Numerical study of two-dimensional laminar flows using contravariant velocity fluxes on collocated grids in general coordinates.

Hao. Xu
University of Windsor

Follow this and additional works at: https://scholar.uwindsor.ca/etd

Recommended Citation

This online database contains the full-text of PhD dissertations and Masters' theses of University of Windsor students from 1954 forward. These documents are made available for personal study and research purposes only, in accordance with the Canadian Copyright Act and the Creative Commons license—CC BY-NC-ND (Attribution, Non-Commercial, No Derivative Works). Under this license, works must always be attributed to the copyright holder (original author), cannot be used for any commercial purposes, and may not be altered. Any other use would require the permission of the copyright holder. Students may inquire about withdrawing their dissertation and/or thesis from this database. For additional inquiries, please contact the repository administrator via email (scholarship@uwindsor.ca) or by telephone at 519-253-3000ext. 3208.
INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI
A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor MI 48106-1346 USA
313/761-4700  800/521-0600
NUMERICAL STUDY OF 2-D LAMINAR FLOWS USING CONTRAVARIANT VELOCITY FLUXES ON COLLOCATED GRIDS IN GENERAL COORDINATES

by

Hao Xu

A Thesis
Submitted to the Faculty of Graduate Studies and Research through the Department of Mechanical and Materials Engineering in Partial Fulfilment of the Requirements for the Degree of Master of Applied Science at the University of Windsor

Windsor, Ontario, Canada

1996
The author has granted a non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of this thesis in microform, paper or electronic formats.

The author retains ownership of the copyright in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author’s permission.

L’auteur a accordé une licence non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de cette thèse sous la forme de microfiche/film, de reproduction sur papier ou sur format électronique.

L’auteur conserve la propriété du droit d’auteur qui protège cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

0-612-31010-8
© Hao Xu 1996
To

My beloved parents
ABSTRACT

A numerical methodology has been developed to solve steady laminar flows in two-dimensional domains of arbitrary shape using body-fitted coordinates. The finite volume procedure was used to discretize the governing equations. Contravariant velocity fluxes were used as the dependent variables in the momentum equations, which retain a strongly conservative form in general curvilinear coordinates. Since non-staggered grids were used, a momentum interpolation was employed to suppress the pressure oscillation. A hybrid differencing scheme was employed to treat the convection-diffusion terms in the momentum equations. The capability of this method to predict flow characteristics in complex geometries was demonstrated by solving four typical fluid flow problems. They are laminar flows between two concentric cylinders, laminar flows through a channel with gradual expansion, laminar flows inside a tube with a constriction and separated flows in a lid-driven cavity. The SIMPLEC algorithm was adopted in the first three problems while both SIMPLE and SIMPLEC algorithms were adopted in the last problem. Comparisons with exact solutions, bench-mark solutions, experimental data and results from other researchers were performed.
ACKNOWLEDGMENTS

I wish to express my sincere gratitude to my advisor, Dr.C. Zhang, for her excellent guidance, encouragement and support throughout this study.

I also wish to thank the other committee members, Dr. R.M. Barron and Dr. P.N. Kaloni, for their helpful suggestions.

I deeply appreciate the encouragement and the support provided by my parents during this work.

The facilities and assistance provided by the University of Windsor Computer Center are also acknowledged. The work was financially supported through Natural Sciences and Engineering Research Council of Canada Grant Number OGP0105727.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>DEDICATION</td>
<td>iv</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>v</td>
</tr>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>vi</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>vii</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>ix</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>x</td>
</tr>
<tr>
<td>NOMENCLATURE</td>
<td>xiii</td>
</tr>
<tr>
<td>CHAPTER I INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Background</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Motivation</td>
<td>5</td>
</tr>
<tr>
<td>1.3 Objectives of the Present Work</td>
<td>6</td>
</tr>
<tr>
<td>1.4 Outlines of the Thesis</td>
<td>7</td>
</tr>
<tr>
<td>CHAPTER II LITERATURE REVIEW</td>
<td>9</td>
</tr>
<tr>
<td>2.1 Choice of the Velocity Components as the Dependent Variables</td>
<td>9</td>
</tr>
<tr>
<td>2.2 Grid Arrangement</td>
<td>14</td>
</tr>
<tr>
<td>CHAPTER III MATHEMATICAL FORMULATION</td>
<td>17</td>
</tr>
<tr>
<td>3.1 Basic Tensor Analysis</td>
<td>17</td>
</tr>
<tr>
<td>3.2 Transformation of Derivatives</td>
<td>24</td>
</tr>
<tr>
<td>3.3 Governing Equations and Their Transformation</td>
<td>27</td>
</tr>
<tr>
<td>CHAPTER IV SOLUTION PROCEDURE</td>
<td>35</td>
</tr>
</tbody>
</table>
4.1 The Control-Volume Approach 35
4.2 General Discretization Equation 39
4.3 Discretization Equations for Momentum and Continuity Equations 44
4.4 Numerical Algorithm 48
4.5 Momentum Interpolation Method 54
4.6 Treatment of the Boundary Conditions 58
4.7 Overall Solution Procedure 60

CHAPTER V COMPUTATIONAL RESULTS 67

5.1 Laminar Flows between Two Concentric Cylinders 67
5.2 Laminar Flows through a Channel with Gradual Expansion 70
5.3 Laminar Flows inside a Tube with a Constriction 72
5.4 Separated Flows in a Lid-driven Cavity 75

CHAPTER VI CONCLUSIONS AND RECOMMENDATIONS 106

6.1 Conclusions 106
6.2 Recommendations 109

REFERENCES 111

VITA AUCTORIS 116
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>Average Errors (%) in Velocity and Pressure for $\theta = -15^\circ$</td>
<td>81</td>
</tr>
<tr>
<td>5.2</td>
<td>Average Errors (%) in Velocity and Pressure for $\theta = 0^\circ$</td>
<td>81</td>
</tr>
<tr>
<td>5.3</td>
<td>Average Errors (%) in Velocity and Pressure for $\theta = 15^\circ$</td>
<td>82</td>
</tr>
<tr>
<td>5.4</td>
<td>Average Errors (%) in Velocity and Pressure for $\theta = 22.5^\circ$</td>
<td>82</td>
</tr>
<tr>
<td>5.5</td>
<td>Comparison of Wall Pressure Values for $Re = 10$</td>
<td>83</td>
</tr>
<tr>
<td>5.6</td>
<td>Comparison of Wall Pressure Values for $Re = 100$</td>
<td>83</td>
</tr>
<tr>
<td>5.7</td>
<td>Geometries of Model Stenosis</td>
<td>84</td>
</tr>
<tr>
<td>5.8</td>
<td>Comparison of Separation Length</td>
<td>84</td>
</tr>
<tr>
<td>5.9</td>
<td>Comparison of Reattachment Length</td>
<td>84</td>
</tr>
</tbody>
</table>
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>The general curvilinear coordinates</td>
<td>34</td>
</tr>
<tr>
<td>4.1</td>
<td>Locations of the control-volume faces</td>
<td>63</td>
</tr>
<tr>
<td>4.2</td>
<td>One-dimensional case in the computational space</td>
<td>64</td>
</tr>
<tr>
<td>4.3</td>
<td>Grid arrangement</td>
<td>65</td>
</tr>
<tr>
<td>4.4</td>
<td>The computational space</td>
<td>66</td>
</tr>
<tr>
<td>5.1a</td>
<td>Flow between two concentric cylinders: geometry</td>
<td>85</td>
</tr>
<tr>
<td>5.1b</td>
<td>Flow between two concentric cylinders: domain discretization for $\theta = 15^\circ$</td>
<td>85</td>
</tr>
<tr>
<td>5.2a</td>
<td>Velocity distribution along the diagonal for $\theta = -15^\circ$</td>
<td>86</td>
</tr>
<tr>
<td>5.2b</td>
<td>Velocity distribution along the diagonal for $\theta = 0^\circ$</td>
<td>86</td>
</tr>
<tr>
<td>5.2c</td>
<td>Velocity distribution along the diagonal for $\theta = 15^\circ$</td>
<td>87</td>
</tr>
<tr>
<td>5.2d</td>
<td>Velocity distribution along the diagonal for $\theta = 22.5^\circ$</td>
<td>87</td>
</tr>
<tr>
<td>5.3a</td>
<td>Pressure distribution along the diagonal for $\theta = -15^\circ$</td>
<td>88</td>
</tr>
<tr>
<td>5.3b</td>
<td>Pressure distribution along the diagonal for $\theta = 0^\circ$</td>
<td>88</td>
</tr>
<tr>
<td>5.3c</td>
<td>Pressure distribution along the diagonal for $\theta = 15^\circ$</td>
<td>89</td>
</tr>
<tr>
<td>5.3d</td>
<td>Pressure distribution along the diagonal for $\theta = 22.5^\circ$</td>
<td>89</td>
</tr>
<tr>
<td>5.4a</td>
<td>Flow through a channel with gradual expansion: geometric configuration</td>
<td>90</td>
</tr>
<tr>
<td>5.4b</td>
<td>Flow through a channel with gradual expansion:</td>
<td></td>
</tr>
</tbody>
</table>
5.5a Pressure distribution on the wall: Re = 10
5.5b Pressure distribution on the wall: Re = 100
5.6 Velocity vectors for Re = 10
5.7 Isobars for Re = 10
5.8a Flow inside a tube with a constriction: geometric configuration
5.8b Flow inside a tube with a constriction: domain discretization
5.9 Axial variation of the centerline velocity, Re = 40, model M3
5.10 Wall pressure, Re = 40, model M3
5.11 Velocity vectors, Re = 40, model M3
5.12 Isobars, Re = 40, model M3
5.13a Separated flow in a lid-driven cavity: geometric configuration
5.13b Separated flow in a lid-driven cavity: domain discretization for $\beta = 60^\circ$
5.14a Variation of the centerline velocity profiles: $\beta = 45^\circ$, u-component
5.14b Variation of the centerline velocity profiles: $\beta = 45^\circ$, v-component
5.14c Variation of the centerline velocity profiles: $\beta = 30^\circ$, u-component
5.14d Variation of the centerline velocity profiles: $\beta = 30^\circ$, v-component
5.15a Convergence properties at $\beta = 90^\circ$ for SIMPLE:
$\alpha_u = 0.6$, $\alpha_u = 0.7$

5.15b Convergence properties at $\beta = 90^\circ$ for SIMPLE:
$\alpha_u = 0.8$, $\alpha_u = 0.9$

5.15c Convergence properties at $\beta = 60^\circ$ for SIMPLE:
$\alpha_u = 0.6$, $\alpha_u = 0.7$

5.15d Convergence properties at $\beta = 60^\circ$ for SIMPLE:
$\alpha_u = 0.8$, $\alpha_u = 0.9$

5.15e Convergence properties at $\beta = 45^\circ$ for SIMPLE:
$\alpha_u = 0.6$, $\alpha_u = 0.7$

5.15f Convergence properties at $\beta = 45^\circ$ for SIMPLE:
$\alpha_u = 0.8$, $\alpha_u = 0.9$

5.15g Convergence properties at $\beta = 30^\circ$ for SIMPLE:
$\alpha_u = 0.6$, $\alpha_u = 0.7$

5.15h Convergence properties at $\beta = 30^\circ$ for SIMPLE:
$\alpha_u = 0.8$, $\alpha_u = 0.9$

5.16a Convergence properties at $\beta = 90^\circ$ for SIMPLEC

5.16b Convergence properties at $\beta = 60^\circ$ for SIMPLEC

5.16c Convergence properties at $\beta = 45^\circ$ for SIMPLEC

5.16d Convergence properties at $\beta = 30^\circ$ for SIMPLEC
## NOMENCLATURE

<table>
<thead>
<tr>
<th>2-D</th>
<th>two-dimensional</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-D</td>
<td>three-dimensional</td>
</tr>
<tr>
<td>$A_e$, $A_w$, $A_n$, $A_s$, $A_p$</td>
<td>coefficients in the discretization equations for momentum and pressure-correction equations</td>
</tr>
<tr>
<td>$b_p^u$, $b_p^v$</td>
<td>source terms in the discretization equations for Cartesian velocity components (Eqs.(4.21) and (4.22))</td>
</tr>
<tr>
<td>$b_{CURV}^u$, $b_{CURV}^v$</td>
<td>source terms given by Eqs.(4.27) and (4.31) in the discretization equations for contravariant velocity fluxes</td>
</tr>
<tr>
<td>$b_{np}^u$, $b_{np}^v$</td>
<td>all the source terms except the pressure gradient terms in the discretization equations for contravariant velocity fluxes</td>
</tr>
<tr>
<td>$b_p^u$, $b_p^v$</td>
<td>source terms given by Eqs.(4.25e) and (4.29e) in the discretization equations for contravariant velocity fluxes</td>
</tr>
<tr>
<td>$b^\phi$</td>
<td>source term given by Eq.(4.12) in Eq.(4.9)</td>
</tr>
<tr>
<td>$b^\phi_{NO}$</td>
<td>source term given by Eq.(4.14) in Eq.(4.9)</td>
</tr>
<tr>
<td>$b_p^\phi$</td>
<td>source term given by Eq.(4.19) in the</td>
</tr>
</tbody>
</table>

xiii
discretization equation for $\phi$

$B_{11}, B_{12}, B_{21}, B_{22}$

coefficients of the pressure gradient terms in the discretization equations for contravariant velocity fluxes

$\hat{B}_{11}, \hat{B}_{12}, \hat{B}_{21}, \hat{B}_{22}$

given by $B_{11}, B_{12}, B_{21}, B_{22}$ divided by $A_p$, respectively

$C_{11}, C_{12}, C_{21}, C_{22}$

coefficients of the pressure gradient terms in a unified form in the discretization equations for contravariant velocity fluxes

$D$

diffusional conductance

$D^0$

non-orthogonal diffusional conductance given by Eq.(4.15)

$e_\xi, e_\eta$

covariant base vectors in general curvilinear coordinates

$e^\xi, e^\eta$

contravariant base vectors in general curvilinear coordinates

$f$

scalar function

$f^+$

geometric interpolation factor

$F$

flow rate through cell faces

$g$

defined by Eq.(3.11)

$g'$

defined by Eq.(3.15)

$g^i$

contravariant metric components
\( g_{ij} \)  
\( H_u \)  
\( i, j \)  
\( J \)  
\( J^x, J^y \)  
\( J^\xi, J^\eta \)  
\( J_a \)  
\( k \)  
\( L \)  
\( m_p \)  
\( n \)  
\( P \)  
\( POW \)  
\( r \)  
\( R \)  
\( R_0 \)  
\( R_1 \)  
\( R_2 \)  
\( Re \)  

Covariant metric components

Term on the right-hand side of the discretization equation (4.58) for contravariant velocity fluxes

Unit vectors in Cartesian coordinates

Total flux of convection and diffusion

Components of \( J \) in Cartesian coordinates

Components of \( J \) in general curvilinear coordinates

Jacobian of the transformation

Unit vector normal to the 2-D plane

Cavity length

Mass residual of a control volume in the pressure-correction equation

Indicates the governing equation in Cartesian or cylindrical coordinates

Pressure

Power-law scheme

Position vector

Tube radius

Unconstricted tube radius

Inner radius of a cylinder

Outer radius of a cylinder

Reynolds number
\[ \text{Res} \quad \text{residual} \]
\[ \overline{\text{Res}} \quad \text{normalized residual} \]
\[ S^\phi \quad \text{source term in the governing equation for } \phi \]
\[ S^\phi_c, \ S^\phi_p \quad \text{linearized terms of } S^\phi \]
\[ \text{SOU} \quad \text{second-order upwind scheme} \]
\[ u, \ v \quad \text{Cartesian or cylindrical velocity components} \]
\[ \overline{u} \quad \text{average velocity at the inflow boundary} \]
\[ U, \ V \quad \text{contravariant velocity fluxes} \]
\[ U^0, \ V^0 \quad \text{given by Eqs.}(4.25a)-(4.25d) \text{ and } (4.29a)-(4.29d) \text{ respectively} \]
\[ u_L \quad \text{cavity lid velocity} \]
\[ \mathbf{V} \quad \text{velocity vector} \]
\[ V_\xi, \ V_\eta \quad \text{covariant components of velocity vector} \]
\[ V^\xi, \ V^n \quad \text{contravariant components of velocity vector} \]
\[ \text{Vol} \quad \text{cell volume} \]
\[ x, \ y \quad \text{Cartesian and cylindrical coordinates} \]
\[ x_0 \quad \text{half length of the constriction in a tube} \]
\[ x_{out} \quad \text{outflow boundary coordinate in } x \text{ direction in a channel} \]
\[ x_r \quad \text{reattachment length} \]
\[ x_s \quad \text{separation length} \]
\[ x^* = \frac{x}{R_0} \]
γ_i \hspace{2cm} \text{lower boundary coordinate in y direction in a channel}

