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NUMERICAL EXPERIMENT ON
DIRECT DIFFERENTIATION METHOD

by

Yiqiu Huang

A Thesis
Submitted to the Faculty of Graduate Studies and Research
through the Department of Mathematics and Statistics in
Partial Fulfilment of the Requirements for the
Degree of Master of Science at the
University of Windsor

Windsor, Ontario, Canada
1994
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ABSTRACT

This thesis deals with the study of the finite element analysis and direct differentiation method applied in the optimum design of structures. The penalty function technique and Newton's method are employed to solve the optimization problem with inequality constraint.

The thesis also focuses on the sensitivity analysis, the effects of penalty parameters and convergence to the exact optimal solution. Several benchmark problems are used to verify the algorithms discussed.
Dedicated to:

My parents and my wife
ACKNOWLEDGEMENTS

I wish to express my sincere thanks to my supervisor, Dr. N.G. Zamani, for his encouragement, knowledge, patience and support throughout the course of the study.

I would like to thank the members of my committee, Dr. R.M. Barron and Dr. B. Budkowska, for their offering valuable suggestions and comments. I would also like to thank Dr. R. Caron, Head of the Department of Mathematics and Statistics, for providing graduate assistantship and research facilities. In addition, I would like to thank Dr. An Changfa and Dr. Zhang Shuxin for their helpful suggestions in computer graphics software selection.
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CHAPTER 1
INTRODUCTION

1.1 LITERATURE REVIEW.

The availability of inexpensive and powerful computers together with advancement of numerical methods have resulted in considerable research activity in the area of optimum structural design. Optimum structural design is a process where an objective function representing cost, effort or the risk factor associated with a design is minimized. Some optimization methods can be traced to the days of Newton, Lagrange and Cauchy. In the broad sense, the major developments in the area of numerical methods in optimization were initiated in United Kingdom during the 1960’s. A comprehensive review of which can be found in Rao (1984) and Hestenes (1980). An optimization tool used in structural design is known as mathematical programming. Practical engineering problems involve complicated geometries, loading and material properties and are not amenable to an analytical solution. With the development of numerical methods, increasingly complicated problems arising in science and engineering have been solved. In general, there are well-established numerical methods for the solution of the field equations: finite differences, finite elements and boundary elements. These methods all result in a system of simultaneous algebraic equations yielding approximate values of the unknowns at a discrete number of points in the continuum. In this thesis, we only concentrate on the finite element method.

The finite element method (FEM) in its modern form first appeared in the literature during the 1940’s in the field of structural engineering and was presented in the publications by Hrennikoff (1941). Along with the development of high-speed digital computers in the early 1950’s, the work of Turner, Clough, Martin and Topp (1956) prompted increased research in the subject. The application of the
minimum potential energy in the finite element method quickly led to its use in a variety of engineering areas. The leading second generation publications were due to Zienkiewicz and Cheung (1965) and Visser (1965). Using a weighted residual procedure, the range of applications for FEM was significantly expanded. Among the leaders were Szabo and Lee (1969) and Zienkiewicz (1971). The minimization of the total potential energy and the Galerkin's method (or the Principle of Virtual Work) are the two basic approaches in deriving the FEM equations. The fundamental concept of the finite element method is that any continuous quantity, such as temperature, pressure, or displacement, can be approximated by a discrete model composed of a set of piecewise continuous functions.

With the advancement of the theory of optimization and the finite element method, design engineers are no longer only satisfied with solving the field equations, but they are also interested in the optimal design issues which have economic implications. The last two decades have witnessed considerable activity in optimum design using the finite element method and sensitivity analysis. It is becoming increasingly important to effectively use an FEA tool in the engineering design cycle. The pioneering work in this area was done by Zienkiewicz and Campbell (1973) and Ramakrishnan and Francavilla (1974). These researchers were mainly concerned with shape optimization using FEA. In such problems, the shape of a structure is optimized according to some desired objectives and subject to given physical constraints. The design variables may involve configuration parameters, nodal locations, boundary shapes, etc.

Mathematical programming, a major ingredient in optimum structural design, is generally divided into two categories:

(1) Nongradient techniques where only the function values are used.

(2) Gradient-based methods where the derivatives of the objective functions,
constraint functions, as well as the function values are employed.

Comparing these two methods, it is well known that the latter is more efficient and accurate. It is not surprising that the former method is not as computationally intensive. In practice, the conjugate gradient method, Davidon-Fletcher-Powell method and Newton's method are widely employed. See e.g. Rao (1981) and Hestenes (1980).

The design sensitivity analysis plays an important role in the optimum design analysis. Generally speaking, since the objective functions and constraints are nonlinear and implicit in the design variables, it is difficult to calculate the exact derivatives. The implicit differentiation method (direct differentiation method) is used in structural optimization where the sensitivities are computed numerically. The advantage of this method is its generality and simplicity, whereas the disadvantage is the possible round-off error.

The first work in design sensitivity analysis using the finite element and optimization theory was done by Zienkiewicz and Campbell (1973) and Ramakrishnan and Francavilla (1974). Their studies focused on the structural shape optimization. Zienkiewicz and Campbell have advocated the use of sequential linear programming whereas Ramakrishnan and Francavilla used the penalty function method and sequential unconstrained minimization techniques (SUMT). The latter authors adopted a very accurate procedure to evaluate the stress derivatives which is important for their optimization analysis. Since these works, a great deal of research has been conducted on optimum design and sensitivity analysis.

The early 1980's experienced a great deal of activity in this research area. Botkin (1981) and Botkin and Bennet (1985) studied the shape optimization of plate and shell structures for 2D and 3D. Bennett and Botkin (1984) were the first to combine the structural shape optimization with adaptive mesh refinement to get more ac-
accurate estimates of the true solution. Botkin et. al. (1981,1984,1985) successfully used the implicit differentiation method in their work to compute the sensitivities. In this scheme, one directly differentiates the discretized equilibrium equations with respect to the design variables and then uses iterative methods to solve the resulting system for the sensitivities.

Haftka (1981) has discussed numerous techniques for thermal sensitivity analysis. In his paper, he has shown that the choice of the most efficient technique depends on the ratio of the number of temperature constraints to the number of design variables.

Pederson (1983) successfully used finite elements and sensitivity analysis for eigenvalue problems in elastic solids. Choi and Haug (1983) have developed a unified theory of structural design sensitivity for linear elastic structures, using a variational formulation for structural analysis. This method allows one to take the total derivative, or material derivative, of the variational state equation and to use an adjoint variable method for design sensitivity analysis. The main advantage is that one can compute the needed derivatives analytically with no approximations. The disadvantage of this approach is that the formulation requires evaluating accurate stress quantities on the boundary which are often difficult to obtain. Dems and Mroz (1984) applied the stress, strain and displacement functionals and variational approach to obtain an adjoint system for structural optimization and sensitivity analysis.

Bendsoe et. al. (1985) employed modern methods of functional analysis and the principle of minimum potential energy to obtain general results for sensitivity analysis in an elasticity problem with unilateral constraints.

Wang et. al. (1985) have introduced an efficient sensitivity formulation for shape optimization of continuum structures using a limited number of master nodes to
characterize the surface of a set of isoparametric finite elements and adopt their coordinates as design variables of the shape optimization. This paper has also derived an analytical formulation for calculating design sensitivities.

Rajan and Budiman (1987) used a semi-analytical method to study the two-dimensional plane elasticity finite element for optimal design. The adjoint variable differentiation, finite difference and the semi-analytical methods are compared for efficiency and accuracy.

Based on the knowledge of sensitivity analysis in finite elements, considerable work on sensitivity analysis has been done in the boundary element method. These works have been mainly done by Kane and Saigal (1988), Saigal et al. (1989, 1990), Meric (1988), Barone and Yang (1988), Kane and Wang (1990), Amaya and Aoki (1990), Kishimoto et al. (1990). Most of the research conducted by Kane, Saigal and Barone were of the shape optimization form. They have extended the design sensitivity analysis of FEM to the boundary element concept. Zamani and Chuang (1987) and Kishimoto et al. (1990) considered the optimal design of galvanic corrosion cells for 2D problems. Amaya and Aoki (1992) have extended this technique to the 3D boundary element method for the design of cathodic protection systems.
1.2 HIGHLIGHTS OF THE THESIS.

Most of the previous work involved complicated physical problems which were very difficult to understand and impossible to isolate the different components from each other. Also, these works did not discuss clearly the accuracy and the nature of convergence to the exact optimal solution. The objective of this thesis is to develop an elementary and clear account of the application of the finite element analysis and sensitivity formulation in the optimum design of structures. The benchmark problems are sufficiently simple to isolate and exhibit the parameters that contribute to the overall optimization process. We begin with simple examples which can be solved analytically and numerically. In a step by step fashion, we look into the details of the computational strategies such as calculation of the derivatives, iteration times, effect of the different penalty parameters and the nature of the convergence to the optimal solution. To keep the size of the thesis manageable and develop a good understanding of the important issues involved, only one dimensional second and fourth order boundary value problems are treated.
CHAPTER 2

STATEMENT OF THE OPTIMIZATION PROBLEM

2.1 FORMULATION.

An optimization or a mathematical programming problem can be stated as follows:

Find the vector $X$

$$X = \{x_1, x_2, \ldots, x_n\}^T$$  \hspace{1cm} (2.1)

which achieves

$$\min_{X} f(X)$$  \hspace{1cm} (2.2)

subject to the constraints

$$g_i(X) \leq 0 \quad i = 1, 2, \ldots, m$$  \hspace{1cm} (2.3)

$$l_j(X) = 0 \quad j = 1, 2, \ldots, p$$  \hspace{1cm} (2.4)

$$x^l_k \leq x_k \leq x^u_k \quad k = 1, 2, \ldots, n$$  \hspace{1cm} (2.5)

In this context, $X$ is called the design variable vector. All of the parameters which can uniquely determine an engineering design will be written as a function of $X$ according to the specifics of the practical problems. $f(X)$ is called the objective or the cost function, $g_i(X)$ and $l_j(X)$ are the inequality and the equality constraint functions respectively. The number of variables $n$ and the number of constraints $m$ and $p$ need not be related in any way. The constraints $g_i \leq 0$ and $l_j = 0$ define a feasible region in the $n$ dimensional design space. The problem stated in (2.1)-(2.5) is called a constrained optimization problem. Some optimization problems do not involve any constraints and can be stated as:

Find $X = \{x_1, x_2, \ldots, x_n\}^T$  \hspace{1cm} (2.6)
which achieves
\[
\min_X f(X) \quad (2.7)
\]
Such cases are called unconstrained optimization problems. The inequalities (2.5) where the design variables have the upper and lower bounds \( z_k^u \) and \( z_k^l \) are the side constraints. In such cases, the design variables cannot be chosen arbitrarily and they have to satisfy certain requirements. The inequality constraint (2.3) and the equality constraint (2.4) are generally implicit in nature. When considering the optimization problem with only inequality constraints \( g_i(X) \leq 0 \), the set of values \( X \) that satisfy the equation \( g_i(X) = 0 \) form a hypersurface in the design space called a constraint surface. The constraint surface divides the design space into two regions, namely, \( g_i(X) < 0 \) and \( g_i(X) > 0 \). The points lying in the region \( g_i(X) < 0 \) are feasible or acceptable whereas those points in the region \( g_i(X) > 0 \) are infeasible or unacceptable.
2.2 TECHNIQUES FOR CONSTRAINED OPTIMIZATION.

The classical methods of differential calculus can be used to find unconstrained maxima and minima of a function of several variables. Generally, these methods assume that the function is twice differentiable with respect to the design variables and that the derivatives are continuous. For problems with constraints, the Lagrange multiplier method is frequently used. But this method, in general, leads to a set of nonlinear simultaneous equations which may be difficult to solve. For problems with inequality constraints, we have two types of approaches: direct method and indirect method.

The direct approach includes the complex method, cutting plane method and method of feasible direction. The indirect approach uses the penalty function method. Research has shown that optimization based on penalty function is reliable and easy to implement, therefore this thesis employs such an approach.

