Design of 2-D stable recursive digital filter satisfying prescribed magnitude and group delay responses.

Sandeep P. Golikeri

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LA THÈSE A ÉTÉ MICROFILMÉE TELLE QUE NOUS L'AVONS RECEUE
DESIGN OF 2-D STABLE RECURSIVE DIGITAL FILTER
SATISFYING PRESCRIBED MAGNITUDE
AND GROUP DELAY RESPONSES

by
Sandeep P. Golikeri

A Thesis
submitted to the Faculty of Graduate Studies
through the Department of Electrical Engineering
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ABSTRACT

This thesis presents two methods for the design of 2-Dimensional (2-D) stable recursive digital filters satisfying a given magnitude and group delay responses. The first design method uses the properties of positive definite matrices while the second method uses the properties of derivatives for generating an even or odd part of Hurwitz polynomial. Both these techniques are used to generate a very strict Hurwitz Polynomials (VSHP) which will be then assigned to the denominator of the 2-D analog reference filters. Bilinear transformations are then applied to the transfer functions of the 2-D analog reference filters to obtain the discrete version of the filters. Parameters of the discrete transfer function can be used as variables of optimization to minimize the least mean square error of the desired and the designed magnitude and group delay responses of the filters. The methods are illustrated by several examples.
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TABLE OF CONTENTS

ABSTRACT. ........................................ iii
ACKNOWLEDGEMENTS. ................................ iv
LIST OF TABLES. ................................. vii
LIST OF FIGURES ................................. viii
LIST OF SYMBOLS ................................. ix

CHAPTER

I. INTRODUCTION ................................. 1
  1.1 Introduction. ............................. 1
  1.2 Digital Filtering ......................... 3
  1.3 Types of Digital Filter ................. 3
  1.4 Characterization of 2-D Analog Filters and Recursive Digital Filters ........................ 5
  1.5 Organization of Thesis ................... 6

II. STABILITY AND STABILIZATIONS ............... 8
  2.1 Definitions of Stability ................. 8
  2.2 Shanks' Stability Theorem ............... 10
      2.2.1 Alternative Stability Formulation .......... 11
      2.2.2 Modification of Shanks' Theorem .......... 12
  2.3 Huang's Stability Test ................... 13
  2.4 Ansell's Stability Test ................. 14
  2.5 Anderson and Jury Stability Test ....... 15
  2.6 Maria and Fahmy Method ................. 17
  2.7 Comparison of Stability Tests .......... 17
  2.8 Stabilization Techniques ............... 18
      2.8.1 Value of Function and Infinity ....... 19
      2.8.2 Definition of Singularity .......... 20
      2.8.3 Two Variable Hurwitz Polynomials .......... 22
      2.8.4 Test Procedure for VSHP .......... 26
  2.9 Conclusion. ............................. 26

III. GENERATION OF VSHP USING MATRIX FACTORIZATION TECHNIQUE ......................... 27
  3.1 Introduction. ............................. 27
  3.2 Mathematical Background ................ 27
<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Values of the Parameters of the Designed Low Pass Filter</td>
<td>45</td>
</tr>
<tr>
<td>2. Values of the Parameters of the Designed Low Pass Filter</td>
<td>64</td>
</tr>
<tr>
<td>3. Coefficients of Band Pass Filter</td>
<td>65</td>
</tr>
<tr>
<td>4. Coefficients of Fan Filter</td>
<td>66</td>
</tr>
</tbody>
</table>
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>3-D plot of the amplitude response of the designed low pass filter.</td>
<td>46</td>
</tr>
<tr>
<td>1b</td>
<td>3-D plot of the group delay response w.r.t. w.</td>
<td>47</td>
</tr>
<tr>
<td>1c</td>
<td>3-D plot of group delay response w.r.t. w₂</td>
<td>48</td>
</tr>
<tr>
<td>2</td>
<td>N-variable passive network.</td>
<td>51</td>
</tr>
<tr>
<td>3a</td>
<td>3-D plot of amplitude response of the designed low pass filter.</td>
<td>68</td>
</tr>
<tr>
<td>3b</td>
<td>Group delay response w.r.t. w₁</td>
<td>69</td>
</tr>
<tr>
<td>3c</td>
<td>Group delay response w.r.t. w₂</td>
<td>70</td>
</tr>
<tr>
<td>4a</td>
<td>3-D plot of amplitude response of the designed band pass filter.</td>
<td>71</td>
</tr>
<tr>
<td>4b</td>
<td>Group delay w.r.t. w₁</td>
<td>72</td>
</tr>
<tr>
<td>4c</td>
<td>Group delay w.r.t. w₂</td>
<td>73</td>
</tr>
<tr>
<td>5a</td>
<td>3-D plot of amplitude response of the designed fan filter.</td>
<td>74</td>
</tr>
<tr>
<td>5b</td>
<td>Group delay response w.r.t. w₁</td>
<td>75</td>
</tr>
<tr>
<td>5c</td>
<td>Group delay response w.r.t. w₂</td>
<td>76</td>
</tr>
<tr>
<td>6</td>
<td>Plot to verify the stability of low pass filter designed using matrix approach method.</td>
<td>81</td>
</tr>
<tr>
<td>7</td>
<td>Plot to verify the stability of filters designed with derivative approach method:</td>
<td></td>
</tr>
<tr>
<td>7a</td>
<td>Low pass filter</td>
<td>82</td>
</tr>
<tr>
<td>7b</td>
<td>Band pass filter</td>
<td>83</td>
</tr>
<tr>
<td>7c</td>
<td>Fan filter</td>
<td>84</td>
</tr>
</tbody>
</table>
LIST OF SYMBOLS

\[ \sum_{i=1}^{n} a_i \quad \text{represents } a_1 + a_2 + \ldots + a_n \]

\[ \prod_{i=1}^{n} a_i \quad \text{represents } a_1 \cdot a_2 \cdot \ldots \cdot a_n \]

\( \in \quad \text{represents "is member of"} \)

\( \cap \quad \text{represents "the intersection of"} \)

\[ \bigcap_{i=1}^{n} (|z_i|=1) \quad \text{represents } (|z_1|=1) \cap (|z_2|=1) \cap \ldots \cap (|z_n|=1) \]

\( \forall \quad \text{represents "for all"} \)
CHAPTER I

INTRODUCTION

1.1

One of the significant problems in communications is the processing of signals which have been passed through a system which has resulted in a deterioration in their quality. The need for such processing resulted, in early days, from the distortion produced in telephone links over relatively long distances. The whole theory of analogue signal processing developed from this need and led to classical filters and other passive networks design and later developed into the more modern aspects of active network synthesis.

More recently, digital signal processing has been introduced for the treatment either of analogue signals which have been sampled periodically to produce a set of discrete pulses, or of signals which from transmission to reception are in discrete form. Such signals may be considered to be represented by a sequence of numbers representing the value of the signal at successive instants of time. Once this conceptual approach to time sequences has been accepted, it may be realized that digital signal processing may be con-
considered in the simpler manner as a means whereby an array of numbers may be modified according to some selected laws to generate an output array. These arrays may be of one or more dimensions.

In the recent years, there has been rapid growth of interest in the area of one and two dimensional signal processing. Examples of one dimensional signals are speech and ECG, whereas examples of two dimensional signal processing are pictorial data or non-pictorial data. In the case of speech, for example, it may be used to extract the information corresponding to the identity of a speaker, whereas in ECG it may be used for data compression. In the case of two dimension, it may be used for image enhancement of low quality images such as X-rays, photographs, and for compensation of linear optical degradations. They are also used for preprocessing data in a 2-D pattern recognition system [1].

Such arrays need to be processed in order to remove anomalous signals and also to enhance certain aspects of data. For example, in medicine, X-ray films may be processed to remove a low spatial frequency variation, thereby enhancing the sudden variations of abnormal conditions. In the case of scanned isotope detection, the processor may be required to remove the distortions on the image produced by the finite width of scanning lines and other distortion which occur as a result of finite resolution of the gamma rays scanner. In electron microscopy, the purpose may be
to reduce the low frequency background noise which is inherent in such processes. In the field of geophysical prospecting, echoes from boundaries of geological strata of a detonation are detected by linear array of detectors placed in line with the source. Desired echoes will be reviewed by the detectors from various changes in geological strata, whereas undesired echoes may occur from multiple reflections and random signals may be generated by wind noise [1].

It is conceptually simple to appreciate that arrays are now no longer restricted to two dimensions but may be extended to three dimensions. Practical examples are not presented, as this is beyond the scope of this thesis.

1.2 Digital Filtering

Any computational process which transforms an input sequence of numbers to an output sequence of numbers in a prescribed manner is called digital filtering.

1.3 Types of Digital Filter

There are basically two types of digital filters:
1) recursive digital filters (IIR); and 2) non-recursive digital filters (FIR).

Much work has been carried out in both types of filters, with each one having its own advantages and disadvantages. In general, to design a filter satisfying prescribed fre-
frequency spectrum, recursive filters require much less multiplications than non-recursive digital filters, hence they are cheaper to implement and have faster response time. This advantage lies in the fact that the feedback loop is used to modify the input of the filter with the weighted samples of previous output yielding an infinite impulse response with the requirement of only a finite number of computational steps per output point. An inherent stability problem, associated with the feedback loop, is one of the main disadvantages of this filter [2]. Another disadvantage in recursive filters in general they do not offer a linear phase characteristic, which is not a problem in non-recursive filters [2]. It is shown by Huang [3] that phase characteristics should be given as much importance as magnitude characteristics. Also, in many applications, linear phase is important, where dispersion due to non-linear phase characteristics is harmful, specifically in image processing problems.

Non-recursive digital filters on the other hand generate the output from weighted samples of input only. Although this process is inherently more inefficient than recursive filtering, certain techniques like fast fourier transforms or number theoretic transforms can be implemented to speed up the process.
1.4 Characterization of 2-D Analog Filters and Recursive Digital Filters

A 2-D analog filter is characterized by its transfer function.

\[
H_a(s_1, s_2) = \frac{A(s_1, s_2)}{B(s_1, s_2)} = \frac{M_1 i_1 \left[i_1 = 0\right] + M_2 i_2 \left[i_2 = 0\right]}{M_1 i_1 + M_2 i_2 + b(i_1, i_2) s_1 + s_2} \quad (1.1)
\]

where \(A\) and \(B\) are polynomials in \(s_1\) and \(s_2\). The design problem is to obtain the polynomial coefficients \(a(i_1, i_2)\) and \(b(i_1, i_2)\) such that:

i) \(H_a\) approximates the given response,

ii) the designed filter is stable, that is [4]

\[
B(s_1, s_2) \neq 0 \text{ for } \sum_{i=1}^{2} \text{Re } s_i > 0 \quad (1.2)
\]

Similarly, a 2-D recursive digital filter is characterized by its \(z\)-transfer function

\[
H_d(z_1, z_2) = \frac{N(z_1, z_2)}{D(z_1, z_2)} = \frac{M_1 i_1 \left[i_1 = 0\right] + M_2 i_2 \left[i_2 = 0\right]}{M_1 i_1 + M_2 i_2 + d(i_1, i_2) z_1 + z_2} \quad (\star.3)
\]

where \(N\) and \(D\) are polynomials in \((z_i = \exp s_i T, i = 1, 2)\) and
the design problem is to obtain polynomials co-efficients 
n(i_1, i_2) and d(i_1, i_2) such that z transfer function evaluated 
on the unit circles in z_i, i=1,2 plane approximates the de-
sired response of the filter, besides maintaining the 
stability of the filter. The latter condition requires [1]

\[ D(z_1, z_2) \neq 0 \text{ for } |z_1| \geq 1 \] (1.4)

The approximation can be carried out in analog or digital 
domains by using any suitable optimization method, but the 
difficulty lies in maintaining the stability of the designed 
filter, inhibiting the widespread application of recursive 
filters, in spite of their significant computational advan-
tages over non-recursive digital filters.

The overall design procedure involves two steps: a) 
the approximation of magnitude followed by b) group delay 
equalization which compensates for non-linearities in the 
group delay characteristics of the magnitude only filter.

1.5 Organization of Thesis

This thesis will first review the problems of stability 
and stabilization of two dimensional digital filters in 
Chapter II. It will also deal with some basic definitions 
and test procedures for Strict Hurwitz Polynomials (SHP) 
and Very Strict Hurwitz Polynomials (VSHP).

Two different approaches for the generation of VSHP 
are discussed in Chapters III and IV. Chapter III is based
on properties of positive semi-definite matrices. Also a
design method based on the above is presented for a low
pass filter satisfying prescribed magnitude and group delay
responses.

Chapter IV deals with the properties of even or odd
parts of n-variable Hurwitz Polynomial. This method was
implemented in designing a low pass filter, band pass
filter and fan filter satisfying prescribed magnitude and
group delay responses.
CHAPTER II

STABILITY AND STABILIZATIONS

2.1 Definitions of Stability

The most commonly used definition of stability is bounded input, bounded output (BIBO) stability; a system is defined as being BIBO stable if the output is bounded in response to a bounded input. This has been studied in considerable detail in continuous and one dimensional systems and has more recently been extended to two and multi-dimensional systems.

Consider a bounded sequence

\[ |c(m_1, m_2)| \leq M < \infty \]  \hspace{1cm} (2.1)

This array is absolutely summable if

\[ \sum_{m_1=0}^{M_1} \sum_{m_2=0}^{M_2} |c(m_1, m_2)| \leq N < \infty \]  \hspace{1cm} (2.2)

where \( M \) and \( N \) are positive real numbers.

It is possible to develop the conditions for a two dimensional system to be BIBO stable as follows: The output array \( 0(m_1, m_2) \) is given by the convolution of an input array \( i(m_1, m_2) \) with impulse response of filter \( f(p_1, p_2) \), namely,
\[ 0(m_1, m_2) = \sum_{p_1=0}^{M_1} \sum_{p_2=0}^{M_2} f(p_1, p_2) i(m_1-p_1, m_2-p_2) \]

can be written as

\[ 0(m_1, m_2) = \left| \sum_{p_1=0}^{M_1} \sum_{p_2=0}^{M_2} f(p_1, p_2) i[(m_1-p_1), (m_2-p_2)] \right| \]  

(2.3)

Application of Schwartz's inequality leads to

\[ |0(m_1, m_2)| \leq \sum_{p_1=0}^{M_1} \sum_{p_2=0}^{M_2} |f(p_1, p_2)| \cdot |i[(m_1-p_1), (m_2-p_2)]| \]  

(2.4)

For all values, \( m \) and \( p \) are bounded

\[ |0(m_1, m_2)| \leq m \sum f(p_1, p_2) \]  

(2.5)

Comparison of equation 2.5 with 2.2, for a BIBO stable system, shows that \( f(p_1, p_2) \) must be an absolutely summable array. We have shown above, that it is a necessary and sufficient condition for BIBO stability.

Thus, a necessary and sufficient condition for a network to be BIBO stable is that its impulse response shall be absolutely summable.

An alternative, but less familiar, forms of stability is that in which we require the output sequence to be absolutely summable if the input sequence is absolutely summable (SISO stability). By a similar application of
Schwartz's inequality to the convolution equation (2.3), it may be shown that a necessary and sufficient condition is that impulse response of a system shall be absolutely summable.

Although the above conditions are basic to the definition of stability, they are little value in assessing the stability of a specified network or a system. For this, it is simpler to operate in the frequency or $z$ domain.

