Order properties of the space of continuous affine functions with values in an ordered Banach space.

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ORDER PROPERTIES OF
THE SPACE OF CONTINUOUS AFFINE FUNCTIONS
WITH VALUES IN
AN ORDERED BANACH SPACE

by

MAN HOK WONG

A Dissertation
Submitted to the Faculty of Graduate Studies through the Department of Mathematics in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy at the University of Windsor

Windsor, Ontario

Canada

1973
DEDICATED TO

MY FAMILIES
ABSTRACT

In the first Chapter, after some preliminaries and necessary definitions, we give a short historical development of the space \( A(K) \) of continuous real-valued affine mappings defined on a compact convex set \( K \). This space has many interesting order properties. In the second Chapter, we generalize some results of Chapter One to the space \( A(K,Y) \) of continuous affine mappings with values in the ordered Banach space \( Y \), by using selection theorem techniques. We prove that the space \( A(K,Y) \) can inherit many order properties from \( Y \), such as directedness of the closed unit ball. Our method follows Asimow and Atkinson in (8'). In Chapter Three we extend those order properties in \( A(K,Y) \) to \( L(A^*(K),Y) \) the space of linear weak*-norm continuous mappings from \( A^*(K) \) into \( Y \).

The main result of Chapter Four is that the centre of \( A(K,Y) \) is the largest strongly lattice ordered subspace of \( A(K,Y) \), generalizing a result of Fakhoury. In the last Chapter, we show that the completion of the tensor product of two order-unit Banach space with \( R,S,P \) is also a order-unit Banach space with \( R,S,P \).
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Definitions and Preliminaries

For convenience we shall now list some definitions and results for ordered vector spaces and ordered Banach spaces that have been used in the dissertation.

1. A real vector space $Y$ which is endowed with a partial ordering ($\leq$ reflexive, antisymmetric, transitive) is called an ordered vector space if the following properties hold:

   (1) if $a, b, c \in Y$ and $a \leq b$ then $a + c \leq c + b$
   (2) if $a, b \in Y$ and $\lambda \in \mathbb{R}$ and $a \leq b$ then $\lambda a \leq \lambda b$.

A proper cone $P$ in $Y$ is a subset satisfying the following conditions:

   (1) $P + P \subseteq P$,
   (2) $\lambda P \subseteq P$ ($\forall \lambda \in \mathbb{R}^+$),
   (3) $P \cap (-P) = \{0\}$.

If $P$ satisfied (1) and (2) it is called a wedge.

If $Y$ is an ordered vector space and $P = \{x \in Y ; x \geq 0\}$, then $P$ is a cone and usually called the positive cone of $Y$.

The ordered vector space $Y$ is said to be positively generated by the positive cone $P$ if $Y = P - P$.

$Y$ is said to be directed upwards if for any $x, y$ in $Y$, there is $z$ in $Y$ such that $x, y \leq z$.

It is easy to see that $Y$ is generated if and only if it is directed.
Y is an Archimedean space if \( a \leq b \) for some \( b \) in \( P \) and all \( a \) in \( R^+ \) implies that \( a \leq 0 \).

Y is an almost Archimedean space if \( -a \leq b \leq a \) for some \( a \) in \( P \) and all \( a \in R^+ \) then \( b = 0 \). It is clear that every Archimedean space is an almost Archimedean space.

A base \( B \) of a cone \( P \) is a non-empty convex subset of \( P \) such that if \( a \neq 0 \) in \( P \), there is \( \alpha > 0 \) and \( b \) in \( B \) and the representation \( a = \alpha b \) is unique.

An element \( e \) of \( P \) is said to be an order unit if for any \( a \) in \( Y \), there is \( \alpha > 0 \) such that \( -\alpha e \leq a \leq \alpha e \).

An order interval in \( Y \) is a set of the form:

\[ [a, b] = \{ x \in Y : a \leq x \leq b \} \] for \( a \leq b \).

If \( Y \) has an order unit and a base \( B \), then \( I = [-e, e] \) and \( T = \text{co} (B \cup -B) \) are convex, absorbing and balancing subsets of \( Y \). Thus the Minkowski functional \( q_x \) and \( q_T \) are semi-norms in \( Y \). We have the following two results about the order unit and base.

(a) \( q_x \) is a norm if and only if \( Y \) is almost Archimedean.

(b) \( q_T \) is a norm if and only if \( Y \) is directed and \( T \) is linearly bounded.

\( Y \) is called an order-unit norm space or base-norm space if the norm in \( Y \) is determined either by the order unit or base in the above sense. More information about
order-unit norm or base normed space can be found in Ellis (22).

An approximate order unit in an order vector space is a net \( \{e_\lambda\}_{\lambda \in \Lambda} \) which satisfies:

1. \( e_\lambda \in P \quad (\forall \lambda \in \Lambda) \);
2. \( e_\lambda \preceq e_\mu \) whenever \( \lambda \preceq \mu \);
3. for each \( x \in Y \), there is a positive number \( f_x \) and \( \lambda_x \) such that
   \[ -f_x e_{\lambda_x} \preceq x \preceq f_x e_{\lambda_x}. \]

\( Y \) is an approximate order unit normed space if the Minkowski functional of \( S_Y = \bigcup_{\lambda \in \Lambda} e_\lambda X \), \( e_\lambda \) is a norm in \( Y \), which is true if and only if \( S \) is linearly bounded. See Ng (46).

\( Y \) is said to have Reisz separation property (R.S.P) if and only if for \( a, b, c, d \) in \( Y \) with \( a, b \leq c, d \) then there is \( e \in Y \) such that \( a, b \leq e \leq c, d \).

2. An ordered normed linear space \( Y \) is a normed space and at the same time an ordered vector space with a closed cone \( P \). If \( Y \) is complete, then we call it an ordered Banach space.

An ordered normed linear space \( Y \) is normal; if there is some \( \delta \in \mathbb{R} \) such that for \( a \leq b \leq c \) in \( Y \) we have \( \|b\| \leq \delta \max\{\|a\|, \|c\|\} \).

Davies (14) called an ordered Banach space \( Y \)
a regular space if \( Y \) satisfies the following two conditions:

\((R_1)\): for \( x, y \in Y \) and \( -x \leq y \leq x \) we have \( \|y\| \leq \|x\| \);

\((R_2)\): for \( x' \in Y \) there is \( y \in Y \) with \( x \not\leq y \)

\(-x \leq y \) and \( \|y\| \leq \|x\| \).

In (7) L. Asimow defines an ordered Banach space \( Y \) to be \((\alpha, n)\)-directed if for \( \alpha \geq 1, n > 0 \) and \( x_1, x_2, \ldots, x_n \) in \( B \), there is \( x' \in B \) such that \( x_1, x_2, \ldots, x_n \not\leq x' \). If \( Y \) is \((\alpha', n)\)-directed for all \( \alpha' > \alpha \) then \( Y \) is said to be approximately \((\alpha, n)\)-directed.

A Banach lattice is an ordered Banach space \( Y \) which is vector lattice and satisfies the following conditions:

For any \( x, y \in Y \) with \( \|x\| \leq \|y\| \) we have \( \|x\| \leq \|y\| \), and \( \|y\| = \|y\| \).

An AL-space \( Y \) is a Banach lattice whose norm is additive on the positive cone i.e., for \( -x, y \geq 0 \) we have \( \|x + y\| = \|x\| + \|y\| \).

An AM-space is a Banach lattice which satisfies \( \|x \vee y\| = \|x\| + \|y\| \), for all \( x, y \) in \( Y \).

AL-space and AM-space were due to Kakutani (60, 61).

Effros calls an ordered Banach space a simplex space if its Banach dual is an AL-space (20).

A Banach space has F.2.I.P if and only if every
finite collection of mutually intersecting closed balls has non-empty intersection (39).

3. A point \( x \) in a convex set \( K \) is said extreme if \( x = \frac{1}{2}y + \frac{1}{2}z \) implies \( x = y = z \) for \( y, z \) in \( K \).

The extreme boundary \( \partial K \) of \( K \) is the set of all extreme points in \( K \).

The compact convex set is called a simplex if and only if the space of continuous affine mappings on it has the Reisz separation property.

If \( K \) is a simplex, and \( \partial K \subset K \) is w*-closed, then \( K \) is said to be a Bauer simplex.

The simplex was introduced by G. Choquet in 1956 (14).
Chapter One

Spaces of real-valued affine functions on a compact convex set

M. Krein can be considered as the first person who really investigated ordered Banach spaces. The earliest result about $A(K)$, the space of continuous affine mappings defined on a compact, convex set $K$, can be found in his paper (133) in 1940. He showed that if the positive cone $P$ of an ordered Banach space $E$ has an interior point $u$ and $-u < x < u$ is bounded, then $E$ can be represented as $A(K)$, where $K$ is the state space of $E$. In 1951 (30), R.V. Kadison introduced the order unit and showed the following theorem.