The penalty function method transforms the basic optimization problem into a sequence of unconstrained minimization problems. Suppose we are interested in an optimization problem of the form:

\[ \min_X f(X) \] (2.8)

subject to

\[ g_i(X) \leq 0 \quad i = 1, 2, \ldots, m \] (2.9)

This problem is converted into an unconstrained minimization problem by constructing a revised function of the form:

\[ P_k = P(X, r_k) = f(X) + r_k \sum_{i=1}^{m} G_i[g_i(X)] \] (2.10)

where \( G_i \) is a function of the constraint functions \( g_i \) and \( r_k \) is a positive constant known as the penalty parameter. The second term on the right side of eq.(2.10)
is called the penalty term. If the unconstrained minimization of the $P$-function is repeated for a sequence of values of the penalty parameter $r_k$ ($k = 1, 2, \ldots$), the solution may converge to that of the original problem stated in (2.8)-(2.9). This procedure is called the sequential unconstrained minimization technique (SUMT). The penalty function formulations for inequality constraints can be divided into two categories, namely, the interior and the exterior methods. In the interior formulations, the popular choices of $G_i$ are

$$G_i = -\frac{1}{g_i(X)}$$  \hspace{1cm} (2.11)

or

$$G_i = \log[-g_i(X)]$$  \hspace{1cm} (2.12)

Some of the commonly used forms of the function $G_i$, in the case of exterior penalty function formulations, are

$$G_i = \max[0, g_i(X)]$$  \hspace{1cm} (2.13)

or

$$G_i = (\max[0, g_i(X)])^2$$  \hspace{1cm} (2.14)

In the interior methods, the unconstrained minima of $P_k$ all lie in the feasible region and converge to the solution of (2.8)-(2.9) as $r_k$ is varied in a particular manner. On the other hand, in the exterior methods, the unconstrained minima of $P_k$ all lie in the infeasible region and converge to the desired solution from the outside as $r_k$ is changed in a specified manner. The convergence of the unconstrained minima of $P_k$ is illustrated in Figure 2.1 for the simple problem:

Find $x$ which minimizes

$$f(x) = ax$$  \hspace{1cm} (2.15)
subject to

\[ \beta - x \leq 0 \]  \hspace{1cm} (2.16)

where \( \alpha \geq 0, \beta \geq 0 \) and

\[ P_k = \alpha x + r_k G(\beta - x) \]  \hspace{1cm} (2.17)

It can be seen from Fig. 2.1 (a) that, for the exterior method, the constrained minima of \( P_k \) converge to the optimum point \( x^* = \beta \) as the parameter \( r_k \) is increased sequentially. On the other hand, the interior method in Fig 2.1 (b) converges as the parameter \( r_k \) is decreased sequentially.

In Fig. 2.1 (a), we take \( G = [\max(0, \beta - x)]^2 \), that is

\[ G = (\beta - x)^2 U(\beta - x) \]  \hspace{1cm} (2.18)

where

\[ U(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \]  \hspace{1cm} (2.19)

\( U(x) \) is called unit step function. \( P_1^*, P_2^* \) and \( P_3^* \) are, respectively, the optimum points of \( P \) as \( r \) is taken as \( r_1, r_2 \) and \( r_3 \).

In Fig 2.1 (b), we take

\[ G = -\frac{1}{\beta - x} \]  \hspace{1cm} (2.20)

Currently, there are several algorithms for the minimization of the function (2.10) e.g., conjugate gradient method, Newton's method and Davidon-Fletcher-Powell method.

**Newton's method:**

Consider the following unconstrained problem,

\[ \min_X f(X) \]  \hspace{1cm} (2.21)
Assuming that the objective function $f(X)$ is twice differentiable, the necessary condition for the minima is

$$g(X^*) = \nabla f(X^*) = 0 \quad (2.22)$$

Equation (2.22) is a set of $n$ nonlinear equations which must be solved for $X^*$. The Newton's method is a prime candidate for solving eq. (2.22) and it consists of the following iterative procedure

$$X_{k+1} = X_k - J_k^{-1} g_k \quad (2.23)$$

where

$$g_k = \nabla f(X_k) \quad (2.24)$$

$$J_k = \nabla^2 f(X_k) \quad (2.25)$$

The sequence of points $X_1, X_2, \ldots, X_k, \ldots$ can be shown to converge to the actual solution $X^*$ from any initial point $X_1$ sufficiently close to the solution $X^*$, provided that $J_1$ is nonsingular.

Davidon-Fletcher-Powell method:

The iterative procedure for this method can be stated as follow:

(i) Start with an initial point $X_1$ and a $n \times n$ positive definite symmetric matrix $H_1$. (Usually $H_1$ is taken as the identity matrix I ). Set iteration number to $i = 1$.

(ii) Compute the gradient of the function, $\nabla f_i$, at the point $X_i$ and set

$$S_i = -H_i \nabla f_i \quad (2.26)$$

(iii) Find the optimal step length $\lambda_i^*$ in the direction $S_i$ and set

$$X_{i+1} = X_i + \lambda_i^* S_i \quad (2.27)$$
(iv) Test the new point $X_{i+1}$ for optimality. If $X_{i+1}$ is optimal, terminate the iterative process. Otherwise go to step (v).

(v) Update the $H$ matrix as

$$H_{i+1} = H_i + M_i + N_i$$  \hspace{1cm} (2.28)

where

$$M_i = \lambda_i^* \frac{S_i S_i^T}{S_i^T Q_i}$$  \hspace{1cm} (2.29)

$$N_i = -\frac{(H_i Q_i)(H_i Q_i)^T}{Q_i^T H_i Q_i}$$  \hspace{1cm} (2.30)

and

$$Q_i = \nabla f(X_{i+1}) - \nabla f(X_i) = \nabla f_{i+1} - \nabla f_i$$  \hspace{1cm} (2.31)

(vi) Set the new iteration number $i = i + 1$ and go to (ii).

In this thesis, we only applied Newton’s method to solve the optimization problem.
CHAPTER 3
OPTIMUM DESIGN OF BARS

3.1 THE GOVERNING EQUATIONS AND BOUNDARY CONDITIONS.

Without going into the detailed discussion present in most strength of materials textbooks, cf. Desai (1979), the differential equation governing the equilibrium state for the bar is given by

\[- \frac{d}{dx}(AE \frac{du}{dx}) = f(x) \quad 0 < x < L\]  

(3.1)

Here \(A\) and \(E\) are the constant cross-sectional area and Young’s modulus respectively. The function \(f(x)\) is the distributed axial load and \(u(x)\) is the unknown axial displacement. Finally, \(L\) is the length of the bar. The assumption of constant \(A\) and \(E\) is merely for convenience. Symbolically, the bar without any references to boundary conditions is shown in Figure 3.1.

Naturally, the differential equation (3.1) cannot be solved uniquely without specifying the boundary conditions. Some common end conditions are shown in Figures 3.2-3.5 and mathematically represented below:

a) Clamped at both ends (Fig. 3.2)

\[
\begin{align*}
  u(0) &= 0 \\
  u(L) &= 0
\end{align*}
\]  

(3.2)

b) Clamped at one end and loaded at the other end (Fig. 3.3)

\[
\begin{align*}
  u(0) &= 0 \\
  -AE \frac{du}{dx} \bigg|_{x=L} - f &= 0
\end{align*}
\]  

(3.3)

c) Springs at both ends (Fig. 3.4)

\[
\begin{align*}
  AE \frac{du}{dx} \bigg|_{x=0} - K_L u(0) &= 0 \\
  -AE \frac{du}{dx} \bigg|_{x=L} - K_R u(L) &= 0
\end{align*}
\]  

(3.4)
d) Clamped at one end and spring at the other end (Fig. 3.5)

\[
\begin{align*}
&u(0) = 0 \\
&-AE \left. \frac{du}{dx} \right|_{x=L} - K_R u(L) = 0
\end{align*}
\] (3.5)
3.2 FINITE ELEMENT METHOD FORMULATION

For the sake of simplicity, we will concentrate only on linear variation of displacement within a typical element as shown in Figure 3.6, i.e.,

\[ u(x) \approx N_1(x)u_1 + N_2(x)u_2 \quad (3.6) \]

where \( N_1(x) \) and \( N_2(x) \) are the standard linear shape functions and \( \{u_1, u_2\} \) are the nodal degrees of freedom. We will also assume that \( f(x) \), the distributed load, is either linear or can be approximated by linear variation such as

\[ f(x) \approx N_1(x)f_1 + N_2(x)f_2 \quad (3.7) \]

where \( \{f_1, f_2\} \) are the known nodal force values.

Using the Principle of Virtual Work (Galerkin’s method), the following local \( 2 \times 2 \) equilibrium equations is obtained (Desai (1979)).

\[
\frac{AE}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = h \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} -AE \frac{du}{dx} \bigg|_{x=x_1} \\ AE \frac{du}{dx} \bigg|_{x=x_2} \end{bmatrix} \quad (3.8)
\]

or simply

\[
\begin{bmatrix} K^{(e)} \end{bmatrix} \begin{bmatrix} u^{(e)} \end{bmatrix} = \begin{bmatrix} f^{(e)} \end{bmatrix} + \begin{bmatrix} T^{(e)} \end{bmatrix} \quad (3.9)
\]

The superscript \(^e\) refers to elemental equation, \( h = x_2 - x_1 \) is the element size, \( \{f^{(e)}\} \) and \( \{T^{(e)}\} \) are the load vectors due to the distributed load and the end nodal forces respectively. After the assembly process, the global equilibrium equation is obtained, which is denoted by

\[
[K] \{u\} = \{f\} + \{T\} \quad (3.10)
\]

Upon the imposition of the boundary conditions, system of equations (3.10) can be uniquely solved for the unknown displacements and reaction forces.
3.3 BAR OPTIMIZATION BENCHMARKS. ANALYTICAL SOLUTIONS.

In later sections we will discuss how the finite element calculation and the numerical differentiation algorithm can be used for optimum design purposes. For the time being however, several benchmark optimization problems and their analytical solutions are presented which will be used for verification of the numerical calculations.

BENCHMARK 0:

Consider the axially compressed bar as shown in Figure 3.7 where \( u(L) \) is denoted by \( \delta \) and the exerted load to cause the deformation is \( f \). Suppose the goal is to find \( \delta \) such that the external work given by \( J(\delta) = f\delta \) is minimized. This is a very simple problem that can be solved analytically. The governing boundary value problem for equilibrium is

\[
\begin{align*}
-\frac{AE}{dx^2} u &= 0 & 0 < x < L \\
u(0) &= 0 \\
u(L) &= \delta
\end{align*}
\]

(3.11)

The exact solution to the problem becomes \( u(x) = \delta x/L \). The balance of forces at the right end of the bar requires,

\[
-\frac{AE}{dx} \left. \frac{du}{dx} \right|_{x=L} + f = 0
\]

(3.12)

and therefore substituting the exact solution in (3.12) gives \( f = A^2 E^2 \delta / L \). The expression for \( J \) becomes,

\[
J(\delta) = \frac{AE}{L} \delta^2
\]

(3.13)

The graph of \( J \) vs \( \delta \) is the parabola shown in Figure 3.8 where clearly the minimum occurs at \( \delta = 0 \). This is not unexpected as no displacement requires no external
work.

**BENCHMARK 1:**

We now consider an axially compressed bar which rests against a linearly elastic spring with modulus $K_L$ at the left hand side as shown in Figure 3.9. A more complicated problem is presented here which requires an inequality constraint. Let $\delta$ and $f$ be the displacement and the required force on the right end respectively. Denoting the external work due to the deformation by $J = f\delta$, the objective is to solve the following constrained optimization problem

$$\begin{align*}
\min_{\delta} & \quad J \\
\text{subject to} & \quad u(0) \leq -C^* 
\end{align*} \quad (3.14)$$

where $C^*$ is a fixed positive constant and $u(x)$ is the displacement function. Once again the problem is simple enough to be solved analytically. The boundary value problem is

$$\begin{align*}
- AE \frac{d^2 u}{dx^2} & = 0 \quad 0 < x < L \\
AE \left. \frac{du}{dx} \right|_{x=0} - K_L u(0) & = 0 \\
u(L) & = \delta
\end{align*} \quad (3.15)$$

The exact solution to the above problem is

$$u(x) = \frac{\delta}{K_L L + AE} (K_L x + AE) \quad (3.16)$$

Substituting expression (3.16) in the balance of forces below

$$- AE \left. \frac{du}{dx} \right|_{x=L} + f = 0 \quad (3.17)$$

results in

$$f = AE \frac{K_L \delta}{K_L L + AE} \quad (3.18)$$
We can now find the expression for the external work $J$ as

$$J(\delta) = \frac{AEK_L}{K_LL + AE}\delta^2$$  \hspace{1cm} (3.19)

It is important to note that the constraint $u(0) \leq -C^*$ imposes in turn a constraint on $\delta$ which is

$$\delta \frac{AE}{K_LL + AE} \leq -C^*$$  \hspace{1cm} (3.20)

i.e.

$$\delta \leq -\frac{C^*(K_LL + AE)}{AE}$$  \hspace{1cm} (3.21)

If the right hand side of inequality (3.21) is denoted by $-\delta^*$, equation (3.21) takes the form $\delta \leq -\delta^*$.

Therefore, the original optimization problem (3.14) takes the form

$$\begin{cases} \min \delta \quad J(\delta) \\ \text{subject to } \delta \leq -\delta^* \end{cases}$$  \hspace{1cm} (3.22)

The graph of $J$ together with the feasible region are depicted in Figure 3.10 and clearly the minimum occurs at the boundary of the feasible region where $\delta = -\delta^*$.

Furthermore, the corresponding minimum value for $J$ is

$$J_{\text{min}} = \frac{AEK_L}{K_LL + AE}(\delta^*)^2$$  \hspace{1cm} (3.23)

The optimal exact solution will then be obtained by replacing $\delta = -\delta^*$ in the expression (3.16)

$$u_{\text{opt}}(x) = \frac{-\delta^*}{K_LL + AE}(K_Lx + AE) \equiv -\frac{C^*}{AE}(K_Lx + AE)$$  \hspace{1cm} (3.24)
BENCHMARK 2.

