.1

Let X be base normed.

- (a) If $K(X, Y)$ has a weak order unit, then so does Y .
- (b) If Y is positively generated, and $K(X, Y)$ has a quasi interior point, then so does Y .

Proof.

- (a) Let E be a weak order unit in $K(X, Y)$, $y \in Y_+$ and $y \neq 0$.

We define $F: X_+ \rightarrow Y$ such that $F(x) = \|x\| \cdot y$, $x \in X_+$.

Since X is base normed, F is additive and positive homogeneous on X_+ . We can extend it to X by linearity. Then clearly $F \in K(X, Y)_+$. $F \neq 0$ because $y \neq 0$.

From hypothesis, there is a non-zero $G \in K(X, Y)_+$ such that $G \leq E, F$. Since G is non-zero there exists $x' \in X_+$ such that $x' \neq 0$ with $G(x') \neq 0$. $E \geq G$ implies that $E(x') \neq 0$.

Let $x_0 = x' / \|x'\|$. Then $\|x_0\| = 1$ and therefore $F(x_0) = y$. We set $E(x_0) = y_0$, and $G(x_0) = z$. Then $y_0 \neq 0$, $z \neq 0$ and $z \leq F(x_0), E(x_0)$. i.e. $z \leq y, y_0$. Hence y_0 is a weak order unit in Y .

(b) Let E be a quasi interior point in $K(X, Y)$.

Then $\bigcup_{n=1}^{\infty} [0, nE]$ is dense in $K(X, Y)_+$. Let $y \in Y_+$ and F

be defined as in part (a). Then given $\varepsilon > 0$, there exist non zero G in the order interval $[0, E]$ and $m \in \mathbb{N}$ with

$\|F - mG\| < \varepsilon$. We take x_0 as in (a), $y_0 = E(x_0) \neq 0$,

$F(x_0) = y$ and $z = G(x_0) \neq 0$.

Since $\|F - mG\| = \sup \{ \|(F - mG)(x)\| : \|x\| \leq 1 \} < \varepsilon$.

we have that $\|F(x_0) - mG(x_0)\| < \varepsilon$; i.e. $\|y - mz\| < \varepsilon$.

This implies that $z \in [0, y_0]$; and hence $\bigcup_{n=1}^{\infty} [0, ny_0]$ is dense in Y_+ .

Y being positively generated the linear hull of $[0, y_0]$ is dense in Y , and therefore y_0 is a quasi interior

point of Y_+ .

(The proof is valid for $L(X, Y)$).

Next we look at some sufficient conditions.

Proposition 3.2

Let \tilde{C} be a Bauer simplex, $Y = A(C)$ and X be a countably order complete Banach lattice. If the norm is order-continuous on X and it has a quasi interior point, then $K(X, Y)$ has a weak order unit.

Proof

X^* is a Banach lattice, and it has a weak order unit say e [31: Thm. II.6.6]. Prop. 1.1(5) implies that $A(\tilde{C}, X^*)$ is a Banach lattice. We will show that $A(\tilde{C}, X^*)$ has a weak order unit.

We define $\sigma: \tilde{C} \rightarrow X^*$ by $\sigma(k) = e, \quad k \in \tilde{C}$.

Since σ is a constant map, it is affine and continuous; i.e. $\sigma \in A(\tilde{C}, X^*)$.

If $\pi \in A(\tilde{C}, X^*)_+$ with $\sigma \wedge \pi = 0$, then for $k \in \partial \tilde{C}$ we have:

$$\begin{aligned} 0 &= (\sigma \wedge \pi)(k) = \sigma(k) - (\sigma - \pi)^+(k) \\ &= \sigma(k) - (\sigma(k) - \pi(k))^+ \quad [36: \text{II.3.13}] \\ &= \sigma(k) - [\sigma(k) - (\sigma(k) \wedge \pi(k))] \\ &= \sigma(k) \wedge \pi(k) \\ &= e \wedge \pi(k). \end{aligned}$$

But e being a weak order unit in X^* , $\pi(k) = 0$. Then since \tilde{C} is a simplex, $\pi(h) = 0$ for all h in \tilde{C} . i.e. σ is a weak order unit in $A(\tilde{C}, X^*)$.

If Y is an AM-space with a weak order unit, we have:

Proposition 3.3

Let X be base normed with the R.D.P. and X_+ be the closed convex hull of its extreme rays. If Y is an AM-space with a weak order unit, then $K(X, Y)$ has a weak order unit.