**Theorem 1.** Every ordered normed space $E$ with an order unit can be identified with a dense subspace of $A(K)$ where $K$ is a state space of $E$. If $E$ is complete then $A(K) \cong E$.

In 1964 (17), D.A. Edwards proved the converse of Krein's Theorem. From the proof of Edwards' result, we observed that the compact convex set $K$ can be embedded as a $w^*$-compact base for the positive cone in $A^*(K)$, the Banach dual of $A(K)$ equipped with the weak$^*$ topology. From the above results, it is clear now that the study of the ordered Banach space $A(K)$ is merely the study of
the order-unit Banach spaces. The order theoretic properties of $A(K)$ became more fruitful after G. Choquet introduced the concept of Simplex and the Integral Representation Theorem in 1956 (14). The geometric properties of a simplex were studied by D.G. Kendall in 1961 (32) and E.M. Alfsen in 1964 (2). In 1960 H. Bauer (11) settled the following maximal principle.

Theorem 2. An upper semi-continuous mapping defined on a compact convex set $K$ will attain its maximum on the extreme boundary of $K$.

One important consequence of the Bauer's Maximum Principle is the well known Krein-Milman Theorem which is a particular case of it. In 1962 G. Mokobodzki (43) generalized Dini's Theorem in $A(K)$, he proved:

Theorem 3. Every element in $A(K)$ is the uniform limit on $K$ of continuous affine functions on $A^*(K)$.

In 1961 H. Bauer (10) solved the following Dirichlet problem for simplexes.

Theorem 4. Let $K$ be a simplex; then every element in $C(DK)$ can be extended to an element in $A(K)$ if and only if the extreme boundary of $K$ is closed. Moreover is a AM-space.

In 1966 E. Alfsen (3) extended the Dirichlet
problem to $A(K, K)$ where $K$ is a metrizable compact convex set and $K_1$ is a compact convex subset of a locally convex linear topological space. The metrizability of $K$ can be discarded if $K$ is a Bauer simplex. In 1967 E.G. Effros (21) and 1968 A.J. Lazar (36) had extended this result for the non-metrizable simplices.

**Theorem 5.** Let $K$ be a simplex and $K$, a compact convex subset of a locally convex topological space $E$. Then every element in $C(\partial K; K_1)$ can be extended to an element in $A(K, K_1)$ if and only if for each $f^*$ in $E^*$ the real functions $\hat{f^*}T$ and $\tilde{f^*}T$ are continuous on $K$. Such an extension is unique whenever it exists.

In (20) Effros introduced the structure topology on the closed maximal ideals of a simplex space. In 1970 E.M. Alfsen and T.B. Andersen generalized the structure topology on the extreme boundary of a compact convex set $K$ by introducing the concept of split faces. They generalized Effros' theorem on the extendability of a facially continuous function on the extreme boundary of a simplex $K$ to a function in $A(K)$ to any compact convex set in following theorem.

**Theorem 6.** Let $K$ be a compact convex set, and let $f : \partial K \rightarrow R$ be continuous with respect to the facial topology. Then $f$ can be extended to a unique element in $A(K)$.

In 1965 D.A. Edwards (18) proved a separation theorem defined on a simplex $K$ which stated as follows.
Theorem 7. \( K \) is a simplex if and only if for every pair of lower semi-continuous concave mapping \(-f\) and \(g\) from \( K \) into \( R \cup \{\infty\} \) with \( f \leq g \), then there exist \( h \) in \( A(K) \) such that \( f \leq h \leq g \).

From Edward's Separation Theorem we can deduce that \( K \) is a simplex if and only if \( A(K) \) has the R.S.P. But this result had been established independently by J. Lindenstrauss in 1964 (39) and by Z. Semadeni in 1965 (56).

As a consequence of a Choquet Theorem in (14), \( K \) is a simplex if and only if \( A^*(K) \) is a vector lattice. If \( A(K) \) is a vector lattice then by a result of Reisz \( A^*(K) \) is a vector lattice, hence \( K \) is a simplex but the converse is not true. On the other hand, H. Bauer and J. Lindenstrauss had proved in (39) and (10) that if \( K \) is a Bauer simplex then \( A(K) \) is a vector lattice.

In 1966 E. G. Effros (20) introduced the simplex spaces which he defined as ordered Banach spaces with a closed cone and whose Banach dual is an AL-space. He proved that if \( E \) is a simplex space then it has a representation \( A^*(E'_0) \) where \( A^*(E'_0) = \{ f^* A(E''_0) : f(0) = 0 \} \), and \( E''_0 \) is the positive elements of the closed unit ball of the Banach dual of \( E \).

The following theorem shows several equivalent conditions for simplex spaces.

Theorem 8. If \( E \) is an ordered Banach the following
conditions are equivalent:

1. E is a simplex space
2. \( E \cong A_0(E^*) \)
3. E has a 1-normal cone, R.S.P and directed closed unit ball.

The implication of (1) to (2) was proved by Rogalski (54). The equivalent of (1) & (3) were proved independently by K.P. Ng and E.B. Davies in (46) and (15) respectively.

An ordered Banach space can be represented as a \( A_0(K) \) can be seen from the following theorem due to Asimow (6) and Ng (40).

**Theorem 9.** Let E be an ordered Banach space; then \( E \cong A_0(K) \) if and only if E has a 1-normal positive cone, an open directed unit ball and K is a universal cap.
Chapter Two

The space of affine continuous functions
with values in a Banach space

1 Introduction

We will use the selection theorem technique to obtain
the order properties of \( A(K,Y) \). We show that if \( Y \) satisfies
any one of the following:

(a) The closed unit ball of \( Y \) is directed upwards
(b) \( Y \) is \((\alpha,n)\)-directed
(c) \( Y \) is \((\alpha,n)\)-approximate directed
(d) \( Y \) is regular
(e) \( Y \) is a simplex space
(f) \( Y \) is a predual of \( L\)-space

Then \( A(K,Y) \) also satisfies this condition when \( K \) is a
simplex. If \( K \) is a Bauer simplex and \( Y \) satisfies any
one of

(g) \( Y \) is a Banach lattice
(h) \( Y \) is an AL-space
(i) \( Y \) is an AM-space

Then \( A(K,Y) \) has this property.
2 Notations and Definitions

Let $K$ be a compact convex set in a locally convex Hausdorff space $X$; then we can assume without loss of generality that $K$ is regularly embedded in $X$ in such a way that $X$ is isomorphic to $A^*(K)$ the space of continuous linear functionals on $A(K)$ equipped with the $w^*$-topology. Then $K$ is a base for the positive cone $P$ of $A^*(K)$; See Alfsen (41). Let $Y$ be an ordered Banach space.

A function $f : K \rightarrow Y$ is affine iff it satisfies

$$f(\alpha x + (1 - \alpha)y) = \alpha f(x) + (1 - \alpha)f(y)$$

for $x, y$ in $K$ and $0 \leq \alpha \leq 1$.

Let $A(K,Y)$ be the family of continuous affine functions from $K$ into $Y$. $A(K,Y)$ becomes an ordered Banach space if we equip it with the norm

$$\|f\| = \sup \{ \|f(x)\| : x \in K \}$$

for $f$ in $A(K,Y)$ and the natural ordering $f \geq 0$, iff $f(x) \geq 0$ $(\forall x \in K)$.

By standard arguments, it is not hard to show that the cone induced by this natural ordering is a closed cone.

Our main tool for studying the space $A(K,Y)$ is the selection theorem technique which was developed by E. Michael (42). We give some of the definitions and results concerning selection theorems. Our notations follow (8).
A map or carrier from a convex set \( C \) of some topological linear space into some topological linear space \( Y \) is a function;

\[ T : C \rightarrow 2^Y. \]

\( 2^Y \) is the family of non-empty subsets of \( Y \).

A map \( T \) is said to be affine if \( Tx \) is convex and satisfies

\[ \alpha Tx + (1 - \alpha) Ty \subseteq T(\alpha x + (1 - \alpha) y) \]

for \( x, y \) in \( C \) and \( \alpha \in (0, 1) \).

The map \( T \) is lower semi-continuous if for each open set \( V \) in \( Y \)

\[ \{ x \in C : Tx \cap V \neq \emptyset \} \]

is open in \( C \).

The map \( T \) has an affine continuous selection if there is an continuous affine function from \( C \) into \( Y \) such that

\[ f(c) \in T(c) \]

for any \( c \) in \( C \).