This is essentially the same as the benchmark problem 1 with the major difference that the spring acts nonlinearly following the generalized Hooke's law \( f = -KLx^2 \). As in the previous problem, the objective is to minimize the external work \( J = f\delta \) subject to an endpoint displacement constraint, i.e.,

\[
\begin{align*}
\min_{\delta} J \quad \text{subject to} \quad u(0) \leq -C^* & \quad (3.25)
\end{align*}
\]

Once again, \( \delta \) and \( f \) are the displacement and the required force on the right end of the bar respectively and \( C^* \) is a positive fixed constant.

The equilibrium is described by the boundary value problem below.

\[
\begin{align*}
\begin{cases}
-\frac{AE}{dx^2} \frac{d^2 u}{dx^2} & = 0 \quad 0 < x < L \\
-\frac{AE}{dx} \frac{du}{dx} \bigg|_{x=0} + KL|u(0)|u(0) & = 0 \\
u(L) & = \delta
\end{cases}
\end{align*}
\]

(3.26)

We first observe that because of the constraint \( u(0) \leq -C^* < 0 \), the left boundary condition in eq. (3.26) can be simplified to

\[
-\frac{AE}{dx} \frac{du}{dx} \bigg|_{x=0} - KL[u(0)]^2 = 0
\]

(3.27)

After some algebra, the exact solution can be calculated and represented by

\[
u(x) = \left[ \frac{\delta}{L} - \frac{AE - \sqrt{A^2E^2 - 4LKLA\delta}}{2LK} \right] x
\]

\[
+ \frac{AE - \sqrt{A^2E^2 - 4LKLA\delta}}{2LK} \]

(3.28)

Substituting (3.28) in the force balance equation on the right hand side of the bar,

\[
-\frac{AE}{dx} \frac{du}{dx} \bigg|_{x=L} + f = 0
\]

(3.29)

20
gives

\[ f = \frac{\delta}{L} - \frac{AE - \sqrt{A^2E^2 - 4LK_eAE\delta}}{2L^2K_L} \] (3.30)

The expression for the external work \( J \) becomes

\[ J(\delta) = \left( \frac{\delta}{L} - \frac{AE - \sqrt{A^2E^2 - 4LK_eAE\delta}}{2L^2K_L} \right) \delta \] (3.31)

The constraint \( u(0) \leq -C^* \) translates into

\[ \delta \leq -\frac{(LK_eC^* + AE)C^*}{AE} \] (3.32)

If we denote the right hand side of the inequality (3.32) by \(-\delta^*\), the original optimization problem (3.25) takes the form

\[
\begin{cases}
\min_{\delta} & J(\delta) \\
\text{subject to} & \delta \leq -\delta^*
\end{cases}
\] (3.33)

A plot of \( J(\delta) \) as described by (3.31) indicates that within the feasible region \( \delta \leq -\delta^* \), the function is decreasing and therefore the minimum occurs on the boundary \( \delta = -\delta^* \) similar to Figure 3.10. Of course, in the present benchmark, the function \( J(\delta) \) is no longer quadratic. The minimum \( J \) value is obtained by substituting \( \delta = -\delta^* \) in the expression (3.31) and the corresponding displacement solution \( u(x) \) can be written by substituting \( \delta = -\delta^* \) in (3.28).

**Benchmark 3.**

Although the problem considered here has some similarity to benchmark 2, there exists a major difference when looked at carefully. After transforming the original optimization problem, one will realize that the minimum will no longer occur at the boundary of the feasible domain but it will be achieved at an interior point.
The successful treatment of the problem will prove that this procedure is general enough for different cases. Because of the boundary condition the bar analogy is no longer valid, however for uniformity of presentation, the parameters $A$, $E$ and $K_L$ will still be maintained. Incidentally, such a boundary condition is still in terms of $J = f \delta$, i.e.,

$$\begin{align*}
\min_{\delta} & \quad J \\
\text{subject to} & \quad u(0) \leq -C^*
\end{align*}$$

(3.34)

where $C^*$ is a fixed positive constant. $u(x)$ and $f = AE \frac{du}{dx} |_{x=L}$ are associated with the following boundary value problem.

$$\begin{align*}
AE \frac{d^2u}{dx^2} &= 0 & 0 < x < L \\
AE \frac{du}{dx} |_{x=0} + \frac{K_L}{2} \left[ e^{u(0)+4} + e^{-(u(0)+4)} \right] &= 0 \\
u(L) &= \delta
\end{align*}$$

(3.35)

The exact solution to the boundary value problem (3.35) is $u(x) = ax + b$ where, due to the boundary conditions, the parameters $a$ and $b$ satisfy the system of nonlinear transcendental equations

$$\begin{align*}
AEa + \frac{K_L}{2} \left[ e^{b+4} + e^{-(b+4)} \right] &= 0 \\
aL + b &= \delta
\end{align*}$$

(3.36)

Because of the complexity of the above system, the procedure followed in the previous benchmarks cannot employed and the problem will be solved indirectly.

Recall that $J = f \delta$, $\delta = aL + b$ and $f = a$. Therefore,

$$J = a(aL + b)$$

(3.37)

Substituting the expression for $a$ from the first equation in (3.36) gives

$$J(b) = -\frac{K_L}{2AE} \left[ e^{b+4} + e^{-(b+4)} \right] \left[ -\frac{K_LL}{2AE} (e^{b+4} + e^{-(b+4)}) + b \right]$$

(3.38)
For the sake of concreteness, let us assume that \( A = 1 \text{ in}^2 \), \( E = 10^7 \text{psi} \), \( L = 10 \text{ in} \) and \( C^* = 2 \text{ in} \). Therefore the transformed optimization problem takes the form

\[
\begin{align*}
\min_b & \quad J(b) \\
\text{subject to} & \quad b \leq -2
\end{align*}
\] (3.39)

The graph of \( J \) vs. \( b \) and the feasible region are depicted in Figure 3.11 which indicates a minimum \( J \) value is estimated as \( J_{\text{min}} = 3.877513 \times 10^4 \). Finally, the corresponding exact solution at \( b^* \) is

\[
u(x) = -0.00103763x - 3.72652
\] (3.40)

**BENCHMARK 4.**

The problem considered here is different from the previous benchmarks in many aspects. It involves a bar which is under tension due to a concentrated load at the interior section of the bar at the location \( x = x_f \). The left end of the bar \((x = 0)\) is free and the right end \((x = L)\) is clamped as shown in Figure 3.12. The objective is to minimize the external work subject to a displacement constraint at the left end. If the external work is denoted by \( J = u(x_f)f \), the optimization problem becomes,

\[
\begin{align*}
\min_f & \quad J \\
\text{subject to} & \quad u(0) \leq -C^*
\end{align*}
\] (3.41)

where \( C^* \) is a positive fixed constant. Note that the design variable (the variable with respect to which the minimization is carried out) is the force \( f \) as opposed to the displacement \( \delta \) in the previous benchmarks.
In order to build up the governing equation, we define a new function $H(x - \xi)$ as follow

$$F(x) = \int H(x - \xi) \, dx$$

$$H(x - \xi) = 0 \quad x \neq \xi$$

$$F(\xi+) - F(\xi-) = 1$$

(3.42)

The governing boundary value problem can be written with the help of function $H(x - \xi)$ at $\xi = x_f$,

$$\begin{cases}
-AE \frac{d^2 u}{dx^2} = f \, H(x - x_f) & 0 < x < L, \quad x \neq x_f \\
AE \frac{du}{dx} \bigg|_{x=0} = 0 \\
u(L) = 0
\end{cases}$$

(3.43)

The exact solution of (3.43) consists of a piecewise linear function which can be obtained by imposing the boundary conditions, the displacement continuity at $x_f$ and the derivative jump condition at the same point. The solution can be shown to be

$$u(x) = \begin{cases}
\frac{f}{AE} (L - x_f) & 0 \leq x \leq x_f \\
\frac{f}{AE} (L - x) & x_f \leq x \leq L
\end{cases}$$

(3.44)

The displacement at the point $x_f$ is therefore,

$$u(x_f) = \frac{f}{AE} (L - x_f)$$

(3.45)

and the external work can be calculated from

$$J(f) = \frac{f^2}{AE} (L - x_f)$$

(3.46)

The constraint $u(0) \leq -C^*$, in terms of $f$ translates into,

$$f \leq - \frac{C^* AE}{L - x_f}$$

(3.47)
If the right hand side of the inequality (3.17) is denoted by $-f^*$, the original optimization problem is reformulated as,

$$
\begin{align*}
\min_f \quad & J(f) \\
\text{subject to} \quad & f \leq -f^*
\end{align*}
$$

(3.48)

The graph of $J$ and the feasible region is shown in Figure 3.13. Clearly, as in benchmark 1, the optimum solution is at the boundary of the feasible region $f = -f^*$. The corresponding minimum $J$ value and the exact solution are:

$$
J_{\text{min}} = \frac{(f^*)^2}{AE}(L - x_f)
$$

(3.49)

$$
u(x) = \begin{cases} 
-\frac{f^*}{AE}(L - x_f) & 0 \leq x \leq x_L \\
-\frac{f^*}{AE}(L - x) & x_f \leq x \leq L
\end{cases}
$$

(3.50)
BENCHMARK 5.

The previous problem can be made more interesting by modifying the left end of the bar to involve a linear spring of stiffness $K_L$ as shown in Figure 3.14. The optimization problem is

$$\begin{align*}
\min_f J \\
\text{subject to } u(0) \leq -C^*
\end{align*}$$

where $J = u(x_f)f$ is the external work.

The governing boundary value problem is

$$\begin{align*}
-AE \frac{d^2u}{dx^2} &= fH(x - x_f) &0 < x < L, \quad x \neq x_f \\
AE \frac{du}{dx} \bigg|_{x=0} - K_L u(0) &= 0 \\
u(L) &= 0
\end{align*}$$

The exact solution can be derived to be

$$u(x) = \begin{cases} 
\frac{f(L - x_f)}{AE + K_L L} \left[ \frac{K_L}{AE} x + 1 \right] & 0 \leq x \leq x_f \\
\frac{f}{AE} \frac{K_L x_f + AE}{K_L L + AE} [L - x] & x_f \leq x \leq L
\end{cases}$$

The displacement at the point $x_f$ is

$$u(x_f) = \frac{f}{AE} \frac{K_L x_f + AE}{K_L L + AE} (L - x_f)$$

and therefore the external work is given by

$$J(f) = \frac{L - x_f}{AE} \frac{K_L x_f + AE}{K_L L + AE} f^2$$

The constraint $u(0) \leq -C^*$. in terms of $f$ translates into

$$\frac{f(L - x_f)}{K_L L + AE} \leq -C^*$$
or

\[ f \leq -\frac{AE + KL}{L - x_f} C^* \]  

(3.57)

Denoting the right hand side of (3.57) by \(-f^*)\), the original optimization takes the form

\[
\begin{cases}
\min_f J(f) \\
\text{subject to } f \leq -f^*
\end{cases}
\]  

(3.58)

The graph of \(J\) and the feasible region is as in Figure 3.13. Once again, the optimum solution is at the boundary of the feasible region \(f = -f^*\) and the corresponding minimum \(J\) value and the exact solution are

\[ J_{\text{min}} = \frac{L - x_f}{AE} \frac{KLx_f + AE}{KL + AE} (f^*)^2 \]  

(3.59)

and

\[
\begin{align*}
    u(x) &= \begin{cases} 
    -\frac{f^*}{K_L L + AE} \left[ \frac{K_L}{AE} x + 1 \right] & 0 \leq x \leq x_f \\
    -\frac{f^*}{AE} \frac{KLx_f + AE}{KL + AE} [L - x] & x_f \leq x \leq L 
    \end{cases}
\end{align*}
\]  

(3.60)
3.4 OPTIMIZATION STRATEGY AND SENSITIVITIES.

The previous section revealed that the type of problems encountered in this chapter are all of the form

\[
\begin{aligned}
\min_s & \quad J(s) \\
\text{subject to} & \quad g(s) \leq 0
\end{aligned}
\]  

(3.61)

This is a constrained optimization problem where the side conditions are of the inequality type. Generally speaking, constraints in optimization are handled through the penalty function method or Lagrange multipliers. We have adopted the penalty function approach because of simplicity and generality. A modified objective function \( \tilde{J} \) is defined according to

\[
\tilde{J}(s) = J(s) + \lambda [g(s)]^2 H(g(s))
\]  

(3.62)

where \( \lambda \) is the penalty parameter (usually large) and \( H(z) \) is the unit step function,

\[
H(z) = \begin{cases} 
0 & z \leq 0 \\
1 & z > 0
\end{cases}
\]  

(3.63)

The explanation of (3.62) is as follows. If the variable \( s \) is such that \( g(s) \leq 0 \), the constraint is satisfied and minimizing \( J(s) \) and \( \tilde{J}(s) \) are equivalent. On the other hand if \( g(s) > 0 \), the constraint is violated and

\[
\tilde{J}(s) = J(s) + \lambda [g(s)]^2
\]  

(3.64)

Here minimizing \( \tilde{J} \) and the fact that \( \lambda \) is large ensures that \( g(s) \) is pushed toward zero.

