Plane electromagnetic waves in layered periodic dielectric structures.

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Plane Electromagnetic Waves in
Layered Periodic Dielectric Structures

by

Gregory V. Morozov

A Dissertation Submitted to
the Faculty of Graduate Studies and Research
through the Department of Physics
in Partial Fulfillment
of the Requirements for the Degree of
Doctor of Philosophy
at the University of Windsor

Windsor, Ontario, Canada
May 2001

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To My Mother and My Father
Abstract

Several analytic methods are used to derive and discuss analytic expressions for electromagnetic wave fields in finite one-dimensional layered structures. A new modification is obtained for the so-called multiple reflection method. Special attention is given to layered periodic dielectric structures. The systematic dependence of the reflection coefficient on the parameters characterizing this type of structure is studied in detail, using the two-layered periodic dielectric structure as a typical example. A general method for the construction of the Green's function for finite one-dimensional layered structures is developed. The sum of the total Neumann (Born) series is calculated for sufficiently simple cases of perturbations in the profile of the refractive index. Using the Green's function, the influence of fluctuations of the width of the basic layers on the reflection and transmission of electromagnetic waves propagating through the two-layered periodic dielectric structure is investigated. The results are applied to the design of optical switching systems with periodic dielectric structures as the operating medium.
Acknowledgements

I acknowledge my deep gratitude to my supervisors Dr. Gordon W. F. Drake and Dr. Roman Gr. Maev for their supervision throughout the course of this work. Their insightful guidance and suggestions were greatly appreciated.

Special thanks are also due to Dr. S. N. Stolyarov from Moscow Institute of Physics and Technology for great deal of help and advice in my research.

I wish to thank the Faculty of Graduate Studies, the Department of Physics of the University of Windsor, and the Government of Ontario for providing me with full financial support.

I wish to thank Dr. W. E. Baylis from Physics Department and Dr. N. G. Zamani from Mathematics Department whose lecture courses at University of Windsor greatly influenced my knowledge about theoretical physics and numerical methods.

I am very thankful to Ph.D. students Alex Denissov and Brian O’Neill for their help in computer calculations and technical preparation of this thesis.

Further acknowledgement must go to the research team in the Centre for Imaging Research and Advanced Material Characterization and especially to Dr. Serge Titov, for their advice and support.

Finally, the most thanks I would like to give to my parents without whose constant and huge support this work would never have been possible.
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Chapter 1

Introduction

The propagation of waves through one-dimensional layered dielectric media, especially in the case of layered periodic structures, has long been a topic of interest in various areas of physics and technology, beginning with the well-known paper of Kronig and Penny [1]. There are two reasons for such interest. From a technical point of view, these media have a great importance in many unique applications of optical holography [2], acousto-optics [3], integrated and optical electronics [4, 5, 6], X-ray diffractometry [7, 8], and especially in quantum semiconductor devices (superlattices) [9, 10, 11]. From a mathematical point of view, a one-dimensional structure of homogeneous layers is one of the very few cases where it is possible to find in principle exact analytic solutions. Therefore, although a one-dimensional layered model is invalid in practical cases more often than not, the results obtained in this way are useful in the qualitative analysis and in approximate descriptions of the physics of real structures. For example, techniques developed initially for the electron theory of one-dimensional crystals have been successfully applied to calculate the band structure of electromagnetic waves propagating in either 2D or 3D periodic dielectric structures (photonic band gap crystals) [12, 13]. Moreover, the first structure to have a photonic band gap was found theoretically [14], and only after was it confirmed experimentally [15]. As a result, the one-dimensional model in fact is becoming a basic instrument to improve our understanding of processes occurring in real structures on the one hand, and to develop and test quantitative methods for their analysis on the other.

Despite the fact that there is a large body of literature in the area, including such
well-known monographs as \cite{16, 17, 18, 19, 20}, recent publications \cite{21, 22, 23, 24, 25} demonstrate that there are still many unsolved problems, both theoretical and experimental, concerning the layered periodic structures. In the following chapters of this dissertation some of these problems will be theoretically investigated and, where possible, the results will be broadened for arbitrary finite one-dimensional inhomogeneous layers.

Basically, there are three main goals in this dissertation. The first one is to develop a new modification of the multiple reflection method for an approximate analytic calculation of reflection and transmission coefficients of a layered dielectric structure with a significant number of layers. The second goal is to investigate in a systematic way the reflection and transmission properties of one-dimensional layered periodic dielectric structures with uniform layers in an exact analytic form using a two-layered periodic structure as an example. The third goal is to consider wave propagation in one-dimensional dielectric structures with fluctuations in layer thicknesses. The results show the possibility of using a two-layered dielectric structure as an operating medium for optical switching systems, by shifting the electromagnetic wave from a forbidden to an allowed frequency region. The apparatus of Green's functions will be developed as the most elegant way for solving such a problem. The convergence of the total Born (Neumann) series for a simpler case of the step profile with a single defect in the refractive index will be shown.

The dissertation is organized as follows. In Chapter II we provide researchers acquainted with electromagnetic waves in isotropic layered structures a comprehensive survey of common analytical methods currently in use for calculating fields in such materials, as well as a review of some of the significant papers on the subject. Therefore, despite the fact that the bibliography is by no means complete it is intended to be representative enough to provide a starting point for those who are interested in further research. Moreover, the whole dissertation is written in such a way that it should be understandable to a new graduate student beginning work in this area.

In Chapter III we consider a special case of layered structures - layered periodic
structures. The method of analytic solution for such structures is developed in terms of Floquet-Bloch waves. The comparison with the transfer matrix method for such structures is done. Special attention is given to a two-layered periodic structure (two sub-layers on a unit cell). In particular, we extract exact analytic expressions for the reflection and transmission coefficients for the case of normal angle of incidence in a simple and physically understandable form. Then, we investigate these expressions in a systematic way depending on the structure parameters. A brief review of other analytic methods available for the description of waves in periodic structures is done at the end of the Chapter.

In Chapter IV we develop a general method for the construction of an exact analytical form of the Green's function for one-dimensional problems and show how to make practical use of this function for the calculation of the reflection coefficient. Moreover, we investigate the convergence of the Neumann (Born) series for the reflection coefficient in some simple cases. Then, we apply the results to the calculation of the reflection coefficient of a two-layered periodic structure with fluctuations in the layer thicknesses. After that, with the aid of the results of the previous chapter concerning a two-layered periodic structure, we consider the possibility of shifting of the electromagnetic wave in such structures from a forbidden to an allowed region under the action of an elastic stress. As a result, we suggest theoretical guidelines for the construction of optical switching systems with a two-layered periodic dielectric structure as an operating medium. In particular, we identify the structure parameters, including the limits of fluctuations in them, which are tolerable for such switches.

Finally, the conclusions and suggestions for future work are summarized in Chapter V.

We should note that the calculation of reflection properties of a two-layered periodic dielectric structure with fluctuations in the layer thicknesses is just one example of a wide range of problems that can be solved using the technique developed in Chapter IV. Among other problems we only mention acousto-optical coupling in dielectrics, nonlinear phenomena etc. For such problems the exact solution is not known even in
numerical form. As a result, the Green’s function provides a significant mathematical tool to generate approximate analytical solutions for these problems.

The Gaussian system of units is used throughout all this work. The SI system is not used at all, as it has two serious disadvantages in comparison with the Gaussian system. First, in addition to three basic mechanical quantities, which are the length, the time, and the mass, the SI system has the fourth purely electric basic quantity with self-independent dimension. This is the current strength with the unit of ampere. Second, the Maxwell’s equations are written in the SI system using so-called “rationalized” form that contains no numerical factors. As a result, the dimensions of all four vectors \( \mathbf{E}, \mathbf{D}, \mathbf{B}, \) and \( \mathbf{H} \) are different from each other. Even in vacuum the vector \( \mathbf{E} \) is different from the vector \( \mathbf{D} \), and the vector \( \mathbf{B} \) is different from the vector \( \mathbf{H} \) in the SI system:

\[
\mathbf{D} = \varepsilon_0 \mathbf{E}, \quad \mathbf{B} = \mu_0 \mathbf{H}. \tag{1.1}
\]

The constants \( \varepsilon_0 \) and \( \mu_0 \) have no real physical sense and serve only as artificial dimension coefficients to convert the same physical quantities in vacuum \( \mathbf{E}, \mathbf{D} \) and \( \mathbf{B}, \mathbf{H} \) from one units to others. However, these constants are related as

\[
\varepsilon_0 \mu_0 = \frac{1}{c^2}, \tag{1.2}
\]

i.e. only the combination \( \varepsilon_0 \mu_0 \) has real physical sense. In the linear medium with a dielectric permittivity \( \varepsilon \) and magnetic permittivity \( \mu \) the relations (1.2) take the form

\[
\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu_0 \mu \mathbf{H}, \tag{1.3}
\]

i.e. the dielectric permittivity \( \varepsilon \) and magnetic permittivity \( \mu \) in the SI system are the same as in the Gaussian system.

The basic rules to convert the values of \( \mathbf{E}, \mathbf{D}, \mathbf{B}, \) and \( \mathbf{H} \) from the Gaussian system to the SI system in an arbitrary medium are as follows:

\[
\mathbf{E} \to \sqrt{\frac{4\pi \varepsilon_0}{4\pi \varepsilon}} \mathbf{E}, \quad \mathbf{D} \to \sqrt{\frac{4\pi}{\varepsilon_0}} \mathbf{D}, \tag{1.4}
\]

\[
\mathbf{B} \to \sqrt{\frac{4\pi}{\mu_0}} \mathbf{B}, \quad \mathbf{H} \to \sqrt{4\pi \mu_0} \mathbf{H}.
\]
Chapter 2

Mathematical Methods for Wave Description in Layered Structures

The main purpose of this chapter is to review from a unified standpoint the methods which are currently available for the analysis of wave propagation in one-dimensional layered structures, to expose similarities and differences between these methods and to compare their range of validity. It is certainly impossible to provide an exhaustive review of all methods. We concentrate our attention on the methods that lead to analytical solutions either in an exact or approximate form.

2.1 General Properties of Waves in Layered Structures

First, we give some basic information about the geometry and physical parameters of the problem at hand. A dielectric, nonmagnetic ($\mu = 1$) structure of length $L$ with a position-dependent dielectric permittivity $\varepsilon_L(z)$ is surrounded by two semi-infinite media with constant dielectric permittivities $\varepsilon_0$ and $\varepsilon_f$. In further consideration, for $\varepsilon_f$ we will use sometimes the notation $\varepsilon_{m+1}$, i.e. during this work $m + 1 \equiv f$. Mathematically, the dielectric permittivity $\varepsilon(z)$ of a such profile can be represented in the form

$$
\varepsilon(z) = \begin{cases} 
\varepsilon_0, & z < 0, \\
\varepsilon_L(z), & 0 < z < L, \\
\varepsilon_f, & z > L.
\end{cases}
$$

(2.1)
Let us represent a dielectric permittivity $\varepsilon_L(z)$, $0 < z < L$, by a set of $m$ inhomogeneous layers:

$$
\varepsilon_L(z) = \begin{cases} 
\varepsilon_1(z), & z_0 < z < z_1, \\
\varepsilon_2(z), & z_1 < z < z_2, \\
\varepsilon_j(z), & z_j < z < z_{j+1}, \\
\vdots & \\
\varepsilon_m, & z_{m-1} < z < z_m,
\end{cases}
$$

(2.2)

where $z_0 = 0$, $z_m = L$, $d_0 = 0$, and $d_j = z_j - z_{j-1}$, where $j = 1, 2, \ldots, m$, is the thickness of a layer with a dielectric permittivity $\varepsilon_j(z)$. If a layer $\varepsilon_j(z)$ has absorbing or amplifying properties, its permittivity becomes complex: $\varepsilon(z) = \varepsilon'(z) + i\varepsilon''(z)$. During this work we will suppose that the dielectric permittivity $\varepsilon_0$ is a real quantity, but all other permittivities $\varepsilon_j(z)$, $j = 1, 2, \ldots, m$, and $\varepsilon_f$ are complex in general. Very often, we will additionally suppose that our layers are homogeneous (uniform), emphasizing this by the notation $\varepsilon_j$ instead of $\varepsilon_j(z)$. The geometry of the problem for this case is shown in Figure 2.1.

![Figure 2.1: Multilayered structure of homogeneous dielectric layers](image)

We start our investigation of electromagnetic field in the medium (2.1) with Maxwell’s equations. For monochromatic waves, $\mathbf{E}(x,y,z,t) = \mathbf{E}(x,y,z) \exp(-i\omega t)$, propagating in this medium they have the form [26]

$$
\nabla \times \mathbf{E} = \frac{i\omega}{c} \mathbf{H}, \\
\nabla \times \mathbf{H} = -i\varepsilon(z)\frac{\omega}{c} \mathbf{E},
$$

(2.3)

$$
\nabla \cdot \varepsilon(z) \mathbf{E} = 0, \\
\nabla \cdot \mathbf{H} = 0,
$$

(2.4)
where \( \nabla = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} \) in the Cartesian coordinate system, \( c \) is the velocity of the light in vacuum, and \( \omega \) is an angular frequency of a monochromatic wave. Substituting for \( \mathbf{H} \) from the first equation into the second and taking into account the third, we obtain for \( \mathbf{E} \) the equation

\[
\Delta \mathbf{E} + k^2 \varepsilon(z) \mathbf{E} + \nabla \left( E_z \frac{\partial \ln \varepsilon(z)}{\partial z} \right) = 0, \tag{2.5}
\]

where \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \) in the Cartesian coordinate system, \( k = \frac{\omega}{c} \) is the wave number of an electromagnetic wave in vacuum. Analogous elimination of \( \mathbf{E} \) gives for \( \mathbf{H} \) the equation

\[
\Delta \mathbf{H} + k^2 \varepsilon(z) \mathbf{H} + \frac{\partial \ln \varepsilon(z)}{\partial z} \hat{e}_z \times (\nabla \times \mathbf{H}) = 0, \tag{2.6}
\]

where \( \hat{e}_z \) is the unit vector along the \( z \)-direction. Assuming only plane waves (no spherical or cylindrical waves, for example), due to the symmetry of the problem, without loss of generality we can consider only waves whose direction of propagation lies in the \( xz \)-plane. In such waves all quantities do not depend on \( y \), and the uniformity of the medium in the \( x \)-direction means that the dependence on \( x \) can be taken as being through a factor \( \exp(ik\kappa x) \), where \( \kappa \) is a constant. For \( \kappa = 0 \) the field depends only on \( z \), i.e. we have a wave which is said to pass normally through a layered structure \( \varepsilon_L(z) \). If \( \kappa \neq 0 \) a wave is said to pass obliquely.