In (42) E. Michael investigated the existence of continuous selection for lower semi-continuous maps on topological spaces, Lazar (37) gave necessary and
sufficient conditions for the existence of affine continuous selection for an affine lower semi-continuous map $T$. The following is the Selection Theorem of Lazar.

**Theorem 2.1.** Let $E$ be a Frechet space, $K$ a simplex and

$$
T : K \rightarrow 2^E
$$

an affine lower semi-continuous map such that $T_k$ is closed for each $k$ in $K$. Then there exists an affine continuous selection for $T$.

In (40) Léger gave a elegant short proof of Theorem 2.1. Lazar's proof of the Selection Theorem depends heavily on Proposition 2.5 in (42) which carried over to the affine maps in Lemma 2.3. in (37). It is so useful and important so we mention it as follows.

**Proposition 2.2.** Let $C$ be a convex subset of a topological linear space and let $S,T$ be two lower semi-continuous affine maps from $C$ into $2^Y$, the family of non-empty subsets of the topological linear space $Y$ and let $V$ be a non-empty open convex set in $Y$.

For any $x \in C$ let

$$
R_x = Sx \cap (Tx + V)
$$

If $R_x \neq \emptyset$ for all $x \in C$ then $R$ is a lower semi-continuous affine map from $C$ into $2^Y$. 
3 Order properties of $A(K,Y)$

Proposition 3.1. If $K$ is a simplex and $Y$ is a positively generated, normal ordered Banach space having the R.S.P. then $A(K,Y)$ is a positively generated, normal ordered Banach space with R.S.P.

Proof: It is the Corollary 2.5 in (8).

Proposition 3.2. If $K$ is a compact convex set and $Y$ is an order-unit normed space then $A(K,Y)$ is an order-unit normed space.

Proof: Let $e$ be the order-unit in $Y$ and $h$ be the positive affine function from $K$ into $Y$, such that $h(K) = e$.

Then $h$ is in $A(K,Y)$ and it is an order unit for the natural ordering. Define

$$\|g\| = \inf\{\lambda : -\lambda h \leq g \leq \lambda h, \lambda \in \mathbb{R}^+\}$$

for any $g$ in $A(K,Y)$.

The function $\|\|$ determined by $h$ is a norm in $A(K,Y)$ which is less than the supremum norm. Hence $A(K,Y)$ is an order-unit normed space.
We recall some more results for the affine maps.

Let $K$ be a compact convex set and $Y$ be an ordered Banach space. Let $S$, $T$ be two maps from $K$ into $2^Y$ and $U$ be a subset of $Y$. Denote

$$S \cap (T + U)$$

as the function $x \mapsto Sx \cap (Tx + U)$,

$\overline{T}$ as the function $x \mapsto (Tx)$.

The collection $\mathcal{B}$ of maps from $K$ into $2^Y$ is said to be stable if it satisfies the following conditions:

1. $A(K,Y) \subseteq \mathcal{B}$;
2. If $S$, $T$ in $\mathcal{B}$, and $U$ is a non-empty open convex subset of $Y$ and $Rx = Sx + (Tx + U)$ for $x$ in $K$ then $R$ in $\mathcal{B}$.

Denote $\overline{\mathcal{B}}$ as the collection of all maps $\mathcal{B}$: $K \to 2^Y$ such that for any $\varepsilon > 0$ there is $S_\varepsilon \mathcal{B}$ with

$$T \subseteq S_\varepsilon \subseteq T + B_\varepsilon$$

where $B_\varepsilon$ is the open ball of radius $\varepsilon$ about $0$ in $Y$.

Let $\mathcal{A}(K,Y)$ be the family of maps $T : K \to 2^Y$ having affine continuous selections.

The family of all lower semi-continuous affine maps
is stable. If $K$ is a simplex then the collection of lower semi-continuous affine mappings belongs to $\mathcal{A}(K,Y)$ and if the stable collection $\mathcal{B} \subset \mathcal{A}(K,Y)$ then $\overline{\mathcal{B}} \subset \mathcal{A}(K,Y)$. For a fuller discussion of these results see (8).
In what follows, we will apply selection theorem technique to prove some order theoretic properties of $A(K,Y)$. Before that, we establish the following lemma.

**Lemma 3.3.** If $Y$ is an ordered normed linear space, with a cone $P$ which has a directed upward unit ball

$$B = \{ x : \| x \| \leq 1 \}$$

then there is a constant $\alpha$ such that, for any $a, b$ in $Y$ we have

$$(a + P_a) \cap (b + P_b) \cap B \subseteq (a + P) \cap (b + P) \cap B + B_{\beta}$$

where $\| \beta \| \leq 1$ and $P_{\beta} = P + B_{\beta}$.

**Proof:** Let $x$ in $(a + P_a) \cap (b + P_b) \cap B$, then there are some $p_i$ in $P$ and $b_i$ in $B$, $(i = 1, 2)$ such that

$$x = a + p_1 + b_1 = b + p_2 + b_2$$

Let $x = -b$, $x_i = -b_i$ then $\| x_i \| \leq 1$, $(i = 1, 2)$.

Since $B$ is directed upward, choose $z \geq x$, $x_i$ with

$$\| z \| \leq \alpha \beta \|$$ for some $\alpha$ in $\mathbb{R}$.

Hence

$$x + z \geq a$$ and $$x + z \geq b$$

Therefore $x$ belongs to $(a + P) \cap (b + P) \cap B + B_{\beta}$. 
This completes the proof of the Lemma.

**Theorem 3.4.** Let $K$ be a simplex and $Y$ be an ordered Banach space with cone $P$ such that the unit ball $B = \{ x \in Y : \|x\| \leq 1 \}$ is directed upwards. Then the closed unit ball of $A(K,Y)$ is also directed upwards.

**Proof:** Let $a$ and $b$ in $A(K,Y)$ with $\|a\| \leq M$ and $\|b\| \leq 1$. Define

$$T : K \to 2^Y$$

as

$$Tx = (a(x) + P) \cap (b(x) + P) \cap B$$

Since $B$ is directed therefore $Tx \neq \emptyset$ for every $x$ in $K$.

It is easy to see that $T$ is an affine map.

Let $\mathcal{B}$ be the collection of lower semi-continuous affine maps $K \to 2^Y$. By Lazar's Theorem $\mathcal{B}$ is stable and $\mathcal{B} \subseteq A(K,Y)$. We are going to prove that $T$ has an affine continuous selection. We have already known $\mathcal{B} \subseteq A(K,Y)$. We only to show $T \in \mathcal{B}$.

Given $0 < \epsilon < 1$, define

$$S_\epsilon = (a + P_\alpha^\epsilon) \cap (b + P_\alpha^\epsilon) \cap B$$

where $\alpha$ is the constant in Lemma 3.3. We have $\epsilon \alpha \leq 1$.

We claim that $S_\epsilon \in \mathcal{B}$. $S_\epsilon$ is obviously an affine map.

Let $Tx = B$. Then $T_x$ is a lower semi-continuous.

Since for any open set $W$ in $Y$, the sets
\{ x : B \cap W \neq \emptyset \} = K \quad \text{and} \quad \{ x : B \cap W = \emptyset \} = \emptyset

are open in themselves. Moreover, let

\[ T_2(\,x\,) = T_1(\,x\,) \cap (\,b(x) + P_k \,) \]

Since

\[ P = P + B = \bigcup_{k \in P} (\,P + B \,) \quad \text{is open}. \]

By proposition 2.2, \( T_2 \) is lower semi-continuous.

By substitution

\[ T = T_2 \cap (\,a + P_k \,) \]

is lower semi-continuous. Hence \( T \) is in \( \mathcal{B} \).

By Lemma 3.3.

\[ S_k \subset T + B_k \quad \text{and} \]

\[ T \subset S_k \]

hence, we have \( T \in \mathcal{B} \).

Therefore \( T \) has an affine continuous selection \( c \) such that

\[ a, b \leq c \quad \text{and} \quad \| c \| \leq 1 \]

Therefore, the proof is completed.
**Corollary 3.5.** If $Y$ is an $(\alpha - n)$-directed ordered Banach space, then $A(K,Y)$ is $(\alpha - n)$-directed.

**Proof:** As in Lemma 3.3, let

$$E_\alpha = \{ x \in Y : \| x \| \leq \alpha \};$$

then there is constant $r$ such that for any $a_1, a_2, \ldots, a_\alpha$ in $Y$ we have

$$\bigcap_{i=1}^{\alpha} (a_i + \frac{P}{r}) \cap \bigcap_{j=1}^{\alpha} (a_j + \frac{P}{r})$$

$$\subset \bigcap_{i=1}^{\alpha} (a_i + \frac{P}{r}) \cap \bigcap_{j=1}^{\alpha} (a_j + \frac{P}{r}) \cap B_x$$

where $r \leq \alpha$.