Proof

Proposition 1.2 part. (3) implies that $K(X, Y)$ is a lattice. Let B denote the base of X_+ , and e be the weak order unit in Y . We set $E(x) = ||x||e$, $x \in X_+$; and we extend it to X by linearity since X is positively generated. Then $E \in K(X, Y)_+$.

Let $F \in K(X, Y)_+$ with $E \wedge F = 0$. For $p \in \partial B$ and $G \in K(X, Y)$ we have

$$\begin{aligned} (G \vee 0)(p) &= \sup \{G(u) : 0 \leq u \leq p\} \\ &= \sup \{||u|| \cdot G(p) : 0 \leq u \leq p, \text{ since } p \text{ lies} \\ &\quad \text{on an extreme ray.}\} \\ &\leq \sup \{||u|| \cdot G(p)^+ : 0 \leq u \leq p\} \\ &\leq \sup G(p)^+ \\ &= G(p)^+ \end{aligned}$$

On the other hand

$$G(p)^+ = G(p) \vee 0 \leq \sup \{G(u) : 0 \leq u \leq p\} = (G \vee 0)(p).$$

Thus for $p \in \partial B$, $G^+(p) = G(p)^+$, and therefore

$$0 = (E \wedge F)(p) = E(p) \wedge F(p) = e \wedge F(p).$$

This implies $F(p) = 0$, and since B is the closed convex hull of its extreme boundary, $F(b) = 0$ for all $b \in B$. Hence $F \equiv 0$.

In Prop. III.1.17 we saw that if X is base normed, has the R.D.P.; X_+ is closed convex hull of its extreme rays and Y is a lattice then $L(X,Y)$ is a lattice. If further Y has a weak order unit then the above proof holds in this case as well, and therefore $L(X,Y)$ has a weak order unit.

Proposition 3.4

Let Y be reflexive.

- (a) If $K(Y^*, X^*)$ has a weak order unit then so does $K(X,Y)$.
- (b) If X is α -additive, Y is β -directed, and $K(Y^*, X^*)$ has a quasi interior point, then so does $K(X,Y)$.

Proof

(a) Let T be a weak order unit in $K(Y^*, X^*)$ and $E = T^*|_X$. If $F \in K(X,Y)_+$, $F \neq 0$ then $F^* \in K(Y^*, X^*)_+$ and by a Hahn-Banach argument we can show that $F^* \neq 0$. Therefore there exists $S \in K(Y^*, X^*)_+$ such that $S \leq T, F^*$ and $S \neq 0$.

Let $G = S^*|_X$, then $G \neq 0$ and $G \leq E, F$. Thus E is weak order unit in $K(X,Y)$.

(b) Let A be a quasi interior point in $K(Y^*, X^*)$. Y^* is β -additive and X^* is α -directed. Therefore $K(Y^*, X^*)$ is positively generated by Thm. I.4.6.

Let F be a non zero element in $K(X,Y)$ and $E = A^*|_X$. Then $F^* \neq 0$ and $F^* \in K(Y^*, X^*)$. Since $\bigcup_{n=1}^{\infty} [0, nA]$ is dense in $K(Y^*, X^*)_+$, we have that given $\epsilon > 0$ there is $B \in [0, A]$, $B \neq 0$ and there is $m \in \mathbb{N}$ with $\|F^* - m B\| < \epsilon$.

Let $G = B^*|_X$. Then $G \neq 0$ and $\|F - mG\| < \epsilon$. Since $B \leq A$, $G \leq E$ i.e. $G \in [0, E]$.

Thus $\bigcup_{1}^{\infty} [0, nE]$ is dense in $K(X, Y)_+$ and then the positive generation of $K(X, Y)_+$ implies that the linear span of the order interval $[0, E]$ is dense in $K(X, Y)$.

For an application of 3.4 first we note the following.

A positive regular Borel measure μ on a compact Hausdorff space \mathcal{E} is said to be normal if $\mu(B) = 0$ for each Borel set B of first category in \mathcal{E} . Let $N(\mathcal{E})_+$ denote the set of all positive normal regular Borel measures on \mathcal{E} . We set $N(\mathcal{E}) = N(\mathcal{E})_+ - N(\mathcal{E})_+$. If further \mathcal{E} is extremely disconnected and the union of the supports of the positive normal measures is dense in \mathcal{E} , then \mathcal{E} is called hyperstonian [19].

