1988

Design of 1-D and 2-D recursive digital filters using Hurwitz polynomials.

Henry Jenq Jong, Lee

University of Windsor

Follow this and additional works at: https://scholar.uwindsor.ca/etd

Recommended Citation


https://scholar.uwindsor.ca/etd/1656

This online database contains the full-text of PhD dissertations and Masters' theses of University of Windsor students from 1954 forward. These documents are made available for personal study and research purposes only, in accordance with the Canadian Copyright Act and the Creative Commons license—CC BY-NC-ND (Attribution, Non-Commercial, No Derivative Works). Under this license, works must always be attributed to the copyright holder (original author), cannot be used for any commercial purposes, and may not be altered. Any other use would require the permission of the copyright holder. Students may inquire about withdrawing their dissertation and/or thesis from this database. For additional inquiries, please contact the repository administrator via email (scholarship@uwindsor.ca) or by telephone at 519-253-3000ext. 3208.
NOTICE

The quality of this microform is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us an inferior photocopy.

Previously copyrighted materials (journal articles, published tests, etc.) are not filmed.

Reproduction in full or in part of this microform is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30.

AVIŠ

La qualité de cette microforme dépend grandement de la qualité de la these soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de qualité inférieure.

Les documents qui font déjà l'objet d'un droit d'auteur (articles de revue, tests publiés, etc.) ne sont pas microfilmés.

La reproduction, même partielle, de cette microforme est soumise à la Loi canadienne sur le droit d'auteur, C.C.S. 1970, c. C-30.
DESIGN OF 1-D AND 2-D RECURSIVE DIGITAL FILTERS USING HURWITZ POLYNOMIALS

A Thesis
Submitted to the Faculty of Graduate Studies through the Department of Electrical Engineering in Partial Fulfilment of the Requirements for the Degree of Master of Applied Science at the University of Windsor

by
Henry Jenq Jong Lee

Windsor, Ontario
1988

© Copyright Henry Jenq Jong Lee 1988
Permission has been granted to the National Library of Canada to microfilm this thesis and to lend or sell copies of the film.

The author (copyright owner) has reserved other publication rights, and neither the thesis nor extensive extracts from it may be printed or otherwise reproduced without his/her written permission.

L'autorisation a été accordée à la Bibliothèque nationale du Canada de microfilm er cette thèse et de prêter ou de vendre des exemplaires du film.

L'auteur (titulaire du droit d'auteur) se réserve les autres droits de publication; ni la thèse ni de longs extraits de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation écrite.

To My Mother
ABSTRACT

This thesis presents two methods of generating one-variable Hurwitz polynomial (HP) using properties of positive definite matrices. The problems of designing one-dimensional and two-dimensional stable recursive digital filters satisfying prescribed magnitude and constant group-delay specifications are also studied in this thesis. Specifically, a novel method of designing one-dimensional recursive digital filters is presented and results obtained are compared with a well known conventional design technique.

In the area of two-dimensional filter design several approaches for the design of two-dimensional recursive digital filters with non-separable numerator and separable denominator transfer functions are given. These include quadrantal symmetric, octagonal symmetric and 3-term-separable-denominator (3TSD) 2-D filters. This thesis also consists of a new method of generating two-variable very strict Hurwitz polynomial (VSHP) which is a direct extension of one-variable HP to two-variable. This constitutes a new technique to the design of general 2-D recursive digital filter whose numerator and denominator functions are both non-separable satisfying prescribed magnitude and constant group-delay specifications. In addition, the coefficient sensitivity of the designed filters are examined. The design techniques discussed in this thesis use optimization procedures to calculate the filter coefficients. Illustrative examples are given in order to demonstrate the usefulness of the algorithms.
ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to his supervisors, Dr. M. Ahmadi and Dr. M. Shridhar, for their guidance and encouragement over the course of this research. Also, the critical comments, suggestions and evaluations of Dr. J.J. Soltis and Dr. N.G. Zamani are greatly appreciated. Thanks are also extended to my family for their invaluable moral support and patience throughout my academic career. The help of the computing consultants at the University Computer Center is greatly appreciated.
CONTENTS

Dedication ii
Abstract iii
Acknowledgements iv
Contents v
List of Figures vii
List of Tables xiv

CHAPTER 1 INTRODUCTION 1
1.1 Digital Signal Processing 1
1.2 Characterization and Classification of Digital Filters 2
  1.2.1 One-dimensional Filters 2
  1.2.2 Two-dimensional Filters 4
1.3 Magnitude and Phase Characteristics 6
1.4 Stability 8
  1.4.1 1-D Stability 8
  1.4.2 2-D Stability 10
1.5 Design Techniques Overview 13
  1.5.1 1-D Filter 13
  1.5.2 2-D Filter 15
1.6 Thesis Organization 16

CHAPTER 2 GENERATION OF ONE-VARIABLE HURWITZ POLYNOMIALS AND ITS APPLICATIONS IN 1-D AND 2-D SEPARABLE DENOMINATOR RECURSIVE FILTERS DESIGN 17
2.1 Method I of Generating Hurwitz Polynomial 19
2.2 Method II of Generating Hurwitz Polynomial 21
2.3 Design of Linear Phase Elliptic Digital Filter 23
  2.3.1 Design Example 24
2.4 Proposed 1-D Design Formulation 29
2.5 Method I of Designing 1-D IIR Filters 30
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>Frequency characteristics of general z-transfer function with real coefficients</td>
<td>18</td>
</tr>
<tr>
<td>1b</td>
<td>Quadrantal symmetric frequency characteristics</td>
<td>18</td>
</tr>
<tr>
<td>1c</td>
<td>Octagonal symmetric frequency characteristics</td>
<td>18</td>
</tr>
<tr>
<td>2a</td>
<td>Normalized magnitude response of Elliptic lowpass filter</td>
<td>26</td>
</tr>
<tr>
<td>2b</td>
<td>Normalized group-delay of Elliptic lowpass filter with delay equalizer</td>
<td>27</td>
</tr>
<tr>
<td>3a</td>
<td>Magnitude response of second order lowpass using method I</td>
<td>32</td>
</tr>
<tr>
<td>3b</td>
<td>Group-delay of second order lowpass using method I</td>
<td>33</td>
</tr>
<tr>
<td>4a</td>
<td>Magnitude response of fourth order bandpass using method I</td>
<td>34</td>
</tr>
<tr>
<td>4b</td>
<td>Group-delay of fourth order bandpass using method I</td>
<td>35</td>
</tr>
<tr>
<td>5a</td>
<td>Normalized magnitude response of fourth order lowpass using method I</td>
<td>37</td>
</tr>
<tr>
<td>5b</td>
<td>Normalized group-delay of fourth order lowpass using method I</td>
<td>38</td>
</tr>
<tr>
<td>6a</td>
<td>Magnitude response of second order lowpass using method II</td>
<td>41</td>
</tr>
<tr>
<td>6b</td>
<td>Group-delay of second order lowpass using method II</td>
<td>42</td>
</tr>
<tr>
<td>7a</td>
<td>Magnitude response of fourth order bandpass using method II</td>
<td>45</td>
</tr>
<tr>
<td>7b</td>
<td>Group-delay of fourth order bandpass using method II</td>
<td>46</td>
</tr>
<tr>
<td>8a</td>
<td>Normalized magnitude response of fourth order lowpass using method II</td>
<td>47</td>
</tr>
<tr>
<td>8b</td>
<td>Normalized group-delay of fourth order lowpass</td>
<td>48</td>
</tr>
</tbody>
</table>
using method I

9a Magnitude response of 2-D second order lowpass using method I
58

9b Group-delay 1 of 2-D second order lowpass using method I
57

9c Group-delay 2 of 2-D second order lowpass using method I
58

10a Magnitude response of 2-D second order lowpass using method II
59

10b Group-delay 1 of 2-D second order lowpass using method II
60

10c Group-delay 2 of 2-D second order lowpass using method II
61

11a Magnitude response of 2-D fourth order bandpass using method II
84

11b Group-delay 1 of 2-D fourth order bandpass using method II
65

11c Group-delay 2 of 2-D fourth order bandpass using method II
66

12a Magnitude response of 2-D second order lowpass using method I with zero-phase numerator
70

12b Group-delay 1 of 2-D second order lowpass using method I with zero-phase numerator
71

12c Group-delay 2 of 2-D second order lowpass using method I with zero-phase numerator
72

13a Magnitude response of 2-D second order lowpass using method II with zero-phase numerator
73

13b Group-delay 1 of 2-D second order lowpass using method II with zero-phase numerator
74

13c Group-delay 2 of 2-D second order lowpass using method II with zero-phase numerator
75

14a Magnitude response of 2-D fourth order bandpass using method II with zero-phase numerator
78

14b Group-delay 1 of 2-D fourth order bandpass using method II with zero-phase numerator
79

14c Group-delay 2 of 2-D fourth order bandpass
80
<table>
<thead>
<tr>
<th>Page</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>15a</td>
<td>Magnitude response of 2-D second order lowpass using method I with two-step optimization</td>
</tr>
<tr>
<td>15b</td>
<td>Group-delay 1 of 2-D second order lowpass using method I with two-step optimization</td>
</tr>
<tr>
<td>15c</td>
<td>Group-delay 2 of 2-D second order lowpass using method I with two-step optimization</td>
</tr>
<tr>
<td>16a</td>
<td>Magnitude response of 2-D fourth order bandpass using method I with two-step optimization</td>
</tr>
<tr>
<td>16b</td>
<td>Group-delay 1 of 2-D fourth order bandpass using method I with two-step optimization</td>
</tr>
<tr>
<td>16c</td>
<td>Group-delay 2 of 2-D第四 order bandpass using method I with two-step optimization</td>
</tr>
<tr>
<td>17a</td>
<td>Magnitude response of 2-D second order lowpass using method II with two-step optimization</td>
</tr>
<tr>
<td>17b</td>
<td>Group-delay 1 of 2-D second order lowpass using method II with two-step optimization</td>
</tr>
<tr>
<td>17c</td>
<td>Group-delay 2 of 2-D second order lowpass using method II with two-step optimization</td>
</tr>
<tr>
<td>18a</td>
<td>Magnitude response of 2-D fourth order bandpass using method II with two-step optimization</td>
</tr>
<tr>
<td>18b</td>
<td>Group-delay 1 of 2-D fourth order bandpass using method II with two-step optimization</td>
</tr>
<tr>
<td>18c</td>
<td>Group-delay 2 of 2-D fourth order bandpass using method II with two-step optimization</td>
</tr>
<tr>
<td>19</td>
<td>Contour plot of second order lowpass using method I with octagonal symmetry property</td>
</tr>
<tr>
<td>20a</td>
<td>Magnitude response of 2-D second order lowpass using method I with octagonal symmetry property</td>
</tr>
<tr>
<td>20b</td>
<td>Group-delay 1 of 2-D second order lowpass using method I with octagonal symmetry property</td>
</tr>
<tr>
<td>20c</td>
<td>Group-delay 2 of 2-D second order lowpass using method I with octagonal symmetry property</td>
</tr>
<tr>
<td>21</td>
<td>Contour plot of second order lowpass using method II with octagonal symmetry property</td>
</tr>
<tr>
<td>22a</td>
<td>Magnitude response of 2-D second order lowpass</td>
</tr>
</tbody>
</table>

- IX -
using method II with octagonal symmetry property

22b  Group-delay 1 of 2-D second order lowpass
using method II with octagonal symmetry property

22c  Group-delay 2 of 2-D second order lowpass
using method II with octagonal symmetry property

23  Contour plot of 3TSD filter with 45° rotated
ellipse frequency response specifications

24a  Magnitude response of 3TSD filter with 45°
rotated ellipse frequency response specifications

24b  Group-delay 1 of 3TSD filter with 45° rotated
ellipse frequency response specifications

24c  Group-delay 2 of 3TSD filter with 45° rotated
ellipse frequency response specifications

25  Contour plot of two terms SD filter with 45°
rotated ellipse frequency response specifications

26a  Magnitude response of general two terms SD
filter with 45° rotated ellipse frequency response specifications

26b  Group-delay 1 of general two terms SD filter
with 45° rotated ellipse frequency response specifications

26c  Group-delay 2 of general two terms SD filter
with 45° rotated ellipse frequency response specifications

27  Contour plot of 3TSD filter with non-
quadrantal symmetric frequency band
specifications

28a  Magnitude response of 3TSD filter with
non-quadrantal symmetric frequency band
specifications

28b  Group-delay 1 of 3TSD filter with non-
quadrantal symmetric frequency band
specifications

28c  Group-delay 2 of 3TSD filter with non-
quadrantal symmetric frequency band
specifications

29  Contour plot of general two terms SD filter

- x -
with non-quadrantal symmetric frequency band specifications

30a Magnitude response of general two terms SD filter with non-quadrantal symmetric frequency band specifications 132

30b Group-delay 1 of general two terms SD filter with non-quadrantal symmetric frequency band specifications 133

30c Group-delay 2 of general two terms SD filter with non-quadrantal symmetric frequency band specifications 134

31 N-port lossless frequency independent network 137

32a Magnitude response of 2-D non-separable numerator and denominator lowpass filter 147

32b Group-delay 1 of 2-D non-separable numerator and denominator lowpass filter 148

32c Group-delay 2 of 2-D non-separable numerator and denominator lowpass filter 149

33a Magnitude response of octagonal symmetric lowpass filter using method I with coefficients rounding to four digits 154

33b Group-delay 1 of octagonal symmetric lowpass filter using method I with coefficients rounding to four digits 155

33c Group-delay 2 of octagonal symmetric lowpass filter using method I with coefficients rounding to four digits 156

33d Contour plot of octagonal symmetric lowpass using method I with coefficients rounding to four digits 157

34a Magnitude response of octagonal symmetric lowpass filter using method I with coefficients rounding to three digits 158

34b Group-delay 1 of octagonal symmetric lowpass filter using method I with coefficients rounding to three digits 159

34c Group-delay 2 of octagonal symmetric lowpass 160

- xi -
filter using method I with coefficients rounding to three digits

34d Contour plot of octagonal symmetric lowpass using method I with coefficients rounding to three digits

35a Magnitude response of octagonal symmetric lowpass filter using method I with coefficients rounding to two digits

35b Group-delay 1 of octagonal symmetric lowpass filter using method I with coefficients rounding to two digits

35c Group-delay 2 of octagonal symmetric lowpass filter using method I with coefficients rounding to two digits

35d Contour plot of octagonal symmetric lowpass using method I with coefficients rounding to two digits

36a Magnitude response of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to four digits

36b Group-delay 1 of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to four digits

36c Group-delay 2 of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to four digits

36d Contour plot of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to four digits

37a Magnitude response of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to three digits

37b Group-delay 1 of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to three digits

37c Group-delay 2 of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to three digits
37d Contour plot of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to three digits

38a Magnitude response of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to two digits

38b Group-delay 1 of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to two digits

38c Group-delay 2 of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to two digits

38d Contour plot of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to two digits
<table>
<thead>
<tr>
<th>TABLE</th>
<th>Parameters of the all-pass function used as delay equalizer</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Coefficients of second order 1-D lowpass filter using method I</td>
<td>28</td>
</tr>
<tr>
<td>2</td>
<td>Coefficients of fourth order 1-D bandpass filter using method I</td>
<td>28</td>
</tr>
<tr>
<td>3</td>
<td>Coefficients of fourth order 1-D lowpass filter using method I</td>
<td>40</td>
</tr>
<tr>
<td>4</td>
<td>Coefficients of second order 1-D lowpass filter using method II</td>
<td>40</td>
</tr>
<tr>
<td>5</td>
<td>Coefficients of fourth order 1-D bandpass filter using method II</td>
<td>44</td>
</tr>
<tr>
<td>6</td>
<td>Coefficients of fourth order 1-D lowpass filter using method II</td>
<td>44</td>
</tr>
<tr>
<td>7</td>
<td>Tabulated errors of the three different 1-D design methods</td>
<td>50</td>
</tr>
<tr>
<td>8</td>
<td>Coefficients of second order 2-D SD lowpass filter using method I</td>
<td>55</td>
</tr>
<tr>
<td>9</td>
<td>Coefficients of second order 2-D SD lowpass filter using method II</td>
<td>55</td>
</tr>
<tr>
<td>10</td>
<td>Coefficients of fourth order 2-D SD bandpass filter using method II</td>
<td>63</td>
</tr>
<tr>
<td>11</td>
<td>Coefficients of second order 2-D SD lowpass filter using method I with zero-phase numerator</td>
<td>68</td>
</tr>
<tr>
<td>12</td>
<td>Coefficients of second order 2-D SD lowpass filter using method II with zero-phase numerator</td>
<td>68</td>
</tr>
<tr>
<td>13</td>
<td>Coefficients of fourth order 2-D SD bandpass filter using method II with zero-phase numerator</td>
<td>77</td>
</tr>
<tr>
<td>14</td>
<td>Coefficients of second order 2-D SD lowpass filter using method I with two-step optimization</td>
<td>84</td>
</tr>
<tr>
<td>15</td>
<td>Coefficients of second order 2-D SD bandpass filter using method I with two-step optimization</td>
<td>84</td>
</tr>
<tr>
<td>16</td>
<td>Coefficients of fourth order 2-D SD bandpass filter using method I with two-step optimization</td>
<td>84</td>
</tr>
<tr>
<td></td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>--------</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>Coefficients of second order 2-D SD lowpass filter using method II with two-step optimization</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>Coefficients of fourth order 2-D SD bandpass filter using method II with two-step optimization</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>Coefficients of second order 2-D SD lowpass filter using method I with octagonal symmetry property</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>Coefficients of second order 2-D SD lowpass filter using method II with octagonal symmetry property</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>Coefficients of 3TSD filter with 45° rotated ellipse frequency response specifications</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>Coefficients of general two terms SD filter with 45° rotated ellipse frequency response specifications</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>Coefficients of 3TSD filter with non-quadrantal symmetric frequency band specifications</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>Coefficients of general two terms SD filter with non-quadrantal symmetric frequency band specifications</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>Coefficients of 2-D non-separable numerator and denominator lowpass filter</td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>Tabulated errors of the octagonal symmetric 2-D lowpass filter using method I with coefficients rounding to four, three and two decimal numbers</td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>Tabulated errors of the two terms SD filter with non-quadrantal symmetric frequency band specifications with coefficients rounding to four, three and two decimal numbers</td>
<td></td>
</tr>
</tbody>
</table>
CHAPTER ONE
INTRODUCTION

1.1 Digital Signal Processing

Deterioration in quality of signals passing through a system has been a significant problem in almost every field of science and engineering. The whole theory of analog signal processing has developed from the need for such processing which includes the design of passive networks and later active network synthesis.

Digital signal processing has arisen as a means whereby an input array of numbers may be shaped according to some prescribed manner to produce an output array of numbers. These arrays of numbers can be of one or more dimensions. Acoustics signals are examples of one-dimensional while seismic and geophysical signals are examples of two-dimensional signals. The variables of a system may be time, spatial or any other parameters. For example, in the field of geophysical prospecting echoes from boundaries of geological strata of a detonation are detected by a linear array of detectors placed in line with the source. Desired echoes will be received by the detectors from various changes in geological strata, whereas undesired echoes may occur from multiple reflections, and random signals may also be generated by wind noise [1,2].

Digital filtering is an important branch of digital signal processing. It is in fact a computational process which transforms an input array of numbers to an output array of numbers according to a prescribed rule. A filter is a system whose frequency response takes significant values only
in certain bands of the frequency axis. The problem of filter design is basically a mathematical approximation problem which can be solved either in the analog domain (s-plane) or in the digital domain (z-plane) where the resulting filter is then called an analog or digital filter respectively. Digital filters can be either implemented as software algorithms on a general computer or as special dedicated hardware.

1.2. Characterization and Classification of Digital Filters

Digital filters may be subdivided into two categories, linear and non-linear. This thesis will be concerned exclusively with linear, causal and time (space)-invariant digital filters.

Linear digital filters can be generally classified into two types, namely finite-extent impulse response (FIR) and infinite-extent impulse response (IIR) filters. If the impulse response of a filter is of finite duration, it is called a FIR filter, on the other hand, if the impulse response of a filter is of infinite duration, it is called IIR filter.

1.2.1. One-dimensional filters

A linear time-invariant causal FIR filter can be characterized by the difference equation as

\[ y(n) = \sum_{k=0}^{N-1} h(k)x(n-k) \]  

(1.1)

where \( h(k) \) is the filter impulse response defined over the interval \( 0 \leq k \leq N - 1 \).
\( x(n) \) is the input sequence.
\( y(n) \) is the output sequence.

The \( z \)-transform of a sequence \( x(k) \), for \( -\infty < k < \infty \), is defined as
\[
X(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k}, \quad z = e^{j\omega T}
\]
where \( \omega \) is the continuous frequency variable.
\( T \) is the sampling period.

Therefore, the FIR transfer function is
\[
H(z) = \sum_{k=0}^{N-1} h(k) z^{-k}
\]
(1.3)

From (1.1) it is clear that the output sequence of an FIR filter is a function of the input sequence alone.

A linear time-invariant causal IIR filter can be characterized by the difference equation as
\[
y(n) = \sum_{i=0}^{N} a(i) x(n-i) - \sum_{i=1}^{M} b(i) y(n-i)
\]
(1.4)
where \( a \)'s and \( b \)'s are the filter coefficients and \( b(0) = 1 \).
\( x(n) \) is the input sequence.
\( y(n) \) is the output sequence.