To prove the Corollary, let $a_1, a_2, \ldots, a_\alpha$ in the closed unit ball of $A(K,Y)$, and define

$$T : K \rightarrow 2^Y$$

by

$$T(x) = \bigcap_{i=1}^{\alpha} (a_i(x) + \frac{P}{r}) \cap \bigcap_{j=1}^{\alpha} (a_j(x) + \frac{P}{r}) \cap B_x$$

for any $x$ in $K$.

Then following the same method as in the Theorem, the corollary follows.
Corollary 3.6. If $Y$ is an approximately $(\alpha - n)$-directed space, so also is $A(K,Y)$.

Proof: Follows the same way as in Corollary 3.5.

Corollary 3.7. If $Y$ an approximate order-unit normed space then $A(K,Y)$ is an approximate order-unit normed space.

Proof: In (46), Ng showed that $Y$ is an approximate order-unit normed space if and only if the positive cone of $Y$ is $i$-normal and the open unit ball of $Y$ is directed upwards. We know that the directedness of the closed unit ball of an ordered Banach space implies the directedness of the open unit ball. It is clear now the corollary follows from the Proposition 3.1. and the Theorem.

Corollary 3.6. was proved also by H.Toufic. But our method is different from his. Now, we use his technique to prove the next theorem, although we can prove it by our method. It seems to us that Toufic's method is simpler. Nevertheless, for completeness we will outline the proof of the theorem in our way.
Theorem 3.8. Let $K$ be a simplex and $Y$ be regular; then $A(K,Y)$ is regular.

Proof: Let $a, b$ in $A(K,Y)$ and $-a \leq b \leq a$.

Hence $-a(k) = b(k) = a(k)$ \quad (\forall k \in K)$

Since $Y$ satisfies $R_1$, therefore

$$\|b(k)\| \leq \|a(k)\|$$

and $\|b\| \leq \|a\|$.

By definition $A(K,Y)$ satisfies $R_1$.

Method 1. Out the proof of $R_2$ in $A(K,Y)$.

Since $Y$ satisfies $R_2$ we can show just as in Lemma 3.3, that there is an $\alpha$ such that for any $a$ in $Y$,

$$(a + P) \cap (-a + P) \cap H$$

$$(a + P) \cap (-a + P) \cap H + B_s$$

where

$$H = \{ y \in Y : \|y\| = \|a\|\}.$$

We set up a map $T : K \to 2^Y$ \quad by

$$T(x) = (a(x) + P) \cap (-a(x) + P) \cap H$$

for any $a$ in $A(K,Y)$ and all $x$ in $K$.

Following the same technique as in Theorem 3.4, we can show that $T$ has an affine continuous selection.
Method 2

Let \( a \) in \( A(K,Y) \) and define \( T : K \rightarrow 2^Y \) by

\[
T(x) = \{ y \in Y : y \geq a(x), -a(x), \| y \| \leq \| a(x) \| \}
\]

for any \( x \) in \( K \). Denote \( h_1 = a \) and \( h_2 = -a \).

It is easy to see \( T(x) \neq \emptyset \) and \( T \) is an affine map. The next step is to show \( T \) is a lower semi-continuous.

Let \( x_0 \) be any open set in \( Y \), such that \( T(x_0) \cap V \neq \emptyset \).

If \( y \) in \( T(x_0) \cap V \), there is \( r > 0 \) such that the open ball with radius \( r \) and centre at \( y \) will be completely in \( V \).

Write

\[
h_i(x) = h_i(x_0) + ( h_i(x) - h_i(x_0) ) \quad (i = 1, 2)
\]

Since \( Y \) is a \( R_2 \)-space and by axiom of choice, we can find a function \( f : K \rightarrow Y \), such that

\[
f(x) \geq h_i(x) - h_i(x_0)
\]

and

\[
\| f(x) \| \leq \min \| h_i(x) - h_i(x_0) \|
\]

for all \( x \) in \( K \) and \( i = 1, 2 \).

Since \( h_i \) are continuous, there is a neighborhood \( W \) of \( x \), such that,

\[
\| h_i(x) - h_i(x_0) \| < \varepsilon \quad (\forall x \in W)
\]
where \( \varepsilon = \min \{ -y^T a(x) \| y \| - \| y \| \} \).

By assumption \( y_0 \in T(x_0) \) therefore \( y_0 \geq h^*_0(x_0) \) and

\[ \| y_0 \| \leq \| h^*_0(x_0) \| . \]

Hence

\[ h^*_0(x) = h^*_0(x_0) + (h^*_0(x) - h^*_0(x_0)) \]

\[ \leq h^*_0(x_0) + f(x) \]

\[ \leq y_0 + f(x) . \]

For any \( x \) in \( W \), we have

\[ \| y_0 + f(x) \| \leq \| y_0 \| + \| f(x) \| \]

\[ \leq \| y_0 \| + \| h^*_0(x) - h^*_0(x_0) \| \]

\[ \leq \| y_0 \| + \| a(x) \| - \| y_0 \| \]

\[ = \| a(x) \|. \]

Hence \( y_0 + f(x) \in T(x) \) for any \( x \) in \( W \), and

\[ \| y_0 - (y_0 + f(x)) \| \leq \| f(x) \| \leq \infty \]

i.e., \( y_0 + f(x) \in V \).

Therefore \( y_0 + f(x) \in T(x) \cap V \)

This shows that \( T \) is lower semi-continuous.
and \( T \) is also an affine lower semi-continuous map.

By Lazar's Selection Theorem there is \( g: K \rightarrow Y \) such that \( g(x) \in T(x) \) and

\[
g(x) \geq a(x) \quad \text{and} \quad \|g(x)\| \leq \|a(x)\|.
\]

By the definition of the norm we have \( \|g\| \leq \|a\| \).

This proves that \( A(K,Y) \) is a \( R_{\frac{1}{2}} \)-space. Hence the proof of the theorem is completed.

**Proposition 3.9.** Let \( Y \) be an ordered Banach space with the following property;

\[
(R_1) \quad \text{if } x, y \in Y \text{ there is } z \in Y \text{ such that } x, y \leq z \quad \text{and} \quad \|z\| = \max \{\|x\|, \|y\|\}
\]

Then \( A(K,Y) \) has the property \( R_1 \).

**Proof:** By the same procedure as in the Theorem 3.8.

In (15), Davies proves the following result:

**Proposition 3.10.** An ordered Banach space \( Y \) is a simplex space if and only if it is a Regular space, satisfies \( R_{\frac{1}{2}} \), and has \( R.S.P. \).

Now, we pass to one of our main results.
Theorem 3.11. Let $K$ be a simplex and $Y$ be an simplex space then $A(K,Y)$ is also an simplex space.

Proof: It follows from the Theorem 3.8. and the Proposition 3.9.

The above theorem can be derived from a result in (46). Ng shows that an ordered Banach space is a simplex space if and only if it is $1$-normal has R.S.P. and has a directed upwards unit ball.
The next step is a close look at the extreme boundary of a simplex. Our next theorem is a result due to J.R. Atkinson (9). We give a proof without the help of the concept of centre.

**Lemma 3.12** Let \( K \) be a Bauer simplex and \( Y \) be an ordered Banach space. Then for any \( a \) in \( A(K, Y) \) we have

\[
\|a\| = \sup_{k \in K} \|a(k)\| = \sup_{k \in \partial K} \|a(k)\|
\]

Proof: It is obvious that \( \partial K \subset K \), so

\[
\|a\| \geq \sup_{k \in \partial K} \|a(k)\| \quad (1)
\]

For the converse, let \( k \) in \( co(\partial K) \), then

\[k = \alpha k_1 + (1 - \alpha)k_2\]

for \( k_i \) in \( \partial K \) and \( 0 \leq \alpha \leq 1 \), \( i = 1, 2 \). We have

\[
\|a(k)\| = \|a(\alpha k_1 + (1 - \alpha)k_2)\| \leq \alpha \|a(k_1)\| + (1 - \alpha)\|a(k_2)\|
\]

\[\leq (\alpha + 1 - \alpha)\sup_{k \in \partial K} \|a(k)\|\]

i.e.

\[
\sup_{k \in \partial K} \|a(k)\| \leq \sup_{k \in \partial K} \|a(k)\|
\]

The Krein-Milman Theorem states that \( co(\partial K) = K \).

Therefore

\[
\|a(k)\| \leq \sup_{k \in \partial K} \|a(k)\| \quad \ldots \quad (2)
\]

Combining (1) and (2), we have the required result.
Theorem 3.13. Let \( K \) be a Bauer simplex and \( Y \) be a Banach lattice with a closed cone \( P \). Then \( A(K,Y) \) is a Banach lattice. Moreover, if \( Y \) is an AL-space or an AM-space, \( A(K,Y) \) is also an AL-space or an AM-space.