With the aid of the penalty function, the constrained optimization problem (3.61) is replaced with the unconstrained minimization of \( \tilde{J}(s) \). The necessary first order condition for achieving minimum is

\[
\frac{d\tilde{J}}{ds} = 0
\]  

(3.65)
or $\dot{J}(s) = 0$ which is a nonlinear equation involving $s$. We have used Newton’s method for the solution of (3.65). This is represented by the following iteration procedure

$$s_{n+1} = s_n - \frac{\dot{J}(s_n)}{J''(s_n)}$$  \hspace{1cm} (3.66)

The primes in equation (3.66) refer to differentiation with respect to $s$.

We now explain how the first and the second derivatives of $\dot{J}$ are calculated. This hinges on the fact that the objective function in the present chapter is of the form

$$\dot{J} = f\delta + \lambda[g(s)]^2 H(g(s))$$  \hspace{1cm} (3.67)

where $s = \delta$ or $f$ ($f$ and $\delta$ are the applied force and displacement respectively). Depending on the situation, $f$ or $\delta$ can be used as the design variable. As an example let us assume that $\delta$ is the design variable, i.e.,

$$\dot{J}(\delta) = f(\delta)\delta + \lambda[g(\delta)]^2 H(g(\delta))$$  \hspace{1cm} (3.68)

Differentiating (3.68) with respect to $\delta$ and using the chain rule gives (subscripts signify differentiation)

$$\dot{J}_{,\delta} = f,\delta + f + 2\lambda gg,\delta H(g(\delta))$$  \hspace{1cm} (3.69)

Differentiating again,

$$\dot{J}_{,\delta\delta} = \delta f,\delta + 2f,\delta + 2\lambda \left[g_{,\delta}^2 + gg,\delta\right] H(g(\delta))$$  \hspace{1cm} (3.70)

Repeating the same process with $f$ as the design variable gives

$$\dot{J}_{,f} = f\delta + \delta + 2\lambda gg, f H(g(f))$$  \hspace{1cm} (3.71)

and

$$\dot{J}_{,ff} = f\delta_{,ff} + 2\delta_{,f} + 2\lambda \left[g_{,f}^2 + gg, ff\right] H(g(f))$$  \hspace{1cm} (3.72)
The magnitudes of $\delta_f$ and $f,\delta$ are to be calculated from the finite element calculations. It will be seen that in linear problems, the second derivatives of $\delta$ and $f$ will be zero and therefore simplifying the results considerably. Linear constraints also yield zero second derivative for $g$.

**SENSITIVITIES.**

The procedure used for calculating the design sensitivities, i.e.,

$$
\frac{\partial u_1}{\partial \delta} \cdot \frac{\partial u_2}{\partial \delta} \cdot \frac{\partial u_3}{\partial \delta} \cdots \cdot \frac{\partial u_n}{\partial \delta}
$$

or

$$
\frac{\partial u_1}{\partial f} \cdot \frac{\partial u_2}{\partial f} \cdot \frac{\partial u_3}{\partial f} \cdots \cdot \frac{\partial u_n}{\partial f}
$$

is known as the direct differentiation method. The idea behind this procedure is very simple and begins with the global (or local) equilibrium equation

$$
[K] \{u\} = \{f\} + \{T\}
$$

as presented in section 3.2. Without loss of generality, let us assume that $\delta$ is the design variable (the treatment for $f$ as the design variable is identical). Differentiating both sides of (3.73) with respect to $\delta$ gives

$$
[K]_{,\delta} \{u\} + [K] \{u\}_{,\delta} = \{f\}_{,\delta} + \{T\}_{,\delta}
$$

(3.74)

where the subscripts in equation (3.74) refer to differentiation with respect to $\delta$. For the benchmark problems considered in this chapter, the stiffness matrix $[K]$ is independent of the deformation and therefore, $[K]_{,\delta} = 0$. However, for the sake of generality let us keep the $[K]_{,\delta}$ term. Rearranging the previous equation yields

$$
[K] \{u\}_{,\delta} = \{f\}_{,\delta} + \{T\}_{,\delta} - [K]_{,\delta} \{u\}
$$

(3.75)
Note that the right hand side of (3.75) is now known and the system can be solved to produce the first order design sensitivities. For the second order sensitivities, equation (3.75) can be differentiated and rearranged to produce,

$$\{K\} \{\delta u\} = \{f\} - \{T\} + \{K\} \{\delta u\} - 2 \{K\} \{u\}$$  

(3.76)

Once again, the fact that the right hand side of (3.76) is known can be used to solve (3.76) for \(\{u\} \). It is also important to observe that the equations (3.73), (3.74) and (3.75) involve the factorization of the same matrix \([K]\). In the next section we consider the individual benchmark problems and the details of calculating different derivatives of \(\{f\}\) and \(\{T\}\).
3.5 RESULTS AND CONCLUSIONS.

In this section, the numerical calculation of benchmark problems 1-5 are discussed in detail and comparisons with the analytical solutions derived previously are made. For simplicity the finite element calculations are performed with equally sized elements denoted by 'h'. It is also assumed that there are \( n \) elements and \( n + 1 \) nodes. The benchmark problem 0 has a trivial solution and therefore not pursued.

BENCHMARK 1.

Direct Differentiation Results:

The global finite element system, after imposing the boundary conditions, has the form

\[
\begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 2 & -1 \\
0 & 0 & \cdots & -1 & 1
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n \\
f_0
\end{bmatrix}
= 
\begin{bmatrix}
-K_L u_1 \\
0 \\
\vdots \\
0 \\
f_0
\end{bmatrix}
\]

(3.77)

Differentiating both sides as described in (3.74) leads to

\[
\begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 2 & -1 \\
0 & 0 & \cdots & -1 & 1
\end{bmatrix}
\begin{bmatrix}
u_{1,\delta} \\
u_{2,\delta} \\
\vdots \\
u_{n,\delta} \\
1
\end{bmatrix}
= 
\begin{bmatrix}
-K_L u_{1,\delta} \\
0 \\
\vdots \\
0 \\
f_0
\end{bmatrix}
\]

(3.78)
The above system can be solved for \( \{u_{1,ss}, u_{2,ss}, \cdots, u_{n,ss}, f_{s}\} \). To get the second order sensitivities, equation (3.78) is differentiated with respect to \( \delta \) once more, resulting in

\[
\frac{AE}{h} \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 2 & -1 \\
0 & 0 & \cdots & -1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
u_{1,ss} \\
u_{2,ss} \\
\vdots \\
u_{n,ss} \\
f_{s,ss} \\
\end{bmatrix} = \begin{bmatrix}
-K_L u_{1,ss} \\
0 \\
\vdots \\
0 \\
f_{s,ss} \\
\end{bmatrix} \tag{3.79}
\]

Rearranging the equations in (3.79), one can write it in the form

\[
\frac{AE}{h} \begin{bmatrix}
K_L + 1 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 2 & 0 \\
0 & 0 & \cdots & -1 & -\frac{h}{AE} \\
\end{bmatrix}
\begin{bmatrix}
u_{1,ss} \\
u_{2,ss} \\
\vdots \\
u_{n,ss} \\
f_{s,ss} \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
0 \\
\end{bmatrix} \tag{3.80}
\]

The system (3.80) clearly has the zero vector as its solution and as suggested in the previous section, second order sensitivities vanish is this problem.

We also need the derivatives of the constraint function \( g(s) \). For the present problem

\[
g(u_1) \equiv u_1 + C^* \tag{3.81}
\]

Therefore,

\[
g_s(u_1) = \frac{dg}{du_1} \frac{du_1}{d\delta} = u_{1,s} \tag{3.82}
\]

Differentiating (3.82) with respect to \( \delta \) and noting that the second order sensitivities vanish gives \( g_{ss} = 0 \).
Finite Element Results.

The bar in this question has the following parameters associated with it. \( A = 1 \) in.\(^2\), \( E = 10^7 \) psi, \( K_L = 10^4 \) lb./in., \( L = 10 \) in., and \( C^* = 2 \) in. Substituting these parameters into expressions (3.23) and (3.24) result in the analytical solution.

\[ u(x) = -0.002x - 2. \quad (3.83) \]

and the scaled minimum value \( J_{\text{min}}/K_L = 4.04 \). For FEM calculations, we use 10 equally sized elements to model the bar and three different penalty parameters \( \lambda = 10^2, 10^3 \) and \( 10^4 \) respectively. The stopping criteria is \( \left| \frac{dJ}{d\delta}/\lambda \right| < 10^{-5} \). For the present benchmark, \( \delta \) is the design variable and the initial guess of \( \delta = -1 \) is used for the Newton's method.

The results of the final iterations for the different penalty parameters and the analytical solution (3.83) are depicted in Figure 3.15. Note that the number of iterations necessary to meet the stopping criterion is also considered there. The scaled plot of the function \( J(\delta) \) which is to be minimized along with the analytical and numerical results are described in Figure 3.16. It is interesting that the convergence occurs from within the nonfeasible region. The same plot on a different horizontal scale is represented in Figure 3.17 which shows the convergence more clearly.

**BENCHMARK 2.**

Direct Differentiation Results:

The global finite element system after imposing the boundary conditions takes
the form

\[
\frac{AE}{h} \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 2 & -1 \\
0 & 0 & \cdots & -1 & 1 \\
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n \\
\delta \\
\end{bmatrix} = \begin{bmatrix}
K_L u_1^2 \\
0 \\
\vdots \\
0 \\
f \\
\end{bmatrix}
\] (3.84)

Note that the first entry of the vector on the right hand side is in fact \(-K_L |u_1| u_1\), but because \(u_1 < 0\), it reduces to \(K_L u_1^2\). Differentiating both sides of (3.84) with respect to the design variable \(\delta\) results in the system,

\[
\frac{AE}{h} \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 2 & -1 \\
0 & 0 & \cdots & -1 & 1 \\
\end{bmatrix} \begin{bmatrix}
u_{1,\delta} \\
u_{2,\delta} \\
\vdots \\
u_{n,\delta} \\
1 \\
\end{bmatrix} = \begin{bmatrix}
2K_L u_1 u_{1,\delta} \\
0 \\
\vdots \\
0 \\
f,\delta \\
\end{bmatrix}
\] (3.85)

The above system can be solved for \(\{u_{1,\delta}, u_{2,\delta}, \cdots, u_{n,\delta}, f,\delta\}\) arriving at the first order sensitivities. To obtain the second order sensitivities, equation (3.85) is differentiated for the second time with respect to \(\delta\). The outcome is

\[
\frac{AE}{h} \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 2 & -1 \\
0 & 0 & \cdots & -1 & \frac{h}{AE} \\
\end{bmatrix} \begin{bmatrix}
u_{1,\delta\delta} \\
u_{2,\delta\delta} \\
\vdots \\
u_{n,\delta\delta} \\
0 \\
\end{bmatrix} = \begin{bmatrix}
2K_L u_1^2,\delta + 2K_L u_1 u_{1,\delta\delta} \\
0 \\
\vdots \\
0 \\
f,\delta\delta \\
\end{bmatrix}
\] (3.86)
Rearranging system (3.56) leads to

\[
\begin{bmatrix}
    1 - 2K_L u_1 & -1 & 0 & \cdots & 0 \\
    -1 & 2 & -1 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & 2 & 0 \\
    0 & 0 & \cdots & -1 & -\frac{h}{AE}
\end{bmatrix}
\begin{bmatrix}
    u_{1,\delta} \\
    u_{2,\delta} \\
    \vdots \\
    u_{n,\delta} \\
    f_{,\delta\delta}
\end{bmatrix}
= \begin{bmatrix}
    2K_L u_{1,\delta} \\
    0 \\
    \vdots \\
    0
\end{bmatrix}
\] (3.57)

Unlike the previous benchmark problem, the solution to the system (3.57) is no longer identically zero. The second order sensitivities can therefore be obtained by solving (3.57).

The constraint function represented by (3.25) is given as

\[ g(u_1) \equiv u_1 + C^* \] (3.58)

Therefore

\[ g,\delta(u_1) = \frac{dg}{du_1} \frac{du_1}{d\delta} = u_{1,\delta} \] (3.59)

and

\[ g,\delta\delta(u_1) = \frac{d^2 g}{d\delta^2} = u_{1,\delta\delta} \] (3.60)

Finite Element Results:

The material and geometric properties for the bar under investigation are \( A = 1 \) in\(^2 \), \( E = 13 \) psi, \( K_L = 10^4 \) lb./in., \( L = 10 \) in. and \( C^* = 2 \) in. Inserting these values in equation (3.28) results in the analytical solution

\[ u(x) = -0.004x - 2. \] (3.61)
which corresponds to the scaled minimum value $J_{\text{min}}/K_L = 8.16$.

Once again, 10 equally sized elements are used for modelling purposes and the penalty parameters are chosen to be $\lambda = 10^2$, $10^3$ and $10^4$. The stopping criterion is $\left| \frac{\partial J}{\partial \delta} / \lambda \right| < 10^{-5}$. The initial guess for the design variable is $\delta = -1$ which starts the Newton's method.