There are two independent cases of polarization for \( \kappa \neq 0 \). In the first case, the vector \( \mathbf{E} \) is perpendicular to the plane of wave propagation (\( xz \)-plane), i.e. it is in the \( y \)-direction, and the magnetic field \( \mathbf{H} \) lies in \( xz \)-plane. Such waves are called \( TE \) or \( s \)-waves. For the amplitude \( E(z) \) of the vector \( \mathbf{E} = E(z) \exp[i(k\kappa - \omega t)] \hat{e}_y \), where \( \hat{e}_y \) is the unit vector along the \( y \)-direction, Eq. (2.5) becomes

\[
\frac{d^2 E(z)}{dz^2} + k^2 \left( \varepsilon(z) - \kappa^2 \right) E(z) = 0. \tag{2.7}
\]

Assuming, that we know a fundamental system of solutions \( \alpha_j(z) \), \( \beta_j(z) \) of this equation in each inhomogeneous layer \( \varepsilon_j(z) \), \( j = 1, 2, ..., m \), and using the first of Maxwell's equations (2.3), we can represent electric and magnetic field of \( s \)-wave in each layer
as

\[ E_{jy}(x, z, t) = (A_j \alpha_j(z) + B_j \beta_j(z)) \exp \left[ i(k\kappa x - \omega t) \right], \]

\[ H_{jx}(x, z, t) = \frac{i}{k} \left( A_j \alpha'_j(z) + B_j \beta'_j(z) \right) \exp \left[ i(k\kappa x - \omega t) \right], \quad (2.8) \]

\[ H_{jz}(x, z, t) = \kappa \left( A_j \alpha_j(z) + B_j \beta_j(z) \right) \exp \left[ i(k\kappa x - \omega t) \right]. \]

For semi-infinite media \( \varepsilon_0 \) and \( \varepsilon_f \) Eqs. (2.8) are obviously valid with

\[ \alpha_0(z), \beta_0(z) = \exp(\pm ik_0 z), \]

\[ \alpha_f(z), \beta_f(z) = \exp(\pm ik_f z(z - L)), \quad (2.9) \]

where we introduce wave vectors \( k_0 \) and \( k_f \)

\[ k_0^2 = k^2 \varepsilon_0, \quad k_f^2 = k^2 \varepsilon_f; \]

\[ k_{0z} = k \sqrt{\varepsilon_0 - \kappa^2}, \quad k_{0z} = k \kappa; \quad (2.10) \]

\[ k_{fz} = k \sqrt{\varepsilon_f - \kappa^2}, \quad k_{fz} = k \kappa. \]

Boundary conditions at each interface \( \varepsilon_j/\varepsilon_{j+1} \) require the continuity of tangential components of \( E_{jy} \) and \( H_{jx} \). Therefore, boundary conditions give us a system of \( 2(m + 1) \) equations for determination of \( 2(m + 2) \) coefficients \( A_0, B_0, A_j, B_j, \) and \( A_f, B_f \)

\[ A_0 + B_0 = A_1 \alpha_1(z_0) + B_1 \beta_1(z_0) \]

\[ i k_{0z} (A_0 - B_0) = A_1 \alpha'_1(z_0) + B_1 \beta'_1(z_0) \]

\[ A_1 \alpha_1(z_1) + B_1 \beta_1(z_1) = A_2 \alpha_2(z_1) + B_2 \beta_2(z_1) \]

\[ A_1 \alpha'_1(z_1) + B_1 \beta'_1(z_1) = A_2 \alpha'_2(z_1) + B_2 \beta'_2(z_1) \]

\[ A_j \alpha_j(z_j) + B_j \beta_j(z_j) = A_{j+1} \alpha_{j+1}(z_j) + B_{j+1} \beta_{j+1}(z_j) \quad (2.11) \]

\[ A_j \alpha'_j(z_j) + B_j \beta'_j(z_j) = A_{j+1} \alpha'_{j+1}(z_j) + B_{j+1} \beta'_{j+1}(z_j) \]

\[ A_m \alpha_m(z_m) + B_m \beta_m(z_m) = A_f + B_f \]

\[ A_m \alpha'_m(z_m) + B_m \beta'_m(z_m) = ik_{fz} (A_f - B_f). \]

Two coefficients in this system can be found from physical conditions that will be imposed later in this Chapter.

In the second case of polarization, the vector \( \mathbf{H} \) is in the \( y \)-direction, and \( \mathbf{E} \) lies in the plane of propagation (\( xz \)-plane). Such waves are called \( TM \) or \( p \)-waves. For this
case it is more convenient to use the equation for the amplitude $H(z)$ of the vector $\mathbf{H} = H(z) \exp [i(k\kappa x - \omega t)] \mathbf{e}_y$. According to (2.6) it takes the form

$$
\frac{d}{dz} \left( \frac{1}{\varepsilon(z)} \frac{dH(z)}{dz} \right) + k^2 \left( 1 - \frac{\kappa^2}{\varepsilon(z)} \right) H(z) = 0. \tag{2.12}
$$

Now we can repeat all operations that we have just made for $TE$-waves. Assuming that we know a fundamental system of solutions $\alpha_j(z), \beta_j(z)$ of this equation in each inhomogeneous layer $\varepsilon_j(z)$, and using now the second of Maxwell’s equations (2.3), we can represent magnetic and electric field of a $p$-wave in each layer as

$$
\begin{align*}
H_{jy}(x, z, t) &= (A_j \alpha_j(z) + B_j \beta_j(z)) \exp [i(k\kappa x - \omega t)], \\
E_{jx}(x, z, t) &= -\frac{i}{\kappa \varepsilon_j(z)} (A_j \alpha_j'(z) + B_j \beta_j'(z)) \exp [i(k\kappa x - \omega t)], \\
E_{jz}(x, z, t) &= -\frac{\kappa}{\varepsilon_j(z)} (A_j \alpha_j(z) + B_j \beta_j(z)) \exp [i(k\kappa x - \omega t)]. \tag{2.13}
\end{align*}
$$

For semi-infinite media $\varepsilon_0$ and $\varepsilon_f$ a fundamental system of solutions is again determined by (2.9). The boundary conditions give us the system for amplitudes, which in this case has the form

$$
\begin{align*}
A_0 + B_0 &= A_1 \alpha_1(z_0) + B_1 \beta_1(z_0) \\
\frac{ik_0z}{\varepsilon_0} (A_0 - B_0) &= \left( \frac{A_1 \alpha_1'(z_0) + B_1 \beta_1'(z_0)}{\varepsilon_1(z)} \right) \\
A_1 \alpha_1(z_1) + B_1 \beta_1(z_1) &= A_2 \alpha_2(z_1) + B_2 \beta_2(z_1) \\
\frac{(A_1 \alpha_1'(z_1) + B_1 \beta_1'(z_1))}{\varepsilon_1(z)} &= \left( \frac{A_2 \alpha_2'(z_1) + B_2 \beta_2'(z_1)}{\varepsilon_2(z)} \right) \\
A_j \alpha_j(z_j) + B_j \beta_j(z_j) &= A_{j+1} \alpha_{j+1}(z_j) + B_{j+1} \beta_{j+1}(z_j) \\
\frac{(A_j \alpha_j'(z_j) + B_j \beta_j'(z_j))}{\varepsilon_j(z)} &= \left( \frac{A_{j+1} \alpha_{j+1}'(z_j) + B_{j+1} \beta_{j+1}'(z_j)}{\varepsilon_{j+1}(z)} \right) \tag{2.14}
\end{align*}
$$

$$
\begin{align*}
A_m \alpha_m(z_m) + B_m \beta_m(z_m) &= A_f + B_f \\
\frac{(A_m \alpha_m'(z_m) + B_m \beta_m'(z_m))}{\varepsilon_m(z)} &= \frac{ik_fz}{\varepsilon_f(z)} (A_f - B_f).
\end{align*}
$$

Now let us discuss additional properties of $s$- and $p$-waves in homogeneous layered structures, i.e. we suppose that in (2.2) all $\varepsilon_j(z)$ are constant and $\varepsilon_j(z) = \varepsilon_j, j =
1, 2, ...m. As a result, the fundamental solutions in all layers \( \varepsilon_j \) take the form, similar to (2.9)

\[
\alpha_j(z) = \exp(ik_j(z - z_{j-1})) , \\
\beta_j(z) = \exp(-ik_j(z - z_{j-1})) ,
\]

where we have used phase factors \( \mp \exp(ik_jz_{j-1}) \) only for convenience in further calculations. We can see from (2.15) and (2.13) for TM-waves, or (2.8) for TE-waves that an electromagnetic field in each uniform layer \( \varepsilon_j \), including, of course, semi-infinite media \( \varepsilon_0 \) and \( \varepsilon_f \) (see (2.9)), is represented in terms of two inhomogeneous (damped) plane waves with complex wave vectors \( k_j = k_j' + ik_j'' \). The square of each complex wave vector can be written as

\[
k_j^2 = k_j'^2 - k_j''^2 + 2i k_j' \cdot k_j'' = k^2 \varepsilon_j ,
\]

where \( k_j' \) and \( k_j'' \) are real vectors with directions perpendicular to equiphas and equiamplitude planes respectively in an inhomogeneous plane wave. We can see that \( k_j \) can be real only if \( \varepsilon_j \) is positive and real. Even then, however, \( k_j \) may still be complex if \( k_j' \cdot k_j'' \neq 0 \), corresponding to the case of total internal reflection. Therefore, equiphas and equiamplitude planes of an inhomogeneous plane electromagnetic wave are not parallel to each other in general.

As one can see, we are using the complex representation for the electric and magnetic field of each inhomogeneous (damped) plane wave. For example in the case of TE-waves, \( E^{(+)} = A_j \exp(ik_j(z - z_{j-1})) \exp[i(k \varepsilon x - \omega t)] \hat{e}_y \) and \( E^{(-)} = B_j \exp(-ik_j(z - z_{j-1})) \exp[i(k \varepsilon x - \omega t)] \hat{e}_y \), where \( A_j \) and \( B_j \) are complex numbers. Generally speaking, such a representation is valid only for the linear theory. In this case we can use complex notation through all calculations and extract the real fields only at the very end. If we include into our considerations some non-linear phenomena we have to use the real fields of inhomogeneous plane waves from the beginning of our calculations. For example, we should take \( E^{(+)} \) in the form of

\[
E^{(+)} = \frac{1}{2} [A_j \exp(ik_j(z - z_{j-1})) \exp[i(k \varepsilon x - \omega t)] + \text{complex conjugate}] \hat{e}_y .
\]

Due to the uniformity of the whole stack in the \( x \)-direction, we have for the \( x \)-
components of complex wave vectors $k_j$

$$k_{jz} \equiv k_j \sin(\theta_j) = k_0 \sin(\theta_0) = k\kappa, \quad j = 1, 2, ... m, m + 1,$$ (2.17)

where $\theta_j$, $\theta_0$ here are just formal complex angles. However, as we have mentioned in the beginning of the Section, $\varepsilon_0$ is a real quantity. Therefore, $k_0 = k\sqrt{\varepsilon_0}$ is a real vector (no internal reflections occur in a homogeneous medium). As a result, Eq. (2.17) has a clear physical meaning. First, $\theta_0$ becomes the usual real-value angle of incidence for the incoming wave. Second, it follows that projections $k_{jz}$ are real in all layers, i.e. $k_j''$ is directed along the $z$-axis (waves are damped along the $z$-direction, equiamplitude planes are perpendicular to the $z$-axis). Third, $\theta_j$ becomes a real-quantity angle of refraction for the equiphase plane in the layer $\varepsilon_j$. We can also treat $\theta_j$ as the angle between $k_j'$ and $k_j''$ due to the fact that $k_j''$ is parallel to the $z$-axis. Now, we can express the complex $z$-components of $k_j$ as

$$k_{jz} = k\sqrt{(\varepsilon'_j + i\varepsilon''_j) - \varepsilon_0 \sin^2 \theta_0}, \quad j = 1, 2, ... m, f.$$ (2.18)

As a result, a real-quantity angle $\theta_j$ of refraction for the equiphase plane can be found from the expression

$$\tan \theta_j = \frac{k_{jz}}{k_{jz}' = \frac{\sqrt{\varepsilon_0 \sin \theta_0}}{R(\sqrt{(\varepsilon'_j + i\varepsilon''_j) - \varepsilon_0 \sin^2 \theta_0})}}.$$ (2.19)

Therefore, we have a picture of a wave propagating in the direction $k_j'$ (this direction is determined by (2.19)) and attenuated in the $z$ direction. In the case of total internal reflection, $\theta_j = \pi/2$. Note that there is an approach to apply complex angles for the description of electromagnetic wave propagation [27].