γ_u \hspace{2cm} \text{upper boundary coordinate in y direction in a channel}

\textbf{Greek symbols}

\begin{align*}
\alpha & \quad \text{underrelaxation factor} \\
\alpha_p & \quad \text{pressure underrelaxation factor} \\
\alpha_u, \ \alpha_v & \quad \text{velocity underrelaxation factor} \\
\beta & \quad \text{angle between the side wall of a cavity and the horizontal line} \\
\delta & \quad \text{height of the constriction in a tube} \\
\delta x, \ \delta y & \quad \text{distances between the grid nodes in the physical space} \\
\delta \xi, \ \delta \eta & \quad \text{distances between the grid nodes in the computational space} \\
\Delta x, \ \Delta y & \quad \text{distances between the cell faces in the physical space} \\
\Delta \xi, \ \Delta \eta & \quad \text{distances between the cell faces in the computational space}
\end{align*}
$\varepsilon$  numerical error defined by Eq.(5.1)

$\bar{\varepsilon}$  average value of $\varepsilon$ defined by Eq.(5.2)

$\theta$  angle between one side of a parallelogram shown in Fig.5.1a and the horizontal line

$\mu$  dynamic viscosity of the fluid

$\nu$  kinematic viscosity of the fluid

$\xi, \eta$  general curvilinear coordinates

$\rho$  density of the fluid

$\phi$  dependent variable

$\phi^{(n-1)}$  value of $\phi$ at the level of last iteration

$\omega$  angular velocity

**Superscripts**

* indicates the imperfect values of the velocity and the pressure

* indicates the correction values of the velocity and the pressure
**Subscripts**

- **computed**  computed value
- **cs**  surface enclosing a given control volume
- **exact**  exact value
- **e, w, n, s**  control volume faces (east, west, north, south)
- **E,W,N,S,NE,NW,SE,SW**  grid points (east, west, north, south, northeast, northwest, southeast, southwest)
- **max**  maximum value
- **min**  minimum value
- **nb**  indicates the neighboring points of P
- **P**  nodal location in the control volume

**Other symbols**

- **▽**  operator
- **∀**  control volume
CHAPTER 1

INTRODUCTION

1.1 Background

Most practical fluid flows occur in complex geometrical configurations. Examples can be found in such diverse areas as aerodynamics, meteorology, nuclear reactor design, turbomachinery and physiology.

Generally, there are two methodologies commonly employed in computational fluid dynamics, namely, finite element method and finite difference (volume) method. For fluid flows in complex geometries, a finite element method appears to be a natural choice due to its intrinsic geometrical flexibility. Although this method can correctly approximate boundary shapes and is now being widely used, some difficulties still exist in the implementation of this method, which include the question of equal order interpolation and proper differencing of the convective terms.

Finite difference (volume) method is another way to simulate fluid
flows in complex geometries. The implementation of this method, compared with that of the finite element method, is very easy and straightforward. However, the applications of this method also have certain difficulties and uncertainties. One of the shortcomings is that it requires the boundaries of the physical domain to be regular and to coincide with the coordinate system, otherwise, an elaborate treatment of the boundaries must be made. An alternative is to generate an orthogonal mesh which fits the boundaries. This approach is numerically attractive if a suitable orthogonal mesh can be obtained. However, it may be difficult to find such a mesh, especially for three-dimensional flows. Another alternative is to use a non-orthogonal mesh. When applying this method, a body-fitted coordinate system is defined as a general curvilinear coordinate system in which the boundaries of the physical domain coincide with a portion or all of a curvilinear coordinate line or surface. When partial differential equations are transformed onto such a coordinate system, finite difference representations can be made by using only the neighboring points regardless of the boundary shape or even its movement. Thus all the computations can be done on a fixed rectangular grid in the transformed (computational) space.

The difference among various numerical methods using non-orthogonal grids is in two aspects. One is the choice of the velocity components as the dependent variables in the momentum equations. The
other difference is the grid arrangement. The dependent variables in the
momentum equations in general curvilinear coordinates can be Cartesian
velocity components or non-Cartesian velocity components. The latter
include contravariant and covariant components, physical contravariant
components and physical covariant components, contravariant projections
(resolutes) and covariant projections.

Mathematically, the governing equations in general curvilinear
coordinates are obtained through two different approaches. One is so-
called partial transformation, in which only the independent coordinate
variables are transformed and the dependent variables are left in the
preselected orthogonal coordinate system. Another is complete
transformation, in which both the independent and dependent variables
are transformed. A partial transformation leads to a strongly conservative
form of the Navier-Stokes equations in general curvilinear coordinates,
which uses the Cartesian velocity components as the dependent
variables. However, since the Cartesian velocity vectors in general do not
align with the coordinate direction, this approach may lead to an
increased numerical diffusion when the angles between the velocity
components and the coordinate surfaces become large. A complete
transformation leads to a weakly conservative form of the Navier-Stokes
equations, which uses the non-Cartesian velocity components as the
dependent variables. The non-Cartesian velocity components change their
directions and tend to follow the grid lines. This feature makes them attractive for highly non-orthogonal grids and geometries with strong curvature. Because the completely transformed equations retain only a weakly conservative form, a strongly conservative form of the partially transformed Navier-Stokes equations was proposed by some researchers [1,2]. This strongly conservative form of the Navier-Stokes equations is simple and can be applied with either Cartesian, contravariant, covariant components or projections as the dependent variables.

Another consideration in numerical simulations is the grid arrangement. To avoid the wavy pressure field that satisfies the momentum equations but is physically unrealistic, the common practice is to use a staggered grid arrangement. In the staggered grid arrangement, the scalar quantities are stored at the main grid points, but the velocity components are stored at the faces or the corners of the control volumes. However, the geometrical simplicity of the non-staggered (collocated) grid arrangement is very attractive and this has led to the development of schemes which attempt to eliminate the cause of the pressure oscillations. There are two kinds of such schemes. One is the momentum interpolation, and the other is the pressure gradient interpolation.
1.2 Motivation

The numerical method which employs a finite difference method together with non-orthogonal, body-fitted coordinate system for irregular geometries has been extensively investigated over the past 20 years and is now being widely used in a variety of fluid flow calculations. Cartesian velocity components, whose form of the Navier-Stokes equations in general curvilinear coordinates is obtained through partial transformation, are more likely chosen as the dependent variables than non-Cartesian velocity components, whose form is obtained through complete transformation. As for the grid arrangement, the staggered arrangement is still considered to be an efficient tool to prevent the splitting of the pressure field. Nevertheless, the non-staggered grid arrangement, in which special treatment must be taken to prevent the pressure oscillation, has achieved great success and found wide applications in numerical computations.

Recently, strongly conservative form of the Navier-Stokes equations in general curvilinear coordinates, where complete transformation is avoided and non-Cartesian velocity components are employed as the dependent variables, has been getting more and more attention. Comparison study between different kinds of velocity components (e.g.,
Cartesian velocity components versus covariant velocity projections) can also be found in the open literature. However, to the best of the author’s knowledge, there are no reports on the numerical computations of the fluid flows using contravariant velocity fluxes as the dependent variables on non-staggered grids in general curvilinear coordinate system. Therefore, it is necessary to carry out such study which may result in some new discoveries.

1.3 Objectives of the Present Work

In the present study, contravariant velocity fluxes are used as the dependent variables together with non-staggered grid arrangement. The objectives of this thesis are:

1. to derive the discretized governing equations in general two-dimensional curvilinear coordinate system using the contravariant velocity fluxes as the dependent variables on non-staggered grids. The momentum equations in general curvilinear coordinates retain a strongly conservative form.

2. to develop a modified version of the momentum interpolation method, proposed by Rhie and Chow [3], in order to prevent the splitting of the pressure field.
3. to apply the present method to different types of fluid flow problems and to compare the present results with the analytical solutions, bench-mark solutions, experimental data and the results obtained by other investigators to show the feasibility and superiority of the present method.

1.4 Outlines of the Thesis

This thesis is organized as follows:

In Chapter II, a review of past work in the area of interest is presented. This review includes two aspects. One is the choice of the velocity components as the dependent variables in the momentum equations using general curvilinear coordinates. The other consideration is the grid arrangement with emphasis on the collocated grids.

In Chapter III, basic tensor analysis is explained. The transformation relationship of the derivatives from the physical space to the computational space is presented. Finally, governing equations in the physical space and their transformation into the computational space are given.

Chapter IV deals with the numerical solution procedure. It begins with a slightly different concept of control volume used in this thesis.
The procedure for discretizing the governing equation for any dependent variable, $\phi$, is explained. Then, discretized momentum equations using contravariant velocity fluxes as the dependent variables and discretized continuity equation are presented. Numerical algorithms for solving the discretized equations are also described. The momentum interpolation method proposed by Rhie and Chow [3] is further modified and used in the present work. The treatment of the boundary conditions is briefly discussed. The chapter is closed with the presentation of the overall solution procedure and the discussion of the criterion and residuals required for convergence.

Chapter V illustrates the application of the proposed method to four different types of fluid flow problems. The first one is the laminar flows between two concentric cylinders. The next is the laminar flows through a channel with gradual expansion. The third one is the laminar flows inside a tube with a constriction. The last one is the separated flows in a lid-driven cavity.

In Chapter VI, conclusions are drawn and recommendations for future work are given.
CHAPTER II

LITERATURE REVIEW

As it has been stated in Chapter I, choice of the velocity components as the dependent variables in the momentum equations and the grid arrangement are the two key aspects associated with the use of a general curvilinear coordinate system. A large quantity of literature exists in this area. The purpose of this chapter is to review the previous works that have a direct bearing on the present study. Accordingly, this review is divided into two categories: choice of the velocity components as the dependent variables and the grid arrangement.

2.1 Choice of the Velocity Components as the Dependent Variables

Study and use of the Cartesian velocity components as the dependent variables in the momentum equations have been carried out by numerous investigators [ref. 3-13]. Vinokur [4] was the first one to
Some researchers chose contravariant components as the dependent variables. Demirdzic et al. [14] presented a finite volume technique which solves the semi-strong form of the Navier-Stokes equations in terms of the physical contravariant velocity components. Demirdzic et al. [15] presented a novel and useful procedure for directly transforming the Cartesian tensor forms of the equations into general coordinates. The transformation relations were used to derive the general coordinate version of the ensemble-averaged Navier-Stokes and turbulent model equations in terms of the physical contravariant velocity components. The methodology was applied to the cross flow in a heated tube bank. Yang et al. [16] transformed the conservation equations into those in curvilinear coordinates and then, by using special properties of the geometry, obtained a set of reasonably simple equations for the parallelepiped geometry. Yang et al. [17] furthered their work in [16], and derived the governing equations in non-orthogonal curvilinear coordinates with contravariant velocity components as the dependent variables through a tensor transformation. The application examples were natural enclosures and horizontally closed cylinders with differentially heated ends. Similarly, Zijlema et al. [18] derived two-dimensional governing equations in non-orthogonal curvilinear coordinates. Contravariant velocity components were chosen as the dependent variables to solve the turbulent flow over a sand dune.
Instead of the contravariant components, covariant components were also chosen by some researchers as the dependent variables. For example, Galea and Markatos [19] used covariant velocity resolutes as the dependent variables. Their work was to establish a mathematical model which can describe aircraft cabin fires. Davidson and Hedberg [20] solved the momentum equations for covariant velocity components. They presented a mathematical derivation of the governing equations in the transformed space where a local coordinate system was set up at each grid point. Two problems of laminar flows were solved.

In the above-mentioned work [14-20], the momentum equations using the non-Cartesian velocity components as the dependent variables retain a weakly conservative form. An interesting way of obtaining a strongly conservative form of the momentum equations was given by Karki [1], in which the discretization equations using the covariant velocity projections as the dependent variables were obtained by an algebraic manipulation of the corresponding equations for the Cartesian velocity components. Any reference to the differential form of the conservation equation for the covariant velocity projections was avoided. A variety of two-dimensional incompressible and compressible fluid flows were selected to test the proposed procedure. The procedure proposed by Karki [1] was also adopted by Karki and Patankar [21] to simulate two-dimensional incompressible and compressible flows, ranging from
subsonic to supersonic, by Tamamidis [22] to simulate two-dimensional subsonic turbulent flows and by Tamamidis and Assanis [23] to solve three-dimensional laminar flows in bent ducts. The strongly conservative form of the momentum equations using other non-Cartesian velocity components as the dependent variables can be obtained in the way similar to that used by Karki [1]. Darr and Vanka [24] derived a strongly conservative formulation using contravariant velocity components as the dependent variables. The method was used to study the separated flows in driven trapezoidal cavities on staggered grids. Yang et al. [2] proposed a general strongly conservative formulation of the Navier-Stokes equations in non-orthogonal curvilinear coordinates. In their proposed technique, the differentiation operators were directly applied to the velocity vector itself, instead of velocity components. The formulation has a clear, simple form and can be applied to the Navier-Stokes equations with either Cartesian, contravariant, covariant components or even velocity resolutes as the dependent variables.

In recent years, more and more research has focused on the strongly conservative form of the Navier-Stokes equations in general non-orthogonal curvilinear coordinates. Melaaen [25,26] formulated and compared two finite-volume methods for calculating flows inside complex geometries. One is based on staggered grid arrangement with covariant velocity projections as the dependent variables and the other one is
based on non-staggered grid arrangement with Cartesian velocity components as the dependent variables. Lee and Chiu [27] employed the strongly conservative form of the Navier-Stokes equations using the covariant velocity projections as the dependent variables to simulate pulsatile flows. To obtain the covariant velocities on the control surfaces, the physical contravariant velocity components at the main grid were first interpolated and then converted to their covariant counterparts. Choi et al. [28] chose Cartesian velocity components as the dependent variables. They employed both the contravariant and covariant velocity components as cell velocities to investigate the effect of different cell-face velocities on solution behavior. Choi et al. [29] then extended their work to three-dimensional incompressible flows in complex geometries. Sharatchandra and Rhode [30] proposed a calculation procedure for flows in complex geometries similar to that proposed by Yang et al. [2]. Covariant velocity projections were chosen as the dependent variables.

2.2 Grid Arrangement

There are two kinds of grid arrangements: staggered and non-staggered (collocated). Since staggered grid arrangement is not adopted in the present work, attention of the review is given to the non-
staggered arrangement.

Vanka et al. [31] proposed a scheme in which a cell-by-cell procedure was used rather than solving the pressure-correction equation. At each pressure location, all the surrounding values of the pressure correction were set to zero. This scheme is not widely accepted because of its intrinsic shortcoming. The first successful attempt to utilize the potential advantages of the non-staggered grid arrangement with curved irregular flow boundaries was made by Rhie and Chow [3]. The key idea to eliminate the pressure oscillations is to employ a special interpolation practice called momentum interpolation for evaluating the cell-face velocities. Majumdar [32] presented more general formulations for momentum interpolation which include underrelaxation factors. He found that the converged result for any flow field considered depends on the underrelaxation factor used for the velocity. He also proposed how to implement the momentum interpolation using an iterative algorithm to achieve a unique solution that is independent of the underrelaxation factor. Miller and Schmidt [33] rigorously developed a momentum interpolation and its implementation in the SIMPLEC algorithm. They estimated the degree of dependence of numerical solutions on the underrelaxation factor and obtained a formulation of momentum interpolation which is independent of the underrelaxation factor for a converged solution in Cartesian coordinates. Over the past 10 years,
momentum interpolation has been widely used for computations of fluid flows on non-staggered grids. Such examples can be found in [11,12,23,28]. Recently, Date [34] proposed a so-called pressure gradient interpolation. In this approach, the problem of the pressure oscillation is eliminated by interpolating the pressure gradient terms in the discretized momentum equations and the cell-face velocities are still evaluated by linear interpolation. It seems that this method may be more attractive than the momentum interpolation because a unique solution which is independent of the underrelaxation factor for a given flow field can always be obtained.