Figure 3.18 exhibits the analytical solution (3.91) along with the final iteration results for the three penalty parameters. The numerical approximation corresponding to $\lambda = 10^4$ and the analytical solution can hardly be distinguished from each other. The stopping criterion is met in 3, 4 and 4 iterations respectively. The scaled plot of the function $J(\delta)$ in two different horizontal ranges are depicted in Figures 3.19 and 3.20. As in the previous benchmark, the convergence (in terms of the penalty parameter) occurs from within the nonfeasible region.
**BENCHMARK 3.**

**Direct Differentiation Results:**

This problem is similar to benchmark 2, with two differences. The nonlinearity on the boundary is of transcendental type and the minimum point lies in the interior of the feasible region. The global finite element system, taking the boundary conditions into consideration, is

\[
\frac{AE}{h} \begin{bmatrix}
1 & -1 & 0 & \ldots & 0 \\
-1 & 2 & -1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 2 & -1 \\
0 & 0 & \ldots & -1 & 1
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n \\
\delta
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} K_L \left( e^{u_1+4} + e^{-(u_1+4)} \right) \\
0 \\
\vdots \\
0 \\
f
\end{bmatrix}
\]

(3.92)

Differentiating both sides of (3.92) with respect to the design variable \( \delta \) gives

\[
\frac{AE}{h} \begin{bmatrix}
1 & -1 & 0 & \ldots & 0 \\
-1 & 2 & -1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 2 & -1 \\
0 & 0 & \ldots & -1 & 1
\end{bmatrix} \begin{bmatrix}
u_{1,\delta} \\
u_{2,\delta} \\
\vdots \\
u_{n,\delta} \\
1
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} K_L u_{1,\delta} \left( e^{u_1+4} - e^{-(u_1+4)} \right) \\
0 \\
\vdots \\
0 \\
f,\delta
\end{bmatrix}
\]

(3.93)
The linear system (3.93) can be solved for the first order sensitivities \( \{u_1,\delta, u_2,\delta, \ldots , u_n,\delta, f,\delta \} \). To calculate the second order sensitivities, (3.93) is differentiated with respect to \( \delta \).

\[
\frac{AE}{h} \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 2 & -1 \\
0 & 0 & \cdots & -1 & 1
\end{bmatrix}
\begin{bmatrix}
u_{1,\delta\delta} \\
u_{2,\delta\delta} \\
\vdots \\
u_{n,\delta\delta} \\
f,\delta\delta
\end{bmatrix}
= 
\frac{1}{2} K_L \left[ u_{1,\delta\delta} \left( e^{u_{1,\delta} + i} - e^{-(u_{1,\delta} + i)} \right) + u_{1,\delta}^2 \left( e^{u_{1,\delta} + i} + e^{-(u_{1,\delta} + i)} \right) \right]
\begin{bmatrix}
0 \\
\vdots \\
0 \\
f,\delta\delta
\end{bmatrix}
\] (3.94)

Rearranging (3.94) leads to,

\[
\frac{AE}{h} \begin{bmatrix}
1 - \frac{1}{2} K_L (e^{u_{1,\delta} + i} - e^{-(u_{1,\delta} + i)}) & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 2 & 0 \\
0 & 0 & \cdots & -1 & -\frac{h}{AE}
\end{bmatrix}
\begin{bmatrix}
u_{1,\delta\delta} \\
u_{2,\delta\delta} \\
\vdots \\
u_{n,\delta\delta} \\
f,\delta\delta
\end{bmatrix}
\]
\[
\frac{1}{2} K_L \left[ u_{1,h}^2 \left( c^{u_1+h} + c^{-(u_1+h)} \right) \right] = \begin{cases} 
0 \\
\vdots \\
0 \\
0 
\end{cases}
\] (3.95)

The second order sensitivities are obtained from (3.95). As in the previous two benchmarks, the constraint function is given by

\[
g(u_1) \equiv u_1 + C^* \quad (3.96)
\]

Therefore,

\[
g_{,\delta}(u_1) = \frac{dg}{du_1} \frac{du_1}{d\delta} = u_{1,\delta} \quad (3.97)
\]

and

\[
g_{,\delta\delta}(u_1) = \frac{d^2g}{d\delta^2} = u_{1,\delta\delta} \quad (3.98)
\]

Finite Element Results:

As indicated in section (3.3), although the bar notations \( A, E, L \) and \( K_L \) are used in this benchmark, they do not represent any physical problem. This is simply because of the unrealistic boundary conditions involving the exponential functions. The same values, \( A = 1 \text{ in}^2 \), \( E = 10^7 \) psi, \( K_L = 10^4 \text{ lb/in.} \), \( L = 10 \text{ in.} \) and \( C^* = 2 \text{ in.} \) are used. The analytical solution was presented previously in equation (3.41) as

\[
u(x) = -0.00103763x - 3.72652 
\] (3.99)
which corresponds to the scaled minimum value $J_{\text{min}}/K_L = 3.877513$. In this example we employed 10 equally sized elements and $\lambda = 10^3$ as the penalty parameter. The initial guess for the design variable (to initialize the Newton’s method) was $\delta = -1$. After 5 iterations, the quantity $\left| \frac{dJ}{d\delta} / \lambda \right|$ fell below $10^{-4}$ at which time the process was terminated. The analytical solution and the final iteration solution are depicted in Figure 3.21. The scaled plot of the function $J(\delta)$ which exhibits the local minimum and the approximation to it is presented in Figure 3.22.

**BENCHMARK 4.**

**Direct Differentiation Results:**

The global finite element system, after imposing the boundary conditions, is

$$
\frac{AE}{h} \begin{bmatrix}
1 & -1 & 0 & \ldots & 0 \\
-1 & 2 & -1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 2 & -1 \\
0 & 0 & \ldots & -1 & 1
\end{bmatrix} \begin{bmatrix}
\ddots \\
\vdots \\
\vdots \\
\ddots \\
\ddots \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
\vdots \\
\vdots \\
\vdots \\
0 \\
R_{n+1}
\end{bmatrix}
$$

(3.100)

It was assumed that the concentrated load $f$ (which is also the design variable) was applied at $x_f$ coinciding with an existing mesh point. The last entry on the right side of (3.100), i.e. $R_{n+1}$ is the unknown reaction force at the wall.
Differentiating both sides of (3.100) with respect to the design variable \( f \) yields:

\[
\begin{align*}
\frac{AE}{h} \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 2 & -1 \\
0 & 0 & \cdots & -1 & 1
\end{bmatrix} \begin{bmatrix}
u_{1,f} \\
u_{2,f} \\
\vdots \\
u_{n,f} \\
0
\end{bmatrix} &= \begin{bmatrix}
0 \\
\vdots \\
1 \\
\vdots \\
0
\end{bmatrix} \\
\text{(3.101)}
\end{align*}
\]

This system can be solved for \( \{u_{1,f}, u_{2,f}, \cdots, u_{n,f}, R_{n+1,f}\} \), i.e., first order sensitivities. To arrive at the second order sensitivities, (3.101) is differentiated for the second time with respect to \( f \).

\[
\begin{align*}
\frac{AE}{h} \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 2 & -1 \\
0 & 0 & \cdots & -1 & 1
\end{bmatrix} \begin{bmatrix}
u_{1,ff} \\
u_{2,ff} \\
\vdots \\
u_{n,ff} \\
0
\end{bmatrix} &= \begin{bmatrix}
0 \\
\vdots \\
0 \\
\vdots \\
0
\end{bmatrix} \\
\text{(3.102)}
\end{align*}
\]

Rearranging (3.102) gives:

\[
\begin{align*}
\frac{AE}{h} \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 2 & 0 \\
0 & 0 & \cdots & -1 & -\frac{h}{AE}
\end{bmatrix} \begin{bmatrix}
u_{1,ff} \\
u_{2,ff} \\
\vdots \\
u_{n,ff} \\
R_{n+1,ff}
\end{bmatrix} &= \begin{bmatrix}
0 \\
\vdots \\
0 \\
\vdots \\
0
\end{bmatrix} \\
\text{(3.103)}
\end{align*}
\]
Therefore, the solution of (3.103) indicates that all second order sensitivities are zero.

In this problem, the constraint function is given by,

\[ g(u_1) \equiv u_1 + C^* \]  \hspace{1cm} (3.104)

Therefore

\[ g_f(u_1) = \frac{dg}{du_1} \frac{du_1}{df} = u_{1,f} \]  \hspace{1cm} (3.105)

and

\[ g_{ff} \equiv 0 \]  \hspace{1cm} (3.106)

**Finite Element Results:**

The bar in question has the following parameters associated with it, \( A = 0.1 \) in.\(^2\), \( E = 10^7 \) psi, \( L = 100 \) in., \( C^* = 10 \) in. and \( x_f = 50 \) in. Substituting these parameters into the analytical solution (3.50) gives,

\[ u(x) = \begin{cases} 
-10 & 0 \leq x \leq 50 \\
0.2x - 20 & 50 \leq x \leq 100 
\end{cases} \]  \hspace{1cm} (3.107)

and the scaled minimum objective function value \( J_{\text{min}}/AE = 2 \). For FEM calculations we used 10 equally sized elements and \( \lambda = 10^5, 10^6 \) and \( 10^7 \). The stopping criterion was based on \( \left| \frac{dJ}{df}/\lambda \right| \leq 10^{-8} \). For the present benchmark, \( f \) is the design variable and the initial value \( f = -100 \) lb. was used for the Newton’s method. The results of the final iterations for the different penalty parameters and the analytical solution are presented in Figure 3.23. The scaled plot of \( J(f) \) is given in Figure
3.24. Note that the convergence to the exact value is achieved from outside of the feasible region. Figure 3.25 is the same as Figure 3.24 but on a different horizontal scale.

**BENCHMARK 5.**

**Direct Differentiation Results:**

The previous problem becomes more interesting by placing a spring of stiffness $K_L$ on the right hand side. The global finite element system, after imposing the boundary conditions, takes the form

$$
\frac{AE}{h} \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 2 & -1 \\
0 & 0 & \cdots & -1 & 1 \\
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n \\
0 \\
\end{bmatrix} = \begin{bmatrix}
-K_L u_1 \\
u_2 \\
\vdots \\
u_n \\
R_{n+1} \\
\end{bmatrix} \tag{3.108}
$$

Initially it was assumed that the concentrated load $f$ was applied at $x_f$ which coincided with a node. Naturally, $R_{n+1}$ is the unknown reaction force at the wall. Differentiating (3.108) with respect to the design variable $f$ yields.

$$
\frac{AE}{h} \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 2 & -1 \\
0 & 0 & \cdots & -1 & 1 \\
\end{bmatrix} \begin{bmatrix}
u_{1,f} \\
u_{2,f} \\
\vdots \\
u_{n,f} \\
0 \\
\end{bmatrix} = \begin{bmatrix}
-K_L u_{1,f} \\
u_{2,f} \\
\vdots \\
u_{n,f} \\
R_{n+1,f} \\
\end{bmatrix} \tag{3.109}
$$
The above system of linear equations can be solved for the first order sensitivities \( \{u_{1,s}, u_{2,s}, \ldots, u_{n,s}, f_s\} \). To obtain the second order sensitivities, (3.109) is differentiated again

\[
\frac{AE}{h} \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 2 & -1 \\
0 & 0 & \cdots & -1 & 1
\end{bmatrix} \begin{bmatrix}
u_{1,ff} \\
u_{2,ff} \\
\vdots \\
u_{n,ff} \\
0
\end{bmatrix} = \begin{bmatrix}
-K_L u_{1,ff} \\
\vdots \\
0 \\
\vdots \\
R_{n+1,ff}
\end{bmatrix}
\] (3.110)

It is easy to show that second order sensitivities become zero. As in the previous benchmark problems,

\[
g(u_1) \equiv u_1 + C^* \tag{3.111}
\]
i.e.

\[
g_f(u_1) = \frac{dg}{du_1} \frac{du_1}{df} = u_{1,f} \tag{3.112}
\]
and

\[
g_{ff} = 0 \tag{3.113}
\]

**Finite Element Results:**

The parameters associated with the bar are, \( A = 0.1 \text{ in.}^2 \), \( K_L = 10^4 \text{ lb/in.} \), \( E = 10^7 \text{ psi} \), \( L = 100 \text{ in.} \), \( C^* = 10 \text{ in.} \) and \( x_f = 50 \text{ in.} \). Based on these values, the analytical solution (3.60) becomes

\[
u(x) = \begin{cases}
-0.1x - 10 & 0 \leq x \leq 50 \\
0.3x - 30 & 50 \leq x \leq 100
\end{cases} \tag{3.114}
\]
The scaled minimum objective function is \( J_{\text{min}}/AE = 0 \). The FEM calculations used 10 equally sized elements along with the penalty parameters \( \lambda = 10^7 \) and \( 10^8 \). The stopping criterion was based on \( \left| \frac{df}{df}/\lambda \right| \leq 10^{-9} \). The initial value \( f = -100 \) for Newton’s method was employed. Figure 3.26 demonstrates the analytical solution together with the outcome of final iterations. The scaled plots of \( J(f) \) in two different ranges are shown in Figures 3.27 and 3.28. Convergence is achieved from the outside of the feasible region.

