Dirac theory in the Pauli algebra.

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DIRAC THEORY IN THE PAULI ALGEBRA

By

David E. Kosokowsky

A Thesis
submitted to the Faculty of Graduate Studies and Research
through the
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of the requirements for the degree of
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David E. Kosokowsky, 1991
To my parents, Ann and Gene.
ABSTRACT

A formal analysis of the Clifford algebras of Euclidean three-space and Minkowski spacetime, more commonly referred to as the Pauli and Dirac algebras, is presented. The relationship between these two algebras is explored in detail, culminating in the construction of a rigourous and completely general mechanism by which physical models based on irreducible spinor representations of the Dirac algebra are equivalently realized within the strict algebraic framework of the Pauli algebra. In addition, the Lorentz group and more specifically its connected subgroup are realized within the framework of the Dirac and Pauli algebras through their corresponding universal covering groups via the Clifford group and its subgroups. The salient features of first-quantized Dirac theory are redefined in terms of the more abstract Clifford algebraic structure of the Dirac algebra. The resulting model is then re-expressed within the infrastructure of the Pauli algebra.
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INTRODUCTION

Two significant players in much of this are the Dirac and Pauli algebras. The Dirac algebra $\mathcal{D}$, so named after its first conventional appearance in Dirac's electron theory under the guise of a matrix algebra generated by the famous "gamma matrices" (Bjorken and Drell 1964, Messiah 1976, Itzykson and Zuber 1980, Berestetskii et al 1982, Halzen and Martin 1984, Ryder 1985), is more formally defined as the complexification of the Clifford algebra generated by the Minkowski vector space $\mathbb{M}^4$. The smaller Pauli algebra $\mathcal{P}$, often identified with the algebra generated by the Pauli spin matrices, is a sub-algebra of $\mathcal{D}$ and can be defined as the Clifford algebra of Euclidean three-space. $\mathcal{P}$ can also be realized as the complex quaternions. Lately, there has been much focus on the real Dirac algebra $\mathcal{D}_R$ inspired, primarily, by the work of D. Hestenes (see references above).

This does not preclude the importance of more general Clifford algebras in physics. Indeed, higher finite-
dimensional and even infinite-dimensional Clifford algebras are finding a unique place in the realm of quantum field theory, supersymmetry theory, and grand unification theory.

These observations have spawned an extensive study of Clifford algebras, including their classification (Hermann 1974, Greub 1978, Lounesto 1979, Ilamed and Salingaros 1981, Salingaros 1981[I],1981[II],1982, Poole and Parach 1982, Li et al 1986, Coquereaux 1982,1988). In addition, Clifford algebras have been found to provide an excellent abstract framework for the study of spinors, via the structure of a minimal left ideal, and isometries, such as Lorentz transformations, via the Clifford group (Hamilton 1984[I],1984[II], Lounesto 1986, Bugajska 1986[I],1986[II], Benn and Tucker 1987, Budinich and Trautman 1988). Both are clearly of central importance to physics.

The purpose of this study is three-fold: first, to lay down a formal foundation for interpreting the usual Dirac gamma matrices in terms of the abstract Clifford algebra $\mathcal{D}$; second, to examine the interrelationship between the Dirac and Pauli algebras and to construct a rigorous prescription for expressing physical models, based on irreducible spinor representations of $\mathcal{D}$ or its real subalgebra, wholly in terms of $\mathcal{P}$; and finally, to apply this prescription to the Dirac theory.

Chapter One is essentially mathematical, although many concepts introduced are significant for physics. In Section 1, we introduce the Pauli algebra as the Clifford algebra of Euclidean three-space and expound its fundamental algebraic properties. In Section 2, we first study the real Dirac algebra as the Clifford algebra of Minkowski space-time, then consider its complexification and examine its relationship to the Pauli algebra. Section 3 develops the theory of spinors
in terms of the minimal left ideals of a Clifford algebra, culminating in the analysis of these elements within the structure of the Pauli and Dirac algebras. Here, emphasis is placed on the importance of the irreducible spinor representations of the Clifford algebra on its minimal left ideals which define the corresponding spinor spaces. A detailed analysis is also made of the reducible spinor representations of the even subalgebra of the Dirac algebra and their connection to Weyl spinors.

Section 4 is devoted entirely to the study of Lorentz transformations. We show that the restricted Lorentz group, or more precisely its universal covering group, can be considered as being embedded in the Pauli and Dirac algebras. We then discuss the importance of the so-called Clifford group and its subgroups and show how they provide the necessary mechanism for ascertaining the transformation properties of various physically relevant elements of the Clifford algebra. Finally, Section 5 deals with the principal result of this study; namely, the proof that models based on spinor irreps of the Dirac algebra or its real subalgebra can always be entirely realized within the strict algebraic framework of the Pauli algebra.

As a concrete physical example, Chapter Two is devoted to expounding Dirac theory in terms of the Pauli algebra. Keeping in mind the more traditional formalism found in the physics literature, Section 1 introduces the Dirac equation in terms of the abstract Clifford algebra $\mathcal{D}$ then employs the results of Section 1.5 to re-express it in various forms within the framework of the Pauli algebra. We then examine the configuration taken by simple solutions, first for a fixed spin polarization and later for a completely general spin polarization. In conjunction with this, we present a brief
discussion of the relativistic spin operator using both the Dirac and Pauli algebras. Finally, we consider the massless Dirac equation in \( \mathbb{P} \) and investigate its possible application to the description of neutrinos.

Section 2 is dedicated to the transformation behavior of the Dirac equation under Lorentz transformations, as well as the discrete operations of parity, time reversal and charge conjugation. Again we take the abstract Clifford algebra approach, relating the formalism, where possible, to the more traditional component-wise description commonly found in the literature. We then complete our study in Section 3 by deriving the Dirac current within the structure of the Pauli algebra.

Given the fairly high degree of mathematical abstraction to be found in this study, we have decided to include two appendices dealing with fundamental definitions and theorems pertaining to the work carried out in Chapter One. Since these concepts are treated extensively in the mathematical literature (Hermann 1974, Greub 1978, Benn and Tucker 1987), we have, in the interest of brevity, omitted proofs. We have also added an appendix giving a brief description of how one might modify the various mathematical constructs in order to work exclusively with the real Dirac algebra, including, as a specific example, the conversion of the real Dirac equation as presented by Hestenes into an equivalent equation in the Pauli algebra.
CHAPTER ONE

MATHEMATICAL FRAMEWORK
1.1 The Pauli Algebra

The Pauli algebra \( \mathcal{P} \) is defined as the Clifford algebra generated by real Euclidean 3-space \( \mathbb{R}^3 \), having as symmetric bilinear form the usual dot product of \( \mathbb{R}^3 \):

\[
g(u,v) = u \cdot v \quad (\forall u,v \in \mathbb{R}^3). \tag{1}
\]

Let \( \{ e_i | i \in \mathbb{N}_3 \} \) be an orthonormal basis for \( \mathbb{R}^3 \) with respect to this inner product. Then any \( v \in \mathbb{R}^3 \) may be written in this basis as

\[
v = v^i e_i, \tag{2}
\]

where \( v^i \in \mathbb{R} (\forall i \in \mathbb{N}_3) \). For simplicity, we employ the Einstein summation convention which means that repeated indices are to be summed over. Thus, for example, Eq. (1.1.2) reads

\[
v^i e_i = \sum_{i=1}^{3} v^i e_i. \tag{3}
\]

Using this convention, the dot product of any two vectors \( u,v \in \mathbb{R}^3 \) expressed in terms of the basis elements \( \{ e_i \} \) becomes

\[
u \cdot v = (u^i e_i) \cdot (v^i e_i)
= u^i v^i (e_i \cdot e_i)
= u^i v^i \delta_{ij} = u^i v^i. \tag{4}
\]

Here, \( \delta_{ij} \) is the Kronecker delta which can be considered the metric associated with the dot product of \( \mathbb{R}^2 \). For any \( v \in \mathbb{R}^3 \),

\[
e_i \cdot v = v^i e_i \cdot e_i = v^i \delta_{ij} = v_i. \tag{5}
\]

The \( \{ v^i \} \) and \( \{ v_i \} \) are called the contravariant and covariant components of the vector \( v \in \mathbb{R}^3 \) in the basis \( \{ e_i \} \) respectively. In the case of \( \mathbb{R}^3 \), the contravariant and covariant components are equal:

\[
v^i = v_i \quad (\forall i \in \mathbb{N}_3). \tag{6}
\]
There being no reason to distinguish between contravariant and covariant components when dealing with $\mathbb{R}^4$, we define the dot product of two vectors $u, v \in \mathbb{R}^3$ in terms of components with respect to any orthonormal basis by

$$u \cdot v = u^i v^i.$$  \hfill (7)

Henceforth, all Latin indices shall be assumed elements of $\mathbb{N}_3$ unless otherwise indicated.

By Theorem A2.3, the basis elements $\{e_i\}$ of $\mathbb{R}^n$ generate a basis for $\mathcal{F}$:

$$\mathcal{F} = \text{span}_R \{1, e_i, e_i e_j | 1 \leq i < j \leq 3\}. \hfill (8)$$

where $e = e_1 e_2 e_3$ is the canonical element of $\mathcal{F}$ and

$$\dim_R(\mathcal{F}) = 2^3 - 8. \hfill (9)$$

One then sees that $\mathcal{F}$, as a vector space, may be written as the direct sum

$$\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3. \hfill (10)$$

where

$$\mathcal{F}_0 = \text{span}_R \{1\} \cong \mathbb{R} \hfill (11a)$$

$$\mathcal{F}_1 = \text{span}_R \{e_i\} \cong \mathbb{R}^3 \hfill (11b)$$

$$\mathcal{F}_2 = \text{span}_R \{e_i e_j | i < j\} \hfill (11c)$$

$$\mathcal{F}_3 = \text{span}_R \{e\}. \hfill (11d)$$

The Clifford relation for $\mathcal{F}$ is

$$uv + vu = 2u \cdot v. \hfill (12)$$

where $u, v \in \mathcal{F}_1$. In terms of the basis elements $\{e_i\}$, Eq. (1.1.12) reads

$$e_i e_j + e_j e_i = 2\delta_{ij}. \hfill (13)$$

where multiplication on the right hand side of Eq. (1.1.13) by 1 is to be understood. This result shows that $e$ is in the center of $\mathcal{F}$ and

$$e e_i = \delta_{ij} + \epsilon_{ijk} e_k. \hfill (14)$$
where \( c_{ij} \) is the totally antisymmetric Levi-Civita density for Euclidean 3-space. Using Eq. (1.1.14), Eq. (1.1.8) may be rewritten as

\[
P = \text{span}_R \{ i \cdot e_1 \cdot e_\rho \cdot e | i \cdot N_3 \}
\]

which yields the vector space direct sum decomposition

\[
P = P_0 \oplus P_1 \oplus \rho P_1 \oplus \rho P_0.
\]

\( P_0 \) and \( P_1 \) being defined as before. Thus, any \( \alpha \in P \) may be written in the form

\[
\alpha = \alpha_0 + \alpha + e\alpha' - e\alpha_0'
\]

where \( \alpha_0, \alpha, \alpha' \in P_0, \alpha, \alpha' \in P_1. \)

Since \( e \in P^c \), Eq. (1.1.17) together with the definition of the involutions \( \zeta, r \) and \( \pi \) given in Appendix 2 show that for every \( \alpha \in P \),

\[
\begin{align*}
\zeta(\alpha) &= \alpha_0 - \alpha - e\alpha' - e\alpha_0' \\
r(\alpha) &= \alpha_0 - \alpha - e\alpha' + e\alpha_0' \\
\pi(\alpha) &= \alpha_0 - \alpha + e\alpha' - e\alpha_0'.
\end{align*}
\]

where

\[
\begin{align*}
\zeta(e) &= e_3 e_2 e_1 = -e_1 e_2 e_3 = -e \\
r(e) &= (-1)^3 e_3 e_2 e_1 = e \\
\pi(e) &= (-1)^3 e_1 e_2 e_3 = -e.
\end{align*}
\]

Clearly, \( \pi \) can be considered to be the composition map

\[
\pi = \zeta \circ r = r \circ \zeta,
\]

and thus specifying the action of \( \zeta \) and \( r \) uniquely determines the action of \( \pi \).

Theorem A2.4 and the Wedderburn Structure Theorem given in Appendix 1 show that \( P \) is simple and

\[
P = \mathcal{C}_{3,0}(R) = C(R) \otimes \mathcal{M}_2(R).
\]
Note, however, that \( \mathcal{F} \) is not central simple. In fact, Theorem A2.5 shows that
\[
\mathcal{F}^c = \text{span}_R \{1, e\} \cong C(R),
\] (22)
where the canonical element satisfies
\[
e^2 = -1. \tag{23}
\]
Moreover, the even subalgebra of \( \mathcal{F} \) is isomorphic to \( H(R) \) and is thus a central simple division algebra.

Given Eq. (1.1.21), Definition A2.8 indicates that \( \mathcal{F} \) may be considered complex by extending the ground field to \( C \):
\[
\mathcal{F}(R) \rightarrow \mathcal{F}(C) \cong M_2(C). \tag{24}
\]
One then sees that the Pauli algebra, as a complex Clifford algebra, is isomorphic to the algebra of \( 2 \times 2 \) complex matrices. Theorem A2.6 then shows \( \mathcal{F} \) to be central simple when viewed over \( C \).

A much more convenient means for realizing \( \mathcal{F} \) as a complex Clifford algebra can be obtained by observing that \( \mathcal{F} \) admits a complex structure. Define the map \( J \in \text{END}(\mathcal{F}) \) such that
\[
J(\alpha) = e\alpha \quad \text{(\( \forall \alpha \in \mathcal{F} \))}. \tag{25}
\]
Then \( J^2 = -1 \) and thus \( J^2 = -1d \) which is the condition for a complex structure as outlined in Definition A2.9. Using Theorem A2.7, complex scalar multiplication in \( \mathcal{F} \) is then defined by
\[
(\alpha + i\beta)\alpha = \alpha\alpha + \beta J(\alpha)
= \alpha\alpha + \beta e\alpha
= (\alpha + \beta e)\alpha. \tag{26}
\]
where \( \alpha, \beta \in \mathbb{R}, \alpha \in \mathcal{F} \). Moreover,
\[
\dim_C(\mathcal{F}) = \frac{1}{2} \dim_R(\mathcal{F}) = 4 \tag{27}
\]
which is entirely consistent with Eq. (1.1.24).

In order to fully exploit this property of the Pauli algebra, it is necessary to reevaluate the action of the involutions $\iota$ and $r$ in terms of the complex structure $J$. Let $\alpha, \beta \in \mathbb{R}, \alpha \in \mathcal{P}$. Define a map $^{-}$ on $\mathcal{P}$ as a complex Clifford algebra in terms of the spatial reversion map $r$ of $\mathcal{P}$ as a real Clifford algebra as follows:

$$
(\alpha - i\beta)\bar{\alpha} = r(\alpha \alpha + \beta J(\alpha))
= \alpha r(\alpha) + \beta r(e\alpha)
= \alpha r(\alpha) + \beta r(\alpha)
= \alpha r(\alpha) + \beta J(r(\alpha))
= (\alpha + i\beta)\bar{\alpha}. \quad (28a)
$$

The map $^{-}$ is an involutory anti-automorphism of $\mathcal{P}(\mathbb{C})$ also called spatial reversion.

Similarly, define a map $^{-}$ on $\mathcal{P}$ as a complex Clifford algebra in terms of the main anti-automorphism $\zeta$ of $\mathcal{P}$ as a real Clifford algebra as follows:

$$
[(\alpha + i\beta)\alpha]^{-} = \zeta(\alpha \alpha + \beta J(\alpha))
= \alpha \zeta(\alpha) + \beta \zeta(e\alpha)
= \alpha \zeta(\alpha) + \beta e \zeta(\alpha)
= \alpha \zeta(\alpha) + \beta J(\zeta(\alpha))
= (\alpha - i\beta)\bar{\alpha}^{-}. \quad (28b)
$$

The map $^{-}$ is a conjugate-linear involution on $\mathcal{P}(\mathbb{C})$ called hermitean conjugation.\footnote{By conjugate-linear we mean $(ca)^{-} = c^* a^{-} \ (\forall c \in \mathbb{C}, \alpha \in \mathcal{P}(\mathbb{C}))$, where $^*$ is complex conjugation.} It is important to realize that, although $(ab)^{-} = b^{-} a^{-} \ (\forall a, b \in \mathcal{P}(\mathbb{C}))$, hermitean conjugation is not an anti-automorphism of $\mathcal{P}(\mathbb{C})$. 

\footnote{By conjugate-linear we mean $(ca)^{-} = c^* a^{-} \ (\forall c \in \mathbb{C}, \alpha \in \mathcal{P}(\mathbb{C}))$, where $^*$ is complex conjugation.}
The complex structure of \( \mathcal{F} \) and Eq. (1.1.26) show that in considering \( \mathcal{F} \) to be complex one essentially identifies the canonical element \( e \in \mathcal{F} \) with the complex scalar \( i \in \mathbb{C} \). Such an identification scheme clearly makes sense in view of Eqs. (1.1.22) and (1.1.23).

By defining the ground field to be complex by means of this identification scheme, Eq. (1.1.15) becomes

\[
\mathcal{F} = \text{span}_c \{ 1, e_i \} \tag{29}
\]

and, in analogy with Eqs. (1.1.10) and (1.1.11), the Pauli algebra, as a complex vector space, decomposes into the direct sum

\[
\mathcal{F} = \mathcal{F}_S \oplus \mathcal{F}_V. \tag{30}
\]

where now

\[
\mathcal{F}_S = \text{span}_c \{ 1 \} \cong \mathbb{C} \tag{31a}
\]

\[
\mathcal{F}_V = \text{span}_c \{ e_i \} \cong \mathbb{C}^3. \tag{31b}
\]

Consequently, any \( \alpha \in \mathcal{F} \) may be written in the form

\[
\alpha = \alpha_0 + \alpha. \tag{32}
\]

where \( \alpha_0 \in \mathcal{F}_S, \alpha \in \mathcal{F}_V. \)

Writing Eq. (1.1.32) in terms of the basis \( \{ 1, e_i \} \) and using Eq. (1.1.28b) one sees that

\[
\alpha^* = \alpha_0^* - \alpha^*
\]

\[
= (\alpha_0 1)^* + (a^i e_i)^*
\]

\[
= \alpha_0^* 1 + a^i e_i. \tag{33}
\]

i.e. hermitean conjugation simply complex conjugates the components in the basis \( \{ 1, e_i \} \) while leaving the basis elements invariant. An arbitrary \( \alpha \in \mathcal{F} \) will be termed hermitean if it satisfies the condition \( \alpha^* = \alpha. \)
In contrast to hermitean conjugation, spatial reversion is seen to be C-linear. In fact, Eq. (1.1.28a) shows that for every $a \in \mathcal{F}$,

$$\overline{\overline{a}} = a_0 + \overline{a} = a_0 - a.$$  \hspace{1cm} (34)

One then concludes that spatial reversion — and the spatial reversion map $r$ are completely equivalent irrespective of the choice of ground field.

In summary, these results show that the Pauli algebra may be viewed as either a real or complex Clifford algebra, the latter viewpoint being obtained from the former by extending the ground field to $\mathbb{C}$ and identifying the canonical element with the complex scalar $i \in \mathbb{C}$. Although there are certain advantages to the real scenario (most notably, the canonical action of the main anti-automorphism $\zeta$), we shall, nevertheless, focus primarily on $\mathcal{F}$ as a complex Clifford algebra, this being the most useful perspective for the description of the physical applications we shall consider.

Let $a, b \in \mathcal{F}$ be two arbitrary vectors in $\mathcal{F}$ considered as a complex Clifford algebra. Expanding in terms of the basis \{l.e.,\} and applying the aforementioned identification scheme to Eq. (1.1.14) we obtain the result (Baylis and Jones 1989[I])

$$ab = a \cdot b + i a \times b,$$ \hspace{1cm} (35)

i.e. the algebra product of two vectors can be written as the sum of scalar and vector parts. Here $\times$ is just the usual cross product of Euclidean 3-space. As a corollary to Eq. (1.1.35), one may show that
\[ a \cdot b = \frac{1}{2} (ab - ba) \]  
\[ a \times b = \frac{1}{2i} (ab - ba) . \]

Eq. (1.1.36a) is simply the Clifford relation for \( \mathbb{F} \).

Having decomposed the product of two vectors in \( \mathbb{F} \) into components in \( \mathbb{F}_s \) and \( \mathbb{F}_v \), it is natural to inquire how the product of two arbitrary \( \mathbb{F} \)-elements breaks down. Since every element of \( \mathbb{F} \) as a complex Clifford algebra can be written as the sum of scalar and vector parts, one must have a relation of the form
\[ ab = (ab)_s + (ab)_v . \]

where \((ab)_s \in \mathbb{F}_s, \ (ab)_v \in \mathbb{F}_v, \ (\forall a, b \in \mathbb{F})\). In order to determine the form taken by \((ab)_s\) and \((ab)_v\), consider the spatial reversion of Eq. (1.1.37):
\[ \overline{ab} = (ab)_s - (ab)_v . \]

Adding and subtracting Eqs. (1.1.37) and (1.1.38) then gives
\[ (ab)_s = \frac{1}{2} (ab + \overline{ab}) \]  
\[ (ab)_v = \frac{1}{2} (ab - \overline{ab}) . \]

where \( \overline{ab} = \overline{b} \overline{a} \). One may also expand \( a \) and \( b \) in terms of their scalar and vector parts to obtain
\[ ab = (a_0 + a)(b_0 + b) \]
\[ = a_0 b_0 + a_0 b + b_0 a - ab \]
\[ = a_0 b_0 + a_0 b + b_0 a + \alpha \cdot b + i \alpha \times b . \]

where we have used Eq. (1.1.35). Collecting scalar and vector parts, one then has
\[(ab)_s = a_0 b_0 + \alpha \cdot b \quad (41a)\]
\[(ab)_v = a_0 b + b_0 a + i\alpha \times b. \quad (41b)\]

We shall often utilize the notation (Baylis and Jones 1989[I])
\[a \cdot b = (ab)_s \quad (42)\]
for the scalar part of the product of two \(\mathbb{F}\)-elements \(a\) and \(b\). Using Eq. (1.1.39a), it is fairly simple to show that
\[a \cdot b = b \cdot a = \overline{b} \cdot \overline{a} = \overline{a} \cdot \overline{b} \quad (43a)\]
\[a \cdot (bc) = (ab) \cdot c = 1 \cdot (abc) \quad (43b)\]
for all \(a, b, c \in \mathbb{F}\). Note that Eq. (1.1.43b) implies the scalar part of the product of an arbitrary number of \(\mathbb{F}\)-elements is unchanged under any cyclic permutation of the elements.

An interesting feature of the complex Pauli algebra is the observation that the product of any \(\mathbb{F}\)-element with its spatial reverse is a pure scalar:
\[\overline{a}a = a\overline{a} = (a\overline{a})_s = a \cdot \overline{a} \quad (\forall a \in \mathbb{F}). \quad (44)\]
and decomposing \(a\) in terms of its scalar and vector parts we obtain
\[\overline{a}a = a_0^2 - \alpha^2 = a_0^2 - \alpha \cdot \alpha. \quad (45)\]
The scalar \(\overline{a}a\) is called the modulus of \(a \in \mathbb{F}\). If \(\overline{a}a = 0\), \(a\) is said to be null. If \(\overline{a}a = 1\), \(a\) is said to be unimodular.

Since \(\mathbb{F}\), as a complex Clifford algebra, is isomorphic to \(\mathcal{H}_2(\mathbb{C})\) (Eq. (1.1.24)), one easily constructs a faithful matrix representation for \(\mathbb{F}\). The most well known representation is obtained by identifying the basis elements
with the $2 \times 2$ identity matrix $I$ and the three Pauli matrices
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (46)\n
Denoting $1 = e_0$, $i = e_0$ and letting Greek indices range over $N^0$, one may establish that the $C$-linear map
\[ \sigma : \mathcal{P} \rightarrow M_2(C) \]
\[ \sigma_\mu \rightarrow o_\mu \] (47)
is a faithful matrix irreducible representation (irrep) for $\mathcal{P}$. In fact, $\sigma$ is an algebra isomorphism for $\mathcal{P}$ and $M_2(C)$.

In the $\sigma$-irrep, any $\alpha \in \mathcal{P}$ takes the form
\[ \alpha = \sigma(\alpha) = \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix}. \] (48)

One may also readily establish the following relations (Baylis 1980):
\[ (\alpha)_S = \frac{1}{2} \text{tr}(\alpha). \] (49a)
\[ \bar{\alpha} \alpha = \text{det}(\alpha) = \text{det}(\bar{\alpha}). \] (49b)
\[ \alpha^* = \bar{\alpha} \quad (49c)\n\]
where $\text{tr}$, $\text{det}$ and $^*$ indicate the matrix operations of trace, determinant and transposition respectively. Moreover, given that $\mathcal{P} \cong M_2(C) \cong \text{END}(C^2)$, any $\alpha \in \mathcal{P}$ can be identified with a linear operator on $C^2$. Eq. (1.1.49c) then shows that $\alpha^*$ corresponds to the adjoint of $\alpha$ with respect to the usual inner product on $C^2$.

Since $\mathcal{P}$ is simple, the corollary to Theorem A1.7 shows that all irreps of $\mathcal{P}$ are equivalent. In terms of matrices, this implies that any matrix irrep of $\mathcal{P}$ may be rewritten in the $\sigma$-irrep by means of a similarity transformation. More
importantly, since \( \sigma \) is an isomorphism, any result proved true in the \( \sigma \)-irrep is correspondingly true in the abstract algebra.

A further characteristic of the simplicity of the Pauli algebra is the observation that \( \mathcal{F} \) has an idempotent decomposition (Theorem A1.5). Moreover, in light of Eq. (1.1.47), Theorem A1.8 shows that the number of primitive idempotents forming the decomposition is two.

In order to determine the primitive idempotents of \( \mathcal{F} \), we need the following definition: A vector \( \hat{n} \in \mathcal{F} \) is said to be a unit vector if \( \hat{n}^2 = \hat{n} \cdot \hat{n} = 1 \).

Let \( \hat{n} \) be any unit vector in \( \mathcal{F} \). We claim that the primitive idempotents of \( \mathcal{F} \) take the form

\[
P_\pm = \frac{1}{2} (1 \pm \hat{n}). \tag{50}
\]

As proof, let \( P \in \mathcal{F} \) be idempotent. Then

\[
P^2 = P
\]

\[
\Rightarrow (P_0 + P)(P_0 + P) = P_0 + P
\]

\[
\Rightarrow P_0^2 + 2P_0 P + P^2 = P_0 + P
\]

and equating scalar and vector parts, we obtain

\[
(1 - 2P_0)P = 0, \quad (1 - P_0)P_0 = P^2.
\]

Neglecting the pathological case where \( P = 0 \), these two equations give rise to the following:

i) \( P_0 = 1 \iff P = 0 \).

and we obtain the trivial identity idempotent.

ii) \( P_0 = \frac{1}{2} \Rightarrow P^2 = \frac{1}{4} \Rightarrow P = \pm \hat{n}/2 \Rightarrow P = \frac{1}{2} (1 \pm \hat{n}) \),

where \( \hat{n} \) is any unit vector in \( \mathcal{F} \).
Consequently, any non-trivial idempotent in \( \mathcal{P} \) must take the form presented in Eq. (1.1.50). Now observe that

\[
\begin{align*}
\text{i) } & P_i P_i \sim \frac{1}{4} (1 \pm \mathbf{n})(1 \mp \mathbf{n}) - \frac{1}{4} (1 - \mathbf{n}^2) - 0 \tag{51a} \\
\text{ii) } & P_i \cdot P_i = 1. \tag{51b}
\end{align*}
\]

i.e. \( P_i \) and \( P_i \) are mutually orthogonal idempotents summing to the identity. Thus, by Definition A1.15, the identity idempotent is non-primitive. Given these results, it is fairly simple to show that \( P_i \) and \( P_i \) are indeed primitive idempotents of \( \mathcal{P} \). Theorem A1.5 then shows that the set \((P_i \cdot P_i)\) forms an idempotent decomposition of \( \mathcal{P} \) and thus \( \mathcal{F} \) as a complex vector space, is the direct sum of two minimal left ideals generated by these primitive idempotents, i.e.

\[
\mathcal{F} = \mathcal{F}_i \oplus \mathcal{F}_i. \tag{52}
\]

where \( \mathcal{F}_i = \mathcal{P} \cdot P_i \).

Let \( \mathbf{n} \in \mathcal{P} \) be a unit vector and suppose \( u \in \mathcal{P} \) is non-null. Then \( u \) has an inverse \( u^{-1} \in \mathcal{P} \) given by

\[
u^{-1} = \frac{\mathbf{u}}{(\mathbf{u} \mathbf{u})}. \tag{53}
\]

Note that if \( u \) is unimodular, \( u^{-1} = \bar{\mathbf{u}} \). If \( u \) also satisfies \( u^* = u^{-1} \), then Eqs. (1.1.47) and (1.1.49c) show that \( u \) may be identified with a unitary \( 2 \times 2 \) complex matrix and the element \( \mathbf{n}' = u \mathbf{n} u^{-1} \) is also a unit vector in \( \mathcal{P} \). Moreover, if \( \mathbf{n} \) is hermitean, so is \( \mathbf{n}' \).

As we shall see in later applications, we can, without loss of generality, restrict the primitive idempotents to be hermitean. Consequently, we define the primitive idempotents of \( \mathcal{P} \) to be

\[
\Omega_* = \frac{1}{2} (1 \pm c). \tag{54}
\]
where \( r \mathcal{P} \) is some fixed hermitean unit vector. Note that if one identifies \( \mathcal{P} \) with \( \text{END}(C^2) \), \( \Omega \) and \( \Omega \) become orthogonal projection operators with respect to the usual inner product on \( C^2 \), in agreement with Proposition A1.6.

It is interesting to note that the subset \( \mathcal{P}_{GL} = \{ \alpha \in \mathcal{P} | \alpha \overline{\alpha} \neq 0 \} \subset \mathcal{P} \) forms a group isomorphic to \( GL(2, \mathbb{C}) \). One then sees that the subset \( \mathcal{P}_U = \{ \alpha \in \mathcal{P}_{GL} | \alpha^{-1} = \alpha^{-1} \} \) forms a subgroup of \( \mathcal{P}_{GL} \) isomorphic to \( U(2) \), and the subset \( \mathcal{P}_{SU} \) of \( \mathcal{P}_U \), consisting of all unimodular \( \mathcal{P}_{SU} \)-elements, forms a subgroup of \( \mathcal{P}_U \) isomorphic to \( SL(2) \).

Perhaps of more significance is the observation that the set of all unimodular \( \mathcal{P} \)-elements forms a group isomorphic to \( SL(2, \mathbb{C}) \), making it identifiable with the double covering group of the restricted (proper, orthochronous) Lorentz group \( L \) (Baylis and Jones 1989[I]). We shall discuss these results further in Section 1.4.

Using Theorem A1.7, Proposition A1.8 and the Frobenius Classification Theorem, one sees that \( \mathcal{P} \) as a complex vector space, decomposes into the direct sum
\[
\mathcal{P} = \mathcal{P} \oplus \mathcal{P} \oplus \mathcal{P} \oplus \mathcal{P} \oplus \mathcal{P} \oplus \mathcal{P}.
\]
(55)

where \( \mathcal{P} = \Omega \mathcal{P} \Omega, \mathcal{P} = \Omega \mathcal{P} \Omega, \mathcal{P} = \Omega \mathcal{P} \Omega, \mathcal{P} = \Omega \mathcal{P} \Omega, \mathcal{P} = \Omega \mathcal{P} \Omega. \)

The subspaces \( \mathcal{P} \) and \( \mathcal{P} \) are one-dimensional division subalgebras of \( \mathcal{P} \) both isomorphic to \( \mathbb{C} \). Moreover,

\[\text{Note that, having fixed } P_0 = \Omega, \text{ the minimal left ideals become } P_0 = P \Omega. \]

\[\text{Note that, having fixed } P_0 = \Omega, \text{ the minimal left ideals become } P_0 = P \Omega. \]
Proposition A1.9 shows that they are generated by \( \Omega_+ \) and \( \Omega_- \) respectively:

\[
\mathcal{F}_+ = \text{span}_c(\Omega_+), \quad \mathcal{F}_- = \text{span}_c(\Omega_-).
\]  

(56)

In particular, one may show that for every \( \alpha \in \mathcal{F}_+ \):

\[
\Omega_+ \alpha \Omega_+ = (\alpha_0 - \alpha \cdot e) \Omega_+ \quad \Omega_- \alpha \Omega_- = (\alpha_0 + \alpha \cdot e) \Omega_-.
\]  

(57)

The orthogonality of the primitive idempotents shows the subspaces \( \mathcal{F}_+ \) and \( \mathcal{F}_- \) to be nilpotent of index 2. Using Theorem A1.9, one may infer the existence of nilpotent elements \( \omega_+ \omega_- \in \mathcal{F} \) such that

\[
\mathcal{F}_+ = \text{span}_c(\omega_+), \quad \mathcal{F}_- = \text{span}_c(\omega_-).
\]  

(58)

where

\[
\omega_+ \omega_- = \Omega_+ \Omega_- = \omega_+ \omega_- = 0.
\]  

(59a, 59b, 59c)

Eqs. (1.1.59) then show that

\[
\omega_+ = \frac{1}{2} (e_1 = e_2).
\]  

(60)

where \( e_1 \) and \( e_2 \) are vectors in \( \mathcal{F} \) satisfying

\[
e_1^2 = -e_2^2 = 1
\]  

(61a)

\[
e_1 e_2 = -e_2 e_1 = e.
\]  

(61b)

Since \( \Omega_+ \) and \( \Omega_- \) are assumed hermitean, Eq. (1.1.59a) yields the result:

\[
\omega_+^* = \omega_+ \Rightarrow e_1^* = e_1, \quad e_2^* = -e_2
\]  

(62)

which is entirely consistent with Eq. (1.1.61b).

Given that \( \mathcal{F}_+ \) and \( \mathcal{F}_- \) are minimal left ideals satisfying \( \mathcal{F}_+ \cap \mathcal{F}_- = \{0\} \), Eqs. (1.1.51) and (1.1.55) show that

\[
\mathcal{F}_+ \cap \mathcal{F}_- = \{0\}, \quad \mathcal{F}_+ \cdot \mathcal{F}_- \cdot \mathcal{F}_- \cdot \mathcal{F}_+ = (\mathcal{F}_+ \cdot \mathcal{F}_+ \cdot \mathcal{F}_+ \cdot \mathcal{F}_+).
\]  

(63)
One also sees from Eqs. (1.1.55), (1.1.56) and (1.1.58) that the set \( \{ \Omega, \omega, \omega', \Omega' \} \) forms a basis for \( \mathcal{F} \). where, by Eqs. (1.1.63), \( \{ \Omega, \omega \} \) and \( \{ \Omega', \omega' \} \) form bases for \( \mathcal{P} \) and \( \mathcal{P} \), respectively.

By specifying the elements \( e_1 \) and \( e_2 \) in terms of the basis elements \( \{ e_i \} \) of \( \mathcal{F} \) one may determine the form taken by the \( \Omega \) and \( \omega \) in a matrix representation. In particular, let

\[
e_1 = e_1, \ e_2 = -i e_2 \Rightarrow e = e_1, e_2 = -i e_1, e_2 = e_2. \tag{6.1}
\]

One easily verifies that such a choice for the \( e_1 \) and \( e_2 \) does indeed satisfy Eqs. (1.1.61) and employing the \( \sigma \)-irrep defined in Eq. (1.1.47) we obtain

\[
\Omega = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \omega' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Omega' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{65}
\]

Moreover, Eqs. (1.1.63) show that elements of the minimal left ideals \( \mathcal{P} \) and \( \mathcal{P} \) correspond to matrices whose elements are identically zero in the right and left columns respectively.

Another interesting feature of Eqs. (1.1.64) and the \( \sigma \)-irrep is the observation that

\[
e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{66}
\]

i.e. the set \( \{ e_1, e_2, e \} \) forms a basis for \( \mathcal{M}_2(\mathbb{R}) \) over the reals. As we shall see, this result is independent of the choice of representation.

### 1.2 The Dirac Algebra

In this and later sections, we will begin to make contact with ideas relevant to physics, often introducing terminology inherent to such a viewpoint. Since many of
these concepts are well discussed in the literature (Halzen and Martin 1984, Ryder 1985, Itzykson and Zuber 1980), we shall develop only those of critical importance to this study.

The real Dirac algebra $D_n$ is the Clifford algebra generated by the Minkowski vector space

$$M^4 = \text{span}_\mathbb{R}\{\gamma_\mu | \mu \in \mathbb{N}_0^3\}$$

having bilinear form

$$g(\gamma_\mu, \gamma_\nu) = g_{\mu\nu}.$$  \hspace{1cm} (2)

where $g_{\mu\nu}$ is the Minkowski metric of signature $(1,3)$.

The elements of $M^4$ are called four-vectors. In this basis, any $\nu \in M^4$ can be written as

$$\nu = \nu^\mu \gamma_\mu.$$  \hspace{1cm} (3)

where $\nu^\mu \in \mathbb{R}$ $(\forall \mu \in \mathbb{N}_0^3)$. Thus, given any $u, \nu \in M^4$, we have

$$g(u, \nu) = g(u^\mu \gamma_\mu, \nu^\nu \gamma_\nu)$$

$$= u^\mu \nu^\nu g(\gamma_\mu, \gamma_\nu)$$

$$= u^\mu \nu^\nu g_{\mu\nu}$$

$$= u^\mu \nu^\nu = u_\nu \nu^\nu.$$  \hspace{1cm} (4)

We also see that for every $\nu \in M^4$,

$$g(\nu, \gamma_\mu) = \nu^\nu g(\gamma_\nu, \gamma_\mu) = \nu^\nu g_{\nu\mu} = \nu_\mu.$$  \hspace{1cm} (5)

where now the covariant components $\{\nu_\mu\}$ are related to the contravariant components $\{\nu^\mu\}$ by

$$\nu_0 = \nu^0, \quad \nu_i = -\nu^i.$$  \hspace{1cm} (6)

Here, the change in sign of the three space-like components is due to the signature of $g$. 

These results, together with Eq. (1.1.7), show that for all \( u, v \in \mathbb{M}^4 \)
\[
g(u, v) = u^u v_u = u^0 v_0 + u^i v_i = u_0 v_0 - u \cdot v
\]  
(7)
and thus \( \mathbb{M}^4 \) may be written as the direct sum
\[
\mathbb{M}^4 = \mathbb{R} \oplus \mathbb{R}^3.
\]  
(8)
where
\[
g(u, v) = g(u_0 + u \cdot v_0 + v) = u_0 v_0 - u \cdot v.
\]  
(9)

One immediately sees that \( \mathbb{M}^4 \) may be identified with a subspace of the Pauli algebra considered as a real vector space. Specifically, this subspace consists of all \( \mathbb{P} \)-elements invariant under the main anti-automorphism \( \zeta \). If one considers \( \mathbb{P} \) as a complex vector space, then \( \mathbb{M}^4 \) may be identified with the subset of all hermitean \( \mathbb{P} \)-elements; however, this is not a complex subspace since multiplication by complex scalars will not necessarily preserve hermiticity. So identified, the results of the previous section show that
\[
g(u, v) = u \cdot \bar{v} \quad (\forall u, v \in \mathbb{M}^4 \subset \mathbb{P}).
\]  
(10)

Such a correspondence between four-vectors and hermitean \( \mathbb{P} \)-elements allows for the description of special relativity in terms of the Pauli algebra (Baylis and Jones 1989[I]). Some important physical examples of four-vectors, considered as hermitean \( \mathbb{P} \)-elements, are

---

3 In this study, we choose units such that \( c = \hbar = 1 \) (Halzen and Martin 1984, Ryder 1985, Itzykson and Zuber 1980).
position
\[ x = t + \xi \]  \hspace{1cm} (11a)

momentum
\[ p = \mathcal{E} + p \]  \hspace{1cm} (11b)

vector potential
\[ A = \phi + A \]  \hspace{1cm} (11c)

current density
\[ J = \rho + J. \]  \hspace{1cm} (11d)

The position four-vector \( x \) can be considered to be the \( \mathcal{C}^\infty \) map
\[ x : \mathcal{M}^4 \to \mathcal{M}^4 \]
\[ x \to x(x), \]
where \( \mathcal{M}^4 \) is Minkowski spacetime. One then interprets \( x \) as an assignment to each point \( x = (x^\mu) = (t, x, y, z) \in \mathcal{M}^4 \) an element of \( \mathcal{M}^4 \subset \mathcal{F} \). In fact, by setting \( e_0 = 1 \) and writing the basis for \( \mathcal{F} \) over \( \mathbb{C} \) as \( \{ e_\mu \} \), one may define \( x(x) \) by
\[ x(x) = x^\mu e_\mu. \]  \hspace{1cm} (12)

Similarly, the vector potential and current density can be interpreted as functions of \( x \in \mathcal{M}^4 \), defined so that their components with respect to any basis are appropriately differentiable maps of \( \mathcal{M}^4 \) into \( \mathbb{R} \), i.e. for every \( x \in \mathcal{M}^4 \)
\[ A(x) = A^\mu(x)e_\mu \]  \hspace{1cm} (13a)
\[ J(x) = J^\mu(x)e_\mu. \]  \hspace{1cm} (13b)
where \( A^\mu(x), J^\mu(x) \in \mathbb{R} \) (\( \forall \mu \in \mathbb{N}_3 \)).

One may also define differentiation in \( \mathcal{M}^4 \subset \mathcal{F} \). For simplicity, consider the \( \mathcal{C}^\infty \), \( \mathcal{M}^4 \) and \( \mathbb{R} \)-valued functions on \( \mathcal{M}^4 \), denoted \( \mathcal{F}(\mathcal{M}^4, \mathcal{M}^4) \) and \( \mathcal{F}(\mathcal{M}^4, \mathbb{R}) \) respectively, and let \( f \in \mathcal{F}(\mathcal{M}^4, \mathcal{M}^4) \). Then for every \( x \in \mathcal{M}^4 \) we have that
\[ f(x) = f^\mu(x)e_\mu = f_0(x) + f(x) \in \mathcal{M}^4 \subset \mathcal{F}. \]  \hspace{1cm} (14)

\[ \text{4 Minkowski space-time can be defined as the manifold } \mathbb{R}^4 \text{ endowed with the Minkowski metric } g \text{ (Schutz 1980).} \]
where \( f''(\mathcal{F}(\mathcal{M}^4, \mathbb{R})) (\forall \mu \in \mathcal{N}^3_\mathbb{R}) \). Since \( \mathcal{M}^4 \) is topologically equivalent to \( \mathbb{R}^4 \), the partial derivative of \( f'' \) with respect to the component \( \mathcal{N}^\nu \) of \( \mathcal{N} \in \mathcal{M}^4 \) is given by
\[
\partial_{\nu} f''(\mathcal{N}) = \frac{\partial}{\partial \mathcal{N}^\nu} f''(\mathcal{N}).
\]
(15)

Clearly, the \( \partial_{\nu} \) can be considered maps from \( \mathcal{F}(\mathcal{M}^4, \mathbb{R}) \) into itself. Since this is true for each \( \mu \in \mathcal{N}^0_\mathbb{R} \), we can extend this to a map \( \partial_{\nu}:\mathcal{F}(\mathcal{M}^4, \mathcal{M}^4) \rightarrow \mathcal{F}(\mathcal{M}^4, \mathcal{M}^4) \) such that for every \( \mathcal{N} \in \mathcal{M}^4 \),
\[
\partial_{\nu} f(\mathcal{N}) = (\partial_{\nu} f''(\mathcal{N})) e_\mu \in \mathcal{M}^4 \subset \mathbb{P}.
\]
(16)

One is then lead to define the differential operator
\[
\partial = \partial_{\nu} e_\mu = \partial_i - \nabla
\]
(17)
which operates on the \( f(\mathcal{N}) \in \mathcal{M}^4 \subset \mathbb{P} \) as
\[
\partial f = (\partial_i - \nabla)(f_\mu + f) = \partial_i f_\mu + \partial_i f - \nabla f_\mu - \nabla f.
\]
(18)

where \( \nabla f = \nabla \cdot f + i \nabla \times f \). Here, we have suppressed the \( \mathcal{N} \) dependance for notational simplicity. Thus defined, the differential operator \( \partial \) behaves like an element of \( \mathbb{P} \). Note, however, that if \( f(\mathcal{N}) \in \mathcal{M}^4 \subset \mathbb{P} \), the element \( \partial f(\mathcal{N}) \in \mathbb{P} \) is not back in the subset of \( \mathbb{P} \) identifiable with \( \mathcal{M}^4 \). We are thus lead to generalize our considerations to functions \( f \in \mathcal{F}(\mathcal{M}^4, \mathbb{P}) \) defined by
\[
f(\mathcal{N}) = f''(\mathcal{N}) e_\mu,
\]
where the \( f'' \) are elements of \( \mathcal{F}(\mathcal{M}^4, \mathbb{C}) \), i.e. the \( \mathbb{C} \)-valued functions on \( \mathcal{M}^4 \). One may then define the operator \( \partial \) to be a mapping from \( \mathcal{F}(\mathcal{M}^4, \mathbb{P}) \) into itself, i.e. for all \( \mathcal{N} \in \mathcal{M}^4 \), \( f \in \mathcal{F}(\mathcal{M}^4, \mathbb{P}) \). we set
\[
\partial f(\mathcal{N}) = (\partial f)(\mathcal{N}) \in \mathbb{P}.
\]
(19)

where \( \partial f \in \mathcal{F}(\mathcal{M}^4, \mathbb{P}) \). Identifying \( \mathcal{M}^4 \) with the set of hermitean \( \mathbb{P} \)-elements, one immediately sees that \( \mathcal{F}(\mathcal{M}^4, \mathcal{M}^4) \subset \mathcal{F}(\mathcal{M}^4, \mathbb{P}) \), enabling us to deal with \( f \in \mathcal{F}(\mathcal{M}^4, \mathcal{M}^4) \) as before. Note that the peculiar form taken by the
differential operator is due to the fact that $\frac{\partial}{\partial \lambda} = \gamma_{\mu}$, i.e. partial differentiation with respect to contravariant components leads to covariant components for the differential operator.