Let \tilde{C} be a Baur simplex then $A(\tilde{C}) = C(\partial\tilde{C})$. If $\partial\tilde{C}$ is hyperstonian then $C(\partial\tilde{C}) = N(\partial\tilde{C})^*$. Thus we can obtain the following result from 3.4 and 3.2:

Proposition 3.5

Let \tilde{C} be a Baur simplex, $\partial\tilde{C}$ be hyperstonian and $X = N(\partial\tilde{C})$. Suppose Y is an order complete reflexive Banach lattice having a weak order unit. Then $K(X, Y)$ has a weak order unit.

Proof

$X^* = A(\tilde{C})$ from above remarks and Y^* is an order complete reflexive Banach lattice having a quasi interior point [35: II.6.5 and II.6.6]. Since the norm is order continuous

in a reflexive Banach lattice, Prop. 3.2 implies that $K(Y^*, X^*)$ has a weak order unit. Then $K(X, Y)$ has a weak order unit by Prop. 3.4(a).

CHAPTER FIVE

COMPACTNESS IN REGULAR ORDERED BANACH SPACES

In this chapter we discuss a relation between order and compactness in ordered Banach spaces. Wickstead has shown that if X has the R.D.P. with a generating cone and the order intervals are norm compact in it, then it is a complete vector lattice and the solid hull of a norm compact set in X is norm compact [35]. It is natural to ask if a similar result holds for those ordered spaces which do not necessarily have the R.D.P. We prove below that this is in fact true for regular ordered spaces. In such spaces the order intervals are norm compact if and only if the order convex cover of a compact set is compact. First we have a few definitions.

A sequence $\{x_n\}$ in an ordered Banach space X will be called relatively uniformly convergent to $x_0 \in X$, if there exists an element $u \in X_+$ and a sequence $\{\lambda_n\}$ of positive real numbers decreasing to zero such that

$$x_n - x_0, x_0 - x_n \leq \lambda_n u, \quad n = 1, 2, \dots$$

A sequence $\{x_n\}$ in X will be called relatively uniformly *-convergent to $x_0 \in X$ if every subsequence of $\{x_n\}$ contains a subsequence that is relatively uniformly convergent to x_0 .

It is known that in a Banach lattice the notions of

norm convergence and relative uniform \ast -convergence are identical [29: IV.2.4]. We show now that this is also true for another class of ordered Banach spaces which may not have the lattice order.

Proposition 5.1

Let X be a regular ordered Banach space. A sequence $\{x_n\}$ in X norm converges to $x_0 \in X$ if and only if $\{x_n\}$ is relatively uniformly \ast -convergent to x_0 .

Proof

Let $\{x_n\}$ be relatively uniformly convergent to x_0 . There exists $u \in X_+$ and a sequence $\{\lambda_n\}$ of real numbers which decrease to zero and

$$x_n - x_0, x_0 - x_n \leq \lambda_n u \quad n = 1, 2, \dots$$

$$\text{i.e. } -\lambda_n u \leq x_n - x_0 \leq \lambda_n u.$$

Since X satisfies (R_1) , we have

$$||x_n - x_0|| \leq |\lambda_n| \cdot ||u||.$$

Thus $\{x_n\}$ converges to x_0 in norm.

Since a sequence $\{y_n\}$ in X converges in norm to x_0 if and only if every subsequence of $\{y_n\}$ has a subsequence that converges in norm to x_0 , we see that relative uniform \ast -convergence implies norm-convergence.

Conversely let $\{x_n\}$ converge to zero in norm, and $\{y_m\}$ be a subsequence of $\{x_n\}$. Since X satisfies (R_2) , there are $z_m \in X_+$ such that $-y_m, y_m \leq z_m$ and $||z_m|| \leq ||y_m|| + \varepsilon$

for a given $\varepsilon > 0$. Thus $z_m \rightarrow 0$.

Let $B_n = \{x \in X: ||x|| < \frac{1}{2^n}\}$, then there exists a subsequence $\{z_{m_k}\}$ of $\{z_m\}$ such that $k \cdot z_{m_k} \in B_k$ for all k .

Now for given $p, q > 0$,

$$\begin{aligned} || \sum_{k=p+1}^{p+q} k z_{m_k} || &\leq \sum_{k=p+1}^{p+q} ||k z_{m_k}|| \\ &< \frac{1}{2^{p+1}} + \dots + \frac{1}{2^{p+q}} \\ &< \frac{1}{2^p}, \end{aligned}$$

which implies that $\{\sum_{k=1}^p k z_{n_k}\}$ is a Cauchy sequence in X

and hence converges to an element $z \in X_+$.