We also experimented with the case where the position of the applied concentrated load did not coincide with a mesh point. The value \( x_f = 55 \) in. was chosen and the nodes arranged such that they did not include \( x_f \). We considered only one penalty parameter \( \lambda = 10^6 \) with the stopping criterion \( \left| \frac{df}{df}/\lambda \right| \leq 10^{-9} \). Although 10 elements were still employed for discretizing the bar, the three elements immediately surrounding the point \( x_f \) were of different sizes. If we denote the size of the element containing \( x_f \) by \( h \), the results as \( h \) became smaller are indicated in Figures 3.29 and 3.30. These results indicate that in order to get an accurate approximation, one must ensure that \( x_f \) corresponds to a mesh point.
CHAPTER 4

OPTIMUM DESIGN OF BEAMS

4.1 THE GOVERNING EQUATIONS AND BOUNDARY CONDITIONS.

In this chapter, we will discuss the optimum design of beams by using the finite element method and sensitivities analysis. A beam is a long, slender structural member generally subjected to transverse loading that produces significant bending effects as opposed to twisting or axial effects. This bending deformation is measured by transverse displacement and rotation.

Before we get into further discussion, some sign conventions will be introduced:

1. Moments are positive in the counterclockwise direction.
2. Rotations are positive in the counterclockwise direction.
3. Forces are positive in the positive y direction.
4. Displacements are positive in the positive y direction.

Figure 4.1 shows a beam element with positive nodal displacements, rotations, forces and moments. Figure 4.2 shows the beam theory sign conventions for shear forces and bending moments.

In Figure 4.3, a beam is supported on an elastic foundation which can be represented by a series of linear elastic springs having uniform spring constant $k$.

Equilibrium condition results in the differential equation governing the linear-elastic beam on elastic foundation as [Desai (1979)]:

$$\frac{d^2}{dx^2} \left( EI \frac{d^2 u}{dx^2} \right) + k u = w$$

(4.1)

where $E$ is the Young's modulus, $I$ is the principal moment of inertia, $u(x)$ is the displacement, $w(x)$ is the distributed loading (force/length) and $k$ is spring constant of the elastic foundation. If the elastic foundation is not present, $k = 0$ and equation
(4.1) becomes (see Figure 4.4).

\[
\frac{d^2}{dx^2} \left( EI \frac{d^2 v}{dx^2} \right) = w \tag{4.2}
\]

In Figures 4.3 and 4.4, the boundary conditions are not depicted. In order to solve the beam equation uniquely, we need to specify suitable boundary conditions at the end points. Some common boundary conditions are shown in Figure 4.5 - 4.8 and mathematically represented as following:

a) Clamped at both ends (Figure 4.5):

\[
\begin{align*}
    v(0) &= 0 \\
    \frac{dv}{dx} \bigg|_{x=0} &= 0 \\
    v(L) &= 0 \\
    \frac{dv}{dx} \bigg|_{x=L} &= 0
\end{align*} \tag{4.3}
\]

b) Clamped on the left and loaded on the right end (Figure 4.6):

\[
\begin{align*}
    v(0) &= 0 \\
    \frac{dv}{dx} \bigg|_{x=0} &= 0 \\
    EI \frac{d^2 v}{dx^2} \bigg|_{x=L} &= 0 \\
    -EI \frac{d^2 v}{dx^2} \bigg|_{x=L} &= p
\end{align*} \tag{4.4}
\]

where \( p \) is external load applied on the right end of the beam.

c) Clamped on the left and loaded on the right in the presence of spring support (Figure 4.7):

\[
\begin{align*}
    v(0) &= 0 \\
    \frac{dv}{dx} \bigg|_{x=0} &= 0 \\
    EI \frac{d^2 v}{dx^2} \bigg|_{x=L} &= 0 \\
    -EI \frac{d^2 v}{dx^2} \bigg|_{x=L} &= p - K_R v(L)
\end{align*} \tag{4.5}
\]
d) Springs at both ends (Figure 4.8):

\[
\begin{align*}
-El \frac{d^2 v}{dx^2} \bigg|_{x=0} &= 0 \\
El \frac{d^2 v}{dx^2} \bigg|_{x=0} &= -K_L v(0) \\
El \frac{d^2 v}{dx^2} \bigg|_{x=L} &= 0 \\
-El \frac{d^2 v}{dx^2} \bigg|_{x=L} &= -K_R v(L)
\end{align*}
\]
4.2 **FINITE ELEMENT FORMULATION.**

We assume that the transverse displacement variation within a typical element as shown in Figure 4.9 can be represented as

\[ v(\tilde{x}) \approx d_1 N_1(\tilde{x}) + \phi_1 N_2(\tilde{x}) + d_2 N_3(\tilde{x}) + \phi_2 N_4(\tilde{x}) \]  \hspace{1cm} (4.7)

where \( N_1(\tilde{x}) \), \( N_2(\tilde{x}) \), \( N_3(\tilde{x}) \) and \( N_4(\tilde{x}) \) are the standard cubic shape functions and \( \{d_1, \phi_1, d_2, \phi_2\} \) are the nodal degrees of freedom. The cubic variation is only an approximation. \( d_1 \) and \( \phi_1 \) are the displacement and rotation at node 1 and \( d_2 \) and \( \phi_2 \) are the displacement and rotation at node 2. Furthermore, (Desai (1979))

\[ N_1(x) = \frac{(2x^3 - 3x^2 L + L^3)}{L^3} \]
\[ N_2(x) = \frac{(x^3 L - 2x^2 L^2 + xL^3)}{L^3} \]
\[ N_3(x) = \frac{(-2x^3 + 3x^2 L)}{L^3} \]  \hspace{1cm} (4.8)
\[ N_4(x) = \frac{(x^2 L - x^2 L^2)}{L^3} \]

where \( L \) is the length of the element.

We will now use Galerkin’s method to formulate the beam element stiffness equations. The basic differential equation (4.1) with transverse loading \( w(x) \) is

\[ \frac{d^2}{dx^2} \left( EI \frac{d^2 v}{dx^2} \right) + kv = w \]  \hspace{1cm} (4.9)

Applying Galerkin’s criterion, one has

\[ \int_0^L \left( EI \frac{d^4 v}{dx^4} + kv - w \right) N_i d\tilde{x} = 0 \]  \hspace{1cm} (4.10)

where the shape functions \( N_i \) are defined by (4.8). Finally, we get the local 4 \( \times \) 4 equilibrium equation (Desai (1979))

\[ \left[ \left[ K^{(e)} \right] + \left[ B^{(e)} \right] \right] \left[ \phi^{(e)} \right] = \left[ F^{(e)} \right] + \left[ T^{(e)} \right] \]  \hspace{1cm} (4.11)

50
where

\[ K_{ij}^{(e)} = \int_0^{L_e} EI \frac{d^2 N_i}{dx^2} \frac{d^2 N_j}{dx^2} dx \quad 1 \leq i \leq 2, \quad 1 \leq j \leq 2 \]

\[ B_{ij}^{(e)} = \int_0^{L_e} K N_i N_j dx \quad 1 \leq i \leq 2, \quad 1 \leq j \leq 2 \]

\[ \{d^{(e)}\} = \{d_1^{(r)}, \phi_1^{(r)}, d_2^{(r)}, \phi_2^{(r)}\}^T \]

\[ \{T^{(e)}\} = \{f_1^{(r)}, m_1^{(r)}, f_2^{(r)}, m_2^{(r)}\}^T \]

\[ \{F^{(e)}\} = \begin{bmatrix} \int_0^{L_e} w(\hat{x}) N_1(\hat{x}) d\hat{x} \\ \int_0^{L_e} w(\hat{x}) N_2(\hat{x}) d\hat{x} \\ \int_0^{L_e} w(\hat{x}) N_3(\hat{x}) d\hat{x} \\ \int_0^{L_e} w(\hat{x}) N_4(\hat{x}) d\hat{x} \end{bmatrix} \]

(4.12)

where the superscripts 'e' represents a standard element. The nodal shear forces and moment are illustrated in Figure 4.10. \{F^{(e)}\} and \{T^{(e)}\} are the load vectors due to the distributed load and the concentrated nodal forces and \(L_e = x_2 - x_1\) is the element size. Assembling the element equation, we can obtain the global equilibrium system as follow:

\[ \{[[K] + [B]]\} \{d\} = \{F\} + \{T\} \]

(4.13)

Imposing the boundary conditions, system of eqs. (4.13) can be uniquely solved for the unknown displacements, rotations, reaction forces and moments.
4.3 BEAM OPTIMIZATION PROBLEMS, ANALYTICAL SOLUTIONS.

In this section, we will discuss several benchmark optimization problems for beams together with their analytical solutions which will be used for verification purposes.

BENCHMARK 0.

Consider the beam as shown in Figure 4.11 where $v(L)$ is denoted by $\delta$ and the exerted load to cause the deformation is $p$. Suppose that the goal is to find $p$ such that the external work given by $J(p) = p\delta$ is minimized. This problem can be solved analytically. The governing equation and the associated boundary conditions are,

\[
\begin{align*}
\frac{d^2}{dx^2} \left( EI \frac{d^2 v}{dx^2} \right) &= w \\
\left. \frac{dv}{dx} \right|_{x=0} &= 0 \\
\left. EI \frac{d^2 v}{dx^2} \right|_{x=L} &= 0 \\
\left. -EI \frac{d^3 v}{dx^3} \right|_{x=L} &= p
\end{align*}
\]  

(4.14)

The exact solution to the problem is

\[
v(x) = \frac{-p}{6EI} x^3 + \frac{pL}{2EI} x^2
\]  

(4.15)

Thus, $\delta = v(L) = pL^3/3EI$ and the expression for $J$ becomes,

\[
J(p) = p\delta = \frac{L^3}{3EI} p^2
\]  

(4.16)

Clearly, the minimum occurs at the point $p = 0$. This means that no external force requires no external work. The graph of $J$ vs $p$ is shown in Figure 4.12.
BENCHMARK 1.

Now, we consider the linear-elastic beam as shown in Figure 4.11. The problem presented here has an inequality constraint. Let $p$ and $v(L)$ be the required force and displacement at the right end respectively. Denoting the external work due to the deformation by $J = pv(L)$, the objective is to solve the following constrained optimization problem,

$$\begin{align*}
\begin{cases}
\min_{p} & J \\
\text{subject to } & v'(L) \leq -C^* 
\end{cases}
\end{align*}$$

(4.17)

where $C^*$ is a fixed positive constant. $v(x)$ is the displacement function and $v'(x)$ is the rotation function. Benchmark 0 gave the exact solution to the boundary value problem with no constraints imposed. However, one should formulate the rotation constraint presented in (4.17). We begin with,

$$v(x) = -\frac{p}{6EI}x^3 + \frac{pL}{2EI}x^2$$

(4.18)

The expression for $J$ is,

$$J(p) = \frac{L^3}{3EI}p^2$$

(4.19)

The constraint $v'(L) \leq -C^*$ translates into,

$$\frac{pL^2}{2EI} \leq -C^*$$

(4.20)

i.e.,

$$p \leq -\frac{2C^*EI}{L^2}$$

(4.21)

If we denote the right hand side of inequality (4.20) by $-p^*$, then the original optimization problem (4.17) takes the form,

$$\begin{align*}
\begin{cases}
\min_{p} & J \\
\text{subject to } & p \leq -p^* 
\end{cases}
\end{align*}$$

(4.22)
The graph of $J$ together with the feasible region are depicted in Figure 4.13 and clearly the minimum occurs at the boundary of the feasible region where $p = -p^*$. Furthermore, the corresponding minimum value for $J$ is

$$J_{\text{min}} = \frac{4EI}{3L} (C^*)^2$$  \hfill (4.23)

The optimal exact solution will be obtained by replacing $p = -p^*$ in the expression (4.18),

$$v(x) = -\frac{p^*}{6EI} x^3 + \frac{-p^*L}{2EI} x^2$$  \hfill (4.24)

where $p^* = 2C^*EI/L^2$.