To the author's knowledge, for strongly conservative form of the Navier-Stokes equations, previous research has been mainly limited to the use of covariant velocity components as the dependent variables with either a staggered or a collocated grid arrangement. Papers dealing with contravariant velocity components chosen as the dependent variables together with staggered grid arrangement are relatively few in number. No attempt has been made to study the solution behavior using contravariant velocity fluxes as the dependent variables on collocated grids. Especially, no work has been done to apply the modified Rhie and Chow's momentum interpolation scheme to the strongly conservative form of the Navier-Stokes equations which uses the contravariant velocity fluxes as the dependent variables in general curvilinear coordinates.
CHAPTER III

MATHEMATICAL FORMULATION

In this chapter, basic concepts of tensor analysis in connection with the present work and the transformation relations of the derivatives from the physical space to the computational space are presented. This is followed by the steady two-dimensional governing equations, which are written in Cartesian and cylindrical coordinate systems. The procedure for transforming the governing equations from the physical domain to the computational domain is also presented.

3.1 Basic Tensor Analysis

Let $\xi$ and $\eta$ denote two independent, continuous and single-valued scalar point functions. The functions $\xi$ and $\eta$ can be taken as coordinates. Such coordinates are called General Curvilinear Coordinates. If $r$ denotes the position vector of the point $P(\xi, \eta)$ with respect to an arbitrary origin, then $r$ can be considered as a function of $\xi$ and $\eta$. The partial derivatives of $r$ with respect to $\xi$ and $\eta$ are
known as the covariant base vectors of the curvilinear coordinate system, which can be denoted by $e_\xi$ and $e_\eta$ as shown in Fig.3.1. That is

$$
e_\xi = \frac{\partial r}{\partial \xi}, \quad e_\eta = \frac{\partial r}{\partial \eta} \quad (3.1)$$

Naturally, $r$ is also the function of the Cartesian coordinates $x$ and $y$, and $r = xi + yj$. Thus, Eq.(3.1) can be written as

$$
e_\xi = \frac{\partial x}{\partial \xi}i + \frac{\partial y}{\partial \xi}j, \quad e_\eta = \frac{\partial x}{\partial \eta}i + \frac{\partial y}{\partial \eta}j \quad (3.2)$$

where $i$ and $j$ are the unit vectors in the Cartesian coordinate system.

In the two-dimensional plane, the contravariant base vectors, $e^\xi$ and $e^\eta$, are introduced which are reciprocal to the covariant base vectors, $e_\xi$ and $e_\eta$, that is

$$
e^\xi = \frac{e_\eta \times k}{e_\xi(e_\eta \times k)}, \quad e^\eta = \frac{k \times e_\xi}{e_\xi(e_\eta \times k)} \quad (3.3)$$

where $k$ is the unit vector normal to the two-dimensional plane. Through mathematical manipulation, Eq.(3.3) becomes

$$
e^\xi = \left(\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\right)/Ja, \quad e^\eta = \left(-\frac{\partial y}{\partial \xi} + \frac{\partial x}{\partial \xi}\right)/Ja \quad (3.4)$$
where

\[ Ja = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} \] (3.5)

Ja is the Jacobian of the transformation. \( \mathbf{e}^\xi \) and \( \mathbf{e}^\eta \) can also be expressed as

\[ \mathbf{e}^\xi = \frac{\partial \xi}{\partial x} \mathbf{i} + \frac{\partial \xi}{\partial y} \mathbf{j}, \quad \mathbf{e}^\eta = \frac{\partial \eta}{\partial x} \mathbf{i} + \frac{\partial \eta}{\partial y} \mathbf{j} \] (3.6)

Comparing Eq.(3.6) with Eq.(3.4), we have

\[ \frac{\partial \xi}{\partial x} = \frac{\partial y}{\partial \eta} / Ja \] (3.7a)

\[ \frac{\partial \xi}{\partial y} = -\frac{\partial x}{\partial \eta} / Ja \] (3.7b)

\[ \frac{\partial \eta}{\partial x} = -\frac{\partial y}{\partial \xi} / Ja \] (3.7c)

\[ \frac{\partial \eta}{\partial y} = \frac{\partial x}{\partial \xi} / Ja \] (3.7d)

The usefulness of the base vectors is that they can be used to express a vector in terms of its components in the curvilinear coordinate...
system. This will be shown at the later part of this section. The dot products between covariant and contravariant base vectors are

\[ \mathbf{e}^\xi \cdot \mathbf{e}_n = \mathbf{e}_\xi \cdot \mathbf{e}^n = 0 \]  
\[ \mathbf{e}^\xi \cdot \mathbf{e}_n = \mathbf{e}_n \cdot \mathbf{e}^n = 1 \]  

(3.8a)  
(3.8b)

The covariant metric components \( g_{ij} \) are defined by the following equation

\[ g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \]  

(3.9)

where \( i = 1,2 \) and \( j = 1,2 \). The subscripts \( i \) and \( j \) indicate the coordinates.

Using Eqs. (3.9) and (3.2), we obtain

\[ g_{11} = \mathbf{e}_\xi \cdot \mathbf{e}_\xi = x_\xi^2 + y_\xi^2 \]  
\[ g_{12} = \mathbf{e}_\xi \cdot \mathbf{e}_n = x_\xi x_n + y_\xi y_n = g_{21} \]  
\[ g_{22} = \mathbf{e}_n \cdot \mathbf{e}_n = x_n^2 + y_n^2 \]  

(3.10a)  
(3.10b)  
(3.10c)

where \( x_\xi, x_n, y_\xi \) and \( y_n \) denote the partial derivatives of \( x \) and \( y \) with respect to \( \xi \) and \( \eta \). Let

\[ g = g_{11} g_{22} - g_{12}^2 \]  

(3.11)
Substituting Eqs.(3.10a) to (3.10c) into Eq.(3.11) and comparing with Eq.(3.5), we have

\[Ja = \sqrt{g}\]  
(3.12)

The contravariant metric components \(g^{i\bar{i}}\) are defined by

\[g^{i\bar{j}} = e^i e^j\]  
(3.13)

where the superscripts also indicate the corresponding coordinates, and \(i=1, 2\) and \(j=1, 2\). Substituting Eq.(3.4) into Eq.(3.13) and comparing with Eqs.(3.10a) to (3.10c), \(g^{i\bar{j}}\) can be expressed in terms of \(g_{ij}\) as follows

\[g^{11} = g_{22}(Ja)^2\]  
(3.14a)

\[g^{12} = -g_{12}(Ja)^2 = g^{21}\]  
(3.14b)

\[g^{22} = g_{11}(Ja)^2\]  
(3.14c)

Let

\[g' = g^{11}g^{22} - (g^{12})^2 = \frac{1}{g}\]  
(3.15)
Knowing the two different sets of base vectors, namely, the covariant and contravariant base vectors, any vector, for example the velocity vector \( \mathbf{V} \), can be expressed as follows

\[
\mathbf{V} = V^i \mathbf{e}^i = V_\xi \mathbf{e}^\xi + V_\eta \mathbf{e}^\eta \quad (3.16a)
\]
\[
\mathbf{V} = V^j \mathbf{e}_j = V^\xi \mathbf{e}_\xi + V^\eta \mathbf{e}_\eta \quad (3.16b)
\]

\( V_i \) and \( V^i \) are known as the covariant components and contravariant components of the velocity vector \( \mathbf{V} \), respectively. Dotting Eq.\((3.16a)\) with \( \mathbf{e}_i \) and Eq.\((3.16b)\) with \( \mathbf{e}^i \), and utilizing Eqs.\((3.8a)\) and \((3.8b)\), we obtain

\[
V_i = \mathbf{V} \cdot \mathbf{e}_i \quad (3.17a)
\]
\[
V^i = \mathbf{V} \cdot \mathbf{e}^i \quad (3.17b)
\]

Usually \( V_i \) and \( V^i \) do not have the same magnitude as \( \mathbf{V} \) since both the contravariant base vector \( \mathbf{e}^i \) and the covariant base vector \( \mathbf{e}_i \) are not unit vectors. The relation between the \( V_i \) and \( V^i \) can be obtained by taking the dot product of Eq.\((3.16a)\) with \( \mathbf{e}^i \) and the dot product of Eq.\((3.16b)\) with \( \mathbf{e}_i \).
\[ V^i = g^{ij} V_j \]  
\[ V_i = g_{ij} V^j \]  

On the other hand, \( V \) can be expressed in the Cartesian coordinate system as

\[ V = u i + v j \]  

where \( u \) and \( v \) stand for the Cartesian components of \( V \) in \( x \) and \( y \) directions. Substituting Eq.(3.2) into Eq.(3.16b) and comparing with Eq.(3.19), we obtain the following relation

\[ u = V^\xi x_\xi + V^\eta x_\eta \]  
\[ v = V^\xi y_\xi + V^\eta y_\eta \]  

Solving for \( V^\xi \) and \( V^\eta \), we have

\[ V^\xi = (u y_\eta - v x_\eta) / Ja \]  
\[ V^\eta = (v x_\xi - u y_\xi) / Ja \]  

Using Eqs.(3.7a) to (3.7d) and Eqs.(3.21a), (3.21b), \( V^\xi \) and \( V^\eta \) can also be expressed as
\[ V^\xi = \xi_u \nu + \xi_u v \quad (3.22a) \]
\[ V^\eta = \eta_u \nu + \eta_u v \quad (3.22b) \]

where \( \xi_u, \xi_v, \eta_u \) and \( \eta_v \) denote the partial derivatives of \( \xi \) and \( \eta \) with respect to \( x \) and \( y \).

Before ending this section, we give the expressions for the divergence of a velocity vector and the volume of the element in general curvilinear coordinates. To save the space, they are not derived but will be used in this thesis.

\[ \nabla \cdot V = \frac{1}{Ja} \left[ \frac{\partial (JaV^\xi)}{\partial \xi} + \frac{\partial (JaV^\eta)}{\partial \eta} \right] \quad (3.23) \]

\[ \text{Vol} = \left| \frac{\partial r}{\partial \xi} \times \frac{\partial r}{\partial \eta} \right| \, d\xi \, d\eta = J \, ad\xi \, d\eta \quad (3.24) \]

### 3.2 Transformation of Derivatives

The governing equations presented in the following section contain various derivatives. These derivatives can be calculated easily and accurately in the computational space. The relations between the derivatives in the physical \((x, y)\) space and the derivatives in the transformed \((\xi, \eta)\) space are presented in this section. Since the intent
here is to provide a quick reference for further derivations, most of the algebraic development is omitted.

Let \( f(x, y) \) be a twice continuously differentiable scalar function of \( x \) and \( y \).

The derivatives are transformed using the chain rule. The first derivatives, \( f_x \) and \( f_y \), in the physical space can be evaluated in the computational space via the following equations

\[
\begin{align*}
  f_x &= f_\xi \xi_x + f_\eta \eta_x \\
  f_y &= f_\xi \xi_y + f_\eta \eta_y
\end{align*}
\]  

(3.25a)  

(3.25b)

where the subscripts \( x, y, \xi \) and \( \eta \) denote the partial derivatives of any variable or function with respect to corresponding coordinates. \( \xi_x, \xi_y, \eta_x \) and \( \eta_y \) can be evaluated using Eqs.(3.7a) to (3.7d). Therefore, Eqs.(3.25a) and (3.25b) become

\[
\begin{align*}
  f_x &= (f_\xi y_\eta - f_\eta y_\xi)/Ja \\
  f_y &= (-f_\xi x_\eta + f_\eta x_\xi)/Ja
\end{align*}
\]  

(3.26a)  

(3.26b)

Now, terms such as \( f_\xi, f_\eta, x_\xi, y_\xi, x_\eta, \) and \( y_\eta \) can easily be calculated in the transformed space by utilizing the finite difference representations.

The second derivatives of \( f \) with respect to \( x \) and \( y \) are calculated
by differentiating Eqs.(3.26a) and (3.26b). For example

\[ f_{xx} = \frac{[(f_x)_\eta y_\eta - (f_x)_\xi y_\xi]}{Ja} \] 

(3.27)

Substituting Eq.(3.26a) into Eq.(3.27) and rearranging it, one can get the second derivative of \( f \) with respective to \( x \), which is given by Eq.(3.29a) below. In this process, some new terms appear. These are the derivatives of the Jacobian of the transformation, which can easily be calculated. For example

\[ (Ja)_\xi = (x_\xi y_\eta - x_\eta y_\xi)_\xi = x_{\xi\eta}y_\eta + x_\xi y_{\eta\xi} - x_{\eta\xi}y_\xi - x_\eta y_{\xi\xi} \] 

(3.28)

Now it can be seen that all the derivatives are first and second derivatives with respect to \( \xi \) and \( \eta \) and thus can be calculated in the computational space. The second derivatives of \( f \) in the physical space can be calculated in the computational space as follows

\[
\begin{align*}
  f_{xx} &= (y_\eta^2 f_{\xi\xi} - 2y_\xi y_\eta f_{\xi\eta} + y_\xi^2 f_{\eta\eta})/Ja^2 \\
  &= [(y_\eta^2 y_{\xi\xi} - 2y_\xi y_\eta y_{\xi\eta} + y_\xi^2 y_{\eta\eta})(x_\eta f_\xi - x_\xi f_\eta) \\
  &+ (y_\eta^2 x_{\xi\xi} - 2y_\xi y_\eta x_{\xi\eta} + y_\xi^2 x_{\eta\eta})(y_\xi f_\eta - y_\eta f_\xi)]/Ja^3 \\
  f_{yy} &= (x_\eta^2 f_{\xi\xi} - 2x_\xi x_\eta f_{\xi\eta} + x_\xi^2 f_{\eta\eta})/Ja^2
\end{align*}
\] 

(3.29a)

26
\[ \begin{align*}
+ \left[ (x_\eta^2 y_{\xi\xi} - 2x_\xi x_\eta y_{\xi\eta} + x_\eta^2 y_{\eta\eta})(x_\eta f_\xi - x_\xi f_\eta) \right. \\
+ \left. (x_\eta^2 x_{\xi\xi} - 2x_\xi x_\eta x_{\xi\eta} + x_\eta^2 x_{\eta\eta})(y_\xi f_\eta - y_\eta f_\xi) \right]/Ja^3 \quad (3.29b)
\end{align*} \]

\[ \begin{align*}
f_{xy} &= \left[ (x_\xi y_\eta + x_\eta y_\xi)f_{\xi\eta} - x_\xi y_\eta f_{\xi\xi} - x_\eta y_\xi f_{\xi\eta} \right]/Ja^2 \\
+ \left[ (x_\xi y_{\eta\eta} - x_\eta y_{\xi\eta})/Ja^2 + (x_\eta y_\eta(Ja)_\xi - x_\xi y_\eta(Ja)_\eta)/Ja^3 \right]f_\xi \\
+ \left[ (x_\eta y_{\xi\xi} - x_\xi y_{\eta\eta})/Ja^2 + (x_\xi y_\xi(Ja)_\eta - x_\eta y_\xi(Ja)_\xi)/Ja^3 \right]f_\eta \quad (3.29c)
\end{align*} \]

Eqs. (3.26a), (3.26b), (3.29a), (3.29b) and (3.29c) will be used to transform the governing equations from the physical space to the computational space.

### 3.3 Governing Equations and Their Transformation

The governing equations are written in Cartesian and cylindrical coordinate systems. Because it is considerably easier to solve the transformed equations in the computational space than to solve the governing equations in the physical space, the governing equations must be first transformed from the physical space to the computational space.

The governing equations for two-dimensional, steady-state, incompressible flows are continuity equation and momentum equations written as
Continuity Equation

\[
\frac{\partial (\rho u)}{\partial x} + \frac{\partial (y^n \rho v)}{y^n \partial y} = 0 \quad (3.30)
\]

Momentum Equations

\[
\frac{\partial (\rho uu)}{\partial x} + \frac{\partial (y^n \rho vu)}{y^n \partial y} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x} (\mu \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y} (y^n \mu \frac{\partial u}{\partial y}) \\
+ \frac{\partial}{\partial x} (\mu \frac{\partial u}{\partial y}) + \frac{\partial}{y^n \partial y} (y^n \mu \frac{\partial v}{\partial x}) \quad (3.31)
\]

\[
\frac{\partial (\rho vv)}{\partial x} + \frac{\partial (y^n \rho vv)}{y^n \partial y} = -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x} (\mu \frac{\partial v}{\partial x}) + \frac{\partial}{\partial y} (y^n \mu \frac{\partial v}{\partial y}) \\
+ \frac{\partial}{\partial x} (\mu \frac{\partial v}{\partial y}) + \frac{\partial}{y^n \partial y} (y^n \mu \frac{\partial u}{\partial x}) - \frac{2\mu \nu n}{y^2} \quad (3.32)
\]

where \(n = 1\) for the cylindrical coordinate system, \(n = 0\) for the Cartesian coordinate system, \(\rho\) is the constant density of the fluid, \(\mu\) is the viscosity of the fluid, \(u\) and \(v\) are the velocity components in \(x\) and \(y\) directions in the physical space, respectively, and \(P\) is the pressure.