It is customary in the literature to allow the metric tensor $g_{\mu \nu}$ to act on the $M^4$ basis vectors $(\gamma_{\mu})$ as well as components:

$$\gamma_{\mu} = g_{\mu \nu} \gamma^{\nu}. \quad (20)$$

where

$$\gamma^{\nu} = g^{\nu \lambda} \gamma_{\lambda} = g^{\nu \lambda} g_{\lambda \rho} \gamma^{\rho} = g^{\nu \lambda} \delta_{\rho}^{\lambda}. \quad (21)$$

Here,

$$g^{\nu \lambda} g_{\lambda \rho} = g^{\nu \rho} = \delta_{\nu}^{\rho}. \quad (22)$$

where $\delta_{\nu}^{\rho}$ is the four-dimensional Kronecker delta. Given any $u \in M^4$, we then have

$$u = u^{\nu} \gamma_{\nu} = u^{\nu} g_{\nu \rho} \gamma^{\rho} = u_{\nu} \gamma^{\nu}. \quad (23)$$

Actually, the $\{\gamma^{\nu}\}$ form a basis for the dual space of $M^4$:

$$M^4 = \text{span}_R \{\gamma^{\nu}\} \quad (24)$$

defined as the vector space consisting of all $R$-linear maps from $M^4$ into $R$. Since $M^4$ is finite-dimensional, the $R$-linear map

$$\varphi : M^4 \rightarrow M_4^4 \quad u \rightarrow u_{\cdot} \cdot$$

where $u_{\cdot}(v) = g(u, v)$. is a vector space isomorphism. Since the action of $M^4$ on $M^4$ is uniquely specified by the bilinear form $g$, one generally identifies $M^4$ with $M^4$ in the manner indicated by Eqs. (1.2.21), (1.2.22) and (1.2.23). In particular, by defining

$$\gamma^{\nu}(v_{\cdot}) = g^{\nu \rho} = \delta_{\nu}^{\rho}.$$

one sees that for every $u_{\cdot} \in M^4$ and $v \in M^4$
u. \cdot v = u. \cdot v = u. \cdot v = u. \cdot v = g(u, v).

Clearly, Eq. (1.2.21) together with the signature of the metric tensor shows that

\gamma^0 = \gamma_0, \quad \gamma^1 = -\gamma_1. \quad (25)

It is convenient to further extend the relationship between \( M^4 \) and \( \mathcal{F} \). Again letting \( c_0 = 1 \) and writing the basis for \( \mathcal{F} \) as \( \{e_\mu\} \), we see that

\begin{equation}
\text{e}_\mu \cdot \text{e}_\nu = g_{\mu \nu}. \quad (26)
\end{equation}

In particular, letting \( a, b \in \mathcal{F} \) be expanded in terms of the basis \( \{e_\mu\} \) we have

\begin{equation}
a \cdot b = (a^\mu e_\mu) \cdot (b^\nu e_\nu) = a^\mu b^\nu e_\mu \cdot e_\nu = a^\mu b^\nu = a^\mu b_\mu. \quad (27)
\end{equation}

One also observes that

\begin{equation}
e_\mu \cdot e_\nu = \delta_{\mu \nu} \quad (28)
\end{equation}

and since \( g^\mu_\nu = \delta^\mu_\nu \). it is natural to extend the action of the Minkowski metric to the basis elements of \( \mathcal{F} \), i.e. we set

\begin{equation}e^\mu = g^\mu_\nu e_\nu = \text{e}_\mu. \quad (29)
\end{equation}

One then sees that for any \( a \in \mathcal{F} \),

\begin{equation}
a = a^\mu e_\mu = a^\mu g^\mu_\nu e_\nu = a_\nu e^\nu = a_\nu \text{e}_\nu. \quad (30)
\end{equation}

where the contravariant and covariant components of \( a \in \mathcal{F} \) are found by

\begin{equation}
a^\mu = a \cdot \text{e}_\mu, \quad a_\mu = a \cdot \text{e}_\mu. \quad (31)
\end{equation}

As a final example, Eq. (1.2.28) shows that for all \( a, b \in \mathcal{F} \)
\[ u \cdot b = \left( u^\mu b^\nu e_\mu \cdot e_\nu \right) \]
\[ = \left( u^\mu b^\nu \gamma_\mu \right) \gamma_\nu \]
\[ = u^\nu b^\mu \gamma_\mu \gamma_\nu \]
\[ = u^0 b^0 - \alpha^i b^i \]
\[ = u^0 b_0 + \alpha^i b^i. \]  

(32)

These results allow for a much more transparent connection between \( M^4 \), written in the standard component notation, and the component-free notation of the Pauli algebra. See Baylis and Jones (1989[I]) for a more detailed discussion.

Since \( M^4 \) generates the real Dirac algebra, Theorems A2.3 and A2.5 show that the \( M^4 \) basis elements \( \{ \gamma_\mu \} \) may be used to generate a basis for \( \mathcal{D}_R \). Specifically,

\[ \mathcal{D}_R = \text{span}_R \{ 1, \gamma_\mu, \gamma_\mu \gamma_\nu, \gamma_\mu \gamma_\nu \gamma_\lambda, \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\sigma \; | \; 0 \leq \mu < \nu < \lambda < \sigma \}. \]

(33)

where the canonical element \( \gamma = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \) satisfies

\[ \gamma^2 = -1. \]

(34)

Consequently, the real Dirac algebra, as a vector space, decomposes into the direct sum

\[ \mathcal{D}_R = \mathcal{D}_R^0 \oplus \mathcal{D}_R^1 \oplus \mathcal{D}_R^2 \oplus \mathcal{D}_R^3 \oplus \mathcal{D}_R^4. \]

(35)

where

\[ \mathcal{D}_R^0 = \text{span}_R \{ 1 \} = \mathbb{R} \]  

(36a)

\[ \mathcal{D}_R^1 = \text{span}_R \{ \gamma_\mu \} = M^4 \]

(36b)

\[ \mathcal{D}_R^2 = \text{span}_R \{ \gamma_\mu \gamma_\nu \; | \; \mu < \nu \} \]

(36c)

\[ \mathcal{D}_R^3 = \text{span}_R \{ \gamma_\mu \gamma_\nu \gamma_\lambda \; | \; \mu < \nu < \lambda \} \]

(36d)

\[ \mathcal{D}_R^4 = \text{span}_R \{ \gamma \}. \]

(36e)
The Clifford relation for $\mathcal{D}_R$ is
\[ u^\mu v^\nu = 2g(u,v), \]  
where $u \cdot v \in \mathcal{D}_R^1 = M^4$. In terms of the basis elements, the Clifford relation reduces to
\[ g_{\mu \nu} = \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) \]  
and using the definition of the Minkowski metric, one sees that
\[ \gamma_0^2 = 1, \quad \gamma_i^2 = -1 \]  
\[ \gamma_\mu \gamma_\nu = -\gamma_\nu \gamma_\mu \quad (\gamma_\mu \neq \gamma_\nu). \]  

These results show $\mathcal{D}_R^2$ to be a six-dimensional subspace of $\mathcal{D}_R$ identifiable with the space of real antisymmetric second-rank tensors. In fact, one may rewrite Eq. (1.2.36c) as
\[ \mathcal{D}_R^2 = \text{span}_R \{ \gamma_{\mu \nu} | \mu, \nu \in N^2_2 \}. \]  

where
\[ \gamma_{\mu \nu} = \frac{1}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) = \gamma_\mu \gamma_\nu - g_{\mu \nu}. \]  

Any element of $\mathcal{D}_R^2$ may then be written in the form
\[ \frac{1}{2}d^{\mu \nu} \gamma_{\mu \nu}, \]  
the real tensor components being given by
\[ d^{\mu \nu} = a^\mu b^\nu - a^\nu b^\mu. \]

The subspace $\mathcal{D}_R^3$ is seen to be four-dimensional over $R$ and can be identified with the space of real antisymmetric third-rank tensors. Letting $\epsilon_{\mu \nu \lambda \delta}$ be the usual
four-dimensional analog of the totally antisymmetric Levi-Civita density, we see that
\[ \nu_{\mu, \nu, \lambda, \alpha} \nu_{\mu} \wedge \nu_{\nu} \wedge \nu_{\lambda} \wedge \nu_{\alpha} = (\forall \mu, \nu, \lambda, \alpha) (N_3^j) \]
and thus \( \mathcal{D}_R^j \) may be rewritten as
\[ \mathcal{D}_R^j = \text{span}_R \{ \nu_{\mu} \mid \mu \in N_2^0 \}. \]
Note that
\[ \nu_{\mu} \wedge - \nu_{\mu} \nu \quad (\forall \mu \in N_2^0). \]
The subspace \( \mathcal{D}_R^j \) is termed the space of pseudovectors while \( \mathcal{D}_R^+ \), generated by the canonical element \( \nu \), is called the space of pseudoscalars.

Using these results, one may rewrite the basis for \( \mathcal{D}_R \) as
\[ \mathcal{B} = \{ \underline{1}, \nu_{\mu}, \nu_{\nu}, \nu_{\mu, \nu} \mid \mu, \nu \in N_3^j \}. \]
and any \( d \in \mathcal{D}_R \) will be written in this basis as
\[ d = d_0 \underline{1} + d_1 \nu_{\mu} + \frac{1}{2} d_2 \nu_{\mu, \nu} + d_3 \nu_{\mu, \nu} + d_4 \nu. \]
the coefficients all being real. Clearly,
\[ \text{dim}_R(\mathcal{D}_R) = 16 \]
which is in complete agreement with Theorem A2.2.

Theorems A2.4 and A2.5 show that \( \mathcal{D}_R \) is central simple and
\[ \mathcal{D}_R = C_{1, 3}(R) \cong H(R) \otimes \mathcal{M}_2(R). \]
Moreover, Proposition A2.4 shows that the even subalgebra of \( \mathcal{D}_R \) is isomorphic to the real Pauli algebra:
\[ \mathcal{D}_R^e \cong C_{2, 0}(R) = \mathcal{P}(R). \]

---

5 Here, we choose our phase convention so that \( \epsilon_{123} = +1 \).
Recall from the previous section that the even subalgebra of \( \mathcal{F} \), considered as a real Clifford algebra, was isomorphic to the quaternions:

\[
\mathcal{H} = \text{span}_\mathbb{R} \{1, e_1\} = \mathcal{F}'(\mathbb{R}). \tag{52}
\]

Employing the complex structure of \( \mathcal{F} \), one may then identify \( \mathcal{H} \) with a real subset of \( \mathcal{F} \), considered as a complex Clifford algebra, closed under the algebra product of \( \mathcal{F} \). In particular, \( \mathcal{H} \) may be identified with the subset

\[
\mathcal{F}_H = \text{span}_\mathbb{R} \{1, i e_1\} = \{a \in \mathcal{F} | a^* = \overline{a}\} \subset \mathcal{F}. \tag{53}
\]

\( \mathcal{F}_H \) is certainly closed in the sense that

\[
(ab)^* = b^* a^* = \overline{b \overline{a}} = \overline{ab} \quad (\forall a, b \in \mathcal{F}_H).
\]

One also sees that

\[
\mathcal{M}_2(\mathbb{R}) \subset \mathcal{M}_2(\mathbb{C}) \equiv \mathcal{F}.
\]

In fact, Theorem A2.4 shows that \( \mathcal{M}_2(\mathbb{R}) \) is isomorphic to the Clifford algebra \( C_{1,1}(\mathbb{R}) \) which is generated by a two-dimensional real vector space \( \nu = \text{span}_\mathbb{R} \{v_1, v_2\} \) having bilinear form \( g_\nu \) given by

\[
g_\nu(v_1, v_1) = g_\nu(v_2, v_2) = 1, \quad g_\nu(v_1, v_2) = 0.
\]

i.e.

\[
C_{1,1}(\mathbb{R}) = \text{span}_\mathbb{R} \{1, v_1, v_2, v_1 v_2\}.
\]

Comparing with Eqs. (1.1.63), the \( \mathcal{F} \)-elements \( \{1, e_1, e_2, e\} \) are seen to behave in exactly this way and thus constitute a canonical basis for \( \mathcal{M}_2(\mathbb{R}) \) over the reals. Letting

\[
\mathcal{F}_X = \text{span}_\mathbb{R} \{1, e_1, e_2, e\} \subset \mathcal{F}. \tag{54}
\]

Eq. (1.2.50) shows that \( \mathcal{D}_R \) may be identified with \( \mathcal{F}_H \cap \mathcal{F}_X \) which is a real subspace of \( \mathcal{F} \otimes \mathcal{F} \) closed under the induced algebra product.

At this point the reader may well ask why \( \mathcal{D}_R \) is not rather identified with a subalgebra of \( \mathcal{F} \otimes \mathcal{F} \) by viewing \( \mathcal{F} \) as a real Clifford algebra. The answer lies in the presence and interpretation of the element \( i \) in quantum theory. For
example, the components of the four-momentum quantum mechanical operator for a free particle are given by

\[ p_\mu = i \gamma_\mu. \]  

(55)

where \( i \) is almost universally taken to be the unit imaginary of the complex field \( \mathbb{C} \). An alternative approach can be found in the work of Hestenes, where the ground field is taken to be \( \mathbb{R} \) and \( i \) is identified with \( \sqrt{2} \gamma_1 \in \mathcal{D}_N \) (Hestenes 1966, 1982). Naturally, such an approach has serious ramifications for the interpretation of quantum mechanics and we shall not attempt to assess the merit of it; rather we shall proceed with the usual interpretation and consider \( i \) to be the unit imaginary of the complex field \( \mathbb{C} \).

One is thus lead to consider the complexification of \( \mathcal{D}_n \) which we term simply the Dirac algebra \( \mathcal{D} \):

\[ \mathcal{D} = \mathbb{C} \hat{\otimes} \mathcal{D}_n. \]  

(56)

Using Theorem A2.8, \( \mathcal{D} \) is seen to be generated by \( \mathcal{M}_4^2 = \mathbb{C} \hat{\otimes} \mathcal{M}_4^4 \).

and extending the ground field to \( \mathbb{C} \) we have that \( \mathcal{D} = \mathbb{C} \hat{\otimes} \mathcal{D}_R \subset \mathbb{C} \hat{\otimes} \mathbb{C}_{1,3}(\mathbb{R}) \cong \mathbb{C}_4(\mathbb{C}) \). Theorem A2.6 then shows that

\[ \mathcal{D} \cong \mathcal{M}_4(\mathbb{C}). \]  

(57)

i.e. the Dirac algebra is central simple and can be identified with the algebra of \( 4 \times 4 \) complex matrices.

Theorem A2.6 also shows that the even subalgebra of \( \mathcal{D} \) is semi-simple, being the algebra direct sum of \( \mathcal{M}_2(\mathbb{C}) \) with itself:

\[ \mathcal{D}^e \cong \mathcal{M}_2(\mathbb{C}) \hat{\otimes} \mathcal{M}_2(\mathbb{C}). \]  

(58)

---

6 For a discussion of Hestenes' approach, see Appendix 3.
Moreover, this decomposition corresponds to an algebra direct sum of $\mathcal{D}^*$ into two isomorphic simple ideals:

$$\mathcal{D}^* = \mathcal{D}^- \oplus \mathcal{D}^+. \quad (59)$$

where the elements

$$\varphi_s = \frac{1}{2}(1 \pm \gamma_5) \quad (60a)$$

are primitive idempotents of $\mathcal{D}^-$. The $\gamma_i$ and $\gamma_5$ are called the chirality projection operators and the element

$$\gamma_5 = \gamma^5 = -i\gamma$$

(60b)

is termed the chirality operator. Note that

$$\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma^0\gamma^1\gamma^2\gamma^3,$$

where we have used Eq. (1.2.25). One then sees that $\gamma_5$, as defined by Eq. (1.2.60b), is identical to the usual definition given in the literature.

Using Eq. (1.2.50), Theorem A2.8 and the associativity of the $R$-bilinear tensor product, we see that

$$\mathcal{D} = C \otimes \mathcal{D}_R$$

$$= C \otimes (H(R) \otimes \mathcal{M}_2(R))$$

$$= (C \otimes H(R)) \otimes \mathcal{M}_2(R)$$

$$= (C \otimes \mathcal{M}_2(R)) \otimes \mathcal{M}_2(R)$$

and thus, using Eq. (1.1.21), we obtain the result

$$\mathcal{D} \cong \mathcal{F} \otimes \mathcal{M}_2(R). \quad (61)$$

By considering $\mathcal{F}$ to be complex, one may, through Definition A2.8, consider $\mathcal{D}$ to be complex; however, the tensor product remains strictly $R$-bilinear. A much more convenient means for relating $\mathcal{D}$ as a complex Clifford algebra to $\mathcal{F}$ is provided by the observation (see Appendix 1) that

$$\mathcal{M}_4(C) \cong \mathcal{M}_2(C) \otimes \mathcal{M}_2(C).$$
Eqs. (1.1.24) and (1.2.57) then show that the Dirac algebra may be identified with the tensor product of \( \mathcal{F} \) with itself:

\[
\mathcal{D} = \mathcal{F} \otimes \mathcal{F},
\]

where now the tensor product is \( \mathbb{C} \)-bilinear. Moreover, the algebra direct sum decomposition of \( \mathcal{D} \) given in Eq. (1.2.58) may be rewritten as

\[
\mathcal{D} = \mathcal{F} \oplus \mathcal{F}.
\]

As we shall see, Eq. (1.2.62) allows for a more elegant description of Dirac theory. Moreover, its more general character should prove useful in extending our later results to more complicated theories, in particular \( SU(2) \times U(1) \) electroweak theory.

Since we are considering \( \mathcal{D} \) to be complex, the follow-up remarks to Definition A2.8 show that

\[
\mathcal{D} = \text{span}_c \{ I, \gamma_\mu \gamma_\nu, \gamma \gamma_\mu, \gamma | \mu, \nu \in \mathbb{N}_3 \}.
\]

where \( \mathcal{D} \) is generated by

\[
M_c^4 = \text{span}_c \{ \gamma_\mu \}.
\]

Moreover, analogous to Eqs. (1.2.35) and (1.2.36), the Dirac algebra, as a complex vector space, decomposes into the direct sum

\[
\mathcal{D} = \mathcal{D}_0 \oplus \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3 \oplus \mathcal{D}_4.
\]

where

\[
\mathcal{D}_0 = \text{span}_c \{ I \} \cong \mathbb{C}
\]

\[
\mathcal{D}_1 = \text{span}_c \{ \gamma_\mu \} \cong M_c^4
\]

\[
\mathcal{D}_2 = \text{span}_c \{ \gamma_\mu \gamma_\nu \}
\]

\[
\mathcal{D}_3 = \text{span}_c \{ \gamma \gamma_\mu \}
\]

\[
\mathcal{D}_4 = \text{span}_c \{ \gamma \}.
\]
Again, these subspaces satisfy the properties of their real counterparts in \( D_\mathbb{R} \) except that now the ground field is taken to be \( \mathbb{C} \). Clearly,

\[
\dim_\mathbb{C}(D) = 16.
\]  

(68)

Although we have complexified \( D_\mathbb{R} \) in order to properly deal with quantities such as the four-momentum operator, we shall focus much of our attention on the real subset of \( D \) identifiable with \( D_\mathbb{R} \); that is, the subset of \( D \) having real components in the basis \( \mathcal{B} \) defined in Eq. (1.2.47).

Eq. (1.2.62) yields the important result that the Dirac algebra may be identified with \( \mathcal{F} \otimes \mathcal{F} \). The next requirement is a specific algebra isomorphism allowing for such an identification. The search for an isomorphism is greatly simplified by the fact that the algebraic properties of \( D \) are completely determined by Eqs. (1.2.38) and (1.2.39). Consequently, we need only find a mapping \( \psi : D \rightarrow \mathcal{F} \otimes \mathcal{F} \) satisfying

\[
\psi(\gamma_\mu)\psi(\gamma_\nu) + \psi(\gamma_\nu)\psi(\gamma_\mu) = 2g_{\mu\nu}.
\]  

(69)

To this end, let

\[
\psi(\gamma_0) = 1 \otimes e_1, \quad \psi(\gamma_i) = e_i \otimes e_2.
\]

\( e_1,e_2 \in \mathcal{F} \) being the elements defined in Eqs. (1.1.61). One readily establishes that \( \psi \) as defined above, satisfies Eq. (1.2.69). Moreover, since \( \{1,e_1,e_2,1\} \) also forms a basis for \( \mathcal{F} \) over \( \mathbb{C} \) (see the comment following Definition A2.8), one easily verifies that \( \psi \) maps the basis \( \mathcal{B} \) of \( D \) to a basis for \( \mathcal{F} \otimes \mathcal{F} \). Consequently, \( \psi \) may be used to identify \( D \) with \( \mathcal{F} \otimes \mathcal{F} \) and we set

\[
\gamma_0 = 1 \otimes e_1, \quad \gamma_i = e_i \otimes e_2.
\]  

(70)

Clearly, \( 1 = 1 \otimes 1 \) and Eqs. (1.2.21) and (1.2.29) give
\[ \gamma^0 = 1 \otimes e_1 = \gamma_0 \]  
\[ \gamma^1 = e_1 \otimes e_2 = \gamma_1 \]  
\[ \gamma^{+} = e_1 \otimes e_2 - e_2 \otimes e_1 = -\gamma, \]  

in agreement with (1.2.25).

Given Eq. (1.2.57), one easily constructs a faithful matrix representation for \( \mathcal{D} \). Recall the \( \sigma \)-irrep defined in Eq. (1.1.47). Theorems A1.2 and A1.3 show that the C-linear map

\[ \Sigma: \mathbb{F} \otimes \mathbb{F} \to \mathcal{M}_2(\mathbb{C}) \otimes \mathcal{M}_2(\mathbb{C}) = \mathcal{M}_4(\mathbb{C}) \]
\[ a \otimes b \to \Sigma(a \otimes b) = \sigma(a) \otimes \sigma(b) \]  

is a faithful irrep of \( \mathcal{D} \). Moreover, since \( \mathcal{M}_n(\mathbb{C}) \cong \text{END}(\mathbb{C}^n) \) \( (\forall n \in \mathbb{N}) \). Theorem A1.3 together with Eqs. (1.1.24) and (1.2.57) show that \( \Sigma \) is an algebra isomorphism for \( \mathcal{D} \) and \( \mathcal{M}_4(\mathbb{C}) \). Writing \( e_1, e_2 \in \mathbb{F} \) in terms of the basis elements \( \{ e_u \} \) of \( \mathbb{F} \) as specified by Eqs. (1.1.66), the \( \{ \gamma_u \} \) are seen to be represented in the \( \chi \)-irrep by

\[ \gamma_0 = \Sigma(\gamma_0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_i = \Sigma(\gamma_i) = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}. \]  

This irrep of \( \mathcal{D} \) is termed the Weyl representation. By setting \( e_1 = e_3, e_2 = ie_2 \Rightarrow e = e_1 e_2 = e_1 \) (which also satisfies Eqs. (1.1.61)) one obtains the so-called standard representation.

These results may be used to extend the action of hermitean conjugation to \( \mathcal{D} \) in a natural way. Let \( d \in \mathcal{D} \). Then \( d \) may be written as a finite linear combination of elements from \( \mathbb{F} \otimes \mathbb{F} \):

\[ d = \sum_{i \in I} a_i \otimes b_i, \]  

where \( I \) is some countably finite index set and \( a_i, b_i \in \mathbb{F} \) \( (\forall i \in I) \). We then define the hermitean conjugate of \( d \) to be
\[ d^* = \sum_{i \in I} a_i^* \otimes b_i^*. \]  \hfill (75)

Observe that

\[
[(a_1 \otimes b_1)(a_2 \otimes b_2)]^* = (a_1 a_2 \otimes b_1 b_2)^*
\]

\[
= (a_1 a_2)^* \otimes (b_1 b_2)^*
\]

\[
= a_2^* a_1^* \otimes b_2^* b_1^*
\]

\[
= (a_2 \otimes b_2)^*(a_1 \otimes b_1)^*
\]

for every \(a_1, a_2, b_1, b_2 \in \mathcal{F}.\) Since \(-\) on \(\mathcal{F}\) is a conjugate-linear involution, one readily sees that the above defined extension to \(\mathcal{D}\) is also a conjugate-linear involution. Note that, so defined, hermitean conjugation in \(\mathcal{D}\) is also equivalent to taking the complex-conjugate transpose in the matrix theory. Moreover, since \(\mathcal{D} \cong M_4(\mathbb{C}) \cong \text{END}(\mathbb{C}^4),\) the element \(d^* \in \mathcal{D}\) also corresponds to the adjoint of \(d \in \mathcal{D}\) with respect to the usual inner product on \(\mathbb{C}^4.\)

Using Eq. (1.1.64) and the hermiticity of the basis elements \(\{e_\mu\}\) of \(\mathcal{F}.\) one sees from Eqs. (1.2.70) that

\[ y_0^* = y_0, \quad y_\nu^* = -y_\nu. \]  \hfill (76)

Eqs. (1.2.76) clearly show that the subset of \(\mathcal{D}\) identifiable with \(\mathcal{D}_R\) does not correspond to the set of hermitean \(\mathcal{D}\)-elements. In order to determine this subset, define a map \(-: \mathcal{D} \to \mathcal{D}\) by

\[ \tilde{d} = (y_0 \xi(d) y_0)^* \]  \hfill (forall \(d \in \mathcal{D}\)).  \hfill (77)

where \(\xi\) is the main anti-automorphism of \(\mathcal{D}\) (see Definition A2.7). By expanding any \(d \in \mathcal{D}\) in terms of the basis \(\mathcal{B}:\)

\[ d = d_0 l + d_1^r y_\mu + \frac{1}{2} d_2^r y_{\mu \nu} + d_3^r y y_\mu + d_4^r y. \]  \hfill (78)

the coefficients now being complex, one may show that

\[ \tilde{d} = d_0^* l + d_1^r y_\mu + \frac{1}{2} d_2^r y_{\mu \nu} + d_3^r y y_\mu + d_4^r y. \]  \hfill (79)
Consequently, \( \tilde{\alpha} \) is a conjugate-linear involution on \( \mathcal{D} \) which simply complex-conjugaates the components of any \( d \in \mathcal{D} \) in the basis \( \mathcal{E} \). One then concludes that the subset of \( \mathcal{D} \) identifiable with \( \mathcal{D}_R \) with respect to the basis \( \mathcal{E} \) is

\[
\mathcal{D}_R = \{ d \in \mathcal{D} \mid \tilde{d} = d \} \subset \mathcal{D}.
\]  

(80)
i.e. every element of \( \mathcal{D}_R \) has real components with respect to \( \mathcal{E} \). Clearly, if \( d \in \mathcal{D} \) satisfies \( \tilde{d} = -d \), then \( d = id_1 \), where \( d_1 \in \mathcal{D}_R \). Consequently, any \( d \in \mathcal{D} \) may be written

\[
d = \frac{1}{2}(d + \tilde{d}) = \frac{1}{2}(d - \tilde{d}) = d_1 + id_2.
\]

where \( d_1 = \frac{1}{2}(d + \tilde{d}), d_2 = \frac{1}{2}(d - \tilde{d}) \in \mathcal{D}_R \).

As we have seen, the Dirac algebra may be identified with \( \mathbb{F} \otimes \mathbb{F} \), the identification being provided by Eqs. (1.2.70). We now wish to compute all the elements of the basis \( \mathcal{E} \) in terms of this correspondence.

First observe that for all \( a, b \in \mathbb{F} \) Eqs. (1.1.54) and (1.1.62) yield

\[
a \otimes (1 - b \otimes c) = (a + b) \otimes e_+ - (a - b) \otimes e_-
\]

(81a)

\[
a \otimes (c_1 + b \otimes c_2) = (a + b) \otimes \omega_+ - (a - b) \otimes \omega_-
\]

(81b)

Now let \( u \in \mathcal{D}_R \approx \mathbb{M}_2^\mathbb{C} \). Expanding in terms of the \( \{ e_\mu \} \) and employing Eqs. (1.2.70) one sees that

\[
u^\ast e_\mu = u_0 e_0 + u' e_1
\]

\[
= u_0 (1 \otimes e_1) + u' (e_1 \otimes e_2)
\]

\[
= u_0 \otimes e_1 + u \otimes e_2.
\]

(82)

Eq. (1.2.81b) then gives
\[ u''\gamma_\mu = (u_0 + u) \otimes \omega_+ + (u_0 - u) \otimes \omega_- \]
\[ = u \otimes \omega_+ + \bar{u} \otimes \omega_- \quad (83) \]
\[ = u'' \rho_\mu \otimes \omega_+ + u'' \bar{\rho}_\mu \otimes \omega_- \]
\[ = u'' (\rho_\mu \otimes \omega_+ + \bar{\rho}_\mu \otimes \omega_-) \]

and thus we obtain the result
\[ \gamma_\mu = \rho_\mu \otimes \omega_+ + \bar{\rho}_\mu \otimes \omega_- \quad (84b) \]

Eqs. (1.1.61) and (1.2.41) then show that
\[ \gamma_{\mu \nu} = \bar{\gamma}_{\mu \nu} \otimes \Omega_+ + \gamma_{\mu \nu} \otimes \Omega_- \quad (84c) \]

where
\[ e_{\mu \nu} = \frac{1}{2} (e_\mu e_\nu - e_\nu e_\mu) \quad (85) \]

The \( \mathcal{F} \)-element \( e_{\mu \nu} \) is antisymmetric and using the properties of the \( \{e_\mu\} \) one sees that
\[ e_{0i} = -e_i \quad e_{ij} = -\varepsilon_{ijk} e_k \quad (86) \]

The \( \rho_\mu \) are also seen to satisfy
\[ \bar{\rho}_{\mu \nu} = -e_{\mu \nu} = e_{\nu \mu} \quad (87a) \]
\[ e_{\nu \mu} = e_{\mu \nu} = -e_{\nu \mu} \quad (87b) \]
\[ \Rightarrow \bar{\rho}_{\mu \nu} = e_{\mu \nu} \quad (87c) \]

where we have used Eq. (1.2.29).

Since
\[ \gamma = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \]
\[ = (1 \otimes e_1) (e_1 \otimes e_2) (e_2 \otimes e_2) (e_3 \otimes e_2) \]
\[ = (1 \otimes e_1) (e_1 e_2 e_3 \otimes -e_2) \]
\[ = -i \otimes e_1 \quad (88) \]

Eq. (1.2.81a) shows that
\[ \gamma = -i \otimes \Omega_+ + i \otimes \Omega_- \quad (84e) \]
Eqs. (1.2.84b) and (1.1.61) then yield
\[ \gamma_{\nu} = i\sigma_\nu \otimes \omega_- - i\tilde{\sigma}_\nu \otimes \omega_+ \] 
(81d)

Clearly, \( I = 1 \otimes \Omega_+ \oplus 1 \otimes \Omega_- \) and hence, in summary, the elements of \( \mathcal{B} \) are given by
\[ I = 1 \otimes \Omega_+ \oplus 1 \otimes \Omega_- \] 
(81a)

\[ \gamma_{\nu} = i\sigma_\nu \otimes \omega_+ - i\tilde{\sigma}_\nu \otimes \omega_- \] 
(81b)

\[ \gamma_{\nu} = i\sigma_\nu \otimes \Omega_+ + i\tilde{\sigma}_\nu \otimes \Omega_- \] 
(81c)

\[ \gamma_{\nu} = i\sigma_\nu \otimes \omega_- - i\tilde{\sigma}_\nu \otimes \omega_+ \] 
(81d)

\[ \gamma = -i\sigma \otimes \Omega_+ + i\sigma \otimes \Omega_- \] 
(81e)

The chirality operator is
\[ \gamma_5 = -i\gamma = -1 \otimes \Omega_+ \oplus 1 \otimes \Omega_- \] 
(89)

and employing Eq. (1.2.60a), the chirality projection operators become
\[ \gamma_\mu = i \Omega_{\mu} \] 
(90)

Since
\[ \mathcal{D}^- = \text{Span}_c \{ I, \gamma_{\nu}, \gamma \}, \quad \mathcal{D}^- \text{-Span}_c (\gamma_{\mu}, \gamma_{\nu}) \] 
(91)

Eqs. (1.2.84) show that
\[ \mathcal{D}^- = (\mathcal{F} \otimes \Omega_+ \oplus \mathcal{F} \otimes \Omega_-) \Theta_\gamma (\mathcal{F} \otimes \omega_+ \oplus \mathcal{F} \otimes \omega_-) \] 
(92a)

\[ \mathcal{D}^- = (\mathcal{F} \otimes \omega_+ \oplus \mathcal{F} \otimes \omega_-) \Theta_\gamma (\mathcal{F} \otimes \omega_+ \oplus \mathcal{F} \otimes \omega_-) \] 
(92b)

where \( \Theta_\gamma \) denotes a vector space direct sum. Moreover,
\[ (\mathcal{F} \otimes \Omega_+ \oplus \mathcal{F} \otimes \Omega_-)(\mathcal{F} \otimes \Omega_+) = (\mathcal{F} \otimes \Omega_+ \oplus \mathcal{F} \otimes \Omega_-)(\mathcal{F} \otimes \Omega_-) = 0 \] 
(93)

and letting \( \Theta_\Delta \) denote an algebra direct sum, we have
\[ \mathcal{D}^- = \mathcal{F} \Theta_\Delta \mathcal{F} \]

which is simply Eq. (1.2.63).
An important result provided by Eqs. (1.2.84) is that any \( d \in \mathcal{D}_K \) can be written in the form
\[
d = \bar{\Omega} \otimes \Omega - a \otimes \omega + \bar{\omega} \otimes \omega - d \otimes \Omega,
\]
where
\[
\bar{\Omega} \otimes \Omega + a \otimes \Omega \in \mathcal{D}_K^r \tag{95a}
\]
\[
b \otimes \omega + \bar{\omega} \otimes \omega \in \mathcal{D}_K^r. \tag{95b}
\]

To see that this is so, first observe that Eqs. (1.2.84) may be rewritten as
\[
1 = \bar{\Omega} \otimes \Omega + 1 \otimes \Omega.
\]
\[
\gamma_\mu = e_\mu \otimes \omega + \bar{e}_\mu \otimes \omega
\]
\[
\gamma_{\nu \mu} = \bar{e}_{\nu \mu} \otimes \Omega + e_{\nu \mu} \otimes \Omega
\]
\[
\gamma_{\nu \mu} = (ie_\mu) \otimes \omega + (ie_\nu) \otimes \omega
\]
\[
\gamma = i \otimes \Omega + i \otimes \Omega.
\]

Now let \( d \in \mathcal{D}_K \). Then \( d \) takes the form given in Eq. (1.2.48), the coefficients all being real. Using Eqs. (1.2.86) and the antisymmetry of the \( e_{\nu \mu} \) we see that
\[
\frac{1}{2} d_{\nu \mu} e_{\nu \mu} = \frac{1}{2} (d_{0} e_0 + d_{1} e_1 + d_{2} e_2)
\]
\[
= d_{0} e_0 - \epsilon_{ijk} i d_{2} e_k
\]
\[
= -(d_{0} + \epsilon_{ijk} i d_{2}) e_k \in \mathcal{D}_K.
\]

Letting \( d_2 = -(d_{0} e_0 + \epsilon_{ijk} i d_{2}) e_k \), \( d_0 = d_0 + i d_4 \), we have
\[
d = (d_0 \otimes \Omega + d_1 \otimes \omega) + (d_2 \otimes \omega + d_3 \otimes \omega) + (-d_2 \otimes \omega + d_3 \otimes \Omega) +
\]
\[
+ (i d_4 \otimes \omega + i d_3 \otimes \omega) + (-i d_4 \otimes \Omega + i d_3 \otimes \Omega)
\]
\[
= (d_{04} - d_2) \otimes \Omega + (d_1 + i d_3) \otimes \omega + (d_2 - i d_3) \otimes \omega + (d_0 + d_2) \otimes \Omega.
\]
and setting \(a = a_0 + it\), \(b = a_1 + i\alpha\), we obtain Eq. (1.2.94) as required. Notice that \(d\) as defined in Eq. (1.2.94), contains sixteen independent, real parameters. Consequently, by restricting the ground field to \(\mathbb{R}\) and identifying the elements of \(\mathbb{E}\) with elements of \(\mathbb{F} \otimes \mathbb{F}\) as specified by Eqs. (1.2.84), we may identify the set

\[
\text{span}_\mathbb{R} \mathbb{E} = \left\{ d = a^- \otimes \Omega + b \otimes \omega, -b^- \otimes \omega, -a \otimes \Omega, \{a, b \in \mathbb{F}\} \right\}
\]

with the real Dirac algebra.

Note that a general \(D\)-element does not take the form given by Eq. (1.2.94). The problem lies in the presence of the conjugate-linear involution \(-\) in Eq. (1.2.84c). In particular, we see that

\[
\frac{1}{2} d^{\mu\nu} \gamma_{\mu\nu} = \frac{1}{2} d^{\mu\nu} \left( \bar{e}_{\mu\nu} \otimes \Omega + e_{\mu\nu} \otimes \Omega \right)
\]

\[
= \left( \frac{1}{2} d^{\mu\nu} \bar{e}_{\mu\nu} \right) \otimes \Omega - \left( \frac{1}{2} d^{\mu\nu} e_{\mu\nu} \right) \otimes \Omega.
\]

where the coefficients \(d^{\mu\nu}\) are now considered complex. If we employ Eq. (1.2.43), we obtain

\[
\frac{1}{2} a^{\mu\nu} e_{\mu\nu} = \frac{1}{4} (a^{\mu} b^\nu - a^\nu b^\mu) (e_{\mu} \bar{e}_{\nu} - e_{\mu} \bar{e}_{\nu})
\]

\[
= \frac{1}{2} (a b - b a) = (a b)
\]

and thus

\[
\frac{1}{2} d^{\mu\nu} \gamma_{\mu\nu} = (a b) \otimes \Omega - (a b) \otimes \Omega.
\]

Clearly, \((a b) = (a b)\) iff \(a\) and \(b\) are hermitian, i.e. iff the coefficients \(d^{\mu\nu}\) are real.
Nevertheless, Eq. (1.2.94) may be generalized to arbitrary \( D \)-elements. In particular, let \( d \in D \). Then we may write
\[
d = d_1 - id_2.
\]
(96)
where \( d_1, d_2 \in D_R \). Setting
\[
d_i = \overline{a}\bar{\otimes} \Omega + b_2 \otimes \Omega + \overline{b} \otimes \Omega + a_1 \otimes \Omega.
\]
for \( i \in \{1, 2\} \) we obtain
\[
d = \overline{a} \otimes \Omega + b_1 \otimes \Omega + \overline{b} \otimes \Omega + a_1 \otimes \Omega.
\]
(97)
where
\[
a_* = a_1 \pm ia_2, \quad b_* = b_1 \pm ib_2.
\]
(98)
Since \( d = d_1 - id_2 = d_1 - id_2 \), we also see that
\[
d = \overline{a} \otimes \Omega + b_1 \otimes \Omega + \overline{b} \otimes \Omega + a_1 \otimes \Omega.
\]
(99)
One may easily establish the equivalence of Eqs. (1.2.99) and (1.2.77) using the definition of \( \zeta \) given in Appendix 2.

In the Weyl representation, any \( d \in D_R \) takes the form
\[
d = d^{(+)} + d^{(-)} = \begin{pmatrix} \overline{a} & \overline{b} \\ b & a \end{pmatrix} \quad \{a, b \in \mathbb{F}\}.
\]
(100a)
where
\[
d^{(+)} = \begin{pmatrix} \overline{a} & 0 \\ 0 & a \end{pmatrix}, \quad d^{(-)} = \begin{pmatrix} 0 & \overline{b} \\ b & 0 \end{pmatrix}.
\]
(100b)
It is important to keep in mind that these are \( 4 \times 4 \) matrices, the elements \( a = \sigma(a) \), \( \overline{a} = \sigma(\overline{a}) \), \( b = \sigma(b) \) and \( \overline{b} = \sigma(\overline{b}) \) being \( 2 \times 2 \) matrices in the \( \sigma \)-irrep of \( \mathbb{F} \).

Now let \( d \in D_R \) be given by Eq. (1.2.94). The even part \( d^{(+)} \in \mathbb{D}_R \) of \( d \) is then given by (1.2.95a), and applying the chirality projection operators we have that
\[
\gamma_+ d^{(+)} = a \otimes \Omega = 0 \otimes a
\]
(101a)
\[
\gamma_- d^{(-)} = \overline{a} \otimes \Omega = \overline{a} \otimes 0.
\]
(101b)
where we have used the canonical isomorphism

$$
\mu: (\mathcal{F} \otimes \Omega_+) \otimes (\mathcal{F} \otimes \Omega_-) \rightarrow \mathcal{F} \otimes \mathcal{F}
$$

$$
\overline{a} \cdot \Omega_+ + a \cdot \Omega_-. \rightarrow \overline{a} \cdot \Theta a.
$$

(102)

More generally, we see that

$$
\gamma_+ d = a \cdot \Omega_+ + b \cdot \omega_+.
$$

(103a)

$$
\gamma_- d = \overline{a} \cdot \Omega_- + \overline{b} \cdot \omega_-.
$$

(103b)

where we have used Eqs. (1.1.59).

Since $\gamma_+ \gamma_- = 1$, any $d \in \mathcal{D}$ can be written in the form

$$
d = \gamma_+ d + \gamma_- d.
$$

(104)

The elements $\gamma_+ d$ and $\gamma_- d$ are termed the right and left hand components of $d$ respectively. Eqs. (1.2.101) yield the observation that if $a \in \mathcal{F}$ is associated with the right-hand component of $d^{(e)} \in \mathcal{D}_R$, then the corresponding left-hand component is associated with $\overline{a} \cdot \mathcal{F}$. Eqs. (1.2.102) generalize this result to all $\mathcal{D}_R$-elements. We shall explore this relationship further in Section 1.3.

Since the Dirac algebra is simple (and thus semi-simple), Theorems A1.5, A1.8 and the Wedderburn Structure Theorem show that it contains four primitive idempotents forming an idempotent decomposition. Given our identification of $\mathcal{D}$ with $\mathcal{F} \otimes \mathcal{F}$, one is immediately lead to consider the elements

$$
\Omega_+ = \Omega_+ \otimes \Omega_+, \Omega_- = \Omega_- \otimes \Omega_-, \Omega_- = \Omega_+ \otimes \Omega_-. (105)
$$

as possible candidates for this decomposition. That these elements are mutually orthogonal idempotents of $\mathcal{F} \otimes \mathcal{F} \equiv \mathcal{D}$ is clear. Moreover, we see that
\[ \Omega_{--} + \Omega_{-} + \Omega_{-} + \Omega_{--} = \Omega_{-} \oplus \Omega_{-} + \Omega_{-} \oplus \Omega_{-} + \Omega_{-} \oplus \Omega_{-} \]
\[ = (\Omega_{-} + \Omega_{-}) \oplus (\Omega_{-} + \Omega_{-}) \oplus \Omega_{-} \]
\[ = (\Omega_{-} + \Omega_{-}) \oplus (\Omega_{-} + \Omega_{-}) \]
\[ = \mathbf{1} \oplus \mathbf{1} = \mathbf{1}. \]

Thus, it remains to show the \( \Omega_{--}, \Omega_{-}, \Omega_{-} \) and \( \Omega_{--} \) are all primitive. Now since \( \Omega_{-} \) and \( \Omega_{-} \) are primitive idempotents of \( F \), Propositions A1.8 and A1.9 together with the follow-up remarks to the Wedderburn Structure Theorem show that \( \Omega_{-}F\Omega_{-} \) and \( \Omega_{-}F\Omega_{-} \) are division subalgebras of \( F \) both isomorphic to \( C \). In fact, for every \( a, b \in F \), we had that
\[ \Omega_{-} a \Omega_{-} = \alpha \Omega_{-}, \quad \Omega_{-} b \Omega_{-} = \beta \Omega_{-}. \]
where \( \alpha = (a_{0} + \alpha \cdot e), \beta = (b_{0} - b \cdot e) \in C \). Let \( a \oplus b \in F \oplus F \). Then
\[ \Omega_{-} (a \oplus b) \Omega_{-} = (\Omega_{-} \oplus \Omega_{-}) (a \oplus b) (\Omega_{-} \oplus \Omega_{-}) \]
\[ = \Omega_{-} a \Omega_{-} \oplus \Omega_{-} b \Omega_{-} \]
\[ = \alpha \Omega_{-} \oplus \beta \Omega_{-} \]
\[ = \alpha \beta \oplus (\Omega_{-} \oplus \Omega_{-}) = \lambda \Omega_{-} \Omega_{-}. \]

where \( \lambda = \alpha \beta \in C \). In a similar fashion, one may show that
\[ \Omega_{-} (a \oplus b) \Omega_{-} = \lambda \Omega_{-} \]
\[ \Omega_{-} (a \oplus b) \Omega_{-} = \lambda \Omega_{-} \]
\[ \Omega_{-} (a \oplus b) \Omega_{-} = \lambda \Omega_{-} \]

where \( \lambda = \alpha \beta \lambda = \alpha \beta \lambda = \alpha \beta \in C \). One then sees that the subspaces
\[ \Omega_{-} \Omega_{-}, \quad \Omega_{-} \Omega_{-}, \quad \Omega_{-} \Omega_{-}, \quad \Omega_{-} \Omega_{-} \]
are division subalgebras of \( D \) isomorphic to \( C \) and thus, by Propositions A1.8 and A1.9, the elements \( \Omega_{-}, \Omega_{-}, \Omega_{-} \) and \( \Omega_{-} \) are primitive idempotents of \( D \).
Theorem A1.5 then shows that the Dirac algebra decomposes as a vector space into the direct sum
\[ \mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_- \oplus \mathcal{D}_{++} \oplus \mathcal{D}_{--} \]  
(106)
where the subspaces
\[ \mathcal{D}_+ = \mathcal{D}_+ \mathcal{N}_+, \quad \mathcal{D}_- = \mathcal{D}_- \mathcal{N}_-, \quad \mathcal{D}_{++} = \mathcal{D}_{++} \mathcal{N}_{++}, \quad \mathcal{D}_{--} = \mathcal{D}_{--} \mathcal{N}_{--} \]  
(107)
are minimal left ideals of \( \mathcal{D} \).

Note that in the Weyl representation the minimal left ideals of \( \mathcal{D} \) correspond to the four columns of the matrix algebra \( M_4(\mathbb{C}) \approx \mathcal{D} \), where \( \Sigma(\mathcal{D}_+) \) is the left-most column, \( \Sigma(\mathcal{D}_-) \) is the next column to the right, and so on, culminating in the identification of \( \Sigma(\mathcal{D}_-) \) with the right-most column.

1.3 Spinors

The most general definition of spinors applicable to this study is in terms of minimal left ideals of a Clifford algebra. Results outlined in Appendix 1 show that the left regular representation of a Clifford algebra induces a faithful irrep of the algebra on a minimal left ideal. If the Clifford algebra is simple, this induced irrep is termed the spinor representation of the Clifford algebra and the minimal left ideal is called the space of spinors. By Theorem A1.7, the choice of a different minimal left ideal yields an equivalent representation.

If, on the other hand, the Clifford algebra is non-simple, the results of Appendix 2 show that it is semi-simple, being the algebra direct sum of two isomorphic simple ideals. In this case, the regular representation induces a faithful representation on a left ideal which is the algebra direct sum of two minimal left ideals, one lying in each simple component. This induced rep is termed the
spinor representation of the non-simple Clifford algebra and the left ideal is called the space of spinors. The minimal left ideals forming this left ideal are termed semi-spinor spaces and the mapping induced by the regular representation on a minimal left ideal is called a semi-spinor representation of the Clifford algebra. The kernel of a semi-spinor rep is the simple component algebra which does not contain the associated semi-spinor space. One then concludes that the spinor representation of a non-simple Clifford algebra is reducible, being the sum of two inequivalent semi-spinor representations.