**BENCHMARK 2.**

The problem considered here is more complicated and different from the previous two benchmarks in many aspects. The left end of the beam is clamped, the right end rests on a linear elastic spring with modulus $K_R$. The deformation is due to a concentrated load $w$ at the interior section of the beam at the location $x = x_w$ and an external load $p$ at right end of the beam is shown in Figure 4.14. The objective is to minimize the function $J = wv(x_w)$ subject to the rotation constraint at the right end $(x = L)$. Note that the total external work is $(wv(x_w) + pv(L))$. The optimization problem is.

$$\begin{aligned}
\min_w & \quad J(w) \\
\text{subject to} & \quad v'(L) \leq -C^*
\end{aligned}$$  \hfill (4.25)

where $C^*$ is a positive fixed constant. The design variable is the external load $w$ where the other concentrated load $p$ is kept fixed.
The governing boundary value problem can be obtained with the help of function $H(x - x_w)$ as following.

\[
\begin{align*}
\frac{d^2}{dx^2}EI\frac{d^2 v}{dx^2} &= wH(x - x_w) \quad 0 < x < L, \quad x \neq x_w, \\
v(0) &= 0 \\
\left. \frac{dv}{dx} \right|_{x=0} &= 0 \\
\left. EI\frac{d^2 v}{dx^2} \right|_{x=L} &= 0 \\
\left. -EI\left(\frac{d^2 v}{dx^3}\right) \right|_{x=L} &= p - K_R v(L)
\end{align*}
\] (4.26)

The exact solution of (4.26) is a piecewise cubic function. Imposing the boundary conditions and the smoothness property of the function at point $x_w$, we can determine the coefficients of the piecewise cubic polynomial. Clearly, the displacement, rotation and the moment are continuous at $x_w$. The exact solution can be obtained as follow:

\[
v(x) = \begin{cases} 
  a_1 x^3 + a_2 x^2 & 0 \leq x \leq x_w \\
  b_1 x^3 + b_2 x^2 + b_3 x + b_4 & x_w \leq x \leq L
\end{cases}
\] (4.27)

where

\[
\begin{align*}
a_1 &= b_1 - \frac{w}{6EI} \\
a_2 &= b_2 + \frac{wx_w}{2EI} \\
b_1 &= -\frac{\{6pEI - K_RW_x^2 [3L - x_w]\}}{12EI (3EI + K_R L^3)} \\
b_2 &= \frac{\{6pEI - K_RW_x^2 [3L - x_w]\}}{4EI (3EI + K_R L^3)} L \\
b_3 &= \frac{wx_w}{2EI} \\
b_4 &= -\frac{wx_w^3}{6EI}
\end{align*}
\] (4.28)
The displacement at the point \( x_w \) is therefore,

\[
v(x_w) = b_1 x_w^3 + b_2 x_w^2 + b_3 x_w + b_4 \tag{4.29}
\]

where \( b_1, b_2, b_3 \) and \( b_4 \) are defined in (4.28). Thus, the objective function \( J \) becomes

\[
J(w) = wv(x_w) = w(b_1 x_w^3 + b_2 x_w^2 + b_3 x_w + b_4) \tag{4.30}
\]

The constraint \( v'(L) \leq -C^* \) in term of \( w \) becomes,

\[
v'(L) = 3b_1 L^2 + 2b_2 L + b_3 = b_2 L + b_3
\]

or,

\[
v'(L) = \frac{6EI - K_R w x_w^2 [3L - x_w]}{4EI (3EI + K_R L^3)} L^2 + \frac{w x_w^2}{2EI} \leq -C^* \tag{4.31}
\]

Assuming that \( 6EI + L^2 x_w - K_R L^3 > 0 \), from (4.31) we can get,

\[
w \leq -\frac{4EI (3EI + K_R L^3) C^* + 6pEIL^2}{x_w^2 (6EI + x_w L^2 - K_R L^3)} \tag{4.32}
\]

Denoting the right hand side of the inequality (4.32) by \(-w^*\), (4.31) becomes,

\[
w \leq -w^* \tag{4.33}
\]

Therefore, the original optimization problem (4.25) takes the form

\[
\begin{align*}
\min_{w} \quad & J(w) \\
\text{subject to} \quad & w \leq -w^*
\end{align*} \tag{4.34}
\]

The graph of \( J \) and the feasible region is similar to beam problem in the benchmark 1 where the optimum solution is at the boundary of the feasible region \( w = -w^* \) and the corresponding minimum \( J \) is,

\[
J_{\min} = J(-w^*) \tag{4.35}
\]

The optimal exact solution is obtained by replacing \( w = -w^* \) in the expressions (4.27) and (4.28).
BENCHMARK 3.

Here, we introduce a more complicated optimum design problem. The beam rests on an elastic foundation which can be represented by a series of linear elastic springs with a spring constant \( k \). The left end is fixed and the right end is free. The distributed load is \( w(x) \) as shown in Figure 4.15. The objective is to maximize the potential energy of the spring foundation subject to a displacement constraint at the right end of the beam. The design variable is the spring constant. If the potential energy of the spring foundation is denoted by \( J \), then

\[
J = \frac{1}{2} \int_0^L kv^2(x) \, dx
\]

The optimization problem becomes

\[
\begin{align*}
\max_k J(k) \\
\text{subject to } v(L) \leq -C^*
\end{align*}
\]

where \( C^* \) is a positive fixed constant. \( k \) is the spring constant which is used as the design variable.

The governing boundary value problem can be written as follow,

\[
\begin{align*}
\frac{d^2}{dx^2} EI \frac{d^2 v}{dx^2} + kv &= w(x) & 0 < x < L \\
v(0) &= 0 \\
\frac{dv}{dx} \bigg|_{x=0} &= 0 \\
\left. EI \frac{d^2 v}{dx^2} \right|_{x=L} &= 0 \\
\left. -EI \frac{d^3 v}{dx^3} \right|_{x=L} &= 0
\end{align*}
\]

In view of the complexity of the formulated problem, a closed form analytical solution cannot be obtained. An accurate numerical solution will be used for verification purposes.
4.4 OPTIMIZATION STRATEGY AND SENSITIVITIES.

The previous section indicated that the type of problems encountered in this chapter have the following different forms.

\[
\begin{aligned}
\min \quad & J(s) \\
\text{subject to} \quad & g(s) \leq 0
\end{aligned}
\]  \quad (4.39)

or

\[
\begin{aligned}
\max \quad & J(s) \\
\text{subject to} \quad & g(s) \leq 0
\end{aligned}
\]  \quad (4.40)

These are constrained optimization problems where the side conditions are of the inequality type. The optimization strategy for (4.39) is the same as in the bar problem. We construct a modified objective function based on the original objective function as,

\[
\hat{J}(s) = J(s) + \lambda [g(s)]^2 H(g(s))
\]  \quad (4.41)

where \( \lambda \) is the penalty parameter and \( H(z) \) is the unit step function defined in the previous chapter. For the problem (4.40), we define the modified objective function as follow,

\[
\hat{J}(s) = J(s) - \lambda [g(s)]^2 H(g(s))
\]  \quad (4.42)

In order to optimize equations (4.41) and (4.42), we use Newton’s method as follow,

\[
s_{n+1} = s_n - \frac{\hat{J}'(s)}{\hat{J}''(s)}
\]  \quad (4.43)

The primes in the above equation refer to differentiation with respect to \( s \). The first and second derivatives of \( \hat{J} \) are calculated as in the bar problems.
The direct differentiation method is used to calculate the following design sensitivities:

\[
\frac{\partial r_1}{\partial \delta} \cdot \frac{\partial r_2}{\partial \delta} \cdot \frac{\partial r_3}{\partial \delta} \cdots \cdot \frac{\partial r_n}{\partial \delta}
\] (4.44)

where \( \delta \) is the design variable. The nature of \( \delta \) depends on the specific problem at hand. For example, in benchmark 1, \( \delta \) is the external load \( p \).

The next step is to apply the direct differentiation method to the global equilibrium system of the beam given below

\[
\{[K] + [B]\} \{v\} = \{F\} + \{T\}
\] (4.45)

Differentiating both sides of (4.45) with respect to \( \delta \) gives,

\[
\{[K] + [B]\} \{v\},_\delta + \left\{[K],_\delta + [B],_\delta\right\} \{v\} = \{F\},_\delta + \{T\},_\delta
\] (4.46)

where the subscript in equation (4.46) refers to differentiation with respect to \( \delta \).

For the benchmark problems 1 and 2 considered in this chapter, the matrix \([B]\) is zero matrix, so \([B],_\delta = [0]\). Also, the stiffness matrix \([K]\) is independent of the deformation and therefore, \([K],_\delta = [0]\). However, for the sake of generality, let us carry the terms \([B]\), \([B],_\delta\) and \([K],_\delta\) throughout. Rearranging the equation (4.46) yields,

\[
\{[K] + [B]\} \{v\},_\delta = \{F\},_\delta + \{T\},_\delta - \left\{[K],_\delta + [B],_\delta\right\} \{v\}
\] (4.47)

The right hand side of equation (4.47) is known and therefore the system can be solved to obtain the first order design sensitivities. For the second order sensitivities, equation (4.47) is differentiated with respect to \( \delta \) and rearranged to produce,

\[
\{[K] + [B]\} \{v\},_{\delta \delta} = \{F\},_{\delta \delta} + \{T\},_{\delta \delta}
\]
- \left\{ [K]_{r,r} + [B]_{r,r} \right\} \{v\} - 2 \left\{ [K]_{r,r} + [B]_{r,r} \right\} \{v\}_r \\

(4.48)

Once again, the right hand side of (4.48) can be obtained and is known, so the system can be solved to compute the second order design sensitivities. We can see that equations (4.45), (4.47) and (4.48) involve the factorization of the same matrix \([K] + [B]\). In the next section, we will consider the individual benchmark problems and the details of calculating different derivatives of \{F\} and \{T\}. 
4.5 RESULTS AND CONCLUSIONS.

The optimization method and sensitivities strategy are parallel to the developments in the bar problem. We use the penalty function approach and Newton’s method to arrive at the final optimal results. In this section, we introduce the numerical calculations of the benchmark problems 1-3 in detail and compare the numerical solutions in benchmarks 1 and 2 with their analytical solutions derived previously. For simplicity, the finite element calculations are performed with equally sized elements denoted by \( h \). It is also assumed that there are \( n \) elements and \( n + 1 \) nodes. Benchmark 0 has a trivial solution and therefore not discussed further.

BENCHMARK 1.

Direct Differentiation Results:

The global finite element system, after imposing the boundary conditions, has the following form,

\[
[K] \{d\} = \{F\} \quad (4.49)
\]

where

\[
\{F\} = \{f_1, m_1, 0, 0, \ldots, 0\}^T
\]

\[
\{d\} = \{0, 0, d_2, \phi_2, \ldots, d_{n+1}, \phi_{n+1}\}^T \quad (4.50)
\]
\[
[K] = \frac{EI}{h^3} 
\]
\[
\begin{bmatrix}
12 & 6h & -12 & 6h \\
4h^2 & -6h & 2h^2 \\
24 & 0 & -12 & -6h \\
8h^2 & -6h & 2h \\
\vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]

Differentiating both sides of equation (4.49) with respect to \( \delta \) (\( \delta = p \)) leads to

\[
[K] \{d\} ,_\delta = \{F\} ,_\delta 
\]

(4.51)

where matrix \([K]\) is defined in (4.50) and

\[
\{F\} ,_\delta = \{f_{1,\delta}, m_{1,\delta}, 0.0, \ldots, 1.0\}^T
\]

\[
\{u\} ,_\delta = \{0.0, d_{2,\delta}, \phi_{2,\delta}, \ldots, d_{n+1,\delta}, \phi_{n+1,\delta}\}^T
\]

(4.52)

The above system can be solved for \(\{d, \phi_{1,\delta}, \ldots, d_{n+1,\delta}, \phi_{n+1,\delta}\}\) and \(\{f_{1,\delta}, m_{1,\delta}\}\).

To obtain the second order sensitivities, equation (4.51) is differentiated with respect to \(\delta\), resulting in

\[
[K] \{d\} ,_{\delta\delta} = \{F\} ,_{\delta\delta}
\]

(4.53)

where \([K]\) is defined in (4.50) and

\[
\{F\} ,_{\delta\delta} = \{f_{1,\delta\delta}, m_{1,\delta\delta}, 0.0, \ldots, 0.0\}^T
\]

\[
\{d\} ,_{\delta\delta} = \{0.0, d_{2,\delta\delta}, \phi_{2,\delta\delta}, \ldots, d_{n+1,\delta\delta}, \phi_{n+1,\delta\delta}\}^T
\]

(4.54)
The system (4.53) clearly has the zero solution vector for \( \{d\}_{n+1} \), and therefore the second order sensitivities vanish in this problem.

We also need to compute the derivative of the constraint function \( g(s) \). For the present problem,

\[
y(\phi_{n+1}) = \phi_{n+1} + C^* \tag{4.55}
\]

Therefore,

\[
g_s(\phi_{n+1}) = \frac{dg}{d\phi_{n+1}} \frac{d\phi_{n+1}}{ds} = \phi_{n+1,s} \tag{4.56}
\]

The second order sensitivities vanish giving

\[
g_{ss} \equiv 0 \tag{4.57}
\]

**Finite Element Results:**

The beam in question has the following parameters associated with it, \( E = 3.0 \times 10^7 \text{ psi} \), \( I = 200 \text{ in.}^4 \), \( L = 100 \text{ in.} \), and \( C^* = 10^{-2} \text{ rad} \). Substituting these parameters in expressions (4.21) and (4.24) results in the analytical solution for displacement,

\[
v(x) = \frac{1}{3 \times 10^5} x^3 - \frac{1}{10^4} x^2 \tag{4.58}
\]

Differentiating (4.58) with respect to \( x \) gives the analytical expression for rotation,

\[
r(x) = \frac{1}{10^6} x^2 - \frac{2}{10^4} x \tag{4.59}
\]

For FEM calculations, we use 5 equally sized elements to model the beam and three different penalty parameters \( \lambda = 10^5, 5 \times 10^7 \text{ and } 10^9 \). The stopping criteria
is $\left| \frac{dJ}{d\delta} / \lambda \right| < 10^{-2}$. For the present benchmark, $\delta$ is the design variable and the initial guess of $\delta = -100$ is used for the Newton's method.

The results of the final iterations for the different penalty parameters and the analytical solutions (4.58) and (4.59) are depicted in Figures 4.16 and 4.17, respectively. The number of iterations necessary to meet the stopping criterion is 1, 1 and 2 respectively.