Eqs. (3.30) to (3.32) can be written in a general form as

\[
\frac{\partial (\rho \phi)}{\partial x} + \frac{\partial (y^n \rho v \phi)}{y^n \partial y} = \frac{\partial}{\partial x} (\mu \frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y} (y^n \mu \frac{\partial \phi}{\partial y}) + S^\phi \quad (3.33)
\]

where \(\phi\) is a dependent variable, and

\[
\phi = 1 \quad \text{for continuity equation}
\]
\[ \phi = u \text{ or } v \quad \text{for } u- \text{ or } v-\text{momentum equation} \]

\( S^\phi \) denotes the source term, where

\[ S^\phi = 0 \quad \text{for continuity equation} \]

\[ S^\phi = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}(\mu \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(y^n \mu \frac{\partial v}{\partial x}) \quad \text{for } u-\text{momentum equation} \]

\[ S^\phi = -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}(\mu \frac{\partial u}{\partial y}) + \frac{\partial}{\partial y}(y^n \mu \frac{\partial v}{\partial y}) - \frac{2\mu \nu}{y^2} \quad \text{for } v-\text{momentum equation} \]

Since only laminar flows are considered, the viscosity, \( \mu \), is the molecular viscosity which is taken as a constant in this study. Therefore, the terms in the momentum equations

\[ \frac{\partial}{\partial x}(\mu \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(y^n \mu \frac{\partial v}{\partial x}) \]

and

\[ \frac{\partial}{\partial x}(\mu \frac{\partial u}{\partial y}) + \frac{\partial}{\partial y}(y^n \mu \frac{\partial v}{\partial y}) - \frac{\mu \nu}{y^2} \]

are zero. This is obtained by utilizing the continuity equation. However, in order to enhance the stability of numerical computation, it is better to retain these terms in their corresponding equations.

In order to save space, only the transformation of the governing equations from the Cartesian coordinate system to the general curvilinear
coordinate system is given here. Interested readers can find the transformation of the governing equations from the cylindrical coordinate system to the general curvilinear coordinate system in [35]. Eq.(3.33) can be written in the following more compact form

\[ \nabla \cdot \mathbf{J} = S^\phi \]  

(3.34)

where \( \mathbf{J} \) is the total flux made up of convection flux and diffusion flux, and is given by

\[ \mathbf{J} = \rho \mathbf{V} \phi - \mu \nabla \phi \]  

(3.35)

In Cartesian coordinates (physical space), \( \mathbf{J} \) can be expressed as

\[ \mathbf{J} = J_x \mathbf{i} + J_y \mathbf{j} \]  

(3.36)

where

\[ J_x = \rho u \phi - \mu \frac{\partial \phi}{\partial x} \]  

(3.37a)

\[ J_y = \rho v \phi - \mu \frac{\partial \phi}{\partial y} \]  

(3.37b)

According to Eqs.(3.26a) and (3.26b),
\[
\frac{\partial \phi}{\partial x} = (\Phi_\eta y_\eta - \Phi_\eta y_\xi) / J a \quad (3.38a)
\]
\[
\frac{\partial \phi}{\partial y} = (-\Phi_\xi x_\eta + \Phi_\eta x_\xi) / J a \quad (3.38b)
\]

In general curvilinear coordinates, \( J \) can be expressed as

\[
J = J^\xi e_\xi + J^\eta e_\eta \quad (3.39)
\]

According to Eqs. (3.21a) and (3.21b),

\[
J^\xi = (J^\xi y_\eta - J^\eta x_\eta) / J a \quad (3.40a)
\]
\[
J^\eta = (J^\eta x_\xi - J^\xi y_\xi) / J a \quad (3.40b)
\]

Now combining Eqs. (3.38a), (3.38b), (3.37a), (3.37b), (3.40a) and (3.40b), we have

\[
J^\xi = [\rho u \Phi y_\eta - \mu (\Phi_\xi y_\eta^2 - \Phi_\eta y_\xi y_\eta)] / J a
- \rho v \Phi x_\eta + \mu (\Phi_\eta x_\xi x_\eta - \Phi_\xi x_\eta^2) / J a \quad (3.41a)
\]
\[
J^\eta = [\rho v \Phi x_\xi - \mu (\Phi_\eta x_\xi^2 - \Phi_\xi x_\xi x_\eta)] / J a
- \rho u \Phi y_\xi + \mu (\Phi_\xi y_\xi y_\eta - \Phi_\eta y_\xi^2) / J a \quad (3.41b)
\]

According to Eq. (3.23), the divergence of \( J \) in a general curvilinear
coordinate system is

\[ \nabla \cdot J = \left[ \frac{\partial (J a J^\xi)}{\partial \xi} + \frac{\partial (J a J^n)}{\partial \eta} \right] J a \]  

(3.42)

Therefore, in the computational space, Eq.(3.34) becomes

\[ \left[ \frac{\partial (J a J^\xi)}{\partial \xi} + \frac{\partial (J a J^n)}{\partial \eta} \right] J a = S^\Phi(\xi, \eta) \]  

(3.43)

Substituting Eqs.(3.41a) and (3.41b) into Eq.(3.43) and rearranging it, Eq.(3.43) becomes

\[ \frac{\partial (U \phi)}{\partial \xi} + \frac{\partial (V \phi)}{\partial \eta} = \frac{\partial}{\partial \xi} \left[ \frac{\mu}{J a} (g_{22} \Phi_{\xi} - g_{12} \Phi_{\eta}) \right] \\
+ \frac{\partial}{\partial \eta} \left[ -\frac{\mu}{J a} (-g_{12} \Phi_{\xi} + g_{11} \Phi_{\eta}) \right] + J a S^\Phi(\xi, \eta) \]  

(3.44)

where \( U = \rho (u y_{\eta} - v x_{\eta}), \ V = \rho (v x_{\xi} - u y_{\xi}) \)

\[ g_{22} = x_{\eta}^2 + y_{\eta}^2, \ g_{12} = x_{\xi} x_{\eta} + y_{\xi} y_{\eta}, \ g_{11} = x_{\xi}^2 + y_{\xi}^2 \]

U and V are called contravariant velocity fluxes and will be used as the dependent variables in the momentum equations. Eq.(3.44) is the transformed general governing equation.

Comparing the expressions for U and V in Eq.(3.44) with Eqs.(3.21a) and (3.21b), we can see that \( V^\xi \) and U, \( V^n \) and V differ
from each other by a factor \( \rho \). The contravariant velocity fluxes are similar to the contravariant velocity components and are also components along the \( \xi \) and \( \eta \) coordinates. They stand for the flow fluxes through the faces of the control volumes. From the view point of numerical computation, it is more convenient to use \( U \) and \( V \) as the dependent variables than to use \( V^{\xi} \) and \( V^{\eta} \) if the SIMPLE series algorithms are employed to solve the discretization equations. This will be shown in Chapter IV.

The source terms in the momentum equations can be transformed by using Eqs.(3.26a) to (3.26b) and Eqs.(3.29a) to (3.29c). For example, the pressure gradient terms can be transformed as

\[
- \frac{\partial P}{\partial x} = -(P_{\xi} y_{\eta} - P_{\eta} y_{\xi})/Ja \tag{3.45a}
\]

\[
- \frac{\partial P}{\partial y} = (P_{\xi} x_{\eta} - P_{\eta} x_{\xi})/Ja \tag{3.45b}
\]

The transformation of other source terms can easily be obtained by using the second derivative expressions.
Figure 3.1  The general curvilinear coordinates
CHAPTER IV

SOLUTION PROCEDURE

This chapter deals with the discretization of the transformed equations given in Chapter III. Only the discretization equations for Cartesian velocity components are presented. The discretized momentum equations using contravariant velocity fluxes can be obtained through the algebraic manipulation of the discretized momentum equations using Cartesian velocity components. The coupling between the momentum and continuity equations is done by using the SIMPLE [36] or SIMPLEC [37] algorithm. Rhie and Chow’s original momentum interpolation method [3] is further modified and used to eliminate the pressure oscillation on non-staggered grids. The treatment of boundary conditions is briefly explained. Finally, the overall solution procedure is presented and the residual for convergence is explained.

4.1 The Control-Volume Approach

As it has been stated in section 1.1, there are mainly two
methods used to discretize a differential equation, namely, finite element method and finite volume method. In the present study, the latter is adopted to discretize the transformed equation in body-fitted coordinates. In finite volume method, the computational domain is divided into a set of quadrilateral volumes and the conservation laws are expressed in an integral form for each of these control volumes. The control-volume approach has the advantage of preserving conservation properties, which is desirable for engineering applications where overall balance of mass, momentum and other quantities are often of prime importance.

There are two schemes to locate the control-volume faces in relation to the grid points. One is to place the control-volume faces midway between neighboring grid points. This is shown in Fig.4.1, where the dashed lines indicate the control volume faces. In this scheme, the grid points are first located and then the control volume boundaries are drawn. The grid point does not lie at the geometric center of the control volume that surrounds it if the grids are nonuniform. This scheme is denoted as Practice A in [36]. Another is to draw the control-volume boundaries first and then place a grid point at the geometric center of each control volume. In this scheme, when the control-volume sizes are nonuniform, their faces do not lie midway between the grid points. This scheme is shown in Fig.4.1 and denoted as Practice B in [36]. In Practice A, there are “half” control volumes
around the boundary grid points. Special discretization equation may be needed for the near-boundary control volumes. In Practice B, the boundary grid points are placed on the faces of the near-boundary control volumes, i.e., the control volumes surrounding the grid points on the boundaries have zero thickness. There is no need for a special discretization equation for the near-boundary control volumes. Obviously, Practice B is better than Practice A.

Let $\delta \xi$ and $\delta \eta$ denote the distance between the grid points in $\xi$ and $\eta$ directions in the computational space, respectively, while $\Delta \xi$ and $\Delta \eta$ denote the distance between the faces of the control volumes. $\delta x$, $\delta y$, $\Delta x$ and $\Delta y$ have the same meanings in the physical space. As it will be seen in next section, $\delta \xi$, $\delta \eta$, $\Delta \xi$ and $\Delta \eta$ will appear in the discretized transformed equations instead of $\delta x$, $\delta y$, $\Delta x$ and $\Delta y$. It would be, of course, convenient for $\delta \xi$, $\delta \eta$, $\Delta \xi$ and $\Delta \eta$ to all be unity no matter how much $\delta x$, $\delta y$, $\Delta x$ and $\Delta y$ are in the physical space. However, it is not possible for all the $\Delta \xi$ and $\Delta \eta$ to be unity if all the $\delta \xi$ and $\delta \eta$ are kept to be unity. The near-boundary control-volumes actually are only half of the interior ones if Practice A is used, as shown in Fig.4.2 for the one-dimensional case. Thus, special discretization equations are needed for these “half” control-volumes. In order to overcome this shortcoming, we can make the near-boundry control-volumes larger than the interior ones. This is shown in Fig.4.2

37
for the one-dimensional case, where $\delta \xi = 1$ everywhere while $\Delta \xi = 1.5$ for interior control-volumes and $\Delta \xi = 1.5$ for near-boundary control-volumes. This practice, used in the present study, is a combination of Practice A with Practice B, in which the distances between the grid points are first determined (in computational space $\delta \xi = \delta \eta = 1$ everywhere), then the control-volume faces are drawn and the thickness of the control volumes surrounding the grid points on the boundaries is zero.

Fig. 4.3 shows the grid arrangement adopted in this thesis with left and lower sides being assumed to be boundaries, where $\delta \xi = \delta \eta = 1$ and $\Delta \xi = \Delta \eta = 1$ except for near-boundary control-volumes where $\Delta \xi = \Delta \eta = 1.5$. The control-volume faces pass midway between the grid points except on the boundaries where the control-volume faces pass through the boundary points. Since this is a non-staggered grid arrangement, the components of velocity, either Cartesian components or contravariant fluxes, and the pressure are all stored at the same grid points. The grid in the computational space is shown in Fig. 4.4, where the uppercase letters, P, E, W, N and S, denote the grid points while the lowercase letters, e, w, n and s, denote the cell faces relative to point P.
4.2 General Discretization Equation

The discretization equation of the transformed general governing equation, Eq.(3.44), can be obtained by integrating it over the control-volume on the $\xi - \eta$ plane.

$$
\int_V \int [\frac{\partial (U\Phi)}{\partial \xi} + \frac{\partial (V\Phi)}{\partial \eta}] d\xi d\eta = \int_V \int [\frac{\partial}{\partial \xi} \left( \frac{\mu}{Ja} (g_{22}\Phi_\xi - g_{12}\Phi_\eta) \right) d\xi d\eta \\
+ \int_V \int [\frac{\partial}{\partial \eta} \left( \frac{\mu}{Ja} (-g_{12}\Phi_\xi + g_{11}\Phi_\eta) \right) d\xi d\eta \\
+ \int_V JaS^\Phi(\xi, \eta) d\xi d\eta \tag{4.1}
$$

where $\forall$ denotes the control volume. Applying Green's theorem, we obtain

$$
\oint_{cs} (U\Phi d\eta - V\Phi d\xi) + \oint_{cs} \left[ \frac{\mu}{Ja} (g_{22}\Phi_\xi + g_{12}\Phi_\eta) \right] d\eta \\
- \oint_{cs} \left[ \frac{\mu}{Ja} (g_{12}\Phi_\xi - g_{11}\Phi_\eta) \right] d\xi = \int_V JaS^\Phi(\xi, \eta) d\xi d\eta \tag{4.2}
$$

where $cs$ is the surface of the given volume $\forall$. Now we discretize the Eq.(4.2) over control-volume cell in the computational space as shown in Fig.4.4. The line integral consists of four parts

$$
\oint_{cs} = \int_{12} + \int_{23} + \int_{34} + \int_{41} \tag{4.3}
$$
where

\[ \int_{12} = \left[ -V\Phi \Delta \xi - \frac{\mu}{Ja}(g_{12}\Phi_\xi - g_{11}\Phi_\eta)\Delta \xi_{12} \right] \quad (4.4a) \]

\[ \int_{23} = \left[ U\Phi \Delta \eta + \frac{\mu}{Ja}(-g_{22}\Phi_\xi + g_{12}\Phi_\eta)\Delta \eta_{23} \right] \quad (4.4b) \]

\[ \int_{34} = \left[ -V\Phi \Delta \xi - \frac{\mu}{Ja}(g_{12}\Phi_\xi - g_{11}\Phi_\eta)\Delta \xi_{34} \right] \quad (4.4c) \]

\[ \int_{41} = \left[ -U\Phi \Delta \eta + \frac{\mu}{Ja}(-g_{22}\Phi_\xi + g_{12}\Phi_\eta)\Delta \eta_{41} \right] \quad (4.4d) \]

Therefore Eq.(4.2) can be written as

\[
[U\Phi \Delta \eta + \frac{\mu}{Ja}(-g_{22}\Phi_\xi + g_{12}\Phi_\eta)\Delta \eta_{41}^{23} + \left[ -V\Phi \Delta \xi - \frac{\mu}{Ja}(g_{12}\Phi_\xi - g_{11}\Phi_\eta)\Delta \xi_{34}^{12} = S^\Phi Ja \Delta \xi \Delta \eta \right]
\]

Here we assume that the values at the points e, w, n and s represent those on the surfaces 23, 41, 34 and 12, respectively. Thus, Eq.(4.5) becomes

\[
[U\Phi \Delta \eta]_w^n + [V\Phi \Delta \xi]_s^n = \left[ \frac{\mu}{Ja}(g_{11}\Phi_\eta - g_{12}\Phi_\xi)\Delta \xi]_s^n + \left[ \frac{\mu}{Ja}(g_{22}\Phi_\xi - g_{12}\Phi_\eta)\Delta \eta]_w^n + S^\Phi Ja \Delta \xi \Delta \eta \right]
\]