Of particular interest is the case where the even subalgebra of a simple Clifford algebra is non-simple. In this case, the irreducible spinor representation of the Clifford algebra induces a faithful, reducible representation of the even subalgebra. This reducible rep is the sum of two inequivalent irreps whose kernels are the different simple ideals of the even subalgebra. We shall again term these irreps of the even subalgebra semi-spinor representations.

Since the Pauli algebra is simple, these results show that the C-linear maps \( \rho_+ : \mathcal{F} \to \text{END}(\mathcal{F}_+) \) and \( \rho_- : \mathcal{F} \to \text{END}(\mathcal{F}_-) \) defined by

\[
\rho_+ (\alpha)(\Psi_+) = \rho_+ \Psi_+ = \alpha \Psi_+ \quad (1a)
\]

\[
\rho_- (\alpha)(\Psi_-) = \rho_- \Psi_- = \alpha \Psi_- \quad (1b)
\]

for all \( \alpha \in \mathcal{F} \), \( \Psi_+ = \Psi \Omega \), \( \in \mathcal{F}_+ \). are equivalent, irreducible spinor representations of \( \mathcal{F} \) on the spinor spaces \( \mathcal{F}_+ \) and \( \mathcal{F}_- \). The irreducibility may be demonstrated by making use of the \( \sigma \)-irrep together with the defining relations presented in
Eq. (1.1.64). In order to show the equivalence, consider the C-linear map
\[ \psi : \mathcal{F}_+ \rightarrow \mathcal{F}_- \]
\[ \psi \Omega_+ \rightarrow \psi \omega_- \] (1)

Since \( \{ \Omega_+, \omega_- \} \) form bases for \( \mathcal{F}_+ \) and \( \mathcal{F}_- \) respectively, we see that
\[ \psi(\Omega_+) = \psi(\Omega_+ \omega_-) = \Omega_+ \omega_- - \omega_- \] (2a)
\[ \psi(\omega_-) = \psi(\omega_- \Omega_+) = \omega_- \omega_+ = \Omega_+ \] (2b)

where we have also used Eqs. (1.1.59). Moreover,
\[ \psi(\rho_+ (\alpha)(\psi \Omega_+)) = \psi(\alpha \psi \Omega_+) = \alpha \psi \omega_- = \rho_+ (\alpha)(\psi \omega_-) \]
\[ = \rho_+ (\alpha)(\psi \Omega_+) \]
for all \( \alpha \in \mathcal{F}_- \). \( \psi \Omega_+ \in \mathcal{F}_- \). Consequently, \( \psi \) is a linear isomorphism intertwining the action of the spinor reps \( \rho_+ \) and \( \rho_- \), thus establishing their equivalence by Definition A1.16. For convenience, we shall define these spinor irreps, generically, as \( \rho_n : \mathcal{F} \rightarrow \text{END}(\mathcal{F}_n) \) where \( \mathcal{F}_n \equiv \mathcal{F} \Omega \) and \( \Omega \in \{ \Omega_+, \Omega_- \} \). We call the \( \mathcal{F}_n \) Pauli spinor spaces, and their elements will be termed Pauli spinors.

Consider now the Dirac algebra. As was seen in the previous section, there are four minimal left ideals \( \mathcal{D}_-, \mathcal{D}_- \mathcal{D}_- \mathcal{D}_- \) and \( \mathcal{D}_- \) of \( \mathcal{D} \). Since \( \mathcal{D} \) is simple, the previous results show that the regular rep induces four equivalent, irreducible spinor reps \( \rho_- : \mathcal{D} \rightarrow \text{END}(\mathcal{D}_-) \), \( \rho_- : \mathcal{D} \rightarrow \text{END}(\mathcal{D}_-) \), \( \rho_- : \mathcal{D} \rightarrow \text{END}(\mathcal{D}_-) \), \( \rho_- : \mathcal{D} \rightarrow \text{END}(\mathcal{D}_-) \) of \( \mathcal{D} \) defined by
\[ \rho_- (d)(\psi_-) = \rho_- \psi_- = d \psi_- \]
\[ \rho_- (d)(\psi_-) = \rho_- \psi_- = d \psi_- \] (4a)
\[ \rho_- (d)(\psi_-) = \rho_- \psi_- = d \psi_- \]
\[ \rho_- (d)(\psi_-) = \rho_- \psi_- = d \psi_- \] (4b)
The minimal left ideals are termed Dirac spinor spaces and the elements \( \psi = \psi \Omega \in \mathcal{D} \), \( \psi = \psi \Omega \in \mathcal{D} \), \( \psi = \psi \Omega \in \mathcal{D} \), \( \psi = \psi \Omega \in \mathcal{D} \) are called Dirac spinors. Again, we shall denote these spinor irreps as \( \rho_{\Omega} : \mathcal{D} \to \text{END}(\mathcal{D}_{\Omega}) \). where \( \Omega, \Omega' \in \{\Omega, \Omega\} \).

The results of the previous section together with Theorems A1.1 and A1.2 show that the spinor irreps of \( \mathcal{D} \) may be expressed in terms of the spinor irreps of \( \mathcal{P} \). In particular, we have

\[
\rho_{\Omega} = \rho \Omega \otimes \rho \Omega \quad (\forall \Omega, \Omega' \in \{\Omega, \Omega\}). \tag{5}
\]

For example, \( \rho_{\Omega} \) is given by

\[
\rho_{\Omega} : \mathcal{P} \otimes \mathcal{P} \to \text{END}(\mathcal{P} \otimes \mathcal{P})
\]

\[
\alpha \otimes b \to \rho_{\alpha \otimes b} = \rho_{\alpha} \otimes \rho_{b}. \quad (6a)
\]

where, for all \( a, b \in \mathcal{P} \), \( \phi \Omega \in \mathcal{P} \), \( \chi \Omega \in \mathcal{P} \),

\[
\rho_{\Omega}(a \otimes b)(\phi \Omega \otimes \chi \Omega) = \rho_{\alpha \otimes b}(\phi \Omega \otimes \chi \Omega)
\]

\[
= (a \otimes b)(\phi \Omega \otimes \chi \Omega)
\]

\[
= a\phi \Omega \otimes b\chi \Omega
\]

\[
= \rho_{a} \phi \Omega \otimes \rho_{b} \chi \Omega
\]

\[
= \rho_{a}(\phi \Omega) \otimes \rho_{b}(\chi \Omega). \quad (6b)
\]

\( \rho_{\Omega}, \rho_{\Omega} \), and \( \rho_{\Omega} \) being defined in a similar fashion.

The irreducibility of each Dirac spinor rep follows from the irreducibility of the Pauli spinor reps and Theorem A1.2. To show the equivalence of these spinor irreps of \( \mathcal{D} \), define the \( \mathbb{C} \)-linear maps
\[ \psi_+ : \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F}. \]
\[ \psi_- : \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F}. \]
\[ \psi_\uparrow : \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F}. \]
\[ \psi_\downarrow : \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F}. \]

such that for every \( \Phi, \chi \in \mathcal{F} \).
\[ \psi_+ (\Phi \otimes \chi) = \psi(\Phi \otimes \chi) = \phi \omega \otimes \chi. \]
\[ \psi_- (\Phi \otimes \chi) = \psi^{-1}(\Phi \otimes \chi) \cdot \text{id}(\chi) = \phi \omega \otimes \chi. \]
\[ \psi_\uparrow (\Phi \otimes \chi) = \psi(\Phi \otimes \chi) = \phi \omega \otimes \chi. \]
\[ \psi_\downarrow (\Phi \otimes \chi) = \psi^{-1}(\Phi \otimes \chi) \cdot \text{id}(\chi) = \phi \omega \otimes \chi. \]

where \( \psi^{-1} \) is the inverse of the linear isomorphism \( \psi : \mathcal{F} \rightarrow \mathcal{F} \),
and \( \text{id} \) is the identity map. This clearly shows the
\( \psi_+ \), \( \psi_- \), \( \psi_\uparrow \) and \( \psi_\downarrow \) to be linear isomorphisms and thus the
Dirac spinor spaces are pair-wise isomorphic as vector spaces. It remains to show they are intertwining operators.
To this end, consider, for example, \( \psi_- \). We see that
\[ \psi_- (\rho_+ (\alpha \otimes b) (\Phi \otimes \chi)) = \psi_- (\rho_+ (\alpha \otimes \omega \otimes \chi \omega)). \]
\[ = \alpha \Phi \omega \otimes b \chi \omega. \]
\[ = \alpha \Phi \omega \otimes b \chi \omega. \Omega. \]
\[ = \rho_+ (\alpha \otimes b) (\Phi \omega \otimes \chi \omega \cdot \Omega). \]
\[ = \rho_+ (\alpha \otimes b) (\Phi \omega \otimes \chi \omega). \]
\[ = \rho_+ (\alpha \otimes b) (\psi_- (\Phi \otimes \chi \omega)) \]

for all \( \alpha, b \in \mathcal{F}, \Phi, \chi \in \mathcal{F}, \chi \omega \in \mathcal{F} \), and thus \( \psi_- \) intertwines the
action of \( \rho_- \) and \( \rho_- \). The three remaining maps may also be
shown to be intertwining operators for their respective
spinor reps and thus all four spinor irreps of \( \mathcal{D} \) are
equivalent.
An important property of the Dirac algebra is that irreducible representations of $\mathcal{D}$ induce irreducible representations of $\mathcal{D}_R$. This is because the division algebra occurring in the Wedderburn decomposition of the real Dirac algebra is the quaternions (see Eq. (1.2.50)). In fact, it may be shown that a complex irrep of the complexification of a real Clifford algebra, having as division algebra either $\mathbb{C}$ or $\mathbb{H}$ in its Wedderburn decomposition, induces a complex irrep of the real Clifford algebra (Benn and Tucker 1987). One then concludes that the spinor irreps $\rho_{\alpha\alpha'}$ of $\mathcal{D}$ induce equivalent, complex spinor irreps of $\mathcal{D}_R \subset \mathcal{D}$, defined by simply restricting the $\rho_{\alpha\alpha'}$ to $\mathcal{D}_R \subset \mathcal{D}$.

We shall also denote the induced $\mathcal{D}_R$ spinor irreps by $\rho_{\alpha\alpha'}$, expressly stating upon which Clifford algebra these symbols act when any confusion could arise. Since all physically significant $\mathcal{D}$-elements take the form $d$ or $id$ where $d \in \mathcal{D}_R$, and since the spinor irreps of $\mathcal{D}$ are $\mathbb{C}$-linear, we may focus our attention on the induced spinor irreps of $\mathcal{D}_R$. Indeed, let $d \in \mathcal{D}_R \subset \mathcal{D}$, $\psi \in \mathcal{D}_\alpha$, and consider the spinor irrep $\rho_{\alpha\alpha'}$ of $\mathcal{D}$. Clearly,

$$\rho_{\alpha\alpha'}(id)(\psi) = i\rho_{\alpha\alpha'}(d)(\psi)$$

and since $\rho_{\alpha\alpha'}(d) = \rho_{\alpha\alpha'}(id)(d)$, we may identify $\rho_{\alpha\alpha'}$ appearing on the right hand side of the above equation with the corresponding spinor irrep of $\mathcal{D}_R \subset \mathcal{D}$. Since this spinor-induced irrep is complex, the left multiplication by $i \in \mathbb{C}$ is well defined. Notice that when making the transition from spinor irreps of $\mathcal{D}$ to spinor irreps of $\mathcal{D}_R$, the complex scalar $i$ must be associated with the Dirac spinor space, i.e.

$$i\rho_{\alpha\alpha'}(d)(\psi) = i(\rho_{\alpha\alpha'}(d)(\psi)) = \rho_{\alpha\alpha'}(d)(i\psi).$$
This is because the spinor irreps of $D_R \subset D$ are only $R$-linear; that is $\rho_{uu}(id)$ is only meaningful when $\rho_{uu}$ is a spinor irrep of $D$.

Now let $\psi \in D_R$. By Eq. (1.2.94), $\psi$ may be written in the form

$$\psi = \Xi \otimes \omega + \Pi \otimes \omega + \Xi \otimes \omega .$$

(9)

where $\Xi, \Pi \subset \mathcal{H}$. Using Eqs. (1.2.106) and (1.1.59), we may decompose $\psi$ into the sum

$$\psi = \psi_+ + \psi_- + \psi_{--} + \psi_{-+} .$$

(10)

where

$$\psi_+ = \Xi \otimes \omega + \Pi \otimes \omega \in D_+ .$$

(11a)

$$\psi_- = \Xi \otimes \omega + \Pi \otimes \omega \in D_- .$$

(11b)

$$\psi_{--} = \Xi \otimes \omega + \Xi \otimes \omega \in D_{--} .$$

(11c)

$$\psi_{-+} = \Xi \otimes \omega + \Xi \otimes \omega \in D_{-+} .$$

(11d)

Although the element $\psi$ contains sixteen independent real parameters, each $\psi_{\alpha \gamma} \in D_{R \alpha \gamma}$ can be interpreted as containing four independent complex parameters — two coming from each Pauli spinor composing $\psi_{\alpha \gamma}$. Consequently, the subset $D_{R \alpha \gamma} \subset D_{\alpha \gamma}$ is four-dimensional over the complexes and, extending the ground field to $\mathbb{C}$, we may identify $D_{R \alpha \gamma}$ with $D_{\alpha \gamma}$. The $\psi_{\alpha \gamma}$ listed in Eqs. (1.3.11) can now be taken to represent Dirac spinors.

Consider now the even subalgebra of $D_R \subset D$. Recall from Section 1.2 that $D^+$ could be written as

$$D^+ = (\mathcal{F} \otimes \Omega_+) \otimes (\mathcal{F} \otimes \Omega_-) \mathcal{F} \otimes \mathcal{F} .$$

(12)

the elements of $D^+_R$ taking the form

$$d^{(+)} = \overline{a} \otimes \Omega_+ + a \otimes \Omega_- = \overline{a} \otimes \alpha .$$

(13)
This shows that $\mathcal{D}_\uparrow$ when viewed as a complex subspace of $\mathcal{D}_-$ is non-simple and thus the spinor irreps of $\mathcal{D}_\uparrow$ induce reducible reps of $\mathcal{D}_\uparrow$ each being the sum of two inequivalent, irreducible semi-spinor reps.

To see this, consider, for example, the spinor irrep $\rho_-\uparrow$ of $\mathcal{D}_\uparrow$. We first observe, via Eq. (1.3.11c), that the spinor space $\mathcal{D}_-$ can be decomposed as a vector space into the direct sum

$$\mathcal{D}_- = \mathcal{D}_-^{(L)} \oplus \mathcal{D}_-^{(R)}.$$ 

where

$$\mathcal{D}_-^{(L)} = \mathcal{F}_- \otimes \omega_- = \mathcal{F}_-.$$

$$\mathcal{D}_-^{(R)} = \mathcal{F}_- \otimes \Omega_- = \mathcal{F}_-.$$

The elements of these subspaces are also seen to take the form

$$\psi_-^{(L)} = \overline{H}\Omega_- \otimes \omega_- \in \mathcal{D}_-^{(L)} \quad (14a)$$

$$\psi_-^{(R)} = \Xi \Omega_\uparrow \otimes \omega_- \in \mathcal{D}_-^{(R)} \quad (14b).$$

Denoting the induced $\mathcal{D}_\uparrow$-rep by $\rho_-^{(\uparrow)} : \mathcal{D}_\uparrow \to \text{END}(\mathcal{D}_-)$, and using Eqs. (1.3.11c) and (1.3.13), we see that

$$\rho_-^{(\uparrow)}(d_-^{(\uparrow)}(\psi_-)) = d_-^{(\uparrow)}\psi_-$$

$$= (\alpha^- \otimes \Omega_- + \alpha \otimes \Omega_-)(\overline{H}\Omega_- \otimes \omega_- + \Xi \Omega_\uparrow \otimes \omega_-)$$

$$= \overline{H}\Omega_- \otimes \omega_- + a \Xi \Omega_\uparrow \otimes \omega_-$$

$$= \rho_\uparrow(\overline{H}\Omega_- \otimes \omega_-) + \rho_\uparrow(\alpha)(\Xi \Omega_\uparrow \otimes \omega_-)$$

$$= (\rho_\uparrow(\overline{a} \otimes \text{id}) \overline{H}\Omega_- \otimes \omega_-) + (\rho_\uparrow(\alpha) \otimes \text{id})(\Xi \Omega_\uparrow \otimes \omega_-).$$

where id is the identity map. Consequently, $\rho_-^{(\uparrow)}$ is
reducible and can be decomposed into the sum of two reps:
\[ \rho^{(\ell)} = \rho^{(\ell)} \oplus \rho^{(\ell^*)} \]

where, using Eqs. (1.3.12) and (1.3.13),
\[ \rho^{(\ell)} : \mathcal{D}_R \rightarrow \text{END}(\mathcal{D}^{(\ell)}), \quad \rho^{(\ell^*)} : \mathcal{D}_R \rightarrow \text{END}(\mathcal{D}^{(\ell^*)}) \]
\[ \overline{a} \Theta a \rightarrow \rho_a \otimes \text{id} \quad \overline{a} \Theta a \rightarrow \rho_a \otimes \text{id}. \]

Since the spinor rep \( \rho \) of \( \mathcal{P} \) is irreducible, so are \( \rho^{(\ell)} \) and \( \rho^{(\ell^*)} \). Moreover, given that the transformation \( a \rightarrow \overline{a} \) is conjugate-linear, one easily verifies that \( \rho^{(\ell)} \) and \( \rho^{(\ell^*)} \) are inequivalent.

The actions of these semi-spinor irreps are given explicitly by
\[ \rho^{(\ell)}(\overline{a} \Theta a)(\overline{\omega} \otimes \omega) = \overline{a} \overline{\omega} \otimes \omega. \]
\[ \rho^{(\ell^*)}(\overline{a} \Theta a)(\overline{\omega} \otimes \omega) = a \overline{\omega} \otimes \omega. \]

where \( \overline{\omega} \otimes \omega \in \mathcal{D}^{(\ell)} \). Clearly, \( \rho^{(\ell^*)} \) is faithful.

One may use similar arguments to show that the remaining reps of \( \mathcal{D}_R \) induced by the spinor irreps of \( \mathcal{D}_R \) are also faithful and reducible, each being the sum of two inequivalent, irreducible semi-spinor reps. Moreover, given the equivalence of the spinor irreps of \( \mathcal{D}_R \), the induced reps of \( \mathcal{D}_R \) are also seen to be equivalent.

To summarize, the spinor irreps of \( \mathcal{D}_R \) induce four faithful, equivalent, reducible reps of \( \mathcal{D}_R \):
\[ \rho^{(+)} : \mathcal{D}_R \rightarrow \text{END}(\mathcal{D}^{+}) \quad (15a) \]
\[ \rho^{(-)} : \mathcal{D}_R \rightarrow \text{END}(\mathcal{D}^{-}) \quad (15b) \]
\[ \rho^{(+)} : \mathcal{D}_R \rightarrow \text{END}(\mathcal{D}^{-}) \quad (15c) \]
\[ \rho^{(-)} : \mathcal{D}_R \rightarrow \text{END}(\mathcal{D}^{-}) \quad (15d) \]
having decompositions
\[ \rho^{(l)} = \rho^{(l)} \oplus \rho^{(r)} \]  
\[ \rho^{(r)} = \rho^{(l)} \oplus \rho^{(r)} \]  
\[ \rho^{(l)} = \rho^{(l)} \oplus \rho^{(r)} \]  
\[ \rho^{(r)} = \rho^{(l)} \oplus \rho^{(r)} \]  
\[ \rho^{(l)} = \rho^{(l)} \oplus \rho^{(r)} \]  
\[ \rho^{(r)} = \rho^{(l)} \oplus \rho^{(r)} \]  

\text{corresponding to the vector space direct sum decompositions}
\[ \mathcal{D}_p = \mathcal{D}_p^{(l)} \oplus \mathcal{D}_p^{(r)} = (\mathcal{F}_p \otimes \Omega_p) \oplus (\mathcal{F}_p \otimes \omega_p) \]  
\[ \mathcal{D}_p = \mathcal{D}_p^{(l)} \oplus \mathcal{D}_p^{(r)} = (\mathcal{F}_p \otimes \Omega_p) \oplus (\mathcal{F}_p \otimes \omega_p) \]  
\[ \mathcal{D}_p = \mathcal{D}_p^{(l)} \oplus \mathcal{D}_p^{(r)} = (\mathcal{F}_p \otimes \omega_p) \oplus (\mathcal{F}_p \otimes \Omega_p) \]  
\[ \mathcal{D}_p = \mathcal{D}_p^{(l)} \oplus \mathcal{D}_p^{(r)} = (\mathcal{F}_p \otimes \omega_p) \oplus (\mathcal{F}_p \otimes \Omega_p) \]  

The semi-spinor irreps occurring in Eqs. (1.3.16) satisfy all the same properties outlined in the previous example and can be defined generically by
\[ \rho^{(l)}_{\Omega\Omega'} : \mathcal{D}_p^{(l)} \rightarrow \text{END}(\mathcal{D}_p^{(l)}) \]  
\[ \rho^{(r)}_{\Omega\Omega'} : \mathcal{D}_p^{(r)} \rightarrow \text{END}(\mathcal{D}_p^{(r)}) \]  
\[ \bar{a}^{\prime} \otimes a \rightarrow \rho^{(l)}_{\bar{a}^{\prime}} \otimes \text{id} \]  
\[ \bar{a}^{\prime} \otimes a \rightarrow \rho^{(r)}_{\bar{a}^{\prime}} \otimes \text{id} \]  

where \( \Omega, \Omega' \in \{\Omega_p, \Omega_w\} \). The actions of the semi-spinor irreps are given by
\[ \rho^{(l)}_{\bar{a}^{\prime}} (\bar{a}^{\prime} \otimes a)(\psi^{(l)}) = \rho^{(l)}_{\bar{a}^{\prime}} \otimes \text{id} \left( \bar{a}^{\prime} \otimes \Omega_p, \otimes \Omega_p \right) = \bar{a}^{\prime} \otimes \Omega_p \otimes \Omega_p \]  
\[ \rho^{(r)}_{\bar{a}^{\prime}} (\bar{a}^{\prime} \otimes a)(\psi^{(r)}) = \rho^{(r)}_{\bar{a}^{\prime}} \otimes \text{id} \left( \bar{a}^{\prime} \otimes \Omega_r, \otimes \Omega_r \right) = \bar{a}^{\prime} \otimes \Omega_r \otimes \Omega_r \]  
\[ \rho^{(l)}_{\bar{a}^{\prime}} (\bar{a}^{\prime} \otimes a)(\psi^{(l)}) = \rho^{(l)}_{\bar{a}^{\prime}} \otimes \text{id} \left( \bar{a}^{\prime} \otimes \Omega_l, \otimes \Omega_l \right) = \bar{a}^{\prime} \otimes \Omega_l \otimes \Omega_l \]  
\[ \rho^{(r)}_{\bar{a}^{\prime}} (\bar{a}^{\prime} \otimes a)(\psi^{(r)}) = \rho^{(r)}_{\bar{a}^{\prime}} \otimes \text{id} \left( \bar{a}^{\prime} \otimes \Omega_r, \otimes \Omega_r \right) = \bar{a}^{\prime} \otimes \Omega_r \otimes \Omega_r \]  
\[ \rho^{(l)}_{\bar{a}^{\prime}} (\bar{a}^{\prime} \otimes a)(\psi^{(l)}) = \rho^{(l)}_{\bar{a}^{\prime}} \otimes \text{id} \left( \bar{a}^{\prime} \otimes \Omega_l, \otimes \Omega_l \right) = \bar{a}^{\prime} \otimes \Omega_l \otimes \Omega_l \]  
\[ \rho^{(r)}_{\bar{a}^{\prime}} (\bar{a}^{\prime} \otimes a)(\psi^{(r)}) = \rho^{(r)}_{\bar{a}^{\prime}} \otimes \text{id} \left( \bar{a}^{\prime} \otimes \Omega_r, \otimes \Omega_r \right) = \bar{a}^{\prime} \otimes \Omega_r \otimes \Omega_r \]
\[ \rho^{(l)}(\overline{\alpha} \otimes \alpha)(\psi^{(l)}) = \rho_{\overline{\alpha}} \otimes 1 \Omega \otimes \omega \cdot \psi^{(l)} = \overline{\alpha} \cdot \Omega \otimes \omega \cdot \psi^{(l)} \]  
\[ \rho^{(k)}(\overline{\alpha} \otimes \alpha)(\psi^{(k)}) = \rho_{\overline{\alpha}} \otimes 1 \Omega \otimes \omega \cdot \psi^{(k)} = \overline{\alpha} \cdot \Omega \otimes \omega \cdot \psi^{(k)} \]

where \( \psi^{(l)} = \overline{\alpha} \cdot \Omega \otimes \omega \cdot \psi^{(l)} \), \( \psi^{(k)} = \overline{\alpha} \cdot \Omega \otimes \omega \cdot \psi^{(k)} \) and \( \psi^{(l)} = \overline{\alpha} \cdot \Omega \otimes \omega \cdot \psi^{(l)} \), \( \psi^{(k)} = \overline{\alpha} \cdot \Omega \otimes \omega \cdot \psi^{(k)} \).

The kernels of the semi-spinor irreps of \( D \) are given by

\[ \ker(\rho^{(l)}_{\overline{\alpha} \alpha}) = \{ 0 \otimes \alpha! \alpha \in F \} \]
\[ \ker(\rho^{(k)}_{\overline{\alpha} \alpha}) = \{ \overline{\alpha} \otimes 0 \overline{\alpha} \mid \overline{\alpha} \in F \}. \]

where \( \Omega, \Omega' \in \{ \Omega, \Omega' \} \). Comparing with Eqs. (1.2.101), we see that

\[ \ker(\rho^{(l)}_{\overline{\alpha} \alpha}) = \gamma \cdot D \quad \ker(\rho^{(k)}_{\overline{\alpha} \alpha}) = \gamma \cdot D \]

yielding the observation that Eqs. (1.3.16) represent a decomposition of the \( D \)-reps into inequivalent semi-spinor irreps having distinct, opposing chiralities. Consequently, Eqs. (1.3.17) correspond to a decomposition of Dirac spinors into semi-spinors having distinct and opposing chiralities. In particular, letting the chirality projection operators \( \gamma \) and \( \gamma \) defined in Eqs. (1.2.90), act on the Dirac spinors presented in Eqs. (1.3.11), we see that

\[ \gamma \cdot \psi^{(l)} = \overline{\alpha} \cdot \Omega \cdot \omega \cdot \psi^{(l)} \in D^{(l)} \]
\[ \gamma \cdot \psi^{(k)} = \overline{\alpha} \cdot \Omega \cdot \omega \cdot \psi^{(k)} \in D^{(k)} \]
\[ \gamma \psi = \overline{\psi} \Omega \otimes \Omega = \psi^{(r)} \in \mathcal{D}^{(r)} \]  
(28a)

\[ \gamma \psi = \overline{\psi} \Omega \otimes \omega = \psi^{(l)} \in \mathcal{D}^{(l)} \]  
(28b)

We shall term the semi-spinors \( \psi^{(l)} \) and \( \psi^{(r)} \) left and right Weyl spinors respectively.

The Weyl spinors are seen to be eigenstates of the chirality operator \( \gamma_5 \in \mathcal{D} \), i.e.

\[ \gamma_5 \psi^{(r)} = \psi^{(r)}, \quad \gamma_5 \psi^{(l)} = -\psi^{(l)}. \]  
(29)

Moreover, since the semi-spinor spaces of \( \mathcal{D} \) are isomorphic as vector spaces to Pauli spinor spaces, one observes that right Weyl spinors correspond to Pauli spinors of the form \( \phi \Omega \), whereas left Weyl spinors correspond to Pauli spinors of the form \( \overline{\chi} \Omega \). Based on the observations made in Section 1.2 regarding the notion of chirality, we conclude that right-handed \( \mathcal{D}_R \)-elements involve \( \mathcal{P} \)-elements of the form \( \phi \), while left-handed \( \mathcal{D}_L \)-elements involve \( \mathcal{P} \)-elements of the form \( \overline{\chi} \). Consequently, given any \( a \in \mathcal{P} \), the conjugate-linear involution \( a \rightarrow \overline{a} \) is intimately linked to a reversal of chirality in the Dirac algebra.

This distinction between left and right Weyl spinors is also manifested in their behavior under restricted Lorentz transformations which we now discuss.

1.4 Lorentz Transformations

Let \( V \) be an \( n \)-dimensional \( g \)-vector space over a field \( F \) equal to either \( \mathbb{R} \) or \( \mathbb{C} \), and let \( C_V(g) \) be the Clifford algebra generated by \( V \). Set

\[ \mathcal{J} = [V, V] \]  
(1a)

\[ \mathcal{G} = V + \mathcal{J}. \]  
(1b)
where $[ \cdot , \cdot ]$ denotes the commutator (see Appendix 1). One may show (Hermann 1974) that $\mathcal{G}$ is a Lie subalgebra of $\mathfrak{c}(\mathfrak{g})$, and $\mathcal{J}$ is a Lie subalgebra of $\mathcal{G}$ satisfying
\[ [\mathcal{J}, \mathcal{J}] \subset \mathcal{I}. \] (3)
One may also show that
\[ \dim_f(\mathcal{G}) = n(n + 1)/2 \] (3a)
\[ \dim_f(\mathcal{J}) = n(n - 1)/2. \] (3b)

Now define a new vector space
\[ \mathcal{W} = F \oplus \mathcal{I} \] (4)
having bilinear form $\mathcal{g}'$ satisfying
\[ g'(u, v) = g'(u_0 + \tilde{u}, v_0 + \tilde{v}) = u_0 v_0 - g(\tilde{u}, \tilde{v}). \] (5)
where $u_0, v_0 \in F$, $\tilde{u}, \tilde{v} \in \mathcal{I}$. One may show that the map
\[ \rho: \mathcal{G} \rightarrow \text{END}(\mathcal{W}). \]
defined by
\[ \rho(u + [\tilde{u}, \tilde{w}]) = g(\tilde{u}, \tilde{w}) + u_0 \tilde{u} + [[\tilde{u}, \tilde{w}], \tilde{u}]. \] (6)
is a Lie algebra homomorphism of $\mathcal{G}$ into the Lie algebra (under the commutator) of $F$-linear maps $\mathcal{W} \rightarrow \mathcal{W}$. More importantly, $\rho$ can be shown to be an isomorphism between $\mathcal{G}$ and the Lie algebra of derivations of $\mathcal{g}'$ on $\mathcal{W}$, i.e.
\[ \mathcal{G} \cong o_{\mathcal{W}}(\mathcal{g}'). \] (7)
where
\[ o_{\mathcal{W}}(\mathcal{g}') = \{ D \in \text{END}(\mathcal{W}) | g'(D(u), v) + g'(u, D(v)) = 0 \ (\forall u, v \in \mathcal{W}) \}. \] (8)

Define the set
\[ O_{\mathcal{W}}(\mathcal{g}') = \{ A \in \text{AUT}(\mathcal{W}) | g'(A(u), A(v)) = g'(u, v) \ (\forall u, v \in \mathcal{W}) \}. \] (9)
where \( \text{AUT}(\mathfrak{h}') \subset \text{END}(\mathfrak{h}') \) is the subset of all invertible \( \mathbb{F} \)-linear maps \( \mathfrak{h}' \to \mathfrak{h}' \). The elements of \( O_{\mathbb{F}}(g') \) are termed the automorphisms of \( \mathfrak{h}' \) preserving the bilinear form \( g' \). One may show that \( O_{\mathbb{F}}(g') \) is a Lie group having associated Lie algebra \( o_{\mathbb{F}}(g') \).\(^7\)

If \( \mathfrak{h}' \) is an \( n \)-dimensional real vector space having bilinear form \( g \) with signature \( (p,q) \), then \( O_{\mathbb{F}}(g) = O(p,q) \), where \( O(p,q) \) is a linear Lie group that can be identified with the group of invertible \( n \times n \) real matrices preserving the bilinear form \( g \). Letting \( \mathbb{F} = \mathbb{R} \) and taking \( g \) to have signature \( (p,q) \), where \( p+q = n \), we then see that \( G \subset C_{p,q}(\mathbb{R}) \) is isomorphic to the Lie algebra of \( O(q+1,p) \) while \( \mathcal{J} \) is isomorphic to the Lie algebra of \( O(p,q) \).

Identifying \( O(p,q) \) with a matrix group, one defines the proper subgroup \( SO(p,q) \) to be the set of \( O(p,q) \)-elements having determinant \( +1 \). It may be shown that \( O(p,q) \) and \( SO(p,q) \) have the same dimension (Isham 1989). Moreover, they have the same Lie algebra which we denote \( so(p,q) \).

Now let \( \mathcal{V} = \mathbb{R}^3 \). Then \( \mathcal{C}_{\mathcal{V}}(g) = C_{3,0}(\mathbb{R}) \) which is isomorphic to the Pauli algebra (considered as a real Clifford algebra). Consequently, the Lie algebra \( \mathcal{G} \) of \( \mathcal{F} \) is seen to be isomorphic to \( so(1,3) \) which is the Lie algebra of the Lorentz group \( O(1,3) \) while \( \mathcal{J} \) is isomorphic to \( so(3) \) which is the Lie algebra of the rotation group \( O(3) \).

In order to make contact with these groups within the algebraic framework, we are lead to consider their connected subgroups as well as their associated universal covering groups. In particular, there exists a bijective

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correspondence between an abstract Lie algebra \( \mathfrak{g} \) and the universal covering group \( UC[G] \) of the associated Lie group \( G \) via the exponential map

\[
\exp : \mathfrak{g} \to UC[G]
\]

\[
\alpha \to \exp(\alpha).
\]  

One is then able to relate \( UC[G] \) to the connected subgroup \( G_0 \) of \( G \) via a surjective group homomorphism \( \varphi : UC[G] \to G_0 \) such that

\[
UC[G]/\ker(\varphi) \cong G_0.
\]  

where \( \ker(\varphi) \) is the normal subgroup of \( UC[G] \) consisting of those elements which are mapped under \( \varphi \) to the identity element of \( G_0 \).

Using Eqs. (1.4.1), the Lie algebra \( \mathfrak{g} \) of \( P \) may be written as

\[
\mathfrak{g} = R^3 \oplus [R^3 \cdot R^3].
\]  

where

\[
\mathcal{J} = [R^3 \cdot R^3] \subset \mathfrak{g}.
\]  

Here, \( R^3 \) is to be interpreted as a subspace of \( P \) (considered as a real vector space). By using the canonical basis \( \{e_i\} \) of \( R^3 \) we may determine a basis for \( \mathcal{J} \). In particular, we see that for \( i \neq j \)

\[
[e_i, e_j] = e_i e_j - e_j e_i = 2e_i e_j = 2\varepsilon_{ijk}ee_k.
\]  

---

Employing the complex structure of $\mathcal{F}$ and setting $J_i = -\frac{i}{2} e_i, \quad (\forall i \in \mathbb{N}_3)$ we then have

$$[J_i, J_j] = -\epsilon_{ijk} J_k. \quad (15)$$

The $\{J_i\}$ are clearly seen to form a basis for $\mathcal{J}$ over $\mathbb{R}$ and one may identify the real Lie algebra $\mathcal{J}$ with the subset of $\mathcal{F}$ composed of elements of the form $-i \bar{0}/2$, where $0 \in \mathcal{F}$ is an hermitean vector:

$$\mathcal{J} = \left\{-i \bar{0}/2 | \bar{0}^* = 0 \in \mathbb{F}, \bar{0} \right\} \subset \mathcal{F}. \quad (16)$$

Eq. (1.4.14) also shows that $\left[ \frac{1}{2} e_i, \frac{1}{2} e_j \right] = -\epsilon_{ijk} J_k$ and thus any element of $\mathcal{G}$ can be written

$$\Lambda = \bar{w}/2 - i \bar{0}/2. \quad (17)$$

$\bar{w}, \bar{0} \in \mathcal{F}$ being hermitean vectors.

Letting $\mathcal{K} = \text{span}_\mathbb{R}\{K_i\}$, where $K_i = \frac{1}{2} e_i, \quad (\forall i \in \mathbb{N}_3)$, we see that

$$\mathcal{G} = \mathcal{K} + \mathcal{J}. \quad (18)$$

where

$$[\mathcal{J}, \mathcal{J}] \subset \mathcal{J} \quad (19a)$$

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K} \quad (19b)$$

$$[\mathcal{J}, \mathcal{K}] \subset \mathcal{K}. \quad (19c)$$

$\mathcal{J}$ is clearly a Lie subalgebra of $\mathcal{G}$ while $\mathcal{K}$ is not. The basis elements of $\mathcal{K}$ are called the generators of boosts while those of $\mathcal{J}$ are termed the generators of rotations.

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9 We have added the minus sign in the definition of the $J_i$ so as to agree with the convention adopted in Baylis and Jones (1989[I]).
Consider now the exponential map

\[ \text{exp}: \mathcal{J} \rightarrow \mathcal{U}[O(1,3)] \]

\[ \Lambda \rightarrow \text{exp}(\Lambda). \]

where

\[ \text{exp}(\Lambda) = \sum_{k=0}^{\infty} \frac{\Lambda^k}{k!}. \quad (1.20) \]

Note that, since \( \mathcal{F} \) is isomorphic to a total matrix algebra where the exponential map can be defined, the limit implicit in Eq. (1.4.20) does indeed exist. Using Eq. (1.4.17), we then set

\[ l = \text{exp}(\bar{u}/2 - i\bar{\theta}/2) \quad (21) \]

which is easily seen to be a unimodular element of \( \mathcal{F} \), i.e.

\[ l = 1 + \Lambda - \frac{1}{2!} \Lambda^2 + \frac{1}{3!} \Lambda^3 - \cdots \]

\[ \Rightarrow \quad \bar{l} = 1 - \Lambda - \frac{1}{2!} \Lambda^2 - \frac{1}{3!} \Lambda^3 - \cdots \]

\[ \Rightarrow \quad \bar{l} = \sum_{k=0}^{\infty} \frac{(-\Lambda)^k}{k!} = \text{exp}(-\Lambda) \]

\[ \Rightarrow \quad \bar{l}l = l\bar{l} = 1. \quad (22) \]

Moreover, the condition that an element of \( \mathcal{F} \) be unimodular reduces the number of real independent parameters from eight to six which is the number of real parameters needed to describe \( l \). Thus, since the unimodular \( \mathcal{F} \)-elements form a group isomorphic to \( \text{SL}(2,\mathbb{C}) \), one may identify \( \mathcal{U}[O(1,3)] \) with \( \text{SL}(2,\mathbb{C}) \). It is well known that \( \text{SL}(2,\mathbb{C}) \) is the double covering group of the restricted (proper, orthochronous) Lorentz group \( \mathbb{L} = \text{SO}_0(1,3) \) (Cornwell 1984):

\[ \text{SL}(2,\mathbb{C})/\{-1,1\} \cong \text{SO}_0(1,3). \quad (23) \]
Here, the term orthochronous means that the elements of $SO_u(1,3)$ preserve the direction of time. $SO_u(1,3)$ is the connected subgroup of $O(1,3)$. One then sees that the elements $\pm l \epsilon \mathbb{P}$ correspond to a unique restricted Lorentz transformation.

If $l^\prime = \tilde{l}$, then $l$ must be of the form

$$l = R = \exp(-i\vec{0}/2)$$

(24)

and thus $R \epsilon \mathbb{P}_u \cong SU(2)$. This implies that

$$\mathcal{H} \xrightarrow{\exp} \mathcal{U}C[O(3)] \cong SU(2).$$

(25)

Again, it is well known that $SU(2)$ is the double covering group of the proper rotation group $SO(3)$ (Cornwell 1984):

$$SU(2)/\{-1,1\} \cong SO(3).$$

(26)

$SO(3)$ is then identified as the connected subgroup of $O(3)$ and the elements $\pm R \epsilon \mathbb{P}$ correspond to a unique proper rotation. If $l^\prime = l$, then

$$l = B = \exp(\vec{u}/2).$$

(27)

The elements $\pm B \epsilon \mathbb{P}$ correspond to a unique boost. Since $\mathcal{K}$ is not a subalgebra of $\mathcal{G}$, the boosts do not form a subgroup of $\mathbb{P}$.

By applying Eq. (1.4.20) to Eqs. (1.4.24) and (1.4.27), one may show that

$$R = \cos(\theta/2) - i\sin(\theta/2)$$

(28a)

$$B = \cosh(w/2) - \tilde{u}\sinh(w/2).$$

(28b)

where $0, w \epsilon R$ are the magnitudes of the vectors $\vec{0}, \vec{w} \epsilon \mathbb{P}$, and $\vec{0}$ and $\vec{u}$ are unit vectors in the direction of $\vec{0}$ and $\vec{u}$ respectively. One may also show that any restricted Lorentz transformation can be written as the product of a boost and a rotation (Baylis and Jones 1988).
Now let \( l = \Lambda \). Then \( \mathfrak{C}_1(\mathbb{R}) = \mathfrak{c}_1, \mathbb{R} \oplus \mathfrak{D}_R \) and \( \mathfrak{f} \) is isomorphic to the Lie algebra of the de Sitter group \( O(1,1) \) while \( \mathfrak{f} \) is isomorphic to the Lie algebra of the Lorentz group \( \mathfrak{c}(1,3) \). Using Eq. (1.4.1b) together with the canonical basis \( \{\mathcal{Y}_\mu\} \) of \( \Lambda \), we see that

\[
[Y_\mu, Y_\nu] = Y_\mu Y_\nu - Y_\nu Y_\mu = 2Y_{\mu\nu}.
\]

Hence, for the real Dirac algebra, \( \mathfrak{f} \) is given by

\[
\mathfrak{f} = \text{span}_R \left\{ \frac{1}{2} Y_{\mu\nu} \right\}.
\]

Since \( \mathfrak{D}_R \) can be considered a real subset of \( \mathfrak{D} \), we may use Eqs. (1.2.84c) and (1.2.86) to identify \( \mathfrak{f} \) with a real Lie subalgebra of \( \mathfrak{D}_R \subset \mathfrak{D} \), i.e.

\[
\mathfrak{f} = \{ \Lambda^- \otimes \Omega + \Lambda \otimes \Omega \mid A \in \mathbb{F}_A \subset \mathbb{F} \},
\]

where \( \Lambda = \frac{\bar{u}}{2} - i\tilde{b}/2 \) as before. Although we have embedded \( \mathfrak{f} \) in \( \mathfrak{D} \), it is still seen to be six-dimensional over \( \mathbb{R} \) in agreement with Eq. (1.4.3b).

Given that \( \mathfrak{f} \subset \mathfrak{D}_R \), we may use the canonical isomorphism \( \mu \) defined in Eq. (1.2.102) to rewrite \( \mathfrak{f} \) as

\[
\mathfrak{f} = \{ \Lambda^- \otimes \Lambda \mid A \in \mathbb{F}_A \subset \mathbb{F} \}.
\]

Then, using the fact that \( \alpha \otimes b - \alpha' \otimes b' = (\alpha + \alpha') \otimes (b + b') \) and \( \alpha(\alpha \otimes b) = \alpha \alpha \otimes \alpha b \) for all \( \alpha, \alpha' \in \mathbb{F}_A \subset \mathbb{F} \) and \( \alpha \in \mathbb{R} \), we may apply \( \text{EXP} \) to map \( \mathfrak{f} \) to its universal covering group, i.e.
\[ \exp(\overline{\Lambda} \Theta \Lambda) = \sum_{k=0}^{\infty} \left( \frac{(\overline{\Lambda} \Theta \Lambda)^k}{k!} \right) \]

\[ = \sum_{k=0}^{\infty} \frac{\overline{\Lambda} \Theta \Lambda}{k!} \Theta \sum_{k=0}^{\infty} \frac{(\Lambda)^k}{k!} \Theta \]

\[ = \overline{\exp(\overline{\Lambda} \Theta \Lambda)} \Theta \exp(\Lambda). \quad (33) \]

Again, since \( \mathcal{D} \) is isomorphic to a total matrix algebra, the limit implicit in Eq. (1.4.33) does exist. One easily verifies that

\[ \exp(\overline{\Lambda} \Theta \Lambda) = [\overline{\exp(\Lambda)}]^{\Theta}. \quad (34) \]

and thus \( \exp \) maps \( \overline{\Lambda} \Theta \Lambda \in \mathcal{J} \) to the element \( \overline{\mathcal{J}} \Theta \mathcal{J} \). Finally, employing the inverse of the isomorphism \( \mu \), we set

\[ L = \overline{\mathcal{J}} \Theta \Omega, = \mathcal{J} \Theta \Omega. \quad (35) \]

\( l \) is also seen to be an element of \( \mathcal{D}^*_R \) and since \( \mathcal{J} \neq so(1,3) \), \( \exp \) maps \( \mathcal{J} \) onto \( SL(2, \mathbb{C}) \). Consequently, for \( \mathcal{D}^*_R \) viewed as a real subset of \( \mathcal{D} \), the elements \( \pm L \in \mathcal{D}^*_R \) correspond to a unique restricted Lorentz transformation.

Now consider again the Pauli algebra. We saw that the real Lie algebra \( so(1,3) = \mathcal{G} \subset \mathcal{P} \) was mapped by \( \exp \) onto \( SL(2, \mathbb{C}) = \mathcal{P}_{SL} \subset \mathcal{P} \) while \( so(3) = \mathcal{J} \subset \mathcal{G} \) was mapped onto \( SU(2) = \mathcal{P}_{SU} \subset \mathcal{P}_{SL} \). Here, \( \mathcal{P}_{SL} \) denotes the set of all unimodular \( \mathcal{P} \)-elements while \( \mathcal{P}_{SU} \) denotes the subgroup of \( \mathcal{P}_{SL} \) consisting of elements \( \alpha \in \mathcal{P}_{SL} \) satisfying the condition \( \overline{\alpha} = \alpha^{-1} \) (see Section 1.1). Since \( \mathcal{P}_{SL} \) is contained in \( \mathcal{P} \), one immediately sees that the spinor reps \( \rho_- \) and \( \rho_+ \) of \( \mathcal{P} \) induce equivalent, faithful group reps of \( SL(2, \mathbb{C}) \). Moreover, since the spinor reps are irreducible and all irreps of a simple algebra are equivalent (see Corollary to Theorem A1.7), one may use the \( \alpha \)-irrep defined in Eq. (1.1.47) to show that these induced irreps of \( SL(2, \mathbb{C}) \) are also irreducible. A similar line of reasoning yields two equivalent, faithful group irreps of
$\text{SU}(2)$. Since $\text{SL}(2, \mathbb{C})$ and $\text{SU}(2)$ are the double covering groups of $\text{SO}_0(1,3)=\mathbb{I}$ and $\text{SO}(3)$ respectively, it is logical to ask whether these induced group reps yield reps of $\mathbb{I}$ and $\text{SO}(3)$.