2.2 Shanks' Stability Theorem

For a recursive filter, it has been shown by Shanks [5,6] that the stability of a system is controlled entirely by the properties of the denominator of the transfer function. The conditions imposed on the denominator function $B(z_1,z_2)$ in order to be assumed of stability are that

$$B(z_1,z_2) \neq 0 \forall (z_1, i=1, 2) \in D$$

(2.6)

where $D = \{(z_1, i=1, 2) \bigcap \bigcup_{i-1}^{2} |z_1| \geq 1\}$

Application of this theorem in one dimension is relatively straightforward since the fundamental theorem of algebra states that every real polynomial in a single variable may be factionized into real linear and quadratic factors and thus the location of the roots of the denominator may be obtained, if necessary, in high order systems, to any accuracy, by a computer algorithm.
In two or more dimensions the fundamental theorem of algebra does not hold. In fact, it may be shown that, in the general case, it is not possible to factorize a multi-variable polynomial into first and second order factors. The stability problem thus devolves into the determination of the continuum of values \( z_1, z_2 \) for which \( B(z_1, z_2) = 0 \) and checking whether they be within the domain D.

Shanks' approach was to define the infinite impulse response filter, \( g_{m1}, g_{m2} \), having z transform

\[
G(z_1, z_2) = \frac{1}{B(z_1, z_2)}
\]

He then showed that the stability condition

\[
B(z_1, z_2) \neq 0 \quad (z_i, i=1, 2) \in D
\]

is identical to the condition that \( G(z_1, z_2) \) shall be convolutionally stable, i.e., there exists a stable filter, \( b_{m,n} \) such that the convolution of '\( g_{m,n} \)' with '\( b_{m,n} \)' shall yield a two-dimensional impulse, \( \delta \); thus [5]

\[
g(m,n) * b(m,n) = \delta(m,n)
\]  

(2.7)

2.2.1 Alternative Stability Formulation

Anderson and Jury [7] have proposed an alternative formulation of the stability contention of Equation 2.6. This states that a system is stable if and only if
B(z₁, 0) ≠ 0 ∀ |z₁| ≥ 1

B(z₁, z₂) ≠ 0 ∀ 2 \bigcap_{i=1}^{n} |z_i| = 1 \bigcap |z_n| ≤ 1

(2.8)

This test may be formalized and applied by using the technique of Anderson and Jury [8].

2.2.2 Modification of Shank's Theorem

It has recently been shown by Goodman [9] that the necessity of Shank's theorem fails under certain conditions in which the transfer function numerator as well as denominator are multivariable polynomials. He has shown this for a two variable function, but the limitation is also relevant to a multidimensional system.

A two-dimensional polynomial, although not factorizable into first and second order factors, may be factorized into a set of unique irreducible real polynomials. There may be a number of points, (z₁, z₂) at which the denominator polynomial, B(z₁, z₂) is zero and it is these which control stability. In the majority of cases, the numerator A(z₁, z₂) = 0 also; such point is then termed a non-essential singularity of second kind. The existence of such points modifies Shank's theorem and may show a function to be stable.

A modified function F(z₁, z₂) will represent a stable system if F(z₁, z₂) has no poles in D₁₂ = \{(z₁, z₂) : |z₁| ≥ 1, |z₂| ≥ 1\} and no essential singularities of the second kind, except possibly in R₁₂ = \{(z₁, z₂) : |z₁| = 1, |z₂| = 1\}. 
When \( B(z_1, z_2) \neq 0 \) in \( D_{12} = (z_1, z_2) : |z_1| \geq 1, |z_2| > 1 \) but \( F(z_1, z_2) \) has a non-essential singularities of second kind in \( R_{12} \), it appears that \( F \) may or may not be stable. Examples of either situation are given by Goodman in his article [9].

To summarize Shank's stability theorem, it may be both necessary and sufficient except when essential singularities occur in \( R_{12} \) domain.

2.3 **Huang's Stability Test**

Shank's theorem has proved to be difficult to apply in practice and thus alternative techniques have been proposed for its implementation. Huang [10] put forward a technique which is relatively simple for two dimensional systems and although it might, in principle, be extended to multiple dimensions the application would be incredibly tedious.

He states that a two dimensional function

\[
H(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1, z_2)}
\]

is stable if and only if

1. the map of \( R_1 = \{ z_1 : |z_1| = 1 \) on the \( z_2 \) plane according to the transformation \( B(z_1, z_2) = 0 \) lies outside the domain \( D_2 = \{ z_2 : |z_2| > 1 \} \) and
2. no point in $D_1 = (z_1 : |z_1| > 1)$ maps into the point $z_2 = 0$ by the selection $B(z_1, z_2) = 0$.

To apply this, the circumference of the unit circle $R_1$ is mapped onto the $z_2$ plane and checked to see whether it intersects the unit circle in the $z_2$ plane. In addition, $B(0, z_2) = 0$ must be solved to check whether the magnitude of any root is less than unity.

Despite the simplification, the required computation is still laborious since it involves testing at an infinite number of points.

Huang showed that the test could be alternatively be conducted using the following technique by Ansell [11] which would result in a finite number of steps.

2.4 Ansell's Stability Test

Ansell's theorem [11] effectively transforms the filter function $H(z_1, z_2)$ from the $z$ domain to the $s$ domain via two bilinear transformations.

\[
\begin{align*}
    s_1 &= \frac{1 - z_1}{1 + z_1} \times \frac{2}{T_1} & (2.10a) \\
    s_2 &= \frac{1 - z_2}{1 + z_2} \times \frac{2}{T_2} & (2.10b)
\end{align*}
\]

The stability criteria may now be transformed into $s$ domain as follows:

A filter $H(z_1, z_2)$ is stable if and only if
1. In all real finite \( u_1 \), the complex polynomial in \( s_2, \hat{B}(ju_1, s_2) \), has no zeros in \( \text{Re}(s_2) \geq 0 \), and
2. the real polynomial in \( s_1, \hat{B}(s_1, 1) \), has no zeros in \( \text{Re}(s_1) \geq 0 \).

Condition (2) of Ansell's test is relatively simple to apply using existing one-dimensional stability techniques. Condition (2) is, however, more difficult since it involves a study of the roots of a complex polynomial of a complex variable. It may be put into an alternative form by considering \( B(ju_1, ju_2) \). This may be written as a complex polynomial in \( u_2 \) whose coefficients are real and imaginary functions of \( u_1 \).

The first stability condition of Ansell is now satisfied if all the coefficients are true for all real \( u_i \), \( i = 1, 2 \).

It is also apparent that the same technique may be applied to multidimensional polynomials by successive branching techniques.

2.5 **Anderson and Jury Stability Test**

Anderson and Jury again tackled the significant problem of attempting to bring the stability testing procedure to a simpler form which would facilitate computation. Their techniques originated from the formulation of stability conditions of Equation 2.8.
The first of these is relatively simple since it is a function of only one variable and there are a number of tests for determining whether roots of such polynomials will lie within a unit circle. The method that was implemented was a sequence of polynomials using a simple recurrence equation. The value of these polynomials at \( z = 0 \) are computed and their product obtained. The positivity of this quantity is necessary and sufficient condition for original polynomials to have all zeros outside this unit circle.

The second and subsequent members of Equation (2.8) are more difficult to establish.

Considering the second such polynomial which is a function of two variables \( z_1 \) and \( z_2 \), the test may be carried out in two parts.

One such method, is Schur-Cohn [8] matrix; it involves setting up a matrix formed from the coefficients of the function to be tested. In Schur-Cohn test the positivity of all the eigen values of constructed matrix is assessed. For the stability test we use \( z_2 \) as a variable parameter; the elements of matrix will be function of \( z_1 \). For stability test, the matrix must be definite for all \( z_1 \) such that \( |z_1| = 1 \). This is the problem of self inverse polynomials.

For third and higher order, some branding techniques may be used, but for anything greater than second order filters, the size of matrices becomes great that they may be manipulated with great difficulty.
2.6 Maria and Fahmy Method

The second and subsequent elements of Equation (2.8) have been tested by a technique developed by Maria and Fahmy [12].

They evolved an extended form of the Jury table which is obtained from the coefficients of the original polynomial by a relatively simple technique of computing a succession of 2x2 determinants. The series of critical elements must be non-negative. For two dimensional systems, all the coefficients will be functions of variable $z_1$; however, simplification is achieved since the multiplying coefficients are all real and as the test is applied on the boundary of unit circle where $|z_1| = 1$, we may write $z_1^* = z_1^{-1}$.

For multidimensional systems, the branching process outlined in the case of Anderson and Jury's test may be applied, but as the determinants of higher order than two need to be determined, the computational effort involved is maximized.

2.7 Comparison of Stability Tests

The Ansell stability test suffers from the difficulty of applying bilinear transformation before application of negatively simple test for roots of function in bounded space.

Anderson and Jury remove this drawback, but the matrix for higher order is tedious since order of matrix is equal
to degree of denominator of transfer function.

2.8 Stabilization Techniques

Having obtained a solution to a two dimensional design problem and determined a multivariable transfer function which satisfies the desired specifications, it is naturally a disappointment that the designed filter is unstable.

To overcome this difficulty, a number of techniques exist which may be used to process a transfer function in order to obtain a new and stable function which approximates the desired response. A straightforward method of generating such a function has been to apply double bilinear transformation on a two variable analog function possessing strict Hurwitz Polynomial denominator. However, it has been shown by Goodman [13] that not all two variable analog functions with strict Hurwitz Polynomial denominators upon bilinear transformation result in a stable two dimensional digital transfer function. There exist analog functions which on bilinear transformation result in digital functions possessing singularities of the second kind on the distinguished boundary of the unit bidisk in $z_1, z_2$ plane. The techniques that will be presented here are a class of a two variable Hurwitz polynomials called Very
Strict Hurwitz Polynomials, Rajan et al. [14]. Their application in the generation of two variable functions without non-essential singularities of the second kind is indicated. Some results on the nature of analog two variable function that yield, on bilinear transformation, a two dimensional digital function without any singularities on the unit bidisk. First we define certain mathematical definition.

2.8.1 Value of Function and Infinity

In a one dimension complex plane s, the infinite distant points can be represented by a single point, and the value of any function at this point is found by applying some transformation which transforms the point at infinity to some finite point s" and the value of the transformed function at s' is determined. Often s = 1/u is the transformation that is used and infinity is mapped onto the origin. Using this transformation F(s) at infinity is defined as f(∞) = f_T(0) where F_T(u) = f(1/u).

In two dimensional biplane (s_1, s_2), consisting of two complex planes s_1 and s_2, an infinite distant point can have infinite co-ordinates in any one or both of these planes, and so there exists an infinite number of infinite distant points. Hence these can be classified as follows:

1. s_1 = ∞ and s_2 = finite \hspace{1cm} (2.11a)
2. s_1 = finite and s_2 = ∞ \hspace{1cm} (2.11b)
3. \( s_1 = \infty \) and \( s_2 = \infty \)

Applying the transformation method to each variable, the value of the function at each of the above points is defined as:

\[
\begin{align*}
\text{a. } f(\infty, s_1^\ast) &= f_1(0, s_2^\ast) \text{ where } f_1(u_1, s_2) \\
&= f(1/u, s_2) \mid s_2 \leq \infty \\
\text{b. } f(s_1^\ast, \infty) &= f_2(s_2^\ast, 0) \text{ where } f_2(s_1^\ast, v) \\
&= f(s_1, 1/v) \mid s_1^\ast \leq \infty \\
\text{c. } f(\infty, \infty) &= f_3(0, 0) \text{ where } f_3(u, v) = f(1/u, 1/v)
\end{align*}
\]

we use the term closed right half of the \((s_1, s_2)\) biplane to represent the region \(\{(s_1, s_2) \mid \Re(s_1) \geq 0, \Re(s_2) \geq 0, |s_1| \leq \infty \text{ and } |s_2| \leq \infty\}\).

2.8.2 Definition of Singularity

Given a two variable rational function \( f(s_1, s_2) = \frac{P(s_1^\ast, s_2)}{Q(s_1, s_2)} \) where \( P \) and \( Q \) are mutually prime, may possess two types of singularities and may be defined as follows: 1) non-essential singularity of the first kind: \( f(s_1, s_2) \) is said to possess a non-essential singularity of the first kind at \((s_1^\ast, s_2^\ast)\) if \( Q(s_1^\ast, s_2^\ast) \neq 0 \) and \( P(s_1^\ast, s_2^\ast) = 0 \); 2) non-essential singularity of second kind: \( f(s_1, s_2) \) is said to possess a non-essential singularity of the second kind at \((s_1^\ast, s_2^\ast)\) if \( Q(s_1^\ast, s_2^\ast) = 0 \) and \( P(s_1^\ast, s_2^\ast) = 0 \), as a rational function can have only non-essential singularities.
One method of generating a stable two dimensional digital function has been to apply a double bilinear trans-
formation \( s_1 = \frac{2}{t_1}(1-z_1)/(1+z_1) \) and \( s_2 = \frac{2}{t_2}(1-z_2)/(1+z_2) \) on a
two variable analog function \( T(s_1, s_2) \) with strict Hurwitz denominator. Recently, Goodman [13] has pointed out that in
some cases the 2-D digital function generated using double bilinear transformations possesses non-essential singu-
larities of the second kind on the closed unit bidisk of
\((z_1, z_2)\) biplane. As bilinear transformation maps the centre
\((s_1, s_2)\) biplane on the entire \((z_1, z_2)\) biplane on a one-to-
one basis the behaviour of the function is not altered by
the application of double bilinear transformation. For
instance, \( f_2(z_1, z_2) \) is the transformed function of \( f(s_1, s_2) \)
and \( f_2(z_1, z_2) \) has singularity at \((z_1^*, z_2^*)\) then \( f(s_1, s_2) \)
ought to have the same type of singularity at \((s_1^*, s_2^*)\)
where \((s_1^*, s_2^*)\) is the point corresponding to \((z_1^*, z_2^*)\).
This simply implies that analog functions with strict
Hurwitz polynomial denominators may possess non-essential
singularities at the infinite distant points on the closed
right half of \((s_1, s_2)\) plane. This problem arises because
in the definition of two variable strict Hurwitz polyno-
mials the second kind singularities, which may be present
at infinite distant points are not taken into account.
Naturally, one obvious remedy is to define a two variable
Hurwitz polynomial which avoids this singularity at in-
finite distant points on closed right half of the \((s_1, s_2)\) biplane.

Before defining VSHP, we will define certain terms.

A rational function \(f(s)\) with real coefficients such that \(\text{Re}(f(s)) > 0\) for \(\text{Re}(s) > 0\) is called a positive real function.

A positive real function \(f(s)\) is said to be a strict positive real function if \(\text{Re}(f(s)) > 0\) for \(\text{Re}(s) = 0\).

A positive real function \(f(s)\) is said to be maximum reactive, susceptive if it has neither poles nor zeros on the imaginary axis of the \(s\) plane.

A positive real function \(f(s)\) is called a reactance function if \(\text{Re}[f(s)] = 0\) for \(\text{Re}(s) = 0\).

A two variable rational function \(f(s_1, s_2)\) with real coefficients such that \(\text{Re}(f(s_1, s_2)) \geq 0\) for \(\text{Re}(s_1) > 0\), \(\text{Re}(s_2) > 0\) is called a two variable positive real functions.