Derivatives at cell faces such as $\Phi_\xi|_e$ and $\Phi_\eta|_e$ in the above equation can be determined using central differencing scheme, for
example,

\[ \phi_e|_e = \frac{\phi_E - \phi_P}{\delta \xi} \quad (4.7) \]

\[ \phi_n|_e = \frac{1}{2} \left( \frac{\phi_N - \phi_S}{2 \delta \eta} + \frac{\phi_{NE} - \phi_{SE}}{2 \delta \eta} \right) \]

\[ = \frac{\phi_N - \phi_S + \phi_{NE} - \phi_{SE}}{4 \delta \eta} \quad (4.8) \]

Eq. (4.6) now becomes

\[ F_e \phi_e - F_w \phi_w + F_n \phi_n - F_s \phi_s + (D_e + D_w + D_n + D_s) \phi_P \]

\[ = D_e \phi_E + D_w \phi_W + D_n \phi_N + D_s \phi_S + b^\phi \quad (4.9) \]

where

\[ F_e = (U \Delta \eta)_e, \quad F_w = (U \Delta \eta)_w, \quad F_n = (V \Delta \xi)_n, \quad F_s = (V \Delta \xi)_s \]

\[ (4.10) \]

\[ D_e = \left[ \frac{\mu g_{22} \Delta \eta}{J \alpha \delta \xi} \right]_e, \quad D_w = \left[ \frac{\mu g_{22} \Delta \eta}{J \alpha \delta \xi} \right]_w \]

\[ D_n = \left[ \frac{\mu g_{11} \Delta \xi}{J \alpha \delta \eta} \right]_n, \quad D_s = \left[ \frac{\mu g_{11} \Delta \xi}{J \alpha \delta \eta} \right]_s \quad (4.11) \]

The source term \( b^\phi \) is expressed as
\[ b^\phi = b^\phi_S + b^\phi_{NO} \]  \hspace{1cm} (4.12)

where

\[ b^\phi_S = \text{Ja} S^\phi \Delta \xi \Delta \eta \]  \hspace{1cm} (4.13)

\[ b^\phi_{NO} = (D^0_s - D^0_n) \Phi_E + (D^0_n - D^0_s) \Phi_W + (D^0_w - D^0_e) \Phi_N + (D^0_e - D^0_w) \Phi_S \]
\[ + (D^0_n + D^0_w) \Phi_{NW} - (D^0_s + D^0_w) \Phi_{SW} + (D^0_e + D^0_s) \Phi_{SE} - (D^0_n + D^0_e) \Phi_{NE} \]  \hspace{1cm} (4.14)

\( D^0_e, D^0_w, D^0_n \) and \( D^0_s \) are expressed as

\[ D^0_e = \left[ \frac{\mu g_{12} \Delta \eta}{4 \text{Ja} \delta \eta} \right]_e, \quad D^0_w = \left[ \frac{\mu g_{12} \Delta \eta}{4 \text{Ja} \delta \eta} \right]_w \]

\[ D^0_n = \left[ \frac{\mu g_{12} \Delta \xi}{4 \text{Ja} \delta \xi} \right]_n, \quad D^0_s = \left[ \frac{\mu g_{12} \Delta \xi}{4 \text{Ja} \delta \xi} \right]_s \]  \hspace{1cm} (4.15)

In above equations, \( F \) is the flow rate through a control-volume surface, \( D \) is the orthogonal diffusional conductance and \( D^0 \) is the non-orthogonal diffusion conductance. \( b^\phi_{NO} \) denotes the source term due to the non-orthogonality of the coordinate system. This term disappears in the orthogonal coordinates where \( g_{12} = 0 \).

Discretization of the convection term \((F_s \Phi_s - F_w \Phi_w + F_n \Phi_n - F_e \Phi_e)\) in Eq.(4.9) is of considerable importance to the accuracy and stability for numerical computations. Central differencing scheme may fail when the mesh Reynolds number
is less than two. An effective but simple scheme to avoid the numerical instability is the upwind scheme, which is unconditionally stable. However, it may lead to large truncation error. Hybrid scheme [36] combines the accuracy of the central differencing scheme and the stability of the upwind scheme. In this thesis, hybrid scheme [36] is employed to discretize the convection-diffusion terms. The resulting algebraic equation is

\[ A_p \phi_p = A_E \phi_E + A_W \phi_W + A_N \phi_N + A_S \phi_S + b_P^\phi \]  \hspace{1cm} (4.16)

where

\[ A_E = \max(-F_e, D_e - \frac{F_e}{2}, 0), \quad A_W = \max(F_w, D_w + \frac{F_w}{2}, 0) \]
\[ A_N = \max(-F_n, D_n - \frac{F_n}{2}, 0), \quad A_S = \max(F_s, D_s + \frac{F_s}{2}, 0) \]  \hspace{1cm} (4.17)

\[ A_p = A_E + A_W + A_N + A_S - S_P^\phi Ja \Delta \xi \Delta \eta \]  \hspace{1cm} (4.18)

\[ b_P^\phi = Ja S_C^\phi \Delta \xi \Delta \eta + b_{NO}^\phi \]  \hspace{1cm} (4.19)

\( S^\phi \) in Eq.(4.13) has been linearized and is expressed as

\[ S^\phi = S_C^\phi + S_P^\phi \phi_p \]  \hspace{1cm} (4.20)

where \( S_P^\phi \) must be negative.
4.3 Discretization Equations for Momentum and Continuity Equations

In this section, we will first give the discretization equations for Cartesian velocity components. Then we will derive the discretization equations for contravariant velocity fluxes, $U$ and $V$, from those for Cartesian velocity components. Meanwhile, discretization equation for continuity equation will also be presented.

If the dependent variables for the momentum equations are the Cartesian velocities, the corresponding coefficients ($A_E$, $A_W$, $A_N$ and $A_S$) in their discretization equations will be identical (e.g., the $A_E$ in the $u$-component discretization equation is equal to the $A_E$ in the $v$-component discretization equation) except for the external source terms. Using Eq.(4.16), we have the following discretization equations for $u$ and $v$ momentum equations:

\[
A_p u_p = A_E u_E + A_W u_W + A_N u_N + A_S u_S + b_p^u \quad (4.21)
\]

\[
A_p v_p = A_E v_E + A_W v_W + A_N v_N + A_S v_S + b_p^v \quad (4.22)
\]

where
\[ b_p^u = -(P_n y_n - P_n x_n) \Delta \xi \Delta \eta + (\text{transformed form of}) \]
\[ \left[ \frac{\partial (\mu \frac{\partial u}{\partial x})}{\partial x} + \frac{\partial (\mu \frac{\partial v}{\partial y})}{\partial y} \right] Ja \Delta \xi \Delta \eta + b_{NO}^u \]

\[ b_p^v = (P_n x_n - P_n x_n) \Delta \xi \Delta \eta + (\text{transformed form of}) \]
\[ \left[ \frac{\partial (\mu \frac{\partial u}{\partial y})}{\partial y} + \frac{\partial (\mu \frac{\partial v}{\partial y})}{\partial y} \right] Ja \Delta \xi \Delta \eta + b_{NO}^v \] (4.23)

In Eq.(4.23), the transformed pressure gradient terms were given in Eqs.(3.45a) and (3.45b), and \( b_{NO}^u \) and \( b_{NO}^v \) are expressed by Eq.(4.14) with \( \phi \) replaced by \( u \) and \( v \), respectively. Both the transformed forms of \( \left[ \frac{\partial (\mu \frac{\partial u}{\partial x})}{\partial x} + \frac{\partial (\mu \frac{\partial v}{\partial y})}{\partial y} \right] \) and \( \left[ \frac{\partial (\mu \frac{\partial u}{\partial y})}{\partial y} + \frac{\partial (\mu \frac{\partial v}{\partial y})}{\partial y} \right] \) were briefly discussed in section 3.3. The pressure gradients such as \( P_n \) at point \( P \) can be discretized using central differencing scheme for the interior nodes. For the near-boundary nodes, second-order backward or forward differencing scheme can be used.

The discretization equation for \( U \) equation can be obtained by multiplying Eq.(4.21) by \( (\rho y_n)_{\rho} \) and Eq.(4.22) by \( -(\rho x_n)_{\rho} \) and adding them up

\[ A_p U_p = A_E U_E^0 + A_W U_W^0 + A_N U_N^0 + A_S U_S^0 + b_p^u \] (4.24)

where

\[ U_E^0 = (\rho y_n)_{\rho} u_E - (\rho x_n)_{\rho} v_E \] (4.25a)
\[ U_w^0 = (py_n)_{\rho} u_w - (px_n)_{\rho} v_w \]  
\[ U_n^0 = (py_n)_{\rho} u_n - (px_n)_{\rho} v_n \]  
\[ U_s^0 = (py_n)_{\rho} u_s - (px_n)_{\rho} v_s \]  
\[ b_p^u = (py_n)_{\rho} b_p^u - (px_n)_{\rho} b_p^v \]

The quantities with the superscript 0 are the velocity components parallel to \( U_0 \). The source term \( b_p^u \) includes the pressure gradients, the viscous forces and the source terms due to the non-orthogonality of the coordinate system. Eq.(4.24) would not be suitable for a field solution because different variables exist on the two sides of the equation. This difficulty can easily be removed by introducing the “actual” neighbors [1] in Eq.(4.24) as follows

\[ A_p U_p = A_E U_E + A_W U_W + A_N U_N + A_S U_S + b_p^u + b_{\text{CURV}} \]  

where

\[ b_{\text{CURV}}^u = A_E(U_E^0 - U_E) + A_W(U_W^0 - U_W) \]

\[ + A_N(U_N^0 - U_N) + A_S(U_S^0 - U_S) \]

\[ b_{\text{CURV}}^u \] represents the effect of the curvature.

The discretization equation for \( V \) equation can be obtained in a similar way. Multiplying Eq.(4.21) by \(- (py_\xi)_{\rho}\) and Eq.(4.22) by \((px_\xi)_{\rho}\) and
adding them up, we have

\[ A_p V_p = A_E V_E^0 + A_W V_W^0 + A_N V_N^0 + A_S V_S^0 + b_p^v \quad (4.28) \]

where

\[ V_E^0 = (\rho x_\xi)_p v_E - (\rho y_\xi)_p u_E \quad (4.29a) \]
\[ V_W^0 = (\rho x_\xi)_p v_W - (\rho y_\xi)_p u_W \quad (4.29b) \]
\[ V_N^0 = (\rho x_\xi)_p v_N - (\rho y_\xi)_p u_N \quad (4.29c) \]
\[ V_S^0 = (\rho x_\xi)_p v_S - (\rho y_\xi)_p u_S \quad (4.29d) \]
\[ b_p^v = (\rho x_\xi)_p b_p^v - (\rho y_\xi)_p b_p^u \quad (4.29e) \]

The final resulting discretization equation is

\[ A_p V_p = A_E V_E + A_W V_W + A_N V_N + A_S V_S + b_p^v + b_{\text{CURV}}^v \quad (4.30) \]

where

\[ b_{\text{CURV}}^v = A_E(V_E^0 - V_E) + A_W(V_W^0 - V_W) \]
\[ + A_N(V_N^0 - V_N) + A_S(V_S^0 - V_S) \quad (4.31) \]

The discretization equation for continuity equation can be obtained from Eq.(4.6) by setting \( \phi = 1 \) and \( S^\phi = 0 \).
\[(U\Delta n)_e - (U\Delta n)_w + (V\Delta \xi)_n - (V\Delta \xi)_s = 0 \quad (4.32)\]

### 4.4 Numerical Algorithm

The SIMPLE algorithm and its variant SIMPLEC algorithm have proven to be powerful tools to solve the discretization equations. In this thesis, most of the problems are solved using the SIMPLEC algorithm. A brief description of these two algorithms combined with the present method is given in this section.

The resulting velocity field obtained by solving the momentum equations, Eqs.(4.26) and (4.30), will not satisfy the continuity equation, Eq.(4.32), unless the correct pressure field is employed. Such an imperfect velocity field, denoted by \(U^*\) and \(V^*\), results from the solutions of Eqs.(4.26) and (4.30) based on a guessed pressure field \(P^*\).

\[
\begin{align*}
A_P U_P^* &= \sum A_{nb} U_{nb}^* + b_{nP}^U + (B_{11}P^*_\xi)_P + (B_{12}P^*_n)_P \quad (4.33) \\
A_P V_P^* &= \sum A_{nb} V_{nb}^* + b_{nP}^V + (B_{21}P^*_\xi)_P + (B_{22}P^*_n)_P \quad (4.34)
\end{align*}
\]

where the subscript \(nb\) indicates the neighboring points of \(P\). \(b_{nP}^U\) and \(b_{nP}^V\) stand for the source terms except the pressure gradients in Eqs.(4.26) and (4.30). \(B_{11}, B_{12}, B_{21}\) and \(B_{22}\) are coefficients, given by
\begin{align}
B_{11} &= -\rho(x^2_\eta + y^2_\eta)\Delta\xi\Delta\eta = -\rho g_{22}\Delta\xi\Delta\eta \\
B_{12} &= \rho(x_\xi x_\eta + y_\xi y_\eta)\Delta\xi\Delta\eta = \rho g_{12}\Delta\xi\Delta\eta \\
B_{21} &= B_{12} \\
B_{22} &= -\rho(x^2_\xi + y^2_\xi)\Delta\xi\Delta\eta = -\rho g_{11}\Delta\xi\Delta\eta
\end{align}

Eqs.(4.35a) to (4.35d) can be derived from Eqs.(4.23), (4.25e) and (4.29e).

Let the primed variables denote the corrections to these imperfect values, then

\begin{align}
P &= P' + P' \\
U &= U' + U' \\
V &= V' + V'
\end{align}

Utilizing Eqs.(4.36a) to (4.36c) and subtracting Eq.(4.33) from Eq.(4.26), and Eq.(4.34) from Eq.(4.30), we have

\begin{align}
A_pU'_{P} &= \sum A_{nb}U'_{nb} + (B_{11}P'_{\xi})_P + (B_{12}P'_{\eta})_P \\
A_pV'_{P} &= \sum A_{nb}V'_{nb} + (B_{21}P'_{\xi})_P + (B_{22}P'_{\eta})_P
\end{align}

In the SIMPLE algorithm [36], \( \sum A_{nb}U'_{nb} \) and \( \sum A_{nb}V'_{nb} \) are dropped from Eqs.(4.37) and (4.38) because these terms will be zero when the
convergence is reached. Thus, Eqs.(4.37) and (4.38) become

\[ U_p' = \frac{(B_{11}P_{e\xi})_p}{A_p} + \frac{(B_{12}P_{e\eta})_p}{A_p} \]  
(4.39)

\[ V_p' = \frac{(B_{21}P_{e\xi})_p}{A_p} + \frac{(B_{22}P_{e\eta})_p}{A_p} \]  
(4.40)

In the SIMPLEC algorithm [37], first, \( \sum A_{nb}U_p' \) and \( \sum A_{nb}V_p' \) are subtracted from Eqs.(4.37) and (4.38), respectively. Then terms \( \sum A_{nb}U_{nb}' - \sum A_{nb}U_p' \) and \( \sum A_{nb}V_{nb}' - \sum A_{nb}V_p' \) are dropped from their corresponding equations. Thus, Eqs.(4.37) and (4.38) become

\[ U_p' = \frac{(B_{11}P_{e\xi})_p}{A_p - \sum A_{nb}} + \frac{(B_{12}P_{e\eta})_p}{A_p - \sum A_{nb}} \]  
(4.41)

\[ V_p' = \frac{(B_{21}P_{e\xi})_p}{A_p - \sum A_{nb}} + \frac{(B_{22}P_{e\eta})_p}{A_p - \sum A_{nb}} \]  
(4.42)