Alternatively, an IIR filter can also be defined by the \( z \)-transform relationships from which the \( z \)-transfer function may be obtained.
\[
H(z) = \sum_{k=0}^{\infty} h(k) z^{-k} = \frac{\sum_{i=0}^{N} a(i) z^{-i}}{\sum_{i=0}^{M} b(i) z^{-i}}
\]
(1.5)

The output sequence of an IIR filter is a function of input as well as past values of the output sequences (1.4).
A 1-D analog filter can be described by its transfer function

\[ H(s) = \frac{\sum_{i=0}^{N} a(i)s^i}{\sum_{i=0}^{N} b(i)s^i} \] (1.6)

where the coefficients of the filter \( a \) and \( b \) are real while \( s \) is complex Laplace variable.

1.2.2. Two-dimensional filters

A 2-D linear shift-invariant, causal FIR filter can be characterized by the difference equation as

\[ y(m,n) = \sum_{k=-\infty}^{L-1} \sum_{l=-\infty}^{L-1} h(k,l)x(m-k,n-l) \] (1.7)

where \( x(m,n) \) and \( y(m,n) \) are the input and output arrays respectively.

A two-dimensional z-transform can be defined as

\[ X(z_1,z_2) = \sum_{i=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x(i,l)z_1^{-l}z_2^{-i} \] (1.8)

where \( z_k = e^{j\omega_k T} \), \( k=1,2 \)

and \( T = T_1 = T_2 \), that is, the sampling periods in both dimensions are assumed to be equal.

The 2-D corresponding transfer function is of the form

\[ H(z_1,z_2) = \sum_{i=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(i,l)z_1^{-l}z_2^{-i} \] (1.9)

Similarly, the difference equation that characterizes a two-dimensional linear shift-invariant causal IIR filter can be written as
\[ y(m,n) = \sum_{i=0}^{M_1} \sum_{j=0}^{M_1} a(i,j)x(m-i,n-j) - \sum_{i=0}^{M_2} \sum_{j=0}^{M_2} b(i,j)y(m-i,n-j) \]

with \( b(0,0) = 1 \).

(1.10)

The transfer function of the 2-D IIR filter can be equally obtained from the difference equation (1.10) as

\[ H(z_1, z_2) = \frac{\sum_{i=0}^{M_1} \sum_{j=0}^{M_1} a(i,j)z_1^{-i}z_2^{-j}}{\sum_{i=0}^{M_2} \sum_{j=0}^{M_2} b(i,j)z_1^{-i}z_2^{-j}}, \quad z_k = e^{j\omega_k} \]

(1.11)

and the 2-D analog counterpart is characterized by

\[ H(s_1, s_2) = \frac{\sum_{i=0}^{M_1} \sum_{j=0}^{M_1} a(i,j)s_1^{-i}s_2^{-j}}{\sum_{i=0}^{M_2} \sum_{j=0}^{M_2} b(i,j)s_1^{-i}s_2^{-j}} \]

(1.12)

where \( s_k, \quad k=1,2 \), are complex variables.

It is obvious that the above 2-D recursive filter belongs to a particular type of filter whose impulse response is spread over the first quadrant of the 2-D plane. In this thesis, only this type of filter is considered and it is called quarter plane recursive filter as opposed to the asymmetric half plane recursive filter. It can be assumed, without any loss of generality, that \( M_1 = M_1 = M_2 = M_2 \).

The transfer function of the 2-D recursive digital filter can take two sub-classes of the general form shown in equation (1.11), namely, the separable product transfer function,

\[ H(z_1, z_2) = H_1(z_1) \cdot H_2(z_2) \]

\[ = \left[ \begin{array}{c} \sum_{i=0}^{M} a(i)z_1^{-i} \\ \sum_{i=0}^{N} b(i)z_1^{-i} \end{array} \right] \left[ \begin{array}{c} \sum_{i=0}^{M} c(i)z_1^{-i} \\ \sum_{i=0}^{N} d(i)z_1^{-i} \end{array} \right] \]

(1.13)
and the separable denominator non-separable numerator transfer function

\[ H(z_1, z_2) = \frac{N(z_1, z_2)}{D(z_1)D(z_2)} \]

\[ H(z_1, z_2) = \frac{\sum_{i=0}^{N} a_i z_1^{-i} z_2^{-l}}{\sum_{i=0}^{N} b_i z_1^{-i}} \left[ \sum_{i=0}^{N} c_i z_2^{-i} \right] \]  

(1.14)

1.3. Magnitude and Phase Characteristics:

The transfer function of a 1-D digital filter can be expressed as

\[ H(e^{j\omega T}) = H(z) \bigg|_{z=e^{j\omega T}} \]

\[ H(e^{j\omega T}) = \mathcal{H}(\omega) e^{j\Phi(\omega)} \]  

(1.15)

where

\[ \mathcal{H}(\omega) = |H(e^{j\omega T})| = \text{Magnitude or gain of the filter.} \]

\[ \Phi(\omega) = \text{Phase shift of the filter.} \]

The group-delay of the filter is a measure of the average spatial or time delay of a filter as a function of frequency.

The group-delay is defined as

\[ \tau(\omega) = -\frac{\partial \Phi(\omega)}{\partial \omega} = -jz \frac{\partial \Phi(\omega)}{\partial z} \bigg|_{z=e^{j\omega T}} \]  

(1.16)

and it can be rewritten as

\[ \tau(\omega) = -\Re \left\{ \frac{z}{H(z)} \cdot \frac{\partial H(z)}{\partial z} \right\} \bigg|_{z=e^{j\omega T}} \]  

(1.17)

As an extension of the 1-D case, a 2-D digital filter transfer function can be written as
\[ H(e^{j\omega_i}, e^{j\omega_i}) = H(z_i, z_i) \mid z_i = e^{j\omega_i} \quad i = 1, 2. \]

\[ = |H(\omega_i, \omega_i)| \cdot e^{j\theta(\omega_i, \omega_i)} \quad (1.18) \]

where \[ |H(\omega_i, \omega_i)| = \text{gain or magnitude response}. \]
\[ \theta(\omega_i, \omega_i) = \text{phase response}. \]

and the group-delays are defined as

\[ \tau_i(\omega_i, \omega_i) = -\frac{\partial \theta(\omega_i, \omega_i)}{\partial \omega_i}, \quad i = 1, 2. \]

\[ (1.19) \]

\[ \tau_i(\omega_i, \omega_i) = -\Re \left\{ \frac{\frac{\partial}{\partial z_i}}{H(z_i, z_i)} \cdot \frac{\partial H(z_i, z_i)}{\partial z_i} \right\} \mid z_i = e^{j\omega_i} \quad i = 1, 2. \]

The 'filter design' problem is to find the filter coefficients such that the filter's response approximates a prescribed behavior. Both FIR and IIR filters have their own advantages and disadvantages associated with the design techniques and implementations. The choice between an FIR and IIR filter depends upon the application and the computational facilities available to the designer. In terms of hardware complexity or computational speed, it is generally true that a given specification can be met most efficiently with an IIR filter, that is, relatively lower order filter is required [3, 27]. The phase component of a filter plays an important role in certain applications such as image processing, where lines and edges are destroyed if the system has nonlinear phase response [4, 5]. The non-recursive filters can be readily designed to possess constant group-delay over the entire baseband. In contrast, stable, IIR filters with exact linear phase cannot be designed. Therefore, it has been of great interest to design
recursive filters with constant group-delay specifications in the passband. The iterative procedures using optimization techniques to design IIR filters overcome the loss of flexibility in the attainable filter response of closed-form design formulas. It will be shown that virtually any form of frequency selective specifications can be obtained as opposed to only standard filters such as lowpass, highpass, or bandpass. In this thesis, the design of 1-D and 2-D IIR filters with constant group-delay is to be illustrated.

1.4. Stability

From the difference equations (1.4, 1.10), it is obvious that the present response of an IIR filter depend upon the present and past N values of the excitation as well as the past N values of the output. Therefore, the response of the system can become arbitrarily large independent of the magnitude of the input signal and the filter may become unstable. A system is said to be BIBO stable if and only if for any bounded input, the output is bounded [6].

1.4.1. 1-D stability

There are various techniques in determining the stability of the filters, such as, Jury test [3], the Inners test [7], Schur-Cohn [8], Schussler’s theorem [9]. A digital filter is said to be stable if and only if any bounded excitation results in a bounded response [3]. A causal, linear, time-invariant filter can be described by the convolution equation
\[ y(nT) = \sum_{k = 0}^{\infty} h(kT) x(nT - kT) \]  

where

- \( x(nT) \) is the excitation
- \( y(nT) \) is the response
- \( h(nT) \) is the impulse response

If

\[ |y(nT)| \leq \sum_{k = 0}^{\infty} |h(kT)| |x(nT - kT)| \]

and

\[ |x(nT)| \leq M < \infty \quad \text{for all } n \]

then

\[ |y(nT)| \leq \infty \quad \text{and} \quad |x(nT)| < \infty \]

Hence, for

\[ \sum_{k = 0}^{\infty} |h(kT)| < \infty \quad \text{and} \quad |x(nT)| < \infty \]

then

\[ |y(nT)| < \infty \quad \text{for all } n. \]

That is, the condition for a stable system is that its impulse response to be absolutely summable [3].

Since the impulse response of a FIR filter is only defined over a bounded limit, this type of filter is always stable. Although the above condition is the basic definition of stability, it is simpler to assess the stability of a system in the frequency domain. In general, the filter is first designed and the stability check is then required. It is particularly disappointing to perform a long and tedious stability test only to find out that the designed filter is unstable. If the designed filter is not stable, stabilization [13,14] is to be carried out. Therefore it is always preferred to design a filter whose stability is guaranteed in nature. For an IIR filter described by its transfer function (1.5),

\[ H(z) = \frac{A(z)}{B(z)} = \frac{\sum_{k = 0}^{W} a_k z^{-k}}{\sum_{k = 0}^{N} b_k z^{-k}}, \quad z = e^{j\omega T} \]
to be stable \([10,11]\),

\[
B(z) = 0 \quad \forall \quad |z| > 1.
\]  

(1.21)

Similarly, for an analog filter characterized by (1.6),

\[
H(s) = \frac{N(s)}{D(s)} = \sum_{i=0}^{N} \frac{a(i)s}{\sum_{j=0}^{M} b(j)s}
\]

to be stable \([10,12]\),

\[
D(s) = 0 \quad \forall \quad \Re\{s\} > 0.
\]  

(1.22)

Hence an analog system is stable if and only if the denominator of its irreducible transfer function is a Hurwitz polynomial which satisfies condition (1.22). Then the discrete function without poles outside the unit circle, which satisfies condition (1.21), can be obtained, on bilinear transformation, from the analog filter since all roots on the left-hand s-plane are mapped inside the unit circle on the z-plane. This approach avoids the use of stabilization procedure which replaces the poles outside the unit circle in the z-plane by their mirror images \([31]\). It should be noted that the phase characteristics of the stabilized filter is essentially distorted as a result of the poles replacements.

1.4.2. 2-D stability

"Similar to the 1-D case, a necessary and sufficient condition for a linear, shift-invariant system to be BIBO stable is that its impulse response be absolutely summable.

\[
\sum_{n_{1}=-\infty}^{\infty} \sum_{n_{2}=-\infty}^{\infty} |h(n_{1},n_{2})| = S_{1} < \infty
\]

A 2-D analog filter described by (1.12),
H(s_1,s_2) = \frac{N(s_1,s_2)}{D(s_1,s_2)} = \frac{\sum_{k=0}^{k=N} \sum_{l=0}^{l=N} a(i,l) s_1^k s_2^l}{\sum_{k=0}^{k=N} \sum_{l=0}^{l=N} b(i,l) s_1^k s_2^l}

where N(s_1,s_2) and D(s_1,s_2) are polynomials in s_1 and s_2, is stable if and only if [15]

\[ D(s_1,s_2) \neq 0 \quad \text{for} \quad \bigcap_{s_1, s_2} \mathcal{X}(s_1) \neq 0. \quad (1.23) \]

Similarly, a 2-D digital filter (1.11)

\[ H(z_1,z_2) = \frac{N(z_1,z_2)}{D(z_1,z_2)} = \frac{\sum_{k=0}^{k=N} \sum_{l=0}^{l=N} a(i,l) z_1^{-k} z_2^{-l}}{\sum_{k=0}^{k=N} \sum_{l=0}^{l=N} b(i,l) z_1^{-k} z_2^{-l}} \]

is stable if and only if [15]

\[ D(z_1,z_2) \neq 0 \quad \text{for} \quad \bigcap_{z_1, z_2} |z_1|, |z_2| \geq 1. \quad (1.24) \]

The stability of a one-dimensional recursive filter is merely related to the locations of its poles. In two-dimensional case, however, the stability of a system is related to the zeros of both D(z_1,z_2) and N(z_1,z_2). This is due to the possibility of the existence of non-essential singularities of the second kind on the unit circle, which in turn, comes from the fact that the two-variable polynomial factorization is not always possible. Consider the following example from Goodman [16],

\[ G_1(z_1,z_2) = \frac{P_1}{Q} = \frac{(1 - z_1^{-1})^6(1 - z_2^{-1})^8}{2 - z_1^{-1} - z_2^{-1}} \]

\[ G_1(z_1,z_2) = \frac{P_2}{Q} = \frac{(1 - z_1^{-1})(1 - z_2^{-1})}{2 - z_1^{-1} - z_2^{-1}} \]

Both transfer functions G_1 and G_2 have mutually prime
numerator and denominator polynomials and \( Q \neq 0 \) in the region 
\[ \left\{ \left( z_1, z_2 \right) : |z_1| \geq 1, \ |z_2| \geq 1 \right\} \] 
except at \( z_1 = z_2 = 1 \). Goodman has shown that \( G_i \) is BIBO stable while \( G \) is not, even though both functions have nonessential singularities of the second kind at \( z_1 = z_2 = 1 \) \[16\]. In general, there is no straightforward means of determining whether a transfer function which has no poles outside the unit bidisk and which has a non-essential singularity of the second kind on the unit bicircle is stable or not \[16,22\].

In case of the transfer function \( (1.11) \) having \( N(z_1, z_2) = 1 \), the issue of nonessential singularities of the second kind is avoided and there exist stability theorems developed by Shank \[17\], Huang \[18\], Decarlo and Strintzis \[19,20,21\] which can be utilized. It is noted by King \[23\] that the stabilization techniques given in \[17,24,25\] may fail in certain systems and a repeating check on stability is needed.

One simple way of avoiding two-dimensional stability problem is to restrict the 2-D transfer function to a sub-class having separable denominator polynomial which is a cascade of two one-dimensional polynomial in \( z_1 \) and \( z_2 \) as shown in the equation \( (1.14) \). This class of 2-D IIR filters will be considered in the following chapter. For a general non-separable numerator and non-separable denominator 2-D recursive digital filter, the stability can only be guaranteed if the denominator polynomial satisfies condition \( (1.24) \), that is, the transfer function should not possess any kind of singularities outside the closed unit bidisk.

It is well known that \[26\], in analog domain, a very strict Hurwitz polynomial (VSHP) does not possess any singularities in the region 
\[ \left\{ (s_1, s_2) : Re\{s_1\} \geq 0, \ Re\{s_2\} \geq 0, \right\} \]
\(|s_1| \leq \infty\) and \(|s_1| \leq \infty\). A method of generating two-variable VSHP will be shown in later chapter which will be used in designing 2-D analog reference filters hence assuring the stability of the designed systems. The 2-D digital functions without any kind of singularities outside the closed unit bidisk can then be generated from the 2-D analog transfer functions by the application of double bilinear transformation.

1.5. Design Techniques Overview

The scope of this section is to briefly discuss the frequency domain design techniques for 1-D and 2-D recursive digital filters. As in analog filters, the approximation step in the design of digital filters is to obtain a realizable transfer function satisfying prescribed specifications.

1.5.1. 1-D filter

In general, there are two approaches to the design of stable, realizable recursive digital filters, namely direct and indirect methods. The direct methods involve design techniques that carry out the approximation problems in the z-domain. In the indirect methods, on the other hand, an analog filter approximation is first used to design the normalized analog transfer function and the standard filters can then be designed by applying simple transformations such as lowpass to lowpass, lowpass to bandpass and lowpass to highpass transformations. The discrete version of the filter can be obtained through the use of one of the mapping
procedures such as mapping of differentials, impulse-invariant, matched z-transform or bilinear transformation. Detailed design procedures can be found in [3,11,27,34]. Among the transformations the use of impulse invariant or bilinear transformation is to be preferred over others. Unfortunately, the phase response of Tschebyshev, Butterworth and Elliptic approximations are inherently nonlinear. One popular remedy is the use of allpass functions to compensate for the nonlinearity of the phase response. One way of designing recursive filters is direct closed form design in the z-domain. With the given desired response of the filter the approximation can be obtained directly by carefully selecting the pole and zero locations [28]. Other direct design techniques, often called computer-aided design techniques, include the use of iterative approaches based upon linear or nonlinear programming. Chotterra and Jullien [30] have modified the linear programming approach of Rabiner et al. [29] by including the stability constraint and linear phase in the design process. Steiglitz has proposed a design procedure based upon minimization of the mean square error in the frequency domain [31,32] and Deczky generalized the procedure in a number of ways [33]. A direct method suggested by Ramachandran et al. [35] makes use of the stability test which is due to Schussler's [36] in the design procedures. This method guarantees both the linear phase response and stability. A new method of designing linear phase recursive filters using HP's is to be presented in the later chapter. The designed filters are stable in nature and neither stability check nor stabilization procedure is required.
1.5.2. 2-D Filter

The 2-D recursive filters can be classified into three classes, the product separable (1.13), the separable denominator (SD) type (1.14) and the general non-separable numerator and denominator filters (1.12). The separable product design technique was first suggested by Hall in 1870 [37]. Although the product separable filter transfer function can be expressed as the product of two 1-D filters and the desired filter can be easily designed using 1-D design techniques, this type of filters possess rectangular shape passband and stopband regions. This particular class of filters will not be considered in this thesis. The SD filters can be designed to have circular symmetric magnitude responses [38-40]. The separable denominator reduces the stability problem associated with 2-D filters to that of the 1-D. As an extension of [35], Ahnadi et al. [40] implemented the design of SD filter with the stability test from [36]. It will be shown in this thesis that quadrantal or even octagonal symmetry can be achieved with this type of filters. The design technique for general non-separable filters may include s-plane rotation [41], two-variable reactance function technique [42], and the complex transformation [43]. Linear programming approach in the design of 2-D recursive filters is an extension of the 1-D case [53]. A numerous nonlinear programming approaches to the 2-D recursive filters have also been developed [46-51]. The complex transformation $z = e^{j\gamma} \frac{\alpha_1}{\beta_1} \frac{\alpha_2}{\beta_2}$ where $\gamma = 0$, $\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} = 1$ will be used in the design of 3TSD filters which has been reported to enhance the non-circular and non-quadrantal frequency response [44]. A general approach for the design of stable 2-D recursive digital filters is to assign a two-variable
VSHP to the denominator of a 2-D analog filter and the transfer function is discretized by the application of the double bilinear transformation. The spatial design techniques are more appropriate when the impulse response of the desired filter is available [17,45].

1.6. Thesis Organization

Chapter II of this thesis presents the methods of generating one-variable Hurwitz polynomials (HP) and its applications in the one-dimensional and two-dimensional separable denominator recursive digital filters design. In this chapter, different design techniques of 1-D and 2-D SD IIR filters are given. In addition, a class of 3-term-separable-denominator (3TSD) filter is also considered. Chapter III includes the generation of two-variable VSHP and its applications in the two-dimensional non-separable filters design. In order to illustrate the usefulness of the proposed design techniques, each design method is accompanied by various examples. At the end of chapter III the sensitivity of the designed filters to the effect of coefficient quantization in general is discussed. The final chapter, reviews the advantages and disadvantages of the different design techniques covered in earlier chapters and concludes this thesis.
CHAPTER TWO

GENERATION OF ONE-VARIABLE HURWITZ POLYNOMIALS AND ITS
APPLICATIONS IN 1-D AND 2-D SEPARABLE DENOMINATOR
RECURSIVE FILTERS DESIGN

The stability of a rational function of $s$ depends upon
the location of the roots of its characteristic equation.
The condition for roots with negative real parts can only be
guaranteed if the polynomial is a HP. In this chapter, two
different methods of generating one-variable HP's using the
properties of positive definite (or semi-definite) matrices
are given and their applications in the design of 1-D and 2-D
SD IIR filters with constant group-delay are illustrated.
The advantage of designing 2-D filters with separable
denominators is that the need of 2-D stability test is
eliminated while maintaining the flexibility of the
non-separable 2-D filters. The HP's generated in this
chapter can be used to form the denominator of this
particular class of 2-D filters. Three methods of designing
2-D SD filters are to be introduced here. As will be shown
later, the designed SD filters have different symmetrical
properties on the 2-D frequency plane. The first approach
defines a general class of 2-D transfer function with real
coefficients and hence the magnitude response in the first
quadrant is symmetrical to the third quadrant and the second
quadrant is symmetrical to the fourth [38] as illustrated in
figure 1a. The design methods II and III lead to quadrantal
symmetry in the frequency responses which implies identical
magnitude responses in all four quadrants shown in figure 1b.
Compared to the first method, the second and third are more
efficient because the number of discrete frequency points

- 17 -
a: Frequency characteristic of general z-transfer function with real coefficients

b: Quadrantal symmetric frequency characteristics

c: Octagonal symmetric frequency characteristics

Figure 1
over which the responses are to be evaluated are much less (only one quadrant of the frequency plane instead of the whole half plane). In addition, the design of 2-D linear phase octagonal symmetric IIR filter whose magnitude response is symmetrical for each octant of the frequency plane is outlined. This property is depicted in figure 1c. Finally a particular type of separable denominator 2-D filter called 3TSD is also presented here.