Thus $z_{n_k} \leq \frac{1}{k} z$ for all k ; which means that $\{z_{n_k}\}$

converges relatively uniformly to zero. This further implies that $\{x_n\}$ is relatively uniformly $*$ -convergent to zero.

Next we use this result to derive a characterization of ordered spaces which have norm-compact order-intervals.

Proposition 5.2

Let X be regular. The following are equivalent:

- (a) Order-intervals are norm-compact in X .
- (b) The order convex cover of a compact subset of X , is itself compact.

Proof

(a) \Rightarrow (b). Let A be a compact subset of X , $B = [A]$, and $\{x_n\}$ be a sequence in B . There exist sequences $\{z_n\}$

and $\{y_n\}$ in A such that $z_n \leq x_n \leq y_n$, $n = 1, 2, \dots$.

Since A is sequentially compact $\{z_n\}$ and $\{y_n\}$ have convergent subsequences, say $u_k \rightarrow u$, and $v_k \rightarrow v$.

Prop. 5.1 implies that there exist subsequences $\{p_i\}$ of $\{u_k\}$, and $\{q_i\}$ of $\{v_k\}$ which are relatively uniformly convergent to u and v respectively; i.e. there exist $s, t \in X_+$ and sequences $\{\sigma_i\}$ and $\{\lambda_i\}$ of positive real numbers, both decreasing to zero such that

$$p_i - u, u - p_i \leq \sigma_i s, \text{ and}$$

$$q_i - v, v - q_i \leq \lambda_i t.$$

$$\text{Thus } q_i = v + (q_i - v) \leq v + \lambda_i t \leq v + \lambda_i t$$

$$\text{and } u - \sigma_i s \leq u - \sigma_i s \leq u + (p_i - u) = p_i.$$

Let $\{w_i\}$ be the subsequence of $\{x_n\}$, corresponding to $\{p_i\}$ and $\{q_i\}$, i.e. $p_i \leq w_i \leq q_i$.
Therefore $u - \sigma_1 \cdot s \leq w_1 \leq v + \lambda_1 \cdot t$.

Since order interval $[u - \sigma_1 \cdot s, v + \lambda_1 \cdot t]$ is compact, there exists a convergent subsequence of $\{w_i\}$ and hence of $\{x_n\}$, which implies that B is sequentially compact.

(b) \Rightarrow (a):

Every order interval $[x, y]$ is the order convex cover of the compact set $\{x, y\}$.

Similarly we can show that if the notions of order convergence and norm convergence are identical in X (e.g. R^n) then the following are equivalent:

- (1) Order intervals are norm compact in X .
- (2) The order convex cover of a norm compact subset of X is norm compact.
- (3) If Y has R.D.P. with a generating cone and X is order complete with a normal cone, then $K(Y, X)$ is order complete.

(1) implies (2) is similar to Prop. 5.2. (2) implies (3) is similar to [35: Prop. 6] and (3) implies (1) is the same as [35: Prop. 6].

From 5.2 and II.2.1 we get another case where $K(X, Y)$ is an order ideal in $L(X, Y)$.

Proposition 5.3

Let X have a generating cone, and Y be a regular space in which the order intervals are norm compact. Then $K(X, Y)$ forms an order ideal in $L(X, Y)$.

REFERENCES

1. ALFSEN, E. M. "Compact convex sets and boundary integrals." Springer Verlag, New York, 1971.
2. ANDO, T. "On fundamental properties of a Banach space with a cone," Pacific J. Math. 12 (1962) 1163-69.
3. ASIMOW, L. "Directed Banach spaces of affine functions," Trans. Amer. Math. Soc. 143 (1969) 117-32.
4. ASIMOW, L. and ATKINSON, H. "Dominated extensions of continuous affine functions with range on ordered Banach space," Quart. J. Math. Oxford (2) 23 (1972) 383-89.
5. BIRNBAUM, D.A. "Preregular maps between Banach lattices," Bull. Australian Math. Soc. 11; 231-254 (1974).
6. BONSALL, F.F. "The decomposition of continuous linear functionals into non-negative components," Proc. Univ. Durham Philos. Soc. Ser A 13 (1957) 6-11.
7. DAVIES, E.B. "On the Banach duals of certain spaces with the Riesz decomposition property," Quart. J. Math. Oxford (2) 18, 109-111 (1967).
8. DAVIES, E.B. "The structure and ideal theory of the predual of a Banach lattice," Trans. Amer. Math. Soc. 131 (1968) 544-555.
9. DAY, M.M. "Normed linear spaces," Springer-Verlag, Berlin (1958).