Because \( \sum A_{nb}U_{nb}' - \sum A_{nb}U_p' \) and \( \sum A_{nb}V_{nb}' - \sum A_{nb}V_p' \) are much smaller than \( \sum A_{nb}U_{nb}' \) and \( \sum A_{nb}V_{nb}' \), respectively, \( U_p' \) and \( V_p' \) are much less affected by their neighboring velocity corrections in Eqs.(4.41) and (4.42) than in Eqs.(4.39) and (4.40) respectively. Therefore, SIMPLEC algorithm is more reasonable than SIMPLE algorithm. However, the denominator, \( A_p - \sum A_{nb} \), will be zero if \( S_p^\phi = 0 \) (see Eq.(4.18)). This very likely will occur in the momentum equations. Fortunately, this can be avoided if
an underrelaxation factor is used in the discretization equation for \( \phi \) to ensure the convergence. Thus, Eq.(4.16) is expressed as

\[
\frac{A_p \phi_p}{\alpha} = \sum A_{nb} \phi_{nb} + b_p \phi + (1-\alpha) \frac{A_p \phi^{(n-1)}}{\alpha} \tag{4.43}
\]

where \( \alpha \) is the underrelaxation factor. \( \phi_p^{(n-1)} \) is the value of \( \phi_p \) at the level of last iteration. Applying Eq.(4.43) to Eqs.(4.33) and (4.34), and repeating the same procedure for obtaining Eqs.(4.41) and (4.42), we have

\[
U_p = \frac{(B_{11} P_{\xi})_P}{A_p/\alpha - \sum A_{nb}} + \frac{(B_{12} P_{\eta})_P}{A_p/\alpha - \sum A_{nb}} \tag{4.44}
\]

\[
V_p = \frac{(B_{21} P_{\xi})_P}{A_p/\alpha - \sum A_{nb}} + \frac{(B_{22} P_{\eta})_P}{A_p/\alpha - \sum A_{nb}} \tag{4.45}
\]

Since \( \alpha \) is between 0 and 1, \( \frac{A_p}{\alpha} \) is always greater than \( \sum A_{nb} \). For convenience, we express Eqs.(4.39) and (4.44), Eqs.(4.40) and (4.45) in the following unified form

\[
U_p = (C_{11} P_{\xi})_P + (C_{12} P_{\eta})_P \tag{4.46}
\]

\[
V_p = (C_{21} P_{\xi})_P + (C_{22} P_{\eta})_P \tag{4.47}
\]

where
\[ C_{11} = \frac{B_{11}}{A_P}, \quad C_{12} = \frac{B_{12}}{A_P} = C_{21}, \quad C_{22} = \frac{B_{22}}{A_P} \quad \text{for SIMPLE} \]
\[ (4.48a) \]
\[ C_{11} = \frac{B_{11}}{A_P / \alpha - \sum A_{nb}}, \quad C_{12} = \frac{B_{12}}{A_P / \alpha - \sum A_{nb}} = C_{21}, \quad C_{22} = \frac{B_{22}}{A_P / \alpha - \sum A_{nb}} \quad \text{for SIMPLEC} \]
\[ (4.48b) \]

Now the corrected velocity field can be expressed as

\[ U = U' + C_{11} P_{\xi}' + C_{12} P_{\eta}' \quad (4.49) \]
\[ V = V' + C_{21} P_{\xi}' + C_{22} P_{\eta}' \quad (4.50) \]

To avoid solving the pressure-correction equation for 9 points (P, N, S, E, W, NE, NW, SE, SW), we neglect the non-orthogonality terms in Eqs.(4.49) and (4.50). Eqs.(4.49) and (4.50) now become

\[ U = U' + C_{11} P_{\xi}' \quad (4.51) \]
\[ V = V' + C_{22} P_{\eta}' \quad (4.52) \]

Substituting these two equations into the discretization equation for continuity equation (Eq.(4.32)) leads to

\[ (C_{11} P_{\xi}' \Delta \eta)_e - (C_{11} P_{\xi}' \Delta \eta)_w + (C_{22} P_{\eta}' \Delta \xi)_n - (C_{22} P_{\eta}' \Delta \xi)_s + m_P = 0 \]
\[ (4.53) \]
where

\[
  m_P = (U \cdot \Delta \eta)^{e} - (U \cdot \Delta \eta)^{w} + (V \cdot \Delta \xi)^{n} - (V \cdot \Delta \xi)^{s} \tag{4.54}
\]

Derivatives at cell faces such as \( P_{\xi}^{\cdot} \mid_{e} \) can be discretized using central differencing scheme. Thus, Eq.(4.53) can be written as

\[
  A_{p}P_{P}^{\cdot} = A_{E}P_{E}^{\cdot} + A_{W}P_{W}^{\cdot} + A_{N}P_{N}^{\cdot} + A_{S}P_{S}^{\cdot} + m_{P} \tag{4.55}
\]

where

\[
  A_{E} = \frac{C_{11} \Delta \eta}{\delta \xi}^{e}, \quad A_{W} = \frac{C_{11} \Delta \eta}{\delta \xi}^{w}, \quad A_{N} = \frac{C_{22} \Delta \xi}{\delta \eta}^{n}, \quad A_{S} = \frac{C_{22} \Delta \xi}{\delta \eta}^{s} \tag{4.56a}
\]

\[
  A_{p} = A_{E} + A_{W} - A_{N} + A_{S} \tag{4.56b}
\]

Eq.(4.55) is the discretization pressure-correction equation. After this equation is solved, velocities are corrected using Eqs.(4.36b) and (4.36c). The pressure is corrected using the following equation

\[
  P = P^{\cdot} + \alpha_{p}P^{\cdot} \tag{4.57}
\]

where \( \alpha_{p} \) is the pressure underrelaxation factor. For SIMPLE algorithm, this factor is usually less than 1. ForSIMPLEC algorithm, this factor
could be 1.

4.5 Momentum Interpolation Method

So far it seems that we are now able to solve the governing equations without any difficulty in the computational space. However, there still exists a serious problem associated with the non-staggered grid arrangement. The problem is that storing both the pressure and velocities at the same grid point will cause non-physical oscillation or so-called red-black checkerboard splitting of the pressure field. This undesirable behavior stems from the fact that the resulting equations couple the pressure and velocities only at alternate nodes if a linear interpolation is used to express the gradients of pressure in the momentum equations and the variations of velocity in the continuity equation. A solution to this problem is to use a staggered arrangement in which the pressure is located at the main grid point and the velocities at the cell faces (or cell corners). Staggered location removes the necessity for interpolations of pressure in the momentum equations and of velocities in the continuity equation. However, a staggered grid arrangement has its own disadvantages. The calculation of the coefficients and geometric interpolation factors is time-consuming when different control volumes are
used for different variables. The advantage of utilizing the non-staggered
grid arrangement is obvious. Nevertheless, a special kind of interpolation
is needed to evaluate the cell-face velocities. Rhie and Chow [3]
proposed a scheme based on momentum interpolation. In this scheme,
momentum equations are solved at main grid points for Cartesian
velocity components and the cell-face velocities are obtained by the
interpolation of the momentum equations on the neighboring nodes. In
the present study, the values of U and V at the cell faces are
calculated to determine the coefficients, \( A_e, A_w, A_n \) and \( A_s \), in the
momentum discretization equations in general curvilinear coordinates. The
following modified version of Rhie and Chow’s scheme is derived to
evaluate the values of U and V at the cell faces.

Divided by \( A_p \), Eq.(4.26) can be expressed as

\[
U_p = (H_U)_p + (\hat{B}_{11}P_{\xi})_p
\]  (4.58)

where \( H_U \) stands for all the terms on the right-hand side of Eq.(4.26)
except the pressure gradient in \( \xi \) direction. \( \hat{B}_{11} \) denotes the coefficient
\( B_{11} \) in Eq.(4.33) divided by \( A_p \). At point E, Eq.(4.58) becomes

\[
U_E = (H_U)_E + (\hat{B}_{11}P_{\xi})_E
\]  (4.59)
At point \( e \), which lies on the cell face between points \( P \) and \( E \), Eq.(4.58) becomes

\[
U_e = (\hat{H}_U)_e + (\hat{B}_{1i}P_{\xi})_e
\]

(4.60)

Linearization assumption is now introduced to evaluate \((H_U)_e\) as follows

\[
(H_U)_e = f^\cdot(H_U)_E + (1-f^\cdot)(H_U)_P
\]

(4.61)

where \( f^\cdot \) is the geometric interpolation factor and is defined in terms of the distance between nodal points

\[
f^\cdot = \bar{Pe}l(\bar{Pe} \pm \bar{eE})
\]

(4.62)

where \( \bar{Pe} \) denotes the distance between the points \( P \) and \( e \) while \( \bar{eE} \) is the distance between the points \( e \) and \( E \). Substituting Eq.(4.61) into Eq.(4.60) leads to

\[
U_e = f^\cdot(H_U)_E + (1-f^\cdot)(H_U)_P + (\hat{B}_{1i}P_{\xi})_e
\]

(4.63)

Combining Eqs.(4.58), (4.59) and (4.63), we obtain
\[ U_e = f^* U_E + (1-f^*) U_P + \left[ (\hat{B}_{11} P_{\xi} - f^* (\hat{B}_{11} P_{\xi} E)^{-(1-f^*)} (\hat{B}_{11} P_{\xi} P) \right] \quad (4.64) \]

We can obtain the expression for \( V \) at cell face \( n \) in a similar manner

\[ V_n = f^* V_N + (1-f^*) V_P + \left[ (\hat{B}_{22} P_{\eta} - f^* (\hat{B}_{22} P_{\eta} N)^{-(1-f^*)} (\hat{B}_{22} P_{\eta} P) \right] \quad (4.65) \]

where

\[ f^* = \frac{\overline{Pn}}{(\overline{Pn} + \overline{nN})} \quad (4.66) \]

\( \overline{Pn} \) denotes the distance between the points \( P \) and \( n \), and \( \overline{nN} \) is the distance between the points \( n \) and \( N \). Terms in the square parentheses in Eqs.(4.64) and (4.65) account for the effects of the pressure gradients in \( \xi \) and \( \eta \) directions, respectively.

Eqs.(4.64) and (4.65) are used to evaluate the contravariant velocity fluxes \( U \) and \( V \) at the cell faces. If \( (\hat{B}_{11})_E \) and \( (\hat{B}_{11})_P \), \( (\hat{B}_{22})_N \) and \( (\hat{B}_{22})_P \) are assumed to be approximately equal to \( (\hat{B}_{11})_e \) and \( (\hat{B}_{11})_n \), respectively, Eqs.(4.64) and (4.65) will become the expressions for Rhie and Chow’s original scheme. Also, there are no velocity underrelaxation factors in Eqs.(4.64) and (4.65). This makes them simpler. Miller and Schmidt [33] obtained a formulation of momentum interpolation which gives a converged solution in Cartesian coordinates.
independent of the underrelaxation factors. Eqs.(4.64) and (4.65) can be considered as an extension of such formulation to general curvilinear coordinates.

4.6 Treatment of the Boundary Conditions

In this section, the treatment of the boundary conditions will be discussed. In general, fluid flow boundary conditions can be broadly classified into four categories.

(1) inflow

(2) impermeable wall

(3) symmetry line

(4) outflow

At an inflow boundary, the flow enters the computational domain and normally the values of the various variables are known. These values can be substituted into the discretization equations for the nodes next to the inflow boundaries and thus nothing special needs to be done.

At an impermeable wall, the no-slip condition can be imposed for the velocities, i.e., the velocity of the fluid at the wall must be equal to the velocity of the wall. If the wall is stationary, the velocity of the
fluid at the wall is zero. The velocity values are directly substituted into the discretization equations for the nodes next to the wall boundaries.

At a symmetry line, the normal component of the velocity and the normal gradient of the parallel component of the velocity are zero. Suppose that the symmetry line is the line of \( \eta = \text{const} \), then

\[
V = 0 
\quad (4.67a)
\]
\[
\frac{\partial U}{\partial \eta} = 0 
\quad (4.67b)
\]

Eq.(4.67a) can be substituted into the discretization equations for the nodes next to the symmetry lines. Assuming that point \( N \) is on the symmetry line, Eq.(4.67b) is discretized as

\[
U_N = U_P 
\quad (4.68)
\]

Substituting Eq.(4.68) into Eq.(4.26), the discretization equation for \( U_P \) becomes

\[
(A_P - A_N)U_P = A_EU_E + A_WU_W + A_SU_S + b_P^U + b_{\text{CURV}}^U 
\quad (4.69)
\]

Hence, no boundary values are needed. Note that the treatment of the derivative at the boundary in this way can be made only when the grid
arrangement is such that the thickness of the control volume around the node on the boundary is zero.

At an outflow boundary, the fluid leaves the computational domain. The most commonly used practice at an outflow boundary is to assume that the diffusive flux is zero and the total flux is purely convective in nature. This assumption is equivalent to setting the streamwise gradients to zero at the outflow. Suppose that the outflow line is the line of $\xi=\text{const}$ and flow is in the positive $\xi$ direction. Then the coefficient $A_\xi$ in the momentum equations can be set to zero. An alternative is to use the values of last iteration at the outflow boundary and substitute them into the discretization equations for the nodes next to the outflow boundary.

### 4.7 Overall Solution Procedure

The overall solution procedure consists of the following sequence of operations:

1. Generate the grid using transfinite interpolation. This method can be found in [38], which is not repeated in this thesis.

2. Initialize all the variables and also calculate the cell-face velocity fluxes by using the initial velocity fluxes on the grid nodes.
3. Calculate the coefficients and source terms for the momentum equations.

4. Solve the momentum discretization equations to obtain the new velocity fluxes, $U$ and $V$, on the grid nodes. An Alternating Direction Implicit (ADI) solver is used for this purpose.

5. Calculate the cell-face velocity fluxes by using the momentum interpolation method.

6. Solve the pressure-correction equation by using the new values of velocity fluxes. Also, ADI solver is used.

7. Update the pressure and velocity flux fields.

8. Repeat the steps 3-7 until the convergence is reached.

The convergence is considered to be reached when the residuals for each equation are below a critical value. In this work, this value is $10^{-4}-10^{-8}$. The residuals, which provide a measure of the degree to which each equation is satisfied throughout the flow field, are computed for each discretization equation by summing the imbalance in the equation for all cells in the domain. The residual, $\text{Res}$, can be obtained by using Eq.(4.16)

$$\text{Res} = \sum |A_E\phi_E + A_W\phi_W + A_N\phi_N + A_S\phi_S + b_p\phi - A_p\phi_p| \quad (4.70)$$

Here $\text{Res}$ is the non-normalized residual. The normalized residual, $\overline{\text{Res}}$,
is defined as

$$\bar{\text{Res}} = \sum \frac{|A_E \phi_E + A_W \phi_W + A_N \phi_N + A_S \phi_S + b_p^\phi - A_p \phi_p|}{\sum |A_p \phi_p|}$$  \hspace{1cm} (4.71)

Eqs.(4.70) and (4.71) are the residuals for velocity fluxes. The residual for pressure as the imbalance in the continuity equation is

$$\text{Res} = \sum |[F_e - F_w + F_n - F_s]|$$  \hspace{1cm} (4.72)

The normalized pressure residual is defined as the ratio of the residual at N\textsuperscript{th} iteration to the residual at the second iteration.

$$\bar{\text{Res}} = \frac{\text{Res}_{\text{iteration } N}}{\text{Res}_{\text{iteration } 2}}$$  \hspace{1cm} (4.73)

Eq.(4.73) implies that if the imbalance in continuity equation at the second iteration is quite small, the normalized residual will be relatively large. In this case, it is better to use non-normalized residual.
Figure 4.1 Locations of the control-volume faces
(a) Practice A; (b) Practice B

63
Figure 4.2 One-dimensional case in the computational space
(a) Practice A;  (b) Practice used in this thesis
Figure 4.3  Grid arrangement
Figure 4.4 The computational space
CHAPTER V

COMPUTATIONAL RESULTS

In order to test the proposed numerical procedure, four test problems are solved using the present procedure. They are

1. Laminar Flows between Two Concentric Cylinders
2. Laminar Flows through a Channel with Gradual Expansion
3. Laminar Flows inside a Tube with a Constriction
4. Separated Flows in a Lid-driven Cavity

The numerical results will be compared with the exact solutions, bench-mark solutions, experimental data and the results obtained by other researchers to show the feasibility and the superiority of the proposed procedure. The first three problems are solved using the SIMPLEC algorithm. The last problem is solved using both SIMPLE and SIMPLEC algorithms.

5.1 Laminar Flows between Two Concentric Cylinders

This problem is shown in Fig.5.1a. In the present calculation, the inner cylinder has radius $R_1$ and is stationary. The outer cylinder has
radius $R_2 = 2R_1$ and rotates about its axis at a constant angular velocity $\omega$. This is a one-dimensional problem in cylindrical-polar coordinates, for which the analytical solution exists and is independent of Reynolds number [39]. The angular component of the velocity and the pressure vary only in the radial direction. The radial component of the velocity is zero. However, this problem becomes two-dimensional in Cartesian coordinates.