**BENCHMARK 2.**

**Direct Differentiation Results:**

The global finite element system, after imposing the boundary conditions, takes the form

$$[K]\{d\} = \{F\} + \{T\} \tag{4.60}$$

where

$$\{T\} = \{0, 0, \ldots, w, 0, \ldots, 0\}^T$$

$$\{F\} = \{f_1, m_1, 0, 0, \ldots, p - Kd_{n+1}, 0\}^T \tag{4.61}$$

$$\{d\} = \{0, 0, d_2, \phi_2, \ldots, d_{n+1}, \phi_{n+1}\}^T$$
\[
\begin{bmatrix}
12 & 6h & -12 & 6h \\
4h^2 & -6h & 2h^2 \\
24 & 0 & -12 & -6h \\
8h^2 & -6h & 2h & \\
\end{bmatrix}
\]

\[ [K] = \frac{EI}{h^2} \]

It was assumed that the concentrated load \( w \) (which is also the design variable) was applied at \( x_w \) which coincide with an existing mesh point. Here we assumed that the design variable is \( \delta \) (\( \delta = w \)). Differentiating equation (4.50) with respect to the design variable \( \delta \) results in the system,

\[ [K] \{d\},\delta = \{F\},\delta + \{T\},\delta \quad (4.62) \]

where \([K]\) is defined in (4.61) and

\[ \{T\},\delta = \begin{bmatrix} 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \end{bmatrix}^T \]

\[ \{F\},\delta = \begin{bmatrix} f_{1,\delta} & m_{1,\delta} & 0 & 0 & \cdots & -Kd_{n+1,\delta} & 0 \end{bmatrix}^T \quad (4.63) \]

\[ \{d\},\delta = \begin{bmatrix} 0 & 0 & d_{2,\delta} & \phi_{2,\delta} & \cdots & d_{n+1,\delta} & \phi_{n+1,\delta} \end{bmatrix}^T \]

The above linear system can be solved for \( \{d_{1,\delta}, \phi_{1,\delta}, \cdots, d_{n+1,\delta}, \phi_{n+1,\delta}\} \) and \( \{f_{1,\delta}, m_{1,\delta}\} \) arriving at the first order sensitivities. To obtain the second order sensitivities, equation (4.62) is differentiated again with respect to \( \delta \). The outcome is

\[ [K] \{d\},\delta\delta = \{F\},\delta\delta + \{T\},\delta\delta \quad (4.64) \]
where \([K]\) is defined in equation (4.61) and

\[
\{T\}_h = \{0.0, \ldots, 0.0, 0, 0, 0\}^T
\]

\[
\{F\}_h = \{f_1, f_2, m_1, m_2, 0, 0, \ldots, -Kd_{n+1,h}, 0\}^T
\]

\[
\{d\}_h = \{0, 0, d_{2,h}, \phi_2, \ldots, d_{n+1,h}, \phi_{n+1,h}\}^T
\]  

(4.65)

The system (4.64)-(4.65) clearly has the zero vector as its solution and therefore the second order derivatives vanish in this problem.

The constraint equation represented by (4.25) is given as

\[
g(\phi_{n+1}) \equiv \phi_{n+1} + C^* \tag{4.66}
\]

Therefore,

\[
g_h(\phi_{n+1}) = \frac{dg}{d\phi_{n+1}} \frac{d\phi_{n+1}}{d\delta} = \phi_{n+1,h} \tag{4.67}
\]

and

\[
g_{hh}(\phi_{n+1}) = 0 \tag{4.68}
\]

**Finite Element Results:**

The beam in this example has the following parameters, \(E = 30 \times 10^6 psi, I = 200 \text{ in.}^3, L = 100 \text{ in.}, x_w = 40 \text{ in.}, C^* = -10^{-2} \text{ rad.} \text{ and } p = -2000 \text{ lb.} \). Substituting these parameters in the right hand side of (4.32) gives \(-w^*\). In turn substitution of \(w = -w^*\) in the analytical solution (4.27)-(4.28) gives the following optimal solution.

\[
v(x) = \begin{cases} 
    a_1 x^3 + a_2 x^2 & 0 \leq x \leq x_w \\
    b_1 x^3 + b_2 x^2 + b_3 x + b_4 & x_w \leq x \leq L
\end{cases} \tag{4.69}
\]
where,

\[ a_1 = 1.915215 \times 10^{-6} \]
\[ a_2 = -5.651601 \times 10^{-4} \]
\[ b_1 = 3.2015 \times 10^{-8} \]
\[ b_2 = -9.60452 \times 10^{-6} \]
\[ b_3 = -9.0395 \times 10^{-3} \]
\[ b_4 = 1.205273 \times 10^{-1} \]

In order to obtain the rotation function, (4.69) is differentiated with respect to \( x \).

\[
r(x) = \begin{cases} 
3a_1 x^2 + 2a_2 x & 0 \leq x \leq x_w \\
3b_1 x^2 + 2b_2 x + b_3 & x_w \leq x \leq L 
\end{cases} \quad (4.70)
\]

where \( a_1, a_2, b_1, b_2 \) and \( b_3 \) are defined above. For FEM calculations, 5 equally sized elements are used to model the beam and three different penalty parameters \( \lambda = 10^5 \), \( 10^8 \) and \( 10^9 \). The stopping criteria \( \left| \frac{d^2 f}{dx^2} / \lambda \right| < 10^{-3} \). For the present benchmark, \( \delta (\delta = w) \) is the design variable and the initial guess of \( \delta = -100 \) is used for the Newton’s method.

The results of the final iterations for the different penalty parameters and the analytical solution (4.69) and (4.70) are depicted in Figures 4.18 and 4.19 respectively. The number of iterations necessary to meet the stopping criterion is 1 for the three penalty parameters.

**BENCHMARK 3.**

**Direct Differentiation Results:**
This problem is more complicated than the previous two benchmark problems.

The global finite element system, after imposing the boundary conditions, takes the form

\[
\{[K] + [B]\} \{d\} = \{F\} + \{T\}
\]  

(4.71)

where

\[
[K] = \frac{EI}{h^3}
\]

and

\[
\begin{bmatrix}
12 & 6h & -12 & 6h \\
4h^2 & -6h & 2h^2 \\
24 & 0 & -12 & -6h \\
8h^2 & -6h & 2h & \\
\vdots & \vdots & \vdots & \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
sym & & & & \\
& 4h^2 & -6h & 2h^2 & \\
& 12 & -6h & \\
& 4h^2 & & \\
\end{bmatrix}
\]
\[
[B] = \frac{kh}{420}
\]

\[
\begin{bmatrix}
156 & 22h & 54 & -13h \\
4h^2 & 13h & -3h^2 \\
312 & 0 & 54 & -13h \\
8h^2 & 13h & -3h^2 \\
\end{bmatrix}
\]

\[
\text{sym}
\begin{bmatrix}
8h^2 & 13h & -3h^2 \\
156 & -22h \\
4h^2
\end{bmatrix}
\]

\[
\{F\} = \{f_1, m_1, 0, 0, \cdots, 0, 0\}^T
\]

\[
\{T\} = \text{kw} \left\{ \frac{1}{2}, \frac{h}{12}, 1.0, \cdots, \frac{1}{2}, -\frac{h}{12} \right\}^T
\]

\[
\{d\} = \{0, 0, d_2, \phi_2, \cdots, d_{n+1}, \phi_{n+1}\}^T
\]

(4.72)

It is assumed that \(\delta = k\) is design variable. Differentiating both sides of (4.71) with respect to \(\delta\) gives (\([K]\) is independent of \(\delta\)),

\[
\{(K) + [B]\} \{d\}_{,\delta} = \{F\}_{,\delta} + \{T\}_{,\delta} - [B]_{,\delta} \{d\}
\]

(4.73)

where matrices \([K]\) and \([B]\) are defined as in (4.72) and

\[
\{T\}_{,\delta} = \{0, 0, 0, 0, \cdots, 0, 0\}^T
\]

\[
\{F\}_{,\delta} = \{f_1, \delta, m_1, \delta, 0, 0, \cdots, 0, 0\}^T
\]

\[
\{d\}_{,\delta} = \{0, 0, d_2, \phi_2, \cdots, d_{n+1}, \phi_{n+1}\}^T
\]

(4.74)
\[
\begin{bmatrix}
156 & 22h & 54 & -13h \\
4h^2 & 13h & -3h^2 \\
312 & 0 & 54 & -13h \\
8h^2 & 13h & -3h^2 \\
\end{bmatrix}
\]

\[\begin{array}{c}
\frac{h}{420} \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}\]

\[\text{sym}\]

\[
\begin{bmatrix}
8h^2 & 13h & -3h^2 \\
156 & -22h \\
4h^2 \\
\end{bmatrix}
\]

The above system (4.73), (4.74) can be solved for the first order sensitivities \{d_{2,\delta}, \phi_{2,\delta}, \cdots, d_{n+1,\delta}, \phi_{n+1,\delta}, f_{1,\delta}, m_{1,\delta}\}. To calculate the second order sensitivities, equation (4.73) is differentiated with respect to \(\delta\) and since matrices \([K]\) and \([B]_\delta\) are independent of \(\delta\), we have

\[
\{(K) + [B]\} \{d\}_{,\delta} = \{F\}_{,\delta} - 2[B]_\delta \{d\}_\delta \tag{4.75}
\]

where \([K]\) and \([B]\) are defined in (4.72) and \([B]_\delta\) is defined in (4.74). Furthermore,

\[
\{F\}_{,\delta} = \{f_{1,\delta}, m_{1,\delta}, 0, 0, \cdots, 0, 0\}^T
\]

\[
\{d\}_{,\delta} = \{0, 0, d_{2,\delta}, \phi_{2,\delta}, \cdots, d_{n+1,\delta}, \phi_{n+1,\delta}\}^T \tag{4.76}
\]

The second order sensitivities are obtained by solving (4.75). As in the previous two benchmarks, the constraint equation is given by

\[
g(d_{n+1}) \equiv d_{n+1} + C^* \tag{4.77}
\]

Therefore,

\[
g_{,\delta}(d_{n+1}) = \frac{dg}{dd_{n+1}} \frac{dd_{n+1}}{d\delta} = d_{n+1,\delta} \tag{4.78}
\]
and,

\[ g_{n+1} \sigma d_{n+1} = \frac{d^2 y}{d \delta^2} = d_{n+1,ss} \] (4.79)

**Finite Element Results:**

The beam in this case has the following parameters, \( E = 30 \times 10^6 \) psi, \( I = 200 \) in.\(^2 \), \( L = 100 \) in., \( w = 900 \) lb./in.. Here two separate cases of \( C^* = 1.5 \) in. and 0.5 in. are considered.

For FEM calculations, 5 equally sized elements are used to model the beam and three different penalty parameters \( \lambda = 10^2, 10^3 \) and \( 10^4 \) are taken. The stopping criteria is \( \left| \frac{\partial J}{\partial \delta} / \lambda \right| < 10^{-2} \). For the present benchmark, \( \delta = k \) is the design variable and the initial guess of \( \delta = -800 \) is used for the Newton's method.

**Case 1.** \( C^* = 1.5 \)

The results of the final iterations for different penalty parameters are depicted in Figure 4.20. The number of iterations necessary to meet the stopping criterion is 2, 2 and 5. Looking at the results, we can see that the optimum point occurs at the \( k = k^* \) where \( v(L) = -C^* \). This means that the maximum is obtained on the boundary of the feasible region.

**Case 2.** \( C^* = 0.5 \)

In this situation, the results of final iterations for the three penalty parameters are the same. The constraint condition is satisfied. This means that the maximum is obtained in the feasible region. The result is depicted in Figure 4.21.

As indicated previously, this benchmark problem does not have closed form
analytical solution. We have therefore used a numerical calculation in order to obtain the graph of $J(k)$ vs $k$ which is given by Figure 4.22. There is clearly a global maximum in the vicinity of $k = 750 \text{ lb./in.}$ which corresponds to $v(L) \approx -1 \text{ in.}$.
CHAPTER 5

CONCLUSIONS

This thesis presented an elementary account of the topic of optimum structural design. The emphasis was to explore the effects of the penalty parameters, the nature of convergence and some other practical aspects of the optimum design problem. The thesis dealt only with simple examples (one dimensional bar and beam problems) to see clearly how the finite element method and the direct differentiation process can be used effectively to handle the optimization sensitivity analysis in structures.

The penalty function method has proven to be an effective and accurate tool for constrained optimization. The choice of the penalty function has an impact on the nature of convergence to the optimum solution. The exterior method where $G(s) = \max[0, s]$ was used in this thesis. The convergence of the numerical solution to the optimal solution was seen as the penalty parameter $\lambda$ was increased to infinity. One can also use the interior method where $G(s) = -1/s$, but the penalty parameter $\lambda$ should be decreased to zero.

Newton's method was the primary tool applied in the thesis. In such a method, the initial value has to be reasonable in order to achieve convergence. In higher dimensional problems, a more suitable algorithm such as Davidson-Fletcher-Powell method has to be used.

One can extend these ideas to 2D and 3D optimum designs where more than one design variable is used. Although the focus of the present thesis was on the finite element method, other techniques such as boundary elements and finite differences can also be used for design optimization problems.
References


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Figure 4.19, The Analytical Solution and Final Iterations for Rotation, Beam's Benchmark 2.
Figure 4.20, The Results of Final Iterations For Case 1. Beam's Benchmark 3.

Figure 4.21, The Results Of Final Iterations for Case 2. Beam's Benchmark 3.
Figure 4.22 The Potential Energy of Spring Foundation $J(K)$ vs the Design Variable $K$. 
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