A two variable positive real function \(f(s_1, s_2)\) such that \(f(s_1, s_2) = -f(-s_1, -s_2)\) is called a two variable reactance function.

A two variable polynomial \(Q(s_1, s_2)\) is an even polynomial if \(Q(s_1, s_2) = Q(-s_1, -s_2)\) and is an odd polynomial if \(Q(s_1, s_2) = -Q(-s_1, -s_2)\).

2.8.3 Two Variable Hurwitz Polynomials

In a survey on the stability of multidimensional polynomials [4], Jury has discussed the existence of more
than one type of two variable Hurwitz polynomials. In the study of properties of two variable reactance functions, Ansell [1] defined a two variable Hurwitz polynomial in the narrow sense as against the two variable Hurwitz polynomials in the broad sense, which is similar to the one variable Hurwitz polynomial. Finding this definition on NHP inadequate, as Ansell did not consider the points at infinity, Huang [10], in his study of stability analysis, modified this definition so as to avoid zeros in the imaginary axes of the \( (s_1, s_2) \) biplane and called the resulting polynomial a strict Hurwitz polynomial. As stated earlier, the non-essential singularities of the second kind were not considered, and this caused the difficulty reported by Goodman. As one is interested in the closed right half of the \( (s_1, s_2) \) biplane in the study of the stability of transfer functions, it is desirable to include the behaviour of the polynomial at infinite distant points in the definition of Hurwitz polynomials. Rajan et al. [14] modified the definition so as to avoid the non-essential singularities of second kind. To distinguish this class of polynomials from the earlier classes, these are called very strict Hurwitz (VSH) polynomials. In all, then, there are four types of Hurwitz polynomials and their definitions are stated below in a slightly different form in terms of singularities rather than zeros as has been the common
practice. This has been done so as to facilitate a uniform definition for all the four types of polynomials, differing only in the region of analyticity. In the following definitions, $D(s_1, s_2)$ is a polynomial in $s_1$ and $s_2$ and $\text{Re}(s)$ refers to the real part of $s$.

$D(s_1, s_2)$ is a broad sense Hurwitz polynomial (BHP) if $1/D(s_1, s_2)$ does not possess any singularities in the region.

$\{(s_1, s_2) | \text{Re}(s_1) > 0, \text{Re}(s_2) > 0, |s_1|^{>\infty}$ and $|s_2|^{<\infty}\}$

$D(s_1, s_2)$ is a narrow sense Hurwitz polynomial if $1/D(s_1, s_2)$ does not possess any singularities in the region.

$\{(s_1, s_2) | \text{Re}(s_1) > 0, \text{Re}(s_2) > 0, |s_1|^{<\infty}$ and $|s_2|^{<\infty}\}$ $\cup \{(s_1, s_2) | \text{Re}(s_1) = 0, \text{Re}(s_2) > 0, |s_1|^{<\infty}$ and $|s_2|^{<\infty}\}$

In this definition of narrow sense Hurwitz polynomial [14] slightly modified Ansell's definition so as to include points at infinity respectively.

$D(s_1, s_2)$ is strict Hurwitz polynomial if $1/D(s_1, s_2)$ does not possess any singularities in the region.

$\{(s_1, s_2) | \text{Re}(s_1) \geq 0, \text{Re}(s_2) \geq 0, |s_1|^{<\infty}$ and $|s_2|^{<\infty}\}$

$D(s_1, s_2)$ is a very strict Hurwitz polynomial if $1/D(s_1, s_2)$ does not possess any singularities in the region.

$\{(s_1, s_2) | \text{Re}(s_1) \geq 0, \text{Re}(s_2) \geq 0, |s_1|^{<\infty}$ and $|s_2|^{<\infty}\}$

It is worthwhile to point out at this point that the question of which Hurwitz polynomials are required from the
point of view of two dimensional stability has not been resolved. From the two dimensional filter point of view, we may say that SHP is necessary, but VSHP ensures stability. A few examples are presented here.

**Example 1**

\[ D(s_1, s_2) = s_1 + s_1 s_2 \]. It can be verified that \[ 1/D(s_1, s_2) = l/s_1 + s_1 s_2 \] does not have singularity in the region specified in the definition. However, it has first kind singularity at \( s_1 = 0 \) and for any \( s_2 \). Hence, \( s_1 + s_1 s_2 \) is broad sense Hurwitz polynomial.

**Example 2**

\[ D_2(s_1, s_2) = 1 + s_1 s_2 \]. \( D(s_1, s_2) \) is a narrow sense Hurwitz polynomial.

**Example 3**

\[ D_3(s_1, s_2) = 1 + s_1 + s_2 \]. \( 1 + s_1 + s_2 \) is an SHP but not a VSHP as at \( s_1 = \infty \) and \( s_2 = \infty \) we have \( D_3(\infty, \infty) = 0/0 \) and \( D_3(s_1, s_2) \) has as second kind singularity at \( (\infty, \infty) \). In a similar way, we can show \( 1 + s_1 + s_1 s_2 \) and \( 1 + s_2 + s_1 s_2 \) are SHPs but not VSHPs. \( D_4(s_1, s_2) = 1 + s_1 + s_2 + s_1 s_2 \) can be shown to be VSHP. Procedure to test BHP, NHP and SHP is well established. Test procedure to check a polynomial as a VSHP is given below.
2.8.4 Test Procedure for VSHP

Step 1
Test \( D(s_1, s_2) \) to be a SHP. For this [10] and [4] can be used. If it is SHP, then go to step 2.

Step 2
a) \( D(\infty, s_2^*) \neq 0/0 \) for \( \text{Re}(s_2^*) = 0 \) and \( |s_2^*| < \infty \) (2.12a)
b) \( D(s_1^*, \infty) \neq 0/0 \) for \( \text{Re}(s_1^*) = 0 \) and \( |s_1^*| < \infty \) (2.12b)
c) \( D(\infty, \infty) \neq 0/0 \) (2.12c)

If conditions (2.12a) - (2.12c) are satisfied, \( D(s_1, s_2) \) is a VSHP; otherwise it is not. The above conditions follow directly from the definition of a VSHP. The sufficiency of condition (2.12a) in testing for \( \text{Re}(s_2^*) \geq 0 \) rather than for \( \text{Re}(s_2^*) > 0 \) as required by the definition is an SHP. Similarly, sufficiency condition for (2.12b) can be proved.

2.9 Conclusion

In this chapter, we have defined stability and the problems arising after application of double bilinear trans-function. In order to avoid existence of non-essential singularities of second kind using [14] we have defined a new type of Hurwitz polynomial known as a very strict Hurwitz polynomial. After outlining some preliminary definitions, some test procedures for VSHP are studied.
CHAPTER III

GENERATION OF VSHP USING MATRIX FACTORIZATION TECHNIQUE

3.1 Introduction

This chapter presents a method of designing a stable 2-D recursive digital filter satisfying prescribed magnitude and group delay responses. The technique that is presented here is from a well known matrix approach method [15] using properties of positive definite matrices and their applications in generating a 2-variable very strict Hurwitz polynomials (VSHP) which will be assigned to the denominator of the 2-D analog reference filter. Bilinear transformation is then applied to a transfer function of the desired 2-D analog reference filter to obtain a discrete version of the 2-D filter. Parameters of the discrete 2-D filters are then used as the variables of optimization to minimize the mean square error of the desired and designed magnitude and group delay responses of the filter [23].

3.2 Mathematical Background

It is well known that a positive definite or a positive semi-definite matrix is always physically realizable. It is further known that any positive definite matrix P can always
be decomposed as a product of two matrices $QQ^T$ where $Q$ is either an upper triangular or a lower triangular matrix [7]. Therefore the matrix

$$D = A\Gamma A^T s_1 + B\Delta B^T s_2 + G$$  \hspace{1cm} (3.1)$$

where $A$ and $B$ are upper triangular matrices, $\Gamma$ and $\Delta$ are diagonal matrices and $G$ is a skew symmetric matrix, is realizable as a two variable reactance network. Therefore, $\det D$, constitutes either the even part or the odd part of a two variable Hurwitz Polynomial (H.P.), depending on whether the order is even or odd. It has to be noted however, that some of the elements of $\Gamma$ and $\Delta$ can be zero.

Since $D$ is a physically realizable matrix, we have

$$B(s_1, s_2) = \det D + K_1 \frac{\delta (\det D)}{\delta s_1} + K_2 \frac{\delta (\det D)}{\delta s_2}$$  \hspace{1cm} (3.2)$$

as a two-variable Hurwitz polynomial, where $K_1$ and $K_2$ are non-negative constants. Similarly other HP's can be formed [20], using higher order derivatives.

Before converting $B(s_1, s_2)$ into a VSHP, several properties of $\det D$ (and hence those of $B(s_1, s_2)$) shall be studied. In this chapter, the order of $D$ is considered to be even and similar treatment can be given, if the order of $D$ is odd.
3.3 Properties of Determinant of D When the Order of D is Even

Let us assume the matrices A, $\Gamma$, B, $\Delta$ and G are as follows:

$$A = \begin{bmatrix} 1 & a_{12} & a_{13} & a_{1,2m} \\ 0 & 1 & a_{23} & a_{2,2m} \\ 0 & 0 & 1 & a_{3,2m} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.2a)$$

$$\Gamma = \begin{bmatrix} \gamma_1 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \gamma_{2m} & 0 \end{bmatrix} \quad (3.2b)$$

$$B = \begin{bmatrix} 1 & b_{12} & b_{13} & b_{1,2m} \\ 0 & 1 & b_{23} & b_{2,2m} \\ 0 & 0 & 1 & b_{3,2m} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.2c)$$
\[
\begin{array}{cccc}
\delta_1 & 0 & 0 & 0 \\
0 & \delta_2 & 0 & 0 \\
0 & 0 & \delta_3 & 0 \\
\end{array}
\]

\[
\Delta = \begin{bmatrix}
\delta_1 & 0 & 0 & 0 \\
0 & \delta_2 & 0 & 0 \\
0 & 0 & \delta_3 & 0 \\
\end{bmatrix}
\]

\[
G = \begin{bmatrix}
0 & g_{12} & g_{13} & g_{1,2m} \\
-g_{12} & 0 & g_{23} & g_{2,2m} \\
-g_{1,2m} & -g_{2,2m} & -g_{3,2m} & 0 \\
\end{bmatrix}
\]

(3.2d)

(3.2e)

The following properties for \( \det \Delta \) can be observed:

1. The highest term of \( s_1 \) will be \( \gamma_1 \gamma_2 \cdots \gamma_{2m-1} s_1^{2m} \). Similarly, the highest term of \( s_2 \) will be \( \delta_1 \delta_2 \cdots \delta_{2m-1} s_2^{2m} \).

Since \( B(s_1, s_2) \) contains these terms, it is essential that some \( \gamma \)'s and some \( \delta \)'s must be equal to zero in order that \( B(s_1, s_2) \) can be VSHP [14]. Therefore, in such a case, the starting matrices \( A \) and \( B \) are necessarily Positive semi-definite.
3.1 Equation

\[
D = \begin{bmatrix}
A_{11} & A_{12} & \Gamma_{11} & [0] & A_{11}^T & [0] \\
[0] & A_{22} & [0] & \Gamma_{22} & A_{12}^T & A_{22}^T \\
\end{bmatrix}_{s_1}
\]

\[
+ \begin{bmatrix}
B_{11} & B_{12} & \Delta_{11} & [0] & B_{11}^T & [0] \\
[0] & B_{22} & [0] & \Delta_{22} & B_{12}^T & B_{22}^T \\
\end{bmatrix}_{s_2}
\]

\[
+ \begin{bmatrix}
G_{11} & G_{12} \\
-G_{12}^T & G_{22} \\
\end{bmatrix}
\]

(3.3)

where \(A_{11}, B_{11}, G_{11}, \Gamma_{11}\) and \(\Delta_{11}\) are matrices of order \(K \times K\), \(A_{22}, B_{22}, \Gamma_{22}, \Delta_{22}\) and \(G_{12}\) are matrices of order \((2m - K) \times (2m - K)\).

\(A_{12}, B_{12}\) and \(G_{12}\) are matrices of order \(K \times (2m - K)\). In addition, it shall be noted that \(A_{11}, A_{22}, B_{11}\) and \(B_{22}\) are upper-triangular matrices, and \(G_{11}\) and \(G_{22}\) are skew-symmetric matrices.
Hence, D can be rewritten as

\[
D = \begin{vmatrix}
A_{11} \Gamma_{11} A_{11}^T + A_{12} \Gamma_{22} A_{12}^T & A_{12} \Gamma_{22} A_{22}^T \\
A_{22} \Gamma_{22} A_{12}^T & A_{22} \Gamma_{22} A_{22}^T \\
\end{vmatrix}
\]

\[
+ \begin{vmatrix}
B_{11} \Delta_{11} B_{11}^T + B_{12} \Delta_{22} B_{12}^T & B_{12} \Delta_{22} B_{22}^T \\
B_{22} \Delta_{22} B_{12}^T & B_{22} \Delta_{22} B_{22}^T \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
G_{11} & G_{12} \\
-G_{12}^T & G_{22} \\
\end{vmatrix}
\]

(3.4)

As can be immediately perceived, there exist a large number of possibilities for making some of the \(\gamma\)'s and \(\delta\)'s zero. However, it is preferable to keep the degrees of \(s_1\) and \(s_2\) same, as this will influence the filter response, particularly regarding the symmetry.
(3) Let $\Delta_{22} = \Gamma_{22} = 0$. Under such a condition, $D$ becomes

$$D = \begin{vmatrix}
A_{12} \Gamma_{22} A_{12}^T + B_{11} \Delta_{11} B_{11}^T s_2 + G_{11} & G_{12} \\
- G_{12}^T & G_{22}
\end{vmatrix}$$

(3.5)

Then det $D$ can be written as

$$\det D = \left| A_{12} \Gamma_{22} A_{12}^T s_1 + B_{11} \Delta_{11} B_{11}^T s_2 + G_{11} \right|$$

$$\left| G_{22} + G_{12}^T \left[ A_{12} \Gamma_{22} A_{12}^T s_1 + B_{11} \Delta_{11} B_{11}^T s_2 \\
+ G_{11} \right]^{-1} G_{12} \right|$$

(3.6)

Though the expression appears to be complicated, the orders of $s_1$ and $s_2$ can be determined easily. However, the disadvantage of this choice is that the elements of the matrices $A_{12}$, $A_{22}$, $B_{12}$ and $B_{22}$ do not appear in det $D$ and this is not desirable.