Proof: We have already known that \( A(K,Y) \) is an ordered Banach space with respect to the supremum norm and the natural ordering. In order to prove \( A(K,Y) \) is a vector lattice, it suffice to show that, for any \( a \) in \( A(K,Y) \), \( a^+ \) exists. Define

\[
b(k) = (a(k))^+
\]

for any \( k \) in \( \mathcal{K} \).

Since \( \mathcal{Y} \) is a vector lattice and the lattice operations are continuous on \( Y \), therefore \( b \) is well-defined and continuous on \( K \). By a theorem of Alfsen, \( b \) can be extended to \( \tilde{b} \) in \( A(K,Y) \).

We claim that \( \tilde{b} \geq 0 \), \( a \) and

\[
\mathcal{B}_b = a^+.
\]

By definition of \( b \), we have

\[
b \leq \tilde{b} \mid_{\mathcal{OK}} = 0.
\]

For any \( x, y \) in \( K \) and \( 0 \leq \lambda < 1 \), we have

\[
\tilde{b}(\lambda x + (1 - \lambda)y) = \lambda \tilde{b}(x) + (1 - \lambda)\tilde{b}(y)
\]

\[
= \lambda b(x)^+ + (1 - \lambda)b(y) \geq 0.
\]
Therefore \( \tilde{b} \geq 0 \) on \( \text{co}(\emptyset K) \).

By Krein-Milman Theorem, we have \( \tilde{b} \geq 0 \) on \( K \).

We can see from the following that \( \tilde{b} \geq a \), for any \( k \) in \( \text{co}(\emptyset K) \), we have

\[
(\tilde{b} - a)k = \tilde{b}(k) - a(k)
= \tilde{b}(\lambda x + (1 - \lambda)y) - a(\lambda x + (1 - \lambda)y)
= \lambda \tilde{b}(x) + (1 - \lambda)\tilde{b}(y) - \lambda a(x) - (1 - \lambda)a(y)
= \lambda b(x) + (1 - \lambda)b(y) - \lambda a(x) - (1 - \lambda)a(y)
= a(x)^+ + (1 - \lambda)(a(y)^+) - \lambda a(x) - (1 - \lambda)a(y)
\geq 0
\]

If there is some \( c \) in \( A(K,Y) \) such that \( c \geq a, \emptyset \).

We want to show \( c \geq b \). For any \( k \) in \( \emptyset K \), we have

\[
c(k) \geq a(k), 0
\]

Since \( Y \) is a vector lattice, therefore

\[
c(k) \geq (a(k))^+ \quad \text{i.e.} \quad c(k) \geq b(k)
\]

By Krein-Milman Theorem we obtain \( c \geq b \) on \( K \).
hence

\[ c \geq \tilde{b} \quad \text{i.e.} \quad \tilde{b} = a \lor \tilde{0} = a^+ \]

We have shown that \( A(K,Y) \) is a vector lattice.

We want to show the supremum norm \( \| \| \) on \( A(K,Y) \) is a lattice norm. In fact for any \( a \) in \( A(K,Y) \) we have

\[ \| \| a \| = \| a^+ + a^- \| \]

\[ = \sup_{k \in K} \| (a^+ + a^-)_k \| \]

\[ = \sup_{k \in K} \| (a^+ - a^-)_k \| \]

\[ = \sup_{k \in K} \| a(k) + \tilde{a}(k) \| \]

\[ = \sup_{k \in K} \| (a(k))^+ + (a(k))^\neg \| \]

\[ = \sup_{k \in K} \| a(k) \| \]

\[ = \sup_{k \in K} \| a(k) \| \]

\[ = \| a \| \]

If \( a \) and \( b \) are in \( A(K,Y) \) with \( a < b \) then

\[ \| a \| = \sup_{k \in K} \| a(k) \| \]

\[ = \sup_{k \in K} \| a(k) \| \]

\[ = \sup_{k \in K} \| b(k) \| \]

\[ = \sup_{k \in K} \| b(k) \| \]

\[ = \| b \| \quad \text{i.e.} \quad \| a \| \leq \| b \| \]
Therefore $A(K,Y)$ is a Banach lattice.

Moreover, let $f, g$ in $A(K,Y)$ and $f, g \geq 0$

\[ \| f + g \| = \sup \{ \| (f + g)x \| : x \in K \} \]

\[ = \sup \{ \| f(x) + g(x) \| : x \in K \} \]

\[ = \sup_{x \in K} \| f(x) \| + \sup_{x \in K} \| g(x) \| \]

\[ = \| f \| + \| g \| \]

This proves $A(K,Y)$ is an AL-space; similarly for AM-space.

Corollary 1.14 If $Y$ is a vector lattice which is also a base-normed space, then so also is $A(K,Y)$.

Proof: D.A. Edwards had shown that $Y$ is an AL-space if and only if $Y$ is a vector lattice and the norm on $Y$ is a base-norm, which shows that the corollary is clearly can be read out from the Theorem 3.13.
4 Predual of $L$-space

A Banach space $X$ is said be a $L$-space if it is isometric to a space $L(\mu)$ for measure $\mu$ on a locally compact space. If the dual of the Banach space $X$ is a $L$-space, then $X$ is said to be a pre-dual of $L$-space.

In (25) H. Pakhoury showed following:

**Theorem 4.1.** If $X$ and $Y$ are two preduals of an $L$-space, then $A(K,Y)$ is a predual of $L$-space, where $K$ is the closed unit ball of $X$.

We claim that above result is also true in $A(K,Y)$.

**Theorem 4.2.** Let $K$ be a simplex and $Y$ be a predual of an $L$-space. Then $A(K,Y)$ is a predual of an $L$-space.

**Proof:** In (39), J. Lindenstrauss showed that $Y$ is a predual of an $L$-space if and only if it has $\text{4,2,I.P.}$ By a theorem of Lazar (38), he showed that if $Y$ has $\text{n.2.I.P.}$ then $A(K,Y)$ has $\text{n.2.I.P.}$ From these two results, the theorem follows.
Chapter Three

Order properties of the linear operators

1 Introduction

In general the cone \( \mathcal{P} \subset L(X,Y) \) of positive continuous linear transformations from an ordered topological vector space \( X \) into another ordered topological vector space \( Y \) may not be generated. H.H. Schaefer had shown that if the cone in \( X \) is weakly normal and the cone in \( Y \) is generated, then \( \mathcal{P} - \mathcal{P} \) is dense in \( L(X,Y) \) with respect to the topology of simple convergence. In this chapter we present a sufficient condition that \( \mathcal{P} \) is actually a generating cone, when \( X \) and \( Y \) are assumed to be some kind of ordered Banach spaces.

It is quite well known that the space \( L(X,Y) \) of continuous linear transformation from Banach lattice \( X \) into Banach lattice \( Y \) may not be a Banach lattice. We will show \( L(X,Y) \) is a Banach lattice if we impose some more conditions on \( X \). W.M. May (41) had shown some sufficient conditions that \( L(X,Y) \) is a Banach lattice. But his approach is quite different to ours.
2 Preliminary results in $L(X,Y)$

The family $\mathcal{P}$ of all positive linear transformations from an ordered vector space $Y$ is obviously a wedge where $X$, $Y$ are ordered by the cones $P$, $Q$ respectively. The following proposition shows a sufficient condition for $\mathcal{P}$ to be a cone.

**Proposition 2.1.** If $X$ is generated by $P$, then $\mathcal{P}$ is a cone in $L(X,Y)$.

**Proof:** Let $f$ in $\mathcal{P} \cap (-\mathcal{P})$ then there is $g$ in $\mathcal{P}$ such that $f = -g$. Therefore, for any $x$ in $P$

$$f(x) = -g(x) = 0.$$

But

$$g(x) > 0$$

and $Q$ is proper. The only case can happen is that

$$f(x) = g(x) = 0,$$

thus, for all $x$ in $P$, we have

$$f(x) = 0.$$

Since $X = P - P$, hence for all $x$ in $X$

$$f(x) = 0 \quad \text{i.e. } f = 0$$

Hence $\mathcal{P}$ is a cone.
Proposition 2.2. Let \( X, Y \) be ordered normed linear spaces such that \( X \) satisfies \( R_2 \) and \( Y \) satisfies \( R_1 \). Then the continuous linear transformations \( L(X,Y) \) satisfies \( R_1 \).