Unfortunately, the answer is no. Consider first the case of $\text{SU}(2)$. It is known from angular momentum theory that to every $j \in \{0, 1/2, 1, 3/2, \ldots\}$ there exists a $2j+1$-dimensional irrep $D_j$ of the Lie group $\text{SU}(2)$ on a complex vector space having orthonormal basis labelled by $(\phi_n^m|j, -j; j+1, \ldots, -1, j)$ (Gilmore 1974). These irreps of $\text{SU}(2)$ provide reps of $\text{SO}(3)$ iff $j$ is an integer. One may readily show that the spinor-induced irreps of $\text{SU}(2) \ltimes \mathbb{F}_\text{sl} \subset \mathbb{F}$ correspond to the irrep $D_j^{1/2}$ and thus do not provide reps of $\text{SO}(3)$. Since $\text{SO}(3)$ is a subgroup of $\text{SO}_0(1,3)$, the same is true for the case of $\text{SL}(2, \mathbb{C})$. For suppose the spinor-induced irreps of $\text{SL}(2, \mathbb{C}) \ltimes \mathbb{F}_\text{sl} \subset \mathbb{F}$ did provide reps of $\text{SO}_0(1,3)$. Since a group rep automatically induces a representation of any subgroup, we would obtain a spinor-induced rep of $\text{SO}(3)$, in contradiction to the previous result.\footnote{10 We should point out, however, that tensor product reps of $\text{SU}(2)$ and $\text{SL}(2, \mathbb{C})$ do provide irreps of $\text{SO}(3)$ and $\text{SO}_0(1,3)$ respectively via Clebsch-Gordan decomposition.}

Consequently, the spinor irreps of $\mathbb{F}$ induce group irreps of $\text{SU}(2)$ and $\text{SL}(2, \mathbb{C})$ but do not provide reps of $\text{SO}(3)$ or $\text{SO}_0(1,3)=\mathbb{I}$. However, this is not as catastrophic as one might think. After all, $\text{SL}(2, \mathbb{C})$, which contains $\text{SU}(2)$ as a subgroup, covers $\mathbb{I}$ and although we are unable to construct single-valued representations of $\mathbb{I}$ on the Pauli spinor spaces, we should, nevertheless, be able to discern how Pauli spinors behave under restricted Lorentz
transformations and, subsequently, proper rotations. For the sub-case of \( \text{SL}(2) \) and proper rotations this is indeed possible. However, for the more general case of \( \text{SL}(2, \mathbb{C}) \) and the restricted Lorentz group, one is unable to explicitly deduce this information solely based on the spinor irreps of the Pauli algebra, being instead forced to look at \( \text{SL}(2, \mathbb{C}) \) from the point of view of the Dirac algebra. In order to elaborate on this, we need to introduce what is termed the Clifford group.

Let \( C_\gamma(g) \) be the Clifford algebra generated by a \( g \)-vector space \( \gamma \) defined over either \( \mathbb{R} \) or \( \mathbb{C} \). The Clifford group \( \Gamma \) of \( C_\gamma(g) \) is defined as the set of all invertible elements \( s \in C_\gamma(g) \) satisfying the condition

\[
sus^{-1} \in \gamma \quad (\forall \nu \in \nu). \tag{36}
\]

One easily checks that \( \Gamma \) does indeed form a group under multiplication. In addition, the subset \( \Gamma^- = \{ s \in \Gamma \mid s \in C_\gamma^-(g) \} \subset \Gamma \) is seen to form a subgroup of \( \Gamma \).

Now define the map

\[
\chi: \Gamma \to \text{AUT}(C_\gamma(g))
\]

\[
s \mapsto \chi(s). \tag{37a}
\]

where

\[
\chi(s)(\alpha) = sus^{-1} \quad (\forall \alpha \in C_\gamma(g)). \tag{37b}
\]

\( \chi \) is a group rep of \( \Gamma \) called the vector representation. The importance of the vector representation is that it maps the Clifford group onto the orthogonal group \( O_\nu(g) \), i.e.

\[
g(\chi(s)(u), \chi(s)(v)) = \frac{1}{2} (sus^{-1} ss^{-1} + ss^{-1} sus^{-1})
\]

\[
= \frac{1}{2} s(uv + vu) s^{-1}
\]

\[
= g(u, v) ss^{-1} = g(u, v)
\]
for all \( \mu, \tau \in \Gamma \) and \( \chi \in \Gamma \). The follow-up remarks to Theorem A2.3 show that \( \chi \) is completely reducible, corresponding to the vector space direct sum decomposition of \( \mathfrak{c}_{i}(s) \) into the \( \dim(\Gamma') = \dim(\Gamma) + 1 \) minimal invariant subspaces \( \mathfrak{c}_{i}(s) = \{ \mu, \mathfrak{n}_{\dim(\Gamma')} \} \) (see Appendix 2). \( \chi \) is also seen to be non-faithful, i.e. given any \( s \in \Gamma \) and invertible \( c \in \mathfrak{c}_{i}(s) \), the elements \( \chi(s) \) and \( \chi(cs) \) correspond to the same orthogonal transformation. For this reason, one normalizes \( \Gamma \) according to the following conventions.

Let \( s \in \Gamma \) and consider the product \( \zeta(s)s \), where \( \zeta \) is the main anti-automorphism defined in Appendix 2. Clearly, \( \zeta(s) \) is invertible if \( s \) is invertible and \( \zeta(s)^{-1} = \zeta(s^{-1}) \). Letting \( \Gamma \) and using the fact that \( \zeta \) maps \( \mathfrak{c}_{i}(s) \to \mathfrak{c}_{i}(s) \) for all \( \mu, \mathfrak{n}_{\dim(\Gamma)} \), we see that

\[
\begin{align*}
\zeta(ss^{-1}) &= ss^{-1} \\
\Rightarrow (\zeta(s))^{-1}v\zeta(s) &= ss^{-1} \\
\Rightarrow v\zeta(s)s &= \zeta(s)s.
\end{align*}
\]

i.e. \( \zeta(s)s \) is an element of the center of \( \mathfrak{c}_{i}(s) \). Since the product of two even or odd elements is always even, \( \zeta(s)s \) must be an even element of the center. Theorems A2.5 and A2.6 then show \( \zeta(s)s \) to be a non-zero element of the ground field of \( V \).

Now take \( V \) to be an \( n \)-dimensional real vector space having bilinear form \( g \) with signature \( (p, q) \) where \( p + q = n \). One may define an \( (\mathbb{R} - \{0\}) \)-valued norm on \( \Gamma \):

\[
\mathcal{N}(s) = \zeta(s)s \quad (\forall s \in \Gamma). \quad (38)
\]

Since \( \mathcal{N}(s_{1}s_{2}) = \zeta(s_{1}s_{2})s_{1}s_{2} = \zeta(s_{2})\zeta(s_{1})s_{1}s_{2} = \mathcal{N}(s_{1})\mathcal{N}(s_{2}) \) for all \( s_{1}, s_{2} \in \Gamma \). \( \mathcal{N} \) is a group homomorphism of \( \Gamma \) into \( \mathbb{R} - \{0\} \) and one
may use this norm to form subgroups of $\Gamma$. These are the
groups $\text{PIN}, \text{SPIN}$ and $\text{SPIN}_0$, defined by

\[ \text{PIN}(p,q) = \{ s \in \Gamma | \mathcal{N}(s) = 1 \} \]  \hspace{1cm} (39a)

\[ \text{SPIN}(p,q) = \text{PIN}(p,q) \cap \Gamma^* \]  \hspace{1cm} (39b)

\[ \text{SPIN}_0(p,q) = \{ s \in \text{SPIN}(p,q) | \mathcal{N}(s) = 1 \}. \]  \hspace{1cm} (39c)

Clearly, $\chi(\Gamma) = \chi(\text{PIN})$ and PIN may be viewed as a "normalized"
$\Gamma$. One may show that

\[ \chi(\text{PIN}(p,q)) = \begin{cases} O(p,q) & \{ n \text{ even} \} \\ \text{SO}(p,q) & \{ n \text{ odd} \} \end{cases} \] \hspace{1cm} (40)

\[ \chi(\text{SPIN}(p,q)) = \text{SO}(p,q) \] \hspace{1cm} (41)

\[ \chi(\text{SPIN}_0(p,q)) = \text{SO}_0(p,q). \] \hspace{1cm} (42)

Moreover, $\text{PIN}(p,q), \text{SPIN}(p,q)$ and $\text{SPIN}_0(p,q)$ are the double
covering groups of $O(p,q), \text{SO}(p,q)$ and $\text{SO}_0(p,q)$ respectively
(Benn and Tucker 1987).

The fundamental significance of the Clifford group lies
in the fact that irreps of the Clifford algebra induce
irreps of $\Gamma$ and its subgroups. In particular,

The irreducible spinor (semi-spinor) reps of a simple
(semi-simple) Clifford algebra induce irreps of $\text{PIN}$.

The irreducible spinor (semi-spinor) reps of the simple
(semi-simple) even subalgebra induce irreps of $\text{SPIN}_0$.

Now consider again the Pauli algebra. Since $\mathcal{F}$, as a
real Clifford algebra, is equal to $\mathbb{C}_{3,0} (R)$. Eqs. (1.4.40) and
(1.4.41) give

\[ \chi(\text{PIN}(3,0)) = \chi(\text{SPIN}(3,0)) = \text{SO}(3). \] \hspace{1cm} (43)

where $\text{PIN}(3,0)$ and $\text{SPIN}(3,0)$ are the double covering groups
of $O(3)$ and $\text{SO}(3)$ respectively. Moreover, since $\text{SO}(3)$ is
the connected subgroup of $O(3), \text{SO}_0(3) = \text{SO}(3)$ and thus
where we have used Eq. (1.4.26). Consequently, in all cases, the vector representation maps onto $\text{SO}(3)$.

By viewing $\mathcal{F}$ as a complex Clifford algebra by means of the complex structure given in Eq. (1.1.25), one may employ Eqs. (1.1.53), (1.4.36), (1.4.38) and (1.4.39) to show that

$$\mathcal{N}(s) = s^* s = \bar{s} s = 1$$

$$\Rightarrow \quad s^{-1} = \bar{s} / (\bar{s} s) = \bar{s} s$$

for all $s \in \text{PIN}(3,0)$. Consequently, $\bar{s} - s$ and thus $s \in \text{PIN}(3,0) = \mathcal{P}_s = \text{SU}(2)$. The elements of $\text{PIN}(3,0)$ are then given by Eq. (1.4.24) so that $\star R = \pm \exp (-i \vec{\Omega}/2)$; $\text{PIN}(3,0)$ corresponds to a proper rotation.

Since the spinor irreps of $\mathcal{F}$ were seen to induce group irreps of $\mathcal{P}_s = \text{SU}(2)$, one concludes that for every $\phi \in \mathcal{F}$ the Pauli spinors $\phi \Omega_+ \in \mathcal{F}$ transform under $\text{PIN}(3,0)$ according to the rule

$$\phi \Omega_+ \rightarrow R \phi \Omega_+.$$

where $R \in \text{PIN}(3,0) \subset \mathcal{F}$. In particular, the Pauli spinors $\phi \Omega_-$ and $\bar{ \Omega}_- \Omega_+ \Omega_-$ transform the same under $\text{PIN}(3,0)$. If they didn't, we would not be able to use the spinor irreps of $\mathcal{F}$ to induce group irreps of $\text{PIN}(3,0)$. This result actually hinges on the fact that $\bar{R} \cdot R$ for every $R \in \text{PIN}(3,0)$.

The vector representation and the spinor transformation equation (1.4.45) allow for the description of proper rotations entirely within the framework of the Pauli algebra. Notice, too, that the Clifford group tells us nothing about $\text{SL}(2,C)$ and the restricted Lorentz group. Thus, although the spinor irreps of $\mathcal{F}$ induce equivalent group irreps of $\text{SL}(2,C)$, the Clifford algebra approach does
not provide sufficient information to tie this in with a formalism describing restricted Lorentz transformations wholly in terms of $\mathcal{P}$.

Consider now the real Dirac algebra $\mathcal{D}_R$. Again viewing it as a real subset of $\mathcal{D}$. The (real) Lie algebra $\mathcal{J}$ of $\mathcal{D}_R$ was seen to be isomorphic to the Lie algebra of the Lorentz group $O(1,3)$. Under $\exp$, $\mathcal{J}$ was mapped onto the subset $\mathcal{D}_{sl} = \{ L - \bar{L} \otimes \Omega + L \otimes \bar{\Omega} \mid L \in \mathbb{R} \}$ which could be identified with $\text{SL}(2, \mathbb{C})$. Since the spinor irreps of $\mathcal{D}$ induce equivalent, complex spinor irreps of $\mathcal{D}_R$. the spinor-induced reducible reps of $\mathcal{D}^-$ induce equivalent, reducible reps of $\mathcal{D}_R$. each being the sum of two inequivalent, irreducible semi-spinor irreps. Given that $\mathcal{D}_{sl} \subset \mathcal{D}_R$, we subsequently obtain four equivalent, faithful, reducible group reps of $\mathcal{D}_{sl} \approx \text{SL}(2, \mathbb{C})$ where each is the sum of two inequivalent irreps. We now show that an analysis of the Clifford group of $\mathcal{D}_R$ provides a means for describing restricted Lorentz transformations within the framework of the Dirac algebra. This, in turn, will lead us to a corresponding formalism for the Pauli algebra.

Since $\mathcal{D}_R = C_{1,3}(\mathbb{R})$. Eqs. (1.4.40) through (1.4.42) give

$$\chi(\text{PIN}(1,3)) = O(1,3)$$

$$\chi(\text{SPIN}(1,3)) = \text{SO}(1,3)$$

$$\chi(\text{SPIN}_0(1,3)) = \text{SO}_0(1,3).$$

(46a) (46b) (46c)

where $\text{PIN}(1,3), \text{SPIN}(1,3)$ and $\text{SPIN}_0(1,3)$ are the double covering groups of $O(1,3), \text{SO}(1,3)$ and $\text{SO}_0(1,3)$ respectively. Here, we have used the fact that $n+3=4$ is even.

Now since $\text{SPIN}_0(1,3) \subset \mathcal{D}_R$, any $s \in \text{SPIN}_0(1,3)$ must take the form

$$s = s_0 I + \frac{1}{2} s^\nu \gamma_{\nu\nu} + s_3 \gamma.$$  

(47)
where the coefficients are all real and
\[ N(s) = \zeta(s)s = 1 \]  
(48)
\[ s \in \mathbb{S} \setminus \mathbb{S}_- \quad (\forall \mu, \nu \in \mathbb{S}_0) \]  
(49)

Using Eqs. (1.2.95a) and (1.2.96), we may rewrite Eq. (1.4.47) as
\[ s = \bar{s} \ominus \Omega_+ \varepsilon \ominus \Omega_- \]  
(50)

where \( s = (s_0 + is_4) - (s_0^{04} + c_{14} vs_4) e_4 \). Using the definition of \( \zeta \), given in Appendix 2, one sees that
\[ \zeta(s) = s_0 l - \frac{1}{2} s_{2}^v \gamma_{v} - s_4 \gamma \]
\[ = s_0 l - \frac{1}{2} s_{2}^v \gamma_{v} \ominus - s_4 \gamma \]
\[ - s_0 \ominus \Theta \ominus \Theta \ominus \Theta \]  
(51)

and thus
\[ \zeta(s) s = (s_0 \ominus \Theta \ominus \Theta \ominus \Theta \ominus \Theta) (s_0 \ominus \Theta \ominus \Theta \ominus \Theta \ominus \Theta) \]
\[ = (\bar{s}s) \ominus \Theta \ominus \Theta \ominus \Theta \ominus \Theta \]
\[ = (\bar{s}s) \ominus \Theta \ominus \Theta \ominus \Theta \ominus \Theta \]  
(52)

Clearly, Eq. (1.4.48) holds iff \( \bar{s}s = 1 \), i.e. iff \( s \in \mathcal{P} \) is
unimodular. One then sees that SPIN\(\theta\) holds iff \( \bar{s}s \in \mathcal{P} \) is
since \( \text{SL}(2, \mathbb{C}) \) is the double covering group of \( \text{SO}\_\mathbb{R}(1, 3) \). Eq.
(1.4.46c) shows that SPIN\(\theta\) holds iff \( \bar{s}s \in \mathcal{P} \).

Consequently, Dirac spinors are seen to transform under
SPIN\(\theta\) according to the rule
\[ \text{SPIN}\_\mathbb{R}(1, 3) \]
\[ \psi \rightarrow L\psi \]  
(53)

and using Eqs. (1.3.19) through (1.3.24) together with the canonical vector space isomorphism between the Dirac
semi-spinor and Pauli spinor spaces one concludes that the Pauli spinors $\Phi \Omega$ and $\bar{\mathcal{N}}^* \Omega$ transform as

$$
\Phi \Omega \rightarrow l \Phi \Omega, \quad \bar{\mathcal{N}}^* \Omega \rightarrow l^* \bar{\mathcal{N}}^* \Omega,
$$

where $l \in \mathcal{F}_{\mathcal{S}_l} \subset \mathcal{F}$.

The vector representation and the spinor transformation equation (1.4.53), together, provide a formalism for discussing restricted Lorentz transformations within the framework of the Dirac algebra. Moreover, since $\mathcal{F}_{\mathcal{S}_l} = \mathcal{S}_l(\mathbb{C})$ corresponds to a unique element of $\mathcal{L}$, the transformations (1.4.54) imply that the Pauli spinors $\Phi \Omega$ and $\bar{\mathcal{N}}^* \Omega$ behave differently under $\mathcal{L}$. This is actually a manifestation of the inequivalence of the semi-spinor irreps of $\mathcal{D}_R$, i.e. the map $\mathcal{X} \rightarrow \bar{\mathcal{X}}^*$ is a conjugate-linear involution in $\mathcal{F}$. We now see why the spinor irreps of $\mathcal{F}$, alone, do not provide sufficient information to describe the behavior of Pauli spinors under the restricted Lorentz group. In order to complete our formalism for describing $\mathcal{L}$ within the framework of the Pauli algebra, we need to determine how more general $\mathcal{F}$-elements transform. In particular, we need to ascertain how four-vectors transform.

To this end, consider again the Dirac algebra. Given Eq. (1.4.46c), one immediately sees that any four-vector $u \in \mathcal{M} \subset \mathcal{D}_R \subset \mathcal{D}$ transforms under $SO_0(1,3) = \mathcal{L}$ according to the rule

$$
u \rightarrow L \nu L^{-1}, \tag{55}$$

where $L \in \mathcal{D}_S = \text{Spin}(1,3)$ is given by Eq. (1.4.35). Since $\zeta(1,l) = 1$ for every $L \in \mathcal{D}_S$, Eq. (1.4.51) gives

$$
L^{-1} \tau_{-} = \zeta(1) = L^* \tau_{+} \Omega + \bar{L} \tau_{-} \bar{\Omega}.
$$

Eqs. (1.2.83) and (1.1.59) then yield
\[
\begin{align*}
1.\omega^{-1} &= \left( L^* \otimes \Omega_+ + L \otimes \Omega_+ \right) \left( u \otimes \omega_+ - \bar{u} \otimes \omega_+ \right) (\lambda^* \otimes \Omega_+ + \bar{\lambda} \otimes \Omega_+ ) \\
&= L \otimes \omega_+ + \bar{L} \otimes \omega_+ \\
&= (L \otimes \omega_+) \otimes \omega_+ + (\bar{L} \otimes \omega_+) \otimes \omega_+ .
\end{align*}
\]

and thus the four vector \( u \in \mathcal{F} \) is seen to transform as

\[
u \mapsto L \otimes \omega_+. \quad (58)
\]

Notice that \( L \otimes \omega_+ \) is hermitean (since \( u \) is hermitean), i.e. \( L \otimes \omega_+ \) is a four vector in \( \mathcal{F} \). Moreover, this transformation is seen to preserve the Minkowski scalar product in \( \mathcal{F} \) defined by Eq. (1.2.10), i.e.

\[
u \cdot \bar{v} \mapsto (L \otimes \omega_+) \cdot (\bar{L} \otimes \omega_+ )
\]

\[
= \frac{1}{2} (L \otimes \omega_+ \bar{L} + \bar{L} \otimes \omega_+ \bar{L}) \\
= \frac{1}{2} L (u \bar{v} + \bar{u} v) \bar{L} \\
= L (u \bar{v}) \bar{L} \\
= (u \bar{v}) \bar{L} = u \cdot \bar{v}. \quad (59)
\]

for all \( u, v \in \mathcal{M}^+ \subset \mathcal{F} \). This result also demonstrates that

\[
u \mapsto (u \bar{v})_\nu \rightarrow L (u \bar{v})_\nu \bar{L}. \quad (60)
\]

Pauli elements transforming in this fashion are termed six-vectors (Baylis 1980, Baylis and Jones 1989[I]).

Thus far, we have focused solely on restricted Lorentz transformations. This has actually been forced upon us from the outset by our use of the Lie algebras \( \mathcal{G} \) and \( \mathcal{J} \). embedded within the Clifford algebras, to arrive at their associated
connected Lie groups. However, for the Dirac algebra, Eq. (1.4.46a) shows that by expanding our analysis to PIN(1,3) we may also deal with elements of the full Lorentz group.

It is known that the Lorentz group has four connected components:

$$O(1,3) = L^1 : \cup L^2 : \cup L^3 : \cup L^4 .$$

(61)

where $L^i = SO_0(1,3)$ is the component connected to the identity \(^{11\text{Coquereaux 1988}}\). One may show that every element of PIN(1,3) can be written as $s = u_1 u_2 \cdots u_4$, with $u_i \in M^4 \subset \mathbb{R}$ (with $N_1$). One then defines the four components of $O(1,3)$ as follows:

If $k$ is even and $\zeta(s)s > 0$, then $\chi(s) \in L^1$.\(^{\text{\textbullet}}\)

If $k$ is odd and $\zeta(s)s > 0$, then $\chi(s) \in L^1$. An example is the space inversion transformation $\mathbb{P} = \gamma_0$ defined by

$$\gamma_\mu \rightarrow \gamma_\mu^\mathbb{P} = \chi(\mathbb{P}) \gamma_\mu$$

$$= \mathbb{P} \gamma_\mu \mathbb{P}^{-1}$$

$$= \gamma_0 \gamma_\mu \gamma_0 = \gamma_\mu^\mathbb{P} .$$

(62)

If $k$ is odd and $\zeta(s)s < 0$, then $\chi(s) \in L^1$. An example here is the time inversion transformation $\mathbb{T} = \gamma_0$ defined by

$$\gamma_\mu \rightarrow \gamma_\mu^{\mathbb{T}} = \chi(\mathbb{T}) \gamma_\mu$$

$$= \mathbb{T} \gamma_\mu \mathbb{T}^{-1}$$

$$= \gamma_0 \gamma_\mu \gamma_0 = -\gamma_\mu .$$

(63)

\(^{\text{11}}\) $L^i$ is the only component forming a subgroup.
If $k$ is even and $\xi(s)$ is even, then $\chi(s)$ is odd. This component contains the spacetime inversion operator $\Theta = P T = \gamma$ given by
\[
\gamma_{\mu} \rightarrow \gamma'_{\mu} = \chi(\Theta) \gamma_{\mu}
\]
\[
= \Theta \gamma_{\mu} \Theta^{-1}
\]
\[
= -\gamma \gamma_{\mu} \gamma = -\gamma_{\mu}.
\]
(61)

Observe that
\[
\zeta(\gamma_0) \gamma_0 = \gamma_0 \gamma_0 = I
\]
(65a)
\[
\zeta(\gamma \gamma_0 \gamma) \gamma = \gamma_0 \gamma \gamma_0 \gamma = -I
\]
(65b)
\[
\zeta(\gamma) \gamma = \gamma \gamma = I.
\]
(65c)

and thus $P, T$ and $\Theta$ are all elements of $\text{Pin}(1,3)$. Since the spinor-induced irreps of $\mathcal{D}_k \subset \mathcal{D}$ induce irreps of $\text{Pin}(1,3)$, any Dirac spinor $\psi$ transforms under $P, T$ and $\Theta$ as
\[
\psi \rightarrow \psi' = P \psi = \gamma_0 \psi
\]
(66a)
\[
\psi \rightarrow \psi' = T \psi = \gamma \gamma_0 \psi
\]
(66b)
\[
\psi \rightarrow \psi' = \Theta \psi = \gamma \psi.
\]
(66c)

Note that since $\text{Pin}(1,3)$ is the double covering group of $\text{O}(1,3)$, we could have equally defined $P$ and/or $T$ to include an extra minus sign. Such a choice, however, is purely a matter of convention and hence we shall arbitrarily fix $P = \gamma_0, T = \gamma \gamma_0$. 
To see what is happening in terms of the Pauli algebra, consider, for example, the Dirac spinor $\psi_\tau$. Using Eqs. (1.3.11c) and (1.2.84), one may show that

\begin{align*}
\left(\bar{\Pi}^\tau \Omega \otimes \omega + \Xi \Omega \otimes \Omega \right)^2 &= \Xi \Omega \otimes \omega + \bar{\Pi}^\tau \Omega \otimes \Omega. \\
\left(\bar{\Pi}^\tau \Omega \otimes \omega + \Xi \Omega \otimes \Omega \right)^4 &= -i\Xi \Omega \otimes \omega + i\bar{\Pi}^\tau \Omega \otimes \Omega. \\
\left(\bar{\Pi}^\tau \Omega \otimes \omega + \Xi \Omega \otimes \Omega \right)^6 &= -i\bar{\Pi}^\tau \Omega \otimes \omega + i\Xi \Omega \otimes \Omega.
\end{align*}

One clearly sees that space and time inversion reorder the Pauli spinors within the tensor product structure of the Dirac algebra. In addition, $\bar{T}$ is seen to multiply the Pauli spinors by plus or minus $i$, while $P$ leaves them unchanged. This demonstrates why one is unable to deal with the full Lorentz group strictly in terms of spinor irreps of the Pauli algebra. Under spacetime inversion however, the right and left Weyl spinors forming $\psi_\tau$ are simply multiplied by $-i$ and $-i$ respectively. Identical results may be obtained for $\psi$ belonging to the other three Dirac spinor spaces. One may then use the canonical isomorphism between Weyl and Pauli spinors to obtain the corresponding transformations in $\mathcal{F}$.

These non-restricted Lorentz transformations are intimately related to the more physical operations of parity, time reversal and charge conjugation which we will discuss in Chapter 2.

1.5 The Main Result

We now come to the fundamental result of this study; namely, to show how physical models based on the Dirac algebra and its spinor representations may be expressed entirely within the framework of the Pauli algebra.
Recall that the spinor irreps of $\mathcal{D}$ could be written generically as

$$\rho_{\mu\nu} : \mathcal{F} \otimes \mathcal{F} \to \text{END}(\mathcal{F} \Omega \otimes \mathcal{F} \Omega')$$

$$\alpha \otimes b \to \rho_{\mu} \otimes \rho_{\nu}.$$

(1)

where $\mathcal{D} = \mathcal{F} \otimes \mathcal{F}$ and $\Omega, \Omega' \in \{\Omega_+, \Omega_-, \Omega_0\}$. The Dirac spinor spaces $\mathcal{D}_{\Omega, \Omega'} = \mathcal{F} \Omega \otimes \mathcal{F} \Omega'$ are seen to satisfy

$$\dim_c(\mathcal{D}_{\Omega, \Omega'}) = \dim_c(\mathcal{F} \Omega) \dim_c(\mathcal{F} \Omega')$$

$$= \dim_c(\mathcal{F} \Omega) + \dim_c(\mathcal{F} \Omega')$$

$$= \dim_c(\mathcal{F}).$$

(2)

where we have used Eq. (1.1.52) together with Theorems A2.6, A1.8 and the corollary to Theorem A1.7. Consequently, the Dirac spinor spaces are all isomorphic as complex vector spaces to $\mathcal{F}$.

Define the C-linear map

$$\rho : \mathcal{F} \otimes \mathcal{F} \to \text{END}(\mathcal{F})$$

$$\alpha \otimes b \to \rho_{\mu} \circ \rho_{\nu}.$$

(3\text{a})

where

$$\rho_{\mu} \circ \rho_{\nu}(\psi) = \rho_{\mu} \circ \rho_{\nu} \psi = \alpha \psi \overline{b} \quad (\forall \psi \in \mathcal{F}).$$

(3\text{b})

Since spatial reversion is an anti-automorphism of $\mathcal{F}$, one easily verifies that $\rho$ is indeed a C-linear map. Moreover, we see that
\[ \rho((a \otimes b)(a' \otimes b'))(\Psi) = \rho(a a' \otimes b b')(\Psi) = \rho_{a a'} \rho_{b b'}(\Psi) = \rho_{a a'} \psi_{b b'} = \alpha' a' \psi_{b b'} = \alpha' a' \psi_{\bar{b} \bar{b}} = \rho_a \rho_{a'} ^{\Psi} \psi_{\bar{b} \bar{b}} = \rho_a \rho_{a'} ^{\Psi} \psi_{\bar{b} \bar{b}} = \rho(a \otimes b)(a' \otimes b')(\Psi) \]

for all \(a, a', b, b', \Psi \in \mathcal{F}\), and thus \(\rho\) defines a representation of \(\mathcal{D} = \mathcal{F} \otimes \mathcal{F}\) in \(\mathcal{F}\).

We now would like to find a linear isomorphism \(\phi: \mathcal{D}_{\alpha \alpha} \rightarrow \mathcal{F}\) intertwining the action of \(\rho_{\alpha}\) and \(\rho\). To this end, define the \(C\)-linear map

\[ \phi_{\alpha \alpha} : \mathcal{F} \mathcal{F} \rightarrow \mathcal{F} \]

\[ \phi_{\alpha \alpha}(\Omega \otimes \Omega') = \phi_{\alpha \alpha}(\overline{\Omega} \overline{\Omega'}) \quad (4) \]

where \(\phi_{\alpha \alpha}(\overline{\Omega} \overline{\Omega'}) = \phi_{\alpha \alpha}(\overline{\Omega} \overline{\Omega'}) \). One immediately sees that \(\phi_{\alpha \alpha}\) corresponds to the zero map when \(\Omega = \Omega'\), i.e.

\[ \phi_{+}(\phi_{\alpha \alpha} \otimes \phi_{\alpha \alpha}) = \phi_{\alpha \alpha} \overline{\Omega_+} \overline{\Omega_+} = \phi_{\alpha \alpha} \overline{\Omega_+} \overline{\Omega_+} = 0 \quad (5) \]

for all \(\phi_{\alpha \alpha} \otimes \phi_{\alpha \alpha} \in \mathcal{F} \mathcal{F} \otimes \mathcal{F} \mathcal{F} \). If, on the other hand, \(\Omega \neq \Omega'\), we obtain

\[ \phi_{+}(\phi_{\alpha \alpha} \otimes \phi_{\alpha \alpha}) = \phi_{\alpha \alpha} \overline{\Omega_+} \overline{\Omega_+} = \phi_{\alpha \alpha} \overline{\Omega_+} \overline{\Omega_+} = \phi_{\alpha \alpha} \overline{\Omega_+} \overline{\Omega_+}. \quad (6) \]

One may show that \(\phi_+\) and \(\phi_-\) do form linear isomorphisms (in particular, they map bases to bases). Moreover, we see that
\[ \phi_c (\rho, (u \otimes b)(\phi \Omega, \Theta \setminus \Omega)) = \phi_c (u \phi \Omega, \Theta \setminus \Omega) \]

\[ + \alpha \phi \Omega, \Theta \setminus \Omega \]

\[ - \rho_u \rho^k \phi \Omega, \Theta \setminus \Omega \]

\[ - \rho (u \otimes b)(\phi_c (\phi \Omega, \Theta \setminus \Omega)) \]

for all \( u, b \in F, \phi \Omega, \Theta \setminus \Omega \in F \otimes \mathcal{F} \) and thus \( \phi_c \) intertwines the action of \( \rho \) and \( \rho \). In a similar fashion, one may show that \( \phi_c \) is an intertwining operator for \( \rho \) and \( \rho \) and hence \( \rho \) is equivalent to both \( \rho \) and \( \rho \)\(^{12}\). Since the spinor irreps of \( \mathcal{D} \) induce (complex) spinor irreps of \( \mathcal{D}_K \) on the same carrier spaces, these results carry over equally to the spinor irreps of \( \mathcal{D}_K \). Consequently, given spinor irreps of the Dirac algebra or its real subalgebra on the Dirac spinor spaces \( \mathcal{D}_c \) or \( \mathcal{D}_c \); these results allow for the construction of an equivalent model expressed entirely within the framework of the Pauli algebra. It is this fact which leads to the formulation of Dirac theory in terms of the Pauli algebra.

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12 Actually, the equivalence of \( \rho \) and \( \rho \) immediately establishes the equivalence of \( \rho \) and \( \rho \).
CHAPTER TWO

DIRAC THEORY
2.1 The Dirac Equation

The Dirac equation for a free, spin-1/2 particle of mass \( m \) is

\[
\gamma^\mu p_\mu \psi(x) - m \psi(x),
\]

(1)

where \( p_\mu = i \partial_\mu \) is the four-momentum operator and \( \psi \) is a Dirac spinor field.¹

Eq. (2.1.1) arose out of Dirac's search for a relativistically covariant wave equation linear in the time derivative and satisfying the energy-momentum relation

\[
p^\mu p_\mu = E^2 - p \cdot p = m^2.
\]

(2)

the classical four-momentum \( p_\mu \) being identified with the quantum mechanical operator \( i \partial_\mu \) via the Schrödinger correspondence rule. Here, \( E = p_\alpha = i \partial_\alpha \) and \( p = -i \nabla \) are the energy and linear momentum operators respectively.

The first attempt to describe relativistic quantum theory resulted in the Klein-Gordon equation

\[
(\Box - m^2) \phi(x) = 0.
\]

(3)

where \( \Box = \gamma^\nu \gamma_\nu - \gamma^r \gamma_r \) and \( \phi \) is a complex scalar field. Eq. (2.1.3) is nothing more than a quantum mechanical statement of Eq. (2.1.2).

The problem with the Klein-Gordon equation is that it does not yield a positive-definite probability density. In addition, it yields both positive and negative-energy solutions. Although one may ignore the negative-energy solutions when dealing with free particles, the inclusion of interactions allows for energy exchange between the particle

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¹ We shall deal with the concept of a spinor field later in this section.
and external fields with the result being the possibility of the particle cascading down to infinite negative-energy states.

The first difficulty stems from the fact that the Klein-Gordon equation is of second order. This is what motivated Dirac to construct a wave equation linear in the time derivative. In fact, Dirac first proposed a Schrödinger-like equation of the form

\[ \mathcal{H} \psi(t, \mathbf{x}) = E \psi(t, \mathbf{x}). \]  

(4)

with \( E \) being identified with the energy operator \( i \hbar \). For consistency, the Hamiltonian \( \mathcal{H} \) was also assumed to be linear of the form

\[ \mathcal{H} = \alpha_0 p_0 - \beta m. \]  

(5)

where the \( \alpha_i \) and \( \beta \) were taken to be algebraic quantities defined in such a way as to satisfy Eq. (2.1.2), i.e.

\[ \mathcal{H}^2 \psi(t, \mathbf{x}) = E^2 \psi(t, \mathbf{x}) \]

\[ = (p_\mathbf{p}, -m^2) \psi(t, \mathbf{x}). \]

This, in turn, lead to the defining equations

\[ \alpha_i \alpha_j + \alpha_j \alpha_i = 2 \delta_{ij}, \]

\[ \beta \alpha_i + \alpha_i \beta = 0 \]

\[ \beta^2 = 1. \]

(6a)

(6b)

(6c)

In solving these equations, the \( \alpha_i \) and \( \beta \) were found to take the form of \( 4 \times 4 \) complex matrices. Consequently, \( \psi(t, \mathbf{x}) \) had to have the structure of a four-component column vector. In order to properly interpret \( \psi(t, \mathbf{x}) \) as a wavefunction, its components were taken to be complex-valued, square-integrable functions on \( \mathbb{R}^3 \), the time \( t \) being treated as an independent parameter. The inner product was defined according to both the usual inner product on \( \mathbb{C}^4 \) and the usual inner product on \( L^2 \). Since \( \mathcal{H} \), by definition, had to
be self-adjoint with respect to this inner product, the \( \alpha \) and \( \beta \) had to be self-adjoint with respect to the usual inner product on \( \mathbb{C}^4 \).

In making contact with Eq. (2.1.1), we may use the fact that \( \mathcal{D} = \mathcal{M}_i(\mathbb{C}) = \text{END}(\mathbb{C}^4) \). Since hermitean \( \mathcal{D} \)-elements correspond to self-adjoint elements of \( \text{END}(\mathbb{C}^4) \) let

\[
\beta = \gamma_0, \quad \alpha = \gamma_0 \gamma_i.
\]

Notice that, \( \gamma_0^* = \gamma_0 \), \( (\gamma_0 \gamma_i)^* = -\gamma_0 \gamma_i \gamma_0 \gamma_i \), \( \forall \gamma_i \in \mathcal{D} \) and thus the \( \alpha \) and \( \beta \) are indeed hermitean. Using Eq. (1.2.38), we see that

\[
\alpha \gamma_i \alpha = \gamma_0 \gamma_i \gamma_0 \gamma_i = \gamma_0 \gamma_i \gamma_0 \gamma_i = 2b_i,
\]

\[
\beta \alpha - \alpha \beta = \gamma_0 \gamma_i \gamma_0 \gamma_i = 0.
\]

Clearly, \( \beta^2 = \gamma_0^2 = I \) and thus the \( \alpha \) and \( \beta \) as defined in Eqs. (2.1.7), do satisfy Eqs. (2.1.6). Substituting Eqs. (2.1.7) into Eq. (2.1.4) and using the fact that \( \chi = \left( \begin{array}{c} 1 \\ \chi \end{array} \right) \) we obtain

\[
(\gamma_0 \gamma_i \rho_i - \gamma_0 \rho_i) \psi(x) = \mathcal{L} \psi(x)
\]

\[
\Rightarrow (\gamma_i \rho_i - m) \psi(x) = \gamma_0 \rho_0 \psi(x)
\]

\[
\Rightarrow (\gamma_0 \rho_0 - \gamma_i \rho_i) \psi(x) = m \psi(x)
\]

\[
\Rightarrow \gamma^i \rho_i \psi(x) = m \psi(x)
\]

which is indeed Eq. (2.1.1).

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2 It is known that the operators \( \rho_i = i \partial_i \) are self-adjoint on \( L^2 \) (Jordan 1964).
We would now like to interpret the Dirac equation in terms of an irreducible spinor representation of $\mathcal{D}$ on one of the Dirac spinor spaces. In order to accomplish this, we must first define what we mean by a Dirac spinor field. Now since the Dirac spinor spaces are all isomorphic as vector spaces to $\mathbb{C}^4$, there exists no difficulty, from the purely algebraic viewpoint, in identifying $\psi(\lambda)$ with a Dirac spinor. However, we must also be able to properly interpret $\psi(\lambda)$ as a wavefunction. To this end, let $\psi: \mathbb{H}^4 \rightarrow \mathcal{D}_{\text{ana}}$ be an appropriately differentiable mapping of $\mathbb{H}^4 \cong \mathbb{R}^4$ into the Dirac spinor space $\mathcal{D}_{\text{ana}}$, such that its components with respect to a basis $\{\Gamma^\mu|\mu \in \mathbb{N}^3\}$ for $\mathcal{D}_{\text{ana}}$ are appropriately differentiable functions of $\mathbb{H}^4$ into $\mathbb{C}$, i.e.

$$\psi(\lambda) = \psi'(\lambda) \Gamma^\mu \in \mathcal{D}_{\text{ana}}.$$  

We have already seen how one may incorporate differentiable, complex-valued functions within the framework of the Pauli algebra and the extension to including such elements in the Dirac algebra, especially given Eqs. (1.2.70), should be obvious. If we now demand that the components of $\psi$ be square-integrable on $\mathbb{R}^3 \subset \mathbb{H}^4$, we arrive at a workable formalism for identifying wavefunctions with Dirac spinors. It is in this sense that $\psi$ will be taken to define a Dirac spinor field. Of course, it remains to construct an inner product on $\mathcal{D}_{\text{ana}}$ consistent with the usual inner product on $\mathbb{C}^4$. This is indeed possible, although we shall defer its development until Section 2.3.

It should be pointed out that the definition we have given for a Dirac spinor field is not, rigourously speaking, correct. Indeed, although the Dirac equation does allow for the construction of a positive-definite probability
density, it still yields negative-energy solutions. As in the case of the Klein-Gordon equation, these negative-energy solutions cannot be ignored when including interactions. How then does one interpret the negative-energy solutions? As was asserted at the start of this chapter, the Dirac equation describes particles of spin-1/2. Consequently, Dirac predicted the vacuum to be composed of an "infinite sea" of negative-energy spin-1/2 particles; that is, all possible negative-energy states were assumed to be occupied so that, according to the Pauli exclusion principle, no particle of initial positive energy could decay down to a negative-energy state. Notice that one cannot apply this technique to the Klein-Gordon equation since, as it stands, it may describe particles of arbitrary spin.

This idea of Dirac, first formulated around 1930, lead invariably to the prediction of antiparticles, which were observed experimentally about two years later. In achieving this success, however, the concept of the Dirac equation as a single particle equation could no longer be rigourously maintained and one was forced to reinterpret the wavefunction $\psi(x)$ as a second-quantized field operator acting on a Hilbert space of state vectors. Interestingly, one may also rescue the Klein-Gordon equation for the description of spin-0 particles, such as the pion, by a similar re-interpretation of its wavefunction.

This procedure of second-quantizing the wavefunction requires the introduction of quantum field theoretic techniques beyond the scope of this study. We should point out, however, that the Feynman-Stückelberg formalism of identifying negative-energy particles running backwards in

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3 We shall consider this in Section 2.3.
time with positive-energy antiparticles running forwards in
time does allow for the construction of the propagator. The
propagator is an essential ingredient of quantum field
theory and ultimately yields the all-important Feynman rules
used in calculating scattering amplitudes.4

Since many formal characteristics of Dirac theory can be
derived without explicit reference to quantum field theory,
we shall deal with the Dirac equation in its first-quantized
form, foregoing, for the present, a true quantum field
theoretic description.

The final point we must address centers on interpreting
the four-momentum operator within the Clifford algebra
structure. Recall that we complexified the real Dirac
algebra in order to properly consider quantities like the
four-momentum operator; and although we saw how one may deal
heuristically with the differential operator in the Pauli
algebra, the formal definition of a Clifford algebra cannot
encompass the inclusion of such elements. This is because
the components of the differential operator do not satisfy
the axioms required of an element belonging to the ground
field of the Clifford algebra.

A solution to this dilemma is to first let each operator
\( i\partial_\mu \) act independently on the spinor field \( \psi \) to produce four
new Dirac spinors \( \psi_\mu = (i\partial_\mu \psi)(x) \) \((\mu \in \mathbb{N}_0^4)\) and then view the
quantity \( \gamma^\mu \psi_\mu (x) \) as a sum with \( \gamma^\mu \) acting on \( \psi_\mu (x) \) via the
left regular representation. In particular, consider Eq.
(2.1.8). Since each component \( \psi^\nu \) of \( \psi \) is an appropriately
differentiable function of \( \mathcal{M}^4 \) into \( \mathbb{C} \), \( \psi_\mu = i\partial_\mu \psi^\nu \) can be
interpreted as satisfying the properties of a Dirac spinor

4 For a description of the propagator, see Bjorken and
Drell (1964).
component for each \( \mu \in \mathbb{N}^\alpha_j \). Consequently, \( \psi_\mu(\chi) = ((i\gamma_\mu \psi)(\chi))(\gamma_\chi \psi_\mu(\chi)) = \psi_\mu(\chi) \gamma_\chi \) is a Dirac spinor for each \( \mu \in \mathbb{N}^\alpha_j \), and since \( \gamma_\mu \in \mathcal{D} \) \((\forall \mu \in \mathbb{N}^\alpha_j)\), we may interpret \( \gamma_\mu \psi_\mu(\chi) \) as

\[
\gamma_\mu \psi_\mu(\chi) = \sum_{\mu} \rho_{\mu \alpha} (\gamma_\mu)(\psi_\mu(\chi)).
\]

where \( \psi_\mu(\chi) \in \mathcal{D}_{\alpha \alpha} \) \((\forall \mu \in \mathbb{N}^\alpha_j)\) (see Section 1.3).

Given this interpretation, the Dirac equation becomes

\[
\gamma_\mu \psi_\mu(\chi) = m \psi(\chi). \tag{9}
\]

This clearly allows us to interpret Eq. (2.1.1) in terms of the spinor irreps of \( \mathcal{D} \): however, Eq. (2.1.9) masks the fact that the \( \delta_\mu \) do behave, from the physical point of view, like the components of a four-vector. In particular, they transform under \( L^\dagger \) as the covariant components of a four-vector.\(^5\) The \( i \) is included in the definition of the four-momentum operator in order to make it self-adjoint with respect to the underlying \( L^2 \) structure.

What we are really dealing with in the Dirac equation are two algebraic structures: an algebra of operators describing the fundamental variables of the system and a Clifford algebra describing the intrinsic variables of the system. In fact, following the formalism of Messiah\(^6\), we may view the Dirac spinor \( \psi(\chi) \) as the configuration-space representation of an abstract state vector \( \psi(i) \) belonging

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5 We shall consider this in Section 2.2.

to the state-space $\mathcal{E}^{(0)} \otimes \mathcal{E}^{(1)}$, where $\mathcal{E}^{(0)}$ is the orbital-variable space and $\mathcal{E}^{(1)}$ the intrinsic-variable space:

$$\psi(\Lambda) = \psi^\mu(\Lambda) \Gamma_\mu$$

$$= \psi^\mu(l, \Lambda) \Gamma_\mu$$

$$- <\Lambda | \psi^\mu(l)> \Gamma_\mu$$

$$- <\Lambda | \psi(l)> .$$

Note that we have used simple juxtaposition in place of the tensor product symbol (as is commonly done in the literature). We then have

$$|\psi(t)> = |\psi^\mu(t)> \Gamma_\mu.$$

where $|\psi^\mu(t)> \in \mathcal{E}^{(0)}$ and $\Gamma_\mu \in \mathcal{E}^{(1)}$ for all $\mu \in \mathbb{N}_0^3$. In this formalism, time is again taken as a simple parameter while the spatial coordinates are included among the dynamical variables of the system.

If we now blur this distinction between time and space as demanded by relativity, we may set up a correspondence between the orbital-variable space $\mathcal{E}^{(0)}$ and the C-valued functions of $\mathcal{M}^3$ which are square-integrable on $\mathbb{R}^3 \subset \mathcal{M}^4$. It is this space upon which the components of the four-momentum operator $p_\mu$ act.

In analyzing the intrinsic-variable space, we clearly see that $\mathcal{E}^{(1)}$ may be taken to correspond with the Dirac spinor space $\mathcal{D}_{20}$. Moreover, in analogy with Messiah's decomposition of $\mathcal{E}^{(1)}$ into the tensor product space $\mathcal{E}^{(1)} = \mathcal{E}^{(s)} \otimes \mathcal{E}^{(s)}$, $\mathcal{D}_{20}$, as we have already seen, is to be identified with $\mathcal{P}\mathcal{O} \otimes \mathcal{P}\mathcal{O}'$, where $\mathcal{P}\mathcal{O}$ and $\mathcal{P}\mathcal{O}'$ are Pauli spinor spaces. The space $\mathcal{E}^{(s)} \otimes \mathcal{P}\mathcal{O}$ allows for the description of spin-1/2 fermions and can be taken as spin space while the presence of $\mathcal{E}^{(s)} \otimes \mathcal{P}\mathcal{O}'$ attests to the existence of both positive and negative-energy solutions leading to its
interpretation as particle-antiparticle space. Clearly, \( E^{(1)} \sim \mathcal{D}_{u\alpha} \) is the space upon which the \( \gamma_\mu \) act. Moreover, since \( \gamma_\mu \in \mathcal{D}_K (\bigvee_\mu (N_3^\alpha)) \), we may interpret \( \mathcal{D}_{u\alpha} \) as the carrier space for a complex spinor irrep of \( \mathcal{D}_K \subset \mathcal{D} \). As we shall see, viewing the Dirac equation in terms of these two algebraic structures provides valuable insight into the nature of angular momentum.