2.1. Method I of Generating HP

The even or odd part of a HP can be generated by taking determinant of a positive definite matrix, and, by associating the matrix determinant with its partial derivatives, a HP can be obtained. Consider a positive definite (or semi-definite) matrix, $M$, which can be decomposed in the following form [53]

$$M = A \Gamma A^T s + C \quad (2.1)$$

where

$$A = \begin{bmatrix}
1 & a_1 & a_2 & \cdots & a_n \\
0 & 1 & a_2 & \cdots & a_n \\
0 & 0 & 1 & \cdots & a_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix} \quad (2.2)$$

$$\Gamma = \begin{bmatrix}
\gamma_1 & 0 & 0 & \cdots & 0 \\
0 & \gamma_1 & 0 & \cdots & 0 \\
0 & 0 & \gamma_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \gamma_n
\end{bmatrix} \quad (2.3)$$
\[ G = \begin{bmatrix}
  0 & g_{11} & g_{12} & g_{14} & g_{1n} \\
  -g_{11} & 0 & g_{22} & g_{24} & g_{2n} \\
  -g_{12} & -g_{22} & 0 & g_{24} & g_{2n} \\
  -g_{1n} & -g_{2n} & 0 & 0 & 0
\end{bmatrix} \]  

(2.4)

and \( \Delta^T \) denotes the transpose of \( \Delta \).

It is also known that a positive definite (or semi-definite) immittance matrix is always physically realizable as a reactance network. The determinant of \( \mathbf{M} \) constitutes either the even or odd part of a HP depending upon its order \( n \).

Therefore, a HP can be formed as follows

\[ D(s) = \det \mathbf{M} + k \frac{\partial (\det \mathbf{M})}{\partial s} \]  

(2.5)

where \( k \) is a positive constant.

The above idea is best illustrated by an example.

For a second order matrix \( \mathbf{M} \),

\[ \mathbf{M} = \begin{bmatrix}
  1 & s \\
  0 & 1
\end{bmatrix} \begin{bmatrix}
  \gamma_i^2 & 0 \\
  0 & \gamma_i^2
\end{bmatrix} \begin{bmatrix}
  s & 0 \\
  0 & s
\end{bmatrix} + \begin{bmatrix}
  0 & 0 \\
  0 & 0
\end{bmatrix} \]

\[ \mathbf{M} = \begin{bmatrix}
  (\gamma_i^2 + s \gamma_i^2) \mathbf{s} & \gamma_i^2 \mathbf{s} + g \\
  \gamma_i^2 \mathbf{s} - g & \gamma_i^2 \mathbf{s}
\end{bmatrix} \]

\[ \therefore \det \mathbf{M} = \gamma_i^2 \mathbf{s}^2 + g^2 \]

Hence \[ D(s) = \det \mathbf{M} + k \frac{\partial (\det \mathbf{M})}{\partial s} \]

Let \( k = 2 \), \[ D(s) = \gamma_i^2 \mathbf{s}^2 + 4 \gamma_i^2 \mathbf{s} + g^2 \]  

(2.6)

which is a HP of order 2.

Higher order of HP can also be generated either from higher order matrices in (2.1) or by using two or more lower order HP's in cascade. For example, by cascading two second
order HP's a fourth order HP can be formed.

Let $\det M_1 = \gamma_1^2 \gamma_2^2 s^4 + g_1^2$

$\det M_1 = \gamma_2 \gamma_4 s^3 + g_2^2$

then

$D_1(s) = \gamma_1^2 \gamma_2^2 s^4 + 4 \gamma_1^2 \gamma_2 s^2 + g_1^2$  \hspace{1cm} (2.7)

and

$D_2(s) = \gamma_2 \gamma_4 s^3 + 4 \gamma_1 \gamma_2 s + g_2^2$  \hspace{1cm} (2.8)

Therefore, a fourth order HP, $D_3(s)$, is the product of $D_1(s)$ and $D_2(s)$,

$D_3(s) = D_1(s) D_2(s) \hspace{1cm} (2.9)$

2.2. Method II of Generating HP

Another way of generating one-variable HP is discussed here. By adding a resistance matrix to equation (2.1) a HP can be formed without the need for calculation of partial derivatives. This advantage will be of greater significance in the generation of two-variable VSHP which will be shown in the next chapter.

Consider a matrix $L$,

$L = A \Gamma A^T s + G + R \Sigma R^T \hspace{1cm} (2.10)$

where $A$ : upper triangular matrix shown in equation (2.2).

$\Gamma$ : diagonal matrix with non-negative elements of the form shown in equation (2.3).

$G$ : skew-symmetric matrix shown in equation (2.4).

and

\[
R = \begin{bmatrix}
1 & r_{10} & r_{14} & r_{1n} \\
0 & 1 & r_{24} & r_{2n} \\
0 & 0 & 1 & r_{3n} \\
0 & 0 & 0 & 1
\end{bmatrix}
\hspace{1cm} (2.11)
\]
\[
\mathbf{L} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\gamma_i^1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\boldsymbol{s} + \begin{bmatrix}
\delta & q \\
q & 0
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
1 & r \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\sigma_i^2 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
r & 1
\end{bmatrix}
\]

\[
\mathbf{L} = \begin{bmatrix}
(\gamma_i^1 + \sigma_i^2)\gamma_i^1 s + (\sigma_i^2 + r^2 \sigma_i^2) \\
\sigma_i^2 \gamma_i^1 s + r\sigma_i^2 - q
\end{bmatrix}
\]

Hence, the HP, \(D(s)\) is

\[
D(s) = \det \mathbf{L} = \gamma_i^1 \gamma_i^1 s^2 + [ \sigma_i^2 \gamma_i^1 + \sigma_i^2 \gamma_i^1 + (a - r)^2 \sigma_i^2 \gamma_i^1 ]s + (\sigma_i^2 \sigma_i^2 + g_i^2)
\]

Similar to the first method of generating HP discussed above, higher order HP can be formed by cascading two or more second-order HP's or by starting with higher order matrices in equation (2.10).

Hence, a fourth order HP can be generated as follows.

\[
D_3(s) = D_1(s) D_2(s)
\]

where

\[
D_i(s) = \gamma_i^1 \gamma_i^1 s^2 + [ \sigma_i^1 \gamma_i^1 + \sigma_i^1 \gamma_i^1 + (a_i - r_i) \sigma_i^1 \gamma_i^1 ]s + (\sigma_i^1 \sigma_i^1 + g_i^1)
\]
and \( i = 1,2. \)

The generation of HP's will be shown to be useful in the design of 1-D and 2-D recursive filters in the following sections.

2.3. Design of Linear Phase Elliptic Digital Filter

In the indirect digital filter design approach, the prototype analog filter is discretized using analog to digital transformation techniques. Butterworth, Tschebyshev and Elliptic are inherently nonlinear-phase approximations while Bessel approximation is formulated to have near constant delay in the passband region. But the phase response of the transformed digital filters including Bessel approximation is necessarily nonlinear due to the warping effect of the bilinear transformation. Therefore the design of digital filter with constant group-delay has been traditionally accomplished in two separate steps:

1. An analog filter is designed to meet the desired amplitude specification using any of the classical filters such as Elliptic, Butterworth, Tschebyshev, etc. The designed filter is then discretized by the application of bilinear transformation.

2. In order to compensate for the nonlinear phase response of the filter, the transfer function is cascaded with a digital allpass function whose coefficients are to be optimized based upon the desired group-delay characteristic.

The overall transfer function of the filter with equalizer is

\[ H(z) = H_0(z) \cdot H_c(z) \]
where \[ H_p(z) = H_p(s) \bigg| s = \frac{1}{T} \frac{(z - 1)}{(z + 1)} \]

Note that \( H_a(s) \)'s coefficients can be obtained from standard tables in [34].

and

\[ H_c(z) = \prod_{i=1}^{n} \frac{z^i + a_i z^i + b_i}{1 + a_i z^i + b_i z^i} \]

where \( a_i \)'s and \( b_i \)'s satisfy the inequality condition,

\[ b > \max \left\{ 1, (a_i - 1), -(a_i + 1) \right\} \quad (2.16) \]

In fact, instead of cascading a digital allpass function which requires the incorporation of the condition (2.16) as shown above, an analog allpass function can be used with bilinear transformation. This avoids the use of constraint optimization hence improving the design simplicity. The analog allpass function is shown below.

\[ H_a(s) = \prod_{i=1}^{n} \frac{s^i - a_i s^i + b_i}{s^i + a_i s^i + b_i} \quad (2.17) \]

where \( a_i \)'s and \( b_i \)'s are positive numbers.

Then the desired digital transfer function is

\[ H(z) = H(s) \bigg| s = \frac{1}{T} \frac{(z - 1)}{(z + 1)} \]

where \[ H(s) = H_c(s) \cdot H_a(s) \]

2.3.1. Design example

To illustrate the design technique, an Elliptic lowpass filter with linear phase characteristic is considered.
\[ |H_1(\omega)| = \begin{cases} 
1 & \text{for } 0 < \omega < 0.7 \text{ rad/sec} \\
0 & \text{for } 1.4 \leq \omega \leq \omega_s / 2.
\end{cases} \]

where \( \omega_s = 2\pi \text{ rad/sec} \) and \( T = 1 \).

From the above specifications, the required transition ratio can be determined.

\[ \omega_t = \omega_s / \omega_c = 1.4 / 0.7 = 2.0 \]

Assuming acceptable passband ripple be 2 dB and stopband attenuation be -20 dB, from the standard tables [34], a second order analog transfer function that satisfies \( \omega_t = 2.0 \) is given below:

\[ H_a(s) = \frac{0.100103(s^2 + 3.60961)}{s^2 + 0.537326s + 0.454891} \]

Note that the actual transition ratio is

\[ \omega_r = 1.94332 < 2. \]

\[ \begin{align*}
\omega_c &= \frac{1}{\sqrt{\omega_r}} = 0.717 \\
\omega_s &= \frac{\omega_r}{\sqrt{\omega_c}} = 1.394
\end{align*} \]

This transfer function is cascaded with a second order allpass function giving the overall filter transfer function as

\[ H(s) = H_a(s) \cdot H_b(s) \quad (2.18) \]

where

\[ H_a(s) = \frac{s^2 - as + b}{s^2 + as + b} \]

Now the discrete version of the desired filter can be obtained by applying bilinear transformation to the equation (2.18). Note that the resulting filter is a fourth order filter. The filter coefficients \( a \) and \( b \) are to be obtained through the use of any suitable nonlinear optimization procedure.
Figure 2a: Normalized magnitude response of Elliptic lowpass filter
Figure 2b: Normalized group-delay of Elliptic lowpass filter with delay equalizer
### Table 1: Parameters of the all-pass function used as delay equalizer.

<table>
<thead>
<tr>
<th>All-pass function parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 0.1053e+02$</td>
</tr>
<tr>
<td>$b = 0.4451e+01$</td>
</tr>
</tbody>
</table>

### Table 2: Coefficients of second order 1-D lowpass filter using method I.

<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_0 = 0.4058e+01$</td>
<td>$\gamma_1 = 0.2998e+01$</td>
</tr>
<tr>
<td>$n_1 = 0.1622e+00$</td>
<td>$\gamma_2 = 0.3081e+00$</td>
</tr>
<tr>
<td>$n_2 = 0.9171e-01$</td>
<td>$g = 0.2006e+01$</td>
</tr>
</tbody>
</table>

### Table 3: Coefficients of fourth order 1-D bandpass filter using method I.

<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_0 = -0.1540e+02$</td>
<td>$\gamma_1 = 0.3120e+01$</td>
</tr>
<tr>
<td>$n_1 = -0.2847e+01$</td>
<td>$\gamma_2 = 0.4331e+00$</td>
</tr>
<tr>
<td>$n_2 = -0.4714e+02$</td>
<td>$g_1 = 0.3055e+01$</td>
</tr>
<tr>
<td>$n_3 = -0.1258e+01$</td>
<td>$\gamma_3 = 0.2655e+01$</td>
</tr>
<tr>
<td>$n_4 = -0.1268e+00$</td>
<td>$\gamma_4 = 0.4331e+00$</td>
</tr>
<tr>
<td></td>
<td>$g_2 = 0.4019e+01$</td>
</tr>
</tbody>
</table>
1 and figure 2a,b show the values of a and b, normalized magnitude and group-delay responses respectively.

2.4. Proposed 1-D Design Formulation

The design method used here involves the application of the optimization techniques in the course of searching for coefficients of the filter transfer function such that it approximates the given specifications.

The objective function in the optimization is set to be the sum of squares of the errors between the ideal and designed frequency responses of the filter. The cost function is the errors between both magnitude responses and group-delay responses of the ideal and designed filters.

\[ E(\omega_c, \mathbf{c}) = \frac{1}{I_p} \sum_{\omega_c \in I_p} \{ E^2_{\text{M}}(\omega_c) + E^2_{\tau}(\omega_c) \} + \frac{1}{I_s} \sum_{\omega_c \in I_s} E^2_{\text{M}}(\omega_c) \]  \hspace{1cm} (2.19)

where \( I_p \) is the set of discrete frequency points in the passband region.
\( I_s \) is the set of discrete frequency points in the stopband region.
\( \mathbf{c} \) is the coefficient vector.

and

\[ E_{\text{M}}(\omega_c) = |H_i(e^{j\omega_c \tau})| - |H_p(e^{j\omega_c \tau})| \] \hspace{1cm} (2.20)

\[ E_{\tau}(\omega_c) = \tau_I \tau - \tau(\omega_c) \] \hspace{1cm} (2.21)

where \( |H_i(e^{j\omega_c \tau})| \) represents magnitude responses of the ideal filter.
\( |H_p(e^{j\omega_c \tau})| \) represents magnitude responses of the designed filter.
\( \tau_I \tau \) represents the ideal group-delay; \( \tau_I \) is defined to
be in the region of the order of the filter and T is the sampling period.

\( \tau(\omega) \) is the designed group-delay which is defined in equation (1.17).

From the above it is obvious that the group-delay response of a filter is required to be constant only in the passband region. In fact, weighting factors can be used in the cost function in order to emphasize either passband magnitude, group-delay or stopband magnitude in the design process.

2.5. Method I of Designing 1-D IIR Filters

An analog filter is characterized by the transfer function

\[
H_c(s) = \frac{N(s)}{D(s)} = \frac{\sum_{k=0}^{N} a_k s^k}{\sum_{i=0}^{N} b_i s^i}
\]  

(2.22)

Correspondingly, a recursive digital filter transfer function can be obtained through the use of bilinear transformation, that is,

\[ H_d(z) = H_c(s) \bigg|_{s = \frac{2}{T} \frac{z-1}{z+1}} \]

The objective of the design problem is to obtain the polynomial coefficients of the transfer function which, evaluated on the unit circle in the z-plane, approximates the desired response while maintaining the stability of the filter. The stability of the filter is guaranteed if the condition of equation (1.21), that is, all poles of the transfer function are inside the unit circle in the z-plane, is satisfied. This can be achieved by assigning a HP to the
denominator of the analog transfer function (2.22) and it is then transformed to digital using bilinear transformation.

The following outlines the design procedures:

1. Form an analog reference filter with proper order N (2.22)
2. Generate a HP using procedures given in section 2.1.
3. Assign the resulting one-variable HP from step 2 to \( D(s) \) in (2.22) and discretize the final transfer function using bilinear transformation.
4. Optimize the filter coefficients using (2.19, 20, 21) as cost function.

In order to compare the performances of different design techniques the same example used in the Elliptic filter design will be used here. Other examples are also given to support the validity of the technique.

2.5.1. Design examples

(i) Design a 1-D linear phase lowpass filter with the following specifications:

\[
|H_1(\omega)| = \begin{cases} 
1 & \text{for } 0 \leq \omega < 1 \text{ rad/sec} \\
0 & \text{for } 2.5 \leq \omega \leq \omega_s/2.
\end{cases}
\]

where \( \omega_s = 10 \text{ rad/sec} \) and \( T = \pi/5 \text{ sec} \).

Assuming second order filter, the reference analog transfer function is

\[
H_a(s) = i \frac{P_0}{\pi} \frac{n_0 s^4}{D(s)}
\]  

(2.23)

where \( D(s) \) is defined in equation (2.6).

The filter coefficients are calculated based upon the cost function described by equations (2.19, 20, 21) where \( \tau \) is equal to 2. Table 2 gives the resulting coefficients of the
Figure 3a: Magnitude response of second order lowpass using method I
Figure 3b: Group-delay of second order lowpass using method I
Figure 4a: Magnitude response of fourth order bandpass using method I
Figure 4b: Group-delay of fourth order bandpass using method I
transfer function and figure 3a,b are the magnitude response and group-delay of the filter.

(ii) Design a bandpass filter with the following specifications:

\[
|H_{i}(\omega)| = \begin{cases} 
0 & \text{for } 0 \leq \omega \leq 1 \text{ rad/sec} \\
1 & \text{for } 2 \leq \omega \leq 3 \text{ rad/sec} \\
0 & \text{for } 4 \leq \omega \leq \omega_s/2.
\end{cases}
\]

where \(\omega_s = 10 \text{ rad/sec} \) and \(T = \pi/5 \text{ sec.} \) and constant group-delay.

In order to have an equivalent performance to the lowpass filter previously designed, it is required to double the order of the filter. Since a fourth order polynomial is required in this case, two second order HP's can be cascaded as explained before.

\[
H_{o}(s) = \frac{\ell_{0} \cdot n \cdot s^{r}}{D(s)} \quad (2.24)
\]

where \(D(s)\) is defined in equations \((2.7,8,9)\).

This procedure avoids the complexity involved in taking determinant of higher order matrix but increases the nonlinearity of the denominator polynomial \(D(s)\). The objective function used here is shown in \((2.19,20,21)\) where \(\tau_i\) is set to be four. Table 3 shows the coefficients of the bandpass filter. Transfer function and figure 4a,b illustrates the magnitude and group-delay responses of the designed filter.

(iii) Design a lowpass with constant group-delay and

\[
|H_{i}(\omega)| = \begin{cases} 
1 & \text{for } 0 \leq \omega < 0.7 \text{ rad/sec} \\
0 & \text{for } 1.4 \leq \omega \leq \omega_s/2.
\end{cases}
\]

where \(\omega_s = 2\pi \text{ rad/sec} \) and \(T = 1 \text{ sec.} \)
Figure 5a: Normalized magnitude response of fourth order lowpass using method I
Figure 5b: Normalized group-delay of fourth order lowpass method I
Note that this is the example used in the Elliptic filter design. Assuming fourth order filter, the transfer function is given in (2.24). In this case, \( \tau \) is used as one of the parameters of the optimization and this increases significantly the degree of flexibility in the process. Table 4 and figure 5a,b show the set of coefficients, the magnitude and group-delay responses of the filter respectively.

2.6. Method II of Designing 1-D IIR Filters

Similar to the first approach, the transfer function of the analog reference filter has the form shown in equation (2.22) with the denominator being defined by equations (2.10-2.13). This again guarantees that the poles of designed analog reference filter have negative real parts. Bilinear transformation is used to discretize the analog transfer function hence ensuring stability of the digital filter. It is to be noted that the optimization process is carried out in the \( z \)-domain which excludes the warping effect of the bilinear transformation. The explicit design procedures are the same as in section 2.5 except that the generation of one-variable HP is accomplished using procedures given in section 2.2. Same examples as in previous section are used here in order to compare the filter performances.

2.6.1. Design examples

(i) Design a 1-D linear phase lowpass filter with the following specifications:
### Table 4: Coefficients of fourth order 1-D lowpass filter using method I.

<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_0 = 0.8622e+01$</td>
<td>$\gamma_1 = 0.5000e+00$</td>
</tr>
<tr>
<td>$n_1 = 0.1016e+00$</td>
<td>$\gamma_2 = 0.8953e+00$</td>
</tr>
<tr>
<td>$n_2 = 0.1568e+01$</td>
<td>$g = 0.9000e+00$</td>
</tr>
<tr>
<td>$n_3 = 0.4687e-02$</td>
<td>$\gamma_3 = 0.1552e+01$</td>
</tr>
<tr>
<td>$n_4 = 0.2109e-01$</td>
<td>$\gamma_4 = 0.1000e+01$</td>
</tr>
<tr>
<td></td>
<td>$g = 0.3126e+01$</td>
</tr>
</tbody>
</table>

### Table 5: Coefficients of second order 1-D lowpass filter using method II.

<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_0 = 0.7033e+01$</td>
<td>$\gamma_1 = 0.1320e+01$</td>
</tr>
<tr>
<td>$n_1 = 0.3961e+00$</td>
<td>$\gamma_2 = 0.1442e+01$</td>
</tr>
<tr>
<td>$n_2 = 0.1010e+00$</td>
<td>$\sigma_1 = 0.1052e+01$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_2 = 0.1294e+01$</td>
</tr>
<tr>
<td></td>
<td>$a = -0.4833e-01$</td>
</tr>
<tr>
<td></td>
<td>$r = 0.5997e+00$</td>
</tr>
<tr>
<td></td>
<td>$g = -0.2285e+01$</td>
</tr>
</tbody>
</table>
Figure 6a: Magnitude response of second order lowpass using method II.
Figure 6b: Group-delay of second order lowpass using method II
\[ |H_1(\omega)\| = \begin{cases} 1 & \text{for } 0 \leq \omega < 1 \text{ rad/sec} \\ 0 & \text{for } 2.5 \leq \omega \leq \omega_s/2. \end{cases} \]

where \( \omega_s = 10 \text{ rad/sec} \) and \( T = \pi/5 \text{ sec} \).