Proof: Let \( T \) and \( S \) in \( L(X,Y) \) with \( -T \leq S \leq T \) and suppose \( x \) in \( X \) such that \( \| x \| \leq 1 \). If \( z \geq x, 0 \) then:

\[
S(x) = S(x - z) + S(z) \\
= -S(z - x) + S(z) \\
\leq T(z - x) + T(z) \\
= T(2z - x).
\]

Since \( X \) satisfies \( R_2 \), there is \( y \) in \( X \) such that \( x \leq y \) and \( -x \leq y \) with

\[
\| y \| \leq \| x \| \leq 1.
\]

Let \( z = \frac{1}{2}(x + y) \), then

\[
z \geq x, 0
\]

and

\[
S(x) \leq T(y).
\]

By interchanging \( x \) and \( -x \) we have
\[ -S(x) \leq T(y) \]

Since \( Y \) satisfies \( R \), we have

\[ \| S(x) \| \leq \| T(y) \| \leq \| T \| \]

Hence

\[ \| S \| \leq \| T \| \]

Thus the condition of \( R \) of \( L(X,Y) \) is proved.

The above Proposition is a generalization of a result due to E.B. Davies (15). Our proof is similar to his.

The following propositions can be found in Peressini (47).

**Proposition 2.3.** If \( X \) is a nuclear space ordered by a normal cone and \( Y \) is an order complete Banach lattice, then \( L(X,Y) \) is positively generated.

**Proposition 2.4.** If \( X \) is an ordered locally convex space with a weakly normal cone and \( Y \) is a generated ordered locally convex space. Then \( L(X,Y) = \overline{\mathcal{P}} - \overline{\mathcal{P}} \) for the topology of pointwise convergence.

Proposition 2.3 and 2.4 are due to H.H. Shaefer (58).
3 The Space $L(A^*(K), Y)$

We have observed in the last chapter that if $K$ is a simplex, then $A^*(K)$ is vector lattice and the cone $P$ of $A^*(K)$ has the base $K$. In this section, we will assume that $K$ is a simplex, $Y$ is an ordered Banach space, and $L(A^*(K), Y)$ denoted as the family of $w^*$-norm continuous linear transformations from $X$ into $Y$. We pass to our first important theorem in this chapter.

Theorem 3.1. Any element $a$ in $A(K, Y)$ has a unique extension in $L(A^*(K), Y)$.

Proof: Let $a$ in $A(K, Y)$, define

$$T_a(x) = \Lambda a(k)$$

if $x$ in $P$ and $x = \Lambda k$, where $\Lambda$ in $R^+$ and $k$ in $K$.

$$T_a(x) = T_a(x^+) - T_a(x^-)$$

if $x$ in $A^*(K)$ and $x = x^+ - x^-$. Since $A^*(K)$ is a vector lattice, the value of $T_a$ is unique. Hence $T_a$ is well-defined, and $T_a$ is a unique extension of $a'$. The linearity of $T_a$ can be seen from the following:

Let $x$ in $A^*(K)$,
then
\[ x = \lambda y_i - \mu z_i \]
for some \( \lambda \) and \( \mu \) in \( \mathbb{R}^+ \) and \( y_i \) and \( z_i \) in \( K \).

Thus
\[
T_a(x) = \begin{cases} 
\alpha[\lambda a(y_i) - \mu a(z_i)] = \alpha T_a(x) \text{ if } x \geq 0 \\
\alpha[\mu a(z_i) - \lambda a(y_i)] = \alpha T_a(x) \text{ if } x < 0 
\end{cases}
\]

Therefore \( T_a \) is homogeneous.

Let \( x_i \) and \( x_2 \) in \( A*(K) \), then
\[
x_i = \lambda_i y_i - \mu_i z_i
\]
\[
= x_i^+ - x_i^- \quad \quad i = 1, 2
\]
for some \( \lambda_i \) and \( \mu_i \) in \( \mathbb{R}^+ \) and \( y_i, z_i \) in \( K \).

Since
\[
x_i + x_2 = \lambda_i y_i + \lambda_2 y_2 - (\mu_i z_i + \mu_2 z_2)
\]
\[
= (\lambda_i + \lambda_2)\frac{\lambda_i y_i + \lambda_2 y_2}{\lambda_i + \lambda_2} - (\mu_i + \mu_2)\frac{\mu_i z_i + \mu_2 z_2}{\mu_i + \mu_2}
\]

By the fact that \( a \) is affine.
we have

\[ T_a(x_1, x_2) = (\lambda_1 + \lambda_2) \left( \frac{\lambda_1 y_1}{\lambda_1 + \lambda_2} + \frac{\lambda_2 y_2}{\lambda_1 + \lambda_2} \right) \]

\[ - (\lambda_1 + \lambda_2) \left( \frac{\lambda_1 z_1}{\lambda_1 + \lambda_2} + \frac{\lambda_2 z_2}{\lambda_1 + \lambda_2} \right) \]

\[ = (\lambda_1 + \lambda_2) \left( \frac{\lambda_1 y_1}{\lambda_1 + \lambda_2} + \frac{\lambda_1 z_1}{\lambda_1 + \lambda_2} \right) \]

\[ - (\lambda_1 + \lambda_2) \left( \frac{\lambda_2 y_2}{\lambda_1 + \lambda_2} + \frac{\lambda_2 z_2}{\lambda_1 + \lambda_2} \right) \]

\[ = \lambda_1 a(y_1) + \lambda_2 a(y_2) - \lambda_1 a(z_1) - \lambda_2 a(z_2) \]

\[ = \lambda_1 a(y_1) - \lambda_1 a(z_1) + \lambda_2 a(y_2) - \lambda_2 a(z_2) \]

\[ = T_a(\lambda_1 y_1) - T_a(\lambda_1 z_1) + T_a(\lambda_2 y_2) - T_a(\lambda_2 z_2) \]

\[ = T_a(\lambda_1 y_1) - T_a(\lambda_1 z_1) + T_a(\lambda_2 y_2) - T_a(\lambda_2 z_2) \]

\[ = T_a(x_1^+) - T_a(x_1^-) + T_a(x_2^+) + T_a(x_2^-) \]

\[ = T_a(x_1) + T_a(x_2) \]

Therefore, the linearity of \( T_a \) is proved.

The last step is to prove the \( \ast \)-continuity of \( T_a \).

It suffices to show that \( T_a \) is \( \ast \)-continuous on \( P \).

Let \( x \) in \( P \); there is \( \lambda > 0 \) and \( k \) in \( K \) such that \( x = \lambda k \).

Since \( T_a(x) = \lambda a(k) \), for any neighborhood \( N \) of \( \lambda a(k) \)

there is a \( \ast \)-neighborhood \( V \) of \( \lambda a(k) \) such that

\[ \lambda a(V) \subset N \]
Let $W = V \cap K$, then $\lambda W$ is a neighborhood of $\lambda k$.

Therefore

$$T_a(\lambda W) = \lambda T_a(W)$$

$$= \lambda a(W)$$

$$\subseteq \lambda a(V)$$

$$\subseteq N.$$ 

We have proved $T_a$ is continuous at $x$, so that it is continuous on $P$. This completes the proof of the theorem.
Theorem 3.2. If \( Y \) is a positively generated ordered Banach space, then \( L(A^\ast(K), Y) \) is also positively generated.

Proof: Let \( T \) in \( L(A^\ast(K), Y) \) and denote
\[
a = T|_K
\]
then \( a \) is obviously in \( A(K, Y) \). By a theorem of Asimow and Atkinson (8), there is \( \tilde{a} \) in \( A(K, Y) \), such that
\[
\tilde{a} \geq 0, a
\]
By theorem 3.1, we can extend \( \tilde{a} \) to \( \tilde{T}_a \) in \( L(A^\ast(K), Y) \) and
\[
\tilde{T}_a \geq 0, T
\]
In fact, let \( k \) in \( K \), then \( k = \lambda k \), for some \( \lambda \in \mathbb{R}^+ \) and \( k \) in \( K \). We have
\[
\tilde{T}_a(k) = \tilde{T}_a(\lambda k,)
\]
\[
= (\lambda \tilde{a}(k))
\]
\[
\geq \lambda a(k)
\]
\[
= \lambda T(k)
\]
\[
= T(k)
\]
We have shown the directedness of \( L(A^\ast(K), Y) \). Hence it is positively generated.
Corollary 3.3. If \( T_1, T_2 \) are in \( L(\mathcal{A}^*(K), Y) \), then there exists \( T_3 \) in \( L(\mathcal{A}^*(K), Y) \), such that \( T_3 \succeq T_1, T_2 \) whenever \( Y \) is directed upwards.

Theorem 3.4. If \( Y \) is an ordered Banach space having R.S.P. Then \( L(\mathcal{A}^*(K), Y) \) has R.S.P.