10. DUNFORD, N. and SCHWARTZ, J.T. "Linear operators," Part I. New York (1958).
11. EDWARDS, D.A. "On the homeomorphic affine embedding of a locally compact cone into a Banach dual space endowed with the vague topology," Proc. London Math. Soc (3) 14 (1964) 399-414.
12. ELLIS, A.J. "The duality of partially ordered normed linear spaces," J. London Math. Soc. 39 (1964) 730-44.
13. ELLIS, A.J. "Linear operators in partially ordered normed vector spaces," J. London Math. Soc. 41 (1966) 323-32.
14. GROSBERG, J. and KREIN, K. "Sur la décomposition des fonctionnelles en composantes positives," Doklady Akad. Nauk S.S.S.R. (N.S) 25 (1939) 723-26.
15. JAMESON, G. "Ordered linear spaces," Berlin-Heidelberg, New York: Springer 1970.
16. KAKUTANI, S. "Concrete representations of abstract L-spaces and the mean ergodic theorem," Ann. of Math. 42, 523-537 (1941).
17. KAKUTANI, S. "Concrete representations of Abstract M-spaces, Ann. of Math. 42. (1941) 994-1024.
18. KRENGEL, U. "Remark on the modulus of compact operators," Bull. Amer. Math. Soc. 72, 132-133 (1966).
19. LACEY, H.E. "The isometric theory of classical Banach spaces," Springer-Verlag: New York-Heidelberg, Berlin (1974).
20. LAZAR, A.J. "Spaces of affine continuous functions on simplexes," Trans. Amer. Math. Soc. 134 (1968) 503-525.

21. LINDENSTRAUSS, J. "Extension of compact operators,"
Mem. Amer. Math. Soc. 48 (1964).
22. MAY, W.D.; and CHIVUKULA, R.R. "Lattice properties
of $L(X,Y)$," Duke Math. J. Vol. 39(2), 345-351 (1972).
23. NACHBIN, L. "Topology and order," Van Nostrand,
Princeton (1965).
24. NAMIOKA, I. "Partially ordered linear topological
spaces," Mem. Amer. Math. Soc. 24 (1957).
25. NG, K.F. "The Duality of Partially ordered Banach
spaces," Proc. London Math. Soc. (3) 19 (1969) 269-
288.
26. NG, K.F. "A note on Partially ordered Banach spaces,"
J. London Math. Soc. (2) 1 (1969) 520-524.
27. NG, K.F. "A duality theorem on partially ordered normed
spaces," J. London Math. Soc. (2) 3, (1971) 403-4.
28. ORNO, P. "On Banach lattices of operators," Israel. J.
Math. 19; 264-265 (1974).
29. PERESSINI, A.L. "Ordered topological vector spaces,"
Harper and Row, New York (1967).
30. SCHAEFER, H.H. "Topological vector spaces," Springer-
Verlag, Berlin-Heidelberg, New York (1971).
31. SCHAEFER, H.H. "Banach lattices and positive operators,"
Spring-Verlag, Berlin-Heidelberg, New York (1974).
32. SHIELDS, P.C. "Weakly compact positive operators on
summable functions," Proc. Amer. Math. Soc. (2) 37,
(1973) 456-458.
33. WESTON, "The decomposition of a continuous linear
Functional into non-negative components," Math. Scand.

- 5 (1957) 54-56.
34. WICKSTEAD, A.W. "Spaces of linear operators between partially ordered Banach spaces," Proc. London Math. Soc (3) 28 (1974) 141-158.
35. WICKSTEAD, A.W. "Compact subsets of partially ordered Banach spaces," Math. Ann. 212 (1975) 271-284.
36. WONG, M.H. "Order properties of the space of continuous affine functions with values in an ordered Banach space," Thesis, University of Windsor.

VITA AUCTORIS

Name

MUHAMMAD NASIR CHAUDHARY

Place and year of birth

Samundari, Pakistan, 1944

Education

M.A. Maths. Punjab Univ.
Lahore, Pakistan. 1965.
M.Sc. Maths. Univ. of Toronto
Toronto, Canada. 1970.
Faculty of Graduate Studies
Univ. of Windsor, Canada
1973-76.

Experience

Engineering University,
Lahore, Pakistan.
Lecturer 1965-68..
Assistant Prof. 1970-73.

Mathematics Department,
University of Windsor;
Teaching Assistant 1973-76.