Assuming second order filter, the reference analog transfer function is given in (2.23) where \( D(s) \) is defined in equation (2.14). \( \tau_1 \) is equal to 2 in (2.21). Table 5 gives the resulting coefficients of the transfer function and figure 6a,b are the magnitude response and group-delay of the filter.

(ii) Design a bandpass filter with the following specifications:

\[ |H_2(\omega)\| = \begin{cases} 0 & \text{for } 0 \leq \omega \leq 1 \text{ rad/sec} \\ 1 & \text{for } 2 \leq \omega \leq 3 \text{ rad/sec} \\ 0 & \text{for } 4 \leq \omega \leq \omega_s/2. \end{cases} \]

where \( \omega_s = 10 \text{ rad/sec} \) and \( T = \pi/5 \text{ sec} \) and constant group-delay.

The analog reference transfer function of a fourth order filter is given in (2.24) where \( D(s) \) is defined in equation (2.15). \( \tau_1 \) is set to be four. Table 6 shows the coefficients of the bandpass filter transfer function and figure 7a,b illustrates the magnitude and group-delay responses of the designed filter.

(iii) Design a lowpass with constant group-delay and

\[ |H_3(\omega)\| = \begin{cases} 1 & \text{for } 0 \leq \omega < 0.7 \text{ rad/sec} \\ 0 & \text{for } 1.4 \leq \omega \leq \omega_s/2. \end{cases} \]

where \( \omega_s = 2\pi \text{ rad/sec} \) and \( T = 1 \text{ sec} \).

Note that this is the example used in both the Elliptic filter design and the previous proposed method. Assuming fourth order filter, the transfer function is given in
<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_3 = 0.6295e+01$</td>
<td>$\gamma_{11} = 0.6954e+00$</td>
</tr>
<tr>
<td>$n_1 = 0.3570e+01$</td>
<td>$\gamma_{12} = 0.1347e+01$</td>
</tr>
<tr>
<td>$n_2 = 0.1807e+02$</td>
<td>$\sigma_{11} = 0.1441e+01$</td>
</tr>
<tr>
<td>$n_3 = 0.3827e+00$</td>
<td>$\sigma_{12} = 0.5336e+00$</td>
</tr>
<tr>
<td>$n_4 = 0.6109e-01$</td>
<td>$a_1 = 0.9045e+00$</td>
</tr>
<tr>
<td></td>
<td>$r_1 = -0.2047e+00$</td>
</tr>
<tr>
<td></td>
<td>$g_1 = -0.4215e+01$</td>
</tr>
</tbody>
</table>

Table 6: Coefficients of fourth order 1-D bandpass filter using method II.

<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_0 = 0.2114e+01$</td>
<td>$\gamma_{11} = 0.1717e+01$</td>
</tr>
<tr>
<td>$n_1 = -0.1062e+00$</td>
<td>$\gamma_{12} = 0.1123e+01$</td>
</tr>
<tr>
<td>$n_2 = -0.1706e+01$</td>
<td>$\sigma_{11} = 0.9553e+00$</td>
</tr>
<tr>
<td>$n_3 = -0.2938e+00$</td>
<td>$\sigma_{12} = 0.9344e+00$</td>
</tr>
<tr>
<td>$n_4 = -0.1891e+00$</td>
<td>$a_1 = 0.1194e+01$</td>
</tr>
<tr>
<td></td>
<td>$r_1 = 0.8219e+00$</td>
</tr>
<tr>
<td></td>
<td>$g_1 = 0.7688e+00$</td>
</tr>
</tbody>
</table>

Table 7: Coefficients of fourth order 1-D lowpass filter using method II.
Figure 7a: Magnitude response of fourth order bandpass using method II
Figure 7b: Group-delay of fourth order bandpass using method II
Figure 8a: Normalized magnitude response of fourth order lowpass using method II
Figure 8b: Normalized group-delay of fourth order lowpass using method II
(2.24) and \( D(a) \) is defined in (2.15). Again, \( \tau_1 \) is used as one of the optimization parameters. Table 7 and figure 8a,b show the optimized coefficients, the magnitude and group-delay responses of the filter respectively.

2.7. Comparisons of 1-D Design Methods

From the filter characteristics of the common example in all three approaches obtained above it is appropriate to conclude that all three techniques yield satisfactory responses. The question now is which of the three methods outperforms the others and which approach has the simplest design procedures. In order to critically evaluate the different design techniques, it is essential to analyze the deviation from the desired response especially in the passband region. Table 8 gives the maximum errors of the magnitude in the passband, stopband and the group-delay responses of the three techniques. It is more than obvious that the group-delay response of method I is far better than those of the other methods but it has the highest passband magnitude response deviation. Method II has the best passband magnitude response with relatively small error in the group-delay response while the Elliptic filter has smallest stopband magnitude error. One reason for method II to have relatively better performance than the first method is the presence of much larger number of coefficients in its denominator polynomial (seven compared to three). At the first glance it may seem that method II, due to larger number of optimization parameters, will have slower convergence time than method I. It is found to have the opposite effect. It is important to note that the phase characteristics of the
<table>
<thead>
<tr>
<th>Maximum Error</th>
<th>Passband Magnitude</th>
<th>Stopband Magnitude</th>
<th>Passband Group-delay</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method I HP using Derivative</td>
<td>27.66</td>
<td>16.7</td>
<td>0.37</td>
</tr>
<tr>
<td>Method II HP using Resistance</td>
<td>14.1</td>
<td>17.06</td>
<td>2.88</td>
</tr>
<tr>
<td>Elliptic with delay equalizer</td>
<td>25.6</td>
<td>12.6</td>
<td>11.26</td>
</tr>
</tbody>
</table>

Table 8: Tabulated errors of the three different 1-D design methods (based on a fourth order lowpass filter).
Elliptic digital filter, even after phase compensation, is highly nonlinear. One remedy here is to cascade more second-order sections of the allpass function. In conclusion, it has been shown that single-step algorithms for designing linear phase stable 1-D IIR filter are successfully implemented and their performances can be far better than the conventional approaches.

2.8. Proposed 2-D Design Formulation

Similar to the 1-D design approaches, the proposed design procedure used in 2-D filter design is the iterative approach using suitable nonlinear optimization techniques. The objective function defined in the optimization is the sum of squares of the errors in the frequency responses of the designed and ideal filters.

The mean square error of the designed magnitude response can therefore be written as

\[ E(\omega_1, \omega_2, c) = \sum_{\omega_1, \omega_2} E_\omega(\omega_1, \omega_2) \]

(2.25)

where \( I_p \) is the set of discrete frequency points in both the passband and stopband regions.

\( c \) is the 2-D transfer function coefficient vector.

And

\[ E_\omega(\omega_1, \omega_2) = |H_i(e^{j\omega_1^T}, e^{j\omega_2^T})| - |H_o(e^{j\omega_1^T}, e^{j\omega_2^T})| \]

(2.26)

where

\[ |H_i(e^{j\omega_1^T}, e^{j\omega_2^T})| \] represents magnitude responses of the ideal filter.

and \( |H_o(e^{j\omega_1^T}, e^{j\omega_2^T})| \) represents magnitude responses of the
designed filter.

This formulation is suitable for designing 2-D filter with prescribed magnitude response regardless of the phase characteristics.

The mean square error between the magnitude responses as well as group-delays of ideal and designed filters is then given as

\[
E(\omega_1, \omega_2, \Omega) = \sum_{\Omega} \sum_{\mu} \left\{ E_M^\mu(\omega_1, \omega_2, \Omega) + E_T^\mu(\omega_1, \omega_2, \Omega) + E_\tau^\mu(\omega_1, \omega_2, \Omega) \right\} + \sum_{\Omega} \sum_{\mu} E_I^\mu(\omega_1, \omega_2) \tag{2.27}
\]

where \( I_p \) is the set of discrete frequency points in the passband region.

\( I_s \) is the set of discrete frequency points in the stopband region.

and \( E_\tau^\mu(\omega_1, \omega_2) = \tau_1 T - \tau_k(\omega_1, \omega_2) \tag{2.28} \)

for \( k = 1, 2 \).

where \( \tau_1 T \) represents the ideal group-delay; \( \tau_1 \) is defined be in the proximity of the order of the filter and \( T \) (\( T_i = T_s = T \)) is the sampling period.

\( \tau_k(\omega_1, \omega_2), k=1,2 \), are the designed group-delay with respect to \( \omega_1 \) and \( \omega_2 \) which are defined in equation (1.19).

Similar to the 1-D case, the weighting functions can be used to emphasize either the magnitude in the passband, stopband or the group-delays during the optimization process. The 2-D design formulation given in this section will be used to design both separable and non-separable denominator 2-D filters.
2.9. Method I of Designing 2-D SD IIR Filter

Assuming the following analog transfer function

\[ H(s_1, s_2) = \frac{\sum_{i=0}^{N} \sum_{j=0}^{N} n(i,1)s_1^i s_2^j}{\sum_{i=0}^{N} \sum_{j=0}^{N} b(i,1)s_1^i s_2^j} = \frac{N(s_1, s_2)}{D(s_1, s_2)}, s_k = j\omega_k, \quad k = 1, 2. \]

In order to simplify the 2-D stability problem, the denominator of the filter can be further assumed to be separable. That is, the denominator is the product of two one-variable polynomials of \( z_1 \) and \( z_2 \), and the transfer function can be rewritten as

\[ H(s_1, s_2) = \frac{N(s_1, s_2)}{D_1(s_1)D_2(s_2)} \quad (2.20) \]

where \( D_1(s_1) \) and \( D_2(s_2) \) can assume either form of HP's generated in previous sections (see equation (2.5) and (2.13)) and the discrete version is defined as

\[ H(z_1, z_2) = H(s_1, s_2) \bigg|_{s_k = T'(z_k+1)} = 2(z_k-1), \quad k = 1, 2. \]

The design procedures for linear phase filter are as follows:
1. Assume the analog reference filter be of the form shown in (2.29).
2. Generate a one-variable HP using either procedures given in section 2.1 or 2.2.
3. Assign the resulting HP's to \( D_1(\omega_k) \) and \( D_2(\omega_k) \). Discretize the transfer function using bilinear transformation.
4. Optimize the filter coefficients based upon the cost function in (2.26-2.28).

To prove the usefulness of the approach, examples are
given below using both 'forms' of one-variable HP's in the
design procedure.

2.9.1. Design examples

(i) Design a linear phase 2-D lowpass filter using
method I of generating one-variable HP. The filter
specifications are given below.

\[ |H_i(\omega_1, \omega_2)| = \begin{cases} 1 & \text{for } 0 \leq \sqrt{\omega_1^2 + \omega_2^2} < 1 \text{ rad/sec} \\ 0 & \text{for } 2.5 \leq \sqrt{\omega_1^2 + \omega_2^2} \leq \omega_s/2. \end{cases} \]

where \( \omega_s = 10 \text{ rad/sec} \) and \( T = \pi/5 \text{ sec} \).
Assuming second order filter the analog reference transfer
function is

\[ H(s_1, s_2) = \frac{\int_{\rho_0}^{\rho_1} n(i,j) s_1^i s_2^j}{D_i(s_1)D_i(s_2)} \tag{2.30} \]

where \( D_i(s_i) \) is given by equations (2.7,8) with \( i = 1,2 \).
The formulation used in the optimization process is given by
equation (2.26-2.28).

It is noted that the above filter specifications force the
frequency response to be circular. The formulated transfer
function indicates that the filter response is not
necessarily quadrantal symmetric, therefore it is required to
calculate the error function over the whole half plane in
order to ensure that the requirements are met. Table 9 and
figure 9a,b,c illustrate the 2-D filter coefficients and the
magnitude and group-delays responses respectively.

(ii) Design the lowpass filter given in example (i)
using the second method of generating one-variable HP.
The transfer function is the same as in (2.30) with \( D_i(s_i) \)
<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n(0,0) = 0.1255e+03 )</td>
<td>( \gamma_1 = -0.6444e+00 )</td>
</tr>
<tr>
<td>( n(0,1) = 0.8984e+01 )</td>
<td>( \gamma_2 = 0.4091e+01 )</td>
</tr>
<tr>
<td>( n(0,2) = 0.3230e+01 )</td>
<td>( \gamma_3 = 0.5499e+01 )</td>
</tr>
<tr>
<td>( n(1,0) = 0.9080e+01 )</td>
<td>( \gamma_4 = 0.8223e+00 )</td>
</tr>
<tr>
<td>( n(1,1) = 0.8797e-01 )</td>
<td>( g_1 = 0.1247e+01 )</td>
</tr>
<tr>
<td>( n(1,2) = 0.3679e+00 )</td>
<td>( g_2 = 0.2135e+01 )</td>
</tr>
<tr>
<td>( n(2,0) = 0.3276e+01 )</td>
<td>( )</td>
</tr>
<tr>
<td>( n(2,1) = 0.3665e+00 )</td>
<td>( )</td>
</tr>
<tr>
<td>( n(2,2) = 0.5723e-01 )</td>
<td>( )</td>
</tr>
</tbody>
</table>

**Table 9:** Coefficients of second order 2-D SD lowpass filter using method I.

<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n(0,0) = 0.3521e+01 )</td>
<td>( \gamma_{11} = 0.9463e+00 )</td>
</tr>
<tr>
<td>( n(0,1) = 0.3079e+00 )</td>
<td>( \gamma_{12} = 0.1332e+01 )</td>
</tr>
<tr>
<td>( n(0,2) = 0.8977e-01 )</td>
<td>( \gamma_{13} = 0.9669e+00 )</td>
</tr>
<tr>
<td>( n(1,0) = 0.3522e+00 )</td>
<td>( \gamma_{21} = 0.8530e+00 )</td>
</tr>
<tr>
<td>( n(1,1) = 0.4166e-01 )</td>
<td>( \gamma_{22} = 0.1104e+01 )</td>
</tr>
<tr>
<td>( n(1,2) = 0.4407e-01 )</td>
<td>( a_1 = 0.7562e+00 )</td>
</tr>
<tr>
<td>( n(2,0) = 0.9387e-01 )</td>
<td>( a_2 = 0.8891e+00 )</td>
</tr>
<tr>
<td>( n(2,1) = 0.6587e-01 )</td>
<td>( r_1 = 0.1228e+01 )</td>
</tr>
<tr>
<td>( n(2,2) = -0.1615e-02 )</td>
<td>( r_2 = 0.1174e+01 )</td>
</tr>
</tbody>
</table>

**Table 10:** Coefficients of second order 2-D SD lowpass filter using method II.
Figure 9a: Magnitude response of 2-D second order lowpass using method I
Figure 9b: Group-delay 1 of 2-D second order lowpass using method I
Figure 8c: Group-delay 2 of 2-D second order lowpass using method I
Figure 10a: Magnitude response of 2-D second order lowpass using method II
Figure 10b: Group-delay 1 of 2-D second order lowpass using method II
Figure 10c: Group-delay 2 of 2-D second order lowpass using method II
given by equation (2.15), i = 1,2. Optimization process is carried out over the half-plane. The filter coefficients are given in table 10 and figures 10a,b,c illustrate the magnitude and group-delays of the filter.

(iii) Design a linear phase bandpass filter with the following specifications using the second method of generating one-variable HP:

$$|H_i(\omega_1, \omega_2)| = \begin{cases} 
0 & \text{for } 0 \leq \sqrt{\omega_1^2 + \omega_2^2} \leq 1 \text{ rad/sec} \\
1 & \text{for } 2 \leq \sqrt{\omega_1^2 + \omega_2^2} \leq 3 \text{ rad/sec} \\
0 & \text{for } 4 \leq \sqrt{\omega_1^2 + \omega_2^2} \leq \omega_c/2.
\end{cases}$$

where $\omega_c = 10$ rad/sec and $T = \pi/5$ sec.

The order of the filter is assumed to be four. The analog transfer function is

$$H(s) = \frac{\prod_{i=0}^{4} n(i, j) s^i}{\prod_{i=0}^{4} D_i(s)}$$

where $D_i(s) = D_{i1}(s)D_{i2}(s)$,

$$D_{i1}(s) = D_{i1}(s_1)D_{i1}(s_2)$$

and

$$D_{i1}(s_1) = \gamma_{i1}^1 \gamma_{i1}^1 s_1^1 + \left( c_{i1}^1 \gamma_{i1}^1 + c_{i1}^1 \gamma_{i1}^1 + (a_{i1}^1 - r_{i1}) \right) s_1^{i1} \gamma_{i1}^1 s_1^{i1} + (c_{i1}^1 \gamma_{i1}^1 + g_{i1}^1)$$

$$D_{i2}(s_2) = \gamma_{i2}^2 \gamma_{i2}^2 s_2^2 + \left( c_{i2}^2 \gamma_{i2}^2 + c_{i2}^2 \gamma_{i2}^2 + (a_{i2}^2 - r_{i2}) \right) s_2^{i2} \gamma_{i2}^2 s_2^{i2} + (c_{i2}^2 \gamma_{i2}^2 + g_{i2}^2)$$

with $i = 1,2$.

The objective function for minimization is given by equation (2.26-2.28) and any suitable nonlinear optimization technique can be used here. Table 11 is the set of optimized coefficients and figures 11a,b,c give the magnitude response and group-delays of the designed filter.
<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n(0,0) = -0.1382e+02$</td>
<td>$\gamma_{11} = 0.1158e+01$</td>
</tr>
<tr>
<td>$n(0,1) = 0.2315e+01$</td>
<td>$\gamma_{12} = 0.5708e+00$</td>
</tr>
<tr>
<td>$n(0,2) = -0.4038e+02$</td>
<td>$\sigma_{11} = 0.9816e+00$</td>
</tr>
<tr>
<td>$n(0,3) = 0.1248e+01$</td>
<td>$\sigma_{12} = 0.8527e+00$</td>
</tr>
<tr>
<td>$n(0,4) = -0.6859e-01$</td>
<td>$\sigma_{12} = 0.1009e+01$</td>
</tr>
<tr>
<td>$n(1,0) = -0.1332e+02$</td>
<td>$a_{11} = 0.1079e+01$</td>
</tr>
<tr>
<td>$n(1,1) = 0.5730e+01$</td>
<td>$a_{11} = 0.1916e+01$</td>
</tr>
<tr>
<td>$n(1,2) = 0.8926e+01$</td>
<td>$r_{11} = 0.1889e+01$</td>
</tr>
<tr>
<td>$n(1,3) = 0.8326e-01$</td>
<td>$r_{11} = 0.1037e+01$</td>
</tr>
<tr>
<td>$n(1,4) = -0.3352e-01$</td>
<td>$g_{11} = 0.9390e+00$</td>
</tr>
<tr>
<td>$n(2,0) = -0.2584e+02$</td>
<td>$g_{11} = 0.1932e+01$</td>
</tr>
<tr>
<td>$n(2,1) = -0.2355e+01$</td>
<td>$\gamma_{13} = 0.7529e+00$</td>
</tr>
<tr>
<td>$n(2,2) = 0.1358e+01$</td>
<td>$\gamma_{13} = 0.8944e+00$</td>
</tr>
<tr>
<td>$n(2,3) = -0.2725e+00$</td>
<td>$\gamma_{13} = 0.7066e+00$</td>
</tr>
<tr>
<td>$n(2,4) = 0.7945e-01$</td>
<td>$\gamma_{13} = 0.9986e+00$</td>
</tr>
<tr>
<td>$n(3,0) = 0.9566e+00$</td>
<td>$\sigma_{13} = 0.8216e+00$</td>
</tr>
<tr>
<td>$n(3,1) = 0.4507e+00$</td>
<td>$\sigma_{13} = 0.1180e+01$</td>
</tr>
<tr>
<td>$n(3,2) = 0.4480e+00$</td>
<td>$\sigma_{13} = 0.1517e+01$</td>
</tr>
<tr>
<td>$n(3,3) = 0.2560e-01$</td>
<td>$a_{11} = 0.1310e+01$</td>
</tr>
<tr>
<td>$n(3,4) = -0.3876e-02$</td>
<td>$r_{11} = 0.1210e+01$</td>
</tr>
<tr>
<td>$n(4,0) = -0.6379e-01$</td>
<td>$g_{11} = 0.2060e+01$</td>
</tr>
<tr>
<td>$n(4,1) = -0.1087e+00$</td>
<td>$g_{11} = -0.8707e+00$</td>
</tr>
<tr>
<td>$n(4,2) = 0.5206e-01$</td>
<td></td>
</tr>
<tr>
<td>$n(4,3) = -0.1082e-01$</td>
<td></td>
</tr>
<tr>
<td>$n(4,4) = 0.1363e-02$</td>
<td></td>
</tr>
</tbody>
</table>

Table 11: Coefficients of fourth order 2-D SD bandpass filter using method II.
Figure 11a: Magnitude response of 2-D fourth order bandpass using method II
Figure 11b: Group-delay 1 of 2-D fourth order bandpass using method II
Figure 11c: Group-delay 2 of 2-D fourth order bandpass using method II
2.10. Method II of Designing 2-D SD IIR Filter

Assume the following transfer function of a 2-D SD analog filter

\[
H(s_i, s_j) = \frac{\sum_{i=0}^{N} \sum_{j=0}^{N} a(i,j)s_i^is_j^j}{D_i(s_i)D_j(s_j)}
\]

The one-variable HP's generated earlier can be assigned to the denominator of the transfer function. The digital filter can be obtained through bilinear transformation,

\[
H(z_i, z_j) = \frac{\sum_{i=0}^{N} \sum_{j=0}^{N} n(i,j)z_i^iz_j^j}{D_i(z_i)D_j(z_j)}, \quad z_i = e^{j\omega_iT}, \quad k=1,2.
\]

Now by assuming certain symmetries in the numerator coefficients a zero-phase numerator polynomial can be generated.