Now let us consider the refraction and reflection of a single plane inhomogeneous wave at an arbitrary interface $\varepsilon_j/\varepsilon_{j+1}$. In the case of $s$-polarized wave, electromagnetic
fields in the layers \( \varepsilon_j \) and \( \varepsilon_{j+1} \), according to (2.8) and (2.15), can be represented as

\[
E_{jy} = a_j \exp(ik_j z_j - z_{j-1}) \exp(ik \kappa x) \\
+ b_j \exp(-ik_j z_j - z_{j-1}) \exp(ik \kappa x), \\
H_{jz} = -\frac{k_j}{k} \exp(ik_j z_j - z_{j-1}) \exp(ik \kappa x) \\
+ \frac{k_j}{k} b_j \exp(-ik_j z_j - z_{j-1}) \exp(ik \kappa x), \\
E_{j+1,y} = a_{j+1} \exp(ik_{j+1} z_j - z_{j-1}) \exp(ik \kappa x), \\
H_{j+1,z} = -\frac{k_{j+1}}{k} a_{j+1} \exp(ik_{j+1} z_j - z_{j-1}) \exp(ik \kappa x).
\] (2.20)

Due to the standard boundary conditions, \( E_{jy}(z_j - 0) = E_{j+1,y}(z_j + 0) \) and \( H_{jz}(z_j - 0) = H_{j+1,z}(z_j + 0) \). Therefore, the Fresnel reflection \( r_{j,j+1} \) and transmission \( t_{j,j+1} \) coefficients at the interface \( \varepsilon_j / \varepsilon_{j+1} \) equal

\[
r_{j,j+1}^{s} \equiv \frac{E_j^{(-)}(z_j - 0)}{E_j^{(+)}(z_j - 0)} = \frac{b_j \exp(-ik_j d_j)}{a_j \exp(ik_j d_j)} = \frac{k_j - k_{j+1}}{k_j + k_{j+1}}, \\
t_{j,j+1}^{s} \equiv \frac{E_{j+1}^{(+)}(z_j + 0)}{E_j^{(+)}(z_j - 0)} = \frac{a_{j+1} \exp(ik_{j+1} d_j)}{a_j \exp(ik_j d_j)} = \frac{2k_j}{k_j + k_{j+1}},
\] (2.21)

where the index \( s \) indicates that these formulae are valid only for \( s \)-waves. For \( p \)-waves, the analogous procedure leads to

\[
r_{j,j+1}^{p} \equiv \frac{H_j^{(-)}(z_j - 0)}{H_j^{(+)}(z_j - 0)} = \frac{b_j \exp(-ik_j d_j)}{a_j \exp(ik_j d_j)} = \frac{\varepsilon_{j+1} k_j - \varepsilon_j k_{j+1}}{\varepsilon_{j+1} k_j + \varepsilon_j k_{j+1}}, \\
t_{j,j+1}^{p} \equiv \frac{H_{j+1}^{(+)}(z_j + 0)}{H_j^{(+)}(z_j - 0)} = \frac{a_{j+1} \exp(ik_{j+1} d_j)}{a_j \exp(ik_j d_j)} = \frac{2\varepsilon_{j+1} k_j}{\varepsilon_{j+1} k_j + \varepsilon_j k_{j+1}},
\] (2.22)

Further, for Fresnel coefficients for both kinds of waves we will use the notation \( r_{j,j+1} \) and \( t_{j,j+1} \), keeping in mind either (2.21) or (2.22). Note that the so-called Stokes' relations follow immediately

\[
r_{j,j+1} = -r_{j+1,j}, \quad r_{j,j+1}^2 + t_{j,j+1} t_{j+1,j} = 1.
\] (2.23)

Finally, let us discuss the possibility of introducing the quantity \( n_j \), which is called the refractive index, for an absorbing medium. For a non-absorbing medium, \( n_j \) is
defined as the ratio of the velocity of light in vacuum to the phase velocity of light in the medium. Moreover, it can obviously be shown that for a non-absorbing medium, $n_j = \sqrt{\varepsilon_j}$. In the general case of an absorbing medium it is logical to define $n_j'$ as

$$n_j' \equiv \frac{c}{v_j} = \frac{k_j'}{k} = \frac{\sqrt{k_{jx}'^2 + k_{jz}'^2}}{k} = \sqrt{\varepsilon_0 \sin^2 \theta_0 + \left(\Re \sqrt{\varepsilon_j - \varepsilon_0 \sin^2 \theta_0}\right)^2},$$

where $v_j$ is the phase velocity of light in the absorbing medium with complex $\varepsilon_j$. However, such a definition of refractive index is very formal [28] and, in fact, has no physical sense because $n_j'$ not only depends on the nature of the absorbing medium $\varepsilon_j$, but also on the angle of incidence $\theta_0$ and the nature of the non-absorbing medium $\varepsilon_0$. Physically, it means that the phase velocity of light in the absorbing medium $\varepsilon_j$ depends on $\theta_0$ and $\varepsilon_0$.

However, in one specific case it is possible to define a refractive index in a physical way even for an absorbing medium. This is the case where the complex vector $k_j$ has its real and imaginary parts $k_j'$ and $k_j''$ parallel to each other. Consequently, the equiphase and equiamplitudes planes are parallel, and, in an absorbing medium, the plane waves are homogeneous rather than inhomogeneous. For the geometry of our problem this case means normal wave propagation for which $\kappa = 0$ and, as a result, there is no difference between TE- and TM- waves. From (2.24) for this case

$$n_j' = \Re \left(\sqrt{\varepsilon_j}\right) = \Re \left(\sqrt{\varepsilon_j' + i\varepsilon_j''}\right),$$

i.e. it depends only on the nature of the absorbing medium. Moreover, in this case $n_j'$ is the real part of a complex refractive index $n_j$ which we can introduce as

$$n_j = \frac{k_j}{k} = \frac{k_j' + ik_j''}{k} = n_j' + in_j''.$$

From this definition it immediately follows that

$$n_j'^2 - n_j''^2 = \varepsilon_j', \quad 2n_j'n_j'' = \varepsilon_j''.$$

Now, having obtained some physical feeling for the behavior of fundamental solutions inside homogeneous layered structures, we begin the investigation of general
solutions (which are superposition of fundamental ones) inside not only homogeneous but also inhomogeneous layered structures. In particular, we are interested in the reflection and transmission properties of such structures. One can see that the problem of investigation of general solutions is equivalent to the problem of solving the system (2.11) for $TE$-waves, or (2.14) for $TM$-waves. Generally speaking, all classical methods of solving these systems can be divided into two groups. These groups include the transfer matrix method and the global matrix method. In the following sections of this Chapter we briefly analyze each group and, finally, develop another approach. This approach is called the multiple reflection method and is very well-known for one layer, but to the best of my knowledge was not previously studied for multilayered structures.

### 2.2 Transfer Matrix Method

The transfer matrix method works by condensing the multilayered system into a set of equations relating the boundary conditions at the first interface to the boundary conditions at the last interface. In this process one constructs the propagation matrix for a stack of an arbitrary number of layers by extending the solution from one layer to the next while satisfying the appropriate interfacial conditions. The dimension of the matrix depends on how many fundamental solutions are needed to represent a general solution inside each layer. As we have already seen in the previous section, the electromagnetic field in an isotropic layered structure can be divided into two independent (uncoupled) $TE$- and $TM$-modes. Since they are uncoupled, the transfer matrix method involves the manipulation of $2 \times 2$ matrices only. In the case of anisotropic layers the electromagnetic field inside each layer consists of four partial waves (four fundamental solutions). Mode coupling takes place at the interfaces where an incident wave produces waves with different polarizations. As a result, in the general situation of biaxial layers we need a $4 \times 4$ propagation matrix. Different forms of its representation were considered in [29, 30, 31, 32]. However, for uniaxial anisotropic layered structures it is possible to use a reduced $2 \times 2$ propagation matrix.
for both normal [33, 34] and oblique incidence [35]. Moreover, if some conditions concerning electromagnetic fields are valid in the general case of biaxial layered structures it is also possible to use the $2 \times 2$ matrix formulation, as was recently done in [36]. Note that for the description of acoustic wave propagation we need a $4 \times 4$ propagation matrix even for an isotropic stack of solid layers as there is coupling between transverse and longitudinal acoustic fields.

There are four basic modifications of the transfer matrix method for isotropic layered structures: the Thomson-Haskell matrix method, initially proposed in [37, 38], the Abeles matrix method, initially proposed in [39], the impedance method, initially proposed in [40], and the method of recursive coefficients, initially proposed in [41]. We should note that the Thomson-Haskell method was initially proposed for acoustic waves. However, in this work we apply it to the case of electromagnetic wave propagation through a stack of isotropic dielectric layers.

### 2.2.1 Thomson-Haskell Matrix Method

According to the Thomson-Haskell method, matrices relate the mutually perpendicular tangential components of the electric and magnetic fields from layer to neighboring layer. As a result, the reflection and transmission properties of the structure are obtained from the product of $m$ $2 \times 2$ layer matrices and $2 \times 2$ field matrices.

The field matrix, which we denote $[D_j(z)]$, describes the relationship between the coefficients $A_j$, $B_j$ and the amplitudes of the electric $E_{jy}(z)$ and magnetic $H_{jx}(z)$ fields at any location in any layer $\varepsilon_j(z)$, including both half-spaces $\varepsilon_0(z)$ and $\varepsilon_f(z)$. We can express this relationship as

$$
\begin{pmatrix}
E_{jy}(z) \\
H_{jx}(z)
\end{pmatrix} = [D_j(z)] \begin{pmatrix}
A_j \\
B_j
\end{pmatrix},
$$

(2.28)
where for \( TE \) and \( TM \)-waves according to (2.8), (2.13)

\[
[D_j(z)]^{TE} = \begin{pmatrix}
\alpha_j(z) & \beta_j(z) \\
\frac{i}{k} \alpha_j'(z) & \frac{i}{k} \beta_j'(z)
\end{pmatrix},
\]

\[
[D_j(z)]^{TM} = \begin{pmatrix}
-\frac{i}{k \varepsilon_j(z)} \alpha_j'(z) & -\frac{i}{k \mu_j(z)} \beta_j'(z) \\
\alpha_j(z) & \beta_j(z)
\end{pmatrix}.
\]

(2.29)

Introducing the inverse field matrix \([D_j^{-1}(z)]\), we can rewrite the relationship (2.28) as

\[
\begin{pmatrix}
A_j \\
B_j
\end{pmatrix} = [D_j^{-1}(z)] \begin{pmatrix}
E_{jy}(z) \\
H_{jz}(z)
\end{pmatrix},
\]

(2.30)

where

\[
[D_j^{-1}(z)]^{TE} = \frac{1}{W_j} \begin{pmatrix}
\beta_j'(z) & ik \beta_j(z) \\
-\alpha_j'(z) & -ik \alpha_j(z)
\end{pmatrix},
\]

\[
[D_j^{-1}(z)]^{TM} = \frac{1}{W_j} \begin{pmatrix}
-ik \varepsilon_j(z) \beta_j(z) & \beta_j'(z) \\
ike_j(z) \alpha_j(z) & -\alpha_j'(z)
\end{pmatrix}.
\]

(2.31)

Here we have introduced the Wronskian of the fundamental system \( \alpha_j(z), \beta_j(z) \)

\[
W_j = \alpha_j(z) \beta_j'(z) - \alpha_j'(z) \beta_j(z),
\]

(2.32)

which, as we know, does not depend on \( z \). Now we can write the dependence of \( E_{jy}(z_j), H_{jz}(z_j) \) on \( E_{jy}(z_{j-1}), H_{jz}(z_{j-1}) \) for \( j = 1, 2, ..., m \) in a matrix form as

\[
\begin{pmatrix}
E_{jy}(z_j) \\
H_{jz}(z_j)
\end{pmatrix} = [M_j] \begin{pmatrix}
E_{jy}(z_{j-1}) \\
H_{jz}(z_{j-1})
\end{pmatrix},
\]

(2.33)

where

\[
[M_j] = [D_j(z_j)] [D_j^{-1}(z_{j-1})]
\]

(2.34)

is a so-called layer matrix, which relates tangential components of electric and magnetic fields on the boundary points \( z_j \) and \( z_{j-1} \) of the layer \( \varepsilon_j \). For \( TE \)-waves its
elements have the form
\[
[M_j]_{11}^{TE} = \frac{1}{W_j} \left( \alpha_j(z_j) \beta_j'(z_j-1) - \alpha_j'(z_j-1) \beta_j(z_j) \right),
\]
\[
[M_j]_{12}^{TE} = \frac{1}{W_j} ik \left( \alpha_j(z_j) \beta_j(z_j-1) - \alpha_j(z_j-1) \beta_j(z_j) \right),
\]
\[
[M_j]_{21}^{TE} = \frac{1}{W_j} \frac{i}{k} \left( \alpha_j'(z_j) \beta_j'(z_j-1) - \alpha_j'(z_j-1) \beta_j'(z_j) \right),
\]
\[
[M_j]_{22}^{TE} = \frac{1}{W_j} \left( -\alpha_j'(z_j) \beta_j(z_j-1) + \alpha_j(z_j-1) \beta_j'(z_j) \right).
\]

For $TM$-waves the elements of the layer matrix $M_j$ are
\[
[M_j]_{11}^{TM} = \frac{1}{W_j} \frac{\varepsilon_j(z_j-1)}{\varepsilon_j(z_j)} \left( -\alpha_j'(z_j) \beta_j(z_j-1) + \alpha_j(z_j-1) \beta_j'(z_j) \right),
\]
\[
[M_j]_{12}^{TM} = -\frac{1}{W_j} \frac{i}{k \varepsilon_j(z_j)} \left( \alpha_j'(z_j) \beta_j'(z_j-1) - \alpha_j'(z_j-1) \beta_j'(z_j) \right),
\]
\[
[M_j]_{21}^{TM} = -\frac{1}{W_j} i k \varepsilon_j(z_j-1) \left( \alpha_j(z_j) \beta_j(z_j-1) - \alpha_j(z_j-1) \beta_j(z_j) \right),
\]
\[
[M_j]_{22}^{TM} = \frac{1}{W_j} \left( \alpha_j(z_j) \beta_j'(z_j-1) - \alpha_j'(z_j-1) \beta_j(z_j) \right).
\]