Proof: Let \( a, b, c, d \) in \( L(\mathcal{A}^*(K), Y) \) with
\[
 a, b \preceq c, d
\]
then \( a_K, b_K, c_K, d_K \) are in \( A(K, Y) \) and satisfying
\[
 a_K, b_K \preceq (c_K, d_K)
\]
By Theorem 2.4. in (8) there exists \( z \) in \( A(K, Y) \) such that
\[
 a_K, b_K \preceq z \preceq c_K, d_K
\]
By Theorem 3.1., extend \( z \) to \( \tilde{z} \) on \( \mathcal{A}^*(K) \) and with the property
\[
 a, b \preceq \tilde{z} \preceq c, d
\]
We only show \( a \preceq \tilde{z} \), the other cases follow in the way. Let \( x \) in \( P \) there is \( \lambda > 0 \) and \( k \) in \( K \) such that \( x = \lambda k \).
Therefore

\[(\tilde{\ell} - a)|x| = \tilde{\ell}(\lambda k) - a(\lambda k)\]

\[= \lambda(\ell(\overline{k}) - a(k))\]

\[\geq 0\]

This shows that \(a \in \overline{\ell}\), and the Theorem follows.

The simplex \(K\) is a basis of the positive cone \(P\) in \(A^*(k)\), which is positively generated because it is a vector lattice. Hence \(K\) can induce a norm in \(A^*(k)\) which becomes a base-norm space. For every weak*–continuous linear transformation from \(A^*(k)\) into the ordered Banach space \(Y\) is also continuous with respect to the base-norm. Therefore we can define a norm in \(L(A^*(k), Y)\).

For \(f\) in \(L(A^*(k), Y)\), define

\[\|f\| = \sup\{|f(x)| : \|x\| \leq 1, \text{i.e.} \ x \in \text{co}(K \cup K)\}\]

It is not hard to show \(\|\|\) is a norm in \(L(A^*(k), Y)\) which is also a Banach space. With this remark in mind, we pass to the next theorem.
Theorem 3.5. If $Y$ is an ordered Banach space with any one of the following properties,

(a) Normal space.

(b) Having a directed upward unit ball.

(c) $(\sim n)$-directed, $(\sim n)$-approximate directed.

(d) Regular space.

Then $L(A^*(K), Y)$ has this property.

Proof: Let $f, g, h$ in $L(A^*(K), Y)$ such that $f \leq g \leq h$.

We have

$$
\|g\| = \sup \left\{ \|g(x)\| : x \in \text{co}(K \cup -K) \right\}
= \sup \left\{ \|g(x)\| : x \in K \right\}
= \sup \left\{ \max \{\|h(x)\|, \|f(x)\| : x \in K \right\}
= \max \{\|f\|, \|h\| \}
$$

Therefore the proof of (a) is completed.

(b) Let $a, b$ in $L(A^*(K), Y)$ with $\|a\|, \|b\| \leq 1$.

It is obvious that $a|_K, b|_K$ are in $A(K, Y)$.

By Theorem 3.4. in Chapter Two, there is $c$ in $A(K, Y)$.
such that
\[ a, b \leq c \]
and \[ \| c \| \leq 1 \]
By Theorem 3.1, extend \( c \) to \( \tilde{c} \) on \( A^*(K) \), then
\[ a, b \leq \tilde{c} \]
For \( x \) in \( P \), then \( x = \lambda y \) for some \( \lambda > 0 \) and \( y \) in \( K \)
thus
\[ \tilde{c}(x) = \lambda c(y) \]
\[ \geq \lambda a(y), \lambda b(y) \]
\[ = a(\lambda y), b(\lambda y) \]
\[ = a(x), b(y) \]

We have to prove that \( c \) is in the closed unit ball of \( L(A^*(K), Y) \).

\[ \| \tilde{c} \| = \sup \{ \| c(x) \| : x \in \text{co}(K \cup -K) \} \]
\[ = \sup \{ \| c(x) \| : x \in K \} \]
\[ = \sup \{ \| c(x) \| : x \in K \} \]
\[ \leq 1 \]

Hence (b) is proved.

(c) and (d) can be proved in the similar way.
Theorem 3.6. If $Y$ is a simplex space, so also is $L(A^*(K), Y)$.

Proof: By Theorem 3.4 and 3.5.

Theorem 3.7. Let $K$ be a Bauer simplex and $Y$ a Banach lattice. Then so also is $L(A^*(K), Y)$. Similarly, if $Y$ is either an AL-space or an AM-space so also is $L(A^*(K), Y)$.

Proof: Let $a$ in $L(A^*(K), Y)$; then $a|_K$ is in $A(K, Y)$. By Theorem 3.13, in Chapter Two, there is $b$ in $A(K, Y)$ such that,

$$b = a|_K \lor 0$$

By Theorem 3.1, we extend $b$ to $\tilde{b}$ in $L(A^*(K), Y)$ then

$$\tilde{b} \geq 0, a$$

For $p$ in $P$ then $p = \lambda k$ for some $\lambda \in \mathbb{R}^+$ and $k$ in $K$ we have

$$\tilde{b}(p) = \tilde{b}(\lambda k) = \lambda \tilde{b}(k) = \lambda \tilde{b}(k) \geq 0$$
Similarly, we can show $\tilde{b} \geq a$.

For any $f$ in $L(A^*(K), Y)$, such that $f \geq 0$, $a$,

$$\begin{align*}
(f - \tilde{b})^p &= (f - \tilde{b})^{\lambda k} \\
&= \lambda (f(k) - \tilde{b}(k)) \\
&= \lambda (f(k) - b(k)) \\
&\geq 0.
\end{align*}$$

This shows that $\tilde{b} = a \vee 0$ and $L(A^*(K), Y)$ is a vector lattice. Since $A^*(K)$ is a base-normed space, for any $f$ in $L(A^*(K), Y)$, we have

$$\|f\| = \sup_{x \in \mathcal{K} \cup \partial \mathcal{K}} \|f(x)\| = \sup_{x \in \mathcal{K}} \|f(x)\| = \sup_{x \in \partial \mathcal{K}} \|f(x)\|.$$ 

Following the proof as in Theorem 3.13 in Chapter Two, we can show

(a) $\|f\| = \|f\|$ for any $f$ in $L(A^*(K), Y)$.

(b) $\|f\| \leq \|g\|$ for $f \leq g$ in $L(A^*(K), Y)$.

$L(A^*(K), Y)$ is already a Banach space, therefore it is a
Banach lattice. It is easy to prove that $L(A^*(k), Y)$ is an AL-space and AM-space.

In (22) A.J. Ellis shows that if $X$ is a base-normed space and $Y$ is a order-unit space then $L(X, Y)$ is also a order-unit space. He gives a counter example that if $X$ is an order-unit space and $Y$ is a base-norm then $L(X, Y)$ may not be a base-norm space.

**Corollary 3.8.** If $Y$ is a vector lattice and a base-norm space then so also is $L(A^*(k), Y)$. 
Chapter Four

The Centre of $A(K,Y)$

In 1970 Alfsen and Andersen (4) gave sufficient conditions in order that a real-valued continuous function $f$, defined on the extreme boundary of a compact convex set $K$ should have a unique affine continuous extension to $A(K)$.

Theorem 1. If $K$ is a compact convex set and

$$f : \partial K \to \mathbb{R}$$

is continuous with respect to the facial topology on $\partial K$, then $f$ has unique extension in $A(K)$.

The set of such functions in $A(K)$ which are extensions of facially continuous functions on the boundary $\partial K$ is called the centre of $A(K)$. Alfsen and Andersen showed that the centre of $A(K)$ is a Banach lattice with respect to the ordering induced on it by $A(K)$, and that the ordering in the centre is pointwise on $\partial K$.

In 1969 H. Fakhoury (26) introduced the concept of strongly lattice ordered subspace (espace fortement réticulé) in $A(K)$. The following is his definition.

Definition. Let $K$ be a compact convex set. A subspace $H$ of $A(K)$ is strongly lattice ordered, if

(a) $H$ is a vector lattice for the relative order:
(b) If \( f \) and \( g \) are in \( H \) then

\[ (f \sup_{H} g) \kappa = f(\kappa) \vee g(\kappa) \]

for any \( \kappa \) in the extreme boundary of \( K \).

He proved the following result.

**Theorem 2.** Let \( K \) be a simplex, then the centre of \( A(K) \) is the largest strongly lattice ordered subspace of \( A(K) \) containing the constants.

In 1972 (9), H.R. Atkinson generalized the concept of centre of \( A(K) \) to \( A(K,Y) \), where \( Y \) is a Banach space.

**Definition.** The centre of \( A(K,Y) \) is the set

\[ \{ a \in A(K,Y) : a \in C_f(\partial K,Y) \} \]

where \( C_f(\partial K,Y) \) is the set of all the facially continuous functions on the extreme boundary \( \partial K \).

In what follows we answer a conjecture of Atkinson and at the same time we generalize Fakhoury's Theorem.