Let \( n(i,1) = n(-i,1) = n(i,-1) = n(-i,-1) \) \( (2.33) \)

then the transfer function can be rewritten as

\[
H(z_i, z_j) = \frac{\sum_{i=0}^{N} \sum_{j=0}^{N} n(i,j)(z_i^i + \bar{z}_i^i)(z_j^j + \bar{z}_j^j)}{D_i(z_i)D_j(z_j)}
\]

Since

\[
\cos(k\omega T) = \frac{e^{j\omega T} + e^{-j\omega T}}{2}
\]

then the discrete transfer function becomes

\[
H(z_i, z_j) = \frac{\sum_{i=0}^{N} \sum_{j=0}^{N} n'(i,j)\cos(i\omega T)\cos(j\omega T)}{D_i(z_i)D_j(z_j)} \quad (2.35)
\]

As in the first method the stability problem is that of the 1-D case hence the implementation advantages of separable denominator IIR filters can be maintained. It has been shown \( [54] \) that the condition for quadrantal symmetric frequency response is satisfied if the transfer function can be
expressed as

\[
H(z_1, z_2) = \frac{H(z_1, z_2)}{D(z_1, z_2)} = \frac{N_1(z_1, z_2 + z_2^{-1})N_2(z_1 + z_2^{-1}, z_2)}{D_1(z_1)D_2(z_2)}
\]  

(2.36)

Hence it is obvious that the 2-D filter transfer function expressed by equation (2.35) possesses quadrantal symmetric frequency response. It is also expected that the response of the designed filter will have smaller error from the ideal response because the order of the numerator function is double compared to the previous method. The design procedures are similar to the method I with additional symmetry condition (2.33). Three examples are given below to demonstrate the implementation of the technique.

2.10.1. Design examples

(i) Design a linear phase lowpass filter using method I of generating one-variable HP with following specifications:

\[
|H_1(\omega_1, \omega_2)| = \begin{cases} 
1 & \text{for } 0 \leq \sqrt{\omega_1^2 + \omega_2^2} < 1 \text{ rad/sec} \\
0 & \text{for } 2.5 \leq \sqrt{\omega_1^2 + \omega_2^2} \leq \omega_s/2.
\end{cases}
\]

where \(\omega_s = 10 \text{ rad/sec}\) and \(T = \pi/5 \text{ sec}\).

Assume second order filter, the transfer function is

\[
H(z_1, z_2) = \frac{\sum_{i=0}^{1} \sum_{j=0}^{1} n(i,j)\cos(\omega_1 T)\cos(\omega_2 T)}{D_1(z_1)D_2(z_2)}
\]  

(2.37)

where

\[
D_i(z_i) = D_i(s_i) \bigg|_{s_i = \frac{2(z_i - 1)}{s_i + 1}}
\]

and \(D_i(s_i)\) are given in equations (2.7,2.8).

Since the frequency response is quadrantal symmetric, the optimization procedure needs be carried only out in one
<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n(0,0) = 0.1884e+03 )</td>
<td>( \gamma_1 = -0.1300e+01 )</td>
</tr>
<tr>
<td>( n(0,1) = 0.5857e+02 )</td>
<td>( \gamma_2 = 0.4167e+01 )</td>
</tr>
<tr>
<td>( n(0,2) = 0.6362e+02 )</td>
<td>( g_1 = 0.1227e+02 )</td>
</tr>
<tr>
<td>( n(1,0) = 0.5693e+02 )</td>
<td>( \gamma_3 = 0.3582e+00 )</td>
</tr>
<tr>
<td>( n(1,1) = 0.4102e+03 )</td>
<td>( \gamma_4 = 0.3140e+01 )</td>
</tr>
<tr>
<td>( n(1,2) = 0.1952e+03 )</td>
<td>( g_2 = 0.2550e+01 )</td>
</tr>
<tr>
<td>( n(2,0) = 0.4647e+02 )</td>
<td>( )</td>
</tr>
<tr>
<td>( n(2,1) = 0.2318e+03 )</td>
<td>( )</td>
</tr>
<tr>
<td>( n(2,2) = -0.1429e+03 )</td>
<td>( )</td>
</tr>
</tbody>
</table>

Table 12: Coefficients of second order 2-D SD lowpass filter using method I with zero-phase numerator.

<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
<th>( \gamma_1 )</th>
<th>( \gamma_2 )</th>
<th>( \gamma_3 )</th>
<th>( \gamma_4 )</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( r_1 )</th>
<th>( r_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n(0,0) = 0.8888e+00 )</td>
<td>( \nu_1 = 0.9223e+00 )</td>
<td>( \gamma_1 = 0.9192e+00 )</td>
<td>( \gamma_2 = 0.9446e+00 )</td>
<td>( \sigma_1 = 0.1091e+01 )</td>
<td>( \sigma_1 = 0.1083e+01 )</td>
<td>( a_1 = 0.1386e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( r_1 = 0.8536e+00 )</td>
<td>( r_2 = 0.9240e+00 )</td>
</tr>
<tr>
<td>( n(0,1) = 0.8947e+00 )</td>
<td>( \sigma_2 = 0.1103e+01 )</td>
<td>( \sigma_2 = 0.1086e+01 )</td>
<td>( \sigma_2 = 0.1086e+01 )</td>
<td>( \sigma_2 = 0.1086e+01 )</td>
<td>( \sigma_2 = 0.1086e+01 )</td>
<td>( \sigma_2 = 0.1086e+01 )</td>
<td>( \sigma_2 = 0.1086e+01 )</td>
<td>( \sigma_2 = 0.1086e+01 )</td>
<td>( \sigma_2 = 0.1086e+01 )</td>
</tr>
<tr>
<td>( n(0,2) = 0.8701e+00 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
</tr>
<tr>
<td>( n(1,0) = 0.1151e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
</tr>
<tr>
<td>( n(1,1) = 0.1158e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
</tr>
<tr>
<td>( n(1,2) = 0.8661e+00 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
</tr>
<tr>
<td>( n(2,0) = 0.8076e+00 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
</tr>
<tr>
<td>( n(2,1) = 0.8294e+00 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
</tr>
<tr>
<td>( n(2,2) = 0.6052e+00 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
<td>( a_2 = 0.1427e+01 )</td>
</tr>
</tbody>
</table>

Table 13: Coefficients of second order 2-D SD lowpass filter using method II with zero-phase numerator.
Figure 12a: Magnitude response of 2-D second order lowpass using method I with zero-phase numerator.
Figure 12b: Group-delay 2 of 2-D second order lowpass using method I with zero-phase numerator
Figure 12c: Group-delay 2 of 2-D second order lowpass using method I with zero-phase numerator
Figure 13a: Magnitude response of 2-D second order lowpass using method II with zero-phase numerator
Figure 13b: Group-delay 1 of 2-D second order lowpass using method II with zero-phase numerator
Figure 13c: Group-delay 2 of 2-D second order lowpass using method II with zero-phase numerator.
quadrant of the frequency plane. The quadrantal symmetry property ensures the filter response be exactly the same in every quadrant. This advantage of the technique allows significant savings in computer time. Table 12 and figures 12a,b,c give the coefficients of the filter, magnitude and group-delay responses respectively.

(ii) Design the lowpass filter in (i) using method II of generating one-variable HP.

For a second order filter the transfer function is given in (2.37), and \( D_i(z) \) is shown in (2.15) with \( i = 1,2 \). Again, the frequency responses need be calculated only in one quadrant as a consequence of the quadrantal property of the designed filter. Table 13 is the set of coefficients of the filter and figures 13a,b,c are the magnitude and group-delays of the filter.

(iii) Design a bandpass filter using method II of generating one-variable HP:

\[
[H_2(\omega_1, \omega_2)] = \begin{cases} 
0 & \text{for } 0 \leq \sqrt{\omega_1^2 + \omega_2^2} \leq 1 \text{ rad/sec} \\
1 & \text{for } 2 \leq \sqrt{\omega_1^2 + \omega_2^2} \leq 3 \text{ rad/sec} \\
0 & \text{for } 4 \leq \sqrt{\omega_1^2 + \omega_2^2} \leq \omega_z / 2.
\end{cases}
\]

where \( \omega_z = 10 \text{ rad/sec} \) and \( T = \pi / 5 \text{ sec.} \) and constant group-delay.

The analog reference fourth order filter transfer function is

\[
H(z_1, z_2) = \frac{\sum_{i=0}^{4} \sum_{j=0}^{4} n(i,j) \cos(i\omega_1 T) \cos(j\omega_2 T)}{D_1(z_1)D_2(z_2)} \quad (2.38)
\]

where \( D_i(z_1) = D_i(z_2) \left| \begin{array}{c} 2(z_2 - 1) \\ s_i = T(z_i + 1) \end{array} \right|, \quad i = 1,2. \)
<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n(0,0) = 0.1715e+01$</td>
<td>$y_{11} = 0.4047e+00$</td>
</tr>
<tr>
<td>$n(0,1) = 0.7436e+00$</td>
<td>$y_{12} = 0.8721e+00$</td>
</tr>
<tr>
<td>$n(0,2) = -0.3122e+00$</td>
<td>$\sigma_{11} = 0.5580e+00$</td>
</tr>
<tr>
<td>$n(0,3) = 0.5950e+00$</td>
<td>$\sigma_{12} = 0.4604e+00$</td>
</tr>
<tr>
<td>$n(0,4) = -0.8059e+00$</td>
<td>$a_{11} = 0.1144e+01$</td>
</tr>
<tr>
<td>$n(1,0) = 0.3927e+00$</td>
<td>$r_{11} = 0.9895e-01$</td>
</tr>
<tr>
<td>$n(1,1) = -0.9305e+00$</td>
<td>$g_{11} = 0.1104e+01$</td>
</tr>
<tr>
<td>$n(1,2) = -0.2565e+01$</td>
<td>$\gamma_{11} = 0.7752e+00$</td>
</tr>
<tr>
<td>$n(1,3) = 0.6075e+00$</td>
<td>$\gamma_{12} = 0.6082e+00$</td>
</tr>
<tr>
<td>$n(1,4) = 0.2894e+00$</td>
<td>$\gamma_{14} = 0.8419e+00$</td>
</tr>
<tr>
<td>$n(2,0) = -0.6086e+00$</td>
<td>$\sigma_{13} = 0.5968e+00$</td>
</tr>
<tr>
<td>$n(2,1) = -0.2143e+01$</td>
<td>$\sigma_{14} = 0.8708e+00$</td>
</tr>
<tr>
<td>$n(2,2) = 0.7259e+00$</td>
<td>$a_{12} = 0.4646e+00$</td>
</tr>
<tr>
<td>$n(2,3) = 0.9574e+00$</td>
<td>$r_{12} = 0.1381e+01$</td>
</tr>
<tr>
<td>$n(2,4) = -0.1488e+00$</td>
<td>$g_{12} = 0.8618e+00$</td>
</tr>
<tr>
<td>$n(3,0) = 0.1576e+01$</td>
<td>$\gamma_{21} = 0.6049e+00$</td>
</tr>
<tr>
<td>$n(3,1) = -0.2164e+00$</td>
<td>$\gamma_{22} = 0.4072e+00$</td>
</tr>
<tr>
<td>$n(3,2) = 0.9337e+00$</td>
<td>$\sigma_{21} = 0.2646e+00$</td>
</tr>
<tr>
<td>$n(3,3) = -0.4128e-01$</td>
<td>$\sigma_{22} = 0.7505e+00$</td>
</tr>
<tr>
<td>$n(3,4) = -0.8671e+00$</td>
<td>$a_{21} = 0.8946e+00$</td>
</tr>
<tr>
<td>$n(4,0) = -0.1461e+00$</td>
<td>$r_{21} = 0.1026e+01$</td>
</tr>
<tr>
<td>$n(4,1) = -0.8860e+00$</td>
<td>$g_{21} = 0.6463e+00$</td>
</tr>
<tr>
<td>$n(4,2) = 0.9984e+00$</td>
<td></td>
</tr>
<tr>
<td>$n(4,3) = -0.1380e+01$</td>
<td></td>
</tr>
<tr>
<td>$n(4,4) = 0.1120e+01$</td>
<td></td>
</tr>
</tbody>
</table>

**Table 14:** Coefficients of fourth order 2-D SD bandpass filter using method II with zero-phase numerator.
Figure 14a: Magnitude response of 2-D fourth order bandpass using method II with zero-phase numerator.
Figure 14b: Group-delay 1 of 2-D fourth order bandpass using method II with zero-phase numerator
Figure 14c: Group-delay 2 of 2-D fourth order bandpass using method II with zero-phase numerator.
and $D_1(s_i) = D_{i1}(s_i)D_{i2}(s_i)$

$D_i(s_i) = D_{i1}(s_i)D_{i2}(s_i)$

where $D_{i1}(s_i)$, $D_{i2}(s_i)$, $D_{i1}(s_i)$, $D_{i2}(s_i)$ are given in equations (2.31, 2.32).

The objective function in the optimization algorithm is given in (2.28-2.28). Table 14 and figures 14a,b,c give the optimized coefficients, the magnitude and group-delay responses of the designed filter.

2.11. Method III of Designing 2-D SD IIR Filter

Consider the following SD transfer function

$$H(z_1, z_2) = \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} n(i,j)z_1^{-i}z_2^{-j}}{D(z_1)D(z_2)}, \quad z_i = e^{j\omega_T^k}, \quad k=1,2.$$\[2.39\]

where $D(z_i) = D(s_i)\left|_{s_i = \frac{-1}{2} (z_i - 1)}\right.$

Note that $D(s_i)$ is a one-variable polynomial and the denominator of the transfer function is a cascade of two identical 1-D polynomials of $s_i$ and $s_i$ (that is, the coefficients of $D(s_i)$ and $D(s_i)$ are exactly the same).

Now by using the symmetry condition indicated in (2.33), the transfer function is

$$H(z_1, z_2) = \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} n(i,j)\cos(i\omega_T)\cos(j\omega_T)}{D(z_1)D(z_2)}$$

It is to be noted that quadrantal symmetry property is preserved in (2.39).

The design of the above transfer function can be carried out in two steps in order to minimize the computational effort.
Step one: Form a 1-D 'all-pole' function,

\[ H(z) = \frac{1}{D(z)} \]

where \( D(z) \) is the discrete version of \( D(s) \) (through bilinear transformation) which is a one-variable HP generated using either method given in section 2.1 or 2.2. The coefficients of the above function are calculated such that the 1-D mean square error between specified and designed magnitude and phase responses are minimized. The formulation used here is given in equations (2.19-2.21). Note that the designed 1-D filter is guaranteed to be stable.

Step two: By forming 2-D transfer function shown in equation (2.39) with the denominator coefficients left unaltered, only the parameters of zero-phase numerator function are optimized so that the designed filter estimates the desired magnitude response. The 2-D design formulation shown in equations (2.25,2.26) is to be used here. Since the filter possesses quadrantal symmetry properties optimization is to be carried out only in quadrant of the frequency plane.

This approach is feasible because the numerator polynomial is a zero-phase function and the overall phase response is only a function of the denominator polynomial alone. This is a very efficient algorithm when compared with the general approach of the work described in [40]. The number of discrete frequency points reduced from \( M \times M \) to \( M \) in evaluating the error of the group-delay response and this leads to considerable saving in the computation time.

This design technique is supported by examples in the
following section.

2.11.1. Design examples

(i) Consider a 2-D lowpass filter with constant group-delay using method I of generating HP:

\[ |H(z_1, z_2)| = \begin{cases} 
1 & \text{for } 0 \leq \sqrt{\omega_1^2 + \omega_2^2} < 1 \text{ rad/sec} \\
0 & \text{for } 2.5 \leq \sqrt{\omega_1^2 + \omega_2^2} \leq \omega_s/2.
\end{cases} \]

where \( \omega_s = 10 \text{ rad/sec} \) and \( T = \pi/5 \text{ sec} \).

The transfer function of a second order 'all-pole' filter is

\[ H(z) = \frac{1}{D(z)} \]  \hspace{1cm} (2.40)

and

\[ D(z) = D(s) \bigg|_{s = \frac{2}{T} \left(\frac{z-1}{z+1}\right)} \]  \hspace{1cm} i=1,2.

where \( D(s) \) is given in (2.6).

This is a simple 1-D filter design problem. The optimized denominator coefficients are given in table 15. Now, by forming the 2-D transfer function

\[ H(z_1, z_2) = \frac{1}{D(z_1)} \frac{\sum_{i,j} n(i,j) \cos(i\omega_1 T) \cos(j\omega_2 T)}{D(z_1)D(z_2)} \]  \hspace{1cm} (2.41)

where \( D(z_1), D(z_2), i=1,2 \), is the polynomial obtained in the first stage. The optimization process is carried out for the magnitude response alone using (2.25, 2.26) as cost function. This saves the calculations of the group-delays of the filter on the 2-D plane. Since the overall 2-D filter satisfies the quadrantal symmetry condition (2.36), calculation is only carried out in one quadrant. The numerator coefficients are also given in table 15. Magnitude and group-delays are
<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>n(0,0) = 0.4456e-01</td>
<td>γ₁ = 0.2508e+01</td>
</tr>
<tr>
<td>n(0,1) = 0.3371e+00</td>
<td>γ₂ = 0.1831e+00</td>
</tr>
<tr>
<td>n(0,2) = -0.8844e-01</td>
<td>g₁ = 0.1015e+01</td>
</tr>
<tr>
<td>n(1,0) = 0.3434e+00</td>
<td></td>
</tr>
<tr>
<td>n(1,1) = -0.5998e-02</td>
<td></td>
</tr>
<tr>
<td>n(1,2) = 0.4804e+00</td>
<td></td>
</tr>
<tr>
<td>n(2,0) = -0.9120e-01</td>
<td></td>
</tr>
<tr>
<td>n(2,1) = 0.4868e+00</td>
<td></td>
</tr>
<tr>
<td>n(2,2) = -0.3117e+00</td>
<td></td>
</tr>
</tbody>
</table>

Table 15: Coefficients of second order 2-D SD lowpass filter using method I with two-step optimization.

<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>n(0,0) = 0.1689e+02</td>
<td>γ₁ = 0.7713e+00</td>
</tr>
<tr>
<td>n(0,1) = 0.1480e+01</td>
<td>γ₂ = 0.8884e+00</td>
</tr>
<tr>
<td>n(0,2) = -0.4298e+01</td>
<td>g₁ = 0.2378e+01</td>
</tr>
<tr>
<td>n(0,3) = 0.4368e+01</td>
<td>γ₃ = 0.3250e+00</td>
</tr>
<tr>
<td>n(0,4) = -0.2019e+01</td>
<td>γ₄ = 0.9417e+00</td>
</tr>
<tr>
<td>n(1,0) = 0.2777e+01</td>
<td>g₂ = 0.7188e+00</td>
</tr>
<tr>
<td>n(1,1) = -0.1151e+02</td>
<td></td>
</tr>
<tr>
<td>n(1,2) = -0.7745e+01</td>
<td></td>
</tr>
<tr>
<td>n(1,3) = 0.4465e+01</td>
<td></td>
</tr>
<tr>
<td>n(1,4) = -0.2942e+01</td>
<td></td>
</tr>
<tr>
<td>n(2,0) = -0.1216e+01</td>
<td></td>
</tr>
<tr>
<td>n(2,1) = -0.1431e+02</td>
<td></td>
</tr>
<tr>
<td>n(2,2) = 0.9746e+00</td>
<td></td>
</tr>
</tbody>
</table>

Table 16: Coefficients of fourth order 2-D SD bandpass filter using method I with two-step optimization.
Figure 15a: Magnitude response of 2-D second order lowpass using method I with two-step optimization
Figure 15b: Group-delay 1 of 2-D second order lowpass using method I with two-step optimization
Figure 15c: Group-delay 2 of 2-D second order lowpass using method I with two-step optimization
Figure 16a: Magnitude response of 2-D fourth order bandpass using method I with two-step optimization.
Figure 18b: Group-delay 1 of 2-D fourth order bandpass using method I with two-step optimization
Figure 15c: Group-delay 2 of 2-D fourth order bandpass using method I with two-step optimization.
illustrated in figures 15a,b,c. Note that both delays are identical to each other, this is a consequence of the two-step optimization approach which has assumed identical denominator polynomial in \( z_1 \) and \( z_2 \) and hence, the number of denominator parameters is only half of the previous two methods.