The tangential components of electric and magnetic fields must be continuous across each interface inside the stack. Therefore, these components on the right side of the last interface ($m$-interface) are related to those on the left side of the first interface in accordance with
\[
\begin{pmatrix}
E_{1y}(L) \\
H_{1x}(L)
\end{pmatrix} = [M] \begin{pmatrix}
E_{1y}(0) \\
H_{1x}(0)
\end{pmatrix},
\]

where $[M]$ is the matrix product of $m$ layer matrices $[M_j]$
\[
[M] = \prod_{j=m}^{1} [M_j].
\]

For scattering problems it is useful to describe the first and the last interface of the stack in terms of fundamental solutions $\alpha_0(0), \beta_0(0)$ and $\alpha_f(L), \beta_f(L)$ rather than in terms of fields $E_{1y}(x,0), H_{1x}(x,0)$ and $E_{1y}(x,L), H_{1x}(x,L)$. Using again (2.28) and (2.30), we obtain the matrix relation
\[
\begin{pmatrix}
A_f \\
B_f
\end{pmatrix} = [S] \begin{pmatrix}
A_0 \\
B_0
\end{pmatrix},
\]

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where \([S]\) is the system matrix:

\[
[S] = [D_f^{-1}(L)] [M] [D_0(0)].
\]  

(2.40)

In accordance with (2.9), (2.29), and (2.31)

\[
[D_f^{-1}(L)]^{TE} = \begin{pmatrix}
\frac{1}{2} - \frac{k}{2k_fz} \\
\frac{1}{2} & \frac{k}{2k_fz}
\end{pmatrix}, \quad [D_0(0)]^{TE} = \begin{pmatrix}
\frac{1}{k} & 1 \\
-k_0z & k_0z
\end{pmatrix},
\]

\[
[D_f^{-1}(L)]^{TM} = \begin{pmatrix}
\varepsilon_fk & 1 \\
2k_fz & 2k_fz
\end{pmatrix}, \quad [D_0(0)]^{TM} = \begin{pmatrix}
\frac{k_0z}{k} & -\frac{k_0z}{k} \\
\varepsilon_0 & \varepsilon_0
\end{pmatrix}.
\]

(2.41)

In order to solve a typical one-dimensional scattering problem in which the initial wave is incident from the left, we should put \(B_f = 0\) and \(A_0 = 1\). The latter equality means that the incoming field is assumed to be normalized to the unit amplitude. As a result, the amplitude reflection and transmission coefficients for this wave are

\[
R^l = -\frac{S_{21}}{S_{22}}, \quad T^l = \frac{\det[S]}{S_{22}}.
\]

(2.42)

If the initial wave is coming from the right we should put \(A_0 = 0, B_f = 1\). Then, the amplitude reflection and transmission coefficients for this initial wave are

\[
R^r = \frac{S_{12}}{S_{22}}, \quad T^r = \frac{1}{S_{22}}.
\]

(2.43)

It follows from (2.34), (2.38), (2.40), and (2.41) that

\[
\det([S]^{TE}) = \frac{k_0z}{k_fz}, \quad \det([S]^{TM}) = \frac{\varepsilon_fk_0z}{\varepsilon_0k_fz}.
\]

(2.44)

For a layered homogeneous structure, the fundamental system of solutions in each layer \(\varepsilon_j\) is defined by (2.15). As a result, the layer matrices \([M_j]^{TE}\) and \([M_j]^{TM}\) for
$TE$ and $TM$-waves have the form

$$[M_j]^{TE} = \begin{pmatrix}
\cos(k_{jz}d_j) & -i\frac{k}{k_{jz}} \sin(k_{jz}d_j) \\
-i\frac{k_{jz}}{k} \sin(k_{jz}d_j) & \cos(k_{jz}d_j)
\end{pmatrix},$$

$$[M_j]^{TM} = \begin{pmatrix}
\cos(k_{jz}d_j) & i\frac{k_{jz}}{k\varepsilon_j} \sin(k_{jz}d_j) \\
i\frac{k\varepsilon_j}{k_{jz}} \sin(k_{jz}d_j) & \cos(k_{jz}d_j)
\end{pmatrix}. \quad (2.45)$$

Therefore, the elements of the system matrix, $[S] = [D_f^{-1}(L)][M][D_0(0)]$, where $[M] = \prod_{j=m}^1 [M_j]$, for the layered homogeneous structure can be found without any difficulties. After that, the reflection and transmission properties of this structure follow immediately from Eqs. (2.42, 2.43).

### 2.2.2 Abeles Matrix Method

According to the Abeles method [39], matrices relate the coefficients $A_j$, $B_j$ in successive layers. As a result, the reflection and transmission properties of the structure are obtained from the product of $(m + 1) \times 2$ interface matrices. The method was concisely described by Heavens in [42] and reformulated in a more elegant manner in [43, 44].

In accordance with Eqs. (2.28) and (2.30), we have at any interface $z_j$, $j = 0, 1, 2, ...m$

$$[D_j(z_j)] \begin{pmatrix} A_j \\ B_j \end{pmatrix} = [D_{j+1}(z_j)] \begin{pmatrix} A_{j+1} \\ B_{j+1} \end{pmatrix}. \quad (2.46)$$

Therefore,

$$\begin{pmatrix} A_{j+1} \\ B_{j+1} \end{pmatrix} = [I_j] \begin{pmatrix} A_j \\ B_j \end{pmatrix}, \quad (2.47)$$

where $[I_j]$ is the interface matrix

$$[I_j] = [D^{-1}_{j+1}(z_j)] [D_j(z_j)], \quad (2.48)$$

which relates the coefficients $A_j$, $B_j$ in successive layers. Then we can obtain the system matrix (2.40), relating $A_f$, $B_f$ with $A_0$, $B_0$ in the form of the product of
(m+1) interface matrices \( I_j \)

\[
[S] = \prod_{j=m}^{0} [I_j].
\]  

(2.49)

As we can see, the difference between the Thomson-Haskell and Abeles methods is the representation of the system matrix \( S \). In the Thomson-Haskell method we represent it as the product of \( m \) layer matrices \( M_j \) and two field matrices \([D_1(0)], [D_f^{-1}(L)]\). In the Abeles method we represent it as the product of \((m + 1)\) interface matrices \( I_j \).

The elements of the interface matrix \( I_j \) \((j = 0, 1, ..., m)\) for \( TE \)-waves are

\[
[I_{j1}]^{TE} = \frac{1}{W_{j+1}} \left( \alpha_j(z_j) \beta'_{j+1}(z_j) - \alpha'_{j}(z_j) \beta_{j+1}(z_j) \right),
\]

\[
[I_{j2}]^{TE} = \frac{1}{W_{j+1}} \left( \beta_j(z_j) \beta'_{j+1}(z_j) - \beta'_{j}(z_j) \beta_{j+1}(z_j) \right),
\]

\[
[I_{j21}]^{TE} = \frac{1}{W_{j+1}} \left( -\alpha_j(z_j) \alpha'_{j+1}(z_j) + \alpha'_{j}(z_j) \alpha_{j+1}(z_j) \right),
\]

\[
[I_{j22}]^{TE} = \frac{1}{W_{j+1}} \left( -\alpha'_{j+1}(z_j) \beta_j(z_j) + \alpha_{j+1}(z_j) \beta'_{j}(z_j) \right),
\]

(2.50)

and for \( TM \)-waves

\[
[I_{j1}]^{TM} = \frac{1}{W_{j+1}} \left( \alpha_j(z_j) \beta'_{j+1}(z_j) - \frac{\varepsilon_{j+1}(z_j)}{\varepsilon_j(z_j)} \alpha'_{j}(z_j) \beta_{j+1}(z_j) \right),
\]

\[
[I_{j12}]^{TM} = \frac{1}{W_{j+1}} \left( \beta_j(z_j) \beta'_{j+1}(z_j) - \frac{\varepsilon_{j+1}(z_j)}{\varepsilon_j(z_j)} \beta'_{j}(z_j) \beta_{j+1}(z_j) \right),
\]

\[
[I_{j21}]^{TM} = \frac{1}{W_{j+1}} \left( -\alpha_j(z_j) \alpha'_{j+1}(z_j) + \frac{\varepsilon_{j+1}(z_j)}{\varepsilon_j(z_j)} \alpha'_{j}(z_j) \alpha_{j+1}(z_j) \right),
\]

\[
[I_{j22}]^{TM} = \frac{1}{W_{j+1}} \left( -\alpha'_{j+1}(z_j) \beta_j(z_j) + \frac{\varepsilon_{j+1}(z_j)}{\varepsilon_j(z_j)} \alpha_{j+1}(z_j) \beta'_{j}(z_j) \right),
\]

(2.51)

For a layered homogeneous structure these matrices for both \( TE \) and \( TM \) waves take the form

\[
[I_j] = \frac{1}{t_{j+1,j}} \begin{pmatrix}
\exp(ik_{jz}d_j) & \tau_{j+1,j} \exp(-ik_{jz}d_j) \\
\tau_{j+1,j} \exp(ik_{jz}d_j) & \exp(-ik_{jz}d_j)
\end{pmatrix},
\]

(2.52)

where the Fresnel coefficients \( t_{j+1,j} \) and \( \tau_{j+1,j} \) are determined by either (2.21) for \( TE \)-waves or (2.22) for \( TM \)-waves.

As an example of an application of the Abeles method, let us find the reflection and transmission properties of a stack consisting of only one homogeneous layer \( \varepsilon_1 \),
placed between semi-infinite spaces \( \varepsilon_0 \) and \( \varepsilon_f \). The elements of the system matrix \([S]\), which is now \([S] = [I_1][I_0]\), take the form

\[
    S_{11} = \frac{\exp(ik_{1z}d_1) + r_{f1}r_{10}\exp(-ik_{1z}d_1)}{t_{f1}t_{10}}, \\
    S_{12} = \frac{r_{10}\exp(ik_{1z}d_1) + r_{f1}\exp(-ik_{1z}d_1)}{t_{f1}t_{10}}, \\
    S_{21} = \frac{r_{f1}\exp(ik_{1z}d_1) + r_{10}\exp(-ik_{1z}d_1)}{t_{f1}t_{10}}, \\
    S_{22} = \frac{r_{f1}r_{10}\exp(ik_{1z}d_1) + \exp(-ik_{1z}d_1)}{t_{f1}t_{10}}.
\]

(2.53)

Using (2.42) and (2.43), we obtain with the aid of (2.23)

\[
    R^t = \frac{r_{01} + r_{1f}\exp(2ik_{1z}d_1)}{1 + r_{01}r_{1f}\exp(2ik_{1z}d_1)}, \\
    T^t = \frac{t_{01}t_{1f}\exp(ik_{1z}d_1)}{1 + r_{01}r_{1f}\exp(2ik_{1z}d_1)}, \\
    R^r = \frac{r_{f1} + r_{10}\exp(2ik_{1z}d_1)}{1 + r_{01}r_{1f}\exp(2ik_{1z}d_1)}, \\
    T^r = \frac{t_{f1}t_{10}\exp(ik_{1z}d_1)}{1 + r_{01}r_{1f}\exp(2ik_{1z}d_1)}.
\]

(2.54)

2.2.3 Impedance Method

In the literature, the impedance method was extensively used in the fifties and sixties. We only mention such monographs as [18, 45]. But, in fact, this method is just another modification of the transfer matrix method. Moreover, we will show that the impedance method is almost the same as the Thomson-Haskell method.

First of all, let us introduce the concept of impedance. In electrodynamics, the impedance is understood to be the ratio of the tangential components of the electric and magnetic fields. In our coordinate system in each layer \( \varepsilon_j \) the impedance \( Z_j(z) \)
can be written for \( TE \) and \( TM \)-waves as
\[
Z_{j}^{TE}(z) = \frac{E_{jy}}{H_{jx}} = -i k \frac{A_{j} \alpha_{j}(z) + B_{j} \beta_{j}(z)}{A_{j} \alpha_{j}(z) + B_{j} \beta_{j}(z)},
\]
\[
Z_{j}^{TM}(z) = \frac{E_{jx}}{H_{jy}} = -i \frac{A_{j} \alpha_{j}(z) + B_{j} \beta_{j}(z)}{k \varepsilon_{j}(z) A_{j} \alpha_{j}(z) + B_{j} \beta_{j}(z)}.
\] (2.55)

The relation between the impedance at the left surface \( z_{j-1} \) of the layer \( \varepsilon_{j} \) and the impedance at the right surface \( z_{j} \) of the same layer for both \( TE \) and \( TM \)-waves can be easily expressed from (2.33). In terms of the elements of the layer matrix \([M_{j}]\) the relation has the form
\[
Z_{j}(z_{j-1}) = \frac{[M_{j}]_{12} - [M_{j}]_{22} Z_{j}(z_{j})}{-[M_{j}]_{11} + [M_{j}]_{21} Z_{j}(z_{j})}.
\] (2.56)

In the case of a homogeneous layered structure Eq. (2.56) for \( TE \) and \( TM \)-waves takes the form
\[
Z_{j}^{TE}(z_{j-1}) = \frac{Z_{j}^{TE}(z_{j}) + i \frac{k}{k_{jz}} \tan(k_{jz} d_{j})}{1 + i \frac{k_{jz}}{k} \tan(k_{jz} d_{j}) Z_{j}^{TE}(z_{j})},
\]
\[
Z_{j}^{TM}(z_{j-1}) = \frac{Z_{j}^{TM}(z_{j}) - i \frac{k_{jz}}{k \varepsilon_{j}} \tan(k_{jz} d_{j})}{1 - i \frac{k \varepsilon_{j}}{k_{jz}} \tan(k_{jz} d_{j}) Z_{j}^{TM}(z_{j})}.
\] (2.57)

The impedance at the right surface \( z_{j-1} \) of the layer \( \varepsilon_{j-1} \) coincides with the impedance at the left surface \( z_{j-1} \) of the layer \( \varepsilon_{j} \) due to boundary conditions. Therefore, if we know the impedance at the left surface of the semi-infinite homogeneous medium \( \varepsilon_{f} \), using recursively \( m \) times the algorithm (2.57), we can find the impedance at the right surface of the semi-infinite homogeneous medium \( \varepsilon_{0} \). Such an approach was described in [18]). However, if we first define the matrix \([M]\) (2.38), we can relate directly the impedance at the left surface of the semi-infinite homogeneous medium \( \varepsilon_{f} \) with the impedance at the right surface of the medium \( \varepsilon_{0} \) as
\[
Z_{0}(z_{0}) = \frac{M_{12} - M_{22} Z_{f}(z_{m})}{-M_{11} + M_{21} Z_{f}(z_{m})}.
\] (2.58)

As an example of using the impedance method let us describe how to find the reflection coefficient \( R^{l} \). First of all, we should put \( B_{f} = 0 \) in (2.55), then find the
impedance $Z_f(z_m)$, where $z_m \equiv L$, after that, calculate the impedance $Z_0(z_0)$, using either (2.58) or recursively $m$ times (2.56), and, finally, find the reflection coefficient $R^l$, using again (2.55), where we should put $A_0 = 1$, $B_0 = R^l$.