A parallelogram shown in Fig.5.1a is chosen as the computational domain, which is divided into $17 \times 17$ equi-spaced quadrilaterals. The grids are generated by transfinite interpolation which are shown in Fig.5.1b for $\theta = 15^0$. The grids are fixed in space and the flow sweeps through them. Four cases are tested with $\theta$ equal to $-15^0$, $0^0$, $15^0$ and $22.5^0$, respectively. For each case, Reynolds number is set to be 1, 10, 100 and 1000. The boundary conditions are prescribed by using the known analytical solution. The solution in the interior of the computational domain is obtained using the present method. The iterative procedure is terminated when all the normalized residuals for velocity fluxes and pressure are less than $10^{-6}$.

Figs.5.2a-5.2d show the comparison between the analytical and the predicted values of angular velocity while Figs.5.3a-5.3d show the comparison between the analytical and the predicted values of pressure. These values are along the diagonal through the lower left corner to the
upper right corner. In these figures, the velocity, the pressure and the radius are non-dimensionalized by \((2R_1\omega), \rho(2R_1\omega)^2\) and \(R_1\), respectively. The predicted results for velocity are in excellent agreement with the exact ones while those for pressure are in good agreement with the exact ones. The level of agreement for low Reynolds number is better than that for high Reynolds number. Numerical error increases as Reynolds number increases. This phenomenon is not obvious for velocity but can be seen clearly in the figures where the pressure comparison is made.

Karki [1] used the covariant velocity projections and power-law scheme to compute the same flow on staggered 17×17 grids. Tables 5.1-5.4 give the comparisons of average values of the percentage errors in velocity and pressure by using the present approach and the approach used by Karki [1]. The percentage error is defined as

\[
\varepsilon = \frac{\phi_{\text{exact}} - \phi_{\text{computed}}}{\phi_{\text{max}} - \phi_{\text{min}}} \quad (5.1)
\]

where \(\phi\) is either velocity or pressure, and \(\phi_{\text{max}}\) and \(\phi_{\text{min}}\) are the maximum and minimum values of \(\phi\) in the computational domain. The average error is defined as

\[
\bar{\varepsilon} = \sum \varepsilon_i / (\text{total number of internal nodes}) \quad (5.2)
\]
It can be seen clearly that the results obtained by the present method are more accurate than those from Karki except when Reynolds number is very low, i.e., Re = 1.

5.2 Laminar Flows through a Channel with Gradual Expansion

This problem was a test problem for the Fifth Workshop of the International Association for Hydraulic Research Working Group on Refined Modeling of Flow [40]. It has been widely accepted to test the codes based on general curvilinear coordinates.

Fig. 5.4a illustrates the geometry of such an expansion channel. The geometry depends on the values of the Reynolds number. The channel becomes larger and straighter as Reynolds number increases. Since the channel is symmetrical about its axis, only the half-region is chosen as the computational domain as shown in Fig. 5.4b. The lower boundary (solid wall) of the channel is given by the following expression

\[ y_i = \frac{\tanh(2 - 30x/Re) - \tanh(2)}{2} \]

for \( 0 \leq x \leq x_{\text{out}} = \frac{Re}{3} \) \hspace{2cm} (5.3)

The upper boundary (symmetry line) is located at
\[ y_u = 1.0 \] (5.4)

This problem is solved for two cases, i.e., \( \text{Re}=10 \) and \( \text{Re}=100 \). At the inflow boundary, the flow is assumed to be fully developed and the velocity profile is given by

\[ u(0, y) = 3y - \frac{3}{2}y^2 \] (5.5)

For the purpose of comparison with Karki’s results [1] which were obtained based on the \( 17 \times 17 \) grids, the governing equations in this thesis are solved on a \( 17 \times 17 \) grid.

The present predicted wall pressure distributions are compared with the results obtained by Cliffe et al. (cited in [40]) as shown in Fig.5.5a and 5.5b. The solution obtained by Cliffe et al. was considered to be highly accurate and used as the bench-mark solution [40]. It is obvious that the present results are in good agreement with the bench-mark solution for both cases. However, near the inlet for \( \text{Re}=10 \), the present result deviates from the bench-mark solution. This phenomenon can also be observed from the results obtained by other investigators [40]. Also, values of wall pressure distributions at several locations along the wall are listed in Tables 5.5 and 5.6. In comparison with the bench-mark solution, the present approach gives better results than Karki’s, especially
for Re = 10. Velocity vectors and isobar profiles calculated at Re = 10 are visualized in Figs. 5.6 and 5.7, respectively.

5.3 Laminar Flows inside a Tube with a Constriction

The flow characteristics in constricted tubes are important in many practical applications such as off-shore pipelines, arterial stenosis, etc. As a result, a considerable amount of literature, both experimental and numerical, exists on this topic. Young and Tsai [41] studied flow in arterial stenosis experimentally. They used Newtonian fluids in axisymmetric constriction models for the laminar flow. Melaaen [26] carried out numerical calculation of such flows based on $82 \times 22$ grids. He used the covariant velocity projections as the dependent variables on staggered grids, and Cartesian velocity components on non-staggered grids where Rhie and Chow's original momentum interpolation method [3] was adopted. Both power-law scheme (POW) and second-order upwind scheme (SOU) were employed. In this section, the proposed method is used to simulate laminar flows in a tube with a constriction and the results are compared with the experimental and numerical results mentioned above.

The geometrical shape of the axisymmetrical constriction is
specified as a cosine curve shown in Fig.5.8a

\[
\frac{R}{R_0} = 1 - \frac{\delta}{2R_0} (1 + \cos \frac{\pi x}{x_0}) \quad -x_0 \leq x \leq x_0 \quad (5.6)
\]

where \( R_0 \) is the unconstricted tube radius and \( R \) is the local tube radius. The length of the constriction, \( 2x_0 \), and the height of the constriction, \( \delta \), can vary to give different flow situations. Because of the symmetry of the region, only half the region is chosen as the computational domain as shown in Fig.5.8b. In the present work, the geometries corresponding to models M2 and M3 of Young and Tsai [41] as shown in Table 5.7 are used. The Reynolds number based on the definition of \( (2\bar{u}R_0/\nu) \) is set to be 50 and 100 for model M2 and 40 for model M3. \( \bar{u} \) is the average velocity at the inflow boundary, where the flow is assumed to be fully developed, i.e.,

\[
u = 2\bar{u}(1 - \frac{R^2}{R_0^2}) \quad , \quad v = 0 \quad \text{at inlet} \quad (5.7)
\]

All the calculations are started at the dimensionless position \( x^* = x/R_0 = -6 \), while the calculation domain ends at \( x^* = 16 \) and 24 for \( Re = 50 \) and 100 using M2 model, and \( x^* = 12 \) for \( Re = 40 \) using M3 model, respectively. The \( 82 \times 22 \) uniform grids are generated by transfinite interpolation.
When Reynolds number is higher than a certain critical number, the flow in a constricted tube separates and reattaches. Accordingly, comparisons are made for the separation and reattachment points. The present results together with measurements and calculations by other investigators are given in Tables 5.8 and 5.9. The experimental data are taken from [41]. The calculation results from Melaanen [26] are selected for comparison. The results for model M2 using the proposed procedure agree with the experimental data. The present predicted separation lengths for model M2 are as good as those of Melaanen using both power-law scheme and second-order upwind scheme, and the reattachment lengths are better than those of Melaanen using power-law scheme, but are not as good as those using second-order upwind scheme. The present predicted results for model M3 do not agree well with the experimental data. Separation appears too early and reattachment length is overpredicted. Melaanen also got similar results.

The axial variation of the centerline velocity for model M3 is shown in Fig.5.9 while the wall pressure distribution is shown in Fig.5.10. In these figures, the centerline velocity and the wall pressure are non-dimensionalized by \((2\bar{u})\) and \(\rho(2\bar{u})^2\), respectively. Centerline velocity increases and the pressure decreases beyond the minimum area of the constriction. Farther downstream, the pressure recovers slightly and the centerline velocity decreases drastically. Velocity vectors and
isobar profiles for model M3 are visualized in Figs. 5.11 and 5.12, respectively.

5.4 Separated Flows in a Lid-Driven Cavity

In this problem as shown in Fig. 5.13a, separated flows in a lid-driven cavity with a moving top boundary are studied. This kind of flow has served over and over again as a model problem for testing and evaluating a numerical technique. It has also been of much interest from the view points of transition to turbulence and bifurcation phenomena. Furthermore, this kind of flow is solved to investigate the effects of non-orthogonality of the numerical grid on the convergence and the stability of the solutions when the two side walls are inclined ($\beta \neq 90^0$).

The primary configuration concerned has been a square cavity ($\beta = 90^0$). A substantial amount of information concerning the flow in a square cavity is available in the literature. For example, Ghia et al. [42] used a multigrid algorithm with $128 \times 128$ nodes and obtained the key characteristics of the flow with high accuracy. The inclined cavity ($\beta \neq 90^0$) is a more complex configuration. Demirdzic et al. [43] calculated the cavity flows with $\beta = 45^0$ and $30^0$ on the domain of $320 \times 320$ control volumes. Their results can serve as bench-mark solutions for
comparison purposes.

When non-orthogonal grids are employed, the discretization of pressure-correction, $P'$, equation in its full form will result in a matrix which has nonzero coefficients on 9 diagonals in a 2-D case and 19 diagonals in a 3-D case. Since solving this matrix is very complex, expensive and rather impractical especially for 3-D case, most investigators often neglect the non-orthogonal terms in the $P'$ equation. The resulting simplified $P'$ equation has a diagonally dominant coefficient matrix, which is also symmetric with 5 nonzero diagonals in 2-D case and 7 in 3-D case. Peric [44] systematically studied the property of the $P'$ equation in a non-orthogonal grid system. He solved 2-D cavity flows with $\beta=30^\circ, 45^\circ, 60^\circ$ and $90^\circ$ respectively and compared the results obtained from the simplified $P'$ equation with those obtained from the full $P'$ equation. He found that if the grids are not severely non-orthogonal ($\beta \geq 45^\circ$), it is more efficient to use the simplified $P'$ equation than its full form, otherwise, it is necessary to use the full $P'$ equation since either the simplified version does not converge at all or the convergence is too slow. For each given velocity underrelaxation factor, $\alpha_u$ ($\alpha_v = \alpha_u$), the range of the pressure underrelaxation factor, $\alpha_p$, becomes narrower as the skewness of the non-orthogonal grid gets higher if the simplified $P'$ equation is employed. However, this problem does not occur if one uses the full $P'$ equation. For a given skewness
of the grid, the range of $\alpha_p$ value gets narrower as $\alpha_U$ increases. This applies to both the results obtained by using the simplified and the full $P'$ equations. But the range of $\alpha_p$ value gets narrower much faster if the simplified $P'$ equation is used than that if the full one is used. Recently, Cho and Chung [45] proposed a new treatment method for non-orthogonal terms in the $P'$ equation in order to enlarge the ranges of $\alpha_p$ values for convergence, which will ease the difficulty in determining the $\alpha_p$ values. In that new treatment, the non-orthogonal terms in the full $P'$ equation are decomposed into explicit and implicit terms and 5 nodes for 2-D flows and 7 nodes for 3-D flows are used in the coefficient matrix for $P'$ equation. Although this new treatment is superior to the simplified treatment if the grids are significantly non-orthogonal, its $P'$ equation is more complex than the simplified one. In both studies of [44] and [45], SIMPLE algorithm was used to solve the governing equations for Cartesian velocity components on $20 \times 20$ grids with Re = 100. The analysis in [44] and [45], in the investigators’ words, are valid for both staggered and non-staggered grid arrangements.

In this section, three cases of lid-driven cavity flows are presented. In all three cases, Reynolds number is set to be 100 and the grid is generated by using the transfinite interpolation method. As an example, Fig.5.13b shows the domain discretization of the cavity with $20 \times 20$ grids for $\beta = 60^0$. 

77
In the first case, the cavity flows for $\beta=45^0$ and $30^0$ were calculated. For each $\beta$ value, the grid size is chosen to be $81 \times 81$. The present numerical results are compared with the benchmark solutions of Demirdzic et al. [43]. The purpose of the comparison is to demonstrate the feasibility of the present method and to validate the computer code. SIMPLEC algorithm is employed with $\alpha_p=1.0$ and $\alpha_u=0.5$. Figs.5.14a–5.14d show the comparisons between the present predicted results and the benchmark solutions. In these figures, the velocities and the coordinates are non-dimensionalized by $u_L$ and $L$, respectively. $u_L$ and $L$ are the lid velocity and the cavity length respectively, as shown in Fig.5.13a. Note that the coordinates are along the centerline with the origin in the middle of the bottom side of the cavity for the $u$-$y$ figures and the origin in the middle of the left side of the cavity for the $x$-$y$ figures. The present results are in excellent agreement with the benchmark ones.

In the second case, SIMPLE algorithm is employed to solve the discretization equations. The grid size is set to be $20 \times 20$. Ranges of the pressure underrelaxation factor, $\alpha_p$, value for $\beta=90^0$, $60^0$, $45^0$ and $30^0$ are studied. For each $\beta$ value, value of $\alpha_u$ varies from 0.6 to 0.9. The results are compared with those of Cho and Chung [45] which were obtained using the simplified $P'$ equation on $20 \times 20$ grids. Figs.5.15a–5.15h show the comparisons between the present ranges of
α_p values and those of Cho and Chung. It is seen that the present ranges are wider than those of Cho and Chung when the cavity is inclined (β≠90^0). This phenomenon is especially obvious when the cavity is severely inclined (β≤45^0) and α_u≤0.8. On the other hand, there is almost no difference between the present ranges of α_p values and those of Cho and Chung when the cavity is square (β=90^0). Another phenomenon is that the present method needs fewer iterations to reach the convergence than that of Cho and Chung when α_p is higher than 0.2 except when α_u=0.9. In the case of α_u=0.9, the range of α_p value from Cho and Chung does not exceed 0.3 for β=90^0 and 0.2 for β<90^0. Nevertheless, a small value of α_p may not be acceptable for fast convergence according to these two studies. In the study of Cho and Chung, no matter what value β is given, large value of α_u is preferable for fast convergence provided that the convergence can be reached and the convergence rate is not slow. However, the present study indicates that this only applies to the mildly-inclined cavity flows (β>45^0). For severely-inclined cavity flows (β≤45^0), suitable value of α_u may range from 0.7 to 0.8. A common phenomenon is that for the given α_u and β values, there exists an optimum value of α_p which leads to least number of iterations required for convergence. This value, in most cases, is not 1.0.

In the third case, the grid size is also set to be 20×20.
SIMPLEC algorithm is employed to solve the discretization equation. To the author’s knowledge, no literature is available regarding the investigation of the effects of non-orthogonality for cavity flows using the SIMPLEC algorithm. Although the present method enlarges the ranges of $\alpha_p$ values when SIMPLE algorithm is employed, in which non-orthogonal terms in $P'$ equation are neglected, the limit to the ranges of $\alpha_p$ values still exists. However, there is no limit to the ranges of $\alpha_p$ values if SIMPLEC algorithm is employed in combination with the present method. Figs.5.16a–5.16d show the ranges of $\alpha_p$ values for $\beta=90^0$, $60^0$, $45^0$ and $30^0$. For each $\beta$ value, value of $\alpha_u$ varies from 0.5 to 0.9. Each curve is formed by connecting five points at $\alpha_p=0.2$, 0.4, 0.6, 0.8 and 1.0. From each curve, we can see that the larger the $\alpha_p$ value is, the faster the convergence is reached. On each curve, $\alpha_p=1.0$ is the best choice for fastest convergence. This is quite different from the last case where SIMPLE algorithm was employed. Moreover, a smaller value of $\alpha_u$, i.e., 0.6–0.7, is suitable for fast convergence when the cavity becomes significantly inclined. For mildly-inclined cavity flows, a larger value of $\alpha_u$, i.e., 0.7–0.8, is preferable.