(ii) Design a 2-D bandpass filter with constant group-delay based upon method I of generating HP:

\[
|H(z_1, z_2)| = \begin{cases} 
0 & \text{for } 0 \leq \sqrt{\omega_1^2 + \omega_2^2} \leq 1 \text{ rad/sec} \\
1 & \text{for } 2 \leq \sqrt{\omega_1^2 + \omega_2^2} \leq 3 \text{ rad/sec} \\
0 & \text{for } 4 \leq \sqrt{\omega_1^2 + \omega_2^2} \leq \omega_s/2.
\end{cases}
\]

where \( \omega_s = 10 \) rad/sec and \( T = \pi/5 \) sec.

The transfer function of a fourth order 'all-pole' filter is the same as (2.40) with \( D(s) \) defined in (2.7-2.9) and

\[
D(z) = D(s) \bigg|_{s = \frac{2}{T} \frac{(z-1)}{T(z+1)}}, i=1,2.
\]

The optimized denominator coefficients are given in table 16. The second step is to form the 2-D transfer function as

\[
H(z_1, z_2) = \frac{\sum_{i=0}^{4} \sum_{j=0}^{4} n(i,j) \cos(i\omega_1 T) \cos(j\omega_2 T)}{D(z_1)D(z_2)} \tag{2.42}
\]

where \( D(z_i), z_i = 1,2, \) is the polynomial obtained in the first stage. In this second stage calculation is carried out in one quadrant of the 2-D frequency plane. The numerator coefficients are also given in table 16. Magnitude and group-delays are illustrated in figures 16a,b,c.

(iii) Repeat example (i) using method II of generating HP.

The 1-D transfer function of a second order 'all-pole' filter
### Table 17: Coefficients of second order 2-D SD lowpass filter using method II with two-step optimization.

<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>n(0,0) = 0.3994e+00</td>
<td>γ_1 = 0.1114e+01</td>
</tr>
<tr>
<td>n(0,1) = -0.3259e+00</td>
<td>γ_2 = 0.6652e+00</td>
</tr>
<tr>
<td>n(0,2) = -0.2282e+00</td>
<td>σ_i = 0.8644e+00</td>
</tr>
<tr>
<td>n(1,0) = -0.2985e+00</td>
<td>σ_j = 0.8289e+00</td>
</tr>
<tr>
<td>n(1,1) = 0.1343e+01</td>
<td>a = 0.7007e+00</td>
</tr>
<tr>
<td>n(1,2) = 0.4519e+00</td>
<td>r = 0.8148e+00</td>
</tr>
<tr>
<td>n(2,0) = -0.6316e-01</td>
<td>g = 0.6524e+00</td>
</tr>
<tr>
<td>n(2,1) = 0.3714e+00</td>
<td></td>
</tr>
<tr>
<td>n(2,2) = -0.6930e+00</td>
<td></td>
</tr>
</tbody>
</table>

### Table 18: Coefficients of fourth order 2-D SD bandpass filter using method II with two-step optimization.

<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>n(0,0) = -0.3474e+00</td>
<td>γ_{11} = 0.2253e+00</td>
</tr>
<tr>
<td>n(0,1) = 0.5905e+00</td>
<td>γ_{12} = 0.1393e+01</td>
</tr>
<tr>
<td>n(0,2) = -0.1668e+01</td>
<td>σ_{11} = 0.1017e+00</td>
</tr>
<tr>
<td>n(0,3) = 0.1891e-01</td>
<td>σ_{12} = 0.6462e+00</td>
</tr>
<tr>
<td>n(0,4) = -0.1064e+00</td>
<td>a_i = 0.6302e+00</td>
</tr>
<tr>
<td>n(1,0) = 0.2426e+00</td>
<td>r_i = 0.8763e+00</td>
</tr>
<tr>
<td>n(1,1) = 0.5926e+01</td>
<td>g_i = 0.1741e+01</td>
</tr>
<tr>
<td>n(1,2) = -0.2248e+01</td>
<td>γ_{21} = 0.4833e+00</td>
</tr>
<tr>
<td>n(1,3) = 0.1820e+01</td>
<td>γ_{22} = 0.6206e+00</td>
</tr>
<tr>
<td>n(1,4) = -0.4543e+00</td>
<td>σ_{21} = -0.1366e-01</td>
</tr>
<tr>
<td>n(2,0) = -0.1407e+01</td>
<td>σ_{22} = 0.9329e+00</td>
</tr>
<tr>
<td>n(2,1) = -0.2289e+01</td>
<td>a_j = 0.5394e+00</td>
</tr>
<tr>
<td>n(2,2) = -0.1982e+01</td>
<td>r_j = 0.6387e+00</td>
</tr>
<tr>
<td></td>
<td>g_j = 0.8855e+00</td>
</tr>
</tbody>
</table>
Figure 17a: Magnitude response of 2-D second order lowpass using method II with two-step optimization
Figure 17b: Group-delay 1 of 2-D second order lowpass using method II with two-step optimization
Figure 17c: Group-delay 2 of 2-D second order lowpass using method II with two-step optimization
Figure 10a: Magnitude response of 2-D fourth order bandpass using method II with two-step optimization
Figure 18b: Group-delay 1 of 2-D fourth order bandpass using method II with two-step optimization
Figure 18c: Group-delay 2 of 2-D fourth order bandpass using method II with two-step optimization
is shown in (2.40) where \(D(z)\) is the bilinear transformed version of \(D(s)\) which is defined in (2.14). The 2-D transfer function is given in (2.41) where \(D_1(z)\), \(z = 1,2\), is the polynomial obtained in the first stage. The filter coefficients are given in table 17. Magnitude and group-delays are illustrated in figures 17a,b,c.

(iv) Repeat example (ii) using method II of generating HP.

The 1-D transfer function of a fourth order 'all-pole' filter is shown in (2.40) and \(D(s) = D_1(s)D_2(s)\) where \(D_i(s)\) is defined in (2.15), \(i=1,2\). The 2-D transfer function is given in (2.41) where \(D(z)\), \(z = 1,2\), is the polynomial obtained in the first stage. The filter coefficients are given in table 18. Magnitude and group-delays are illustrated in figures 18a,b,c.

2.12. Design of Octagonal Symmetric 2-D IIR Filter

The design of 2-D recursive digital filters possessing octagonal symmetry in the frequency responses has been reported by [38,39]. These design techniques do not guarantee linear phase responses and in addition, stability test as well as stabilization procedures are needed at the last stage of the design. The advantage of the octagonal symmetry property is that the frequency responses approximations are only required in the region \([0', 45']\) of the 2-D frequency plane as shown in figure 1c. This fact can be incorporated in the two-step optimization approach to obtain constant group-delays and it provides an even further saving in computation when compared to the third design method discussed above. In this section a technique in designing
linear phase octagonal symmetric 2-D IIR filter is demonstrated.

Assume

\[ H(z_1, z_2) = \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} n(i, 1)z_1^{-i}z_2^{-j}}{D(z_1)D(z_2)}, \quad z_k = e^{j\omega_k T}, \quad k=1,2. \]

where \[ D(z_k) = D(s_k) \]

\[ s_k = \frac{2(z_k - 1)}{T(z_k + 1)}, \quad k = 1,2. \]

Let \[ n(i, 1) = n(1, i) \] (2.43)

then the transfer function can be written as

\[ H(z_1, z_2) = \frac{\sum_{i=0}^{N} \sum_{j=0}^{N} n'(i, j)\cos(i\omega_1 T)\cos(j\omega_2 T)}{D(z_1)D(z_2)} \] (2.44)

with \[ n'(i, j) = n'(j, i). \]

The conditions for octagonal symmetry are [38]:

1. \[ |H(e^{-j\omega_1 T}, e^{-j\omega_2 T})| = |H(e^{j\omega_1 T}, e^{j\omega_2 T})| \]
2. \[ |H(e^{-j\omega_1 T}, e^{j\omega_2 T})| = |H(e^{j\omega_1 T}, e^{-j\omega_2 T})| \]
3. \[ |H(e^{-j\omega_1 T}, e^{j\omega_2 T})| = |H(e^{j\omega_1 T}, e^{j\omega_2 T})| \]

- Condition 1 is automatically satisfied if the filter coefficients are real numbers.
- Condition 2 implies symmetries in the filter coefficients, that is \( a(i, j) = a(j, i) \).
- Condition 3 requires separable denominator for stable filter.

It is obvious that the transfer function (2.44) satisfies all three conditions.

Since the numerator polynomial is a zero-phase function and the denominator is a product of two 1-D identical polynomials in \( z_1 \) and \( z_2 \), the two-step optimization procedure discussed before can also be incorporated in the design. Therefore the
design procedures are:

1. Form a 1-D 'all-pole' filter

\[ H(z) = \frac{1}{D(z)} \]

where \( D(z) \) is the bilinear transformed version of \( D(s) \) which is a HP generated using methods mentioned in sections 2.1 and 2.2. Optimize the function coefficients based upon equations (2.19-2.21) for both magnitude and group-delay responses.

2. Form the 2-D function (2.44) using denominator coefficients as weighting functions and optimize the numerator coefficients using equations (2.25,2.26). In this case the calculations ought to be carried out only in the range \([0',45']\) of the frequency plane.

This algorithm requires much less computer time than any of the above methods while maintaining linear phase characteristics of the filter. Its validity is justified by following two examples.

2.12.1. Design examples

(i) Design a linear phase 2-D lowpass filter with

\[ |H_1(\omega_{11},\omega_{22})| = \begin{cases} 1 & \text{for } 0 \leq \sqrt{\omega_{11}^2 + \omega_{22}^2} < 1 \text{ rad/sec} \\ 0 & \text{for } 2.5 \leq \sqrt{\omega_{11}^2 + \omega_{22}^2} \leq \omega_s/2 \end{cases} \]

where \( \omega_s = 10 \text{ rad/sec} \) and \( T = \pi/5 \text{ sec} \) using method I of generating HP.

The design procedures for the 1-D 'all-pole' filter are the same as the previous approach. For the 2-D transfer function
Table 19: Coefficients of second order 2-D SD lowpass filter using method I with octagonal symmetry property.

<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>n(0,0) = 0.8931e-01</td>
<td>γ₁ = 0.2509e+01</td>
</tr>
<tr>
<td>n(0,1) = 0.2839e+00</td>
<td>γ₂ = 0.1831e+00</td>
</tr>
<tr>
<td>n(0,2) = -0.2888e+00</td>
<td>g₁ = 0.1015e+01</td>
</tr>
<tr>
<td>n(1,1) = 0.3598e+00</td>
<td></td>
</tr>
<tr>
<td>n(1,2) = 0.7191e+00</td>
<td></td>
</tr>
<tr>
<td>n(2,2) = -0.7820e+00</td>
<td></td>
</tr>
</tbody>
</table>

Table 20: Coefficients of second order 2-D SD lowpass filter using method II with octagonal symmetry property.

<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>n(0,0) = 0.3114e+00</td>
<td>γ₁ = 0.1114e+01</td>
</tr>
<tr>
<td>n(0,1) = -0.3788e+00</td>
<td>γ₂ = 0.6852e+00</td>
</tr>
<tr>
<td>n(0,2) = -0.2761e-01</td>
<td>σ₁ = 0.8844e+00</td>
</tr>
<tr>
<td>n(1,1) = 0.2151e+01</td>
<td>σ₂ = 0.8289e+00</td>
</tr>
<tr>
<td>n(1,2) = -0.2734e-01</td>
<td>a = 0.7007e+00</td>
</tr>
<tr>
<td>n(2,2) = -0.7071e+00</td>
<td>r = 0.6148e+00</td>
</tr>
<tr>
<td></td>
<td>g = 0.6524e+00</td>
</tr>
</tbody>
</table>
Figure 19: Contour plot of second order lowpass using method I with octagonal symmetry property
Figure 20a: Magnitude response of 2-D second order lowpass method I with octagonal symmetry property
Figure 20b: Group-delay 1 of 2-D second order lowpass using method I with octagonal symmetry property
Figure 20c: Group-delay of 2-D second order lowpass using method I with octagonal symmetry property.
Figure 21: Contour plot of second order lowpass using method II with octagonal symmetry property

MAGNITUDE RESPONSE

- 107 -
Figure 22a: Magnitude response of 2-D second order lowpass using method II with octagonal symmetry property.
Figure 22b: Group-delay 1 of 2-D second order lowpass using method II with octagonal symmetry property
Figure 22c: Group-delay 2 of 2-D second order lowpass using method II with octagonal symmetry property
\[ H(z_1, z_2) = \frac{\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} n(i,j) \cos(i\omega_1 T) \cos(j\omega_2 T)}{D(z_1)D(z_2)} \]

where \( n(i,j) = n(j,i) \) and \( D(z_i), z_i = 1, 2 \), is the polynomial obtained in the first stage. The optimization process is carried out for the magnitude response alone using (2.25, 2.26) as cost function. Since the overall 2-D filter satisfies the octagonal symmetry conditions, calculation is only carried out in one octant. Note that the contour plot of the magnitude response shown in figure 19 is truly octagonal symmetric. There are significantly less number of filter coefficients compared to any previous approaches as can be seen from table 19. Magnitude and group-delays are illustrated in figures 20a,b,c.

(ii) Consider the lowpass filter in (i) using method II of generating HP.

Assuming second order filter the designed procedures are the same as in example (i) except the denominator of the analog reference filter which is generated using equation (2.14). The number of discrete frequency points used in the calculation is only half of that needed in examples of section 2.11.1. The filter coefficients are given in table 20. Figure 21 shows the contour plot of the octagonal symmetric response of the designed filter. Magnitude and group-delays are illustrated in figures 22a,b,c.

2.13. Design of Non-Quadrantal, Non-Circular Symmetric IIR Filter

While the above circularly symmetric filters are suitable for signals that do not have any preferred spatial direction,
it is also of interest to consider 2-D filters with non-
circular and non-quadrantal symmetric frequency responses.
It has been reported [44] that instead of having the general
separable two terms (product of two 1-D polynomials in \(z_1\) and
\(z_2\)) in the denominator of the transfer function, a third term
is needed in order to obtain the desired response. This type
of filter is called 3-term-separable-denominator (3TSD)
filter,

\[
H(z_1, z_2) = \frac{N(z_1, z_2)}{D_1(z_1)D_2(z_2)D_3(z_1 z_2)}, \quad z_k = e^{j\omega_k} \quad k=1,2.
\]

This transfer function can be written as

\[
H(z_1, z_2) = \frac{\sum_{i=0}^{N} \sum_{l=0}^{N} n(i, l) z_1^{-i} z_2^{-l}}{D_1(z_1)D_2(z_2)D_3(z_1 z_2)}, \quad z_k = e^{j\omega_k} \quad k=1,2.
\]

(2.45)

where \(D_i(z_i) = D_i(s_i)\)

\[
s_i = \frac{2(z_i - 1)}{T'(z_i + 1)}, \quad i = 1,2.
\]

and \(D_3(z, z_2) = D_3(s_3)\)

\[
s_3 = \frac{2(z_1 z_2 - 1)}{T'(z_1 z_2 + 1)}.
\]

The third term in the denominator of the 2-D filter can be
seen as a special case of the complex transformation design
technique which is used to transform a 1-D to a 2-D filter
[43] where \(z = e^{j\theta} z_1^{a_1}/z_2^{a_2}/z_3^{a_3}\) and \(\theta = 0, a_i = \beta_i = \alpha_i = 1.\) This
type of transformation corresponds to a 45° rotation and it is
well known that the stability and causality of the filter are
not affected by the transformation [43,44].

By assigning one-variable HP 's to the analog transfer
function denominator, the stability of the reference filter
is always guaranteed and hence the digital 2-D filter is also
stable. In fact, since the 2-D transfer function
\[ H(z_1, z_1) = \frac{\sum_{l=0}^{\infty} \sum_{l=0}^{\infty} n(i, l) z_1^{-i} z_1^{-l}}{D_1(z_1) D_2(z_1)} , z_k = e^{\omega_k \tau} \quad k=1,2. \]

does not possess the quadrantal symmetry constraint (2.36), it can be proven that the design of non-quadrantal frequency response filter can be achieved using simple two terms SD transfer function. In order to compare the performance of the two approaches illustrative examples are given in following section.

2.13.1. Design examples

(i) Design the following non-quadrantal symmetric 2-D linear phase filter (45° rotated ellipse),

Original ellipse specification:

\[ |H_1(\omega_1, \omega_1)| = \begin{cases} 
1 & \text{for } \omega_1^2 + (\omega_1/2)^2 \leq 1 \text{ rad/sec} \\
0 & \text{for } \omega_1^2 + (\omega_1/2)^2 \geq 2 \text{ rad/sec} 
\end{cases} \]

45° rotated ellipse specification:

\[ |H_1(\omega_1, \omega_1)| = \begin{cases} 
1 & \text{for } (\omega_1 + \omega_1)^2 + (\omega_1 + \omega_1)^2/4 \leq 2 \\
0 & \text{for } (\omega_1 + \omega_1)^2 + (\omega_1 + \omega_1)^2/4 \geq 4 
\end{cases} \]

Form an analog transfer function

\[ H(s_1, s_1) = \frac{\sum_{l=0}^{\infty} \sum_{l=0}^{\infty} n(i, l) s_1^{-i} s_1^{-l}}{D_1(s_1) D_2(s_1) D_3(s_3)} \quad (2.46) \]

where \( D_i(s_i) \), \( i=1,2,3 \) is defined in equation (2.15). The corresponding 3TSD digital transfer function,

\[ H(z_1, z_1) = \frac{\sum_{l=0}^{\infty} \sum_{l=0}^{\infty} n'(i, l) z_1^{-i} z_1^{-l}}{D_1(z_1) D_1(z_1) D_2(z_1) D_2(z_1)} \quad (2.47) \]
<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n(0,0) = -0.3058e+01 )</td>
<td>( \gamma_{1,1} = 0.1216e+01 )</td>
</tr>
<tr>
<td>( n(0,1) = 0.1208e+01 )</td>
<td>( \gamma_{1,2} = 0.4802e+00 )</td>
</tr>
<tr>
<td>( n(0,2) = -0.7072e-02 )</td>
<td>( \sigma_{1,1} = 0.7490e+00 )</td>
</tr>
<tr>
<td>( n(1,0) = 0.1023e+01 )</td>
<td>( \sigma_{1,2} = 0.6302e+00 )</td>
</tr>
<tr>
<td>( n(1,1) = -0.1178e+01 )</td>
<td>( a_1 = 0.4646e+00 )</td>
</tr>
<tr>
<td>( n(1,2) = 0.4344e-01 )</td>
<td>( r_1 = -0.7476e+00 )</td>
</tr>
<tr>
<td>( n(2,0) = 0.4253e-03 )</td>
<td>( g_1 = 0.7153e+00 )</td>
</tr>
<tr>
<td>( n(2,1) = 0.7166e-01 )</td>
<td>( \gamma_{2,1} = -0.3660e+00 )</td>
</tr>
<tr>
<td>( n(2,2) = 0.1769e-01 )</td>
<td>( \gamma_{2,2} = 0.3491e+00 )</td>
</tr>
<tr>
<td>( \sigma_{2,1} = 0.6971e+00 )</td>
<td>( \sigma_{2,1} = 0.6971e+00 )</td>
</tr>
<tr>
<td>( \sigma_{2,2} = 0.1899e+01 )</td>
<td>( a_2 = 0.5027e+00 )</td>
</tr>
<tr>
<td>( r_2 = -0.1252e+01 )</td>
<td>( g_2 = 0.9850e+00 )</td>
</tr>
</tbody>
</table>

Table 21: Coefficients of 3TSD filter with 45° rotated ellipse frequency response specifications.