### 2.2.4 Method of Recursive Coefficients

This method is similar to the Abeles method in the way that both operate with amplitudes $A_j$ and $B_j$ rather than with fields $E_j$ and $H_j$. The difference between them is similar to the difference between the impedance and Thomson-Haskell method. The core idea of the method of recursive coefficients is to rearrange the basic systems (2.11) and (2.14) for $TE$ and $TM$-waves in such a way that they relate local reflection coefficients in successive layers. The method is extensively used in the literature, see, for example, [41, 45, 46, 47].

First of all let us define the local reflection and transmission coefficients in each layer $\varepsilon_j$ as

\[
\begin{align*}
    r_j &= \frac{B_j}{A_j}, & t_j &= \frac{A_{j+1}}{A_j}. \tag{2.59}
\end{align*}
\]

For both $TE$ and $TM$-waves from (2.11) and (2.14) follow that

\[
\begin{align*}
    r_j &= \frac{r_{j,j+1} + r_{j+1} \exp(2ik_{j+1,z}d_{j+1})}{1 + r_{j,j+1}r_{j+1} \exp(2ik_{j+1,z}d_{j+1})}, \tag{2.60}
\end{align*}
\]

where $r_{j,j+1}$ are Fresnel coefficients either for $TE$-waves (2.21) or for $TM$-waves (2.22). In order to obtain the reflection coefficient $R^l$ from the whole stack we should put $r_m = r_{mf}$ as $B_f = 0$ for this problem. Then, using (2.60) $m$ times we can obtain $r_0$, keeping in mind, that $r_0 = R^l$ under the condition $A_0 = 1$.

In the case of two layers, starting with $r_2 = r_{2f}$ and applying (2.60) twice, we obtain

\[
\begin{align*}
    R_2^l &= \frac{r_{01} + r_{12} \exp(2i\phi_1) + r_{2f} \exp(2i(\phi_1 + \phi_2)) + r_{01}r_{12}r_{2f} \exp(2i\phi_2)}{1 + r_{01}r_{12} \exp(2i\phi_1) + r_{12}r_{2f} \exp(2i\phi_2) + r_{01}r_{2f} \exp(2i(\phi_1 + \phi_2))} \tag{2.61}
\end{align*}
\]

In the case of three layers we should apply (2.60) three times, starting with $r_3 = r_{3f}$. As a result, we have
\[ R_3^i = \begin{bmatrix}
  r_{01} + r_{12} \exp(2i\phi_1) + r_{23} \exp(2i(\phi_1 + \phi_2)) + r_{3f} \exp(2i(\phi_1 + \phi_2 + \phi_3)) \\
  + r_{01} r_{12} r_{23} \exp(2i\phi_2) + r_{01} r_{23} r_{3f} \exp(2i\phi_3) \\
  + r_{01} r_{12} r_{3f} \exp(2i(\phi_2 + \phi_3)) + r_{12} r_{23} r_{3f} \exp(2i(\phi_1 + \phi_3)) \\
  1 + r_{01} r_{12} \exp(2i\phi_1) + r_{12} r_{23} \exp(2i\phi_2) + r_{23} r_{3f} \exp(2i\phi_3) \\
  + r_{12} r_{3f} \exp(2i(\phi_2 + \phi_3)) + r_{01} r_{23} \exp(2i(\phi_1 + \phi_2)) \\
  + r_{01} r_{3f} \exp(2i(\phi_1 + \phi_2 + \phi_3)) + r_{01} r_{12} r_{23} r_{3f} \exp(2i(\phi_1 + \phi_2 + \phi_3))
\end{bmatrix} \]

\[ (2.62) \]

In formulas (2.61) and (2.62) we introduced notation
\[ \phi_j = k_{jz}d_j, j = 1, 2, ..., m, \]
\[ \phi_0 = k_{0z}d_0 \equiv 0, \]

which we will use for brevity in further considerations. Subscripts in the reflection coefficients in formulas (2.61) and (2.62) emphasize the number of layers in the stack. For a stack consisting of more than three layers, the final result would be too cumbersome to write in explicit form. However, in [48], Grook showed how to generalize the explicit expressions (2.61) and (2.62) for the reflection coefficients of two and three layers to the case of an arbitrary number of layers. In my opinion, Grook's method is tedious and artificial.

### 2.3 Global Matrix Method

The Global matrix method is a fundamentally different approach for the description of wave propagation in multilayered media. It was initially proposed by Knopoff [49] for acoustic waves and has been employed by some other researchers in this area [50, 51]. A good overview of the global matrix method can be found in [52]. Although the global matrix method is usually used for acoustic waves, in this work we consider its application to electromagnetic waves.

The main advantage of the method is that it allows us to avoid the problem of the large frequency-thickness product, which we have for the transfer matrix method. The disadvantage is that the global matrix may be huge (if we have a large number of layers) and therefore its solution may be relatively slow. The core idea of the global
matrix method is to assemble directly the system (2.11) for $TE$-waves or (2.14) for $TM$-waves into a single matrix, which represents the complete layered structure.

Let us consider again an arbitrary interface $z_j$, $j = 0, 1, 2, ..., m$. At this interface, according to (2.46), we have

$$[D_{j+1}(z_j)] \begin{pmatrix} A_{j+1} \\ B_{j+1} \end{pmatrix} = [D_j(z_j)] \begin{pmatrix} A_j \\ B_j \end{pmatrix}. \quad (2.64)$$

When the transfer matrix method is used (Abeles modification), the amplitudes $A_{j+1}$, $B_{j+1}$ are expressed in terms of amplitudes $A_j$, $B_j$. But Eq. (2.64), describing the interaction at interface $z_j$ of the waves in the successive layers $\varepsilon_j$ and $\varepsilon_{j+1}$, can also be expressed in a single matrix form as

$$[[D_j(z_j)][-D_{j+1}(z_j)]] \begin{pmatrix} A_j \\ B_j \\ A_{j+1} \\ B_{j+1} \end{pmatrix} = (0). \quad (2.65)$$

Considering all interfaces from $z_j = z_0$ to $z_j = z_m$, we obtain a matrix of $2(m + 1)$ equations and $2(m + 2)$ unknowns

$$\begin{pmatrix} [D_0(z_0)] & [-D_1(z_0)] & \cdots & \cdots & \cdots \\ \cdots & [D_1(z_1)] & [-D_2(z_1)] & \cdots & \cdots \\ \cdots & \cdots & [D_j(z_j)] & [-D_j(z_j)] & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_j \\ V_f \end{pmatrix} = (0) , \quad (2.66)$$

where the wave amplitudes in each layer, $A_j$, $B_j$ ($j = 0, 1, ..., m, f$) are abbreviated simply to a layer vector ($V_j$). Two of the wave amplitudes in (2.66) should be identified as knowns and their coefficients in the equation should be moved to the right hand
side. It is convenient to choose $A_0$ and $B_f$ as knowns. As a result, we have

\[
\begin{pmatrix}
[D_0^B(z_0)] & [-D_1(z_0)] & \cdots & \cdots & \cdots \\
\vdots & [D_1(z_1)] & [-D_2(z_1)] & \cdots & \cdots \\
\vdots & \vdots & [D_j(z_j)] & [-D_j(z_j)] & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & [D_f(z_m)] & [-D_f^A(z_m)] \\
\end{pmatrix}
\begin{pmatrix}
B_0 \\
V_1 \\
V_j \\
A_f \\
A_0 \\
B_f \\
\end{pmatrix}
= \begin{pmatrix}
[-D_0^A(z_0)] & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
[-D_f^A(z_m)] \\
\end{pmatrix}
\begin{pmatrix}
A_0 \\
0 \\
0 \\
0 \\
B_f \\
\end{pmatrix}, \tag{2.67}
\]

where, for example, for $TE$-waves

\[
[D_j^A(z_j)]^{TE} = \begin{pmatrix}
\alpha_j(z_j) \\
i / k \alpha_j'(z_j)
\end{pmatrix}, \quad [D_j^B(z_j)]^{TE} = \begin{pmatrix}
\beta_j(z_j) \\
i / k \beta_j'(z_j)
\end{pmatrix}. \tag{2.68}
\]

The system matrix on the left hand side (2.67) and the sparse matrix on the right hand side are both square and of dimension $2(m + 1)$. If the wave amplitudes for the incoming waves are known then the right hand side may be evaluated immediately, resulting in a vector of known coefficients. As we have already done in previous sections, for the description of reflection and transmission properties of waves incoming from the left we should put $A_0 = 1$, $B_f = 0$, and $A_0 = 0$ and $B_f = 1$ for waves incoming from the right.

For our previous example of two homogeneous layers $\varepsilon_1$, $\varepsilon_2$ placed between half-spaces $\varepsilon_0$ and $\varepsilon_f$, Eq. (2.67) in the case of $TE$-waves, incident from the left, takes form
\[
\begin{pmatrix}
1 & -1 & -1 & 0 & 0 & 0 \\
k_{0z} & k_{1z} & -k_{1z} & 0 & 0 & 0 \\
0 & \exp(i\phi_1) & -\exp(i\phi_1) & -1 & -1 & 0 \\
0 & -\frac{k_{1z}}{k} \exp(i\phi_1) & \frac{k_{1z}}{k} \exp(i\phi_1) & k_{2z} & -\frac{k_{2z}}{k} & 0 \\
0 & 0 & 0 & \exp(i\phi_2) & \exp(-i\phi_2) & -1 \\
0 & 0 & 0 & -\frac{k_{2z}}{k} \exp(i\phi_2) & \frac{k_{2z}}{k} \exp(-i\phi_2) & \frac{k_{2z}}{k}
\end{pmatrix}
\] (2.69)

\[
\begin{pmatrix}
R_2' \\
A_1 \\
B_1 \\
A_2 \\
B_2 \\
T_2'
\end{pmatrix}
= \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
\frac{k_{0z}}{k} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \frac{k_{fz}}{k} & 0
\end{pmatrix}
\times
\begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

Solutions for the vector of wave amplitudes on the left hand side of (2.69) is obtained by inversion (or solution) of the system matrix. The most obvious method of solution is to use Gaussian elimination. It leads us again to the expression (2.62) for the amplitude reflection coefficient \( R_2' \).

## 2.4 Multiple Reflection Method

Another approach to the description of waves in multilayered structures is the so-called multiple reflection method. It is absolutely different from both the transfer and global matrix approaches and has nothing to do with the system (2.11) or (2.14). The basic idea of the method is to obtain the reflection and transmission coefficients for a stack by considering the multiple reflections and transmissions at the boundaries of each layer. For the case of a single homogeneous layer this method is very well-known in the literature and it is often given as the theory of the Fabry-Perot interferometer. Moreover, the method is sometimes used for electron [53, 54] and acoustic [55] waves in one-dimensional structures. However, only in the case of a single layer (square
barrier potential for electron waves) is the summation over all multiply reflected and transmitted waves easily effected. Let us first consider this case.

2.4.1 **Exact Calculations for One Layer**

Let us derive the reflection coefficient $R_1^*$ (see (2.54)) for a single homogeneous layer $\varepsilon_1$, placed between half-spaces $\varepsilon_0$ and $\varepsilon_f$. A wave incident on the first interface can be divided into reflected and transmitted partial waves. Such a division occurs each time when the wave strikes an interface. As a result, the total reflection and transmission coefficients are obtained by summing the partial multiple reflected and multiple transmitted waves (see Figure 2.2).

![Diagram](image)

**Figure 2.2:** Application of a multiple reflection method to one layer

The amplitudes of the successive partial waves reflected into medium $\varepsilon_0$ are given
by \( r_{01} \), \( t_{01} \exp(i\phi_1) r_{1f} \exp(i\phi_1) t_{10} \), \( t_{01} \exp(i\phi_1) r_{1f} \exp(i\phi_1) r_{10} \exp(i\phi_1) r_{1f} \exp(i\phi_1) t_{10} \) etc. Beginning with the second term, we have a geometric series. Summing it up, we obtain the reflected amplitude in the form of

\[
R_1' = r_{01} + \frac{t_{01} t_{10} r_{1f} \exp(2i\phi_1)}{1 + r_{01} r_{1f} \exp(2i\phi_1)} \tag{2.70}
\]

After elementary transformations, we have

\[
R_1' = \frac{r_{01} + r_{1f} \exp(2i\phi_1)}{1 + r_{01} r_{1f} \exp(2i\phi_1)}, \tag{2.71}
\]

which is exactly same as (2.54). Unfortunately, already for two layers the picture of the partial multiple reflected and transmitted waves is getting so complicated, see Figure 2.3, that there is no way to sum all partial waves.