(4) In view of property (3), we can make $\Delta_{11} = \Gamma_{11} = 0$ in Eq. (3.4). Under such a condition, $D$ becomes

$$D = \begin{vmatrix}
A_{12} \Gamma_{22} A_{12}^T & A_{12} \Gamma_{22} A_{22}^T \\
A_{22} \Gamma_{22} A_{12}^T & A_{22} \Gamma_{22} A_{22}^T
\end{vmatrix}$$

$s_1$
\[
\begin{align*}
\begin{pmatrix}
B_{12} & \Delta_{22} & B_{12}^T \\
- & - & - \\
B_{22} & \Delta_{22} & B_{12}^T
\end{pmatrix} & = s_2 \\
\begin{pmatrix}
G_{11} & G_{12} \\
- & - \\
G_{12} & G_{12}
\end{pmatrix} & = [A_1] s_1 + [B_1] s_2 + [G]
\end{align*}
\]

where
\[
A_1 = \begin{pmatrix}
A_{12} & \Gamma_{22} & A_{12}^T \\
- & - & - \\
A_{22} & \Gamma_{22} & A_{12}^T
\end{pmatrix}
\]
\[
B_1 = \begin{pmatrix}
B_{12} & \Delta_{22} & B_{12}^T \\
- & - & - \\
B_{22} & \Delta_{22} & B_{12}^T
\end{pmatrix}
\]
and
\[
G = \begin{pmatrix}
G_{11} & G_{12} \\
- & - \\
-G_{12} & G_{22}
\end{pmatrix}
\]

We shall consider det $A_1$. If the partitioning is such that $A_{12}$ is a non-singular square matrix of order $K$, it is easily verified that

$$
\begin{align*}
\det A_1 &= \left| A_{12} \begin{pmatrix} r_{22} & A_{12}^T \end{pmatrix} \right| - \left( A_{22} \begin{pmatrix} r_{22} & A_{12}^T \end{pmatrix} \right)^{-1} \left( A_{12} \begin{pmatrix} r_{22} & A_{12}^T \end{pmatrix} \right) \\
&= 0 \\
&\quad (3.9)
\end{align*}
$$

Similarly, under the condition $B_{12}$ is a non-singular square matrix of order $K$, det $B_1 = 0$.

This means the terms $s_{11}^K$ and $s_{21}^K$ are absent in det $D$. In fact, this is one of the conditions required that det $D$ shall become a VSHP. This gives us the required clue, for partitioning $A_1$ and $B_1$ suitably.

The foregoing discussion leads to one of the possibilities of partitioning $A_1$ (and hence $B_1$ and $G$) which is discussed below:

Let $A_1$ be written as:
Out of a large number of possibilities, if we make $A_{44} = 1$

$$\gamma_{44} = \gamma_{2m}, \quad A_{34} = a_{2m-1, 2m}$$

(similarly in $B_1$'s and $G$'s) we get
\[ D = \begin{bmatrix} [C^{(a)}_{2m-2}] & [0] \\ \begin{bmatrix} a_{2m-1,2m}^2 & \gamma_{2m} \\
 -0 & -0 \\
 a_{2m-1,2m} & \gamma_{2m} \\
 \end{bmatrix} & s_{1+} \\
 \end{bmatrix} \]

\[ [C^{(b)}_{2m-2}] & [0] \\ \begin{bmatrix} b_{2m-1,2m}^2 & \delta_{2m} \\
 -0 & -0 \\
 b_{2m-1,2m} & \delta_{2m} \\
 \end{bmatrix} & s_{2} \\
 \]

\[ [C^{(g)}_{(2m-2)}] & [0] \\ \begin{bmatrix} 0 & g_{2m-1,2m} \\
 -0 & -0 \\
 -g_{2m-1,2m} & 0 \\
 \end{bmatrix} & (3.12) \]

Where \( C^{(a)}_{2m-1}, C^{(b)}_{2m-2}, \) and \( C^{(g)}_{2m} \) are remaining matrices A, B and G of order \( (2m-1) \).

Under such an identification \( \det D \) becomes

\[
\begin{align*}
\det D &= \left\{ (a_{2m-1,2m} - b_{2m-1,2m})^2 \gamma_{2m} \delta_{2m} s_1 s_2 + g_{2m-1,2m}^2 \right\} \\
&= |[C^{(a)}_{2m-2}] s_1 + [C^{(b)}_{2m-2}] s_2 + [C^{(g)}_{2m-1}]| \\
&= (3.13)
\end{align*}
\]
Making similar identification we can write

\[
\det D = \prod_{i=0}^{2m-2} \left( a_{2m-i-1,2m-i} - b_{2m-i-1,2m-i} \right)^2 \gamma_{2m-i} \delta_{2m-i} s_1 s_2 + g_{2m-i-1,2m-i}^2
\]

(3.14)

If \( a_{2m-i-1,2m-i} \neq b_{2m-i-1,2m-i} \) for \( i=0,1,\ldots, 2m-2 \) we can write

\[
\det D = \prod_{j=0}^{m} \left( C_j s_1 s_2 + g_j \right) = M_{2m}
\]

(3.15)

where \( C_j \) is a positive constant and is a function of the elements \( a \)'s and \( b \)'s and \( g \)'s are the functions of elements \( g \)'s.

If \( a_{2m-i-1,2m-i} = b_{2m-i-1,2m-i} \) for some \( i \)'s, only the order of \( \det D \) will be reduced.

The required VSHP is generated using higher order partial polynomial derivatives. The various steps are given below.

(a) Form

\[
M_{al} = M_{2m} + K_{11} \frac{\delta M_{2m}}{\delta s_1} + K_{21} \frac{\delta M_{2m}}{\delta s_2}
\]

This is known to be a 2-variable Hurwitz Polynomial. This will be of the form

\[
s_1^m \left[ C_m s_2^m + m K_{21} C_m s_2^{m-1} \right] + s_1^{m-1} \left[ K_{11} m C_m s_2^{m-1} \right. \\
+ \left. K_{21} C_{m-1} s_2^{m-2} \right] \ldots (3.16)
\]
(b) Now, form

\[ M_{a2} = M_{a1} + K_{12} \frac{\delta M_{a1}}{\delta s_1} + K_{22} \frac{\delta M_{a1}}{\delta s_2} \]

which involves higher order partial derivatives of \( M_{2m} \).

This will also be a 2-variable Hurwitz polynomial.

(c) This process can be continued till we form

\[ M_{am} = M_{am-1} + K_{1,m-1} \frac{\delta M_{am-1}}{\delta s_1} + K_{2,m-1} \frac{\delta M_{am-1}}{\delta s_2} \]

which can easily be shown to be a VSHP. Other VSHP's can be formed using the other derivatives of \( M_{am} \) and hence the chance is not restrictive.

(5) Instead of choosing the rank of \( A_{44} \) to be unity, we can choose the rank of \( A_{44} \) to be two or more. If the rank of \( A_{44} \) is chosen to be two and the other matrices are chosen accordingly it can be shown that \( \det D = M_{2m} \) will be the product of the factors of the type

\[ (C_{22} s_1^2 s_2^2 + C_{20} s_1^2 + C_{11} s_1 s_2 + C_{02} s_2^2 + C_{00}) \]

Following the steps given earlier (i.e., using higher order partial polynomial derivatives), we can generate VSHP's. Thus, the number of choices are very large.
3.4 Formulation of the Design Problem

In this method a 2-variable VSHP is assigned to the denominator of the transfer function using the method described in the previous section and numerator is left unchanged. Then bilinear transformation is applied to the derived 2-D analog transfer function to obtain the discrete version of the filter. The error between the ideal and the designed magnitude response is calculated using the relationship.

\[ E_m(w_{1m}, w_{2n}) = |H_I(e^{jw_{1m}T}, e^{jw_{2n}T})| - |H_D(e^{jw_{1m}T}, e^{jw_{2n}T})| \]  (3.18)

where \( E_m \) is the error of the magnitude response and \(|H_I|\) and \(|H_D|\) are the magnitude responses of the ideal and the designed filter respectively. Similarly the error between the ideal and the designed group delay response of the filter can be calculated as follows:

\[ E_{\tau_{wi}}(w_{1n}, w_{2n}) = \tau_I - \tau_{wi}(e^{jw_{1m}T}, e^{jw_{2n}T}) \]  (3.19)

where \( \tau_I \) is a constant representing the ideal group delay response of the filter and its value is chosen equal to the order of filter [16] and \( \tau_{wi} \) is group delay response of the designed filter and is calculated using the following equation:
\begin{align*}
\tau_{w_1}(z_1, z_2) &= \frac{\delta \psi(z_1, z_2)}{\delta w_1} \\
&= -\text{Re}\left[ \frac{z_1}{H_D(z_1, z_2)} \cdot \frac{\delta H_D(z_1, z_2)}{\delta z_1} \right]
\end{align*}

(3.20)

where \( \psi \) is the phase response of the filter and \( z_1 = \exp(jw_1 T), l = 1, 2. \)

Using equations (3.18 through 3.20) we can generate the general least mean square error between magnitude and group delay response of the ideal and the designed filter using the relationship.

\begin{align*}
E_G(w_1, w_2) &= a_1 \sum_{m,n} \varepsilon_{I_{ps}}^\prime E_H^2(w_{1m}, w_{2n}) \\
&\quad + a_2 \sum_{m,n} \varepsilon_{I_p}^\prime E_{\tau w_1}^2(w_{1m}, w_{2n}) \\
&\quad + a_3 \sum_{m,n} \varepsilon_{I_p}^\prime E_{\tau w_2}^2(w_{1m}, w_{2m})
\end{align*}

(3.21)

where \( I_{ps} \) is a set of all the discrete points in the passband and stopband of the filter while \( I_p \) is set of all the discrete points in the passband of the filter.

[If we assume
\[
\text{Max of } E_H = \beta_1, \quad \text{Max of } E_{\tau w_1} = \beta_2, \quad \text{Max of } E_{\tau w_2} = \beta_3
\]

then the weighting coefficients \( a_1, a_2, a_3 \) can be chosen to be equal to unity or as follows:

\[
a_1 = \beta_2 \beta_2, \quad a_2 = \beta_1 \beta_3, \quad a_3 = \beta_2 \beta_1
\]
Now any suitable non linear optimization technique can be used to calculate parameters n, a, b, γ and δ of the filter's transformation so as to minimize the $E_a$ in the Eq. 3.20 subject to the following constraints:

(i) $\gamma$ and $\delta$ should be non negative

(ii) det. $G \neq 0$ Equation 3.2e

Condition (i) can be easily overcome by using the following variable substitution

$$\gamma = a^2, \delta = b^2$$

and thereby guaranteeing the positiveness of $\gamma$ and $\delta$ which is required. Condition (ii) can be used as a penalty function in the process of optimization to ensure det $A$ is always non zero.

### 3.5 Design Example

To illustrate the method a 2-D low pass filter is designed using the following specification

$$|H_T(j\omega_1, j\omega_2)| = \begin{cases} 
1.0 \text{ for } 0 & \leq \frac{2}{\sqrt{\omega_1^2 + \omega_2^2}} < 1.0 \text{ rad/sec} \\
0.0 \text{ for } 5 & \geq \sqrt{\omega_1^2 + \omega_2^2} \geq 2.5 \text{ rad/sec}
\end{cases}$$

(3.22)

and a constant group delay equal to the order filter is so chosen. In this example D in Eq. 3.21 is considered to be
\[
D = \begin{bmatrix}
1 & a_1 & a_1^2 & 0 & 1 & 0 \\
0 & 1 & 0 & a_2^2 & a_1 & 1 \\
1 & b_1 & b_1^2 & 0 & 1 & 0 \\
0 & 1 & 0 & b_2^2 & b_1 & 1 \\
0 & s_1 & 0 & 0 & s_2 & s_1 + 0 & g_1 \\
0 & s_2 & b_1 & 1 & -g_1 & 0
\end{bmatrix}
\]

(3.23)

Now using Eqn. (3.2) for \( K_1 = 1 \) and \( K_2 = 0 \) will give

\[
D(s_1s_2) = a_1^2s_1^2 + b_1^2s_2^2 + s_1s_2 + 2a_1a_2s_1 + s_2 + g_1^2
\]

where

\[
\rho = a_2^2 + b_2^2 + a_2^2a_1 + a_2^2b_1 - 2a_2^2a_1b_1
\]

hence

\[
D(s_1s_2) = n_{00} + n_{01} + n_{02}s_2^2 + n_{10}s_1 + n_{20}s_1^2 + n_{11}s_1s_2
\]

In this example the order of the filter is two so \( r_I \) in Eqn. 3.19 equal to two; the optimization scheme used that of Hooke and Jeeves [17]. Table 1 shows the coefficients of 3-D filter while Fig. 1a shows the magnitude response of the designed low pass filter, Fig. 1b-1c shows the group delay response.
3.6 Conclusion

We thus have presented a method for generation of 2-variable VSHP using the properties of positive definite matrices. The derived 2-variable VSHP (or HP) is assigned to the denominator of a 2-D analog reference filter. The discrete version of the filter is obtained by utilization of the double bilinear transformation. The new parameters of the discrete transfer function are then used as variables of optimization to minimize the general least mean square of the error between the amplitude and group delay responses of the ideal and designed filters. The method can be easily implemented using any suitable non-linear optimization technique. This approach is equally applicable to 1-D case and can be extended to the general N-D filters.
Table 1

Values of the Parameters of the Designed Low Pass Filter

<table>
<thead>
<tr>
<th>Numerator's Coefficient</th>
<th>Denominator's Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{00} = 15.1511297$</td>
<td>$b_1 = 83.919$</td>
</tr>
<tr>
<td>$a_{01} = 10.5228539$</td>
<td>$b_2 = 91.5896714$</td>
</tr>
<tr>
<td>$a_{02} = 2.0435381$</td>
<td>$u_1 = 1.15408421$</td>
</tr>
<tr>
<td>$a_{10} = 8.7311573$</td>
<td>$u_2 = 2.61289379$</td>
</tr>
<tr>
<td>$a_{11} = 32.6494038$</td>
<td>$u_3 = 1.42684937$</td>
</tr>
<tr>
<td>$a_{20} = 44.0095007$</td>
<td>$u_4 = 1.69630814$</td>
</tr>
</tbody>
</table>
Fig. 1a. 3-D plot of the amplitude response of the designed low pass filter.
Fig. 1b. 3-D plot of the group delay response w.r.t. $w_A$. 

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Fig. 1c. 3-D plot of group delay response w.r.t. $w_2$. 
CHAPTER IV

GENERATION OF A VSHP USING DERIVATIVE APPROACH METHOD

4.1 Introduction

Multivariable network theory has numerous applications in the design of two and multidimensional recursive digital filters. Ramamoorthy and Bruton [22] have shown how a stable 2-D analog filter can be designed by using the properties of the imittance function of a lossless frequency independent N-Port network. This, however, has failed to deal with the possibility of generating functions with non-essential singularities of the second kind which could result in an unstable discrete filter if a bilinear transformation is applied [17]. We have modified the earlier technique to ensure that the polynomial in the denominator of the filter remains VSHP at all times, hence avoiding any non-essential singularities of the second kind. This can be done by using the constraints for the denominator of the filter to be VSHP as a penalty function in the process of optimization. However, this assurance bears a heavy cost of computational cost.
In this chapter, a method for the generation of a 2-variable VSHP using the properties of the derivatives of even or odd part of Hurwitz Polynomial is presented. The derived VSHP is assigned to the denominator of a 2-D analog reference filter. The discrete version of the filter is obtained by the application of double bilinear transformation. A non-linear optimization technique can be utilized to minimize the mean square error between the desired and the designed magnitude and group delay responses of the filter.

4.2 Properties of Even or Odd Part for Hurwitz Polynomial

The method presented here consists of the following steps:

(i) A suitable even or o-d part of a n-variable Hurwitz polynomial is generated.

(ii) The corresponding derivation giving the odd or even part is associated with it.

(iii) The resulting n-variable Hurwitz polynomial is converted to a 2-variable VSHP [14].

As will be seen later, large possibilities do exist. We start from the basic network theory by taking a n-port gyrator which is terminated by $s_1$-type capacitors.
Fig. 2. N-Variable Passive Network
in $m_1$-ports, by $s_2$-type capacitors in the remaining $n-m_1$ ports is considered. Using this $n$-port gyrator we obtain an admittance matrix that will generate an even or odd part of Hurwitz polynomial.