Having now dealt with these topics, we would like to apply the results of Section 1.5 to write the Dirac equation within the framework of the Pauli algebra. Expressing Eq. (2.1.1) in terms of a spinor irrep \( \rho_{\alpha \alpha} \) of \( \mathcal{D}_K \subset \mathcal{D} \), we have

\[
\rho_{\alpha \alpha} (\gamma_\mu) (i \partial^\mu \psi(x)) = m \psi(x). \quad (10)
\]

where \( i \partial^\mu \psi(x) = (i \partial^\mu \psi)(x) \in \mathcal{D}_{\alpha \alpha} \). Since the map \( \phi_{\alpha \alpha} \) defined in Eq. (1.5.4) is an intertwining isomorphism only for \( \Omega \neq \Omega' \), and since the Dirac spinor spaces are all isomorphic as vector spaces, we shall, without loss of generality, take \( \psi(x) \) to be an element of \( \mathcal{D} \). Eq. (2.1.10) then becomes

\[
\rho_{\ldots} (\gamma_\mu) (i \partial^\mu \psi(x)) = m \psi(x). \quad (11)
\]

where, using Eq. (1.3.11c),

\[
\psi(x) = \bar{\mathcal{H}}(x) \Omega. \theta \omega_+ + \Xi(x) \Omega. \theta \Omega. \quad (12)
\]

Here, \( \bar{\mathcal{H}}(x) \Omega. = \bar{\mathcal{M}}(x) \Omega_+ \) and \( \Xi(x) \Omega_+ \) are Pauli spinors defined in such a way so that their components with respect to any basis for \( \mathcal{F} \) are appropriately differentiable \( \mathbb{C} \)-valued functions of \( \mathcal{M}^4 \) square-integrable on \( \mathbb{R}^3 \subset \mathcal{M}^4 \); that is, \( \Xi \) and \( \mathcal{H} \) are Pauli spinor fields. Applying the isomorphism \( \phi_{\ldots} \) and using the equivalence of \( \rho_{\ldots} \) and \( \rho \) we obtain

\[
\phi_{\ldots} (\rho_{\ldots} (\gamma_\mu) (i \partial^\mu \psi(x))) = m \phi_{\ldots} (\psi(x)) \Rightarrow \rho (\gamma_\mu) (\phi_{\ldots} (i \partial^\mu \psi(x))) = m \phi_{\ldots} (\psi(x)). \quad (13)
\]
Eq. (2.1.13) represents the Dirac equation expressed in terms of the Pauli algebra. In order to make this explicit, consider Eq. (1.5.3). Using the definition of the irrep \( \rho \) given in Eq. (1.5.3) we have
\[
\rho(\gamma_\mu) = \rho(e_\mu \otimes \omega_- + \bar{e}_\mu \otimes \omega_-)
= \rho(e_\mu \otimes \omega_-) + \rho(\bar{e}_\mu \otimes \omega_-)
= \rho_{\epsilon^\mu} \circ \rho_{\omega_\mu} + \rho_{\bar{e}^\mu} \circ \rho_{\omega_\mu}.
\]

Moreover, Eq. (2.1.12) and the definition of \( \phi_- \) show that
\[
\phi_- (i \partial^\mu \psi (x)) = \phi_- (i \partial^\mu (\bar{\Omega}^\dagger (x) \Omega_+ \otimes \omega_- + \bar{\Xi} (x) \Omega_+ \otimes \Omega_-))
= \phi_- (i \partial^\mu \bar{\Omega}^\dagger (x) \Omega_+ \otimes \omega_-) + \phi_- (i \partial^\mu \bar{\Xi} (x) \Omega_+ \otimes \Omega_-)
= i \partial^\mu \bar{\Omega}^\dagger (x) \Omega_+ \otimes \omega_- + i \partial^\mu \bar{\Xi} (x) \Omega_+ \otimes \Omega_-
= i \partial^\mu \phi_- (\bar{\Omega}^\dagger (x) \Omega_+ \otimes \omega_- + \bar{\Xi} (x) \Omega_+ \otimes \Omega_-)
= i \partial^\mu \phi_- (\psi (x)).
\]

where
\[
\phi_- (\psi (x)) = \bar{\Omega}^\dagger (x) \Omega_+ \otimes \omega_- + \bar{\Xi} (x) \Omega_+ \otimes \Omega_-.
\]

Consequently, Eq. (2.1.13) becomes
\[
\left( \rho_{\epsilon^\mu} \circ \rho_{\omega_\mu} + \rho_{\bar{e}^\mu} \circ \rho_{\omega_\mu} \right) \left( i \partial^\mu \psi (x) \right) - m \psi (x)
= e_\mu i \partial^\mu \psi (x) \bar{\omega}_- + \bar{e}_\mu i \partial^\mu \psi (x) \bar{\omega}_- = m \psi (x)
\Rightarrow
\rho \psi (x) \omega_- + \bar{\rho} \psi (x) \omega_- + m \psi (x) = 0.
\]

where \( \rho = p^\mu e_\mu = i \partial^\mu e_\mu = i \partial^\mu \). \( \partial \) being the differential operator defined in Eq. (1.2.17). We shall call \( \rho \) the Pauli algebra four-momentum operator.
Eq. (2.1.14) is the explicit form taken by the Dirac equation in the Pauli algebra. Clearly, we may interpret $\Psi$ as a wavefunction:

$$\Psi(x) = \Xi(x)\Omega - \Pi^*(x)\omega_\downarrow.$$  \hspace{1cm} (15)

with $\Xi$ and $\Pi$ appropriately differentiable functions of $\mathbb{M}^4$ into $\mathbb{F}$. Moreover, the components of $\Psi$ with respect to any basis for $\mathbb{F}$ are to be interpreted as $\mathbb{C}$-valued, square-integrable functions on $\mathbb{R}^3 \subset \mathbb{M}^4$.

We may also interpret elements acting on $\Psi$ to the left as operators in either orbital space, spin space, or both. For example, consider the Pauli algebra four-momentum operator $p^\mu = p^\mu e_\mu$. The $p^\mu$ act on the orbital space $\mathcal{E}^{(o)}$ while the $e_\mu$ act on the spin space $\mathcal{E}^{(s)}$. In general, we shall consider such quantities as operators on orbital-spin space. The elements acting on $\Psi$ to the right can be taken as operators in the particle-antiparticle space $\mathcal{E}^{(s)}$.

So far, we have restricted our attention to free particles. Consider now a charged particle in the presence of an external electromagnetic field described by the four-potential $A_\mu(x)$. The coupling is most simply introduced by means of the $U(1)$ gauge-invariant minimal substitution

$$p_\mu \rightarrow p_\mu - qA_\mu = \pi_\mu,$$

where $q$ is the charge of the particle.\footnote{For an electron, $q = +e$, where $e < 0$. The $\pi_\mu$ are the components of the minimally coupled momentum in contrast to the $p_\mu$ which are the components of the Lagrange canonical momentum called here simply the momentum (Messiah, Vol. II, p. 884).} The Dirac equation in $D$ now takes the form
\[ \gamma''(i\hat{\omega} - \eta \lambda_{n}) \psi - m \psi. \]

where we have dropped the \( \chi \) dependence for notational simplicity. Expressed in terms of the Pauli algebra, this equation becomes

\[ \pi \psi_{\omega} + \bar{\pi} \psi_{\omega} + m \psi = 0. \]

(16)

where

\[ \pi = \rho - q \lambda. \]

(17)

\( A \in \mathcal{F} \) being the vector potential defined in Eq. (1.2.11c).

Now consider substituting Eq. (2.1.14) into (2.1.16). Again suppressing the \( \chi \) dependence, Eqs. (1.1.59) show that

\[ \psi_{\omega} = -\bar{\pi} \Omega. \quad \psi_{\omega} = \bar{\omega}. \]

(18)

and Eq. (2.1.16) becomes

\[ -\pi \bar{\pi} \Omega + \bar{\pi} \bar{\omega} + m \bar{\omega} - m \bar{\pi} \omega = 0. \]

(19)

If we now multiply Eq. (2.1.19) on the right by \( \omega \) we obtain

\[ -\pi \bar{\pi} \Omega + m \bar{\omega} = 0. \]

Similarly, multiplying (2.1.19) on the right by \( \omega \) gives

\[ \bar{\pi} \bar{\omega} - m \bar{\pi} \Omega = 0. \]

Consequently, we see that the Dirac equation may be written as two first-order coupled equations

\[ \bar{\pi} \bar{\omega} = m \bar{\pi} \Omega. \]

(20a)

\[ \pi \bar{\pi} \Omega = m \bar{\omega}. \]

(20b)

Since the Pauli spinor spaces are isomorphic as vector spaces to \( C^{2} \), we may identify \( \bar{\omega} \) and \( \bar{\pi} \Omega \) with two-component column spinors. Comparing with the usual two-component spinor form of the Dirac equation presented in the literature, we see that \( \bar{\omega} \) corresponds to a two-component spinor of rank \( (1,0) \) while \( \bar{\pi} \Omega \) corresponds to
a two-component spinor of rank \((0,1)\)." Moreover, the fact that rank \((1,0)\) and \((0,1)\) spinors transform under inequivalent representations of the restricted Lorentz group is clearly borne out by Eqs. (1.4.54).

Consider now adding and subtracting Eqs. (2.1.20). We then have
\[ n_0(\bar{\xi} + \bar{H}^*) \Omega_+ - \bar{\eta}(\bar{\xi} - \bar{H}^*) \Omega_+ = m(\bar{\xi} + \bar{H}^*) \Omega_+ \]  
\[ n_0(\bar{\xi} - \bar{H}^*) \Omega_+ - \bar{\eta}(\bar{\xi} + \bar{H}^*) \Omega_+ = -m(\bar{\xi} - \bar{H}^*) \Omega_+ \]  
and letting
\[ \varphi \Omega_+ \cdot \frac{1}{\sqrt{2}}(\bar{\xi} + \bar{H}^*) \Omega_+ \quad \chi \Omega_+ \cdot \frac{1}{\sqrt{2}}(\bar{\xi} - \bar{H}^*) \Omega_+ \]  
we obtain
\[ (n_0 - m) \varphi \Omega_+ = -\bar{\eta} \chi \Omega_+ \]  
\[ (n_0 + m) \chi \Omega_+ = -\bar{\eta} \varphi \Omega_+ \]  
Again, we may identify the Pauli spinors \(\varphi \Omega_+\) and \(\chi \Omega_+\) with two-component column spinors and, comparing with the literature, one sees that, for positive-energy solutions, \(\varphi \Omega_+\) and \(\chi \Omega_+\) correspond to the large and small components of the Dirac wavefunction respectively.\(^9\)

Notice that, in deriving Eqs. (2.1.20) and (2.1.23), no mention was made to any particular representation for \(\mathcal{D}\) in contrast to what is normally found in the literature. In fact, Eqs. (2.1.20) are normally derived by making explicit use of the Weyl representation while Eqs. (2.1.23) are obtained using the standard representation. For example,

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8 Berestetskii et al 1982, Ryder 1985, and Baylis and Jones 1989[1].

9 This nomenclature arises after a study of the Dirac equation in the non-relativistic limit.
Halzen and Martin (1984) write the Hamiltonian form of the Dirac equation in the standard rep to derive the free particle equivalent of Eqs. (2.1.23).

Now the Dirac equation in Hamiltonian form is simply

$$\mathcal{H}\psi = E\psi.$$  \hspace{1cm} (21)

where, for a free particle, $\mathcal{H}$ is given by Eq. (2.1.5).

Using Eqs. (2.1.7) and (1.2.70), we have

$$\beta = 1 \otimes c_1, \quad \alpha = c_2 \otimes c$$  \hspace{1cm} (25)

and thus $\mathcal{H}$ may be written as

$$\mathcal{H} = m \otimes c_1 - p \otimes c.$$  \hspace{1cm} (26)

Eq. (2.1.4) then becomes

$$(m \otimes c_1 - p \otimes c)\psi = (E \otimes 1)\psi$$

and, in the Pauli algebra, we obtain

$$p\Psi c - m\Psi c_1 = E\Psi.$$  \hspace{1cm} (27)

Eq. (2.1.27) also follows directly from Eq. (2.1.12). In particular, we see that

$$p\Psi \omega_+ - p\Psi \omega_- + m\Psi = 0$$

$$\Rightarrow p_0\Psi(\omega_+ + \omega_-) + p\Psi(\omega_+ - \omega_-) + m\Psi = 0$$

$$\Rightarrow p_0\Psi c_1 + p\Psi c_2 + m\Psi = 0.$$  \hspace{1cm} (28)

If we now multiply on the right by $c_1$ and use the result $c_2c_1 = -c_1c_2 = -c$, together with the fact that $p_0 = E$, we obtain Eq. (2.1.27) as asserted.

Now define the $P$-elements

$$c_+ = \frac{1}{\sqrt{2}}(c_1 \pm c).$$  \hspace{1cm} (29)

One may show that $c_+$ and $c_-$ are hermitean unit vectors in $P$.

Defining the new wavefunction

$$\phi = \Psi c_+.$$  \hspace{1cm} (29)

Eq. (2.1.27) may be rewritten as
\[ p\Phi_\epsilon c - m \Phi_\epsilon c = E\Phi_\epsilon c. \]

If we now multiply on the right by \( c \) and use the fact that
\[ c.e_1.e_1 = e, c.e_1 = e_1 e = e_1, e_1 c = e \]
\[ \Rightarrow e.c.e = e.c.c.e = e, e = e_1 \]

we obtain the equation
\[ p\Phi_\epsilon c - m \Phi_\epsilon c = E\Phi_\epsilon c, \]

where
\[ \Phi = \Psi c = \left( \tilde{\omega}_+ - \tilde{\Omega}_+ \right) \frac{1}{\sqrt{2}} (e_1 + c) \]

\[ = \frac{1}{\sqrt{2}} \left( \tilde{\omega}_+ - \tilde{\Omega}_+ + \tilde{\omega}_+^* + \tilde{\Omega}_+^* \right) \]

\[ = \frac{1}{\sqrt{2}} (\tilde{\omega}_+ - \tilde{\Omega}_+^*) \Omega_+ + \frac{1}{\sqrt{2}} (\tilde{\omega}_+^* + \tilde{\Omega}_+^*) \omega_+ \]

\[ = \chi \Omega_+ + \phi \omega_+. \]

Clearly, Eq. (2.1.31) may also be taken to represent the Dirac equation. Substituting Eq. (2.1.32) into (2.1.31) we have that
\[ p(\chi \Omega_+ + \phi \omega_+) c_1 - m(\chi \Omega_+ + \phi \omega_+) c = E(\chi \Omega_+ + \phi \omega_+) \]
\[ \Rightarrow p(\chi \omega_+ + \phi \Omega_+) - m(\chi \Omega_+ - \phi \omega_+) = E(\chi \Omega_+ + \phi \omega_+) \]
\[ \Rightarrow p(\chi \omega_+ - \phi \Omega_+) = (E + m) \chi \Omega_+ + (E - m) \phi \omega_+ \]

and multiplying on the right by \( \Omega_+ \) and \( \omega_+ \) then gives the equations
\[ (E + m) \chi \Omega_+ = p \phi \Omega_+ \]
\[ (E - m) \phi \Omega_+ = p \chi \Omega_+ \]

which are immediately seen to be the free particle equivalent of Eqs. (2.1.23).
This example clearly demonstrates that the common practice of choosing different representations for deriving different forms of the Dirac equation is wholly unnecessary when working in the abstract algebra.

We shall now consider solutions to the free particle Dirac equation. In particular, we find that we may construct four solutions — two corresponding to positive energy and two corresponding to negative energy. The existence of both positive and negative-energy solutions is a result of the breakdown of the Dirac equation into two coupled, first order equations. This is possible since particle-antiparticle space may be taken to be two-dimensional over the reals (recall that every Dirac spinor can be written so that all complex coefficients are contained in orbital-spin space). This decomposition of the Dirac equation is achieved by the use of projection-type operators in particle-antiparticle space defined through right multiplication. The existence of two solutions for each sign of the energy is due to the fact that spin space is two-dimensional over $\mathbb{C}$, i.e. for every $\alpha\Omega \in \mathbb{C}^2$, we have $a\tilde{\Omega} = a\Omega + \beta\varphi$ where $a, b \in \mathbb{C}$. This double, two-fold degeneracy of the Dirac equation is a manifestation of its description of both positive and negative-energy spin-$1/2$ fermions. Given that operators on spin space act to the left, the spin space degeneracy implies the existence of some operator $S$ commuting with the Pauli algebra four-momentum operator (or, equivalently, the free particle Hamiltonian in $\mathcal{F}$) such that $\Psi$ (or, equivalently, $\Phi$) are eigenstates of $S$. 
To begin with, consider the free particle equivalent of Eqs. (2.1.20):

\[ p \bar{p} \bar{\Omega} = m \bar{\Pi} \bar{\Omega}. \quad (34a) \]

\[ p \bar{\Pi} \bar{\Omega} = m \bar{\Xi} \bar{\Omega}. \quad (34b) \]

We see that

\[ p \bar{p} \bar{\Xi} \bar{\Omega} = m p \bar{\Pi} \bar{\Omega} = m^2 \bar{\Xi} \bar{\Omega}. \quad (35a) \]

\[ p \bar{p} \bar{\Xi} \bar{\Omega} = m p \bar{\Xi} \bar{\Omega} = m^2 \bar{\Pi} \bar{\Omega}. \quad (35b) \]

Clearly, \( \bar{p} \bar{p} = \bar{\Pi} \bar{\Pi} = -\Box \), so that each component of \( \Psi \), and thus \( \Psi \), is a solution to the Klein-Gordon equation, i.e.

\[ (\Box + m^2) \Psi = 0. \quad (36) \]

Consequently, we may construct plane wave solutions of the form

\[ \Psi(x) = \Psi(p) e^{ip \cdot x}. \quad (37) \]

where \( p \cdot x = \rho \cdot x \) and \( \Psi(p) \) is independent of \( x \). Here, \( \rho \in M^4 \subset \mathcal{F} \) is the actual four-momentum and not the Pauli algebra four-momentum operator. \( \Psi(p) \) is then seen to be a solution to the free particle Dirac equation in momentum space, i.e.

\[ p \Psi(p) \omega + \bar{p} \Psi(p) \omega - m \Psi(p) = 0. \quad (38) \]

It is important to realize that the \( p \) appearing in Eq. (2.1.38) is a true four-vector and not a differential operator. Clearly, \( \Psi(p) \) may be written in the form

\[ \Psi(p) = \Xi(p) \Omega - \bar{\Pi} \Pi(p) \omega. \quad (39) \]

and, substituting Eq. (2.1.39) into (2.1.38) leads to the equations

\[ p \bar{\Xi}(p) \Omega = m \bar{\Pi}(p) \Omega. \quad (40a) \]

\[ p \bar{\Pi}(p) \Omega = m \bar{\Xi}(p) \Omega. \quad (40b) \]
Now assume $\rho > 0$ and let
\[ \tilde{H}^-(\rho)\Omega_+ = \Omega_+ . \]
Using Eq. (2.1.40b), we then have
\[ \tilde{z}(\rho)\Omega_+ = \left( \frac{E + p}{m} \right)\Omega_+ . \]
and one easily verifies that
\[ \Psi(E > 0, \rho, +1/2) = \left( \frac{E + p}{m} \right)\Omega_- - \omega_+ . \]
\[ (41a) \]
is a positive-energy solution of Eq. (2.1.38).
Again assuming $E > 0$, let
\[ \tilde{H}^-(\rho)\Omega_- = \omega_+ . \]
Then
\[ \tilde{z}(\rho)\Omega_- = \left( \frac{E - p}{m} \right)\omega_+ . \]
and we obtain a second positive-energy solution
\[ \Psi(E > 0, \rho, -1/2) = \left( \frac{E - p}{m} \right)\omega_+ - \Omega_- . \]
\[ (41b) \]
For $E < 0$, let
\[ \tilde{z}(\rho)\Omega_- = \Omega_- . \]
Eq. (2.1.40a) then gives
\[ \tilde{H}^-(\rho)\Omega_- = \left( \frac{E - p}{m} \right)\Omega_- , \]
and we obtain the negative-energy solution
\[ \Psi(E < 0, \rho, +1/2) = \Omega_- - \left( \frac{E - p}{m} \right)\omega_+ . \]
\[ (42a) \]
Finally, assume $E < 0$ and let
\[ \tilde{z}(\rho)\Omega_- = \omega_- . \]
Then
\[ \tilde{H}^-(\rho)\Omega_- = \left( \frac{E - p}{m} \right)\omega_- . \]
and we have
\[\psi(E<0,p,-1/2) = \omega_+ \left(\frac{E-p}{m}\right) \Omega_- .\] (42b)

The parameters +1/2 and -1/2 indicate the spin polarization with respect to the unit vector \(\vec{e}_z\) and will be taken to define spin-up and spin-down states respectively.

In looking for plane wave solutions to the free particle Dirac equation we could have started equally with Eqs. (2.1.31). In particular, taking \(\Phi(x) = \Phi(p)\exp(-ip\cdot\vec{e}_z)\). one may show that
\[p\Phi(p)e_1 - m\Phi(p)e = E\Phi(p)\] (43)
\[\Rightarrow (E+m)\chi(p)\Omega_+ = p\Phi(p)\Omega_- \] (44a)
\[(E-m)\Phi(p)\Omega_- = p\chi(p)\Omega_+ \] (44b)

Again, these equations are defined in momentum space so that \(p = E + p\) is the actual four-momentum. Employing techniques identical to those used in solving Eqs. (2.1.40), we obtain the solutions
\[\Phi(E>0,p,+1/2) = \left(\frac{p}{E+m}\right) \Omega_+ + \omega_- \] (45a)
\[\Phi(E>0,p,-1/2) = \left(\frac{p}{E+m}\right) \omega_- - \Omega_+ \] (45b)
\[\Phi(E<0,p,+1/2) = \Omega_+ \left(\frac{p}{E-m}\right) \omega_- \] (46a)
\[\Phi(E<0,p,-1/2) = \omega_- \left(\frac{p}{E-m}\right) \Omega_+ \] (46b)

Eqs. (2.1.45) and (2.1.46) correspond to positive and negative-energy solutions respectively.
Now define new wavefunctions

\[ \Psi(F > 0, \pm \hat{\rho}, \pm 1/2) = \frac{1}{2}(1 \pm \hat{\rho})\Psi(F > 0, \rho, \pm 1/2) \quad (47a) \]

\[ \Psi(F < 0, \pm \hat{\rho}, \pm 1/2) = \frac{1}{2}(1 \pm \hat{\rho})\Psi(F < 0, \rho, \pm 1/2) \quad (47b) \]

\[ \Phi(F > 0, \pm \hat{\rho}, \pm 1/2) = \frac{1}{2}(1 \pm \hat{\rho})\Phi(F > 0, \rho, \pm 1/2) \quad (48a) \]

\[ \Phi(F < 0, \pm \hat{\rho}, \pm 1/2) = \frac{1}{2}(1 \pm \hat{\rho})\Phi(F < 0, \rho, \pm 1/2). \quad (48b) \]

Notice the difference in the arrangement of signs between the positive and negative-energy solutions — in particular, the staggering of the signs in the negative energy case. Since \( \rho \) and \( \bar{\rho} \) commute with \( \frac{1}{2}(1 \pm \hat{\rho}) \), these wavefunctions are also solutions to Eqs. (2.1.38) and (2.1.43) respectively. Moreover, they are immediately seen to be eigenstates of the \( \mathcal{F} \)-element \( \frac{1}{2}\hat{\rho} \) having eigenvalues \( +1/2 \) or \( -1/2 \). The \( \mathcal{F} \)-element \( \frac{1}{2}\hat{\rho} \) is the helicity operator in the Pauli algebra. It is a measure of the spin in the direction of the momentum. We now have a scheme for labelling solutions to the free particle Dirac equation using the helicity and sign of the energy.

In multiplying Eqs. (2.1.42) and (2.1.45) by \( \frac{1}{2}(1 + \hat{\rho}) \) or \( \frac{1}{2}(1 - \hat{\rho}) \) we have effectively polarized the solutions parallel with the direction of the momentum. In generalizing the concept of spin polarization, consider the \( \mathcal{D} \)-element

\[ \psi(\rho, s) = \frac{1}{2m}(m + \gamma^\nu \rho_\nu)^{1/2}(1 + \gamma_5\gamma^\nu s_\nu). \quad (49) \]

One may show \( \psi(\rho, s) \) to be a solution to the free particle Dirac equation in momentum space:

\[ \gamma^\nu \rho_\nu \psi(\rho, s) = m\psi(\rho, s). \quad (50) \]
where : is a space-like four-vector orthogonal to the four-momentum \( p \), i.e.

\[
s^\mu p_\mu = 0 \Rightarrow \bar{p} \cdot s = 0 \quad (51a)
\]

\[
s^\mu s_\mu = -1 \Rightarrow \bar{s} p = -1. \quad (51b)
\]

\( \psi(p, s) \) represents a free particle solution having energy \( E \), momentum \( p \) and spin polarization \( s \).

The problem with \( \psi(p, s) \) is that it does not take the form we have presented for a Dirac spinor. Nevertheless, one may show \( \psi(p, s) \) to be an element of a minimal left ideal of the Dirac algebra.\(^{10,11}\) In fact, one may show that the \( \mathcal{D} \)-elements

\[
P_{-}(p, s) = \frac{1}{2m}(m - \gamma^\mu p_\mu)\frac{1}{2}(1 + \gamma_5 \gamma^\nu s_\nu) \quad (52a)
\]

\[
P_{+}(p, s) = \frac{1}{2m}(m - \gamma^\mu p_\mu)\frac{1}{2}(1 + \gamma_5 \gamma^\nu s_\nu) \quad (52b)
\]

\[
P_{-}(p, s) = \frac{1}{2m}(m + \gamma^\mu p_\mu)\frac{1}{2}(1 + \gamma_5 \gamma^\nu s_\nu) \quad (52c)
\]

\[
P_{+}(p, s) = \frac{1}{2m}(m + \gamma^\mu p_\mu)\frac{1}{2}(1 + \gamma_5 \gamma^\nu s_\nu) \quad (52d)
\]

are primitive idempotents forming an idempotent decomposition of \( \mathcal{D} \).

Now let \( P(p, s) \) be one of the primitive idempotents listed in Eqs. (2.1.52) and let \( S = \mathcal{D} P(p, s) \) be the corresponding minimal left ideal of \( \mathcal{D} \). Based on the results of Section 1.3, we may construct an irreducible spinor representation \( \rho \) of \( \mathcal{D} \) on \( S \) via the left regular representation. Since \( \mathcal{D} \) is simple, all four \( \rho \) are equivalent. More importantly, since \( \mathcal{D} \simeq \mathbb{F} \otimes \mathbb{F} \), each \( \rho \) is equivalent to the spinor irrep \( \rho \).

\(^{10}\) Bjorken and Drell 1964.

\(^{11}\) Ross 1986.
defined in Eq. (1.5.3). Consequently, there exists an isomorphism \( \phi: \mathcal{S} \rightarrow \mathcal{F} \) intertwining the action of \( \rho_3 \) and \( \rho \). The search for such a \( \phi \) is greatly simplified by the use of Schur’s Lemma (see Appendix 1) — since \( \rho \) and \( \rho_3 \) are irreducible, any intertwining map is automatically a linear isomorphism. Given our identification of \( \mathcal{D} \) with \( \mathcal{F} \otimes \mathcal{F} \), we may identify \( \mathcal{S} \) with a subalgebra of \( \mathcal{F} \otimes \mathcal{F} \). Any \( \psi(\mathcal{S}) \) will then take the form

\[
\psi = \sum_{i \in I} \phi_i \otimes X_i. \tag{53}
\]

where \( I \) is some countably finite index set and \( \phi_i, X_i \in \mathcal{F} (\forall i \in I) \). One may readily show that the map \( \phi: \mathcal{S} \rightarrow \mathcal{F} \) defined by

\[
\phi(\sum_{i \in I} \phi_i \otimes X_i) = \sum_{i \in I} \phi_i \otimes X_i = \sum_{i \in I} \phi_i \otimes X_i \tag{54}
\]

intertwines the action of \( \rho_3 \) and \( \rho \). Consequently, \( \phi \) is an intertwining isomorphism and thus \( \rho \) is equivalent to \( \rho_3 \).

We may now apply this result to the solution \( \psi(\rho, s) \) given in Eq. (2.1.49). Comparing with Eqs. (2.1.52), we see that \( \psi(\rho, s) = r_{\rho}(\rho, s) \) and thus \( \psi(\rho, s) \) is trivially an element of \( \mathcal{S}_{\rho} \). Identifying \( \psi(\rho, s) \) with an element of \( \mathcal{F} \otimes \mathcal{F} \) using Eqs. (1.2.84) and (1.2.89), we get

\[
\psi(\rho, s) = \frac{1}{4m} \left[ (m - \rho s) \otimes \Omega_+ + (\rho + ms) \otimes \omega_- + (\rho - m s) \otimes \omega_+ + (m - \rho s) \otimes \Omega_- \right]. \tag{55}
\]

Mapping \( \psi(\rho, s) \) to \( \mathcal{F} \) by means of the isomorphism \( \phi \) defined in Eq. (2.1.54) then yields

\[
\Psi(\rho, s) = \phi(\psi(\rho, s)) = \frac{1}{4m} \left[ (m - \rho s) \Omega_+ - (\rho + ms) \omega_- - (\rho - m s) \omega_+ + (m - \rho s) \Omega_- \right]
\]

\[
= \Xi(\rho, s) \Omega_- - \Xi^*(\rho, s) \omega_+. \tag{56}
\]

where
\begin{align}
\Psi(p,s) &= \frac{1}{\sqrt{m}} \left( (m - p\bar{s})\Omega_+ - (p - ms)\omega_+ \right) \tag{57a} \\
\Pi^\uparrow(p,s) &= \frac{1}{\sqrt{m}} \left( (\bar{p} - ms)\Omega_+ - (m + \bar{p}s)\omega_+ \right). \tag{57b}
\end{align}

Using the energy-momentum relation $\rho\bar{\rho} = \bar{p}\rho = m^2$, one easily verifies $\Psi(p,s)$ to be a solution of Eq. (2.1.38) describing a spin-1/2 particle of energy $E$, momentum $\rho$ and spin polarization $s$.

We may use $\Psi(p,s)$ to create truncated solutions to Eq. (2.1.38). In particular, set

\begin{align}
\Xi(p,s) &= \Xi(p,s,+1/2) - \Xi(p,s,-1/2) \tag{58a} \\
\Pi^\uparrow(p,s) &= \Pi^\uparrow(p,s,+1/2) - \Pi^\uparrow(p,s,-1/2). \tag{58b}
\end{align}

where

\begin{align}
\Xi(p,s,+1/2) &= \frac{1}{4m} (m - p\bar{s})\Omega_+ \tag{59a} \\
\Xi(p,s,-1/2) &= \frac{1}{4m} (p + ms)\omega_+ \tag{59b} \\
\Pi^\uparrow(p,s,+1/2) &= \frac{1}{4m} (\bar{p} - ms)\Omega_+ \tag{59c} \\
\Pi^\uparrow(p,s,-1/2) &= \frac{1}{4m} (m + \bar{p}s)\omega_+. \tag{59d}
\end{align}

One may then show that the wavefunctions

\begin{align}
\Psi(p,s,+1/2) &= \Xi(p,s,+1/2)\Omega_+ - \Pi^\uparrow(p,s,+1/2)\omega_+ \tag{60a} \\
\Psi(p,s,-1/2) &= \Xi(p,s,-1/2)\Omega_+ - \Pi^\uparrow(p,s,-1/2)\omega_+ \tag{60b}
\end{align}

are also solutions to Eq. (2.1.38), and

\begin{align}
\Psi(p,s) &= \Psi(p,s,+1/2) - \Psi(p,s,-1/2). \tag{61}
\end{align}
Now define the operator

\[ S \cdot \frac{1}{\sqrt{m}} \mathcal{W}_\mu s^\nu. \]  \hspace{1cm} (63')

where

\[ \mathcal{W}_\mu = -\frac{1}{2} \gamma_\mu \gamma_\nu \gamma_\rho p^\nu. \] \hspace{1cm} (63)

\( \mathcal{W}_\mu \) is known as the Pauli-Lubanski pseudovector. More generally, \( \mathcal{W}_\mu \) is defined by\(^{12}\)

\[ \mathcal{W}_\mu = -\frac{1}{2} \epsilon_{\mu \nu \rho \sigma} J^{\nu \rho \sigma} p^\sigma. \] \hspace{1cm} (64)

where \( p_\mu = i \partial_\mu \) is the four-momentum operator and

\[ J_{\mu \nu} = (\gamma_\mu p_\nu - \gamma_\nu p_\mu) + \frac{i}{2} \gamma_\mu \gamma_\nu = L_{\mu \nu} + S_{\mu \nu} \] \hspace{1cm} (65)

describes the total angular momentum. Substituting Eq. (2.1.65) into (2.1.64) and using the fact that

\[ \epsilon_{\mu \nu \rho \sigma} (\gamma_\nu p_\rho - \gamma_\rho p_\nu) p^\sigma = (\epsilon_{\mu \nu \rho \sigma} - \epsilon_{\mu \rho \nu \sigma}) \gamma_\nu p^\rho p^\sigma \]

\[ = 2 \epsilon_{\mu \nu \rho \sigma} \gamma_\nu p^\rho p^\sigma \]

\[ = 2 \epsilon_{\mu \nu \rho \sigma} \gamma_\nu p^\rho p^\sigma \]

\[ = -2 \epsilon_{\mu \nu \rho \sigma} \gamma_\nu p^\rho p^\sigma \]

\[ \Rightarrow \epsilon_{\mu \nu \rho \sigma} L^{\nu \rho} p^\sigma = 0. \]

we obtain

\[ \mathcal{W}_\mu = -\frac{1}{2} \epsilon_{\mu \nu \rho \sigma} S^{\nu \rho} p^\sigma. \] \hspace{1cm} (66)

The orbital component has vanished and thus $\mathcal{W}_\mu$ corresponds to intrinsic (i.e. spin) angular momentum. In particular, $\mathcal{W}_\mu$ is orthogonal to $p^\mu$, i.e.

$$\mathcal{W}_\mu p^\mu = -\frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} S^{\nu \rho} p^\sigma p^\mu = 0,$$

so that, in the particle rest frame, $\mathcal{W}_0 = 0$ and

$$\mathcal{W}_i = -\frac{1}{2} \varepsilon_{i\mu\nu} S^{\mu} p^\nu = \frac{m}{\gamma} \varepsilon_{i\mu} \gamma^\mu = \frac{m}{\gamma} (\varepsilon_{i\mu} \gamma^\mu - \delta_{ik} \delta_{kj} e_l \otimes l)$$

$$= \frac{m}{\gamma} (3\delta_{ik} - \delta_{ik} \delta_{kl}) e_l \otimes l$$

$$= \frac{m}{2} \rho_\mu e_l \otimes l.$$  \hspace{1cm} (67)

where we have used Eqs. (1.2.84c), (1.2.86) and (1.2.87c). Consequently,

$$S = -\frac{1}{m} \mathcal{W}_\mu s^\mu = -\frac{1}{2m} \gamma_S \gamma_\mu \gamma_\nu p^\nu s^\mu$$

$$= \frac{1}{2m} \gamma_S (2g_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) p^\nu s^\mu$$

$$= \frac{1}{m} \gamma_S p_\mu s^\mu - \frac{1}{2m} \gamma_S \gamma_\nu \gamma_\mu p^\nu s^\mu$$

$$= \frac{1}{2m} \gamma_\nu p^\nu \gamma_5 \gamma_\mu s^\mu \hspace{1cm} (\because p_\mu s^\mu = 0)$$

$$= \frac{1}{2m} (\rho s \otimes \Omega_- - \rho s \otimes \Omega_+)$$  \hspace{1cm} (68)

represents a generalized, relativistic, spin polarization operator.
Notice from the above derivation that
\[(\gamma_\mu s^a)(\gamma_\nu p^\nu) = (\gamma_\nu p^\nu)(\gamma_\mu s^a)\]
\[\Rightarrow [\gamma_\mu p^\nu, \gamma_\nu s^a] = 0. \tag{69} \]
and since \(\gamma_\mu p^\nu = \gamma_\mu p^\mu\) commutes with itself,
\[\gamma_\mu p^\mu, s^a = 0. \tag{70} \]
Clearly, \(|m, S| = 0\) and thus, in the Dirac algebra, \(S\psi(p, s)\) is also a solution to the free particle Dirac equation in momentum space. One may show that this carries over as well to the Pauli algebra and we find that
\[S\psi(p, s) = \frac{1}{2m}(\bar{p}s\psi(p, s)\Omega_+ + \bar{p}s\psi(p, s)\Omega_-) = \frac{1}{2}\psi(p, s). \tag{71} \]

In the particle rest frame, \(S\) takes the form
\[S = \frac{1}{2m}(Es\otimes \Omega_+ - Es\otimes \Omega_-) = \begin{cases} \frac{1}{2}s \otimes 1 & (E > 0) \\ -\frac{1}{2}s \otimes 1 & (E < 0) \end{cases} \]
where we have used the fact that \(|E| = m\). Consequently, in the Pauli algebra, the rest frame spin polarization operator becomes
\[S = \begin{cases} \frac{1}{2}s & (E > 0) \\ -\frac{1}{2}s & (E < 0) \end{cases}. \tag{72} \]
Taking \(\psi(p, s)\) in the rest frame, we have
\[
\Psi(E,s) = \frac{1}{\sqrt{m}} \left[ (m+E)\Omega. -(E+ms)\omega. -
\right.
\]
\[
\left. -(E+ms)\omega. +(m+E)\Omega. \right] = \frac{1}{4m} \left[ (m-Es) - (E+ms)e_1 \right]
\]
\[
= \begin{cases} 
\frac{1}{2}(1+s)\frac{1}{2}(1-e_1) & (E > 0) \\
\frac{1}{2}(1-s)\frac{1}{2}(1+e_1) & (E < 0)
\end{cases} \quad (73)
\]

and setting
\[
\Psi(E > 0, s) = \frac{1}{2}(1+s)\frac{1}{2}(1-e_1) \quad (74a)
\]
\[
\Psi(E < 0, s) = \frac{1}{2}(1-s)\frac{1}{2}(1+e_1). \quad (74b)
\]

Eqs. (2.1.72) show that
\[
S\Psi(E > 0, \pm s) = \pm \frac{1}{2} \Psi(E > 0, \mp s) \quad (75a)
\]
\[
S\Psi(E < 0, \pm s) = \pm \frac{1}{2} \Psi(E < 0, \mp s). \quad (75b)
\]

Thus, we may also label solutions to the free particle Dirac equation using the sign of the energy and the spin polarization in the rest frame.

Consider now Eq. (2.1.65). It was asserted that $J_{\mu\nu}$ describes the total angular momentum. In actuality, one may show that $J_{\mu\nu}$ represents the total angular momentum. In order for this to be the case, the $J_{\mu\nu}$ must satisfy the correct commutation relations for an angular momentum operator. Moreover, since total angular momentum is conserved for a free particle, the $J_{\mu\nu}$ must commute with the free particle Dirac Hamiltonian.
Now we see that

\[ [\gamma_\mu p^\mu, l_{ab}] = [\gamma_\mu p^\mu, l_{ab}] - [\gamma_\mu p^\mu, S_{ab}] . \]

where

\[ [\gamma_\mu p^\mu, l_{ab}] = \gamma_\mu [p^\mu, \gamma_\alpha p_\beta - \gamma_\alpha p_\beta, \gamma_\alpha p_\beta] = i \delta_a^\mu \gamma_\mu p_\beta - i \delta_b^\mu \gamma_\mu p_\alpha \]

\[ = i (\gamma_\alpha p_\beta - \gamma_\beta p_\alpha) . \]

\[ [\gamma_\mu p^\mu, S_{ab}] = \frac{i}{2} [\gamma_\mu p^\mu, \gamma_\alpha p_\beta + \gamma_\beta p_\alpha] = \frac{i}{2} [\gamma_\mu, \gamma_\alpha \gamma_\beta p] = \frac{i}{2} [\gamma_\mu, \gamma_\beta (\gamma_\alpha p_\beta - \gamma_\beta p_\alpha)] = -i (\gamma_\alpha p_\beta - \gamma_\beta p_\alpha) = -[\gamma_\mu p^\mu, l_{ab}] . \]

Thus,

\[ [\gamma_\mu p^\mu, l_{ab}] = 0 \Rightarrow [\gamma_\mu p^\mu, l_{ab}] = 0 \] (76)

and so, in particular, the \( J_{ij} \) commute with the free particle Hamiltonian.

It remains to show the \( J_{ij} \) exhibit the correct commutation relations. Now observe that

\[ J_{ij} = L_{ij} + S_{ij} \]

\[ = (x_i p_j - x_j p_i) \Theta 1 + \frac{i}{2} (\epsilon_{ij} \Theta \Omega_+ + \epsilon_{ij} \Theta \Omega_-) \]

\[ = (x_i p_j - x_j p_i) \Theta 1 + \frac{i}{2} (-\epsilon_{ik} \Theta e_k) \]

\[ = \left[ (x_i p_j - x_j p_i) + \frac{1}{2} \epsilon_{ik} \Theta e_k \right] \Theta 1 \]

\[ = (L_{ij} + S_{ij}) \Theta 1 = J_{ij} \Theta 1 . \]

where
\[ J_{\mu} = I_{\mu} + S_{\mu}, \]  
(77)

with

\[ I_{\mu} = x_{i} P_{i} - x_{j} P_{j}, \]  
(78a)

\[ S_{\mu} = \frac{1}{2} \epsilon_{\mu \nu \lambda} x_{\lambda}, \]  
(78b)

Clearly, the \( I_{\mu} \) operate on the orbital-variable space while the \( S_{\mu} \in \mathcal{P} \) operate on spin space. Notice that \( J_{\mu} = I_{\mu} \otimes 1 \), so that, in the framework of the Pauli algebra, the \( J_{\mu} \) operate on orbital-spin space while \( 1 = I \otimes 1 \) operates on particle-antiparticle space. Consequently, we may work exclusively with the \( J_{\mu} \).

Now define the operator

\[ J_{k} = \frac{1}{2} \epsilon_{\mu \nu \lambda} J_{\mu} \]
\[ = \frac{1}{2} \epsilon_{\mu \nu \lambda} L_{\nu} + \frac{1}{2} \epsilon_{\mu \nu \lambda} S_{\nu} \]
\[ = \frac{1}{2} \epsilon_{\mu \nu \lambda} (x_{i} P_{i} - x_{j} P_{j}) + \frac{1}{2} \epsilon_{\mu \nu \lambda} \frac{1}{2} e_{i} \]
\[ = \epsilon_{\mu \nu \lambda} x_{i} P_{i} + \frac{1}{2} e_{i}. \]  
(79)

We then see that

\[ J_{i} = L_{i} - S_{i}, \]  
(80)

where \( L_{i} = \epsilon_{\mu \nu \lambda} x_{i} P_{j} \) and \( S_{i} = \frac{1}{2} e_{i} \) are the \( i \)th components of the usual orbital and spin angular momentum operators respectively. Again, the \( L_{i} \) are seen to act in the orbital-variable space \( \mathcal{E}^{(o)} \) while the \( S_{i} \in \mathcal{P} \) act in the spin space \( \mathcal{E}^{(s)} \otimes \mathcal{Q} \). Clearly,

\[ [L_{i}, L_{j}] = i \epsilon_{\mu \nu \lambda} L_{k}, \quad [S_{i}, S_{j}] = i \epsilon_{\mu \nu \lambda} S_{k} \]  
(81)

and since \([L_{i}, S_{j}] = 0 \ (\forall i, j \in \mathbb{N}_{3})\), we have

\[ [J_{i}, J_{j}] = i \epsilon_{\mu \nu \lambda} J_{k}, \]  
(82)
i.e. the $J_i$ satisfy the correct commutation relations for an angular momentum operator.

Given Eq. (2.1.82), we see that the set \{-iJ_i|i \in \mathbb{N}_3\} forms a basis over $\mathbb{R}$ for the Lie algebra of $\text{SU}(2)$, i.e.

$$su(2) = \text{span}_\mathbb{R}\{-iJ_i|i \in \mathbb{N}_3\}. \quad (83)$$

so that any element takes the form

$$0 = -i\theta J_i. \quad (84)$$

where $\theta \in \mathbb{R}$ ($\forall i \in \mathbb{N}_3$). Applying the exponential map $\text{EXP}$ defined in Eq. (1.4.10), we then have

$$R'(\theta) = \exp(\theta)$$

$$= \exp(-i\theta'(L_i+S_i))$$

$$= \exp(-i\theta' L_i)\exp(-i\theta' S_i)$$

$$= R'(\theta)R'(\theta). \quad (85)$$

where

$$R'(\theta) = \exp(-i\theta' L_i) \quad (86a)$$

$$R'(\theta) = \exp(-i\theta' S_i). \quad (86b)$$

Here, $j$ denotes the total angular momentum while $l$ and $s$ denote the orbital and spin angular momenta respectively. Clearly, $s=1/2$.

Since EXP maps any Lie algebra onto its universal covering group, we conclude that $R'(\theta) \in \text{SU}(2)$. Moreover, Eqs. (2.1.81) show that $R'(\theta)$ and $R'(\theta)$ may also be taken as elements of $\text{SU}(2)$.

The $R'(\theta)$ act in the orbital-variable space $\mathcal{E}^{(o)}$ and we may construct an irreducible representation $D'$ of $\text{SU}(2)$ on a $2l+1$-dimensional subspace of $\mathcal{E}^{(o)}$ having orthonormal basis \{\{|l m\rangle| m \in \mathbb{Z}_l\}\}, where the state vectors $|l m\rangle$ are
simultaneous eigenvectors of $i^2 = \sum_1^2 i_1^2$ and $L_z$ (say). In a spherical coordinate representation of the Dirac equation, the $|l \, m \rangle$ would be identified with the spherical harmonics. Since $l$ takes on strictly integral values, the irrep $D'$ also provides a representation of $SO(3)$.