Figure 2.3: Application of a multiple reflection method to two layers
Therefore, we need to develop an approximate scheme which would be able to take into account those partial waves contributing the most to the total reflection or transmission coefficients.

2.4.2 Approximate Calculations for Multilayered Structure

The core physical idea of the suggested scheme is that the partial waves having the least number of reflections inside the stack contribute more to the total reflection coefficient $R_i^t$. This follows from the fact that the Fresnel coefficients at any interface satisfy the condition $|r_{j,j+1}| < 1$. Therefore, in the first approximation we should take into account only terms which are proportional to $r_{j,j+1}$, i.e. we should confine ourselves to a single reflection of the partial wave in each layer. In the second approximation we should also take into account partial waves which have three reflections in the stack, i.e. we should consider also terms proportional to $r_{j,j+1} r_{p,p+1} r_{q,q+1}$, where $j, p, q = 0, 1, 2, ..., m$. In the third approximation we should take into account partial waves having five internal reflections in the stack, and so on.

As a test example, let us consider an application of the method to the already known case of one layer in the stack. For one layer the reflection coefficient in the first approximation takes into account only two terms. First, the contribution of the wave reflected from the front surface of the layer ($\varepsilon_0/\varepsilon_1$ interface). Second, the contribution of the wave penetrating the front surface of the layer, passing through the layer, reflecting from the back surface of the layer, and finally passing through the layer again and leaving through its front surface. Then, $R_1^t \approx r_{01} + t_{01} \exp(i\phi_1) r_{1f} \exp(i\phi_1) = r_{01} + (1 - r_{01}^2) r_{1f} \exp(2i\phi_1) \approx r_{01} + r_{1f} \exp(2i\phi_1)$, where we neglect the term $(-r_{01}^2 r_{1f})$, which is the contribution to the second approximation in accordance with our scheme. As a result the reflection coefficient from one layer in the first approximation of the multiple reflection theory is

$$R_{1}^{(1)} = r_{01} + r_{1f} \exp(2i\phi_1). \quad (2.72)$$

In the second approximation there is also a contribution from the wave penetrating the layer, undergoing two reflections at the back surface and one at the front sur-
face and leaving the layer: \( t_{01} \exp(i\phi_1)r_{1f} \exp(i\phi_1)r_{10} \exp(i\phi_1)r_{1f} \exp(i\phi_1)t_{10} = (1 - r_{01}^2) \exp(4i\phi_1) \approx -r_{01}^2 r_{1f}^2 \exp(4i\phi_1) \), where we neglect the term \( r_{01}^2 r_{1f}^2 \exp(4i\phi_1) \), which is the contribution to the third approximation. Therefore, the reflection coefficient from one layer in the second approximation is

\[
R_1^{(2)} = r_{01} + r_{1f} \exp(2i\phi_1) - r_{01}^2 r_{1f} \exp(2i\phi_1) - r_{01} r_{1f}^2 \exp(4i\phi_1).
\]

(2.73)

The comparison of the results for the reflection coefficient from one layer in the first and second approximation with the exact formula (2.71) is shown on Figure 2.4. We consider the dependence of the reflection coefficient on the wavelength of an incident electromagnetic wave in the optical range. For simplicity we restrict our consideration to the case of normal incidence (all \( \theta_j = 0 \) in (2.17)) and the absence of absorption (all \( n_j'' = 0 \) in (2.26)). The above selection of materials (vacuum, chalcogenide glass, GaAs) realizes a rather extreme situation from the point of view of the suggested method. The reason for this is the relatively big difference in the refractive indexes of the above materials. As a result, the Fresnel coefficients \( r_{01} \) and \( r_{1f} \) are high, and contributions of higher order partial reflected waves to the total reflection coefficient \( R_1' \) are significant. However, as we can see, for one layer even the first approximation is in good agreement with the exact result.
Figure 2.4: Dependence of $R_1^d$ on $\lambda$, $n_0 = 1$, $n_1 = 2.4$, $d_1 = 2.5 \mu m$, $n_f = 3.6$; solid line - first approximation, dashed line - second approximation, gray line - exact result.

Let us now consider the case of two layers ($m = 2$). The reflection coefficient in the first approximation takes into account three terms. First, the contribution of the wave reflected from the front surface of the layer ($\varepsilon_0/\varepsilon_1$ interface). Its contribution is $r_{01}$. Second, the contribution of the wave penetrating the front surface of the first layer, passing through it, reflecting from the back surface of this layer ($\varepsilon_1/\varepsilon_2$ interface), and finally passing through the layer again and leaving through its front surface. Its contribution to the total reflection coefficient is $t_{01} \exp(i\phi_1)r_{12}\exp(i\phi_1)t_{10}$, from which, according to our scheme, the part $r_{12}\exp(2i\phi_1)$ goes toward the first approximation, and the part $-r_{01}^2 r_{12}\exp(2i\phi_1)$ goes toward the second approximation. Third, the contribution of the wave penetrating the front surface of the first layer, passing through it, penetrating the front surface of the second layer, passing through it, reflecting from the back surface of the second layer ($\varepsilon_2/\varepsilon_f$ interface), passing through two layers back, and finally leaving the first layer through its front surface. Its contribution to the total reflection coefficient is $t_{01} \exp(i\phi_1)t_{12} \exp(i\phi_2)r_{2f}\exp(i\phi_2)t_{21}\exp(i\phi_1)t_{10}$.
from which the part $r_{2f} \exp(2i\phi_1) \exp(2i\phi_2)$ goes toward the first approximation, the parts $-r_{01}^2 r_{2f} \exp(2i\phi_1) \exp(2i\phi_2)$ and $-r_{12}^2 r_{2f} \exp(2i\phi_1) \exp(2i\phi_2)$ go toward the second approximation, and the part $r_{01}^2 r_{12} r_{2f} \exp(2i\phi_1) \exp(2i\phi_2)$ goes toward the third approximation. As a result,

$$R_2^{(1)} = r_{01} + r_{12} \exp(2i\phi_1) + r_{2f} \exp(2i\phi_1) \exp(2i\phi_2). \quad (2.74)$$

In the second approximation we should also take into account the contributions from the partial waves which have three reflections in the stack. We can see that there are five such partial waves. As a result,

$$R_2^{(2)} = r_{01} + r_{12} \exp(2i\phi_1) + r_{2f} \exp(2i\phi_1) \exp(2i\phi_2)$$
$$- r_{01}^2 r_{12} \exp(2i\phi_1) - [r_{01}^2 + r_{12}^2] r_{2f} \exp(2i\phi_1) \exp(2i\phi_2)$$
$$- [r_{12} \exp(2i\phi_1) + r_{2f} \exp(2i\phi_1) \exp(2i\phi_2)]^2 r_{01}$$
$$- [r_{2f} \exp(2i\phi_2)]^2 r_{12} \exp(2i\phi_1). \quad (2.75)$$

The comparison of the results for the reflection coefficient from two layers in the first and second approximation with the exact formula (2.61) is shown in Figure 2.5.

![Figure 2.5](image)

Figure 2.5: Dependence of $R_2^l$ on $\lambda$ $n_0 = 1$, $n_1 = 2.4$, $d_1 = 2.5 \mu m$, $n_2 = 1.5$, $d_2 = 2 \mu m$ $n_f = 3.6$; solid line - first approximation, dashed line - second approximation, gray line - exact result
Again, as an example, we have used the case of normal incidence of an optic wave on transparent (no absorption) layers. The selection of materials (vacuum, chalcogenide glass, fused quartz, GaAs) is again testing an unfavorable situation, as the difference in refractive indexes of these materials are high. However, as we can see, the second approximation is in good agreement with the exact formula (2.61).

In the case of an arbitrary number of layers \( m \), summing up all corresponding partial waves we obtain in the first approximation

\[
R_m^{(1)} = \sum_{j=0}^{m} \left[ r_{j,j+1} \prod_{t=0}^{j} \exp(2i\phi_t) \right], \tag{2.76}
\]

and in the second approximation

\[
R_m^{(2)} = \sum_{j=0}^{m} \left[ r_{j,j+1} \prod_{t=0}^{j} \exp(2i\phi_t) \right] - \sum_{j=1}^{m} \left[ \left( \sum_{p=0}^{j-1} r_{p,p+1}^2 \right) r_{j,j+1} \prod_{t=1}^{j} \exp(2i\phi_t) \right] - \sum_{j=0}^{m-1} \left[ \left( \sum_{p=j+1}^{m} r_{p,p+1} \prod_{t=1}^{p} \exp(2i\phi_t) \right)^2 r_{j,j+1} \prod_{t=0}^{j} \exp(2i\phi_t) \right]. \tag{2.77}
\]

We remind the reader that in formulas (2.76) and (2.77) \( \phi_0 = 0 \).

Figure 2.6 illustrates the application of formulas (2.76) and (2.77) to the case of \( m = 5 \) layers with arbitrarily selected refractive indexes in the range \( 1 < n_j < 4 \), that corresponds to the typical values of the refractive indexes of optic materials. The exact curve was obtained from the recursive use of formula (2.60). Again, we can see good agreement between approximate and exact solutions.
Figure 2.6: Dependence of $R_m^i$ on $\lambda$ $n_0 = 1$, $n_1 = 2.4$, $d_1 = 2.5\mu m$, $n_2 = 1.5$, $d_2 = 2\mu m$, $n_3 = 2$, $d_3 = 2.2\mu m$, $n_4 = 3.3$, $d_4 = 2\mu m$, $n_5 = 1.8$, $d_5 = 2.1\mu m$, $n_f = 3.6$; solid line - first approximation, dashed line - second approximation, gray line - exact result

All of the above examples show good agreement between the exact results and the second approximation of the multiple reflection method. Very often there is satisfactory agreement even with the first approximation. However, if the Fresnel reflection coefficients between neighboring layers become too high, the second approximation fails and higher order approximations are needed.
Chapter 3

Layered Periodic Structures

In this chapter we consider a special case of layered structures - layered periodic structures. The chapter is organized as follows. In the first section we consider general properties of electromagnetic fields in an arbitrary periodic dielectric medium. In the second section, with the aid of the transfer matrix method, exact analytic results are obtained for the properties of waves in a two-layered periodic dielectric structure with homogeneous layers. In the third section approximate analytic results are obtained for the properties of an arbitrary periodic medium with a small modulation in the dielectric permittivity. The analysis is done with the aid of the Floquet-Bloch method that follows from the periodicity of the structure and, therefore, was not described in the previous chapter.

3.1 General Properties of Waves in Periodic Structures

For the sake of simplicity, we consider in this chapter the case of electromagnetic waves normally propagating in a periodic dielectric structure. In this case of normal propagation, polarization plays no role (\(TE\) and \(TM\) waves are not distinguishable), and the equation for the electric field \(E(z)\) takes the form

\[
\frac{d^2E(z)}{dz^2} + k^2n^2(z)E(z) = 0, \tag{3.1}
\]

where

\[
E(z, t) = E(z) \exp(-i\omega t). \tag{3.2}
\]
The refractive index \( n(z) \) and the dielectric permittivity \( \varepsilon(z) \) are in general complex, periodic functions

\[
n(z + d) = n(z), \quad n^2(z) = \varepsilon(z).
\]  

(3.3)

Without loss of generality we suppose that \( E(z) \) is directed along the \( y \)-axis. In the case of normal propagation the Fresnel formulas (2.21) take the simpler forms

\[
\begin{align*}
r_{j,j+1} &= \frac{n_j - n_{j+1}}{n_j + n_{j+1}}, & r_{j+1,j} &= \frac{n_{j+1} - n_j}{n_j + n_{j+1}} \\
t_{j+1,j} &= \frac{2n_{j+1}}{n_j + n_{j+1}}, & t_{j,j+1} &= \frac{2n_j}{n_j + n_{j+1}}.
\end{align*}
\]  

(4.4)

Eq. (3.1) with an arbitrary periodic coefficient \( \varepsilon(z) \) is the Hill equation. If \( \varepsilon(z) \) has a harmonic form, Eq. (3.1) is called the Mathieu equation. If \( \varepsilon(z) \) has a piecewise form, we have a layered periodic structure and all the theory developed in Chapter 2 is applicable to this case.

But let us first discuss the general properties of the Hill equation (3.1) which are valid for an arbitrary periodic dielectric permittivity \( \varepsilon(z) \). According to the Floquet theory [56, 57, 58], we can represent the general solution of Eq. (3.1) as a superposition of two travelling Floquet-Bloch waves in the form

\[
E(z) = D_1 E_I(z) + D_2 E_{II}(z),
\]  

(5.5)

where

\[
E_{I,II}(z) = F_{I,II}(z) \exp \left( i \frac{\xi_{1,2}}{d} z \right), \quad F_{I,II}(z) = F_{I,II}(z + d)
\]  

(6.6)

The quantities \( \xi_{1,2} \) are the so-called Bloch phases, related by \( \xi_2 = -\xi_1 \) [58, 59]. They are complex and can be expressed in terms of the real and imaginary parts as \( \xi_{1,2} = \xi_{1,2}^' + i\xi_{1,2}^'' \). The direct Floquet-Bloch wave \( E_I(z) \) gives the solution for the total field \( E_I(z, t) \) in the form

\[
E_I(z, t) = F_I(z) \exp \left( -\frac{\xi_1^''}{d} z \right) \exp \left( i \left( \frac{\xi_1^'}{d} z - \omega t \right) \right).
\]  

(7.7)

This is a periodically modulated in space (\( F_I(z) \) is a periodic function), damped (\( \xi_1^'' \neq 0 \)) electromagnetic wave, travelling (if \( \xi_1^' \neq 0 \)) along the \( z \)-axis with phase velocity \( v_{ph} = \omega d/\xi_1^' \), and having an absorption (or amplification) coefficient \( \xi_1^'' \). The
group velocity of this wave is determined by the standard formula \( v_g = d \partial(\omega)/\partial(\xi_1) \). The backward Floquet-Bloch wave \( E_{II}(z) \) is determined by the analogous form with \( F_I(z) \) and \( \xi_1 \) replaced by \( F_{II}(z) \) and \( \xi_2 \).