**Case A:** The even part of a Hurwitz polynomial as the starting point:

Consider the polynomial $M_{2n}$ given by

$$M_{2n} = \det \left[ \mu I_{2n} + A_{2n} \right] \quad (4.1)$$

where $\mu$ is a diagonal matrix of order $2n$ given by

$$\mu = \text{diag}[\mu_1, \mu_2, \mu_3, \ldots, \mu_{2n}] \quad (4.2)$$

and $A$ is a skew-symmetric matrix of order $2n$ given by

$$A = \begin{bmatrix}
0 & a_{12} & a_{13} & \cdots & a_{1.2n} \\
-a_{12} & 0 & a_{23} & \cdots & a_{2.2n} \\
-a_{13} & -a_{23} & 0 & \cdots & a_{3.2n} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-a_{1.2n} & -a_{2.2n} & \cdots & 0 & 0
\end{bmatrix} \quad (4.3)$$

From the diagonal expansion of the determinant of a matrix [19], $M_{2n}$ can be written as
\[ M_{2n} = \det A + \sum_{1 \leq i_1 < i_2 \leq 2n} \mu_{i_1} \mu_{i_2} \det A_{i_1 i_2} + \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq 2n} \mu_{i_1} \mu_{i_2} \mu_{i_3} \mu_{i_4} A_{i_1 i_2 i_3 i_4} + \ldots + \mu_{1} \mu_{2} \mu_{3} \ldots \mu_{2n} \]

(4.4)

where \( A_{i_1, i_2} \) is the determinant of the submatrix of \( A \) obtained by deleting both the \( i_1 \)th and \( i_2 \)th rows and columns, and is of order \((2n-2)\), \( A_{i_1 i_2 i_3 i_4} \) is the determinant of the submatrix of \( A \) obtained by deleting the \( i_1 \)th, \( i_2 \)th, \( i_3 \)th, and \( i_4 \)th rows and columns and is of order \((2n-4)\), and so on.

The following properties shall be noted:

(i) All odd order terms of the type \( \mu_{i_1}, \mu_{i_1} \mu_{i_2} \mu_{i_3}, \) etc. are absent, since the determinant of an odd order skew-symmetric matrix is zero.

(ii) The degree of any \( \mu_i (i=1, \ldots, 2n) \) is unity.

(iii) The quantities \( \det A, A_{i_1 i_2}, \) etc. are non-negative numbers, since the determinant of an even order skew-symmetric matrix is a perfect square.

Since the matrix \([\mu I_{2n} + A_{2n}]\) is always physically realizable, \( M_{2n} \) represents the even part of a 2n-variable Hurwitz polynomial. Therefore, \( (\partial M_{2n} / \partial \mu_i) / M_{2n} \) is a reactance function \([2]\). As a consequence,

\[ M' = M_{2n} + \sum_{i=1}^{2n} K_j \frac{\partial M_{2n}}{\partial \mu_j} \]

(4.5)
is a 2n-variable Hurwitz polynomial.

From eqn. (4.5), a 2-variable VSHP can be generated by placing some of the \( u \)'s equal to \( s_1 \) and the rest of the \( u \)'s equal to \( s_2 \) and also ensuring that the conditions of a 2-variable VSHP are satisfied [14]. Also, there will be large number of possibilities as can be seen from the following example: Consider

\[
M_4 = \begin{vmatrix}
\mu_1 & a_{12} & a_{13} & a_{14} \\
-a_{12} & \mu_2 & a_{23} & a_{24} \\
-a_{13} & -a_{23} & \mu_3 & a_{34} \\
-a_{14} & -a_{24} & -a_{34} & \mu_4 \\
\end{vmatrix}
\] (4.6)

Using eq. 4.1.

\[
M_4 = \begin{vmatrix}
0 & a_{12} & a_{13} & a_{14} \\
-a_{12} & 0 & a_{23} & a_{24} \\
-a_{13} & -a_{23} & 0 & a_{34} \\
-a_{14} & -a_{24} & -a_{34} & 0 \\
\end{vmatrix} + \mu_1 \mu_2 \begin{vmatrix}
0 & a_{34} \\
-a_{34} & 0 \\
\end{vmatrix}
\]

\[
+ \mu_1 \mu_3 \begin{vmatrix}
0 & a_{24} \\
-a_{24} & 0 \\
\end{vmatrix} + \mu_1 \mu_4 \begin{vmatrix}
0 & a_{23} \\
-a_{23} & 0 \\
\end{vmatrix} + \mu_2 \mu_3 \begin{vmatrix}
0 & a_{14} \\
-a_{14} & 0 \\
\end{vmatrix}
\]

\[
+ \mu_2 \mu_4 \begin{vmatrix}
0 & a_{13} \\
-a_{13} & 0 \\
\end{vmatrix} + \mu_3 \mu_4 \begin{vmatrix}
0 & a_{12} \\
-a_{12} & 0 \\
\end{vmatrix} + \mu_1 \mu_2 \mu_3 \mu_4.
\] (4.7)
Expanding eqn. (4.7), gives

\[ M_4 = \det A_4 + a_{34}^2u_2 + a_{24}^2u_1u_3 + a_{23}^2u_1u_4 + a_{14}^2u_2u_3 \\
+ a_{13}^2u_2u_4 + a_{12}^2u_3u_4 + u_1^2u_2u_3u_4 \]  

(4.8)

where

\[ \det A_4 = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2. \]  

(4.9)

Now using eqn. (4.5) and eqn. (4.8) we get the 4-variable Hurwitz polynomial as

\[ M' = [\det A_4 + a_{34}^2u_1u_2 + a_{24}^2u_1u_3 + a_{23}^2u_1u_4 + a_{14}^2u_2u_3 + a_{13}^2u_2u_4 \\
+ a_{12}^2u_3u_4 + u_1^2u_2u_3u_4] + K_1[a_{34}^2u_2 + a_{24}^2u_3 + a_{23}^2u_4 + u_2^2u_3u_4] \\
+ K_2[a_{34}^2u_1 + a_{14}^2u_3 + a_{13}^2u_4 + u_1^2u_3u_4] + K_3[a_{24}^2u_1 + a_{14}^2u_2 + a_{12}^2u_1 \\
+ u_1^2u_2u_4] + K_4[a_{23}^2u_1 + a_{13}^2u_2 + a_{12}^2u_3 + u_1^2u_2u_3]. \]  

(4.10)

By putting \( u_1 = u_2 = s_1 \) and \( u_3 = u_4 = s_2 \), we get

\[ Q(s_1, s_2) = s_1^2[s_2^2s_2(K_3 + K_4) + a_{34}^2] + s_1[s_2^2(K_1 + K_2) \\
+ s_2(a_{24}^2 + a_{23}^2 + a_{13}^2 + a_{14}^2) + a_{34}^2(K_1 + K_2) \\
+ K_3(a_{24}^2 + a_{14}^2) + K_4(a_{23}^2 + a_{13}^2)] \\
+ [a_{12}^2s_2^2s_2[K_1(a_{24}^2 + a_{23}^2) + K_2(a_{14}^2 + a_{13}^2) \\
+ a_{12}^2(K_3 + K_3)] + (a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23})^2]. \]  

(4.11)
This is a VSHP. However, it can be noticed that the VSHP property is not destroyed if \( a_{24} = 0 \) or \( a_{13} = 0 \) or \( a_{13} = a_{23} = 0 \). Many other combinations are possible. It is also to be noted that certain variables like \( a_{12}, a_{34} \) cannot be made equal to zero.

A similar treatment can be given for any \( k \)-variables VSHP and it can be shown that a large number of variations are possible, as the order of \( M \) increases.

**Case B:** The odd part of a Hurwitz polynomial as the starting point.

Consider the polynomial \( N_{2n+1} \) given by

\[
N_{2n+1} = \det |\mu I_{2n+1} + A_{2n+1}| \quad (4.12)
\]

where \( \mu \) is a diagonal matrix of order \( 2n + 1 \) given by

\[
\mu = \text{diag} [\mu_1, 2, \cdots, 2n+1] \quad (4.13)
\]

\[
A = \begin{bmatrix}
0 & a_{12} & a_{13} & \cdots & a_{1, 2n+1} \\
-a_{12} & 0 & a_{23} & \cdots & a_{2, 2n+1} \\
-a_{13} & -a_{23} & 0 & \cdots & a_{3, 2n+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-a_{1, 2n+1} & -a_{2, 2n+1} & -a_{3, 2n+1} & \cdots & 0 \\
\end{bmatrix} \quad (4.14)
\]
From the diagonal expansion of the determinant of a matrix [21], $N_{2n+1}$ can be written as

$$N_{2n+1} = \sum_{1 \leq i < 2n+1} \mu_i A_i + \sum_{1 \leq i_1 < i_2 < i_3 \leq 2n+1} \mu_{i_1} \mu_{i_2} \mu_{i_3} A_{i_1 i_2 i_3} + \cdots + \mu_1 \mu_2 \mu_3 \cdots \mu_{2n+1}$$

(4.15)

where $A_i$ is the determinant of the submatrix $A$ obtained by deleting the $i$th row and column and is of the order $2n$,

$A_{i_1 i_2 i_3}$ is the determinant of the submatrix $A$ obtained by deleting $i_1$th, $i_2$th and $i_3$th rows and columns and is of the order $(2n-2)$, and so on.

The following properties can be noted:

(i) All even order terms of the type $\mu_1 \mu_2$, $\mu_1 \mu_2$, $\mu_3 \mu_4$, are absent, since the determinant of an odd order skew-symmetric matrix is zero.

(ii) The degree of any $\mu_i$ ($i=1, \ldots, 2n+1$) is unity.

(iii) the quantities $A_i$, $A_{i_1 i_2 i_3}$, etc., are non-negative numbers, since they are determinants of even-order skew-symmetric matrices. Since the matrix

$$[\mu I_{2n+1} + A_{2n+1}]$$

is always physically realizable, $N_{2n+1}$ represents the odd part of a $(2n+1)$-variable Hurwitz polynomial. Therefore, $(\partial N_{2n+1}/\partial \mu_i)/N_{2n+1}$ is a reactance function [22].
As a consequence,

\[ N' = N_{2n+1} + \sum_{j=1}^{2n+1} K_j \frac{\partial N_{2n+1}}{\partial \mu_j} \quad (4.16) \]

is a \((2n+1)\)-variable Hurwitz polynomial.

From eqn. (4.16), a 2-variable VSHP can be obtained by equating some of the \(\mu_i\)'s equal to \(s_1\) and equating the remaining \(\mu_i\)'s equal to \(s_2\), and also making sure to satisfy the conditions of VSHP.

As is to be anticipated, there will be a large number of possibilities, which can be seen from the following example:

Consider

\[ N_5 = |\mu I_5 + A_5| = \begin{vmatrix} \mu_1 & a_{12} & a_{13} & a_{14} & a_{15} \\ -a_{12} & \mu_2 & a_{23} & a_{24} & a_{25} \\ -a_{13} & -a_{23} & \mu_3 & a_{34} & a_{35} \\ -a_{14} & -a_{24} & -a_{34} & \mu_4 & a_{45} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & \mu_5 \end{vmatrix} \quad (4.17) \]

\[ = \mu_1\mu_2\mu_3\mu_4\mu_5 + \mu_1\mu_2\mu_3^2a_{45} + \mu_1\mu_2\mu_4^2a_{35} + \mu_1\mu_2\mu_5^2a_{34} + \mu_1\mu_3\mu_4a_{25} + \mu_1\mu_3\mu_5a_{24} + \mu_1\mu_4\mu_5a_{23} + \mu_2\mu_3\mu_4a_{15} + \mu_2\mu_3\mu_5a_{14} + \mu_2\mu_4\mu_5a_{13} + \mu_3\mu_4\mu_5a_{12} + \mu_1(a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34})^2 \]

\[ + \mu_2(a_{13}a_{45} - a_{14}a_{35} + a_{15}a_{34})^2 \]
\[ + \mu_3 (a_{12}a_{45} - a_{14}a_{25} + a_{15}a_{24})^2 \]
\[ + \mu_4 (a_{12}a_{35} - a_{13}a_{25} + a_{15}a_{23})^2 \]
\[ + \mu_5 (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2. \]

By putting \( \nu_1 = \nu_2 = s_1 \) and \( \nu_3 = \nu_4 = \nu_5 = s_2 \), we obtain
\[ Q(s_1, s_2) = s_1 [s_2^3 + s_2 (K_3 + K_4 + K_5) + s_2 (a_{34}^2 + a_{35}^2 + a_{45}^2) \]
\[ + (K_3 a_{45}^2 + K_4 a_{35}^2 + K_5 a_{34}^2)] \]
\[ + s_1 [s_2^3 (K_1 + K_2) + s_2 (a_{13}^2 + a_{14}^2 + a_{15}^2 + a_{24}^2 + a_{25}^2)] \]
\[ + s_2 [K_1 (a_{23}^2 + a_{24}^2 + a_{25}^2) + K_2 (a_{34}^2 + a_{35}^2 + a_{45}^2) \]
\[ + K_3 (a_{14}^2 + a_{15}^2 + a_{24}^2 + a_{25}^2) + K_4 (a_{13}^2 + a_{14}^2 + a_{23}^2 + a_{25}^2) \]
\[ + K_5 (a_{13}^2 + a_{14}^2 + a_{23}^2 + a_{24}^2)] \]
\[ + [(a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34})^2 + (a_{13}a_{45} + a_{14}a_{35}a_{34})]^2 \]
\[ + s_2 a_{12}^2 + s_2 [K_1 (a_{23}^2 + a_{24}^2 + a_{25}^2) + K_2 (a_{13}^2 + a_{14}^2 + a_{15}^2) \]
\[ + (K_3 + K_4 + K_5) a_{12}^2] \]
\[ + s_2 [(a_{12}a_{45} - a_{14}a_{25} + a_{15}a_{24})^2 + (a_{12}a_{35} - a_{13}a_{25} + a_{15}a_{23})^2 \]
\[ + (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2] \]
\[ + [K_1 (a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34})^2 + K_2 (a_{13}a_{45} - a_{14}a_{35} \]
\[ + a_{15}a_{34})^2 + K_3 (a_{12}a_{24} - a_{14}a_{25} + a_{15}a_{24})^2 + \]
\[ + K_4 (a_{12}a_{35} - a_{13}a_{25} + a_{15}a_{23})^2 + K_5 (a_{12}a_{34} - a_{24}a_{24} \]
\[ + a_{14}a_{23})^2]}. \]
This is a VSHP. It is seen that the VSHP property will be destroyed if \( a_{12} = 0 \) and the VSHP property is preserved if some parameters like \( a_{24}, a_{35} \) are made equal to zero.