On the other hand, the $R'(0) \epsilon F$ act in the spin space $E^{(s)} \otimes F$ and we may construct an irreducible representation $D'$ of $SU(2) = F_{SU} \subset F$ on a $2s + 1$-dimensional subspace of $E^{(s)}$ having orthonormal basis $\{|s \, \sigma \rangle | \sigma \epsilon (-1/2, 1/2)\}$ such that the $|s \, \sigma \rangle$ are simultaneous eigenstates of $S^2$ and $S_3$ (say). Since $s = 1/2$, this irrep does not provide a representation of $SO(3)$. One may readily show that $D' = D^{1/2}$ is equivalent to the spinor-induced group irreps $\rho_{SU}^<$ and $\rho_{SU}^>$ of $SU(2) = F_{SU} \subset F$ obtained by restricting $\rho_-$ and $\rho_+$ to $F_{SU} \subset F$ respectively. In particular, for $D'$, we have chosen the $z$-axis to point in the direction of $c_3$ whereas $z$ coincides with the unit vector $c$ in the case of the spinor irreps. A simple rotation corresponding to a similarity transformation in a matrix representation then yields the desired equivalence.

Given the form of Eq. (2.1.85), we see that the $R^j(0)$ may be interpreted as acting in the tensor product space $E^{(a)} \otimes E^{(a')}$. One may now follow standard techniques from angular momentum theory to construct irreducible representations of the $R^j(0) \epsilon SU(2)$ based on the reducible tensor product representation $D' \otimes D'$ of $SU(2)$. Since $D'$ is equivalent to both $\rho_{SU}^<$ and $\rho_{SU}^>$, one is immediately led to irreducible representations of $SU(2)$, describing the total angular momentum, within the framework of the Pauli algebra.
Before leaving this section, we would like to analyze the Dirac equation for massless particles. To this end, consider Eq. (2.1.38). Setting \( m = 0 \) we have

\[ p \Psi(p) \omega_+ - \bar{p} \Psi(p) \omega_- = 0. \tag{87} \]

Since \( m = 0 \), the energy-momentum relation becomes \( p \bar{p} = k^2 - p^2 = 0 \) so that \( |p| = \pm E \). where the plus sign holds for positive energy while the minus sign holds for negative energy. Consequently, for positive-energy solutions, Eq. (2.1.87) becomes

\[ (1 - \hat{\rho}) \Psi(E > 0, \hat{\rho}) \omega_+ - (1 + \hat{\rho}) \Psi(E > 0, \hat{\rho}) \omega_- = 0 \tag{88a} \]

while for negative-energy solutions we obtain

\[ (1 - \hat{\rho}) \Psi(E < 0, \hat{\rho}) \omega_+ - (1 + \hat{\rho}) \Psi(E < 0, \hat{\rho}) \omega_- = 0. \tag{88b} \]

Multiplying these equations on the right by \( \omega_- \) and rearranging then yields

\[ \hat{\rho} \Psi(E > 0, \hat{\rho}) \Omega_+ = \Psi(E > 0, \hat{\rho}) \Omega_+ \tag{89a} \]

\[ \hat{\rho} \Psi(E < 0, \hat{\rho}) \Omega_- = -\Psi(E < 0, \hat{\rho}) \Omega_- \tag{89b} \]

while multiplication on the right by \( \omega_+ \) gives

\[ \hat{\rho} \Psi(E > 0, \hat{\rho}) \Omega_- = -\Psi(E > 0, \hat{\rho}) \Omega_- \tag{90a} \]

\[ \hat{\rho} \Psi(E < 0, \hat{\rho}) \Omega_+ = \Psi(E < 0, \hat{\rho}) \Omega_+ \tag{90b} \]

Since \( \frac{1}{2} \hat{\rho} \) is the helicity operator in \( \mathbb{F} \), we immediately conclude from these equations that, for positive energy, \( \Psi(E, \hat{\rho}) \Omega_+ \) and \( \Psi(E, \hat{\rho}) \Omega_- \) are positive and negative helicity eigenstates respectively, whereas for negative energy the situation is reversed. If we now add the positive-energy equations (Eqs. (2.1.89a) and (2.1.90a)) we obtain

\[ \hat{\rho} \Psi(E > 0, \hat{\rho}) = \Psi(E > 0, \hat{\rho}) \tag{91a} \]

while addition of Eqs. (2.1.89b) and (2.1.90b) yields

\[ \hat{\rho} \Psi(E < 0, \hat{\rho}) = -\Psi(E < 0, \hat{\rho}) \tag{91b} \]
In order to interpret these results, let $\psi(\Sigma_{\Omega}) (\Omega \neq \Omega')$ be a Dirac spinor. We see that

$$
\phi_{\alpha\alpha}(\gamma_5 \psi) = \phi_{\alpha\alpha}(\rho_{\alpha\alpha}(\gamma_5)(\psi))
= \rho(-1 \otimes e)(\phi_{\alpha\alpha}(\psi))
= \psi e. \tag{92}
$$

where $\gamma_5 = (-1 \otimes e) \otimes \Sigma$ is the chirality operator (see Eqs. (1.2.60b) and (1.2.68)). This shows that the action of the chirality operator in the Dirac algebra corresponds to right multiplication by $e \in F$ in the Pauli algebra. In addition, we see that

$$
\phi_{\alpha\alpha}(\gamma_\mu \psi) = \phi_{\alpha\alpha}(\rho_{\alpha\alpha}(\gamma_\mu)(\psi))
= \rho(1 \otimes \Omega_\mu)(\phi_{\alpha\alpha}(\psi))
= \psi \Omega_\mu. \tag{93}
$$

where $\gamma_\mu$ and $\gamma_\nu$ are the chirality projection operators (see Eq. (1.2.90)). Consequently, the wavefunctions $\psi(E, \rho) \Omega_\mu$ and $\psi(E, \rho) \Omega_\nu$ correspond to left and right Weyl spinors respectively. They are clearly eigenstates of the chirality operator, i.e.

$$
\psi(E, \rho) \Omega_\mu e = \pm \psi(E, \rho) \Omega_\mu. \tag{94}
$$

Moreover, Eqs. (2.1.91) show that the helicity equals the chirality for positive-energy solutions while for negative energy they are opposite. This is entirely consistent with the behavior of $\psi(E, \rho) \Omega_\mu$ and $\psi(E, \rho) \Omega_\nu$ as demonstrated in Eqs. (2.1.89) and (2.1.90).

A possible application for the massless Dirac equation is the description of neutrinos.\textsuperscript{13} Experimental results indicate that only left-hand neutrinos exist in nature.

\textsuperscript{13} Traditionally, the neutrino has been assumed massless although very recent experiments may suggest otherwise.
Consequently, if we associate positive-energy solutions with the neutrino, its behavior in the absence of interactions is described by Eq. (2.1.90a):

\[ \hat{\rho} \Psi_\nu = -\Psi_\nu. \]  

(95)

where \( \Psi_\nu = \Psi(E > 0, \hat{\rho}) \Omega \) represents the neutrino wavefunction in momentum space. Clearly, \( \Psi_\nu \) corresponds to a left Weyl spinor of negative helicity. The negative-energy solution, described by Eq. (2.1.90b), is seen to have positive helicity and can be taken to correspond to a positive-energy antineutrino. Since helicity equals chirality for positive-energy solutions, the antineutrino must be interpreted as being right-handed. This seems to disagree with the chiral behavior of the negative-energy solution in Eq. (2.1.90b). However, as we shall see in the next section, the operation of charge conjugation provides the missing concepts needed to accurately formalize the connection between particle and antiparticle.

2.2 Transformation Properties

We shall now consider the behavior of the Dirac equation under Lorentz transformations as well as the discrete operations of parity, time reversal and charge conjugation.

To begin with, consider the restricted Lorentz group \( L^1 = SO_0(1,3) \). We recall that one of the fundamental requirements of the Dirac equation was its covariance under restricted Lorentz transformations. Now in Section 1.4 we saw that any four-vector \( u \in M^4 \subset \mathcal{P} \) transforms under \( L^1 \) as

\[ u \rightarrow u' = LuL^\dagger. \]  

(1)

where \( L = \exp(\bar{\omega}/2 - i\theta/2) \in \mathcal{P}_{SL} \subset \mathcal{P} \). In the Dirac algebra, the same transformation was seen to take the form

\[ u \rightarrow u' = LuL^{-1}. \]  

(2)
where $\mathfrak{M} \subset \mathcal{D} \subset \mathcal{D}_R$ and $\mathfrak{I} \subset \mathcal{D}_R \subset \mathcal{D}$ are given by Eqs. (1.2.83) and (1.4.35) respectively. However, one may also define restricted Lorentz transformations in terms of components, i.e.

$$u^\nu \to u''^\mu = L^\mu_\nu u^\nu.$$  \hspace{1cm} (3)

where the $L^\mu_\nu$ are the components of an operator $L$ identifiable with an element of $SO_0(1,3)$. The connection with Eqs. (2.2.1) and (2.2.2) is through the group isomorphism given in Eq. (1.4.23). To see what we mean, let $G$ be a Lie group having connected subgroup $G_0$, and let $\phi: UC[G] \to G_0$ be a surjective group homomorphism of the universal covering group of $G$ onto $G_0$. Then there exists an isomorphism $\varphi: UC[G]/\ker \phi \to G_0$ such that the diagram

$$\begin{array}{ccc}
UC[G] & \xrightarrow{\phi} & G_0 \\
\downarrow \Phi & & \\
UC[G]/\ker \phi & \xrightarrow{\varphi} & G_0 
\end{array}$$

commutes, where $\Phi$ denotes the canonical homomorphism. This result is nothing more than the fundamental homomorphism theorem applied to Lie groups. Specifying to the Lorentz group we have

$$\begin{array}{ccc}
SU(2,\mathbb{C}) & \xrightarrow{\phi} & SO_0(1,3) \\
\downarrow \Phi & & \\
SU(2,\mathbb{C})/\mathbb{Z}_{\pm 1} & \xrightarrow{\varphi} & SO_0(1,3) 
\end{array}$$
where \( G = O(1,3) \), \( \mathbb{C}[G] \cong \text{SL}(2, \mathbb{C}) \) and \( \ker \phi = (-1,1) \). As we saw in Section 1.4, both \( \mathbb{P}_{\text{Sl}} \subset \mathbb{P} \) and \( \mathcal{D}_{\text{Sl}} \subset \mathcal{D}_{\text{R}} \subset \mathcal{D} \) may be identified with \( \text{SL}(2, \mathbb{C}) \). Consequently, given \( L \in \mathbb{P}_{\text{Sl}} \), we see that \( \phi(\pm L) = \phi \circ \phi((\pm L)) = \psi(\pm L) = L \). While for \( L \in \mathcal{D}_{\text{Sl}} \) we have \( \phi(\pm L) = \phi \circ \phi((\pm L)) = \psi(\pm L) = L \).

Now let \( u \in \mathbb{M}^1 \subset \mathbb{D}_{\text{R}} \). Then Eq. (2.2.2) gives
\[
L^* = LuL^{-1} = Lu^\gamma L^{-1} = u^\gamma L L^{-1}.
\]
However, using Eq. (2.2.3) we have
\[
u^* = u^\mu v^\nu - L\nu v^\mu - u^\nu L^\mu v\nu.
\]
and thus
\[
L \gamma^\nu L^{-1} = L^\mu \gamma^\mu.
\]
In a similar fashion, Eqs. (2.2.1) and (2.2.3) show that
\[
L \phi v L^{-1} = L^\mu \phi^\mu.
\]
which also may be obtained from Eq. (2.2.4) using Eqs. (1.2.84b) and (1.4.53). Eqs. (2.2.4) and (2.2.5) can be taken as the defining relations for the \( L^\nu \) in terms of \( L \in \mathbb{D}_{\text{Sl}} \) and \( L \in \mathbb{P}_{\text{Sl}} \) respectively.

Our motivation for introducing these results centers on the behavior of the four-momentum operator, as well as Dirac spinor fields, under restricted Lorentz transformations. In the case of the four-momentum operator this is not as transparent as one might imagine since its components involve differential operators with respect to the coordinates of a point \( x \) in Minkowski spacetime. Moreover, the transformation behavior of \( \mathcal{I} \) appearing in the four-momentum operator is not immediately clear since we are implicitly using the group representation of \( \text{Spin}_0(1,3) \) on the real Dirac algebra, induced by the vector representation of \( \text{Pin}(1,3) \), to deduce the behavior of four-vectors in \( \mathcal{D}_{\text{R}} \subset \mathcal{D} \), and subsequently four-vectors in \( \mathbb{P} \), under \( L^* \).
The question concerning the behavior of \( i \) may be answered by making use of the fact that the four-momentum operator acts on Dirac spinor fields, i.e., \( i\sigma_u \) maps spinors to spinors. Now, since the spinor-induced irreps of \( D_R \subset D \) induce complex group irreps of \( \text{PIN}(1,3) \) (see Section 1.4), one immediately concludes that \( i \) commutes with elements of \( \text{PIN}(1,3) \). Consequently, since \( \text{PIN}(1,3) \) is the double covering group of \( O(1,3) \), we conclude that \( i \) is invariant under the full Lorentz group and, more specifically, restricted Lorentz transformations. Parenthetically, although the foregoing discussion has been undertaken solely within the framework of the Dirac algebra, the results of Section 1.5 imply the same conclusion in the Pauli algebra.\(^{14}\)

Armed with this fact, we now claim that the four-momentum operator transforms like a four-vector under \( L \). More specifically, let \( x \in \mathbb{M}^4 \) be a point in Minkowski spacetime defined with respect to some inertial reference frame \( \mathcal{R} \), and let \( x' \in \mathbb{M}^4 \) denote the same point with respect to a new frame \( \mathcal{R}' \) obtained from \( \mathcal{R} \) by a restricted Lorentz transformation \( L \) so that \( x' = Lx \). Then the Pauli algebra four-momentum operator transforms under \( L \) as

\[ p' = LpL^* \tag{6} \]

where the components of \( p' = i\partial' \) involve differential operators with respect to the coordinates of \( x' \in \mathbb{M}^4 \), and \( L \) is related to the components of \( L \) through Eq. (2.2.5).

---

\(^{14}\) Here, it is important to distinguish \( i \in \mathbb{C} \) as an element of the ground field and the canonical element \( e \in \mathcal{F}(\mathbb{R}) \).
To prove this statement, first observe that, in terms of coordinates, \( x' = L x \) is equivalent to the equation \( \lambda' \mu = L^\mu \lambda \). Using the properties of the metric tensor we see that
\[
\lambda' \mu = g_{\mu \nu} \lambda' \nu = g_{\mu \nu} L'_{\nu} \lambda = \lambda_{\mu} L_{\mu} \lambda.
\]
and thus
\[
L_{\mu} \lambda_{\mu} = \delta_{\mu}^{\nu}.
\]
Consider now the Lorentz-invariant scalar \( \delta_{\mu} \lambda' \mu \). Using the chain rule
\[
\delta_{\mu} \lambda' = \left( \frac{\partial x'}{\partial x' \mu} \right) \partial_{\nu}
\]
we see that
\[
\delta_{\mu} \partial_{\nu} x' = \partial_{\mu} x' = \left( \frac{\partial x}{\partial x' \mu} \right) L_{\mu} \partial_{\nu} x
\]
\[
\Rightarrow \left( \frac{\partial x'}{\partial x' \mu} \right) L_{\mu} = \delta_{\mu}^{\nu}.
\]
\[ L^a_{\mu} L^b_{\nu} = \left( \frac{\partial \lambda^a}{\partial x^\Lambda} \right) L^b_{\nu} \]

\[ \Rightarrow \left( L^a_{\mu} - \frac{\partial x^a}{\partial x^\Lambda} \mu \right) L^b_{\nu} = 0 \]

and since this must hold for all \( L^a_{\mu} \), we conclude that

\[ \frac{\partial x^a}{\partial x^\Lambda} \mu = L^a_{\mu}. \]  

Consequently, the differential operators transform under \( \mathcal{L} \) as

\[ \delta^\mu_{\nu} = L^a_{\mu} \delta^a_{\nu}. \]  

i.e., as the covariant components of a four-vector. Again employing the metric tensor, one easily demonstrates that \( \delta^\mu_{\nu} = L^a_{\mu} \delta^a_{\nu} \) and using the Lorentz invariance of \( i \) together with Eq. (2.2.5) we obtain

\[ p^i = i \delta^\mu_{\nu} e_\mu \]

\[ = i L^a_{\nu} \delta^a_{\nu} e_\mu \]

\[ = i \delta^\nu e_\nu L^a = i \sigma \]

which completes the proof.

Since \( x \) and \( x' \) refer to the same spacetime point, albeit, with respect to two inertial frames related by \( \mathcal{L} \), we conclude that the vector potential \( A(x) \in \mathcal{M}^+ \subset \mathcal{P} \) transforms under \( \mathcal{L} \) as

\[ A'(x') = \mathcal{L}(x) A(x). \]  

In terms of components, the equivalent transformation is

\[ A'^a(x') = L^a_{\nu} A^\nu(x). \]
where we have used Eq. (2.2.5). More generally, Eq. (2.2.12) defines what is termed a vector field.\footnote{In the literature, vector fields are more commonly defined in terms of components. See, for example, Barut (1980).} Scalar fields, such as \( \phi(x) \) occurring in the Klein-Gordon equation, satisfy \( \phi'(x') = \phi(x) \).

Of more significance to the task at hand is the transformation behavior of spinor fields. In particular, Eq. (1.4.53) shows that any Dirac spinor field \( \psi'(x') \in D_{uu} \) transforms under \( \mathcal{L} \) as

\[
\psi'(x') = \mathcal{L}_\mathcal{L} \psi(x),
\]

(14)

where \( \mathcal{L} \in \mathcal{D}_{sl} \). Since \( \mathcal{D}_{\Omega} \) is a minimal left ideal, \( \psi'(x') \) is also an element of \( \mathcal{D}_{\Omega'} \). Consequently, one may use Eqs. (1.3.11) and (1.4.54) to show that Pauli spinor fields transform under \( \mathcal{L} \) as

\[
\Phi'(x')\Omega = \mathcal{L}_\mathcal{L} \Phi(x)\Omega
\]

(15a)

\[
\overline{\Phi}^*(x')\Omega = \overline{\mathcal{L}} \overline{\Phi}(x)\Omega.
\]

(15b)

where \( \mathcal{L} \in \mathcal{F}_{sl} \) and \( \Omega \in \{ \Omega_-, \Omega_+ \} \).

If \( \psi(x) \in \mathcal{D}_{\Omega} \) with \( \Omega \neq \Omega' \), we may use Eq. (2.2.14) together with the results of Section 1.5 to deduce the behavior of the wavefunction \( \Psi(x) = \phi_{\Omega}(\psi(x)) \in \mathcal{P} \) under \( \mathcal{L} \). Indeed, we find that

\[
\Psi'(x') = \Psi'(x')\Omega_+ + \Psi'(x')\Omega_-
\]

(16)

\[
= \mathcal{L}_\mathcal{L} \Psi(x)\Omega_+ + \overline{\mathcal{L}} \Psi(x)\Omega_-.
\]
Eqs. (2.2.15) and (1.1.59) then show that
\[ \Psi'(x')\Omega = i\Psi(x)\Omega. \]  \hspace{1cm} (i7a)
\[ \Psi'(x')\Omega = \overline{L}^*\Psi(x)\Omega. \]  \hspace{1cm} (17b)
\[ \Psi'(x')\omega = \overline{L}^*\Psi(x)\omega. \]  \hspace{1cm} (18a)
\[ \Psi'(x')\omega = L\Psi(x)\omega. \]  \hspace{1cm} (18b)

Eqs. (2.2.17) and (2.2.18) shall prove useful when we come to study the transformations properties of the Dirac current in Section 2.3.

Having established the behavior of four-vector and spinor fields under restricted Lorentz transformations, we are now able to address the transformation properties of the Dirac equation. Consider first the Dirac equation in \( D \) describing a charged particle in the presence of an external electromagnetic field, i.e.
\[ \gamma^\mu(i\partial_\mu - qA_\mu(x))\psi(x) = m\psi(x). \]  \hspace{1cm} (19)

Multiplying on the left by \( L\in D_{SL} \) we see that
\[ L\gamma^\mu(i\partial_\mu - qA_\mu(x))\psi(x) = mL\psi(x) \]
\[ \Rightarrow L\gamma^\mu L^{-1}(i\partial_\mu - qA_\mu(x))\psi(x) = mL\psi(x) \]
\[ \Rightarrow \gamma^\mu L^{-1}(i\partial_\mu - qA_\mu(x))L\psi(x) = mL\psi(x) \]
\[ \Rightarrow \gamma^\mu(i\partial_\nu - qA_\nu(x'))\psi'(x') = mL\psi'(x') \]  \hspace{1cm} (20)
which establishes the covariance of the Dirac equation under restricted Lorentz transformations.

To see the corresponding behavior in the Pauli algebra, consider Eqs. (2.1.20). First note that since \( p \) and \( A(x) \) transform as four-vectors under \( L \),
\[ \pi'(x') = L\pi(x)L^*. \]  \hspace{1cm} (21)
where \( \pi(x) = \rho - \eta \lambda(x) \). Multiplying Eq. (2.1.20a) and (2.1.20b) on the left by \( \bar{\tau} \) and \( \eta \) respectively gives
\[
\bar{\tau} \pi(x) \tilde{\eta}(x) \Omega_\eta = m \bar{\tau} \tilde{\eta}(x) \Omega_\eta.
\]
(22a)
\[
\eta \pi(x) \tilde{\eta}(x) \Omega_\eta = m \eta \tilde{\eta}(x) \Omega_\eta.
\]
(22b)
Now since \( \eta \) is unimodular, we have \( \bar{\tau} \eta \bar{\tau} = \tilde{\eta} - 1 \). Moreover, Eq. (2.2.21) shows that
\[
\bar{\tau} \pi(x) \tilde{\eta}(x) \Omega_\eta = \tilde{\tau} \pi(x) \bar{\eta}(x) \tilde{\bar{\tau}} = \tilde{\eta}(x) \Omega_\eta.
\]
(23)
Consequently, Eqs. (2.2.22) may be rewritten as
\[
\bar{\tau} \pi(x) \tilde{\eta}(x) \Omega_\eta = m \bar{\tau} \tilde{\eta}(x) \Omega_\eta.
\]
\[
\eta \pi(x) \tilde{\eta}(x) \Omega_\eta = m \eta \tilde{\eta}(x) \Omega_\eta.
\]
and using Eqs. (2.2.15) we obtain
\[
\bar{\tau} \pi(x) \tilde{\eta}(x) \Omega_\eta = m \tilde{\tau} \eta \Omega_\eta.
\]
(24a)
\[
\eta \pi(x) \tilde{\eta}(x) \Omega_\eta = m \tilde{\eta}(x) \Omega_\eta.
\]
(24b)
which demonstrates the covariance of the Dirac equation in the Pauli algebra.

We could have also established this result using Eq. (2.1.16); however, the procedure turns out to be entirely equivalent to the one carried out for Eqs. (2.1.20). The reason centers on Eq. (2.2.16); i.e., the transformation \( \Psi(x) \rightarrow \Psi'(x') \) inherently introduces an idempotent decomposition of \( \Psi(x) \). The resulting implication for a formal analysis of Eq. (2.1.16) under restricted Lorentz transformations is the necessity of decomposing it into two equations, one residing in \( \mathcal{F}_+ \), the other in \( \mathcal{F}_- \). In particular, multiplying Eq. (2.1.16) on the right by \( \Omega_+ \) and \( \Omega_- \) respectively yields
\[
\pi(x) \Psi(x) \omega_+ + m \Psi(x) \Omega_+ = 0
\]
(25a)
\[
\bar{\pi}(x) \Psi(x) \omega_- + m \Psi(x) \Omega_- = 0.
\]
(25b)
Eq. (2.2.25a) is clearly seen to reside in the minimal left ideal \( \mathcal{P}_- \) while (2.2.25b) resides in \( \mathcal{P}_+ \). One may now demonstrate covariance using the methods outlined previously, i.e.

\[
\begin{align*}
1 \pi(x) \tilde{l} \bar{T} \Psi(x) \omega_+ &+ m \bar{T} \Psi(x) \Omega_- = 0 \\
\tilde{l} \pi(x) l \bar{T} \Psi(x) \omega_- &+ ml \bar{T} \Psi(x) \Omega_+ = 0
\end{align*}
\]

\[
\Rightarrow \quad \pi'(x') \Psi'(x') \omega_- - m \Psi'(x') \Omega_+ = 0 \quad (26a)
\]

\[
\pi'(x') \Psi'(x') \omega_+ + \bar{\pi}'(x') \Psi'(x') \omega_- + m \Psi'(x') \Omega_- = 0. \quad (26b)
\]

so that, upon adding Eqs. (2.2.26), we obtain

\[
\pi'(x') \Psi'(x') \omega_- + \bar{\pi}'(x') \Psi'(x') \omega_+ + m \Psi'(x') = 0. \quad (27)
\]

where we have used the fact that \( \Omega_- + \Omega_+ = 1 \).

Consider now more general Lorentz transformations; in particular, the operations of space and time inversion defined in Section 1.4. We saw in Eqs. (1.4.66) that

\[
\begin{align*}
\psi &\rightarrow \Psi^p = \rho \psi = \gamma_0 \psi \quad (28a) \\
\psi &\rightarrow \Psi^T = \tilde{T} \psi = \gamma_0 \gamma_0 \psi. \quad (28b)
\end{align*}
\]

where \( \psi \in \mathcal{D}_{\infty} \) is any Dirac spinor. Moreover, the \( \mathbb{M}^4 \) basis vectors were seen to transform as

\[
\begin{align*}
\gamma_\mu &\rightarrow \gamma_0 \gamma_\mu \gamma_0 = \gamma^\mu \quad (29a) \\
\gamma_\mu &\rightarrow \gamma_\gamma \gamma_\nu \gamma_\gamma = -\gamma_\mu \quad (29b)
\end{align*}
\]

so that any four-vector \( u \in \mathbb{M}^4 \subset \mathcal{D} \) satisfies

\[
\begin{align*}
u &\rightarrow u^p = \gamma_0 u \gamma_0 \quad (30a) \\
u &\rightarrow u^T = \gamma_\gamma u \gamma_\gamma \gamma_0. \quad (30b)
\end{align*}
\]
Now just as we were able to define restricted Lorentz transformations in terms of components, so may we also define space and time inversion. In particular, given \( u \in M^4 \subset D \) we have

\[
\begin{align*}
  u & \rightarrow u^\gamma = \gamma_0 u \gamma_0 = u^\mu \gamma_0 \gamma_\mu \gamma_0 = u^\mu \gamma_\mu = u_\mu \gamma_\mu \\
  u & \rightarrow u^\tau = \gamma_\mu u^{\gamma \mu} = u^\mu \gamma_\mu \gamma_\nu \gamma_\nu = -u^\mu \gamma_\mu = -u_\mu \gamma_\mu
\end{align*}
\]

and setting

\[
  u^\rho = u^{\rho \mu} \gamma_\mu, \quad u^\tau = u^{\mu \tau} \gamma_\mu
\]

we obtain

\[
\begin{align*}
  u^\rho & \rightarrow u^{\rho \mu} = u_\mu \\
  u^\tau & \rightarrow u^{\mu \tau} = -u_\mu.
\end{align*}
\]

Eqs. (2.2.33) show that the components of the spacetime point \( x \in M^4 \) satisfy

\[
\begin{align*}
  x^\rho = (t, x^\rho) & \rightarrow (t^\rho, -x^\rho) = x^p \\
  x^\tau = (t, x^\tau) & \rightarrow (-t, -x^\tau) = x^\tau.
\end{align*}
\]

Consequently, any Dirac spinor field \( \psi(x) \in D_{\mathbb{C}^2} \) transforms under space and time inversion as

\[
\begin{align*}
  \psi^\rho(x^\rho) & = \gamma_0 \psi(x) \\
  \psi^\tau(x^\tau) & = \gamma \gamma_0 \psi(x).
\end{align*}
\]

where we have used Eqs. (2.2.28).

In addition, the components of any four-vector field \( u(x) = u^\mu(x) \gamma_\mu \in M^4 \subset D \) are seen to transform as

\[
\begin{align*}
  u^\mu(x^\rho) & = u_\mu(x) \\
  u^\tau(x^\tau) & = -u_\mu(x).
\end{align*}
\]

where we have used Eqs. (2.2.31) and (2.2.32).
In applying these results to the Dirac equation, consider first Eq. (2.2.19). Clearly,

\[ \delta \mu = (\partial / \partial x^\mu) \rightarrow (\partial / \partial x'^\mu) = (\partial / \partial x'^\mu) = \delta^\mu = \delta^\mu \]  

(37a)

\[ \delta \mu = (\partial / \partial x^\mu) \rightarrow (\partial / \partial x'^\mu) = (\partial / \partial (-x'^\mu)) = -\delta^\mu = \delta^\mu \]  

(37b)

and since the complex scalar \( i \) is invariant under the full Lorentz group to which \( P \) and \( T \) belong, the components of the four-momentum operator \( p_\mu = i \partial_\mu \) satisfy

\[ p_\mu \rightarrow p'^\mu = p^\mu \]  

(38a)

\[ p_\mu \rightarrow p'^\mu = -p^\mu. \]  

(38b)

The components of the vector potential are also seen to satisfy

\[ A_\mu^p(x^p) = A^\mu(x) \]  

(39a)

\[ A_\mu^T(x^T) = -A^\mu(x). \]  

(39b)

where we have used Eqs. (2.2.33) together with a trivial application of the metric tensor. Eqs. (2.2.29) and (2.2.35) may now be used to show that

\[ \gamma^\mu (i\partial_\mu - qA_\mu^p(x^p))\psi^p(x^p) = m\psi^p(x^p) \]  

(40)

\[ \gamma^\mu (i\partial_\mu - qA_\mu^T(x^T))\psi^T(x^T) = m\psi^T(x^T). \]  

(41)

Now consider Eq. (2.1.16). Taking \( \psi(x) \in \mathcal{D}\Omega \) with \( \Omega \neq \Omega' \), the results of Section 1.5 show that
\[ \psi^r(x^r) = \phi_{\Omega^r} (\psi^r(x^r)) \]
\[ = \phi_{\Omega^r}(\rho_{\Omega^r}(\gamma^r)(\psi(x))) \]
\[ = \rho(1 \otimes e_1)(\phi_{\Omega^r}(\psi(x))) = -\psi(x)e_1 \]
\[ \psi^T(x^T) = \phi_{\Omega^r}(\psi^r(x^r)) \]
\[ = \rho_{\Omega^r}(\rho_{\Omega^r}(\gamma^r)(\psi(x))) \]
\[ = \rho(1 \otimes e_2)(\phi_{\Omega^r}(\psi(x))) = -\psi(x)e_2. \]

and thus the wavefunction \( \psi(x) \in \mathcal{F} \) may be interpreted to transform under space and time inversion as

\[ \psi^r(x^r) = -\psi(x)e_1 \tag{42a} \]
\[ \psi^T(x^T) = -i\psi(x)e_2. \tag{42b} \]

Moreover, given Eqs. (2.2.38) and (2.2.39), \( \pi(x) = \rho - \varphi_4(x) \) can be taken to transform as

\[ \pi^r(x^r) = \pi^r_{\alpha}(x^r)e_\alpha \]
\[ = \pi(x)e_\alpha \]
\[ = \pi^r(x)e_\alpha = \pi(x) \tag{43a} \]
\[ \pi^T(x^T) = \pi^T_{\alpha}(x^T)e_\alpha \]
\[ = -\pi(x)e_\alpha \]
\[ = -\pi^r(x)e_\alpha = -\pi(x). \tag{43b} \]

Multiplying Eq. (2.1.16) on the right by \( e_1 \) then gives

\[ \pi(x)\psi(x)\omega_1 e_1 + \pi(x)\psi(x)\omega_1 e_1 + m\psi(x)e_1 = 0 \]
\[ \Rightarrow \pi(x)\psi(x)e_1\omega_1 + \pi(x)\psi(x)e_1\omega_1 + m\psi(x)e_1 = 0 \]
\[ \Rightarrow \pi^r(x^r)\psi^r(x^r)\omega_1 + \pi^r(x^r)\psi^r(x^r)\omega_1 + m\psi^r(x^r) = 0. \tag{44} \]

Similarly, right multiplication by \( e_2 \) gives
\[ n(x) \Psi(x) \omega \cdot e_2 + \bar{n}(x) \Psi(x) \omega \cdot e_2 + m \Psi(x) e_2 = 0 \]
\[ \Rightarrow -n(x) \Psi(x) e_2 \omega - \bar{n}(x) \Psi(x) e_2 \omega + m \Psi(x) e_2 = 0 \]
\[ \Rightarrow n'(x') \Psi'(x') \omega + \bar{n}'(x') \Psi'(x') \omega + m \Psi'(x') = 0. \tag{45} \]

Again employing the results of Section 1.5, one may show Eqs. (2.2.44) and (2.2.45) to be the equivalent Pauli algebra expressions corresponding to Eqs. (2.2.40) and (2.2.41) respectively. Consequently, they reflect the covariance of the Dirac equation in \( \mathcal{F} \) under space and time inversion.

These results may now be used to demonstrate covariance of the Dirac equation under the spacetime inversion operator \( \Theta \). In particular, for the Pauli algebra, one may show that the transformations

\[ \Psi^\Theta(x^\theta) = i \Psi(x) e \tag{46a} \]
\[ n^\Theta(x^\theta) = -n(x). \tag{46b} \]

where \( x^\theta = -x \), preserve the form of Eq. (2.1.16), i.e.

\[ n^\Theta(x^\theta) \Psi^\Theta(x^\theta) \omega + \bar{n}^\Theta(x^\theta) \Psi^\Theta(x^\theta) \omega + m \Psi^\Theta(x^\theta) = 0. \tag{47} \]

Moreover, for \( \Omega \neq \Omega' \), we see that
\[ \psi''(\lambda') = \phi_{\alpha_0} (\psi''(\lambda'')) \]
\[ = \phi_{\alpha_0} (\rho_{\alpha_0} (\psi'(\lambda))) \]
\[ = \rho (-i \otimes v) (\phi_{\alpha_0} (\psi'(\lambda))) \]
\[ = i \Psi'(\lambda) e \]
\[ = -i (\Psi'(\lambda) e_1) e_2 \]
\[ = -i \Psi'(\lambda) e_2 = \Psi^{PT}(\lambda^{PT}) \quad (48a) \]

\[ n''(\lambda'') = -n(\lambda) \]
\[ = -n(\lambda) \]
\[ = -n'(\lambda') = n^{PT}(\lambda^{PT}) \quad (48b) \]

in agreement with the definition of \( \Theta \) given in Section 1.4.

Our motivation for studying the non-restricted Lorentz transformations centers on their relation to the discrete operations of parity, time reversal (also known as Wigner time reversal) and charge conjugation. In fact, the parity transformation \( P \) coincides with space inversion while time reversal and charge conjugation, denoted \( T \) and \( C \) respectively, combine to give time inversion (Coquereaux 1968).

The only distinction between \( P \) and space inversion as we have defined it lies in the transformation behavior of the wavefunction. In particular,

\[ \psi'(x') = P^{(o)} \psi(x) \quad (49a) \]
\[ \psi'(x') = -P^{(o)} \psi(x)e_1. \quad (49b) \]

where \( P^{(o)} \) is an operator on the underlying orbital-variable space. \( P^{(o)} \) describes the intrinsic parity of the wavefunction. Its effect is merely to alter the wavefunction by a phase factor which may be restricted to \( \pm 1 \).
or \( i \) if we require that four reflections return the (spinor) wavefunction to itself, in analogy with a rotation through \( \frac{3\pi}{2} \) radians. For a more detailed discussion, see Bjorken and Drell (1964).

We now turn our attention to charge conjugation. As was stated in Section 2.1, the concept of antiparticles arose out of Dirac's attempt to explain the existence of negative-energy solutions to his equation. Dirac's idea, known formally as the hole theory, interprets the vacuum as being filled with negative-energy, spin-1/2 particles.

An important consequence of the hole theory is the process of pair production in which a negative-energy particle in the vacuum absorbs sufficient energy to be excited into a positive-energy state. If the particle has charge \( q \), we end up with a positive-energy particle of charge \( q \) and a "hole" in the negative-energy sea which, relative to the vacuum, may be interpreted as a positive-energy antiparticle of charge \(-q\). Conversely, if a hole exists in the negative-energy sea, a positive-energy particle may make a radiative transition down to this negative-energy state. Relative to the vacuum, this process appears as particle-antiparticle annihilation, aptly referred to as pair annihilation.

Although the hole theory introduces difficulties only resolved within the more formal structure of quantum field theory, it does provide an intuitive basis, in conjunction with the Feynman-Stückelberg formalism discussed in Section 2.1, for interpreting the Feynman diagrams commonly used to describe particle interactions.
Consequently, there emerges out of the hole theory a one to one correspondence between negative-energy solutions to the Dirac equation
\[ \gamma^\mu (i\partial_\mu - q A_\mu (x)) \psi(x) = m \psi(x) \]  
(30)
and positive-energy antiparticle solutions to the charge-conjugate equation
\[ \gamma^\nu (i\partial_\nu + q A_\nu (x)) \psi^c(x) = m \psi^c(x). \]  
(31)

In order to make this connection, consider the Dirac equation in the Pauli algebra
\[ \pi(x) \Psi(x) \omega_+ + \overline{\pi(x)} \Psi(x) \omega_- + m \Psi(x) = 0. \]  
(32)
The corresponding charge-conjugate equation takes the form
\[ \pi^c(x) \psi^c(x) \omega_+ + \overline{\pi^c(x)} \psi^c(x) \omega_- + m \psi^c(x) = 0. \]  
(33)
where \( \pi^c(x) = (p - q A(x))^c = p + q A(x) \). The question is what form does \( \psi^c(x) \) take? First note that
\[ \pi^c(x) = p^c - q A^c(x) = -\pi(x) \]  
(34)
where we have used the fact that \( p^c = (i\sigma_\mu)^c = (i\sigma_\mu)^c = -i\sigma_\mu = -p \).

Applying spatial reversion and hermitean conjugation to Eq. (2.2.52) gives
\[ -\pi^c(x) \overline{\psi}(x) \omega_+ - \pi^c(x) \overline{\psi}(x) \omega_- + m \overline{\psi}(x) = 0 \]
and using Eq. (2.2.54) we obtain
\[ \pi^c(x) \overline{\psi}(x) \omega_+ + \overline{\pi^c(x)} \overline{\psi}(x) \omega_- + m \overline{\psi}(x) = 0. \]  
(35)
Comparing with Eq. (2.2.53), we conclude that
\[ \psi^c(x) = \overline{\psi}(x). \]  
(36)

---

16 In taking \( \delta_\mu^c = \delta_\mu \), we are following the convention adopted, for example, in Bjorken and Drell (1964). The reader should be careful to distinguish between hermitean conjugation as defined in the Pauli algebra and the more common quantum-mechanical definition.
Notice that this transformation is conjugate-linear. More significantly, it is seen to be not only representation independent but representation free, in contrast to what is normally found in the literature. Indeed, charge conjugation is generally defined on Dirac spinor fields as

\[ \psi^C(x) = C \psi'(x). \] (57)

where \( C \) is some matrix whose form depends on the choice of representation. In this framework, the spinor \( \psi(x) \) is identified with a column vector in \( \mathbb{C}^4 \) and the \( \{\gamma_\mu\} \) with \( 4 \times 4 \) complex matrices. The matrix \( C \) is required to satisfy the condition

\[ C \gamma_\mu C^{-1} = -\gamma_\mu, \] (58)

which clearly demonstrates its dependence on the choice of representation made for the \( \{\gamma_\mu\} \). Since our ultimate aim has been to discuss Dirac theory within the more abstract Clifford algebra structure, we should like to define charge conjugation in terms of the spinor irreps of the Dirac algebra.

To this end, let \( \psi(x) \in \mathcal{D}_{nn} \) and consider Eq. (2.2.50). Applying the involution \( \sim \) defined in Eq. (1.2.77) we have

\[ \gamma^\nu (-i \partial_\mu - q A_\mu(x)) \psi(x) = m \psi(x). \] (59)

Recall that the effect of \( \sim \) is simply to complex-conjugate the components (with respect to the basis \( \mathcal{B} \)) of any \( \mathcal{D} \)-element. Multiplying Eq. (2.2.59) on the left by \( \gamma \) and using the fact that \( \gamma \) anticommutes with each \( \gamma_\mu \) we obtain

\[ \gamma^\nu (i \partial_\mu + q A_\mu(x)) \gamma \psi(x) = m \gamma \psi(x). \] (60)
Comparing with Eq. (2.2.51), we see that the charge-conjugate spinor can be defined as

\[ \psi^c(x) = \gamma \tilde{\psi}(x). \]  \tag{61}

Given the definition of \( \sim \), one immediately concludes that charge conjugation, as defined in Eq. (2.2.61), is representation independent. Clearly, there is a fundamental difference between this definition and the one given in Eq. (2.2.57). The key to this distinction centers on the fact that \( \sim \) maps the Dirac spinor \( \psi(x) \) to a different minimal left ideal. In particular, if \( \psi \in \mathcal{D}_{\Omega^c} \) with \( \Omega \neq \Omega' \) then \( \tilde{\psi} \in \mathcal{D}_{\Omega^c} \).

To see this, let \( d \) be an arbitrary element of \( \mathcal{D} = \mathcal{F} \otimes \mathcal{F} \). Then \( d \) may be written as in Eq. (1.2.97). Comparing with Eq. (1.2.99), we see that for every \( a \in \mathcal{F} \), the transformations

\[ a \otimes \omega_+ \rightarrow \tilde{a} \otimes \omega_+ \quad a \otimes \omega_- \rightarrow \tilde{a} \otimes \omega_- \]  \tag{62}

suffice to uniquely define \( \sim \). The primitive idempotents of \( \mathcal{D} \) are then seen to transform under \( \sim \) as

\[ \Omega_+ \rightarrow \tilde{\Omega}_+ \quad \Omega_- \rightarrow \tilde{\Omega}_- \]  \tag{63}

\[ \Omega_+ \rightarrow \tilde{\Omega}_- \quad \Omega_- \rightarrow \tilde{\Omega}_+ \]  \tag{64}

Now let \( \psi \in \mathcal{D}_{\Omega^c} \) be any Dirac spinor. Given the results of Section 1.3, there exists \( \tilde{\psi} \in \mathcal{D}_R \subset \mathcal{D} \) such that \( \tilde{\psi} = \tilde{\psi}_{\Omega \Omega^c} = \tilde{\psi}(\Omega \otimes \Omega') \). Since \( (d_1 d_2)^c = \bar{d}_1 \bar{d}_2 \) (\( \forall d_1, d_2 \in D \)) and \( \bar{d} = d \) (\( \forall d \in \mathcal{D}_R \), Eqs. (2.2.64) show that

\[ \tilde{\psi} = \tilde{\phi}_{\Omega \Omega^c} = \tilde{\phi}(\Omega \otimes \Omega') \in \mathcal{D}_{\Omega \Omega^c} \]  \tag{65}

as asserted. More specifically, we see that

\[ \tilde{\phi}_{\Omega \Omega^c} = \phi_{\Omega \Omega^c} \quad \{ \Omega \neq \Omega' \}. \tag{66} \]
In contrast, extending the action of \( \cdot \) to the abstract Dirac algebra in a manner consistent with the usual matrix-dependent definition yields a conjugate-linear involution under which the Dirac spinors spaces are invariant. Central to this extension is the algebra isomorphism

\[ D = C \otimes D_R \cong C \otimes M_4(R) \]  

(67)

obtained via the remark following Theorem A2.8. In fact, whereas \( \cdot \) complex-conjugates the elements of \( C \) leaving the elements of \( D_R \) invariant, \( \cdot \) complex-conjugates \( C \) leaving \( M_4(R) \) invariant. Since the minimal left ideals of \( M_4(R) \) can be identified with its four columns, \( \cdot \) must be interpreted as mapping each Dirac spinor space to itself.

This result is actually a specific application of a more general theorem given, for example, by Benn and Tucker.\(^{18}\) Moreover, as a corollary, one may show that there exists \( D \in D \) such that

\[ DD^\dagger = \bar{d}D \quad (\forall d \in D). \]  

(68)

where \( \bar{D}D = -1 \). Clearly, \( D \) is representation dependent.

Now let \( \psi \in D_{\alpha \alpha} \). Since \( D_{\alpha \alpha} \) is, by definition, a minimal left ideal of \( D \) and since \( \cdot \) maps \( D_{\alpha \alpha} \) to itself, Eq. (2.2.68) shows that

\[ \bar{D}\psi = D\psi^\dagger \in D_{\alpha \alpha}. \]  

(69)

\[ \text{18 Benn and Tucker (1987), pgs. 81-82.} \]
Clearly, right multiplication will preserve the physical content of the Dirac equation in $D$ and thus we may alternatively define $\psi^c(x)$ as

$$\psi^c(x) = \gamma \tilde{\psi}(x) D = \gamma D \psi^c(x).$$  \hfill (70)

Using Eq. (2.2.68) we then have

$$D \gamma^c D^{-1} = \tilde{\gamma}_\mu = \gamma^c_{\mu}$$

$$\Rightarrow \quad \gamma D \gamma^c D^{-1} \gamma^{-1} = \gamma \gamma^c \gamma^{-1}$$

$$\Rightarrow \quad (\gamma D) \gamma^c (\gamma D)^{-1} = -\gamma^c$$

and comparing with Eqs. (2.2.57) and (2.2.58) we conclude that, in the abstract algebra,

$$C = \gamma D.$$  \hfill (71)

Note that for $C$ as defined in Eq. (2.2.70)

$$(\psi^c)^c = \gamma (\gamma \tilde{\psi} D)^c D = \gamma^2 \psi D D = \psi$$  \hfill (72)

so that $C^2 = \text{id}$. On the other hand, Eq. (2.2.61) shows that

$$(\psi^c)^c = \gamma (\gamma \psi)^c = \gamma^2 \psi = -\psi$$  \hfill (73)

and hence in this case $C^2 = -\text{id}$.

The connection with charge conjugation as defined in the Pauli algebra is through the definition given in Eq. (2.2.61). In particular, let $\psi(x) \in D_{-\text{...}}$. Then $\psi^c(x) = \gamma \tilde{\psi}(x) \in D_{-\text{...}}$ where, using Eqs. (2.2.62)

$$\tilde{\psi}(x) = \tilde{E}^c(x) \Omega_+ \Omega_- \Omega_+ \Omega_+ \Omega_- \omega_+.$$  \hfill (74)

Applying the intertwining isomorphism $\phi_-$ gives

$$\phi_-(\tilde{\psi}(x)) = \tilde{E}^c(x) \Omega_- \Omega_+ \Omega_+ \Omega_- \omega_-.$$  \hfill (75)

and using the results of Section 1.5 together with the fact that

$$\Psi(x) = \phi_-(\psi(x)) = \tilde{E}(x) \Omega_+ \Omega_- \Omega_+ \Omega_- \omega_+$$

$$\Rightarrow \quad \Psi^c(x) = \tilde{E}^c(x) \Omega_- \Omega_+ \Omega_+ \Omega_- \omega_+.$$  \hfill (76)
we have

\[
\phi_-(\psi^c(x)) - \phi_-(\rho_-(\gamma)(\tilde{\psi}(x)))
= \rho(\pi(x)\phi_-(\tilde{\psi}(x)))
= -i\left[\bar{c}(x)\Omega_+ - H(x)\omega_+\right]\tilde{c}
= -i\left[\bar{c}(x)\Omega_+ + H(x)\omega_+\right]
= -i\bar{\psi}'(x) = -i\psi^c(x).
\]

(77)

Consequently, apart from an overall phase factor, charge conjugation as defined in \( \mathcal{P} \) does correspond to the definition given in Eq. (2.2.61). Note, however, that \( C^2 = \text{id} \) in \( \mathcal{P} \) while the corresponding definition in \( \mathcal{D} \) satisfies \( C^2 = -\text{id} \).