Therefore, one can see that the peculiarities of wave propagation in periodic structures mainly depend on the dispersion relation \( \xi = \xi(\omega) \). For the sake of simplicity in the notation, we put \( \xi_1 = \xi \). Therefore, \( \xi_2 = -\xi \). From expression (3.7) for the electromagnetic wave travelling in a periodic structure, one can see the basic properties of this wave. There are two physically different regions of parameters for our structure. In the first, \( \xi(\omega) \) is real, i.e. \( \xi'' = 0 \) and the wave \( E_I(z, t) \) propagates without attenuation. Such regions are called allowed bands of frequencies. In the second, \( \xi(\omega) \) is complex, and \( E_I(z, t) \) (direct Floquet-Bloch wave) is exponentially damped, even in the absence of real absorption. Such regions are called forbidden regions. In such regions \( \xi'' \neq 0 \) and the direct Floquet-Bloch wave can not propagate in infinite periodic structures, as typically \( \xi'' > 0 \). However, the case \( \xi'' < 0 \) is realized in an active amplifying laser medium. Besides, under \( \xi'' > 0 \) the direct Floquet-Bloch waves can propagate with absorption in finite periodic structures. Physically, in such structures the accumulated Fresnel reflection from inhomogeneities of the dielectric permittivity \( \varepsilon(z) \) on the period results in an increase of the amplitude of the backward Floquet-Bloch wave at the expense of the forward wave, leading to an increase of the reflection coefficient.

Now let us discuss the relation between the transfer matrix method and the Floquet theory for an arbitrary finite periodic structure with length \( L = Nd \), where \( N \) is the number of periods in the structure. According to the transfer matrix method for the total electromagnetic field inside the structure we have

\[
\begin{pmatrix}
E(L) \\
H(L)
\end{pmatrix}
= [M]
\begin{pmatrix}
E(0) \\
H(0)
\end{pmatrix}
= [C]^N
\begin{pmatrix}
E(0) \\
H(0)
\end{pmatrix},
\]

(3.8)

where \([C]\) is the transfer matrix for a unit cell (one period) of the periodic structure, i.e.

\[
\begin{pmatrix}
E(d) \\
H(d)
\end{pmatrix}
= [C]
\begin{pmatrix}
E(0) \\
H(0)
\end{pmatrix}.
\]

(3.9)
For the fields of the Floquet-Bloch waves $E_{I,II}$ we have
\[
\begin{pmatrix}
E_{I,II}(d) \\
H_{I,II}(d)
\end{pmatrix} = \exp(\pm i\xi) \begin{pmatrix}
E_{I,II}(0) \\
H_{I,II}(0)
\end{pmatrix}.
\] (3.10)

Therefore, the values of the Floquet-Bloch waves at the point $z = 0$ are the eigenvectors of the matrix $[C]$ with the eigenvalues
\[
\rho_{1,2} = \exp(\pm i\xi).
\] (3.11)

The eigenvalue equation for the matrix $[C]$ in terms of its elements has the form
\[
\rho^2 - \rho(C_{11} + C_{22}) + 1 = 0
\] (3.12)
due to the unimodularity ($\det([C]) = 1$) of the transfer matrix $[C]$. As a result, the dispersion equation $\xi(\omega)$ for an arbitrary shape of $n(z)$ with period $d$ has the form
\[
\cos \xi = \frac{1}{2}(C_{11} + C_{22}).
\] (3.13)

Let us express the elements of the transfer matrix $M$ for the whole periodic structure in terms of the elements of the transfer matrix $[C]$ for one period. In accordance with the general matrix theory, the $N^{th}$ power of any unimodular matrix [60] is
\[
[C]^N = [C] \frac{\sin N\xi}{\sin \xi} - [I] \frac{\sin(N - 1)\xi}{\sin \xi},
\] (3.14)
where $[I]$ is the unit matrix. Therefore,
\[
[M] = \frac{\sin N\xi}{\sin \xi} \begin{pmatrix}
C_{11} - \frac{\sin(N - 1)\xi}{\sin N\xi} & C_{12} \\
C_{21} & C_{22} - \frac{\sin(N - 1)\xi}{\sin N\xi}
\end{pmatrix}.
\] (3.15)

Hence, we arrive at the important result that the transfer matrix $[M]$ for an arbitrary finite periodic structure can be written in terms of the unit cell (period) matrix $[C]$, the number of periods $N$, and the Bloch phase $\xi$.

Now let us discuss the reflection and transmission properties of the finite periodic structure placed between two homogeneous semi-infinite media having the same refractive index $n_0$, see Figure 3.1, i.e we suppose that $n_f = n_0$. 

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In accordance with (2.40) and (2.41), the system matrix \([S]\) for such a shape of the refractive index is

\[
[S] = \frac{1}{2} \begin{pmatrix}
M_{11} + M_{22} - \left( n_0 M_{12} + \frac{M_{21}}{n_0} \right) & M_{11} - M_{22} + \left( n_0 M_{12} - \frac{M_{21}}{n_0} \right) \\
M_{11} - M_{22} - \left( n_0 M_{12} - \frac{M_{21}}{n_0} \right) & M_{11} + M_{22} + \left( n_0 M_{12} + \frac{M_{21}}{n_0} \right)
\end{pmatrix}
\]

(3.16)

with \(\det[S] = 1\). Therefore, according to (2.42) and (2.43)

\[
T^t = T^r \equiv T = \frac{2}{M_{11} + M_{22} + \left( n_0 M_{12} + \frac{M_{21}}{n_0} \right)}
\]

(3.17)

and

\[
R^{t,t} = \mp \frac{M_{11} - M_{22} + \left( n_0 M_{12} - \frac{M_{21}}{n_0} \right)}{M_{11} + M_{22} + \left( n_0 M_{12} + \frac{M_{21}}{n_0} \right)}
\]

(3.18)

Using (3.13) and (3.15), we can obtain the transmission and reflection coefficients in terms of the elements of the cell matrix \([C]\) and the Bloch phase \(\xi\)

\[
T = \frac{1}{\cos N\xi - \frac{1}{2} \sin N\xi \left( n_0 C_{12} + \frac{C_{21}}{n_0} \right)}
\]

(3.19)

and

\[
R^{t,t} = \mp \frac{C_{11} - C_{22} + \left( n_0 C_{12} - \frac{C_{21}}{n_0} \right)}{2 \cos \xi + \left( n_0 C_{12} + \frac{C_{21}}{n_0} \right) - 2 \sin(N - 1)\xi \sin N\xi}
\]

(3.20)
If we introduce the transmission $t$ and the reflection coefficients $r^l$, $r^r$ of the unit cell placed between the same semi-infinite spaces with $n = n_0$, see Figure 3.2, we obtain

$$T = \frac{\sin \xi}{\frac{1}{t} \sin N \xi - \sin(N - 1) \xi},$$  \hspace{1cm} (3.21)

$$\frac{R^{l,r}}{T} = \frac{r^{l,r} \sin N \xi}{t \sin \xi}.$$  \hspace{1cm} (3.22)

These results are remarkable because the formulas (3.21) and (3.22) are valid for any shape of the complex refractive index having period $d$. They were first obtained in [61].

![Figure 3.2: Unit-cell refractive index](image)

Formulas (3.19) and (3.20) allow us to find the reflection and transmission properties of a finite periodic structure if we know only the transfer matrix $[C]$ of its unit cell (the Bloch phase $\xi$ can be found in terms of the elements of the matrix $[C]$ as well, see (3.12)). However, a knowledge of $[C]$ is not sufficient to find the electromagnetic fields associated with the Floquet-Bloch waves, see (3.7). The eigenvector equation for the matrix $[C]$ allows us to find only the values of these fields at the point $z = 0$ and, as a result, at the points $z = ld$, $l = 0, 1, 2...N$.

The problem of finding the Floquet-Bloch waves for a periodic structure with uniform layers can be solved in an exact manner using the representation of both direct and backward Floquet-Bloch waves in each homogeneous layer on the period
of the structure in terms of either a superposition of two exponential functions or one sinusoidal function with an indefinite phase. Such an approach was described by us in [62, 63]. However, the final results are very cumbersome even for the simplest case of a two-layered periodic structure. Therefore, in order to find the expressions for the Floquet-Bloch waves inside an arbitrary finite periodic structure, including the case of a structure with uniform layers, it is reasonable to use approximate analytic methods. One of them will be described in the last section of this chapter.

Now let us go back to the theory of the reflection and transmission properties of a periodic structure based on the transfer matrix method. The problem of finding the reflection and transmission can be solved in an exact manner if the layers are uniform, as the expression for the transfer matrix is exact in this case. However, only in the aforementioned case of the two-layered periodic structure the final results are not so cumbersome as to be investigated analytically.

### 3.2 Exact Description of Waves in Two-Layered Periodic Structure

The parameters specifying the problem are as shown in Figure 3.3. We assume a two-layered periodic medium with real refractive indexes $n_1$ and $n_2$ (no absorption) and thicknesses $d_1$ and $d_2$ of the layers such that $d = d_1 + d_2$.

![Figure 3.3: Two-layered periodic structure](image)

Figure 3.3: Two-layered periodic structure
First of all, let us define the parameters which are naturally more suitable for the description of normal wave propagation in two-layered periodic dielectric structures with uniform layers:

$$\Omega = k(n_2d_2 + n_1d_1),$$  \hspace{1cm} (3.23)
$$\Delta = k(n_2d_2 - n_1d_1).$$

Each of these parameters has clear physical meaning: $\Omega$ is a period-average dimensionless wave vector of the light inside the structure, and $\Delta$ is the difference in optical paths of the wave inside each layer.

According to (2.45) the layer matrices have the form

$$[M_2] = \begin{pmatrix}
\cos(kn_2d_2) & -i \frac{n_2}{n_1} \sin(kn_2d_2) \\
-in_2 \sin(kn_2d_2) & \cos(kn_2d_2)
\end{pmatrix},$$  \hspace{1cm} (3.24)
$$[M_1] = \begin{pmatrix}
\cos(kn_1d_1) & -i \frac{n_1}{n_2} \sin(kn_1d_1) \\
-in_1 \sin(kn_1d_1) & \cos(kn_1d_1)
\end{pmatrix}.$$  \hspace{1cm} (3.25)

As a result, the cell matrix $C = [M_2][M_1]$ takes the form

$$[C] = \begin{pmatrix}
\cos \phi_2 \cos \phi_1 - \frac{n_1}{n_2} \sin \phi_2 \sin \phi_1 & -i \frac{n_1}{n_2} \cos \phi_2 \sin \phi_1 - i n_2 \sin \phi_2 \cos \phi_1 \\
-in_1 \cos \phi_2 \sin \phi_1 - in_2 \sin \phi_2 \cos \phi_1 & \cos \phi_2 \cos \phi_1 - \frac{n_2}{n_1} \sin \phi_2 \sin \phi_1
\end{pmatrix},$$

where

$$\phi_2 = kn_2d_2, \hspace{0.5cm} \phi_1 = kn_1d_1.$$  \hspace{1cm} (3.26)

The well-known dispersion equation $\xi(\omega)$ for a two-layered periodic structure [16, 17, 18, 19] follows immediately from (3.13) and (3.25). In terms of $\Omega$, $\Delta$, and the Fresnel coefficient $r_{12}$ it has the form

$$\cos \xi = \frac{1}{1 - r_{12}^2} \cos \Omega - \frac{r_{12}^2}{1 - r_{12}^2} \cos \Delta.$$  \hspace{1cm} (3.27)

Using the condition $|\cos(\xi)| \geq 1$ we can easily find the widths of the forbidden regions:

$$\Delta \Omega_{oddy} = 2 \arccos \left(1 - 2r_{12}^2 \cos^2 \left(\frac{1}{2} \Delta\right)\right),$$  \hspace{1cm} (3.28)
\[ \Delta \Omega_{\text{even}} = 2 \arccos \left( 1 - 2r_{12}^2 \sin^2 \left( \frac{1}{2} \Delta \right) \right), \quad (3.29) \]

where \( \Delta \Omega_{\text{odd}} \) and \( \Delta \Omega_{\text{even}} \) are the widths of odd (with the centers \( \Omega = (2l + 1)\pi, l = 0, 1, 2, \ldots \)) and even (with the centers \( \Omega = 2l\pi \)) forbidden regions. Expressions (3.28) and (3.29) will play an extremely important role in the investigation of the elastic stress influence on the electromagnetic wave propagation in Chapter 4. Also we would like to emphasize that they are exact.

Now let us investigate the reflection and transmission properties of the considered medium. Using formulas (3.19) and (3.20) for the transmission and reflection coefficients of an arbitrary finite periodic structure and expression (3.25) for the unit cell matrix of the two-layered periodic structure, we have in terms of the parameters \( \Omega \) and \( \Delta \)

\[
R^{l,r} = \frac{r_{12}(\cos \Omega - \cos \Delta)}{1 - r_{12}^2} \pm i \left[ \left( \frac{r_{02}}{1 - r_{02}^2} + \frac{r_{01}}{1 - r_{01}^2} \right) \sin \Omega + \left( \frac{r_{02}}{1 - r_{02}^2} - \frac{r_{01}}{1 - r_{01}^2} \right) \sin \Delta \right],
\]

\[
T = \frac{1}{\cos N\xi - i \sin N\xi} \left[ \left( \frac{1}{1 - r_{02}^2} + \frac{1}{1 - r_{01}^2} - 1 \right) \sin \Omega + \left( \frac{1}{1 - r_{02}^2} - \frac{1}{1 - r_{01}^2} \right) \sin \Delta \right].
\quad (3.30)
\]

Now we investigate in detail how the energy reflection coefficient \( |R^l|^2 \) depends on the parameters \( \Omega, \Delta, N, \) and Fresnel coefficients \( r_{12}, r_{01}, r_{02} \). For convenience, we take \( \Omega \) as an independent variable and study the function \( R = R(\Omega) \) with \( \Delta, N, r_{12}, r_{01}, r_{02} \) as parameters.