<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n(0,0) = 0.2069e+02 )</td>
<td>( \gamma_{1,1} = 0.1241e+01 )</td>
</tr>
<tr>
<td>( n(0,1) = 0.4919e-01 )</td>
<td>( \gamma_{1,2} = 0.1346e+01 )</td>
</tr>
<tr>
<td>( n(0,2) = 0.3899e+00 )</td>
<td>( \sigma_{1,1} = 0.1506e+01 )</td>
</tr>
<tr>
<td>( n(1,0) = -0.3678e-01 )</td>
<td>( \sigma_{1,2} = 0.1276e+01 )</td>
</tr>
<tr>
<td>( n(1,1) = 0.7598e+01 )</td>
<td>( a_1 = 0.1425e+01 )</td>
</tr>
<tr>
<td>( n(1,2) = 0.1007e-01 )</td>
<td>( r_1 = 0.1666e+01 )</td>
</tr>
<tr>
<td>( n(2,0) = 0.3786e+00 )</td>
<td>( g_1 = 0.1441e+01 )</td>
</tr>
<tr>
<td>( n(2,1) = 0.9809e-03 )</td>
<td>( \gamma_{2,1} = -0.3660e+00 )</td>
</tr>
<tr>
<td>( n(2,2) = 0.7873e-01 )</td>
<td>( \gamma_{2,2} = 0.3491e+00 )</td>
</tr>
</tbody>
</table>

Table 22: Coefficients of general two terms SD filter with 45° rotated ellipse frequency response specifications.
Figure 23: Contour plot of 3TSD filter with 45° rotated ellipse frequency response specifications.
Figure 24a: Magnitude response of 3TSD filter with 45° rotated ellipse frequency response specifications
Figure 24b: Group-delay 1 of 3TSD filter with 45° rotated ellipse frequency response specifications
Figure 24c: Group-delay 2 of 3TSD filter with 45° rotated ellipse frequency response specifications.
Figure 25: Contour plot of two terms SD filter with 45° rotated ellipse frequency response specifications.
Figure 26a: Magnitude response of general two terms SD filter with 45° rotated ellipse frequency response specifications.
Figure 28b: Group-delay 1 of general two terms SD filter with 45° rotated ellipse frequency response specifications.
Figure 26c: Group-delay 2 of general two terms SD filter with 45° rotated ellipse frequency response specifications.
is the bilinear transformed version of $H(s_1, s_2)$ with

$$D_2(z_1, z_2) = D_3(s_3) \left| s_3 = \frac{2(z_1, z_2 - 1)}{T(z_1, z_2 + 1)} \right.$$ 

Note that the denominator polynomial is essentially an eighth order polynomial. The optimization is carried out over one half of the frequency plane and the cost function is defined by equations (2.26-2.28). Table 21 shows the designed filter coefficients, figures 23 and 24a,b,c are the contour plot of the magnitude response, magnitude and group-delays respectively. It is obvious that the response satisfies the filter non-quadrantal requirements.

For comparison, the same filter specifications can be designed using the usual two terms SD transfer function,

$$H(z_1, z_2) = \sum_{F_0} \sum_{F_0} n'(i, 1)z_1^{-1}z_2^{-1}$$

where $D_i(z_i), i=1,2$ is the bilinear transformed version of equation (2.15). The resulting filter coefficients are given in table 22 and the contour plot of the magnitude is shown in figure 25. Figures 26a,b,c are the magnitude and group-delays of the filter. Both the contour plots illustrated in figures 23 and 25 closely describe the specified 45° rotated ellipse. It should be noted that the denominator polynomial of the 3TSD transfer function (2.47) is of eighth order while the two terms SD (2.48) is only of second order.

(ii) Consider the following non-quadrantal symmetric frequency response filter with constant group-delays:

$$|H_2(\omega_1, \omega_2)| = \begin{cases} 1 & \text{for } \sqrt{(\omega_1 - 1.5)^2 + (\omega_2 - 1.5)^2} \leq 1 \\ 0 & \text{for } \sqrt{(\omega_1 - 1.5)^2 + (\omega_2 - 1.5)^2} > 1.5 \end{cases}$$
<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>n(0,0) = -0.5575e+00</td>
<td>γ_{11} = 0.1059e+01</td>
</tr>
<tr>
<td>n(0,1) = -0.4860e+00</td>
<td>γ_{11} = 0.6184e+00</td>
</tr>
<tr>
<td>n(0,2) = 0.3584e+00</td>
<td>σ_{11} = 0.1430e+01</td>
</tr>
<tr>
<td>n(1,0) = 0.8834e+00</td>
<td>σ_{12} = 0.5627e+00</td>
</tr>
<tr>
<td>n(1,1) = -0.3064e+01</td>
<td>a_1 = 0.2817e+00</td>
</tr>
<tr>
<td>n(1,2) = -0.6850e-01</td>
<td>r_1 = -0.1262e+01</td>
</tr>
<tr>
<td>n(2,0) = 0.6996e+00</td>
<td>g_1 = 0.9012e+00</td>
</tr>
<tr>
<td>n(2,1) = 0.1207e+00</td>
<td>γ_{21} = -0.4503e-01</td>
</tr>
<tr>
<td>n(2,2) = 0.4921e-02</td>
<td>γ_{22} = 0.3073e+00</td>
</tr>
<tr>
<td></td>
<td>σ_{21} = 0.5472e+00</td>
</tr>
<tr>
<td></td>
<td>σ_{22} = 0.1882e+01</td>
</tr>
<tr>
<td></td>
<td>a_3 = 0.5058e+00</td>
</tr>
<tr>
<td></td>
<td>r_3 = -0.1343e+01</td>
</tr>
<tr>
<td></td>
<td>g_3 = 0.8406e+00</td>
</tr>
</tbody>
</table>

\[ γ_{11} = 0.1757e+01 \]
\[ γ_{11} = 0.3215e+00 \]
\[ σ_{11} = 0.7546e+00 \]
\[ σ_{12} = 0.6349e+00 \]
\[ n_1 = -0.9312e+00 \]
\[ r_1 = -0.1629e+01 \]
\[ g_3 = 0.1135e+01 \]

Table 23: Coefficients of 3TSD filter with non-quadrantal symmetric frequency band specifications.
Figure 27: Contour plot of 3TSD filter with non-quadrantal symmetric frequency band specifications

MAGNITUDE RESPONSE

- - - - 0.1
- - - - 0.4
- - - - 0.7
- - - - 1.0
- - - - 0.2
- - - - 0.5
- - - - 0.8
- - - - 0.3
- - - - 0.6
- - - - 0.9

W1 AXIS

W2 AXIS

3.14
1.57
0.00
-1.57
-3.14

3.14
1.57
0.00
-1.57
-3.14
Figure 28a: Magnitude response of 3TSD filter with non-quadrantal symmetric frequency band specifications.
Figure 28b: Group-delay 1 of STSD filter with non-quadrantal symmetric frequency band specifications
Figure 28c: Group-delay 2 of 3TSD filter with non-quadrantal symmetric frequency band specifications
Again, the design of the filter will be shown in both approaches for the sake of comparison. Using a 3TSD analog transfer function (2.46) the corresponding digital transfer function (2.47) can be obtained as mentioned in (i). The filter coefficients are given in table 23. Figures 27 is the contour plot of the filter magnitude response and figures 28a,b,c are the magnitude and group-delays.

Assuming fourth order SD filter, the analog transfer function is

\[ H(s_1, s_2) = \sum_{i=0}^{4} \sum_{j=0}^{4} n(i,j)s_1^i s_2^j / D_1(s_1)D_2(s_2) \]

where \( D_1(s_1) = D_{11}(s_1)D_{14}(s_1) \)
\( D_2(s_2) = D_{21}(s_2)D_{24}(s_2) \)

and

\[ D_{11}(s_1) = \gamma_{11} \sigma_{11}^2 s_1^2 + [ \sigma_{11}^2 \gamma_{11}^2 + \sigma_{11}^2 \gamma_{11}^4 + (a_{11} - r_{11})^2 \sigma_{11}^2 \gamma_{11}^2 ] s_1 \]
\[ + (\sigma_{11}^2 \sigma_{11}^4 + \sigma_{11}^2 \gamma_{11}^2) \]

\[ D_{14}(s_1) = \gamma_{14} \sigma_{14}^2 s_1^2 + [ \sigma_{14}^2 \gamma_{14}^2 + \sigma_{14}^2 \gamma_{14}^4 + (a_{14} - r_{14})^2 \sigma_{14}^2 \gamma_{14}^2 ] s_1 \]
\[ + (\sigma_{14}^2 \sigma_{14}^4 + \sigma_{14}^2 \gamma_{14}^2) \]

with \( i = 1, 2 \).

The digital 2-D SD transfer function is obtained thru bilinear transformation. Table 24 is the set of filter coefficients and figure 29 is the magnitude response contour plot. The magnitude and group-delays are given in figures 30a,b,c. In this case the denominator is a fourth order polynomial and both approaches give satisfactory approximations.
<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n(0,0) = -0.7809e+01$</td>
<td>$\gamma_{i} = 0.9123e+00$</td>
</tr>
<tr>
<td>$n(0,1) = 0.8607e+01$</td>
<td>$\gamma_{i} = 0.4671e+00$</td>
</tr>
<tr>
<td>$n(0,2) = -0.2126e+02$</td>
<td>$\sigma_{i} = 0.7319e+00$</td>
</tr>
<tr>
<td>$n(0,3) = -0.4266e+00$</td>
<td>$\sigma_{i} = 0.8136e+00$</td>
</tr>
<tr>
<td>$n(0,4) = -0.5538e+00$</td>
<td>$a_{1i} = 0.2377e+01$</td>
</tr>
<tr>
<td>$n(1,0) = -0.4237e+01$</td>
<td>$r_{1i} = 0.1583e+01$</td>
</tr>
<tr>
<td>$n(1,1) = 0.3188e+02$</td>
<td>$g_{1i} = 0.1181e+01$</td>
</tr>
<tr>
<td>$n(1,2) = 0.2057e+02$</td>
<td>$\gamma_{i} = 0.1940e+01$</td>
</tr>
<tr>
<td>$n(1,3) = -0.9940e+01$</td>
<td>$\gamma_{i} = 0.9708e+00$</td>
</tr>
<tr>
<td>$n(1,4) = 0.2588e+00$</td>
<td>$\sigma_{i} = 0.6778e+00$</td>
</tr>
<tr>
<td>$n(2,0) = -0.1568e+02$</td>
<td>$\sigma_{i} = 0.1100e+01$</td>
</tr>
<tr>
<td>$n(2,1) = -0.1120e+02$</td>
<td>$a_{1i} = 0.2128e+00$</td>
</tr>
<tr>
<td>$n(2,2) = 0.6910e+01$</td>
<td>$r_{1i} = 0.1187e+01$</td>
</tr>
<tr>
<td>$n(2,3) = 0.2862e+01$</td>
<td>$g_{1i} = 0.2137e+01$</td>
</tr>
<tr>
<td>$n(2,4) = -0.1270e+01$</td>
<td></td>
</tr>
<tr>
<td>$n(3,0) = 0.4061e+01$</td>
<td></td>
</tr>
<tr>
<td>$n(3,1) = -0.9194e+00$</td>
<td></td>
</tr>
<tr>
<td>$n(3,2) = -0.5522e+00$</td>
<td></td>
</tr>
<tr>
<td>$n(3,3) = 0.1347e+01$</td>
<td></td>
</tr>
<tr>
<td>$n(3,4) = 0.2120e+00$</td>
<td></td>
</tr>
<tr>
<td>$n(4,0) = 0.1203e+00$</td>
<td></td>
</tr>
<tr>
<td>$n(4,1) = 0.1123e+01$</td>
<td></td>
</tr>
<tr>
<td>$n(4,2) = -0.7734e+00$</td>
<td></td>
</tr>
<tr>
<td>$n(4,3) = -0.6193e-03$</td>
<td></td>
</tr>
<tr>
<td>$n(4,4) = -0.3885e-01$</td>
<td></td>
</tr>
</tbody>
</table>

Table 24: Coefficients of general two terms SD filter with non-quadrantal symmetric frequency band specifications.
Figure 20: Contour plot of general two terms SD filter with non-quadrantal symmetric frequency band specifications.
Figure 30a: Magnitude response of general two terms SD filter with non-quadrantal symmetric frequency band specifications.
Figure 30b: Group-delay 1 of general two terms SD filter with non-quadrantal symmetric frequency band specifications.
Figure 30c: Group-delay 2 of general two terms SD filter with non-quadrantal symmetric frequency band specifications
CHAPTER THREE
GENERATION OF TWO-VARIABLE VERY STRICT HURWITZ POLYNOMIAL AND ITS APPLICATIONS IN 2-D FILTER DESIGN

In this chapter methods of generating two-variable VSHP's are first reviewed, then a new method for generating such polynomials is proposed and its application in the design of general 2-D (non-separable numerator and non-separable denominator) filter is shown.

3.1. Generation of Two-variable VSHP

In the one-dimensional complex s-plane, the infinite distant points can be represented by a single point [26]. On the other hand, there exists an infinite number of infinite distant points in the two-dimensional biplane \((s, s')\) which consists of two complex \(s\)-plane and \(s\)-plane. Hence, a two-variable rational function \(F(s, s')\) may possess two types of singularities, namely nonessential singularities of the first kind and nonessential singularities of the second kind. The following are some important definitions and theorems that are useful to the generation of VSHP [18, 26].

- **Nonessential Singularity of The First Kind**: A rational function \(F(s, s') = P(s, s')/Q(s, s')\) where \(P(s, s')\) and \(Q(s, s')\) are mutually prime, is said to possess a nonessential singularity of the first kind at \((s', s')\) if \(Q(s', s') = 0\) and \(P(s', s') \neq 0\).

- **Nonessential Singularity of The Second Kind**: A rational function \(F(s, s') = P(s, s')/Q(s, s')\) where \(P(s, s')\) and \(Q(s, s')\) are mutually prime, is said to possess a
nonessential singularity of the second kind at \((s_i', s_i')\) if \(Q(s_i', s_i') = P(s_i', s_i') = 0\).

- Strict Hurwitz Polynomial: \(D(s_i, s_i)\) is a strict HP (SHP) if \(1/D(s_i, s_i)\) does not possess any singularities in the region \(\{(s_i, s_i) / \Re(s_i) \geq 0, \Im(s_i) \geq 0, |s_i| < \infty \text{ and } |s_i| < \infty\}\).

- Very Strict Hurwitz Polynomial: \(D(s_i, s_i)\) is a VSHP if \(1/D(s_i, s_i)\) does not possess any singularities in the region \(\{(s_i, s_i) / \Re(s_i) \geq 0, \Im(s_i) \geq 0, |s_i| < \infty \text{ and } |s_i| < \infty\}\).

- Theorem I: \(D(s_i, s_i) = D_i(s_i, s_i)D_i(s_i, s_i)\) is a VSHP if and only if \(D_i(s_i, s_i)\) and \(D_i(s_i, s_i)\) are both VSHPs.

As shown by Ahmadi and Ramachandran [57], a SHP when applied to the denominator of the filter can be restricted to become a VSHP using proper constraint optimization technique. This approach however bears an expensive computational cost. An improvement of this is to generate directly VSHP using properties of the derivatives of even or odd parts of Hurwitz Polynomials [48]. The method can be summarized in the following steps:

(i) A suitable even (or odd) part of a \(n\)-variable Hurwitz polynomial is generated by taking the determinant of a physically realizable matrix of order \(n\).

(ii) The corresponding odd (or even) part associated with it can be obtained by taking its derivative \(n\) times. Hence the summation of the even (or odd) and odd (or even) part is a \(n\)-variable HP.

(iii) The \(n\)-variable HP can be converted to a two-variable VSHP with a large number of combinations.

It is rather obvious that the procedures involved in this method of generating VSHP are tedious and lengthy since \(n\) derivatives ought to be evaluated for \(n^{th}\) order matrix.
Figure 31: N-port lossless frequency independent network
Another approach to the solution is given in [50]. Consider an n-port gyrator terminated by capacitances $\mu_i$, $i=1,2,\ldots,n$, shown in figure 31. The resulting admittance matrix will be

$$
\mathbf{Y} = \begin{bmatrix}
\mu_1 & \theta_{12} & \theta_{13} & \theta_{14} & \cdots & \theta_{1n} \\
-\theta_{12} & \mu_2 & \theta_{23} & \theta_{24} & \cdots & \theta_{2n} \\
-\theta_{13} & -\theta_{23} & \mu_3 & \theta_{34} & \cdots & \theta_{3n} \\
-\theta_{14} & -\theta_{24} & -\theta_{34} & \mu_4 & \cdots & \theta_{4n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-\theta_{1n} & -\theta_{2n} & -\theta_{3n} & -\theta_{4n} & \cdots & \mu_n
\end{bmatrix}
$$

then the determinant of the matrix $\mathbf{Y}$, under certain conditions upon the capacitances, yields a two-variable VSHP directly. In comparison, this technique is relatively simpler than [49] since it does not require the calculation of the partial derivatives. It should be noted though the appropriate designations of $n$ variables $\mu_i$'s are still necessary in obtaining the desired two-variable VSHP for each case.

An alternative method given in [50] makes use of the properties of positive definite (semidefinite) matrices and does not require the determination of the variables in each case. A positive definite or semi-definite matrix $\mathbf{F}$ can always be decomposed as a product of an upper-triangular or a lower matrix and its transpose [53]. The matrix $\mathbf{F}$ which is realizable as a two-variable reactance network can be written as

$$
\mathbf{F} = \mathbf{A} \Gamma \mathbf{A}^T \mathbf{S}_1 + \mathbf{B} \Delta \mathbf{B}^T \mathbf{S}_1 + \mathbf{C}
$$

(3.1)

where $\mathbf{A}$ and $\mathbf{B}$ are upper-triangular matrices, $\Gamma$ and $\Delta$ are diagonal matrices and $\mathbf{C}$ is a skew-symmetric matrix. Therefore the determinant of $\mathbf{F}$ constitutes either the even part or odd part of a two-variable HP depending upon whether
the order is odd or even. A two-variable HP can then be formed by summing the determinant and its partial derivatives with respect to \( s_1 \) and \( s_2 \), that is,

\[
D(s_1, s_2) = \det(D) + k_1 \frac{\partial (\det D)}{\partial s_1} + k_2 \frac{\partial (\det D)}{\partial s_2}
\]

where \( k_1 \) and \( k_2 \) are positive constants. Ahmadi [50] has shown that before \( D(s_1, s_2) \) can be converted into a VSHP some of the entries of the diagonal matrices in (3.1) must be equal to zero and hence \( \Lambda \) and \( \Xi \) are necessarily positive semidefinite. The matrix \( D \) can be rewritten as

\[
D = \begin{bmatrix}
A_{11} \Gamma_{11} A_{11}^T + A_{12} \Gamma_{12} A_{12}^T & A_{12} \Gamma_{11} A_{12}^T \\
A_{12} \Gamma_{11} A_{12}^T & A_{12} \Gamma_{12} A_{12}^T
\end{bmatrix}
\]

\[
\begin{bmatrix}
A_{21} \Delta_{11} B_{11}^T + A_{22} \Delta_{12} B_{12}^T & B_{11} \Delta_{21} B_{11}^T \\
B_{12} \Delta_{11} B_{11}^T & B_{12} \Delta_{21} B_{12}^T
\end{bmatrix}
\]

\[
\begin{bmatrix}
G_{11} & G_{12} \\
-G_{11}^T & G_{12}
\end{bmatrix}
\]

where \( A_{11}, A_{12}, B_{11}, \) and \( B_{12} \) are upper triangular matrices, \( G_{11}, G_{12}, \) and \( G_{11} \) are skew-symmetric matrices and \( \Gamma_{11}, \Delta_{11}, \Gamma_{12}, \) and \( \Delta_{12} \) are diagonal matrices. In setting certain diagonal elements to zero it is preferrable to keep the degrees of \( s_1 \) and \( s_2 \) the same as this will influence the filter response, particularly regarding the symmetry. Ahmadi proved that \( \Delta_{11} \) to and \( \Gamma_{11} \) can assume zero hence matrix \( D \) becomes

\[
D = \begin{bmatrix}
A_{11} \Gamma_{11} A_{11}^T & A_{12} \Gamma_{12} A_{12}^T \\
A_{12} \Gamma_{11} A_{12}^T & A_{12} \Gamma_{12} A_{12}^T
\end{bmatrix}
\]

\[
\begin{bmatrix}
B_{11} \Delta_{11} B_{11}^T & B_{11} \Delta_{21} B_{11}^T \\
B_{12} \Delta_{11} B_{11}^T & B_{12} \Delta_{21} B_{12}^T
\end{bmatrix}
\]

\[
\begin{bmatrix}
G_{11} & G_{12} \\
-G_{11}^T & G_{12}
\end{bmatrix}
\]
or \[ D = A_i S_i + B_i S_i + G \]
and \( \det A_i = \det B_i = 0 \) is another condition [50] for the \( \det D \) to become a VSHP. Now the VSHP can be generated using higher order partial polynomial derivatives. The steps can be summarized as follows:

(i) Form
\[
D_i = \det(D) + k_i \left( \frac{\partial (\det D)}{\partial S_i} \right) + \left( \frac{\partial (\det D)}{\partial S_i} \right)
\]

(ii) Repeat step (i) until
\[
D_m = \det(D_{m-1}) + k_i \left( \frac{\partial (\det D_{m-1})}{\partial S_i} \right) + k_m \left( \frac{\partial (\det D_{m-1})}{\partial S_i} \right)
\]

where \( 2m \) is the order of matrix \( D \).

It is also noted by the author [50] that a large number of choices in partitioning the matrices are possible so that the generation of VSHP's is not restrictive.

Although the above method of generating VSHP does have its advantages over the others shown in [49,58] the requirement for calculating partial derivatives is still necessary. The calculation is tedious and its complexity increases with the order of the matrices. In the following section a new method of generating VSHP without associating the partial derivatives is introduced.

3.1.1. A new method of generating two-variable VSHP

This technique makes use of the properties of positive definite (semidefinite) matrix as well as the concept of resistance matrices to generate two-variable VSHP. This is in fact a modification of [50] such that a resistance matrix is added to the equation (3.1) and hence the calculation of various derivatives can be avoided.