Having completed our analysis of charge conjugation we now turn our attention to time reversal. In examining time reversal we shall take the approach of Coquereaux; namely, that charge conjugation and time reversal couple to give the Lorentz time inversion operator; that is

\[ T = CT. \]

(78)

Of course, such an approach must also prove to be physically consistent with certain known properties of \( T \) and we shall attempt to verify this.

Consider first time reversal in the Pauli algebra, specifically the behavior of the wavefunction \( \Psi(x) \). Given Eqs. (2.2.42b) and (2.2.56), Eq. (2.2.78) implies

\[ -i\psi(x)e_z = \psi^T(x^T) = \psi^{CT}(x^{CT}). \]

(79)

Now since \( T^2 = C^2 = \text{id} \). we have that

\[ CTCT = C^2 \quad \Rightarrow \quad CT = T^{-1}C. \]

Letting \( \psi^{T^{-1}}(x^{T^{-1}}) = \psi^*(x^*) \). Eq. (2.2.79) becomes
\[-i \Psi(x) e_\gamma = \Psi^{\dagger}(x^{\dagger})\]

\[\Rightarrow \quad -i \Psi(x) e_\gamma = \overline{\Psi^{\dagger}}(x')\]

\[\Rightarrow \quad \Psi(x) = -i \overline{\Psi^{\dagger}}(x') e_\gamma.\]

But \(\Psi(x) = \Psi^{\dagger}(x'^{-T})\) and thus we conclude that

\[\Psi^{T}(x'^{T}) = -i \overline{\Psi^{\dagger}}(x) e_2.\] (80)

Moreover, since \(x\) does not change under charge conjugation,

\[x^T = x^{CT} = x'^T \Rightarrow x'^T = (-i, x)\] (81)

which is consistent with the usual behavior of \(x \in \mathcal{M}^4 \) under

time reversal.

Taking \(\Psi^{T}(x'^{T}) = \Psi^{\dagger}(x')\) we also see that

\[\Psi^{T}(x'^{T}) = -i \overline{\Psi^{\dagger}}(x') e_\gamma\]

\[= -i(-i \overline{\Psi}(x) e_2) e_2\]

\[= -i^2 \Psi(x) e_2^2 = -\Psi(x)\] (82)

and thus \(T^2 = -id\) which is exactly the behavior we expect
when \(\Psi(x)\) describes a single spin-1/2 particle.

Consider now the minimally-coupled momentum \(\pi(x)\). Eqs. (2.2.43b) and (2.2.54) imply

\[-\overline{\pi}(x) = \pi^{T}(x'^{T})\]

\[= \pi^{CT}(x'^{CT})\]

\[= \pi'^{-1} c(x'^{-1} c)\]

\[= -\pi^{\dagger}(x').\]

where \(\pi'(x') = \pi'^{-1}(x'^{-1})\). We then have

\[\pi(x) = \pi'^T(x'^T) = \overline{\pi^{\dagger}}(x')\]

and thus

\[\pi^{T}(x'^{T}) = \overline{\pi}(x).\] (83)
In terms of components this reads
\[ \eta^\tau (x^\tau) = \eta_\mu (x) \cdot (84) \]
Since \( \eta_\mu (x) = i \partial_\mu - q A_\mu (x) \), we have that
\[ A^\mu (x^\mu) = A_\mu (x) \]
and
\[ (i \partial^\mu)^T = -i \partial_\mu \]
\[ \Rightarrow \quad i \partial_t \rightarrow -i \partial_t = i \partial_\tau^, \quad i \nabla \rightarrow i \nabla. \]
which demonstrates the consistency of Eq. (2.2.83) with the behavior we normally expect under time reversal.

Finally, consider the transformation behavior of the Dirac equation. Applying spatial reversion and hermitean conjugation to Eq. (2.2.52) gives
\[ -\bar{\pi}^* (x) \bar{\Psi}^* (x) \omega_\tau - \bar{\pi}^* (x) \bar{\Psi}^* \omega_\tau + m \bar{\Psi}^* (x) = 0. \]
If we now multiply on the right by \( e_2 \) and on the left by \( -i \)
we obtain
\[ \bar{\pi}^* (x) (-i \bar{\Psi}^* (x) e_2) \omega_\tau + \bar{\pi}^* (x) (-i \bar{\Psi}^* (x) e_2) \omega_\tau + m (-i \bar{\Psi}^* (x) e_2) = 0 \]
\[ \Rightarrow \quad \pi (x^T) \Psi (x^T) \omega_\tau + \pi^T (x^T) \Psi (x^T) \omega_\tau + m \Psi^T (x^T) = 0. \quad (85) \]
which clearly demonstrates covariance as expected.

In the Dirac algebra, the condition \( T = CT \) yields two different definitions for the behavior of Dirac spinors under time reversal corresponding to the two alternate definitions we have given for \( C \). Indeed, let \( C \) be defined as in Eq. (2.2.61). Then \( T^2 = -C^2 = id \) and we have \( CT = -T^{-1} C \).
Setting \( \psi' (x') = \psi^{T^{-1}} (x^T) \). Eqs. (2.2.35b) and (2.2.61) show that
\[ \gamma_0 \psi (x) = -\gamma \bar{\psi}' (x') \]
\[ \Rightarrow \quad \psi (x) = -\gamma_0 \bar{\psi}' (x'). \]
But \( \psi^T(x^{'T}) = \psi(x^{'T}) \) and thus
\[
\psi^T(x^{'T}) = -\gamma_0 \tilde{\psi}_0(x^{'T}).
\]
(86)

Alternatively, if \( C \) is defined as in Eq. (2.2.70), we have
\( T^2 = C^2 = \text{id} \) so that \( CT = T^{-1}C \). A similar line of reasoning then yields
\[
\psi^T(x^{'T}) = \gamma_0 \tilde{\psi}(x^{'T}) D
\]
(87)
and using Eqs. (2.2.68) and (2.2.71) we obtain
\[
\psi^T(x^{'T}) = \gamma_0 D \psi^*(x)
\]
\[
= \gamma_0 C \psi^*(x) = \psi^{CT}(x^{CT}).
\]
(88)
where again \( x^{CT} = x^{CT} = x^T = (-t, x^{'T}) \). One may easily demonstrate the covariance of the Dirac equation in \( D \) for both definitions, the components of the minimally-coupled momentum satisfying Eq. (2.2.84) as required.

Notice that for \( T \) as defined in Eq. (2.2.87) we have
\[
(\psi^T)^T = \gamma_0 (\gamma_0 \tilde{\psi} D)^T D
\]
\[
= \gamma_0 \gamma_0 \tilde{\psi} D = -\psi
\]
\[
\Rightarrow \quad T^2 = -\text{id}
\]
(89a)
which is again seen to be consistent with the usual behavior for time reversal in the Dirac theory. However, using the definition given in Eq. (2.2.86) we see that
\[
(\psi^T)^T = -\gamma_0 (-\gamma_0 \tilde{\psi})^* = \psi
\]
\[
\Rightarrow \quad T^2 = \text{id}.
\]
(89b)
which eliminates Eq. (2.2.86) as a viable definition for time reversal in the Dirac algebra. Consequently, based on Eq. (2.2.78), charge conjugation, as defined in Eq. (2.2.61), must also be rejected.
This leads us to the conclusion that charge conjugation and time reversal cannot be defined independent of the choice of representation within the structure of the Dirac algebra. This is in marked contrast to what we have observed in the Pauli algebra. Indeed, our results have demonstrated that $C$ and $T$ are not only representation independent but representation free in $\mathcal{F}$. Moreover both are conjugate-linear transformations in keeping with the usual definitions.

The final test in establishing the validity of $C$ and $T$ as defined in $\mathcal{F}$ is the verification of their consistency with hole theory and the Feynman-Stückelberg formalism.

To begin with, consider the free-particle solutions given in Eqs. (2.1.41) and (2.1.42); in particular, Eq. (2.1.41a). Applying charge conjugation to this solution yields

$$\psi^c(E > 0, p, +1/2) = \overline{\psi}(E > 0, p, +1/2)$$

$$= \left(\frac{E - p}{m}\right) \Omega_+ + \omega_+$$

$$= \omega_+ + \left(\frac{|E| - p}{m}\right) \Omega_+$$

$$= \omega_+ - \left(\frac{-|E| + p}{m}\right) \Omega_-$$

$$= \psi(E < 0, -p, -1/2).$$

(90)

where we have used Eq. (2.1.42b). However, making use of Eq. (2.1.42a) we see that
\[ \psi^I(E < 0, p, +1/2) = -i \overline{\psi}^I(E < 0, p, +1/2)e^\gamma \]

\[ = -i \left( \Omega - \left( \frac{E + p}{m} \right) \omega \right) \gamma \]

\[ = -i \left( \omega - \left( \frac{E - p}{m} \right) \Omega \right) \gamma \]

\[ = -i \psi(E < 0, -p, -1/2) \]  \hspace{1cm} (91)

and thus

\[ \psi^c(E > 0, p, +1/2) = \imath \psi^I(E < 0, p, +1/2). \]  \hspace{1cm} (92)

Since charge conjugation transforms a positive-energy particle into a positive-energy antiparticle, we conclude from Eq. (2.2.92) that, apart from an overall phase factor, a positive-energy, spin-up antiparticle may be identified with a negative-energy, spin-up particle running backwards in time, in agreement with the Feynman-Stückelberg formalism. Moreover, notice that \( T \) reverses the momentum and spin, in agreement with the usual definition given for time reversal.

As a final example, consider the neutrino solutions presented in Section 2.1. We recall that a positive-energy neutrino can be described by the wavefunction

\[ \psi_v = \psi(E > 0, \rho) \Omega. \]  \hspace{1cm} (93)

corresponding to a left Weyl spinor having negative helicity

\[ \beta \psi_v = -\psi_v \]  \hspace{1cm} (94)

(see Eq. (2.1.95)). Applying spatial reversion and hermitean conjugation to Eq. (2.2.94) gives\(^{\dagger}\)

\[ \beta \psi_v^c = \psi_v^c. \]  \hspace{1cm} (95)

---

19 Note that here \( \beta \in \mathbb{R}^2 \subset \mathbb{P} \) so that \( \beta^* = \beta \).
where \( \Psi^c = \Psi^\dagger = \overline{\Psi}(E > 0, \rho) \Omega \) may be taken to describe a positive-energy antineutrino. Eq. (2.2.95) establishes the antineutrino as having positive helicity and, consistent with our previous observation that helicity equals chirality for positive-energy solutions, we see that
\[
\Psi^c = \Psi. \tag{96}
\]
i.e. \( \Psi^c \) corresponds to a right Weyl spinor.

The validity of the Feynman-Stückelberg formalism follows from Eq. (2.1.89b). Indeed, applying spatial reversion and hermitean conjugation to this equation gives
\[
\rho \overline{\Psi}(E < 0, \rho) \Omega = \overline{\Psi}(E < 0, \rho) \Omega.
\]
and multiplying on the right by \( e_2 \) and on the left by \(-i\) we obtain
\[
\rho \Psi^T(E < 0, \rho) \Omega_\epsilon = \Psi^T(E < 0, \rho) \Omega_\epsilon. \tag{97}
\]
Comparing with Eq. (2.2.95) one then concludes that, apart from a possible overall phase factor, \( \Psi^c(E > 0, \rho) \) can be identified with \( \Psi^T(E < 0, \rho) \) as required.

2.3 The Dirac Current

We shall now attempt to construct the Dirac current within the framework of the Pauli algebra using the results presented thus far in this study.

To this end, consider the free particle Dirac equation in \( \mathcal{P} \)
\[
\rho \psi \omega_+ + \overline{\rho} \psi \omega_- + m \psi = 0, \tag{1}
\]
where \( \psi = \Psi(x) \) and \( \rho = i \partial \) is the Pauli algebra four-momentum operator. Multiplying this equation on the left by \( e_1 \psi \) and taking the scalar part gives
\[
(e_1 \psi \overline{(\rho \psi)} \omega_+ + e_1 \psi \overline{(\overline{\rho} \psi)} \omega_- + me_1 \psi \psi) = 0. \tag{2}
\]
where \( \rho \) and \( \overline{\rho} \) operate on \( \Psi \) to the right. Since the scalar part of any product is invariant under cyclic permutation of the elements, Eq. (2.3.2) may be rewritten

\[
((\rho \Psi) \omega \cdot e \cdot \Psi^* + (\overline{\rho} \Psi) \omega \cdot e \cdot \Psi^* + m \Psi e \cdot \Psi^*)_s = 0
\]

and using the fact that \( \omega \cdot e_i = \Omega_i \), we have

\[
((\rho \Psi) \Omega_i \Psi^* + (\overline{\rho} \Psi) \Omega_i \Psi^* - m \Psi e_i \Psi^*)_s = 0. \tag{3}
\]

Alternatively, multiplying Eq. (2.3.1) on the right by \( e_i \) and applying hermitean conjugation gives

\[
\Omega_i (\Psi^* \rho) - \Omega_i (\Psi^* \overline{\rho}) - m e_i \Psi^* = 0. \tag{4}
\]

where we have used the fact that \( \rho^* = -\rho \). Here, \( \rho \) and \( \overline{\rho} \) are to be identified as acting on \( \Psi^* \) to the left. If we now multiply Eq. (2.3.4) on the left by \( \Psi \) and take the scalar part we obtain

\[
(\Psi \Omega_i (\overline{\Psi} \rho) + \Psi \Omega_i (\Psi^* \overline{\rho}) - m \Psi e_i \Psi^*)_s = 0. \tag{5}
\]

Adding Eqs. (2.3.3) and (2.3.5) then yields

\[
((\rho \Psi) \Omega_i \Psi^* + (\overline{\rho} \Psi) \Omega_i \Psi^* + \Psi \Omega_i (\Psi^* \rho) - \Psi \Omega_i (\Psi^* \overline{\rho}))_s = 0
\]

and using the fact that \( (a)_s = (\overline{a})_s \). \( (a + b)_s = (a)_s + (b)_s \) for all \( a, b \in \mathbb{P} \), we obtain

\[
(\overline{\Psi} \cdot \Omega_i (\overline{\Psi} \rho) + (\overline{\rho} \Psi) \Omega_i \Psi^* + (\overline{\rho} \overline{\Psi}) \Omega_i \overline{\Psi} + \Psi \Omega_i (\Psi^* \overline{\rho}))_s = 0
\]

\[\Rightarrow (\overline{\Psi} \cdot \Omega_i (\overline{\Psi} \delta) + (\overline{\delta} \Psi) \Omega_i \Psi^* + (\overline{\delta} \overline{\Psi}) \Omega_i \overline{\Psi} + \Psi \Omega_i (\Psi^* \delta))_s = 0. \tag{6}
\]

Now for every \( a, b \in \mathbb{P} \) we have

\[
\overline{\delta} \cdot (ab) = ((\overline{\delta} a) b + a (b \delta))_s. \tag{7}
\]

where \( \overline{\delta} \) acts to both the left and the right. Comparing with Eq. (2.3.6) we see that
\[ 0 = \left( \partial \psi \right) \Omega . \psi^* - \psi \Omega . \left( \psi^* \partial \right) + \left( \partial \psi^* \right) \Omega . \bar{\psi} + \psi^* \Omega . \left( \bar{\psi} \partial \right) \]

\[ - \left( \left( \partial \psi \right) \Omega . \psi^* - \psi \Omega . \left( \psi^* \partial \right) \right)_s + \left( \left( \partial \psi^* \right) \Omega . \bar{\psi} + \psi^* \Omega . \left( \bar{\psi} \partial \right) \right)_s \]

\[ + \partial \cdot \left( \psi \Omega . \psi^* + \psi^* \Omega . \bar{\psi} \right) \]

\[ = \partial \cdot \left( \psi \Omega . \psi^* + \psi^* \Omega . \bar{\psi} \right) \]

and thus

\[ \partial \cdot j = 0. \]

where

\[ j = \psi \Omega . \psi^* + \psi^* \Omega . \bar{\psi}. \]

Clearly, \( j \) satisfies the continuity equation. In order to definitively identify \( j \) as the Dirac current, we must verify that it behaves like a four-vector in the Pauli algebra. Moreover, \( (j)_s \) should correspond to a probability density consistent with the usual definition given in the literature.

First, notice that \( j \) is hermitean — a fundamental requirement of four-vectors in \( \mathbb{F} \). Now consider its behavior under Lorentz transformations. For restricted Lorentz transformations, Eqs. (2.2.17) show that

\[ \left( \psi \Omega . \psi^* + \psi^* \Omega . \bar{\psi} \right) \]

\[ = (\psi \Omega . \psi^*)^* + (\psi \Omega . \psi^*)^* \]

\[ \rightarrow (L \psi \Omega . \psi^*)^* + (\bar{\Omega} \psi \Omega . \bar{\psi})^* \]

\[ = L \psi \Omega . \psi^* L^* + L \psi \Omega . \bar{\psi} \bar{L} \]

\[ = L \left( \psi \Omega . \psi^* + \psi^* \Omega . \bar{\psi} \right) L^* \]

and thus
Under parity and time reversal Eqs. (2.2.49b) and (2.2.80) show that
\[ j^P = \psi^\dagger \Omega_+ \psi^\dagger + \overline{\psi}^\dagger \Omega_+ \overline{\psi}^\dagger \]
\[ = (-P^{(c)} \psi \Omega_+) \Omega_+ (-P^{(c)} \psi \Omega_+) \Omega_+ (-P^{(c)} \psi \Omega_+) \]
\[ = P^{(c)} P^{(c)} \psi_1 \Omega_+ \psi_1 \psi^\dagger + P^{(c)} P^{(c)} \overline{\psi}_1 \Omega_+ \overline{\psi}_1 \psi^\dagger \]
\[ = \psi \Omega_+ \psi^\dagger + \overline{\psi}^\dagger \Omega_+ \overline{\psi} = \tilde{j} \]

\[ j^T = \psi^T \Omega_+ \psi^T + \overline{\psi}^T \Omega_+ \overline{\psi}^T \]
\[ = (-i \overline{\psi}^T \epsilon_2 \Omega_+ (-i \overline{\psi}^T \epsilon_2 \Omega_+) \Omega_+ (-i \overline{\psi}^T \epsilon_2 \Omega_+) \]
\[ = -i \epsilon_2 \Omega_+ \epsilon_2 \overline{\psi} + \psi \epsilon_2 \Omega_+ \epsilon_2 \psi^\dagger \psi^\dagger \]
\[ = \overline{\psi}^T \Omega_+ \psi^\dagger + \psi \Omega_+ \psi^\dagger = \tilde{j} \]

that is
\[ j \rightarrow j^P = \tilde{j} \], \[ j \rightarrow j^T = \tilde{j} \].

These results clearly establish \( j \) as a four-vector in the Pauli algebra. It remains to show that \((j)_5\) coincides with a probability density. Now one normally finds the Dirac probability density defined as
\[ \rho = \psi^\dagger \psi = \psi^\mu \psi^\mu \].
where \( \psi \) is identified with a column vector in \( \mathbb{C}^4 \). In generalizing Eq. (2.3.13) to the abstract Clifford algebra, let \( \psi \in \mathcal{D}_{\mathcal{D}} \) be any Dirac spinor. Then \( \psi = \psi (\Omega \otimes \Omega') \), where \( \psi \in \mathcal{D}_{\mathcal{D}} \setminus \mathcal{D} \) takes the form
\[ \psi = \overline{\psi}^\dagger \Omega_+ + H_+ \Omega_+ \overline{H}_+ \Omega_+ + \overline{\psi}^\dagger \Omega_+ \].

(14)
Consequently,
\[
\psi^* \psi = (\Omega \otimes \Omega')^* \psi e(\Omega \otimes \Omega')
\]
\[
- (\Omega^* \otimes \Omega'^*) \phi^* \psi (\Omega \otimes \Omega') e \in \mathcal{D}_{\eta \eta'} \quad \mathbb{C}
\]
and using Eq. (2.3.14) we obtain
\[
\psi^* \psi = \Omega \Phi_n \Omega \otimes \Omega'.
\]
where
\[
\Phi_n = \begin{cases} 
T \Xi^* + \Pi^* \Pi & \{\Omega' = \Omega_n \} \\
\Xi^* \Xi + \Pi^* \Pi & \{\Omega' = \Omega \}.
\end{cases}
\]
Clearly, $\psi^* \psi$ is seen to be an element of the division algebra $\mathcal{D}_{\eta \eta'} = \mathbb{C}$ and employing Eqs. (1.1.57) we conclude that
\[
\psi^* \psi = \lambda_{\eta \eta'} \Omega \otimes \Omega'.
\]
where
\[
\lambda_{\eta \eta'} = \begin{cases} 
(\Phi_n)_s + (\Phi_n)_{r^*} e & \{\Omega = \Omega_n \} \\
(\Phi_n)_s - (\Phi_n)_{r^*} e & \{\Omega = \Omega \}.
\end{cases}
\]

Now we recall from Section 2.1 that $\psi$ could be written in the form
\[
\psi = \psi^* \Gamma_\mu.
\]
where the $\psi^* (\forall \mu \in \mathbb{N}_0)$ are appropriately differentiable functions of $\mathcal{M}^4$ into $\mathbb{C}$ and $\{\Gamma_\mu | \mu \in \mathbb{N}_0\}$ is a basis for $\mathcal{D}_{\eta \eta'}$ over $\mathbb{C}$. Since there are four distinct Dirac spinor spaces, let $\{\Gamma_s | s \in \mathbb{N}_0\}$ denote the basis generating the $s$th spinor space, where $s(\Omega_0) = 0$, $s(\Omega_-) = 1$, $s(\Omega_+) = 2$ and $s(\Omega_{--}) = 3$. Define the matrix
\[ [ \Gamma_{\mu} ] = \begin{pmatrix} \Omega \otimes \Omega & \omega \otimes \Omega & \Omega \otimes \omega & \omega \otimes \omega \\ \omega \otimes \Omega & \Omega \otimes \Omega & \omega \otimes \omega & \Omega \otimes \omega \\ \Omega \otimes \omega & \omega \otimes \omega & \Omega \otimes \Omega & \omega \otimes \Omega \\ \omega \otimes \omega & \Omega \otimes \omega & \omega \otimes \Omega & \Omega \otimes \Omega \end{pmatrix}. \] (21)

where \( s \) and \( \mu \) label rows and columns respectively. For example, \( \Gamma_{21} = \omega \otimes \omega \) defines the basis element \( \Gamma_1 \) for the spinor space \( D_2 = D_{\ldots} \). Then, given \( \psi \in D_{\Lambda} = D_s \), we have

\[
\psi^\dagger \psi = (\psi^\dagger \Gamma_{1,\mu}^\dagger) \Gamma_{1,\nu}^\dagger = \psi^\dagger \psi \Gamma_{1,\mu} \Gamma_{1,\nu}^\dagger = \psi^\dagger \psi \delta_{\mu\nu} \Omega \otimes \Omega = \psi^\dagger \psi \Omega \otimes \Omega = \rho \Omega \otimes \Omega \] (22)

and comparing with Eq. (2.3.18) we conclude that

\[
\rho = \lambda_{\Lambda} \Omega. \] (23)

Note that, in Eq. (2.3.22), \( s \) merely indicates to which spinor space \( \psi \) belongs and hence is not summed over. Moreover, since \( D_{\Lambda} = \mathbb{C}^4 \), there is no distinction between contravariant and covariant components.

Eq. (2.3.22) clearly defines the probability density in the abstract Dirac algebra. If we restrict our attention to the case where \( \Omega \neq \Omega' \), we may apply the intertwining isomorphism \( \phi_{\Lambda} \), defined in Section 1.5 to obtain

\[
\phi_{\Lambda}(\psi^\dagger \psi) = \rho \Omega \overline{\Omega} = \rho \Omega \] (24)

which is now seen to be an element of the division algebra \( \mathcal{D}_\Lambda = \mathbb{C} \). One easily demonstrates that

\[
\psi^\dagger \psi = \rho \Omega. \] (25)

where \( \psi = \phi_{\Lambda}(\psi) \) and since
\[(j)_S = \left( \psi \Omega, \psi^* + \overline{\psi^*} \Omega, \overline{\psi} \right)_S\]
\[= (\psi^* \psi \Omega)_S - \left( \Omega, \overline{\psi \psi^*} \right)_S\]
\[= (\psi^* \psi \Omega)_S + (\psi^* \psi \Omega)_S = (\psi^* \psi)_S. \tag{26}\]

we conclude that
\[(j)_S = \rho/2. \tag{27}\]

Consequently, by redefining \(j\) to include an additional factor of 2, \((j)_S\) does coincide with the Dirac probability density.

Having derived the Dirac current in the Pauli algebra it is instructive to see how \(j\) is related to the usual definition
\[j_\mu = \psi^* \gamma_0 \gamma_\mu \psi \tag{28}\]
given in the literature, where \(\psi\) is again identified with a column vector in \(C^4\). In the abstract Clifford algebra \(\psi \in D_{\Sigma 0}\), and using Eqs. (2.3.13) and (2.3.22) we see that
\[\psi^* \gamma_0 \gamma_\mu \psi = j_\mu \Omega \otimes \Omega^* \tag{29}\]
may be taken as the equivalent expression in \(D\).

In order to transform to \(\mathcal{P}\) set \(\Omega \neq \Omega^*\). Then
\[\phi_{\Sigma 0}(\psi^* \gamma_0 \gamma_\mu \psi) = j_\mu \Omega \in \mathcal{P} \tag{30}\]
so that
\[j_\mu = 2(\phi_{\Sigma 0}(\psi^* \gamma_0 \gamma_\mu \psi))_S. \tag{31}\]

But Eqs. (2.3.24) and (2.3.25) imply
\[\phi_{\Sigma 0}(\psi^* \psi) = \psi^* \psi = (\phi_{\Sigma 0}(\psi))^* \phi_{\Sigma 0}(\psi). \tag{32}\]

Consequently, using the results of Section 1.5, we see that
\[ j_u = 2 ((\phi_{au} \psi_0 \gamma_0 \psi))_S \]
\[ = 2 ((\phi_{au} \psi))_* \phi_{au} (\rho_{uu} (\gamma_0 \gamma_0) (\psi))_S \]
\[ = 2 ((\phi_{uu} \psi))_0 \rho (\gamma_0 \gamma_0) (\phi_{uu} \psi))_S \]
\[ = 2 (\psi^* (\gamma (\epsilon_u \Omega - \tilde{\epsilon}_u \Omega_0) \psi)_S \]
\[ = 2 (\psi^* (\epsilon_u \psi \Omega_0 - \tilde{\epsilon}_u \psi \Omega_0)_S \]
\[ = 2 (\psi^* \tilde{\epsilon}_u \psi \Omega_0 - \psi^* \epsilon_u \psi \Omega_0)_S \]
\[ = 2 (\psi \Omega_0 \psi^* \tilde{\epsilon}_u - \tilde{\psi} \Omega_0 \psi^* \epsilon_u)_S \]
\[ = 2 (\psi \Omega_0 \psi^* - \tilde{\psi} \Omega_0 \tilde{\psi}) \tilde{\epsilon}_u \]

and comparing with Eq. (1.2.31) we obtain

\[ j = 2 (\psi \Omega_0 \psi^* - \tilde{\psi} \Omega_0 \tilde{\psi}) \]

(33)

as required

Finally, consider the behavior of \( j \) under charge conjugation. Using Eq. (2.2.56) we see that

\[ \psi^c \Omega_0 \psi^c - \tilde{\psi}^c \Omega_0 \tilde{\psi}^c \]

and hence

\[ j^c = j. \]

Eq. (2.3.34) is consistent with the usual behavior of the first-quantized form of the Dirac current under charge conjugation;\(^ {20} \) however, it is not consistent with the quantum field theoretic description wherein \( j \) changes sign under \( C \). Since we have restricted our attention to the first-quantized regime, such an observation would seem irrelevant. However, we should point out that, in standard treatments of first-quantized Dirac theory, charge

\[ ^{20} \text{See, for example, Halzen and Martin (1984).} \]
conjugation introduces a transposition of the spinor elements comprising the current.\textsuperscript{21} Since the Dirac spinors are ultimately identified as anticommuting operators after second quantization, the standard approach provides an \textit{ad hoc} means for introducing the requisite sign change. Clearly, no such means is seen to exist, for the present framework, within the structure of the Pauli algebra.

This seems to bring into question the validity of Eq. (2.2.56) as a genuine description of charge conjugation in the Pauli algebra. The answer to this question, however, can only truly be answered by rigourously extending the Pauli algebra to quantum field theory.

\textsuperscript{21} This is connected to the fact that charge conjugation is inherently an anti-linear operator in the quantum mechanical sense (Coquereaux 1988).
CONCLUSIONS
In this study we have seen how Clifford algebras provide a comprehensive abstract framework for describing the algebraic properties of first-quantized Dirac theory. Since the Dirac theory is inherently relativistic, the Dirac algebra has traditionally been the Clifford algebra of choice for its description. However, we have clearly demonstrated that the simpler Pauli algebra, with its foundations rooted in the elementary vector analysis of Gibbs, provides an entirely equivalent description that is both compact and component-free. Moreover, in contrast to the usual Dirac algebra approach, the Pauli algebra approach plainly elucidates the orbital-spin, particle-antiparticle character of the theory by dividing operators into two corresponding distinct classes according to their action on the wavefunction $\Psi$.

In our study of the discrete symmetries of parity, time reversal and charge conjugation, we found that time reversal and charge conjugation may be defined as representation independent operators within the framework of the Pauli algebra. In fact, charge conjugation, as defined on the wavefunction $\Psi$, was seen to correspond to the composition of spatial reversion and hermitean conjugation which, over the reals, coincides with the degree automorphism. This is in marked contrast to observations made when analyzing these transformations within the structure of the Dirac algebra. Indeed, charge conjugation in $\mathcal{D}$ was seen to be completely representation dependent. This is clearly a disadvantage since one must choose a matrix representation for $\mathcal{D}$ in order to rigorously define the charge-conjugate spinor. Our observations concerning the behavior of the Dirac current under charge conjugation has introduced some concerns as to the ultimate validity of the definition we have given for $\mathcal{C}$ in the Pauli algebra, when attempting to extend the formalism
to account for the Fermi-Dirac statistics obeyed by the spinor wavefunctions after second-quantization. More work is clearly needed along these lines.

The above paragraph brings up an interesting point in relation to Hestenes' demand for a geometrical interpretation of the $i$ commonly appearing in the Dirac equation. As we have seen, there is a profound correspondence between the complex scalar $i \in \mathbb{C}$ and the canonical element $e_1^* e_2^* e_3^* F$. In Appendix 3, we saw how Hestenes' form of the Dirac equation could easily be transcribed into the Pauli algebra. In this framework, Hestenes interpretation for the action of $i$ was realized as right multiplication by $e_0$. Employing the complex structure of $\mathbb{P}$ then allows for a complex description of Dirac theory with the addition of an extra operator acting on the wavefunction from the right. Since Hestenes' description of relativistic quantum mechanics is ultimately based on the real field, it may be that the Pauli algebra can act as the vehicle for making a meaningful connection between the usual complex quantum mechanical formalism and Hestenes' geometrical interpretation.

Finally, it should be possible to extend our formalism to the electroweak theory of Weinberg and Salam. Indeed, apart from the additional SU(2) gauge structure introduced to describe the weak interaction, the Weinberg-Salam model is inherently based on an irreducible spinor representation of the Dirac algebra. Moreover, the distinct chiral asymmetry of the model, where only left hand components are coupled via the SU(2) gauge group, should allow us to realize the entire theory within the framework of the Pauli algebra. In fact, preliminary studies have shown this to be true, though more work is needed to extend the formalism to encompass the quantum field theoretic aspects of the theory.
APPENDICES
1 Algebraic Structure Theory

In the ensuing discussion, all vector spaces shall be assumed finite dimensional over a field $F$ equal to either $\mathbb{R}$ or $\mathbb{C}$.

Def. 1: An algebra $A$ over a field $F$ is a vector space having an $F$-bilinear map

$$A \times A \rightarrow A$$

$$(a_1,a_2) \rightarrow a_1a_2.$$ $A$ is said to be associative if

$$(a_1a_2)a_3 = a_1(a_2a_3) \quad (\forall a_1,a_2,a_3 \in A).$$

An element $1$ of $A$ is said to be a unit (or identity element) of $A$ if

$$a1 = 1a = a \quad (\forall a \in A).$$

An algebra with unit is called a division algebra if every non-zero element of the algebra has an inverse.

Def. 2: A Lie algebra $A$ over $F$ is a vector space, together with an $F$-bilinear map $[ . , . ] : A \times A \rightarrow A$, called the bracket, such that for every $a,b,c \in A$.

i) $[a,b] = -[b,a] = 0$

ii) $[[a,b],c] + [[c,a],b] + [[b,c],a] = 0$.

Prop. 1: Let $A$ be an associative algebra. Set

$$[a,b] = ab - ba \quad (\forall a,b \in A).$$

Then with this bracket, called the commutator, $A$ becomes a Lie algebra.

Henceforth, the term "algebra" will always be taken to mean an associative algebra with unit.
A well known example of an algebra is the total matrix algebra $\mathcal{M}_n(F)$ which is defined as the abstract algebra isomorphic to the algebra of $n \times n$ matrices over $F$. In particular, if $V$ and $W$ are $n$-dimensional vector spaces over $F$, then by choosing ordered bases for $V$ and $W$, $\mathcal{M}_n(F)$ can be identified with the algebra of all $F$-linear maps $V \to W$.

**Def.3:** Let $A$ be an algebra, $A_0 \subseteq A$. $A_0$ is said to be a subalgebra of $A$ (denoted $A_0 \subseteq A$) if $A_0$ is a linear subspace of $A$ closed under the algebra product of $A$, i.e.

$$a_1, a_2 \in A_0 \implies a_1a_2 \in A_0.$$ 

**Def.4:** Let $A$ be an algebra and consider the subset

$$A^c = \{ a' \in A | aa' = a'a \ (\forall a \in A) \}.$$ 

$A^c$ is a subalgebra of $A$ called the center of $A$. An algebra $A$ satisfying the condition

$$A^c = \text{span}_F \{1\}$$

is called a central algebra.

**Def.5:** Let $A$ be an algebra, $I \subseteq A$. $I$ is said to be a left ideal of $A$ if

$$aI \subseteq I \quad (\forall a \in A).$$

Similarly, $I$ is said to be a right ideal of $A$ if

$$IA \subseteq I \quad (\forall a \in A).$$

A subspace that is both a left and right ideal is called a two-sided ideal or simply an ideal. Every left, right and two-sided ideal is a subalgebra.

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1 We shall find it convenient to identify $\mathcal{M}_n(F)$ with the algebra of $n \times n$ matrices over $F$. 
Def. 6: Let \( V_1 \) and \( V_2 \) be vector spaces over \( F \). An \( F \)-linear mapping \( \phi : V_1 \rightarrow V_2 \) is said to be a linear (or vector space) isomorphism if \( \phi \) is bijective.

Def. 7: Let \( A_1 \) and \( A_2 \) be algebras. An \( F \)-linear mapping \( \phi : A_1 \rightarrow A_2 \) is said to be an algebra homomorphism if

\[
\phi(\alpha_1 \alpha_2) = \phi(\alpha_1) \phi(\alpha_2) \quad (\forall \alpha_1, \alpha_2 \in A_1).
\]

A bijective algebra homomorphism is called an algebra isomorphism. An algebra isomorphism from an algebra to itself is termed an algebra automorphism.

We shall omit the terms "vector space" and "algebra" when dealing with the various "morphisms" if the underlying structure is clear.

Let \( V \) be a vector space over \( F \), and let \( \text{END}(V) \) denote the algebra of \( F \)-linear maps \( V \rightarrow V \) with the algebra product being map composition.²

Def. 8: A linear representation (rep) of an algebra \( A \) in a vector space \( V \) is an algebra homomorphism

\[
\rho : A \rightarrow \text{END}(V)
\]

\[
a \rightarrow \rho(a).
\]

\( \rho \) is said to be faithful if \( \ker(\rho) = 0 \); that is, if \( \rho \) is injective. \( \rho \) is said to be reducible if \( V \) contains an invariant subspace under \( \rho \); that is, there exists a non-trivial proper subspace \( V_0 \) of \( V \) such that \( \rho(A)(V_0) \subset V_0 \). \( \rho \) is said to be irreducible (an irrep) if it is not reducible.

---

² Clearly, \( \text{END}(V) = \mathcal{M}_{\dim(V)}(F) \).
Since $K\otimes D(1)$ and $M_{dim 1}(K)$ are isomorphic as algebras, representations are usually expressed in terms of matrices, the formalism being inherently basis dependent. By viewing an algebra $A$ as a vector space, however, one can define a matrix-free representation of $A$ called the regular (or left regular) representation:

$$\rho_{REG}: A \to \text{END}(A)$$

$$a \to \rho_a,$$

where $\rho_a(a') = aa'$ ($\forall a' \in A$). Since $A$ contains a unit, $\rho_{REG}$ is faithful, and thus $\rho_{REG}(A) \cong A$.

The subspaces of $A$ which are invariant under $\rho_{REG}$ are the left ideals of $A$ and the subspaces in which $\rho_{REG}$ acts irreducibly are the minimal left ideals of $A$, i.e. those left ideals containing no proper non-trivial left ideals.

**Def.9:** Let $A$ and $A'$ be algebras and let $\tau: A \to A'$ be a linear isomorphism satisfying

$$\tau(a_1a_2) = \tau(a_2)\tau(a_1) \quad (\forall a_1, a_2 \in A).$$

Then $A$ and $A'$ are termed opposite algebras, and $\tau$ is called an anti-homomorphism.

If an algebra $A$ and its opposite algebra, denoted $A^{op}$, are isomorphic, then $A^{op}$ may be identified with $A$ and $\tau$ becomes an anti-automorphism.

**Prop.2:** Let $A$ be an algebra. The linear map

$$\rho^{R}_{REG}: A \to \text{END}(A)$$

$$a \to \rho^R_a,$$

where $\rho^R_a(a') = a'a$ ($\forall a' \in A$), is an anti-homomorphism. $\rho^{R}_{REG}$ is also faithful and thus $\rho^{R}_{REG}(A) \cong A^{op}$. 
Since $\rho^R_{\text{REC}}$ is an anti-homomorphism, it is not a representation of $A$. $\rho^R_{\text{REC}}$ may, however, be extended to a rep of $A$ as shown by the following proposition.

Prop.3: Let $A$ be an algebra, $\tau:A \to A$ an anti-automorphism of $A$. Then the linear map $\rho^I_{\text{REC}}:A \to \text{END}(A)$

$$\alpha \to \rho^R_{\tau(\alpha)},$$

where $\rho^R_{\tau(\alpha)}(\alpha') = \alpha' \tau(\alpha)$ ($\forall \alpha' \in A$). is a rep of $A$ called the ($\tau$) right regular representation.

Like $\rho_{\text{REC}}$, $\rho^I_{\text{REC}}$ is also a matrix-free representation in the sense that the representation can be considered entirely "embedded" within the abstract algebra.

Def.10: An algebra $A$ is the direct sum of algebras $A_1$ and $A_2$:

$$A = A_1 \oplus A_2$$

if $A$, as a vector space, is the direct sum of the vector spaces $A_1$ and $A_2$, and

$$A_1 \cdot A_2 = A_2 \cdot A_1 = \{0\}.$$

The product in $A$ is then defined

$$(\alpha_1 \oplus \alpha_2)(\alpha_1' \oplus \alpha_2') = \alpha_1 \alpha_1' \oplus \alpha_2 \alpha_2'.$$

where $\alpha_1, \alpha_1' \in A_1$, $\alpha_2, \alpha_2' \in A_2$. The unit element of $A$ is $I_1 \oplus I_2$, where $I_1$ and $I_2$ are the unit elements of $A_1$ and $A_2$ respectively.

It can be shown that $A_1 \oplus \{0\}$ and $\{0\} \oplus A_2$ may be identified with $A_1$ and $A_2$ respectively, and are ideals of $A$ (Hermann 1974). Furthermore, if $V_1$ and $V_2$ are vector
spaces, then the reps \( \rho_1: A_1 \rightarrow \text{END}(V_1) \), \( \rho_2: A_2 \rightarrow \text{END}(V_2) \) define a rep \( \rho: A \rightarrow \text{END}(V) \) of \( A \) in \( V = V_1 \oplus V_2 \) such that 
\[ \rho(a_1 \otimes a_2)(t_1 \oplus t_2) = \rho_1(a_1)(t_1) \oplus \rho_2(a_2)(t_2), \]
where \( a_1 \in A_1, a_2 \in A_2, t_1 \in V_1, t_2 \in V_2 \). The rep \( \rho \) is clearly reducible: if \( V_1 \) and \( V_2 \) are identified with \( V_1 \oplus \{0\} \) and \( \{0\} \oplus V_2 \), respectively, then \( \rho(\cdot)(V_1) \subseteq V_1, \rho(\cdot)(V_2) \subseteq V_2 \) and we write \( \rho = \rho_1 \oplus \rho_2 \). If \( \rho_1 \) and \( \rho_2 \) are irreducible in \( V_1 \) and \( V_2 \) respectively, we write \( \rho = \rho_1 \oplus \rho_2 \).

Although the same notation is used for both vector space and algebra direct sums, the algebra direct sum implies the added condition that the product of the subalgebras be zero. To distinguish between the two cases, explicit reference will be made to the nature of the direct sum when dealing with vector spaces that are also algebras.

**Def.11:** Let \( A_1 \) and \( A_2 \) be algebras. The tensor product algebra
\[ A = A_1 \otimes A_2 \]
is defined as follows:

(i) As a vector space, \( A \) is the tensor product of the vector spaces \( A_1 \) and \( A_2 \).

(ii) The algebra product in \( A \) is defined
\[ (a_1 \otimes a_2)(a_1' \otimes a_2') = a_1 a_1' \otimes a_2 a_2'. \]
where \( a_1, a_1' \in A_1, a_2, a_2' \in A_2 \).

(iii) The unit element of \( A \) is \( 1_1 \otimes 1_2 \) where \( 1_1 \) and \( 1_2 \) are the units of \( A_1 \) and \( A_2 \) respectively.

**Prop.4:** Let \( A_1 \) and \( A_2 \) be algebras. Then
\[ \dim(A_1 \otimes A_2) = \dim(A_1) + \dim(A_2) \]
\[ \dim(A_1 \oplus A_2) = \dim(A_1) \dim(A_2). \]
It may be shown (Benn and Tucker 1987) that
\[ \mathcal{M}_n(F) \otimes \mathcal{M}_m(F) = \mathcal{M}_{nm}(F). \]

As an example, consider the tensor product of \( \mathcal{M}_2(F) \) with itself. Let
\[
A = \begin{pmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{pmatrix}, \quad B = \begin{pmatrix}
    b_{11} & b_{12} \\
    b_{21} & b_{22}
\end{pmatrix}
\]
be elements of \( \mathcal{M}_2(F) \). Then the tensor product of \( A \) and \( B \) may be defined as
\[
A \otimes B = \begin{pmatrix}
    a_{11}b_{11} & a_{11}b_{12} \\
    a_{21}b_{11} & a_{21}b_{12}
\end{pmatrix},
\]
where
\[
ab_{ij} = \begin{pmatrix}
    a_{11}b_{ij} \\
    a_{21}b_{ij}
\end{pmatrix} \quad (\forall i, j \in \mathbb{N}).
\]

**Thm.1:** Let \( \rho_1: A_1 \to \text{END}(V_1) \), \( \rho_2: A_2 \to \text{END}(V_2) \) be reps of algebras \( A_1 \) and \( A_2 \) in vector spaces \( V_1 \) and \( V_2 \) respectively. Let \( \Lambda = A_1 \otimes A_2 \) be the tensor product algebra, \( V = V_1 \otimes V_2 \) the tensor product vector space. Then the linear map \( \rho: \Lambda \to \text{END}(V) \) defined by
\[
\rho(a_1 \otimes a_2)(v_1 \otimes v_2) = \rho_1(a_1)(v_1) \otimes \rho_2(a_2)(v_2).
\]
where \( a_1 \in A_1, a_2 \in A_2, v_1 \in V_1, v_2 \in V_2, \) is a rep of \( \Lambda \) in \( V \) and we write \( \rho = \rho_1 \otimes \rho_2. \)

**Thm.2:** Let \( A_1 \) and \( A_2 \) be algebras over \( \mathbb{C} \), \( V_1 \) and \( V_2 \) complex vector spaces. If \( \rho_1: A_1 \to \text{END}(V_1) \) and \( \rho_2: A_2 \to \text{END}(V_2) \) are irreps, then \( \rho = \rho_1 \otimes \rho_2 \) is an irrep of \( \Lambda = A_1 \otimes A_2 \) in \( V = V_1 \otimes V_2. \)

---

3 Note that we have here chosen a particular order for the bases comprising the tensor product components.
Thm. 3: Let $V_1$ and $V_2$ be vector spaces, $V = V_1 \otimes V_2$, the tensor product vector space. Then the linear map

$$\mu: \text{END}(V_1) \otimes \text{END}(V_2) \to \text{END}(V)$$

$$\rho_1 \otimes \rho_2 \mapsto \rho,$$

where

$$\rho(v_1 \otimes v_2) = \rho_1(v_1) \otimes \rho_2(v_2) \quad (\forall v_1 \in V_1, v_2 \in V_2),$$

is an algebra isomorphism.

Def. 12: A non-zero element of an algebra is said to be nilpotent if some finite power of it vanishes. The smallest such power is called the index of that element. An algebra is said to be nilpotent of index $n$ if $n$ is the smallest integer such that all products of $n$ terms vanish.

Def. 13: An algebra is said to be semi-simple if it contains no non-trivial nilpotent two-sided ideals. An algebra is said to be simple if it contains no proper non-trivial ideals.

Clearly, every simple algebra is semi-simple.

Def. 14: Let $A$ be an algebra. A non-zero $a \in A$ is said to be an idempotent if

$$a^2 = a.$$

Prop. 5: The unit element is the only non-trivial idempotent in a division algebra.

Thm. 4: Let $I$ be a non-nilpotent minimal left ideal of an algebra $A$. Then $I$ contains an idempotent $\Omega$ such that $I = A\Omega$ and $\Omega$ is said to generate $I$. 
Def.15: Let \( A \) be an algebra, \( \Omega \) an idempotent of \( A \). \( \Omega \) is said to be primitive if it can not be written as the sum of two mutually orthogonal idempotents. Alternatively, \( \Omega \) is primitive iff \( \Omega \Omega \) is a minimal left ideal of \( A \).