A large number of variations are possible as the order of \( N \) increases. In addition, it can be noted that the degrees of \( s_1 \) and \( s_2 \) will never be equal, and they can be made equal only when two sections are property cascaded. Consider the example of Case B. \( N_5 \) in eqn. (4.17) for \( \nu_1 = \nu_2 = \nu_3 = s_1 \) gives and \( \nu_2 = \nu_5 = s_2 \) gives

\[
N_5 = s_1^3 [s_2^2 + (K_4 + K_5) s_2 + a_{45}^2]
+ s_2^2 [s_2 (K_1 + K_2 + K_3) + s_2 (a_{14}^2 + a_{15}^2 + a_{24}^2 + a_{25}^2 + a_{34}^2 + a_{35}^2)
+ [a_{45}^2 (K_1 + K_2 + K_3) + K_4 (a_{15}^2 + a_{25}^2 + a_{35}^2) + K_5 (a_{24}^2 + a_{34}^2)]
+ s_2 (a_{12}^2 + a_{13}^2 + a_{23}^2) + s_2 [K_1 (a_{24}^2 + a_{25}^2 + a_{34}^2 + a_{35}^2)
+ K_3 (a_{14}^2 + a_{15}^2 + a_{24}^2 + a_{25}^2 + a_{34}^2)
+ K_4 + K_5] (a_{12}^2 + a_{13}^2 + a_{23}^2]
+ (a_{23}^2 a_{45}^2 - a_{24}^2 a_{35}^2 + a_{25}^2 a_{34}^2)^2 + (a_{13}^2 a_{45}^2 - a_{14}^2 a_{35}^2 + a_{25}^2 a_{34}^2)^2
+ (a_{12}^2 a_{45}^2 - a_{14}^2 a_{25}^2 + a_{15} a_{24}^2)^2)
+ (s_2^2 (K_1 a_{23}^2 + K_2 a_{12}^2 + K_3 a_{12}^2 + K_5 a_{14}^2)
+ s_2 [(a_{12}^2 a_{35}^2 - a_{13}^2 a_{25}^2 + a_{15} a_{23}^2)^2 + (a_{12}^2 a_{34}^2 - a_{13}^2 a_{24}^2 + a_{14} a_{23}^2)^2]
+ [K_1 (a_{23}^2 a_{45}^2 - a_{24}^2 a_{35}^2 + a_{25}^2 a_{34}^2)^2 + K_2 (a_{13}^2 a_{45}^2 - a_{14}^2 a_{35}^2 + a_{15} a_{34}^2)^2
+ K_3 (a_{12}^2 a_{45}^2 - a_{14}^2 a_{25}^2 + a_{15} a_{24}^2)^2 + K_4 (a_{12}^2 a_{35}^2 - a_{13}^2 a_{25}^2 + a_{15} a_{23}^2)^2
+ K_5 (a_{12}^2 a_{34}^2 - a_{13}^2 a_{24}^2 + a_{14} a_{23}^2)^2)] \tag{4.19}
It is seen that cascading eqns. (4.18) and (4.19) will give polynomial with equal degrees for $s_1$ and $s_2$. This is not the case if the starting matrix is of even order.

4.3 Design Method

As explained in Chapter III a 2-variable VSHP is assigned to the denominator of the transfer function using the method described in the previous section and the numerator is left unchanged. Then bilinear transformation is applied to the derived 2-D analog transfer function to obtain the discrete version of the filter. The error between the ideal and designed magnitude is calculated using the equation shown below:

$$E_m(w_1, w_2) = \left| H_I(e^{jw_1T}, e^{jw_2T}) \right| - \left| H_D(e^{jw_1T}, e^{jw_2T}) \right|$$

(4.20)

where $E_m$ is the error in the magnitude response, $H_I$ and $H_D$ are the ideal and designed magnitude response respectively. Similarly, error between the ideal and designed group delay response of the filter is calculated as follows:

$$E_{\tilde{\tau}}(w_1, w_2) = \tau_{wi} - \tau_{wi}(e^{jw_1T}, e^{jw_2T})$$

(4.21)

where $\tau_{wi}$ is a constant equal to the order of the filter [16]. $\tau_{wi}$ is the designed group delay of the filter and
is a measure of the average delay of the filter as a function of frequency and is defined as follows:

\[
\tau_{w_i}(z_1, z_2) = \frac{\delta \psi(z_1, z_2)}{\delta w_i} = -\text{Re} \left[ \frac{z_1}{\Delta H(z_1, z_2)} \cdot \frac{\delta H(z_1, z_2)}{\delta z_1} \right]
\]

(4.22)

where \( \psi \) is phase response of a filter and \( z_i = \exp(jw_iT) \) \( i = 1, 2; T \) is sampling frequency.

Using Eqs. 4.20 and 4.21 we can formulate an error function between magnitude and group delay responses of the ideal and the designed filter using the equation shown below

\[
E(w_1, w_2) = a_1 \sum_{m,n} I_{ps} E_m^2 (w_{1m}, w_{2n}) + a_2 \sum_{m,n} I_P E_{\tau w_1}^2 (w_{1m}, w_{2n}) + a_3 \sum_{m,n} I_P E_{\tau w_2}^2 (w_{1m}, w_{2n})
\]

(4.23)

where \( I_{ps} \) is a set of all discrete points in the passband and stopband of the filter where \( I_P \) is set of all discrete points in the passband of filter. If we assume

\[
\text{Max of } E_m = \beta_1 \quad \text{Max } E_{\tau w_1} = \beta_2 \quad \text{Max } E_{\tau w_2} = \beta_3
\]

then the weighting coefficients \( a_1, a_2, a_3 \) can be chosen to be equal to unity or as follows:

\[
a_1 = \beta_1 \beta_2, \quad a_2 = \beta_1 \beta_3, \quad a_3 = \beta_1 \beta_3.
\]
4.4 Design Example

To illustrate the method of a low pass, band pass and fan filter is designed using the following filter specifications, are considered.

(a) Low Pass Filter

A low pass filter was designed using the following specifications:

\[
H_I(j\omega_1, j\omega_2) = \begin{cases} 
1.0 & \text{for } 0 \leq \sqrt{\omega_1^2 + \omega_2^2} \leq 1.0 \text{ rad/sec} \\
0.0 & \text{for } 5 \geq \sqrt{\omega_1^2 + \omega_2^2} \geq 2.5 \text{ rad/sec}
\end{cases}
\]

(4.24)

In this example \(Q(s_1, s_2)\) in eqn. (4.11) is assigned to the denominator of the 2D filter's transfer function while the numerator is left unchanged. The minimization used is that of a Fletcher Powell [24] to minimize the least mean square error of the magnitude and group delay responses. Table 2 shows coefficients of the designed low pass filter. Figure 3a shows the magnitude response and Figs. 3b-3c show the group delay response of designed filter.
Table 2

Values of the Parameters of the Designed Low Pass Filter

<table>
<thead>
<tr>
<th>Numerator Coefficients</th>
<th>Denominator Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>a_{00} = 13.3599154</td>
<td>b_{12} = -1.32599163</td>
</tr>
<tr>
<td>a_{01} = 40.529972</td>
<td>b_{13} = -1.65199089</td>
</tr>
<tr>
<td>a_{12} = 1.68998146</td>
<td>b_{14} = 1.82297707</td>
</tr>
<tr>
<td>a_{10} = 61.679346</td>
<td>b_{23} = -88.039942</td>
</tr>
<tr>
<td>a_{11} = -45.0602174</td>
<td>b_{24} = -21.9599962</td>
</tr>
<tr>
<td>a_{12} = 10.179965</td>
<td>b_{34} = 1.40699577</td>
</tr>
<tr>
<td>a_{20} = 1.73297405</td>
<td></td>
</tr>
<tr>
<td>a_{21} = 49.998927</td>
<td></td>
</tr>
<tr>
<td>a_{22} = 22.719972</td>
<td></td>
</tr>
</tbody>
</table>

(b) **Band Pass Filter**

A band pass filter was designed using the following specifications:

\[
|H_i(j\omega_1m, j\omega_2n)| = \begin{cases} 
0.0 & \text{for } 0 \leq \sqrt{\omega_1^2 + \omega_2^2} < 1.5 \\
1.0 & \text{for } 2.5 \leq \sqrt{\omega_1^2 + \omega_2^2} < 3.5 \\
0.0 & \text{for } 5 \leq \sqrt{\omega_1^2 + \omega_2^2} > 4.0 \quad (4.25)
\end{cases}
\]

Table 3 gives the coefficients of the designed filter while Fig. 4a shows the magnitude response and the group delay responses are shown in Figs. 4a–4c respectively.
Table 3
Coefficients of Band Pass Filter

<table>
<thead>
<tr>
<th>Numerator Coefficients</th>
<th>Denominator Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{00} = 28.65$</td>
<td>$b_{12} = -1.874$</td>
</tr>
<tr>
<td>$A_{01} = 11.21$</td>
<td>$b_{13} = 0.923$</td>
</tr>
<tr>
<td>$A_{02} = -1.26$</td>
<td>$b_{14} = -1.822$</td>
</tr>
<tr>
<td>$A_{10} = 11.30$</td>
<td>$b_{23} = -0.48820$</td>
</tr>
<tr>
<td>$A_{11} = -7.967$</td>
<td>$b_{24} = 1.8660$</td>
</tr>
<tr>
<td>$A_{12} = 0.0125$</td>
<td>$b_{34} = 1.8690$</td>
</tr>
<tr>
<td>$A_{20} = 1.321$</td>
<td></td>
</tr>
<tr>
<td>$A_{21} = 0.035$</td>
<td></td>
</tr>
<tr>
<td>$A_{22} = 0.6395$</td>
<td></td>
</tr>
</tbody>
</table>

(c) Fan Filter

A fan filter was designed using the following specifications:

$$H_i(jw_{1m}, jw_{2m}) = \begin{cases} 
1.0 & \text{for } w_2 \leq 0.8w_1 \\
0.0 & \text{for } w_2 \geq 1.2w_1 
\end{cases}$$

Table 4 gives the coefficients of fan filter. Figures 5a to 5c give the magnitude and group delay response of the designed filter respectively.
Table 4

Coefficients of Fan Filter

<table>
<thead>
<tr>
<th>Numerator Coefficients</th>
<th>Denominator Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{00} = 37.84$</td>
<td>$b_{12} = 2.300$</td>
</tr>
<tr>
<td>$a_{01} = -2.180$</td>
<td>$b_{13} = -4.250$</td>
</tr>
<tr>
<td>$a_{02} = -12.66$</td>
<td>$b_{14} = -2.489$</td>
</tr>
<tr>
<td>$a_{10} = -55.83$</td>
<td>$b_{23} = -5.280$</td>
</tr>
<tr>
<td>$a_{11} = 21.39$</td>
<td>$b_{24} = -7.220$</td>
</tr>
<tr>
<td>$a_{12} = -24.40$</td>
<td>$b_{34} = 3.450$</td>
</tr>
<tr>
<td>$a_{20} = 66.14$</td>
<td></td>
</tr>
<tr>
<td>$a_{21} = -33.92$</td>
<td></td>
</tr>
<tr>
<td>$a_{22} = 7.280$</td>
<td></td>
</tr>
</tbody>
</table>
4.5 Conclusion

This chapter presents a method for the design of a 2-D stable digital filter satisfying magnitude and group delay responses. Matrix theory can be utilized to ensure that the denominator of the 2-D analog transfer function does not generate any poles with non-essential singularities of the second kind. This also gives a procedure for n-variable VSHP. In [14] it is stated that certain parameters of optimization can take zero so as to reduce computation cost. In this contribution it is clearly shown only some parameters may take zero value without affecting the VSHP property.
Fig. 4a. Magnitude Response of a Band Pass Filter.
Fig. 4b. Group Delay Response w.r.t. $w_1$. 
Fig. 4c. Group Delay Response w.r.t. $w_2$. 
Fig. 5a. Magnitude Response of a Fan Filter.
Fig. 5b. Group Delay Response w.r.t. \( w_1 \).
Fig. 5c. Group Delay Response w.r.t. $w_2$. 
CHAPTER V

CONCLUSION

5.0

Two dimensional digital filters are used in image processing and seismic data processing for removal of linear noise. It has been shown in the past that the stability of digital filters plays a very important role in their use for such applications. Quite a few techniques have been presented for the design of 2-D digital recursive filters. Some of them started by using an analog reference filter to design these filters and then discretize them by the application of double bilinear transformation. This method, however, had two drawbacks: (i) if the 2-D analog filter possesses non-essential singularities of second kind, this will translate to an unstable filter in the discrete domain; (ii) bilinear transformation process is not phase invariant, that is, if the 2-D analog reference filter has linear phase characteristics, upon the discretization process, the phase is no longer linear.

This thesis discusses two methods for generating a
2-variable VSHP which are free from non-essential singularities of the second kind. VSHP are then assigned to the denominator of an analog reference filter. The discrete version of the filter is obtained does not generate any non-essential singularities of the second kind. Using the parameters of the discrete function as the variables of optimization, the filter can be designed to minimize the least mean squared error of a desired magnitude and group delay responses.
DESIGN OF TWO-DIMENSIONAL FILTERS SATISFYING PRESCRIBED AMPLITUDE AND CONSTANT DELAY RESPONSE

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ABSTRACT

This paper presents a method for the design of 2-dimensional (2D) stable recursive digital filters satisfying a given magnitude and group delay response. This method uses properties of derivatives of even or odd parts of Hurwitz polynomials to generate a very strictly Hurwitz polynomial (VSHP) which will be assigned to the denominator of the 2D analog reference filter. Bilinear transformations is then applied to the transfer function of the 2D analog reference filter to obtain the discrete version of the filter. Parameters of the discrete transfer function can be used as the variables of optimization to minimize the least mean square error of the desired and designed magnitude and group delay response of the filter. The method is illustrated by several examples.

I. INTRODUCTION

2-D digital filters have found numerous applications in the area of digital signal processing. For example in image processing, they can be used for enhancement of low quality images such as X-ray photographs and compensation of linear optical degradation.

It has been shown [1] that these filters can be of better value in image processing application if they have linear phase characteristics. Unfortunately of numbers of existing method for the design of 1-D recursive digital filters very few have addressed design of 2-D filters with linear phase characteristics [2-4]. In this paper, the method reported in [5] is used and modified to design 2-D recursive digital filter satisfying prescribed magnitude and phase response.

II. CHARACTERIZATION OF 2-D ANALOG FILTERS AND RECURSIVE DIGITAL FILTERS

A 2-D analog filter is characterized by its transfer function

\[
H(z_1, z_2) = \frac{B(z_1, z_2)}{A(z_1, z_2)} = \frac{\sum_{i_1=0}^{M_1} \sum_{i_2=0}^{M_2} \frac{b(i_1, i_2)}{z_1^{i_1} z_2^{i_2}}}{\sum_{i_1=0}^{M_1} \sum_{i_2=0}^{M_2} \frac{a(i_1, i_2)}{z_1^{i_1} z_2^{i_2}}}
\]

where \(A\) and \(B\) are polynomials in \(z_1\) and \(z_2\). The design problem is to obtain the polynomial coefficients \(a(i_1, i_2)\) and \(b(i_1, i_2)\) such that

1) \(H\) approximates a given response
2) the filter is stable. That is [6]

\[
B(s_1, s_2) \neq 0 \quad \text{for} \quad s_1 \geq 0
\]

Similarly a 2-D recursive digital filter is characterized by its \(z\)-transfer function

\[
H(z_1, z_2) = \frac{M(z_1, z_2)}{D(z_1, z_2)} = \frac{\sum_{i_1=0}^{M_1} \sum_{i_2=0}^{M_2} \frac{m(i_1, i_2)}{z_1^{i_1} z_2^{i_2}}}{\sum_{i_1=0}^{M_1} \sum_{i_2=0}^{M_2} \frac{d(i_1, i_2)}{z_1^{i_1} z_2^{i_2}}}
\]

where \(M(z_1, z_2)\) and \(D(z_1, z_2)\) are polynomials in \((z_1 = \exp(i_1 \theta_1), (i_1 = 1, 2))\), and the design problem is to obtain polynomial coefficients \(m(i_1, i_2)\) and \(d(i_1, i_2)\) such that the \(z\)-transfer function evaluated on the unit circles in \(z_1\) and \(z_2\) plane approximates to the desired response of the filter. Beside maintaining the stability of the filter, which requires [6]

\[
D(z_1, z_2) \neq 0 \quad \text{for} \quad |z_1| = 1
\]

The approximation can be carried out in \(s\) or \(z\) domain by using any suitable optimization method, but the difficulty lies in maintaining the stability of the designed filter, inhibiting the widespread application of recursive filters, inspite of their significant computational advantages over non-recursive filters.