It should be noted here that for each value of $\beta$, a unique solution, which is independent of the underrelaxation factors ($\alpha_u$ and $\alpha_p$) and the algorithms (SIMPLE or SIMPLEC), can be obtained.
Table 5.1 Average Errors (%) in Velocity and Pressure for $\theta = -15^\circ$

<table>
<thead>
<tr>
<th></th>
<th>Re</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity</td>
<td>Karki [1]</td>
<td>$4.609 \times 10^{-2}$</td>
<td>$5.706 \times 10^{-2}$</td>
<td>0.2845</td>
<td>0.8108</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>$4.378 \times 10^{-2}$</td>
<td>$4.337 \times 10^{-2}$</td>
<td>0.1378</td>
<td>0.6538</td>
</tr>
<tr>
<td>Pressure</td>
<td>Karki [1]</td>
<td>2.317</td>
<td>0.3169</td>
<td>0.5531</td>
<td>1.967</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>2.8156</td>
<td>0.3084</td>
<td>0.2283</td>
<td>1.4210</td>
</tr>
</tbody>
</table>

Table 5.2 Average Errors (%) in Velocity and Pressure for $\theta = 0^\circ$

<table>
<thead>
<tr>
<th></th>
<th>Re</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity</td>
<td>Karki [1]</td>
<td>$1.315 \times 10^{-2}$</td>
<td>$3.546 \times 10^{-2}$</td>
<td>0.3048</td>
<td>0.7353</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>$1.337 \times 10^{-2}$</td>
<td>$1.318 \times 10^{-2}$</td>
<td>0.1558</td>
<td>0.4876</td>
</tr>
<tr>
<td>Pressure</td>
<td>Karki [1]</td>
<td>0.7802</td>
<td>0.2669</td>
<td>1.020</td>
<td>2.661</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>1.1151</td>
<td>0.1313</td>
<td>0.4137</td>
<td>2.1620</td>
</tr>
</tbody>
</table>
Table 5.3 Average Errors (%) in Velocity and Pressure for $\theta = 15^\circ$

<table>
<thead>
<tr>
<th></th>
<th>Re</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity</td>
<td>Karki [1]</td>
<td>$8.831 \times 10^{-3}$</td>
<td>$3.127 \times 10^{-2}$</td>
<td>0.2913</td>
<td>0.7998</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>$1.117 \times 10^{-2}$</td>
<td>$1.143 \times 10^{-2}$</td>
<td>0.0997</td>
<td>0.2502</td>
</tr>
<tr>
<td>Pressure</td>
<td>Karki [1]</td>
<td>0.5585</td>
<td>0.4290</td>
<td>1.261</td>
<td>2.804</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>0.7010</td>
<td>0.1127</td>
<td>0.3629</td>
<td>2.7789</td>
</tr>
</tbody>
</table>

Table 5.4 Average Errors (%) in Velocity and Pressure for $\theta = 22.5^\circ$

<table>
<thead>
<tr>
<th></th>
<th>Re</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity</td>
<td>Karki [1]</td>
<td>$8.852 \times 10^{-3}$</td>
<td>$2.777 \times 10^{-2}$</td>
<td>0.2686</td>
<td>0.8517</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>$1.179 \times 10^{-2}$</td>
<td>$1.179 \times 10^{-2}$</td>
<td>0.0485</td>
<td>0.1579</td>
</tr>
<tr>
<td>Pressure</td>
<td>Karki [1]</td>
<td>0.5806</td>
<td>0.5152</td>
<td>1.226</td>
<td>3.373</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>0.6679</td>
<td>0.1197</td>
<td>0.2836</td>
<td>3.0973</td>
</tr>
</tbody>
</table>
### Table 5.5  Comparison of Wall Pressure Values for Re = 10

<table>
<thead>
<tr>
<th>$x/x_{out}$</th>
<th>Cliffe et al. [40]</th>
<th>Karki [1]</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>-0.665</td>
<td>-0.232</td>
<td>-0.3000</td>
</tr>
<tr>
<td>0.10</td>
<td>-0.3400</td>
<td>-0.280</td>
<td>-0.3269</td>
</tr>
<tr>
<td>0.30</td>
<td>-0.0680</td>
<td>-0.085</td>
<td>-0.0616</td>
</tr>
<tr>
<td>0.50</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.70</td>
<td>0.0500</td>
<td>0.057</td>
<td>0.0491</td>
</tr>
<tr>
<td>0.90</td>
<td>0.0680</td>
<td>0.082</td>
<td>0.0733</td>
</tr>
<tr>
<td>1.00</td>
<td>0.0710</td>
<td>0.091</td>
<td>0.0836</td>
</tr>
</tbody>
</table>

### Table 5.6  Comparison of Wall Pressure Values for Re = 100

<table>
<thead>
<tr>
<th>$x/x_{out}$</th>
<th>Cliffe et al. [40]</th>
<th>Karki [1]</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>-0.2147</td>
<td>-0.1850</td>
<td>-0.2121</td>
</tr>
<tr>
<td>0.10</td>
<td>-0.2275</td>
<td>-0.2420</td>
<td>-0.2382</td>
</tr>
<tr>
<td>0.30</td>
<td>-0.0717</td>
<td>-0.0840</td>
<td>-0.0820</td>
</tr>
<tr>
<td>0.50</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.70</td>
<td>0.0287</td>
<td>0.0273</td>
<td>0.0291</td>
</tr>
<tr>
<td>0.90</td>
<td>0.0294</td>
<td>0.0282</td>
<td>0.0311</td>
</tr>
<tr>
<td>1.00</td>
<td>0.0253</td>
<td>0.0225</td>
<td>0.0264</td>
</tr>
</tbody>
</table>
### Table 5.7 Geometries of Model Stenosis

<table>
<thead>
<tr>
<th>Model</th>
<th>$R_0$ (cm)</th>
<th>$\delta/R_0$</th>
<th>$x_0/R_0$</th>
<th>Percentage reduction in flow area</th>
</tr>
</thead>
<tbody>
<tr>
<td>M2</td>
<td>0.945</td>
<td>2/3</td>
<td>4</td>
<td>89</td>
</tr>
<tr>
<td>M3</td>
<td>0.945</td>
<td>2/3</td>
<td>2</td>
<td>89</td>
</tr>
</tbody>
</table>

### Table 5.8 Comparison of Separation Length

<table>
<thead>
<tr>
<th>Source</th>
<th>Separation Length $x_s/x_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>50 M2</td>
</tr>
<tr>
<td>Experiment [41]</td>
<td>0.33</td>
</tr>
<tr>
<td>Melaaen (POW) [26]</td>
<td>0.33</td>
</tr>
<tr>
<td>Melaaen (SOU) [26]</td>
<td>0.32</td>
</tr>
<tr>
<td>Present</td>
<td>0.34</td>
</tr>
</tbody>
</table>

### Table 5.9 Comparison of Reattachment Length

<table>
<thead>
<tr>
<th>Source</th>
<th>Reattachment Length $x_r/x_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>50 M2</td>
</tr>
<tr>
<td>Experiment [41]</td>
<td>2.28</td>
</tr>
<tr>
<td>Melaaen (POW) [26]</td>
<td>2.10</td>
</tr>
<tr>
<td>Melaaen (SOU) [26]</td>
<td>2.20</td>
</tr>
<tr>
<td>Present</td>
<td>2.18</td>
</tr>
</tbody>
</table>
Figure 5.1a  Flow between two concentric cylinders: geometry

Figure 5.1b  Flow between two concentric cylinders: domain discretization for $\theta = 15^\circ$
Figure 5.2a  Velocity distribution along the diagonal for $\theta = -15^\circ$

Figure 5.2b  Velocity distribution along the diagonal for $\theta = 0^\circ$
Figure 5.2c  Velocity distribution along the diagonal for $\theta = 15^\circ$

Figure 5.2d  Velocity distribution along the diagonal for $\theta = 22.5^\circ$
Figure 5.3a  Pressure distribution along the diagonal for $\theta = -15^\circ$  

Figure 5.3b  Pressure distribution along the diagonal for $\theta = 0^\circ$
Figure 5.3c  Pressure distribution along the diagonal for $\theta = 15^\circ$

Figure 5.3d  Pressure distribution along the diagonal for $\theta = 22.5^\circ$
$y_1(x) = \frac{[\tanh(2 - 30x/Re) - \tanh(2)]}{2}$

Figure 5.4a Flow through a channel with gradual expansion: geometric configuration

Figure 5.4b Flow through a channel with gradual expansion: domain discretization
Figure 5.5a  Pressure distribution on the wall: Re = 10

Figure 5.5b  Pressure distribution on the wall: Re = 100
Figure 5.6  Velocity vectors for Re = 10

Figure 5.7  Isobars for Re = 10
Figure 5.8a  Flow inside a tube with a constriction: geometric configuration
Figure 5.8b  Flow inside a tube with a constriction: domain discretization
Figure 5.9 Axial variation of the centerline velocity, Re = 40, model M3

Figure 5.10 Wall pressure, Re = 40, model M3
Figure 5.11  Velocity vectors, Re=40, model M3

Figure 5.12  Isobars, Re=40, model M3
Figure 5.13a  Separated flow in a lid-driven cavity: geometric configuration

Figure 5.13b  Separated flow in a lid-driven cavity: domain discretization for $\beta=60^\circ$
Figure 5.14a  Variation of the centerline velocity profiles: \( \beta = 45^\circ \), u-component

Figure 5.14b  Variation of the centerline velocity profiles: \( \beta = 45^\circ \), v-component
Figure 5.14c  Variation of the centerline velocity profiles: 
$\beta = 30^\circ$, u-component

Figure 5.14d  Variation of the centerline velocity profiles: 
$\beta = 30^\circ$, v-component
Figure 5.15a Convergence properties at $\beta = 90^\circ$ for SIMPLE:
$\alpha_u = 0.6$, $\alpha_u = 0.7$

Figure 5.15b Convergence properties at $\beta = 90^\circ$ for SIMPLE:
$\alpha_u = 0.8$, $\alpha_u = 0.9$
Figure 5.15c  Convergence properties at $\beta = 60^\circ$ for SIMPLE:
$\alpha_u = 0.6, \alpha_v = 0.7$

Figure 5.15d  Convergence properties at $\beta = 60^\circ$ for SIMPLE:
$\alpha_u = 0.8, \alpha_v = 0.9$
Figure 5.15e  Convergence properties at $\beta = 45^\circ$ for SIMPLE:
$\alpha_u = 0.6$, $\alpha_u = 0.7$

Figure 5.15f  Convergence properties at $\beta = 45^\circ$ for SIMPLE:
$\alpha_u = 0.8$, $\alpha_u = 0.9$
Figure 5.15g  Convergence properties at $\beta=30^\circ$ for SIMPLE: $\alpha_u=0.6, \alpha_u=0.7$

Figure 5.15h  Convergence properties at $\beta=30^\circ$ for SIMPLE: $\alpha_u=0.8, \alpha_u=0.9$
Figure 5.16a Convergence properties at $\beta = 90^\circ$ for SIMPLEC

Figure 5.16b Convergence properties at $\beta = 60^\circ$ for SIMPLEC
Figure 5.16c Convergence properties at $\beta = 45^0$ for SIMPLEC

Figure 5.16d Convergence properties at $\beta = 30^0$ for SIMPLEC
CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

This chapter is divided into two sections. In the first section, conclusions will be drawn from the work in previous chapters. In the second section, recommendations for future work are given.

6.1 Conclusions

In this thesis, a numerical method has been developed to calculate the fluid flows in complex geometries. In this method, the contravariant velocity fluxes were used as the dependent variables in the momentum equations using body-fitted coordinates. The discretization equations for contravariant velocity fluxes retain a strongly conservative form which avoids complete transformation and is derived from the discretization equations for the Cartesian velocity components. Non-staggered grid arrangement was adopted, where the velocity components and the pressure are stored at the same grid point. A modified version of the momentum interpolation method proposed by Rhie and Chow [3] was
developed to prevent the splitting of the pressure field. No underrelaxation factor appears in the present momentum interpolation formulation. In order to carry out numerical calculation, a control-volume approach, which is slightly different from that in [36], was employed to discretize the general convection-diffusion equations and it was shown that non-orthogonality introduces extra diffusion source terms in the discretization equations. Throughout the study, a hybrid differencing scheme was used to treat the convection-diffusion terms in the momentum equations.

Four typical fluid flow problems were solved by using the present method. They are laminar flows between two concentric cylinders, laminar flows through a channel with gradual expansion, laminar flows inside a tube with a constriction, and separated flows in a lid-driven cavity. In the first problem, Reynolds number was set to be 1, 10, 100 and 1000. Comparisons were made between the analytical and the numerical values of angular velocity and pressure. Good agreements were obtained. Furthermore, the average percentage errors in velocity and pressure were assessed and compared with those of Karki [1]. It was shown that the present calculation results are better than those of Karki [1]. In the second problem, calculations were carried out for Re=10 and 100. The predicted wall pressure distributions using the present method were compared with the bench-mark results cited in [40]. Good
agreements were obtained except near the inlet for \( \text{Re} = 10 \). Comparison study between the present results and those of Karki [1] also showed that the former are closer to the bench-mark ones than the latter. In the third problem, the calculations of fluid flows inside a constricted tube were carried out with \( \text{Re} = 50 \) and 100 for one model (M2) and \( \text{Re} = 40 \) for another (M3). The present numerical results were compared against the experimental data and the numerical data from other researchers for the separation lengths and reattachment lengths. More accurate results than those of Melaaen [26] using the power-law scheme were obtained for model M2. However, results for model M3 do not agree well with the experimental data. This is also observed from the calculations by Melaaen. In short, the above three examples illustrate that the present calculation approach behaves better than the approach using the covariant velocity projections on staggered grids with the power-law scheme employed. In the last problem, the comparison shows that excellent agreements were obtained between the present results and the bench-mark solutions [43] for severely-inclined lid-driven cavity flows. The emphasis was placed on the study of the range variation of the pressure underrelaxation factor, \( \alpha_p \), for lid-driven cavity flows where the non-orthogonal terms of \( P' \) equation were omitted. It was shown that the ranges of \( \alpha_p \) values using the present procedure are wider than those of Cho and Chung [45], especially when \( \beta \leq 45^\circ \) and \( \alpha_u \leq 0.8 \), if
SIMPLE algorithm is employed. When the present method is combined with the SIMPLEC algorithm, there is no limit to the range of pressure underrelaxation factor $\alpha_p$ and $\alpha_p = 1.0$ is always preferable for a fast convergence. The unique solution can be obtained for a geometrically-fixed cavity flow in spite of the fact that no velocity underrelaxation factor appears in the present momentum interpolation formulation.

6.2 Recommendations

The present study was limited to two-dimensional, laminar, incompressible fluid flows. The following three suggestions are provided for the refinement and extension of the solution method presented in this thesis.

1. The present method can be easily extended to three-dimensional flows. In three-dimensional case, there are two cross pressure derivatives in each of the momentum equations in a general curvilinear coordinate system. Therefore, non-orthogonal terms in the pressure-correction equation involve 6 cross derivatives of the pressure corrections which can be omitted. This results in a 7-nodal coefficient matrix for the pressure-correction equation.

2. As most of the fluid flows in real life are turbulent, problems
studied by the present method should be extended to the turbulent flow regime. In the simulation of turbulent flows, turbulence model must be employed to account for the effect of the Reynolds stresses on the flow field. A typical and also widely used model is the k-ε model, in which two transport equations for the turbulent kinetic energy and the turbulent kinetic dissipation rate are solved. These equations can be transformed by using the mathematical relations given in sections 3.2 and 3.3.

3. Compressible flow problems should be studied using the proposed method. If density is not a constant, it must be considered separately in the expression for the contravariant fluxes. To evaluate the density at the cell faces, density can be linearized for subsonic flows and upwinded for supersonic flows. The density correction is related to the pressure correction via the equation of state.
REFERENCES


9. C.-N. Yung, T.G. Keith, Jr. and K.J. De Witt, Numerical Simulation


Fluids, Vol.20, pp.621-640, 1995


35. V. Damodaran, Numerical Simulation of Flow through Pipes with Multiple Constrictions and with Moving Walls, Ph.D. Dissertation, University of Windsor, Windsor, ON, 1995


1979


VITA AUCTORIS

Name: Hao Xu  
Sex: Male

1966  Born in Shanghai, China on July 26

1981-1983  Xi’an No.83 High School  
           Xi’an, Shaanxi, China

1983-1987  Xi’an Jiaotong University  
           Xi’an, Shaanxi, China  
           B.E. in Energy and Power Engineering

1987-1994  Xi’an Jiaotong University  
           Xi’an, Shaanxi, China  
           Teaching and Research Assistant in Mechanical Engineering

1994-1996  University of Windsor  
           Windsor, Ontario, Canada  
           M.A.Sc. Candidate in Mechanical Engineering