**Theorem 3.** Let \( K \) be a simplex and \( Y \) be a Banach lattice. If \( H \) is a strongly lattice ordered subspace of \( A(K,Y) \), then

\[ H \subseteq \text{centre of } A(K,Y) \]
Proof: Let $f$ in $H$; we want to prove $f$ is facially continuous. Since $Y$ is a normed space, the norm compact and weakly compact subsets coincide. Therefore $f(K)$ is weakly compact. It suffices to show $u \cdot f$ is in the centre of $A(K)$ for every $u$ in $Y^*$, i.e. $u \cdot f$ is facially continuous. Let

$$L = \{ u \cdot f : u \in C(f(\emptyset K)) \}$$

Then $L$ is a strongly lattice ordered subspace of $A(K)$.

In fact, let $u_1$, $u_2$ in $C(f(\emptyset K))$.

By Proposition 3.3. in (26.), it suffices to show

$$(u_1 \cdot f) \vee (u_2 \cdot f)$$

exists in $A(K)$ and belongs to $L$.

For any $k$ in $\emptyset K$, we have

$$(u_1 \cdot f \vee u_2 \cdot f)k \geq u_1 \cdot f(k), \; u_2 \cdot f(k)$$

$= u_1(f(k)) \vee u_2(f(k))$

i.e.

$$(u_1 \cdot f \vee u_2 \cdot f)k \geq (u_1 \vee u_2) f(k)$$

hence

$$u_1 \cdot f \vee u_2 \cdot f \geq (u_1 \vee u_2) \cdot f \ldots \ldots (1)$$
On the other hand
\[ u_1 \lor u_2 \supseteq u_1, u_2 \]
thus
\[ (u_1 \lor u_2)(f(k)) \supseteq u_1(f(k)) \quad \text{and} \quad u_2(f(k)) \]
for any \( k \) in \( \partial K \),
i.e. \((u_1 \lor u_2)(f(k))\) is an upper bound of \( u_1(f(k)) \) and \( u_2(f(k)) \), so that
\[ (u_1 \lor u_2)(f(k)) \supseteq u_1(f(k)) \lor u_2(f(k)) \]
thus
\[ (u_1 \lor u_2) \circ f \supseteq u_1 \circ f \lor u_2 \circ f \quad \ldots \quad (2) \]
Combining (1) and (2), we have
\[ u_1 \circ f \lor u_2 \circ f = (u_1 \lor u_2) \circ f \]
It shows that \((u_1 \lor u_2) \circ f\) exists in \( A(K) \) and belongs to \( L \). Therefore \( L \) is a strongly lattice ordered subspace of \( A(K) \). Thus
\[ L \subseteq \text{centre of } A(K) \]
i.e. \( u \circ f \) is facially continuous for every \( u \) in \( \mathcal{X}^* \) and \( f \) is facially continuous. Hence
\[ H \subseteq \text{centre of } A(K,\mathcal{X}) \]
Chapter Five

Tensor Products

The tensor product of ordered topological vector spaces have been studied systematically by A.L.Peressini and R.Sherbert in 1960 (48) and also by N.Popá in 1968 and 1969 (50), (51), (52). They have concentrated on the theory of ordered topological tensor product and ordered spaces of bilinear mappings. In 1971 (27) D.H.Fremlin intensively studied the tensor product of Archimedean vector lattices. In 1968 (16) E.B. Davies and G.F.Vincent-Smith and A.J.Lazar (36) introduced the concept of tensor product of simplexes independently. General tensor products of compact convex sets were studied by Z.Semadeni in 1967 (56), Namioka and R.R.Phelps in 1969 (45) and by E.Behrends and G.Wittstock in 1970 (13). In 1973 (59) H.H. Shaefer gave a good account of the tensor product of the Banach lattices.

Let us define the order relation for the tensor product of two ordered vector spaces and some of its order theoretic properties.

Definition. Let \( X, Y \) be two ordered vector spaces having cones \( P, Q \) respectively. The set defined
\[ P \otimes Q = \left\{ \sum_{i=1}^{n} x_i \otimes y_i : x_i \in P, y_i \in Q \right\} \]

is called a projective wedge in \( X \otimes Y \).

In general \( P \otimes Q \) is not a cone, but it is always a wedge. If it is a cone then \( X \otimes Y \) becomes an ordered vector space and called a projective tensor product of \( X \) and \( Y \). We simply call it the tensor product.

The proposition reflects that \( X \otimes Y \) inherits some order properties of \( X, Y \).

**Proposition 1.** Let \( X, Y \) be two vector spaces ordered by the cones \( P \) and \( Q \) respectively. If \( P \) and \( Q \) satisfy:

1. Archimedean.
2. Positively generated or Directed upwards.

Then \( X \otimes Y \) has the same properties.

Proof: It is a direct verification or see (48).

The tensor product of two vector lattices may not be a vector lattice. This can see from an example due to Krenkel. Let \( X = l^2 = Y \) then \( l^2 \otimes l^2 \) is not a vector lattice. In fact, if it is a vector lattice so also is \( (l^2 \otimes l^2)' \). But \( (l^2 \otimes l^2)' = L(l^2, l^2) \) is not generated, shown by Krenkel. Hence the contradiction arises. On the other hand H. Fremlin showed for some particular vector lattices
the tensor product one can yield some meaningful results.
One of his main result in his paper (27) is as follows.

Theorem 2. Let \( X, Y \) be Archimedean vector lattice spaces. Then \( X \otimes Y \) can be embedded as a dense subspace of a unique Archimedean vector lattice.

He showed an example that the above Theorem may fail if the Archimedean condition is discarded. We show that the tensor product of two order-unit Banach spaces having R.S.P., is dense in an order-unit Banach space with R.S.P.

**Definition.** The projective tensor product of two compact convex sets \( K_1 \) and \( K_2 \) is defined to be the pair \((K_1 \otimes K_2, \omega)\) satisfying the universal mapping property, i.e. \( \omega \) is a continuous biaffine mapping \( K_1 \times K_2 \to K_1 \otimes K_2 \) then for every continuous biaffine mapping

\[
\varphi : K_1 \times K_2 \to K_3
\]

where \( K_3 \) is any compact convex set, there is a continuous biaffine mapping

\[
h : K_1 \times K_2 \to K_3
\]

which satisfies \( \varphi = h \cdot \omega \).

It can be shown equivalently that the projective tensor product of two compact convex sets \( K_1, K_2 \) is the state space of the continuous biaffine mappings \( BA(K_1 \times K_2) \) equipped with the \( w^* \)-topology.
E. B. Davies and G. F. Vincent-Smith (16) and A. J. Lazar (36) showed the following result.

**Theorem 3.** If \( K_1 \) and \( K_2 \) are simplexes, then \( K_1 \otimes K_2 \) is also a simplex.

E. B. Davies and G. F. Vincent-Smith (16) also proved

**Theorem 4.** If \( K_1 \) and \( K_2 \) are simplexes, then

\[
\bar{A}(K_1) \otimes \bar{A}(K_2) = BA(K_1 \times K_2).
\]

Namioka and Phelps (45) showed:

**Theorem 5.** Let \( K_1 \) and \( K_2 \) be two compact convex sets and \( A(K) \) has the metric approximation property (M.A.P.) then

\[
\bar{A}(K_1) \otimes \bar{A}(K_2) = BA(K_1 \times K_2).
\]

Behrends and Wittstock showed in (13),

**Theorem 6.** If \( K_1 \) is a simplex and \( K_2 \) is a compact convex set, then

\[
\bar{A}(K_1) \otimes \bar{A}(K_2) = BA(K_1 \times K_2).
\]
Now we come to our last result.

Theorem 7. Let $X$, $Y$ be two order-unit Banach spaces having R.S.P. Then $X \hat{\otimes} Y$ is an order-unit Banach space with R.S.P.

Proof: $X$, $Y$ is order-unit Banach space; see Ellis (23).

Let $K_1$, $K_2$ be the state spaces of $X$, respectively; then by Kadison Representation Theorem we have

$$X \cong A(K_1) \quad \text{and} \quad Y \cong A(K_2)$$

Since $X$, $Y$ have R.S.P., then $K_1$, $K_2$ are simplex sets. By a result of A.Grothendieck $A(K)$ and $A(K)$ have the M.A.P., and they have R.S.P. By a result of Namioka and Phelps we have

$$A(K_1) \hat{\otimes} A(K_2) = BA(K_1 \times K_2)$$

But

$$BA(K_1 \times K_2) = A(K_2, A(K_1))$$

By a Theorem of Asimow and Atkinson, $A(K_2, A(K_1))$ has R.S.P., whenever $A(K)$ has. This completes the proof of the Theorem.
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