III. GENERATION OF 2-VARIABLE VSHP

A 2-variable VSHP can be generated using the method reported in [5]. This method consists of the following steps:

1) a suitable even or odd part of a n-variable Hurwitz Polynomial is generated.
2) The corresponding derivatives giving the odd or even part is associated with it.
iii) The resulting \( n \)-variable Hurwitz polynomial is converted to a 2-variable VSHP.

For example, consider the polynomial \( N_{2n} \) given by

\[
N_{2n} = \det \begin{bmatrix} \mu & \Sigma_{2n} + B_{2n} \end{bmatrix}
\]

(5)

where \( \mu \) is a diagonal matrix of order \( 2n \) given by

\[
\mu = \text{diag} \left[ \mu_1, \mu_2, \mu_3, \ldots, \mu_{2n} \right]
\]

(6)

and \( A \) is a skew-symmetric matrix of order \( 2n \) given by

\[
A_{2n} = \begin{bmatrix} 0 & b_{12} & b_{13} & \ldots & b_{1,2n} \\
-b_{12} & 0 & b_{23} & \ldots & b_{2,2n} \\
-b_{13} & -b_{23} & 0 & \ldots & b_{3,2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-b_{1,2n} & -b_{2,2n} & \ldots & \ldots & 0 \\
\end{bmatrix}
\]

(7)

From the diagonal expansion of the determinant of a matrix [7], \( N_{2n} \) can be written as

\[
N_{2n} = \det B_{2n} + \sum_{1 \leq i_1 < i_2 \leq 2n} \mu_{i_1} \mu_{i_2} \begin{bmatrix} B_{i_1,i_2} & \end{bmatrix}
\]

\[
+ \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq 2n} \mu_{i_1} \mu_{i_2} \mu_{i_3} \mu_{i_4} \begin{bmatrix} B_{i_1,i_2,i_3,i_4} & \end{bmatrix}
\]

(8)

where \( B_{i_1,i_2} \) is the determinant of the submatrix of \( B_{2n} \) obtained by deleting both the \( i_1 \)th and \( i_2 \)th rows and columns, and is of order \((2n-2)\), \( B_{i_1,i_2,i_3,i_4} \) is the determinant of the submatrix of \( B_{2n} \) obtained by deleting \( i_1 \)th, \( i_2 \)th, \( i_3 \)th, and \( i_4 \)th rows and columns and is of order \((2n-4)\), and so on.

The following properties shall be noted:

i) All odd order terms of the type \( \mu_{i_1}, \mu_{i_2} \mu_{i_3}, \) etc. are absent, since the determinant of an odd order skew-symmetric matrix is zero.

ii) The degree of any \( \mu_i (i = 1, \ldots, 2n) \) is unity.

iii) The quantity \( \det B_{2n}, B_{i_1,i_2}, B_{i_1,i_2,i_3,i_4} \) etc. are non-negative numbers, since the determinant of an even order skew-symmetric matrix is a perfect square.

Since the matrix \( \begin{bmatrix} \mu & \Sigma_{2n} + B_{2n} \end{bmatrix} \) is always physically realizable, \( N_{2n} \) represents even part of a 2n-variable Hurwitz polynomial. Therefore, \( \left( \frac{3M_{2n}}{\partial \mu} \right) / M_{2n} \) is a reactance function [8]. As a consequence,

\[
M'_{2n} = M_{2n} + \frac{2n}{2} \sum_{j=1}^{2n} \frac{3M_{2n}}{\partial \mu_j}
\]

(9)

is a 2n-variable Hurwitz polynomial.

From eq. (9), a 2-variable VSHP can be generated by putting some of the \( \mu_j \)s equal to \( s_1 \) and the rest of the \( \mu_j \)s equal to \( s_2 \) with the condition that \( \det A_{2n} \neq 0 \).

IV. FORMULATION OF THE DESIGN PROBLEM

In this method a 2-variable VSHP is assigned to the denominator of the eqn. (1) using eqns. (5) and (9) and numerator is left unchanged. Then bilinear transformation applied to the derived 2-D analog transfer function to obtain the discrete version of the filter.

The error between the Ideal and Design magnitude response is calculated using the relationship

\[
E_D(w_1, w_2) = \left| H_I(jw_1, jw_2) \right| - \left| H_D(jw_1, jw_2) \right|
\]

(10)

where \( E_D \) is the error of the magnitude response and \( |H_I| \) and \( |H_D| \) are amplitude response of ideal and designed filters respectively. \( |H_D| \) can be calculated using

\[
E_D \left( jw_1, jw_2 \right) = \frac{\left| \begin{bmatrix} jw_1 & jw_2 \end{bmatrix} \right|}{A \left( e^{j\theta_1}, e^{j\theta_2} \right)}
\]

\[
\left( -\frac{2}{A} \right) \left( e^{j\theta_1}, e^{j\theta_2} \right) + \frac{\left| \begin{bmatrix} jw_1 & jw_2 \end{bmatrix} \right|}{B \left( e^{j\theta_1}, e^{j\theta_2} \right)}
\]

where \( \text{Re} \{ \cdot \} \) and \( \text{Im} \{ \cdot \} \) are Real Part and Imaginary part of a complex function inside the bracket respectively.

The phase response of the filter can be calculated using

\[
\Phi(z_1, z_2) = \tan^{-1} \frac{\text{Im}(H_D(z_1, z_2))}{\text{Re}(H_D(z_1, z_2))}
\]

(12)

where \( z_i = e^{jw_i} \) for \( i = 1, 2 \), and ultimately the group delay response of the filter which is a measure of the average spatial or time delay of the filter as a function of frequency can be calculated using the equation

\[
\tau_{gD} \left( z_1, z_2 \right) = \frac{\text{Re} \left( \frac{z_1}{H_D(z_1, z_2)} \cdot \frac{3 \text{Im}(H_D(z_1, z_2))}{\delta z_1} \right)}{\text{Re} \left( \frac{z_1}{H_D(z_1, z_2)} \right)}
\]

(13)
for \( z = \exp(j \omega_i) \) and \( i = 1, 2 \)

The error between the ideal and designed group delay response can be calculated using eq. (13)

\[
E_{\omega_i} = (\omega_i, \omega_2) = \tau_i - \tau_{\omega_i} \left[ e^{j \omega_i T}, e^{-j \omega_i T} \right] \tag{14}
\]

\((i = 1, 2)\)

where \( \tau_i \) is a constant representing the ideal group delay response of the filter and its value is chosen equal to the order of the filter [4].

Using eqs. (10) and (14) we can generate the general least mean square error between magnitude and group delay response using the following relationship

\[
E_G(\omega_1, \omega_2) = \sum \sum E^2_{\omega_1, \omega_2} \quad \text{over the passband and stopband regions only}
\]

\( + \sum \sum E^2_{\omega_1, \omega_2} \quad \text{over the passband region only} \tag{15}
\]

Now any suitable non-linear optimization technique can be adapted to calculate parameters \( a \) and \( b \) of the filter's transfer function so as to minimize the \( E_G \) in equation (15).

V. DESIGN EXAMPLES

To illustrate the method a 2-D low-pass filter with the following specification is designed

\[
H_2(j \omega_1, j \omega_2) = \begin{cases} 
1 & \text{for } \sqrt{\omega_1^2 + \omega_2^2} \leq 1 \text{ rad/sec} \\
0 & \text{for } \sqrt{\omega_1^2 + \omega_2^2} \geq 2.5 \text{ rad/sec}
\end{cases}
\]

and constant group delay which will be equal to the order of the chosen filter. In this example matrix \( P_{2n} \) in equation (7) is chosen to be of order 4 x 4 and \( K \)'s in equation (9) are set to unity. Obviously in this example order of the filter will be two so \( \tau_i \) in equation (14) will be two. Optimization technique used in this method is direct search method Kocks and Jeewa [9]. Table 1 shows the values of parameters of the designed filter while Fig. (1) shows the amplitude and Fig. (2a - 2b) show the group delay characteristics of the designed filter.

VI. CONCLUSION

In this paper a method is presented for the design of 2-D digital filters with specified magnitude and group delay responses. This method is based on the properties of the derivatives of a n-variable Hurwitz polynomial and their use in generating a 2-variable VSNP polynomial. This method can be easily used using any unconstant non-linear optimization and requires less memory unlike the one reported in [4]. This method is easily extended to N-D including 1-D case.

VII. ACKNOWLEDGEMENT

The authors are grateful to the Natural Science and Engineering Research Council of Canada for supporting this research.

REFERENCES


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TABLE 1 Values of the Parameters of the Designed Filters

Fig. 1. 3-D Plot of Magnitude Response of the Designed Filter

Fig. 2a. 3-D Plot of the Group Delay Response of the Designed Filter with $\omega_1$ Constant

Fig. 2b. 3-D Plot of the Group Delay Response of the Designed Filter with $\omega_2$ Constant
A NEW METHOD FOR THE DESIGN OF 2-DIMENSIONAL STABLE RECURSIVE DIGITAL FILTERS SATISFYING PRESCRIBED MAGNITUDE AND GROUP DELAY RESPONSE

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Montreal, Quebec, H3G 1M8, Canada

ABSTRACT

This paper presents a technique for the design of 2-Dimensional (2-D) stable recursive digital filters satisfying prescribed magnitude and constant group delay response. This technique uses the properties of positive definite matrices and their application in generating 2-Variable Very Strictly Hurwitz Polynomials (VSHP) which will be assigned to the denominator of a 2-D analog reference filter. Bi-linear transformations is then applied to the transfer function of the 2-D analog reference filter to obtain the discrete version of the 2-D filter.

Parameters of the discrete 2-D filter can be used as the variables of optimization to minimize the least mean square error of the desired and designed magnitude and group delay response of the filter. This technique is illustrated by examples.

I. INTRODUCTION

2-D digital filters are widely used for enhancement of low quality images such as X-ray images and compensation of linear optical degradation.

It has been reported [1] that these filters can be of better value in image processing application if they have linear phase characteristics. Unfortunately of the number of existing methods for the design of 2-D recursive digital filters very few have addressed design of 2-D filters with linear phase characteristics [2-5].

In this paper we first present a method for generating 2-Variable Very Strictly Hurwitz Polynomial (VSHP) using properties of positive definite matrices and then use this to design 2-D stable, recursive digital filters satisfying prescribed magnitude and constant group delay response.

II. CHARACTERIZATION OF 2-D ANALOG AND RECURRENT DIGITAL FILTERS

A 2-D analog filter is characterized by its transfer function

$$H_a(s_1, s_2) = \frac{\prod_{i=0}^{N_1} \prod_{i=2}^{N_2} a(i_1, i_2) s_1^{i_1} s_2^{i_2}}{\prod_{i=0}^{N_1} \prod_{i=2}^{N_2} b(i_1, i_2) s_1^{i_1} s_2^{i_2}}$$

(1)

where A and B are polynomials in $s_1$ and $s_2$. The design problem is to obtain the polynomial coefficients $a(i_1, i_2)$ and $b(i_1, i_2)$ such that:

(i) $H_a$ approximates a given response;
(ii) the filter is stable. That is [6]

$$B(\omega_1, \omega_2) \neq 0$$

$$\sum_{i=1}^{2} \Re s_i > 0$$

(2)

Similarly a 2-D recursive digital filter is characterized by its $z$-transfer function

$$H_d(z_1, z_2) = \frac{\prod_{i=0}^{N_1} \prod_{i=2}^{N_2} n(i_1, i_2) z_1^{i_1} z_2^{i_2}}{\prod_{i=0}^{N_1} \prod_{i=2}^{N_2} d(i_1, i_2) z_1^{i_1} z_2^{i_2}}$$

(3)

where $N$ and $D$ are polynomials in ($z_1 = \exp(s_1 T_1)$, $i=1, 2$), and the design problem is to obtain polynomial coefficients $n(i_1, i_2)$ and $d(i_1, i_2)$ such that the $z$-transfer function evaluated on the unit circles in $z_1$ and $z_2$ plane approximates to the desired response of the filter, besides maintaining the stability of the filter. The latter condition requires [6]

$$D(z_1, z_2) \neq 0$$

(4)

The approximation can be carried out in a or $z$ domain by using any suitable optimization method but the difficulty lies in maintaining the stability of the designed filter, inhibiting the wide spread application of recursive filters, in spite of their significant computational advantages over non-recursive filters.
III. GENERATION OF 2-VARIABLE VSHP

It is known from the matrix algebra that matrix $N$ is a positive definite matrix if

$$ N = [B][\lambda][S]^T \tag{5} $$

where $\lambda$ is a diagonal matrix with non-zero and positive elements, and $B$ is an upper-triangular matrix with unity elements in the diagonal. The general form of $N$ can be written as

$$ N = \begin{bmatrix}
1 & b_{12} & b_{13} & \cdots & b_{1n} \\
0 & 1 & b_{23} & \cdots & b_{2n} \\
0 & 0 & 1 & \cdots & b_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_{1n} & b_{2n} & b_{3n} & \cdots & 1
\end{bmatrix} \begin{bmatrix}
\lambda_1 \\
\vdots \\
\lambda_n
\end{bmatrix} \begin{bmatrix}
\lambda_1 \\
\vdots \\
\lambda_n
\end{bmatrix} $$

(6)

We can construct the matrix $N_1$ as

$$ N_1 = \sum_{i=1}^{2} s_i [B_i][\lambda_i][S_i]^T + C \tag{7} $$

where the order of matrices $B_i$, $\lambda_i$ and $C$ are even and $B_i$, $\lambda_i$, $i = 1, 2$ are upper triangular matrices, $\lambda_i$, $i = 1, 2$ is diagonal matrix with non-negative elements, $C$ is a skew symmetric matrix with the property $det C \neq 0$ and $s_i = 1 = 1, 2$ are complex variables. Then, it is seen that

$$ det N_1 = \det \left[ \sum_{i=1}^{2} s_i [B_i][\lambda_i][S_i]^T + C \right] \tag{8} $$

will constitute the even polynomial of a 2-variable reactance function; that is, for $s_2 = j\omega_2$, the resulting polynomial in $s_1$ will have simple zeros on the imaginary axis of $s_1$. Therefore,

$$ \frac{\partial det N_1}{\partial s_1} \bigg|_{s_1} / det N_1 \text{ for } i = 1, 2 \text{ has no singularities of the second kind.} $$

$$ B(s_1, s_2) = \det N_1 + \sum_{i=1}^{2} k_i \frac{\partial \det N_1}{\partial s_1} \bigg|_{s_1} \tag{9} $$

where $k_i$, $i = 1, 2$ are positive constant.