Thm.5: An algebra \( A \) is semi-simple iff \( A \), as a vector space, is the direct sum of minimal left ideals:
\[
A = \mathbb{I}_1 \oplus \cdots \oplus \mathbb{I}_n
\]
where, for each \( i \in \mathbb{N} \), the \( \mathbb{I}_i \) are generated by primitive idempotents \( \Omega_i \) satisfying
\[
\begin{align*}
&i) \; \Omega_i \Omega_j = \delta_{ij} \Omega_i \\
&\quad \Omega_1 + \cdots + \Omega_n = 1
\end{align*}
\]
The set \((\Omega_1, \ldots, \Omega_n)\) is then said to form an idempotent decomposition of \( A \).

A specific example of an idempotent decomposition for an algebra is provided by the following proposition.

Prop.6: Let \( V \) be a vector space, \((v_1, \ldots, v_n)\) a basis for \( V \). Define \( n \) projection maps \( \Omega_i \in \text{END}(V) \) \((i \in \mathbb{N})\) such that
\[
\Omega_i(v_j) = \delta_{ij} v_i \quad (\forall i, j \in \mathbb{N}).
\]
Then \((\Omega_1, \ldots, \Omega_n)\) is said to form an idempotent decomposition of \( \text{END}(V) \). Moreover, every idempotent decomposition of \( \text{END}(V) \) arises in this way.

Since \( \mathcal{M}_{\dim(V)}(F) \) and \( \text{END}(V) \) are isomorphic as algebras, Proposition A1.6 also applies to \( \mathcal{M}_n(F) \).

Def.16: Let \( A \) be an algebra, \( \rho: A \to \text{END}(V) \), \( \rho': A \to \text{END}(V') \) two reps of \( A \) in vector spaces \( V \) and \( V' \) respectively. \( \rho \)
and \( \rho' \) are said to be equivalent if there exists a vector space isomorphism \( \psi : V \rightarrow V' \) such that
\[
\psi(\rho(u)(v)) = \rho'(u)(\psi(v)) \quad (\forall u \in A, v \in V).
\]

\( \psi \) is then said to intertwine the action of \( \rho \) and \( \rho' \).

An important theorem related to this is Schur's Lemma (Budinich and Trautman 1988).

**Thm.(SL)**: Let \( \rho_1 \) and \( \rho_2 \) be two reps of an algebra \( A \) in vector spaces \( V_1 \) and \( V_2 \) respectively, and let \( \phi : V_1 \rightarrow V_2 \) be an intertwining map. If \( \rho_1 \) is irreducible, then \( \phi \) is injective. If \( \rho_2 \) is irreducible, then \( \phi \) is surjective. If both \( \rho_1 \) and \( \rho_2 \) are irreducible, then \( \phi \) is an isomorphism.

**Thm.6**: Let \( A \) be a semi-simple algebra, \( \rho : A \rightarrow \text{END}(V) \) an irrep of \( A \) in a vector space \( V \). Then \( A \) contains a minimal left ideal \( I \) such that \( \rho \) is equivalent to \( \rho_{\text{REG}} \) acting in \( I \).

**Prop.7**: If all minimal left ideals of a semi-simple algebra are isomorphic, then the algebra is simple.

**Thm.7**: Let \( A \) be a simple algebra, \( I_1 \) and \( I_2 \) two minimal left ideals of \( A \). Then there is an algebra isomorphism \( \mu : I_1 \rightarrow I_2 \) which intertwines the action of \( \rho_{\text{REG}}(A) \). In particular, the induced irreps \( \rho_{\text{REG},I_1} : A \rightarrow \text{END}(I_1) \), \( \rho_{\text{REG},I_2} : A \rightarrow \text{END}(I_2) \) are equivalent.

**Corollary**: All irreps of a simple algebra are equivalent.
Thm. 8: Let $A$ be a simple algebra over $C$, $(\Omega_1, \ldots, \Omega_n)$ an idempotent decomposition of $A$. Let $\rho: A \rightarrow \text{End}(V)$ be an irrep of $A$ in a complex vector space $V$. Then
\[ n = \dim_C(V). \]

Thm. 9: Let $A$ be a simple algebra, $(\Omega_1, \ldots, \Omega_n)$ an idempotent decomposition of $A$. Define the subspaces $A_{i,j} = \Omega_i \cdot \Omega_j$. Then $A_i$, as a vector space, is the direct sum of the $A_i$. Moreover, the $A_i$ satisfy
\[ \text{i) } A_{i,j} \cdot A_{k,l} = \delta_{j,k} A_{i,l}, \]
\[ \text{ii) } n^2 \dim(A_{i,j}) = \dim(A) \quad (\forall i, j \in \{1, \ldots, n\}). \]

Clearly, $A_n < A$ $(\forall i \in N_n)$. Moreover, it may be shown that $\Omega$, is the unit element of $A_n$ and the $A_i$ are all isomorphic as algebras (Benn and Tucker 1987).

As we shall see, a much more convenient means for determining the primitive idempotents of an algebra is provided by the following two propositions.

Prop. 8: Let $A$ be a semi-simple algebra, $\Omega$ an idempotent of $A$. $\Omega$ is primitive iff $\Omega A \Omega$ is a division algebra.

Prop. 9: Let $A$ be an algebra over $C$, $\Omega \in A$ - idempotent. $\Omega$ is primitive iff $(\forall a \in A)(\exists c_a \in C) \cdot \Omega a \Omega = c_a \Omega$.

Thm. 10: An algebra is semi-simple iff it is simple or the algebra direct sum of simple ideals.

We conclude this section with two very important theorems: the Wedderburn Structure Theorem (WST), and the Frobenius Classification Theorem (FCT).
Thm. (WST): An algebra \( A \) is simple iff there exists a division algebra \( D \) and total matrix algebra \( M \) such that \( A \cong D \otimes M \).

Suppose \( A \) is a simple algebra having idempotent decomposition \( (\Omega_1, \ldots, \Omega_n) \). It may be shown that \( D \cong A_i \) \((\forall i \in \mathbb{N}_n)\) (Hermann 1974) and thus, using Theorem A1.8,

\[ \dim(M) = n^2. \]

Consequently, the Wedderburn Structure Theorem shows that any element of a simple algebra \( A \), having idempotent decomposition \( (\Omega_1, \ldots, \Omega_n) \), may be identified with an \( n \times n \) matrix having components in the division algebra \( D \).

Thm. (FCT): The only real associative division algebras, up to isomorphism, are the reals \( \mathbb{R} \), the complexes \( \mathbb{C} \), and the quaternions \( \mathbb{H} \).

Descriptions of these well-known division algebras may be found in Greub 1978, Benn and Tucker 1987, and Hermann 1974.
2 Clifford Algebras

Def.1: Let $\mathcal{V}$ be a vector space over $F$ and set
$\mathcal{T}'(\mathcal{V}) = \bigotimes^\infty \mathcal{V}$ $(\forall \mathcal{V} \subset \mathcal{V})$ with $\mathcal{T}'^0(\mathcal{V}) = F$. The tensor algebra of
\(\mathcal{V}\) is defined as the vector space
$$\mathcal{T}(\mathcal{V}) = \mathcal{T}'(\mathcal{V}) \oplus \mathcal{T}'(\mathcal{V}) \oplus \cdots$$

having an $F$-bilinear mapping
$$\mathcal{T}(\mathcal{V}) \times \mathcal{T}(\mathcal{V}) \to \mathcal{T}(\mathcal{V})$$
which maps $\mathcal{T}'(\mathcal{V}) \times \mathcal{T}'(\mathcal{V}) \to \mathcal{T}'(\mathcal{V})$ according to the rule
$$(v_1 \otimes \cdots \otimes v_i) \otimes (v'_1 \otimes \cdots \otimes v'_i) = v_1 \otimes \cdots \otimes v_i \otimes v'_1 \otimes \cdots \otimes v'_i,
$$
where $(v_1 \otimes \cdots \otimes v_i) \in \mathcal{T}'(\mathcal{V})$, $(v'_1 \otimes \cdots \otimes v'_i) \in \mathcal{T}'(\mathcal{V}).$

Although $\mathcal{T}(\mathcal{V})$ is an infinite-dimensional vector space, every element of $\mathcal{T}(\mathcal{V})$ can be written as a finite linear combination of elements from the $\mathcal{T}'(\mathcal{V})$. Such a restriction is necessary for $\mathcal{T}(\mathcal{V})$ to be well-defined as a vector space.

Def.2: A bilinear form on a vector space $\mathcal{V}$ over a field $F$

is an $F$-bilinear mapping
$$g: \mathcal{V} \times \mathcal{V} \to F.$$

$g$ is said to be symmetric if
$$g(v_1, v_2) = g(v_2, v_1)$$
$(\forall v_1, v_2 \in \mathcal{V}).$

$g$ is said to be non-degenerate if
$$g(v, w) = 0 \quad (\forall v \in \mathcal{V}) \Rightarrow w = 0.$$

One may choose a $g$-orthonormal basis $\{e_i | i \in \mathcal{N}_{\dim(\mathcal{V})}\}$ for $\mathcal{V}$ such that
$$g(e_i, e_j) = \delta_{ij}$$
$(\forall i, j \in \mathcal{N}_{\dim(\mathcal{V})}).$

where $\delta_{ij} = \pm \delta_{ij}$. One often refers to the $g_{ij}$ as the metric of $\mathcal{V}$. Furthermore, the arrangement of signs for the $g_{ij}$
determine the signature of \( \gamma \), usually denoted \((\rho, \eta)\), where \( \rho \) is the number of plus signs and \( \eta \) is the number of minus signs.

Henceforth, all bilinear forms shall be assumed symmetric and non-degenerate unless otherwise indicated. In addition, a vector space \( \lambda' \) shall be termed a \( \gamma \)-vector space if such a bilinear form \( \gamma \) exists on \( \lambda' \).

**Def. 3:** Let \( \lambda' \) and \( \lambda'' \) be vector spaces having bilinear forms \( \gamma \) and \( \gamma' \) respectively. An injective linear map \( \phi: \lambda \to \lambda'' \) is said to be an isometry if

\[
g'(\phi(v_1), \phi(v_2)) = g(v_1, v_2) \quad (\forall v_1, v_2 \in \lambda').
\]

**Prop. 1:** Let \( \lambda' \) be a \( \gamma \)-vector space, \( \mathcal{T}(\lambda') \) the tensor algebra of \( \lambda' \). Then the subalgebra \( I(\gamma) \subset \mathcal{T}(\lambda') \), generated by the set

\[
\{ v_1 \otimes v_2 + v_2 \otimes v_1 - 2g(v_1, v_2) | v_1, v_2 \in \lambda' \}
\]

is an ideal of \( \mathcal{T}(\lambda') \).

**Def. 4:** Let \( \lambda' \) be a vector space having bilinear form \( \gamma \). Let \( \mathcal{T}(\lambda') \) be the tensor algebra of \( \lambda' \), \( I(\gamma) \) the ideal defined in Proposition A2.1. The (non-degenerate) Clifford algebra of the bilinear form \( \gamma \) on \( \lambda' \) is the quotient algebra

\[
\mathcal{C}_{\lambda'}(\gamma) = \mathcal{T}(\lambda') / I(\gamma).
\]

where \( \mathcal{F} \) and \( \lambda' \) are identified with their images in \( \mathcal{C}_{\lambda'}(\gamma) \) under the canonical homomorphism

\[
\Pi: \mathcal{T}(\lambda') \to \mathcal{T}(\lambda') / I(\gamma) = \mathcal{C}_{\lambda'}(\gamma).
\]

We should point out that the most general definition of a Clifford algebra does not include the restriction that \( \gamma \) be
a non-degenerate bilinear form. If this constraint is lifted from the definition of \( g \), the resulting Clifford algebra is termed degenerate.

Since this analysis is confined strictly to the study of non-degenerate Clifford algebras, all bilinear forms shall continue to be assumed symmetric and non-degenerate.

By restricting \( \Pi \) to \( \mathcal{V} \subset \mathcal{V}(\mathcal{M}) \) in Definition A2.4, one obtains the isometry

\[
\iota_{\mathcal{V}} : \mathcal{V} \rightarrow C_{\mathcal{V}}(g)
\]

\[
\iota_{\mathcal{V}}(v) = [v] = v + I(g).
\]

Since \( I(g) = \ker(\Pi) \), one obtains the relation

\[
[v_1][v_2] = [v_2][v_1] = 2g([v_1],[v_2]).
\] (1)

where

\[
g([v_1],[v_2]) = g(v_1,v_2) \quad (\forall v_1,v_2 \in \mathcal{V}).
\]

The isometry \( \iota_{\mathcal{V}} \) giving rise to Eq. (1) is called a Clifford map. Moreover, it is customary to identify \( \iota_{\mathcal{V}}(\mathcal{V}) \subset C_{\mathcal{V}}(g) \) with \( \mathcal{V} \) so that Eq. (1) becomes

\[
u_1 v_2 + v_2 v_1 = 2g(v_1,v_2).
\] (2)

where \( v_1 \) and \( v_2 \) are now considered elements of \( C_{\mathcal{V}}(g) \). Eq. (2) is called the Clifford relation of \( C_{\mathcal{V}}(g) \).

Generally, the structure of a Clifford algebra depends on the signature of \( g \) and the ground field \( F \). Accordingly, when classifying Clifford algebras, the notation \( C_{\mu,\nu}(F) \) will be adopted for the Clifford algebra.
Thm. 1: Let $V$ be a vector space, $V'$ a $g$-vector space. Let 
$\phi: V \to \text{END}(V')$ be a linear map such that for every 
$v_1, v_2 \in V$, 
$$\phi(v_1)\phi(v_2) + \phi(v_2)\phi(v_1) = 2g(v_1, v_2).$$
Then $\phi$ can be extended to an algebra homomorphism 
$p: C_1(g) \to \text{END}(V').$

Clearly, Theorem A2.1 defines a representation of $C_1(g)$.

Thm. 2: Let $V$ be a $g$-vector space and suppose $\dim(V) = n$. Then 
$$\dim[C_1(g)] = 2^n.$$

Thm. 3: Let $V$ be defined as in Theorem A2.2 and let 
$\{v_i | i \in \mathbb{N}_n\}$ be a basis for $V$. Then the $2^n$ elements 
$\{1, v_1, v_2, v_1 v_2, ... v_1 v_2 ... v_n | 1 \leq i < j < k \leq n\}$
form a basis for $C_V(g)$ and $V$ is said to generate 
$C_V(g)$.

Theorem A2.3 shows that $C_V(g)$, as a vector space, may be 
written in the form 
$$C_V(g) = C_V^0(g) \oplus C_V^1(g) \oplus \cdots \oplus C_V^n(g).$$
where 
$$C_V^0(g) = \text{span}_F\{I\} = F$$
$$C_V^i(g) = \text{span}_F\{v_{a_1} \cdots v_{a_i} | 1 \leq a_1 < \cdots < a_i \leq n\} \quad (\forall i \in \mathbb{N}_n).$$

Def. 5: Let $V$ be a $g$-vector space and let $\{e_i | i \in \mathbb{N}_n\}$ be a 
$g$-orthonormal basis for $V$. The element 
$$e = e_1 \cdots e_p e_{p+1} \cdots e_{p+q} \in C_V^p(g)$$
is called the canonical element of $C_V(g)$, where $(p, q)$ 
is the signature of $g$ and $p + q = n.$
It may be shown that $e^{-} = \pm 1$ (Hermann 1974).

**Def. 6:** Let $V$ be a vector space. An linear map $\phi : V \to V$ is said to be an involution if $\phi^2 = \text{id}$, where $\text{id}$ is the identity map on $V$.

**Prop. 2:** Let $V$ be a $g$-vector space, $C_1(g)$ the Clifford algebra generated by $V$. Then the isometry

$$\pi : V \to V$$

$$v \to -v$$

induces an involutory automorphism $\pi$ of $C_1(g)$, called the degree automorphism, mapping $C_1(g) \to C_1(g)$ according to the rule

$$\pi(v_1 \cdots v_i) = (-1)^i v_1 \cdots v_i.$$

The degree automorphism induces a vector space direct sum decomposition:

$$C_g = C^{+}(g) \oplus C^{-}(g).$$

where

$$C^{+}_g = \ker(\pi - \text{id}) = \bigoplus_{i \text{ even}} C^{i}_g$$

$$C^{-}_g = \ker(\pi + \text{id}) = \bigoplus_{i \text{ odd}} C^{i}_g.$$

The elements of $C^{+}_g$ and $C^{-}_g$ are said to be homogeneous of even and odd degree respectively. Furthermore,

$$C^{+}_g C^+ g \subset C^+ g$$

$$C^{-}_g C^- g \subset C^- g$$

$$C^{+}_g C^+ g \subset C^+ g$$

and since $C^0_g \subset C_g$.

$$C^+ g \subset C g.$$

$C^+ g$ is called the even subalgebra of $C g$. 


Def. 7: Let \( V \) be a \( g \)-vector space, \( C_1(g) \) the Clifford algebra generated by \( V \). Define the linear map \( \zeta: C_1(g) \to C_1(g) \) mapping \( C_1(g) \to C_1(g) \) according to the rule
\[
\zeta(v_1 \cdots v_i) = v_1 \cdots v_i.
\]
\( \zeta \) is an involutory anti-automorphism of \( C_1(g) \) called the main anti-automorphism.\(^4\)

Prop. 3: Let \( V \) be a \( g \)-vector space, \( C_1(g) \) the Clifford algebra generated by \( V \). Then the mapping \( r: C_1(g) \to C_1(g) \), defined by
\[
r = \pi \circ \zeta = \zeta \circ \pi,
\]
is an involutory anti-automorphism of \( C_1(g) \) called the spatial reversion map. In particular,
\[
r(v_1 \cdots v_i) = (-1)^i v_i \cdots v_1,
\]
where \( v_1 \cdots v_i \in C_1(g) \).

We now state some general results on the classification of Clifford algebras, beginning with Clifford algebras over the real field \( \mathbb{R} \). All proofs to these theorems may be found in Greub 1978, Benn and Tucker 1987, Salingaros 1982 and Coquereaux 1988.

Let \( V \) be an \( n \)-dimensional real vector space having bilinear form \( g \). The Clifford algebra generated by \( V \) will be denoted \( C_{p,q}(\mathbb{R}) \), where \( (p,q) \) is the signature of \( g \) and \( p+q=n \).

\(^4\) \( \zeta \) is also sometimes referred to as reversion in the Clifford algebra.
Thm. 4: Every Clifford algebra over \( R \) can be written

\[
\mathcal{C}_{p,q}(R) \cong \mathfrak{A}(R) \otimes M_4(R).
\]

where

\[
k'\dim \mathfrak{A}(R) = 2^n.
\]

and the algebra \( \mathfrak{A}(R) \) is determined by \((p-q)\mod 8\).

The following table lists the possibilities for \( \mathfrak{A}(R) \):

<table>
<thead>
<tr>
<th>((p-q)\mod 8)</th>
<th>( \mathfrak{A}(R) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 2</td>
<td>( \mathbb{R} )</td>
</tr>
<tr>
<td>3, 7</td>
<td>( \mathbb{C} )</td>
</tr>
<tr>
<td>4, 6</td>
<td>( \mathbb{H} )</td>
</tr>
<tr>
<td>1</td>
<td>( \mathbb{R} \oplus \mathbb{R} )</td>
</tr>
<tr>
<td>5</td>
<td>( \mathbb{H} \oplus \mathbb{H} )</td>
</tr>
</tbody>
</table>

Here, \( \mathbb{C}(R) \equiv \mathcal{C}_{0,1}(R) \) is the complex numbers, \( \mathbb{H}(R) \equiv \mathcal{C}_{0,2}(R) \) is the quaternions and \( \oplus \) is to be interpreted as an algebra direct sum.

Since \( \mathbb{R} \), \( \mathbb{C} \) and \( \mathbb{H} \) are simple division algebras, Theorem A1.10 and the Wedderburn Structure Theorem show that:

(i) \( \mathcal{C}_{p,q}(R) \) is simple if \((p-q)\mod 8 = 0, 2, 3, 4, 6, 7\)

(ii) \( \mathcal{C}_{p,q}(R) \) is semi-simple if \((p-q)\mod 8 = 1, 5\).

Thus, every real Clifford algebra is either simple or semi-simple depending on the value of \((p-q)\mod 8\). Where, in the latter case, \( \mathcal{C}_{p,q}(R) \) is the algebra direct sum of two isomorphic simple ideals.
Thm. 5: Let \( V \) be an \( n \)-dimensional \( \mathfrak{g} \)-vector space, \( \mathcal{C}_{p,q}(R) \) the Clifford algebra generated by \( \mathfrak{g} \).

a) If \( n \) is even, then \( \mathcal{C}_{p,q}(R) \) is central simple and the even subalgebra \( \mathcal{C}_{p,q}^-(R) \) may be determined by the value of \( e^2 \). Specifically:

(i) If \( e^2 = -I \), then \( \mathcal{C}_{p,q}^-(R) \) is simple and \( \mathcal{C}_{p,q}^-(R) \cong \mathbb{C} \).

(ii) If \( e^2 = +I \), then \( \mathcal{C}_{p,q}^-(R) \) is semi-simple and \( \mathcal{C}_{p,q}^-(R) \cong \mathbb{R} \oplus \mathbb{R} \). In particular,

\[
\mathcal{C}_{p,q}^-(R) = \mathfrak{e}_+ \mathcal{C}_{p,q}^-(R) \oplus \mathfrak{e}_- \mathcal{C}_{p,q}^-(R),
\]

where the \( \mathfrak{e}_+ = \frac{1}{2}(I = e) \) are primitive idempotents of \( \mathcal{C}_{p,q}^-(R) \).

b) If \( n \) is odd, then \( \mathcal{C}_{p,q}(R) \) is central simple and \( \mathcal{C}_{p,q}(R) \) may be determined by the value of \( e^2 \). Specifically:

(i) If \( e^2 = -I \), then \( \mathcal{C}_{p,q}(R) \) is simple and \( \mathcal{C}_{p,q}^+(R) \cong \mathbb{C} \).

(ii) If \( e^2 = +I \), then \( \mathcal{C}_{p,q}(R) \) is semi-simple and \( \mathcal{C}_{p,q}^+(R) \cong \mathbb{R} \oplus \mathbb{R} \). In particular,

\[
\mathcal{C}_{p,q}(R) = \mathfrak{e}_+ \mathcal{C}_{p,q}(R) \oplus \mathfrak{e}_- \mathcal{C}_{p,q}(R),
\]

where the \( \mathfrak{e}_+ = \frac{1}{2}(I - e) \) are primitive idempotents of \( \mathcal{C}_{p,q}(R) \).

Note that, for \( n \) even, \( \mathcal{C}_{p,q}^-(R) = \{ \alpha 1 + \beta e | \alpha, \beta \in R \} \). whereas for \( n \) odd, \( \mathcal{C}_{p,q}^+(R) = \{ \alpha 1 + \beta e | \alpha, \beta \in R \} \).
Prop. 4: Let $C_{p,q}^\ast(R)$ be the even subalgebra of the Clifford algebra $C_{p,q}(R)$. Then

$$C_{p,q}^\ast(R) \cong C_{q,p-1}(R)$$

and $C_{p,q}(R)$ may be written

$$C_{p,q}(R) \cong \mathcal{A}^\ast(R) \otimes M_1(R),$$

where

$$k' \dim[\mathcal{A}^\ast(R)] = 2^{n-1},$$

and the algebra $\mathcal{A}^\ast(R)$ is determined by $(p-q) \mod 8$.

Proposition A2.4 shows that

$$\dim[C_{p,q}^\ast(R)] = \frac{1}{2} \dim[C_{p,q}(R)].$$

The following table summarizes these results.

<table>
<thead>
<tr>
<th>$(p-q) \mod 8$</th>
<th>$\mathcal{A}(R)$</th>
<th>$\mathcal{A}^\ast(R)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\mathbb{R}$</td>
<td>$\mathbb{R} \oplus \mathbb{R}$</td>
</tr>
<tr>
<td>1</td>
<td>$\mathbb{R} \oplus \mathbb{R}$</td>
<td>$\mathbb{R}$</td>
</tr>
<tr>
<td>2</td>
<td>$\mathbb{R}$</td>
<td>$\mathbb{C}$</td>
</tr>
<tr>
<td>3</td>
<td>$\mathbb{C}$</td>
<td>$\mathbb{H}$</td>
</tr>
<tr>
<td>4</td>
<td>$\mathbb{H}$</td>
<td>$\mathbb{H} \oplus \mathbb{H}$</td>
</tr>
<tr>
<td>5</td>
<td>$\mathbb{H} \oplus \mathbb{H}$</td>
<td>$\mathbb{H}$</td>
</tr>
<tr>
<td>6</td>
<td>$\mathbb{H}$</td>
<td>$\mathbb{C}$</td>
</tr>
<tr>
<td>7</td>
<td>$\mathbb{C}$</td>
<td>$\mathbb{R}$</td>
</tr>
</tbody>
</table>

We now consider Clifford algebras over the complex field $\mathbb{C}$. 
Let \( \mathfrak{l} \) be an \( n \)-dimensional complex vector space having bilinear form \( g \). Since \( \mathfrak{l} \) is complex, one can always choose a basis \( \{ e_i \}_{i=1}^n \) of \( \mathfrak{l} \) such that
\[
g(e_i, e_j) = \delta_{ij},
\]
i.e. the bilinear form \( g \) has no inherent signature. Consequently, the structure of the Clifford algebra generated by a complex \( g \)-vector space \( \mathfrak{l} \) can only depend on the dimension of \( \mathfrak{l} \) and the Clifford algebra will be denoted \( C_n(C) \).

**Thm. 6:** Let \( \mathfrak{l} \) be an \( n \)-dimensional complex \( g \)-vector space, \( C_n(C) \) the Clifford algebra generated by \( \mathfrak{l} \).

a) If \( n \) is even, then \( C_n(C) \) is central simple and
\[
C_n(C) \cong M_{2^{n/2}}(C).
\]
Furthermore, \( C_n(C) \) is semi-simple and
\[
C_n^-(C) \cong M_{2^{(n/2) - 1}}(C) \oplus M_{2^{(n/2) - 1}}(C).
\]
In particular,
\[
C_n^+(C) = e_C C_n^-(C) \oplus e_C C_n^-(C).
\]
where the elements
\[
e_e = \begin{cases}
\frac{1}{2} (1 + e) & \{e^2 = +1\} \\
\frac{1}{2} (1 + ie) & \{e^2 = -1\}
\end{cases}
\]
are primitive idempotents of \( C_n(C) \).

b) If \( n \) is odd, then \( C_n(C) \) is central simple and
\[
C_n^-(C) \cong M_{2^{(n-1)/2}}(C).
\]
Furthermore, \( C_n(C) \) is semi-simple and
\[
C_n^-(C) \cong M_{2^{(n-1)/2}}(C) \oplus M_{2^{(n-1)/2}}(C).
\]
In particular,
\[ \mathcal{C}_n(C) = \mathcal{C}_n(C) \oplus \mathcal{C}_n(C) \]

where the elements

\[ e_+ = \begin{cases} 
\frac{1}{2}(I + \alpha) & (\alpha \cdot \gamma = 1) \\
\frac{1}{2}(I - \alpha) & (\alpha \cdot \gamma = -1)
\end{cases} \]

are primitive idempotents of \( \mathcal{C}_n(C) \).

Note that, for \( n \) even, \( \mathcal{C}_n(C) = \{ \alpha I - \beta \} | \alpha, \beta \in C \} \), whereas for \( n \) odd, \( \mathcal{C}_n(C) = \{ \alpha I - \beta \} | \alpha, \beta \in \mathbb{C} \} \).

We conclude this section with a result that relates real and complex Clifford algebras.

**Def. 8:** Let \( V \) be a real vector space. The complexification of \( V \):

\[ \mathcal{C} \otimes V \]

is a \( 2n \)-dimensional real vector space that can be considered an \( n \)-dimensional complex vector space by defining complex scalar multiplication in \( \mathcal{C} \otimes V \) as follows:

\[ c' (c \otimes v) = c' c \otimes v \quad (\forall c, c' \in \mathcal{C}, v \in V) . \]

One immediately sees that every real vector space \( V \) can be included in the vector space \( \mathcal{C} \otimes V \) over \( \mathcal{C} \) in such a way that a basis for \( V \) over \( \mathbb{R} \) is simultaneously a basis of \( \mathcal{C} \otimes V \): that is, given a basis \( \{ v_1, \ldots, v_n \} \) of \( V \) over \( \mathbb{R} \), the \( \mathcal{C} \)-linear map

\[ \phi : \mathcal{C} \otimes V \to \text{span}_{\mathbb{C}} \{ v_1, \ldots, v_n \} \]

\[ c \otimes v \to cv \]
is a vector space isomorphism. Consequently, by extending the ground field to \( C \), one can consider \( l' \) to be a complex vector space. More importantly, if \( l' \) is a real \( \mathfrak{g} \)-vector space, one can consider \( l' \) to be a complex \( \mathfrak{g} \)-vector space by extending the ground field to \( C \) and by extending \( \mathfrak{g} \) to \( C \)-linearity.

In contrast to direct complexification, there are certain cases where a real vector space may be considered complex, as shown by the following definition.

**Def. 9:** An even-dimensional real vector space \( l' \) is said to admit a complex structure if there exists an \( R \)-linear map \( J : l' \rightarrow l' \) such that \( J^2 = -\text{id} \), where \( \text{id} \) is the identity map on \( l' \). The map \( J \) is called the complex structure of \( l' \).

**Thm. 7:** Let \( l' \) be a real vector space admitting a complex structure \( J \). Define complex scalar multiplication in \( l' \) by

\[
(\alpha + i\beta)v = \alpha v + \beta J(v),
\]

where \( \alpha, \beta \in R \), \( v \in l' \). Then \( l' \) becomes well-defined as a complex vector space and

\[
\dim_C(l') = \frac{1}{2} \dim_R(l').
\]

**Thm. 8:** Let \( V \) be an \( n \)-dimensional real \( \mathfrak{g} \)-vector space, \( C_{p,q}(R) \) the Clifford algebra generated by \( V \). Then

\[
C \otimes C_{p,q}(R) = C_{n}(C).
\]

where \( C_{n}(C) \) is generated by \( V_c \).
Theorem A2.8 shows that if \( \mathfrak{c}_p \cdot q(R) \) and \( \mathfrak{c}_p \cdot q(R) \) are two real Clifford algebras satisfying \( p \cdot q = p' \cdot q' \), then
\[
\mathfrak{c} \circ \mathfrak{c}_p \cdot q(R) = \mathfrak{c} \circ \mathfrak{c}_p \cdot q(R).
\]
3 Restriction to the Real Field

In attempting to generalize our results to the real Dirac algebra, we are primarily motivated by the work of D. Hestenes, who has argued extensively for adopting such a viewpoint. In particular, using his STA (SpaceTime Algebra), Hestenes writes the Dirac equation as

$$\gamma^\nu(\partial_\nu \psi i\sigma_3 + v \cdot \mathbf{l}_\nu \psi) = m \psi \gamma_0.$$  \hspace{1cm} (1)

where \(\psi\) is an even multivector and \(i\sigma_3 = \gamma_2 \gamma_1\) corresponds to the \(i\) normally found in the Dirac equation. STA is Hestenes' realization of the Clifford algebra \(C_{1,3}(\mathbb{R}) \cong D^*_R\). Notice that in identifying the wavefunction with an even multivector, corresponding to an element of the even subalgebra \(D^*_R\). Hestenes has departed from the usual spinor representation for the Clifford algebra. This is also borne out by the presence of operators acting on \(\psi\) to the right in Eq. (A3.1).

By adopting the notation \(i\sigma_3\) for the element \(\gamma_2 \gamma_1 \in D^*_R\), Hestenes is implicitly making use of the isomorphism that exists between \(D^*_R\) and the Pauli algebra. Since this isomorphism can only be defined over the reals, the \(i\) in \(i\sigma_3\) must be interpreted as the canonical element \(e = e_1 e_2 e_3\) of \(\mathcal{P}(\mathbb{R})\). It is an interesting fact that Eqs. (1.2.70) may be utilized to demonstrate the connection between \(i\sigma_3\) and \(\gamma_2 \gamma_1\). Indeed, we see that

$$i\sigma_3 = \gamma_2 \gamma_1 = (e_2 \otimes e_2)(e_1 \otimes e_2) = e_1 e_2 \otimes 1 = ee_3 \otimes 1.$$ \hspace{1cm} (2)

which clearly indicates why Hestenes has used \(i\sigma_3\) to denote \(\gamma_2 \gamma_1\).

---

5 In this Appendix, all references to the work of Hestenes can be found in Hestenes 1982.
Going back to Eq. (A3.1), consider left multiplication by $\gamma_0$. This gives
\[
\gamma_0 \gamma^\nu (\epsilon_{\mu} \psi \sigma_{\nu} + \epsilon \mu \psi) = m \gamma_0 \psi \gamma_0.
\]
\[
\gamma_\mu \gamma_0 (\epsilon_{\mu} \psi \sigma_{\nu} + \epsilon \mu \psi) = m \gamma_0 \psi \gamma_0.
\]  \hspace{1cm} (3)

Clearly, $\gamma_\mu \gamma_0$ is an element of the even subalgebra. In fact, Hestenes defines $\gamma_\mu \gamma_0$ as
\[
\sigma_\mu = \gamma_\mu \gamma_0
\]  \hspace{1cm} (4)
and setting $\sigma_0 = 1 = \gamma_0 \gamma_0$. Eq. (A3.3) may be written
\[
\sigma_\mu (\epsilon_{\mu} \psi \sigma_{\nu} - \epsilon \mu \psi) = m \gamma_0 \psi \gamma_0.
\]  \hspace{1cm} (5)

Consider now the wavefunction $\psi$. Since $\psi \in D_1$, the results of Section 1.2 imply
\[
\psi = \psi_0 \gamma_0 + \frac{1}{2} \psi_0 \gamma_0 \gamma_{\mu} \psi_0 \gamma_\mu \psi_0 \gamma_0.
\]  \hspace{1cm} (6)

where $\psi_0, \psi_0 \in \mathbb{R}$ and $\psi_0 \gamma_0 \in \mathbb{R}$ ($\forall \mu, \nu \in \mathbb{N}_0$). Using Eq. (A3.4), we see that
\[
\gamma_{\mu \nu} = \frac{1}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)
\]
\[
= \frac{1}{2} (\gamma_\mu \gamma_0 \gamma_0 \gamma_\nu - \gamma_\nu \gamma_0 \gamma_0 \gamma_\mu)
\]
\[
= \frac{1}{2} (\gamma_\mu \gamma_0 \gamma_0 \gamma_\nu - \gamma_\nu \gamma_0 \gamma_0 \gamma_\mu)
\]
\[
= \frac{1}{2} (\sigma_\mu \sigma_\nu - \gamma_\nu \sigma_\mu).
\]  \hspace{1cm} (7)

Now since $\sigma_\mu$ corresponds to $\sigma_\mu \in \mathcal{P}(\mathbb{R})$, define
\[
\bar{\sigma}_\mu = \sigma_\mu.
\]  \hspace{1cm} (8)

Eq. (A3.7) may then be written
\[
\gamma_{\mu \nu} = \sigma_{\mu \nu}.
\]  \hspace{1cm} (9)

where
\[ u_{\nu} = -\frac{1}{2} (\sigma_{\nu} \sigma_{\nu} - \gamma_{\nu} \sigma_{\mu}). \] (10)

Clearly, \( u_{\nu} \) corresponds to the \( \mathcal{P} \)-element \( e_{\mu} \) defined in Eq. (3.2.85). We also see that
\[ V = V_0 V_1 V_2 V_3 \]
\[ = V_1 V_2 V_0 V_3 \]
\[ = (V_2 V_1)(V_3 V_0) \]
\[ = i\sigma_3 \sigma_3 = i \] (11)

and thus \( \psi \) may be written
\[ \psi = \psi_0 I + \frac{1}{2} \psi_2 \psi_3 \sigma_3 + \psi_4 i \]
\[ = \psi_S + \psi_1 + \psi_2 + \psi_3, \] (12)

where \( \psi_S = \psi_0, \psi_1 = -\psi_0^* \sigma_3, \psi_2 = -\epsilon_{ijk} \psi_3^* i \sigma_3 \) and \( \psi_3 = \psi_4 i \).

Note that
\[ V_0 \psi V_0 = \psi_0 I + \frac{1}{2} \psi_2 \gamma_{\mu} \gamma_{\nu} - \psi_4 \gamma \]
\[ = \psi_S - \psi_1 + \psi_2 - \psi_3 \in \mathcal{D}_R \] (13)

and thus Eq. (3.3.5) resides entirely in \( \mathcal{D}_R = \mathcal{P}(R) \). We may now identify \( \mathcal{D}_R \) with \( \mathcal{P}(R) \) via the \( R \)-linear isomorphism
\[ \phi(\sigma_3) = e, \phi(i \sigma_3) = e. \] (14)

Setting
\[ \Psi = \phi(\psi) \] (15)

and employing the complex structure of \( \mathcal{P}(R) \) to identify the canonical element \( e \) with \( i \in \mathbb{C} \). Eq. (3.3.13) shows that
\[ \phi(V_0 \psi V_0) = \overline{\Psi}. \] (16)
Consequently, within the framework of the Pauli algebra, Hestenes' version of the Dirac equation becomes

\[
\bar{\sigma}^\alpha \sigma_\alpha \psi + e \bar{\sigma}^\alpha \sigma_\alpha \psi = m \bar{\psi}^* \psi.
\]

\[\Rightarrow \quad \bar{p} \bar{\psi}^* \sigma_\alpha \sigma_\alpha \psi = m \psi. \tag{17}\]

Let us now consider a more general scenario. We recall from Section 1.2 that the real Dirac algebra is central simple, where

\[
\mathcal{D}_R = C_{1,0}(R) \simeq \mathbb{H}(R) \otimes \mathbb{M}_2(R). \tag{18}\]

Moreover, since \(\mathbb{H}(R)\) is a division algebra, the number of primitive idempotents forming an idempotent decomposition of \(\mathcal{D}_R\) is two, so that \(\mathcal{D}_R\) decomposes as a vector space into the direct sum

\[
\mathcal{D}_R = \mathcal{D}_{R^+} \oplus \mathcal{D}_{R^-}. \tag{19}\]

where \(\mathcal{D}_{R^+}\) and \(\mathcal{D}_{R^-}\) are minimal left ideals. Clearly, this decomposition corresponds to an idempotent decomposition of \(\mathbb{M}_2(R)\) into two minimal left ideals.

Now we also saw in Section 1.2 that \(\mathbb{H}(R)\) and \(\mathbb{M}_2(R)\) could be identified with subalgebras of \(\mathcal{F}\) considered as a real Clifford algebra. In particular, \(\mathbb{H}(R)\) could be identified with the even subalgebra of \(\mathcal{F}(R)\)

\[
\mathbb{H}(R) = \mathcal{F}^+(R) = \{ H \in \mathcal{F}(R) | \pi(H) = H \}. \tag{20}\]

where \(\pi\) is the degree involution defined in Proposition A2.2, while \(\mathbb{M}_2(R)\) was defined by

\[
\mathbb{M}_2(R) = \text{span}_R \{ 1, e_1, e_2, e \} \subset \mathcal{F}(R). \tag{21}\]

Clearly, the primitive idempotents

\[
\Omega_\ast = \frac{1}{2}(1 \pm e) \tag{22}\]
can be taken as primitive idempotents for $\mathcal{M}_2(R)$ and identifying $\mathcal{D}_H$ with $H(R) \otimes \mathcal{M}_2(R)$ we obtain
\[ \mathcal{D}_{R,\Omega} = H(R) \otimes \mathcal{M}_2(R) \Omega. \] (23)

where $\Omega \in \{\Omega, \Omega^\perp\}$. 

Consider now the irreducible spinor representation
\[ \rho_{\Omega} : \mathcal{D}_H \to \text{END}_R(\mathcal{D}_{R,\Omega}) \] (24a)
of $\mathcal{D}_H = H(R) \otimes \mathcal{M}_2(R)$ on $\mathcal{D}_{R,\Omega}$, where
\[ \rho_{\Omega}(H \otimes M) = \rho_{H \otimes M} \] (24b)
for all $H \in H(R)$ and $M \in \mathcal{M}_2(R)$. The elements of $\mathcal{D}_{R,\Omega}$ correspond to spinors in $\mathcal{D}_H$ which may be written in the form
\[ \psi = \sum_{i \in I} \phi_i \otimes \chi_i \Omega. \] (25)

where $\phi_i \in H(R)$, $\chi_i \in \mathcal{M}_2(R)$ and $I$ is some countably finite index set. Since
\[ \dim_R(H(R)) = 4, \quad \dim_R(\mathcal{M}_2(R) \Omega) = 2. \] (26)
we have
\[ \dim_R(\mathcal{D}_{R,\Omega}) = \dim_R(\mathcal{F}(R)) \] (27)
and thus, as a real vector space,
\[ \mathcal{D}_{R,\Omega} \cong \mathcal{F}(R). \] (28)

Now define the $R$-linear map
\[ \rho : \mathcal{D}_R \to \text{END}_R(\mathcal{F}(R)) \] (29a)
such that
\[ \rho(H \otimes M) = \rho_M \circ \rho_H^{-1}. \]
for all $H \in H(R)$ and $M \in \mathcal{M}_2(R)$. Letting $\Psi \in \mathcal{F}(R)$, we see that
\[ \rho( (H_1 \otimes M_1)(H_2 \otimes M_2))(\Psi) = \rho(H_1 H_2 \otimes M_1 M_2)(\Psi) \]

\[ = \rho_{M_1 M_2} \circ \rho_{H_2} \circ \rho_{H_1}^R(\Psi) \]

\[ = \rho_{M_1 M_2} \circ \rho_{H_2} \circ \rho_{H_1}^R(\Psi) \]

and thus \( \rho \) defines a representation of \( \mathcal{D}_H \) on \( \mathcal{F}(R) \).

Given that \( \mathcal{D}_H \cong \mathcal{F}(R) \), we are lead to construct a vector space isomorphism intertwining the action of \( \rho_\alpha \) and \( \rho \). To this end, define the \( R \)-linear map

\[ \phi_\alpha: \mathcal{D}_R \to \mathcal{F}(R) \]  

(31a)

such that for every \( \psi \in H(R) \), \( \chi \in M_2(R) \).

\[ \phi_\alpha(\psi \otimes \chi) = \chi \omega \psi^{-1}. \]  

(31b)

One may show that \( \phi_\alpha \) is a linear isomorphism; in particular, one may show that it maps bases to bases.\(^6\)

Letting \( H. \psi \in H(R) \) and \( M. \chi \in M_2(R) \), we see that

\[^6\text{This is best seen by adopting the basis } \{1, e_1, e_2, ee\} \text{ for } H(R) \text{ and using the standard basis } \{\Omega, \omega\} \text{ for } M_2(R) \Omega.\]
\[ \phi_\Omega(\rho_\Omega(III \otimes \Lambda)(\psi \otimes \chi \Omega)) = \phi_\Omega(\rho_H \otimes \Lambda \chi \Omega) \]
\[ = \phi_\Omega(H \psi \otimes \Lambda \chi \Omega) \]
\[ = \Lambda \chi \Omega (H \psi)^{-1} \]
\[ = \Lambda (\chi \Omega \psi^{-1})H^{-1} \]
\[ = \rho_H \circ \phi_\Omega(\psi \otimes \chi \Omega) \]
\[ = \rho(III \otimes \Lambda)(\phi_\Omega(\psi \otimes \chi \Omega)). \]

i.e., \(\phi_\Omega\) does intertwine the action of \(\rho_\Omega\) and \(\rho\), thus establishing their equivalence by Definition A1.16.

Consequently, given any physical model based on an irreducible spinor representation of the real Dirac algebra, one is able to construct an equivalent model within the framework of the Pauli algebra.
LIST OF SYMBOLS

\( \mathbb{N}_n \) \hspace{1cm} the set of integers \( \{1, 2, 3, \ldots, n\} \).

\( \mathbb{N}_n^0 \) \hspace{1cm} the set of integers \( \{0, 1, 2, \ldots, n\} \).

\( \mathbb{Z}_n \) \hspace{1cm} the set of integers \( \{-n, -n+1, \ldots, n-1, n\} \).

\( \mathbb{R} \) \hspace{1cm} the field of real numbers.

\( \mathbb{C} \) \hspace{1cm} the field of complex numbers.

\( \mathbb{H} \) \hspace{1cm} the quaternions.

\( \mathbb{R}^n \) \hspace{1cm} \( n \) -dimensional real Euclidean space.

\( \mathbb{C}^n \) \hspace{1cm} \( n \) -dimensional complex Euclidean space.

\( \mathbb{M}^4 \) \hspace{1cm} the Minkowski vector space.

\( \mathcal{M}^4 \) \hspace{1cm} the Minkowski spacetime manifold.

\( \mathcal{C} \) \hspace{1cm} the set of infinitely differentiable functions.

\( L^2 \) \hspace{1cm} the Hilbert space of square integrable functions.

\( \mathcal{M}_n(F) \) \hspace{1cm} the algebra of \( n \times n \) matrices over the field \( F \).

\( \text{END}_F(V) \) \hspace{1cm} the algebra of \( F \)-linear transformations on the vector space \( V \).
\( \text{AUT}_{r}(1) \)  
the algebra of invertible \( F \)-linear transformations on the vector space \( 1 \).

\( \mathcal{U}C[G] \)  
the universal covering group of the Lie group \( G \).

\( G_0 \)  
the connected subgroup of the Lie group \( G \).

\( O(n) \)  
the orthogonal group.

\( SO(n) \)  
the special orthogonal group.

\( O(p,q) \)  
the pseudo-orthogonal group.

\( SO(p,q) \)  
the special pseudo-orthogonal group.

\( SO_0(p,q) \)  
the connected subgroup of \( SO(p,q) \).

\( U(n) \)  
the unitary group.

\( SU(n) \)  
the special unitary group.

\( \text{GL}(n, F) \)  
the general linear group over the field \( F \).

\( \text{SL}(n, F) \)  
the special linear group over the field \( F \).

\( \Gamma \)  
the Clifford group.

\( \text{PIN}(p,q) \)  
the double covering group of \( O(p,q) \).

\( \text{SPIN}(p,q) \)  
the double covering group of \( SO(p,q) \).

\( \text{SPIN}_0(p,q) \)  
the double covering group of \( SO_0(p,q) \).

\( so(n) \)  
the Lie algebra of \( SO(n) \).
so(\mu,\nu) \quad \text{the Lie algebra of } SO(\mu,\nu).

su(n) \quad \text{the Lie algebra of } SU(n).

\otimes^i \quad \text{the tensor product space } l \otimes l \otimes \ldots \otimes l, \text{ where } l \text{ occurs } i \text{ times.}
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David Eugene Kosokowsky was born July 30, 1961 in Windsor, Ontario, Canada. After leaving secondary school in 1977, he spent five years as a member of the general work force. In 1982, he enrolled as a full-time, preliminary-year, undergraduate student in engineering at the University of Windsor. His interests, however, quickly shifted towards physics and mathematics and by 1983 he was enrolled in the honours physics program at the same institution. He received the B.Sc. Degree in physics with mathematics from the University of Windsor in 1987. He then commenced study for the M.Sc. Degree in physics at the University of Windsor under the supervision of Dr. W.E. Baylis, specializing in mathematical physics. He hopes to graduate shortly, and would like to continue towards the Ph.D. Degree in physics.