Consider a symmetric positive definite matrix, \( M \)
\[ M = \Delta \Gamma \Delta^T s_i + \Theta \Delta \Theta^T s_i + \Theta + \Xi \Xi^T \]  \hspace{1cm} (3.2)

where

\[ \Delta = \left[ \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array} \right] \]  \hspace{1cm} (3.3)

\[ \Gamma = \left[ \begin{array}{cccc} \gamma_1^2 & 0 & \cdots & 0 \\ 0 & \gamma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_n^2 \end{array} \right] \]  \hspace{1cm} (3.4)

\[ \Theta = \left[ \begin{array}{cccc} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{array} \right] \]  \hspace{1cm} (3.5)

\[ \Xi = \left[ \begin{array}{c} \xi_1^2 \\ \xi_2^2 \\ \vdots \\ \xi_n^2 \end{array} \right] \]  \hspace{1cm} (3.6)

\[ \Delta = \left[ \begin{array}{cccc} \delta_1^2 & 0 & \cdots & 0 \\ 0 & \delta_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_n^2 \end{array} \right] \]  \hspace{1cm} (3.7)
$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 \\ 0 & \sigma_1^2 & 0 & 0 \\ 0 & 0 & \sigma_2^2 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \sigma_n^2 \end{bmatrix} \quad (3.8)$$

$$G = \begin{bmatrix} 0 & g_{12} & g_{13} & g_{14} & \cdots & g_{1n} \\ -g_{12} & 0 & g_{23} & g_{24} & \cdots & g_{2n} \\ -g_{13} & -g_{23} & 0 & g_{34} & \cdots & g_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -g_{1n} & -g_{2n} & -g_{3n} & \cdots & 0 & g_{nn} \end{bmatrix} \quad (3.9)$$

and superscript T in (3.2) denotes the transpose of a matrix. The determinant of \( M \) can be shown to be a VSHP and hence it is free of nonessential singularities on the closed right-half biplane \((s_1, s_i)\). As discussed in [50] as well as in previous section, certain entries of the diagonal matrices \( I \) and \( \Delta \) are to be set zero in order to obtain a VSHP. This method can be seen to be much simpler than any of the techniques discussed before. From theorem I, the cascade of two or more lower order VSHP's can be used to obtain a higher order VSHP. The idea is best illustrated by a simple example of second order matrices.

Consider

$$M = \Delta \Gamma \Delta^T s_1 + \Delta \Gamma \Gamma^T s_i + G + \Gamma \Gamma^T$$

(3.10)

where, assuming \( \gamma_1 = \delta_1 = 0 \),

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \gamma_1^2 \\ \gamma_1 & 1 \end{bmatrix} s_1 + \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} + \begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 \\ 0 & \sigma_1^2 & 0 & 0 \\ 0 & 0 & \sigma_2^2 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \sigma_1 & 1 \\ 0 & 1 \end{bmatrix} s_1 + \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix} \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}.$$
\[ M = \begin{bmatrix}
\alpha^2 \gamma_1^2 s_1 + b_1 \delta_1^2 s_1 + c_i^2 + r_1^2 \\
\alpha^2 \gamma_1^2 s_1 + b_1 \delta_1^2 s_1 + r_i^2 - q
\end{bmatrix}
\]

\[ \det M = (a - b)^2 \gamma_1^2 \delta_1^2 s_1 s_i + \gamma_1^2 \left[ (a - r)^2 c_i^2 + c_i^2 \right] s_i \\
+ \delta_i^2 \left[ (b - r)^2 c_i^2 + c_i^2 \right] s_i + c_i^2 c_i^2 + q^2 \\
(3.11) \]

It can easily be shown that the resulting two-variable polynomial is indeed a VSHP.
A fourth order VSHP can be generated either by starting with fourth order matrices or by cascading two second order polynomials.

3.2. Design of 2-D General Non-separable IIR Filter

All of the 2-D SD filters discussed so far in this thesis belong to only a sub-class of the general form 2-D filter transfer function,

\[ H(z_1, z_2) = \frac{\sum_{n=0}^{N} \sum_{l=0}^{N} n(i, l) z_1^i z_2^l}{\sum_{n=0}^{N} \sum_{l=0}^{N} b(i, l) z_1^i z_2^l} = \frac{N(z_1, z_2)}{D(z_1, z_2)} \]

A classical problem in the design of 2-D non-separable recursive digital filters is to generate a stable transfer function whose frequency responses behave in a prescribed manner. It has been shown by Goodman [55,56] that a SHP may possess nonessential singularities of second kind at the infinite distant points on the closed right half of the \((s_1, s_2)\) biplane. Hence the use of SHP in the denominator of a two-variable transfer function which, upon double bilinear transformation, may possess nonessential singularities of the
second kind on the unit bidisk of the \((z_1, z_1)\) biplane, does not always guarantee stability [55]. The technique given in this section shows the use of VSHP, generated from the new approach shown in previous section, in the design of 2-D non-separable numerator and denominator filters which approximate the prescribed responses while maintaining the stability. The design procedures are described below.

For

\[
H(s_1, s_2) = \frac{N(s_1, s_2)}{D(s_1, s_2)}
\]

\[
H(s_1, s_2) = \frac{\sum_{l=0}^{N} \sum_{i=0}^{N} n(i, l) s_1^l s_2^l}{\sum_{l=0}^{N} \sum_{i=0}^{N} b(i, l) s_1^l s_2^l}, \quad s_i = j\omega_i T, \quad i=1,2.
\]

where \(N(s_1, s_2)\) is left unaltered and \(D(s_1, s_2)\) can be assumed to be a VSHP generated using the technique given in previous section. The digital transfer function is then obtained by applying bilinear transformation, that is,

\[
H(z_1, z_2) = H(s_1, s_2) \bigg|_{s_k = \frac{2(z_k-1)}{T(z_k+1)}}, \quad k=1,2.
\]

The calculation of the filter parameters are based upon the general least mean square error criterion given in equations (2.26-2.28) such that some prescribed magnitude and phase responses are met. Any suitable non-linear optimization technique can be used here in the search of the filter coefficients. Since the resulting digital filter is a bilinearly transformed version of the stable analog filter, it is therefore guaranteed to be stable. Also, it should be noted that since the optimization procedure is actually carried out in the \(z\)-domain, the inherent warping effect of bilinear transformation does not apply in this case. The
design algorithm is indeed very simple and there is no stability check required. The implementation of the technique is supported by the following example.

3.2.1. Design example

(i) Design a linear phase lowpass filter with

\[ |H_1(\omega_1, \omega_2)| = \begin{cases} 1 & \text{for } 0 \leq \sqrt{\frac{\omega_1^2}{\omega_s^2} + \frac{\omega_2^2}{\omega_s^2}} < 1 \text{ rad/sec} \\ 0 & \text{for } 2.5 \leq \sqrt{\frac{\omega_1^2}{\omega_s^2} + \frac{\omega_2^2}{\omega_s^2}} \leq \omega_s/2. \end{cases} \]

where \( \omega_s = 10 \text{ rad/sec} \) and \( T = \pi/5 \text{ sec} \).

Assuming fourth order filter, the analog reference transfer function is

\[ H(s_1, s_1) = \frac{\frac{1}{i\pi s_1} \frac{1}{i\pi s_1} n(i, l) s_1^1 s_1^1}{\frac{1}{i\pi s_1} \frac{1}{i\pi s_1} b(i, l) s_1^1 s_1^1} = \frac{\frac{1}{i\pi s_1} \frac{1}{i\pi s_1} n(i, l) s_1^1 s_1^1}{D(s_1, s_1)} \]

where \( D(s_1, s_1) = D_1(s_1, s_2)D_4(s_1, s_1) \) and

\[ D_1(s_1, s_1) = (a_1 - b_1)^i \gamma_2^i \delta_2^i s_1^1 s_1^1 + \gamma_1^i \left[ (a_1 - r_1)^i \sigma_1^i + c_1^i \right] s_1^1 \]

\[ + \delta_1^i \left[ (b_1 - r_1)^i \sigma_1^i + c_1^i \right] s_1^1 + c_1^i \sigma_1^i + g_1^i \]

\[ D_4(s_1, s_1) = (a_1 - b_1)^i \gamma_4^i \delta_4^i s_1^1 s_1^1 + \gamma_4^i \left[ (a_1 - r_1)^i \sigma_4^i + c_4^i \right] s_1^1 \]

\[ + \delta_4^i \left[ (b_1 - r_1)^i \sigma_4^i + c_4^i \right] s_1^1 + c_4^i \sigma_4^i + g_4^i \]

Note that \( D_1(s_1, s_1) \) and \( D_4(s_1, s_1) \) are two second order VHSP's of the form given in equation (3.11). The denominator of the transfer function is a fourth order VHSP which is the product of two second order polynomials. A fourth order VSHP can also be derived from a fourth order matrix as mentioned above. The numerator is left unchanged as a general two-variable polynomial since there is no restrictions on the
<table>
<thead>
<tr>
<th>Numerator coefficients</th>
<th>Denominator coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n(0,0) = 0.5770e+01$</td>
<td>$a_1 = 0.1544e+01$</td>
</tr>
<tr>
<td>$n(0,1) = -0.5555e+00$</td>
<td>$\gamma_2 = 0.8066e+00$</td>
</tr>
<tr>
<td>$n(0,2) = 0.1512e+00$</td>
<td>$b_1 = -0.1468e+00$</td>
</tr>
<tr>
<td>$n(1,0) = -0.8914e+00$</td>
<td>$\delta_1 = 0.1137e+01$</td>
</tr>
<tr>
<td>$n(1,1) = -0.2500e+00$</td>
<td>$g_1 = 0.1427e+01$</td>
</tr>
<tr>
<td>$n(1,2) = -0.3086e-01$</td>
<td>$r_1 = -0.3180e+00$</td>
</tr>
<tr>
<td>$n(2,0) = 0.1520e+00$</td>
<td>$\sigma_1 = 0.9822e+00$</td>
</tr>
<tr>
<td>$n(2,1) = -0.1016e-01$</td>
<td>$\sigma_2 = 0.5414e+00$</td>
</tr>
<tr>
<td>$n(2,2) = 0.3906e-02$</td>
<td>$\sigma_3 = 0.4891e+00$</td>
</tr>
</tbody>
</table>

Table 25: Coefficients of 2-D non-separable numerator and denominator lowpass filter.
Figure 32a: Magnitude response of 2-D non-separable numerator and denominator lowpass filter
Figure 32b: Group-delay 1 of 2-D non-separable numerator and
denominator lowpass filter
Figure 32c: Group-delay 2 of 2-D non-separable numerator and denominator lowpass filter
zeroes of the 2-D filter. The digital transfer function is obtained using double bilinear transformation. The cost function used in the optimization process is given in (2.26-2.28) and the calculation is carried out over the half frequency plane. Table 25 and figures 32a,b,c give the filter coefficients and the frequency responses.

3.3. Measurement of Deviation of Designed Filters Responses To Coefficient Quantization

The realization of digital filters, in software as well as hardware, requires the numbers be stored in registers with finite word length. Therefore, among other quantization errors, the filter frequency characteristics are subject to the effect of coefficient quantization and this type of error is normally called the coefficient-quantization error. The quantization of the transfer function coefficients may result in the significant deviation of the frequency response from the desired response and the filter may actually fail to meet the prescribed description. In the floating point arithmetics, one simple way to analyze the performance of a filter is to consider rounding the mantissa of the numbers.

In order to prove that the designed filters have acceptably low deviations of the frequency responses to the rounding of their coefficients, the general least square error, defined in equations (3.12-3.14), of the responses are calculated.

\[
E(\omega_i, \omega_j, c) = \sum \sum \sum \sum \left( E^1_{\kappa}(\omega_i, \omega_j) + E^1_{\xi_1}(\omega_i, \omega_j) + \right.
\]

\[
\left. E^1_{\xi_2}(\omega_i, \omega_j) \right) + \sum \sum \sum \sum E^1_{\kappa}(\omega_i, \omega_j)
\]  (3.12)
where \( I_p \) is the set of discrete frequency points in the passband region.

\( I_s \) is the set of discrete frequency points in the stopband region.

and

\[
E_K(\omega_1, \omega_2) = |H_i(e^{j\omega_1 T}, e^{j\omega_2 T})| - |H_d(e^{j\omega_1 T}, e^{j\omega_2 T})|
\]  

(3.13)

where

\[ |H_i(e^{j\omega_1 T}, e^{j\omega_2 T})| \]

represents magnitude responses of the ideal filter.

and \[ |H_d(e^{j\omega_1 T}, e^{j\omega_2 T})| \]

represents magnitude responses of the designed filter.

and

\[
E_{\tau_k}(\omega_1, \omega_2) = \tau_i T - \tau_k(\omega_1, \omega_2)
\]  

(3.14)

for \( k = 1, 2 \).

where \( \tau_i T \) represents the ideal group-delay; \( \tau_k \) is defined in the proximity of the order of the filter and \( T \) \( (T_1 = T_2 = T) \) is the sampling period.

\( \tau_k(\omega_1, \omega_2) \), \( k=1, 2 \), are the designed group-delay with respect to \( \omega_1 \) and \( \omega_2 \) which are defined in equation (1.19).

The percentage deviation, defined in equation (3.15), are then used as performance indices to give a qualitative measure of the designed filters.

\[
\text{Percentage deviation} = \left| \frac{E(\omega_1, \omega_2, C) - E_n(\omega_1, \omega_2, C)}{E(\omega_1, \omega_2, C)} \right| \times 100
\]  

(3.15)

where \( E(\omega_1, \omega_2, C) \) is the least mean square error defined in (3.12-3.14) and \( E_n(\omega_1, \omega_2, C) \) is the corresponding mean square error of the filter with coefficients rounding to \( n \) decimal
points. Two examples of the above designed filters are particularly chosen here and they are the octagonal symmetric lowpass linear phase IIR filter (example (i) of section 2.12.1) and the non-quadrantal symmetric linear phase filter (example (ii) of section 2.13.1). Figures 33a,b,c,d, 34a,b,c,d, and 35a,b,c,d are the frequency response plots for the octagonal symmetric lowpass filter with optimized coefficients rounding to four, three and two decimal points respectively. Figures 36a,b,c,d, 37a,b,c,d and 38a,b,c,d are the corresponding plots for the non-quadrantal symmetric filter with rounding coefficients. Tables 26 and 27 are the tabulated percentage errors of the filters with quantized coefficients. As can be seen from the tables, the errors are by and large insignificant by any standard. It can therefore be concluded that, in general, the designed filters have extremely low coefficient-quantization error which is one of the vital factors in filter design practicality and feasibility.
### Table 26: Tabulated errors of the octagonal symmetric 2-D lowpass filter using method I with coefficients rounding to four, three and two decimal numbers.

<table>
<thead>
<tr>
<th>Number of decimal points</th>
<th>Percentage deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>four</td>
<td>0.288</td>
</tr>
<tr>
<td>three</td>
<td>3.042</td>
</tr>
<tr>
<td>two</td>
<td>3.773</td>
</tr>
</tbody>
</table>

### Table 27: Tabulated errors of the two terms SD filter with non-quadrantal symmetric frequency band specifications with coefficients rounding to four, three and two decimal numbers.

<table>
<thead>
<tr>
<th>Number of decimal points</th>
<th>Percentage deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>four</td>
<td>0.002</td>
</tr>
<tr>
<td>three</td>
<td>0.177</td>
</tr>
<tr>
<td>two</td>
<td>13.722</td>
</tr>
</tbody>
</table>
Figure 33a: Magnitude response of octagonal symmetric lowpass filter using method I with coefficients rounding to four digits.
Figure 33b: Group-delay 1 of octagonal symmetric lowpass filter using method I with coefficients rounding to four digits
Figure 33c: Group-delay of octagonal symmetric lowpass filter using method 1 with coefficients rounding to four digits.
Figure 33d: Contour plot of octagonal symmetric lowpass using method I with coefficients rounding to four digits.
Figure 34a: Magnitude response of octagonal symmetric lowpass filter using method I with coefficients rounding to three digits
Figure 34b: Group-delay 1 of octagonal symmetric lowpass filter using method I with coefficients rounding to three digits
Figure 34c: Group-delay of octagonal symmetric lowpass filter using method I with coefficients rounding to three digits.
Figure 34d: Contour plot of octagonal symmetric lowpass using method I with coefficients rounding to three digits.
Figure 35a: Magnitude response of octagonal symmetric lowpass filter using method I with coefficients rounding to two digits.
Figure 35b: Group-delay 1 of octagonal-symmetric lowpass filter using method 1 with coefficients rounding to two digits.
Figure 35c: Group-delay 2 of octagonal symmetric lowpass filter using method I with coefficients rounding to two digits
Figure 35d: Contour plot of octagonal symmetric lowpass using method I with coefficients rounding to two digits.
Figure 36a: Magnitude response of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to four digits.
Figure 36b: Group-delay 1 of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to four digits
Figure 36c: Group-delay Z of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to four digits.
Figure 36d: Contour plot of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to four digits.
Figure 37a: Magnitude response of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to three digits.
Figure 37b: Group-delay 1 of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to three digits.
Figure 37c: Group-delay 2 of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to three digits
Figure 37d: Contour plot of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to three digits.
Figure 38a: Magnitude response of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to two digits
Figure 38b: Group-delay 1 of SD filter with non-quadrantally symmetric frequency band with coefficients rounding to two digits.
Figure 38c: Group-delay 2 of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to two digits.
Figure 38d: Contour plot of SD filter with non-quadrantal symmetric frequency band with coefficients rounding to two digits.
CHAPTER FOUR
CONCLUSIONS

This thesis is concerned with the design techniques for linear phase recursive digital filters in one and two dimensions. It should be noted that the design techniques in this thesis, which use the unconstrained optimization procedures to search for the coefficients such that the filter approximates the desired response, lend themselves to the flexibility of designing virtually any type of filter besides standard filters such as lowpass, bandpass or highpass filters. Based upon the properties of positive definite (or semidefinite) matrices, two methods of generating one-variable HP's of arbitrary order and their applications in the 1-D linear phase filter design have been developed. These new approaches ensure the stability of the designed filters, while most existing techniques do not. The first method requires the use of partial derivatives as a procedure in deriving the desired polynomial while the second method makes use of an additional resistance matrix and avoids the need for partial derivatives, hence simplifying the filter design procedure. It is worth mentioning that the second method does require more parameters than the first method. This may be regarded as a disadvantage since larger number of coefficients generally means more computational complexity, which is obviously undesirable. However, through many examples, it has been observed that the large number of coefficients actually provide higher degree of freedom and flexibility in the process of optimization hence reducing the constraints involved. Both methods nevertheless give satisfactory results. Critical comparisons made between the
Elliptic filter compensated with delay equalization and the proposed designs showed that the new techniques lead to an overall better performance. Another advantage of the proposed techniques over the well known conventional approaches is that the desired frequency responses are obtained through a single step optimization procedure.

The application of one-variable HP's has been extended to the design of first-quadrant 2-D SD linear phase IIR filters. Methods of designing 2-D SD IIR filters include general two terms SD transfer function which possesses circular symmetry response and SD filter with zero-phase numerator polynomial. The SD transfer function designs retain much of the flexibility of non-separable filters and yet offer the implementation advantages of separable IIR filters. The zero-phase numerator transfer function possesses quadrantal symmetric property and it does, in general, give better filter approximation than the general SD filter. Also, a new algorithm for designing 2-D SD filters has been developed and it has proven to be very effective and efficient. This technique involves a two-stage optimization procedure which does not require the evaluation of group-delays in 2-D. As a result of further investigation, the two-step design technique has been modified to include the implementation of the octagonal symmetric frequency response linear phase IIR filter which has significantly less number of parameters when compared to any other approaches discussed in this thesis. This has been achieved with certain symmetry conditions on the numerator parameters and it leads to even further savings in computation time. A new type of 2-D filter called 3TSD filter has also been studied and its performance in approximating non-quadrantal symmetric response filter was compared to the general two terms SD filter. These types of
filters are suitable for processing signals which do not have any preferred spatial direction while preserving the merits of simple 1-D stability problem. It is interesting to find out that both methods give comparable desired responses and it can be concluded that the additional term in the denominator of the 3TSO transfer function, in contrast to the discussion in [44], is not really needed.

Due to the absence of a fundamental theorem of algebra for multidimensional polynomials, the stability concerning 2-D polynomials, which is the classical problem of 2-D filter design, is quite different and much more complex than that of the 1-D case. A new method of generating two-variable VSHP, which excludes all singularities on the right-half biplane \((s_1, s_2)\), has been presented. The new method has been shown to outperform all others given in [49,50,57,58] in terms of simplicity. Its usefulness in the design of general non-separable numerator and denominator 2-D IIR linear phase filters has been well justified.

In practice, the implementation of digital filters requires finite word length storage and this represents a problem in the filter design. The designed filters, in general, have noticeable low sensitivity to coefficient quantization.
REFERENCES


29. L. R. Rabiner, N.Y. Graham, M.D. Helms, "Linear


M. Ahmadi, V. Ramachandran, "A New Method of Generating 2-Variable VSHP and Its Application in The Design of 2-D Recursive Digital Filter With Prescribed Magnitude and


VITA AUCTORIS

Henry Jenq Jong Lee graduated from W.D. Lowe S.S. of Windsor, Ontario in 1982 with the Ontario S.S.G.D. and S.S.H.G.D. He was an Ontario Scholar 1982. He received his B.A.Sc. degree in Electrical Engineering from the University of Windsor in 1986 and is a candidate for the degree of M.A.Sc. in Electrical Engineering. Publications out of his research work presented in this thesis include:


Upon graduation he will join Allied-Signal Aerospace Company, Garrett Canada as a Control Systems Design Engineer.