IV. FORMULATION OF THE DESIGN PROBLEM

In this method a 2-variable VSHP is assigned to the denominator of the eqn.(1) using eqn.(9) and the numerator is left unchanged. Then bi-linear transformation is applied to the derived 2-D analog transfer function to obtain the discrete version of the filter.

The error between the ideal and the designed magnitude response is calculated using the relationship

$$ P_m(\omega_1, \omega_2) = \begin{bmatrix} H_I(e^{-j\omega_1 T}, e^{-j\omega_2 T}) \\ H_D(e^{-j\omega_1 T}, e^{-j\omega_2 T}) \end{bmatrix} $$

(10)

where $P_m$ is the error of magnitude response and $H_I$ and $H_D$ are amplitude response of ideal and designed filter respectively.

The error between the ideal and designed group delay response can be calculated using the following relationship

$$ E_g(\omega_1, \omega_2) = \tau_I - \tau_D \begin{bmatrix} e^{-j\omega_1 T} \\ e^{-j\omega_2 T} \end{bmatrix} $$

(11)

where $\tau_I$, $\tau_D$ is a constant representing the ideal group delay response of the filter and its value is chosen equal to the order of the filter [2], and $\tau_D$ is calculated using the following equation:

$$ \tau_D(z_1, z_2) = -\frac{\partial \psi(z_1, z_2)}{\partial \omega_1} $$

$$ = -\frac{z_1}{R_D(z_1, z_2)} \frac{\partial H_D(z_1, z_2)}{\partial \omega_1} \tag{12} $$

where $\psi$ is the phase response of the filter and $z_i = \exp{j\omega_i T}$, $i = 1, 2$.

Using eqns.(10) and (11) we can generate the general least mean square error between magnitude and group delay response of the ideal and designed filter using the relationship

$$ E_m(\omega_1, \omega_2) = \sum_{m,n} \sum_{n=1}^{p} E_m^2(\omega_1, \omega_2) $$

$$ + \sum_{m,n} \sum_{n=1}^{p} E_m^2(\omega_1, \omega_2) $$

(13)
where \( I_p \) is set of all discrete points in the passband and stopband of the filter while \( I_s \) is set of all discrete points in the passband of the filter.

Now any suitable non-linear optimization technique can be used to calculate parameters \( a \) and \( b \) and \( \lambda \) of the filter's transformation so as to minimize the \( E_p \) in eqn.(13) subject to the following constraints

1. \( \lambda \) should be non-negative; 2. \( \det C \neq 0 \).

Condition 1 can be easily overcome by using the following variable substitution

\[
\lambda = \mu^2
\]

and thereby guaranteeing the positiveness of \( \lambda \). Condition 2 can be used as penalty function in the process of optimization to ensure \( \det C \) is always non-zero.

### V. DESIGN EXAMPLES

To illustrate the method a 2-D lowpass filter with the following specification is designed.

\[
| H_1(j\omega_1 , j\omega_2) | = \begin{cases} 
1 & \text{for} \quad \sqrt{\omega_1^2 + \omega_2^2} \leq 1 \text{ rad/sec} \\
0 & \text{for} \quad \sqrt{\omega_1^2 + \omega_2^2} \geq 2.5 \text{ rad/sec}
\end{cases}
\]

and constant group delay which will be equal to the order of the chosen filter. In this example \( N_1 \) in eqn.(7) is considered to be

\[
\begin{bmatrix}
1 & b_2 & 0 & 0 \\
0 & 1 & b_2 & 0 \\
0 & 0 & 1 & b_2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Now using eqn.(9) for \( k_1 = 1 \) and \( k_2 = 0 \) will give

\[
B(s_1,s_2) = k_1^2 s_1^2 + k_2^2 s_2^2 - x s_1 s_2 + 2 b_2^2 s_1 + x s_2 + c^2
\]

where

\[
x = b_2^2 u_4 + b_2^2 u_4 + b_2^2 u_4 + b_2^2 u_4 + b_2^2 u_4 + b_2^2 u_4 + b_2^2 u_4 + b_2^2 u_4
\]

and using eqn.(1) we can write

\[
\lambda(s_1,s_2) = a_{00} + a_{01} s_2 + a_{02} s_2^2 + a_{10} s_1
\]

In this example, the order of the filter is two so \( \gamma \) in eqn.(11) will be two. Optimization technique used in this method is Hooke's and Jeeves[7].

### REFERENCES


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**TABLE (1) VALUES OF THE PARAMETERS OF THE DESIGNED FILTER**

Fig. (1) 3-D Plot of the Amplitude Response of the Designed Filter

Fig. (2a) Group Delay Response With Respect to $\omega_1$

Fig. (2b) Group Delay Response With Respect to $\omega_2$
DESIGN OF 2-D RECURSIVE DIGITAL FILTERS SATISFYING PRESCRIBED MAGNITUDE AND GROUP DELAY CHARACTERISTICS

S. Golikeri, V. Ramachandran*, M. Ahmadi

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Windsor, Ontario, N9B 3P4, Canada

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Montreal, P.Q., H3G 1M8, Canada

ABSTRACT

This paper presents a method for the design of 2-dimensional (2-D) stable recursive digital filters satisfying a given magnitude and group delay response. This method uses properties of derivative of even or odd parts of Hurwitz polynomial (VSPH) which will be assigned to the denominator of the 2-D analog reference filter. Bilinear transformations are then applied to the transfer function of the 2-D analog reference filter to obtain the discrete version of the filter. Parameters of the discrete transfer function can be used as the variables of optimization to minimize the least mean square error of the desired and designed magnitude and group delay response of the filter. The method is illustrated by several examples.

I. INTRODUCTION

2-D digital filters have found numerous applications in the area of digital signal processing. For example in image processing, they can be used for enhancement of low quality images such as X-ray photographs and compensation of linear optical degradation.

It has been shown [1] that these filters can be of better value in image processing application if they have linear phase characteristics. Unfortunately of numbers of existing method for the design of 1-D recursive digital filters very few have addressed design of 2-D filters with linear phase characteristics [2-4]. In this paper, the method reported in [5] is used and modified to design 2-D recursive digital filter satisfying prescribed magnitude and phase response.

II. CHARACTERIZATION OF 2-D ANALOG FILTERS AND RECURSIVE DIGITAL FILTERS

A 2-D analog filter is characterized by its transfer function

\[ H(z_1, z_2) = \frac{M_1 z_1 + M_2 z_2}{D(z_1, z_2)} \]

where \( M_1 \) and \( M_2 \) are polynomials in \( z_1 \) and \( z_2 \). The design problem is to obtain polynomials \( M_1 \) and \( M_2 \) and \( D(z_1, z_2) \) such that the transfer function evaluated on the unit circles in \( z_1 \) and \( z_2 \) plane approximates the desired response of the filter, besides maintaining the stability of the filter. The latter condition requires [6]

\[ |z_1| < 1 \]

Similarly a 2-D recursive digital filter is characterized by its z-transfer function

\[ H(z_1, z_2) = \frac{N(z_1, z_2)}{D(z_1, z_2)} \]

where \( N(z_1, z_2) \) and \( D(z_1, z_2) \) are polynomials in \( z_1 = \exp(j\omega_1) \) and \( z_2 = \exp(j\omega_2) \). The design problem is to obtain polynomial coefficients \( a(i_1, i_2) \) and \( b(i_1, i_2) \) such that the z-transfer function evaluated on the unit circles in \( z_1 \) and \( z_2 \) plane approximates the desired response of the filter, besides maintaining the stability of the filter. The latter condition requires [6]

\[ D(z_1, z_2) \neq 0 \text{ for } i_1, i_2 = 0 \]
The approximation can be carried out in a or z domain by using any suitable optimization method, but the difficulty lies in maintaining the stability of the designed filter, inhibiting the widespread application of recursive filters, in spite of their significant computational advantages over non-recursive filters.

III. GENERATION OF 2-VARIABLE VSNP

A 2-variable VSNP can be generated using the method reported in [5]. This method consists of the following steps:

(i) a suitable even or odd part of a n-variable Burwitz Polynomial is generated.
(ii) The corresponding derivatives giving the odd or even part is associated with it.
(iii) The resulting n-variable Burwitz polynomial is converted to a 2-variable VSNP.

For example, consider the polynomial $M_{2n}$ given by

$$M_{2n} = \det \begin{bmatrix} \mu I_{2n} + B_{2n} \end{bmatrix} \quad (5)$$

where $\mu$ is a diagonal matrix of order $2n$ given by

$$\mu = \text{diag} \left[ \mu_1, \mu_2, \mu_3, \ldots, \mu_{2n} \right] \quad (6)$$

and $A$ is a skew-symmetric matrix of order $2n$ given by

$$A_{2n} = \begin{bmatrix} 0 & b_{12} & b_{13} & \cdots & b_{1,2n} \\ -b_{12} & 0 & b_{23} & \cdots & b_{2,2n} \\ -b_{13} & -b_{23} & 0 & \cdots & b_{3,2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_{1,2n} & -b_{2,2n} & \cdots & 0 & 0 \end{bmatrix} \quad (7)$$

The following properties shall be noted:

(i) All odd order terms of the type $\mu_1, \mu_2, \mu_3, \ldots, \mu_{2n}$, etc. are absent, since the determinant of an odd order skew-symmetric matrix is zero.

(ii) The degree of any $\mu_i (i = 1, \ldots, 2n)$ is unity.

(iii) The quantity $\det \begin{bmatrix} B_{2n}, B_{1,1}, B_{2,1}, \ldots, B_{2,2n} \end{bmatrix}$, etc. are non-negative numbers, since the determinant of an even order skew-symmetric matrix is a perfect square.

Since the matrix $[M_{2n} + B_{2n}]$ is always physically realizable, $M_{2n}$ represents even part of a 2-variable Burwitz polynomial. Therefore, $(3M_{2n}/3\mu_j)/M_{2n}$ is a reactance function [8]. As an consequence,

$$M' = M_{2n} + \sum_{j=1}^{2n} \frac{3M_{2n}}{\partial \mu_j} \frac{\partial}{\partial \mu_j} \quad (9)$$

is a 2-variable Burwitz polynomial.

From eq. (9), a 2-variable VSNP can be generated by putting some of the $\mu$'s equal to $s_1$ and the rest of the $\mu$'s equal to $s_2$ with the condition that $\det A_{2n} \neq 0$.

IV. FORMULATION OF THE DESIGN PROBLEM

In this method a 2-variable VSNP is assigned to the denominator of the equation (1) using eqs. (5) and (9) and numerator is left unchanged. Then bilinear transformation applied to the derived 2-D analog transfer function to obtain the discrete version of the filter,

The error between the Ideal and Design magnitude response is calculated using the relationship

$$\Sigma_n (\omega_1, \omega_2) = |H_I (j\omega_1, j\omega_2)| - |H_D (j\omega_1, j\omega_2)| \quad (10)$$

where $E_n$ is the error of the magnitude response and $|H_I|$ and $|H_D|$ are amplitude response of Ideal and designed filters respectively. $|H_D|$ can be calculated using

$$H_D(e^{j\omega_T}, e^{j\omega_T}) = \frac{-A(e^{j\omega_T}, e^{j\omega_T})}{B(e^{j\omega_T}, e^{j\omega_T})}$$

where $Re(\cdot)$ and Im(\cdot) are Real Part and Imaginary part of a complex function inside the bracket respectively.
The phase response of the filter can be calculated using
\[ \psi(z_1, z_2) = \tan^{-1} \frac{\text{Im}[H_D(z_1, z_2)]}{\text{Re}[H_D(z_1, z_2)]} \]  
(12)

where \( z_i = e^{j\omega T} \) for \( i = 1, 2 \), and ultimately the group delay response of the filter which is a measure of the average spatial or time delay of the filter as a function of frequency can be calculated using the equation
\[ \tau_{\omega_1}(z_1, z_2) = \frac{3}{\omega_1} \frac{\text{Re}[H_D(z_1, z_2)]}{\text{Re}[H_D(z_1, z_2)]} \]  
(13)

for \( z = e^{j\omega_1 T} \) and \( i = 1, 2 \)

The error between the ideal and designed group delay response can be calculated using eq. (13)
\[ E_{\tau_{\omega_1}}(\omega_1, \omega_2) = \tau_{\omega_1} - \tau_{\omega_2} \]

(14)

where \( \tau_\omega \) is a constant representing the ideal group delay response of the filter and its value is chosen equal to the order of the filter [4].

Using eqs. (10) and (14) we can generate the general least mean square error between magnitude and group delay response using the following relationship
\[ E_G(\omega_1, \omega_2) = \sum \sum E_m^2(\omega_1, \omega_2) + \sum \sum E_T^2(\omega_1) \]

(15)

over the passband and stopband regions
\[ + \sum \sum E_T^2(\omega_2) \]

only

Now any suitable non-linear optimization technique can be adapted to calculate parameters \( A \) and \( B \) of the filter's transfer function so as to minimize the \( E_G \) in equation (15).

V. DESIGN EXAMPLES

To illustrate the method a 2-D low-pass filter with the following specification is designed
\[ H_1(j\omega_1, j\omega_2) \}
\[ \begin{cases} 1 & \text{for } \sqrt{\omega_1^2 + \omega_2^2} \leq 1 \text{ rad/sec} \\ 0 & \text{for } \sqrt{\omega_1^2 + \omega_2^2} \geq 2.5 \text{ rad/sec} \end{cases} \]

and constant group delay which will be equal to the order of the chosen filter. In this example matrix \( E_{\tau_\omega} \) in equation (7), is chosen to be of order \( 4 \times 4 \) and \( K's \) in equation (9) are set to unity. Obviously in this example order of the filter will be two so \( \tau_\omega \) in equation (14) will be two. Optimization technique used in this method is direct search method [8] in [9]. Table 1 shows the values of the parameters of the designed filter while Fig. (1) shows the magnitude response and Fig. (2a-b) shows the phase delay characteristics of the designed filter.

VI. CONCLUSION

In this paper a method is presented for the design of 2-D digital filters with specified magnitude and group delay response. This method is based on the properties of the derivatives of a \( n \)-variable Hurwitz polynomial and their use in generating a 2-variable VSP polynomial. This method can be easily used using any unconstrained non-linear optimization and requires less memory unlike the one reported in [4]. This method is easily extended to N-D including 1-D case.

VII. ACKNOWLEDGEMENT

The authors are grateful to the Natural Science and Engineering Research Council of Canada for supporting this research.

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TABLE 1 Values of the Parameters of the Designed Filters

Fig. 1 3-D Plot of Magnitude Response of the Designed Filter

Fig. 1a 3-D Plot of the Group Delay Response of the Designed Filter with \( \omega_1 \) Constant

Fig. 2b 3-D Plot of the Group Delay Response of the Designed Filter with \( \omega_2 \) Constant

REFERENCES


VITA AUCTORIS

SANDEEP P. GOLIKERI

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<tr>
<td>1979</td>
<td>Worked as Electronic Engineer in International Power Semiconductors.</td>
</tr>
<tr>
<td>1982-</td>
<td>Design Engineer at Rotoflex International Inc., present Mississauga, Ontario, Canada.</td>
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<tr>
<td>1984</td>
<td>Candidate for the Degree of Master of Applied Science in Electrical Engineering at University of Windsor, Windsor, Ontario, Canada.</td>
</tr>
</tbody>
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