First, we consider the function \( |R^l|^2 = |R^l(\Omega)|^2 \) for different values of \( \Delta \). Suppose that \( n_0 = 2, n_1 = 1.5, n_2 = 2.5 \). It means that \( r_{12} = -1/4, r_{01} = 1/7, r_{02} = 1/9 \). Let \( N = 10 \). Figure 3.4(a) shows a plot of the graph of \( |R^l(\Omega)|^2 \) with \( \Delta = 0 \), i.e. when the basic layers have equal optical thickness. As can be seen, the behavior of the curve is quite different in the forbidden (\( |\cos \xi| > 1 \)) and allowed (\( |\cos \xi| > 1 \)) regions. In
the forbidden regions, $|R^i|^2$ is almost constant and for given $r_{12}$, $r_{01}$, and $r_{02}$ it very nearly reaches unity. In the allowed regions, its dependence on $\Omega$ is oscillatory with increasing amplitude near the boundaries with the forbidden regions. With $\Delta = 0$ there are only odd forbidden regions. Figure 3.4(b) shows the effect of introducing a difference in optical thickness of the basic layers. Then even forbidden regions appear. With $\Delta = \pi/2$, the widths of even and odd forbidden regions become equal (Figure 3.4(c)). With $\Delta = \pi$ (Figure 3.4(d)), the odd forbidden regions disappear, leaving only even forbidden regions, which now have their maximum widths. If $\Delta$ is further increased, the pattern proceeds in the opposite direction so that when $\Delta = 2\pi$ we have returned to the initial picture corresponding to $\Delta = 0$. The cycle then repeats with period $\Delta = 2\pi$. 
Figure 3.4: Dependence of $|R^i(\Omega)|^2$ on $\Delta$, for $n_0 = 2$, $n_1 = 1.5$, $n_2 = 2.5$, $N = 10$. (a) $\Delta = 0$, (b) $\Delta = \pi/4$, (c) $\Delta = \pi/2$, (d) $\Delta = \pi$.

Now we consider the influence on the reflection coefficient of Fresnel interaction at the boundaries of the structure. For this we fix $\Delta = 0$ and increase $n_0$ until $n_0 = 4$, corresponding to $r_{01} = 5/11$, $r_{02} = 3/13$ (Figure 3.5). We can see (compare with Figure 3.4(a)) that increasing $r_{01}$ and $r_{02}$ changes the behavior of the reflection coef-
ficient in the allowed regions, making the amplitude of oscillations higher. However, the behavior of the reflection coefficient in the forbidden regions does not change.

![Graph showing the dependence of $|R^l(\Omega)|^2$ on $\Omega$ for $n_0 = 4$, $n_1 = 1.5$, $n_2 = 2.5$, $N = 10$, $\Delta = 0$.]

Figure 3.5: Dependence of $|R^l(\Omega)|^2$ for $n_0 = 4$, $n_1 = 1.5$, $n_2 = 2.5$, $N = 10$, $\Delta = 0$.

In order to follow the influence on $|R^l|^2$ of the optical modulation of the structure, which is the Fresnel coefficient $r_{12}$ in this particular case, and the number of periods $N$, we minimize Fresnel interaction at the boundaries by taking $n_0^2 = n_1 n_2$, and take $n_1$ and $n_2$ to be 1.5 and 1.6 respectively, while keeping $N = 10$ (Figure 3.6(a)). We can see that decreasing $r_{12}$ causes the widths of the allowed regions to increase, with $|R^l|^2$ tending to zero. $|R^l|^2$ in the forbidden regions is also diminished, but it remains relatively high compared with $|R^l|^2$ in the allowed regions. However, if we increase the number of periods $N$, for example up to $N = 50$, keeping all other parameters the same, we can see (Figure 3.6(b)) that $|R^l|^2$ in the forbidden regions almost reaches unity again. Thus, the most important and physically interesting influence of the number of periods on the reflection coefficient is the following: even though the optical modulation $r_{12}$ may be very small, there are always some narrow forbidden regions in which the reflection coefficient practically reaches unity if the number of periods is sufficiently great.
The above examples illustrate a high sensitivity of the reflection and transmission of the electromagnetic wave to the parameters of a two-layered periodic structure. This property makes it possible to use the two-layered periodic dielectric structure as the basic medium for optical switching devices. For this we should shift by some way an electromagnetic wave with wavelength $\lambda$ from a forbidden region, where reflection is close to 100%, to the edge of the allowed region, where reflection is not more than 15%. As the simplest possibility for the creation of such a shift, we will consider in Chapter IV the result of applying a constant elastic stress inside the structure.

3.3 Floquet-Bloch Method

The approximate expressions for the solutions (3.6) of Eq. (3.1) with an arbitrary periodic coefficient (3.2) can be found with the aid of two basic methods: the Floquet-Bloch method and the coupled-wave theory. The first method can be divided into two physically different approaches: the Raman-Nath multi-wave diffraction method
when both direct and backward Floquet-Bloch waves (3.6) can be represented as a superposition of many diffracted waves, and the Bragg two-wave diffraction method when each of the Floquet-Bloch waves can be represented as two travelling waves with constant amplitudes. The coupled-wave theory and its geometro-optical modification are closely related to the Bragg two-wave diffraction approach. The solution is taken in the form of a superposition of two counter-propagating waves as in the Bragg two-wave diffraction approach, but with slowly variable amplitudes. The coupled-wave theory was initially proposed by Kogelnik in [64]. Its geometro-optical modification was developed in [65]. A detailed review of the coupled-mode theory and the comparison between the results obtained with the aid of the classical Kogelnik method and with the aid of the geometro-optical modification were done in [66, 67]. In this work we consider only the Floquet-Bloch method. The difference of our description from the classical approach [16, 57, 68, 69] is that we construct both the first and the second Floquet-Bloch waves rather than only the first.

The essence of the Floquet-Bloch method is as follows. In accordance with (3.6) we can represent the first Floquet-Bloch wave $E_I(z)$ in the form

$$E_I(z) = \exp \left( \frac{i \xi_1}{d} z \right) \sum_{l=-\infty}^{l=+\infty} F_{l,I} \exp \left( \frac{2\pi i}{d}lz \right),$$

(3.32)

where $F_{l,I}$ are unknown coefficients that determine the form of the periodic function $F_I(z)$. The Fourier expansion of the periodic dielectric permittivity $\varepsilon(z)$ has the form

$$\varepsilon(z) = \sum_{m=-\infty}^{m=+\infty} \varepsilon(m) \exp \left( \frac{2\pi i}{d} mz \right),$$

(3.33)

in which

$$\varepsilon(m) = \frac{1}{d} \int_{z=0}^{z=d} \varepsilon(z) \exp \left( -\frac{2\pi i}{d} mz \right) dz.$$  

(3.34)

As in the previous section we suppose that the dielectric permittivity $\varepsilon(z)$ is real, i.e. there is no absorption. Substituting (3.31) and (3.32) in (3.1), multiplying by $\exp (-i(2\pi/d)l'z)$ and integrating over the period $d$, we obtain the following infinite set of equations for the coefficients $F_{l,I}$:

$$- \left( \xi_1 + \frac{2\pi}{d} l' \right)^2 F_{l',I} = -k^2 \sum_{l=-\infty}^{l=+\infty} \varepsilon_{l'-l} F_{l,I},$$

(3.35)
where \( l' = 0, \pm 1, \pm 2, \ldots \). The second Floquet-Bloch wave \( E_2(z) \) is determined by an analogous procedure with the Bloch phase \( \xi_1 \) replaced by \( \xi_2 \) and with the unknown coefficients \( F_{l, I} \) replaced by \( F_{l, II} \). Thus, the system (3.35) describes both Floquet-Bloch waves \( E_1(z) \) and \( E_2(z) \). To emphasize this point, we may drop the subscripts on the Bloch phase \( \xi \) and the unknown amplitudes \( F_l \). Separating out the \( l' = 0 \) term and returning to the old summation indexes, we can write (3.35) in the form

\[
\left[ k^2 \varepsilon_{(0)} - \left( \frac{\xi}{d} + \frac{2\pi}{d} \right)^2 \right] F_l = -k^2 \sum_{m=-\infty}^{\infty} \varepsilon_{(l-m)} F_m (1 - \delta_{lm}), \tag{3.36}
\]

where \( l = 0, \pm 1, \pm 2, \ldots \), \( \delta_{lm} \) is the Kronecker symbol, and the factor \( (1 - \delta_{lm}) \) annuls the \( m = l \) term. The system (3.36) is exact. If we equate its determinant to zero, we obtain the dispersion equation for the Bloch phase \( \xi \equiv \xi_i \). The unknown coefficients \( F_m \) can be expressed in terms of \( F_0 \) either by the method of continued fractions [16, 57] or by the well-known methods available for the evaluation of infinite matrices [16, 57]. The coefficient \( F_0 \) plays the role of a normalization constant and without any loss of a generality can be taken as unity. In practice, instead of the infinite set of equations, we solve a finite set that is obtained from (3.36) by discarding the higher-order harmonics. The order of the approximate set of equations is determined by the required precision of the final result. The system (3.36) is particularly suitable for numerical calculations. However, we can obtain from (3.35) approximate analytical solutions as well, using perturbation theory.

### 3.3.1 Raman-Nath Multi-Wave diffraction

To construct the approximate solution of (3.35), we consider its right-hand side as a perturbation that is possible for a small modulation of the dielectric permittivity \( \varepsilon(z) \), when \( |\varepsilon_{(m)}| \ll \varepsilon_{(0)} \) for all \( m \neq 0 \). The zero-order Fourier harmonic \( \varepsilon_{(0)} \) is real according to (3.34). If, at the same time, we suppose that \( F_l \ll F_0 \), and if we solve (3.36) by the method of successive approximations, we obtain a perturbation-theory series for the Bloch phase \( \xi \) and the amplitudes \( F_l \). For the zero-order approximation
we have
\[ \xi^{(0)} = \pm kd \varepsilon^{1/2}_{(0)}, \]  
\[ F_{l}^{(0)} = \delta_{l0}. \]  
(3.37)

In the first approximation we have
\[ \xi^{(1)} = \pm kd \varepsilon^{1/2}_{(0)} \left[ 1 + \left( \frac{d}{\lambda \varepsilon^{1/2}_{(0)}} \right)^2 \sum_{l=1}^{\infty} \frac{|\varepsilon_{l}|^2}{l^2 - (4d^2 \varepsilon_{(0)}/\lambda^2)} \right], \]  
\[ F_{l}^{(1)} = \left( \frac{d}{\lambda} \right) \frac{\varepsilon_{l}}{l^2 \pm 2l (d/\lambda) \varepsilon^{1/2}_{(0)}}. \]  
(3.38)

The upper sign in (3.37) and (3.38) corresponds to the first Bloch wave \( E_{1}(z) \) and the lower sign corresponds to the second Bloch wave \( E_{2}(z) \). As we can see the solution method is similar to the Born approximation that is widely used in quantum theory of scattering [70]. It is based on the expansion of the field in powers of a small parameter that involves the perturbation \( \Delta \varepsilon \equiv |\varepsilon_{(0)}| \) and the ratio of the size of the scatterer to the wavelength. In our case the small parameter is
\[ q = \frac{kd}{|\varepsilon_{(0)}|^{1/2}} \Delta \varepsilon. \]  
(3.39)

It is small if either \( \lambda >> d \) or \( \Delta \varepsilon << \varepsilon_{(0)} \).

According to (3.32) the total field associated with each of the Floquet-Bloch waves is a superposition of plane waves with the amplitudes \( F_{l} \) and wave vectors \( k_{l} = (\pm \xi/d + 2\pi/d l) \). When \( q << 1 \), each of the Floquet-Bloch waves in the zero-order approximation consists of one plane wave with unit amplitude and the wave vector \( \pm k \varepsilon^{1/2}_{(0)} \). As a result, the total field (3.5) inside the structure in the zero-order approximation is
\[ E^{(0)} = D_{1} \exp \left( i k \varepsilon^{1/2}_{(0)} \right) + D_{2} \exp \left( -i k \varepsilon^{1/2}_{(0)} \right). \]  
(3.40)

Then, in the first approximation the secondary waves with small amplitudes that are excited by these initial waves on permittivity inhomogeneities produce a weak wave background. This defines the approach called Raman-Nath multi-wave diffraction.
3.3.2 Bragg Diffraction

It is clear from the solutions given by (3.38) that the condition $F_l << F_0$ fails to be valid for numbers $l = \mp M$, $M = 1, 2..., \text{ where again the upper sign should be used for the first Bloch wave and the lower sign should be used for the second. Physically, these numbers satisfy the Bragg condition}

$$k\varepsilon_{(0)}^{1/2} = \frac{\pi}{d} M \quad \text{or} \quad M\frac{\lambda}{\varepsilon_{(0)}^{1/2}} = 2d.$$  \hspace{1cm} (3.41)

Then, the amplitudes $F_{\mp M}$ of the $(\mp M)$-th harmonic can become equal to or greater than the amplitudes $F_0$ of the leading components with the wave vector $k\varepsilon_{(0)}^{1/2}$ for the first Bloch wave and with the wave vector $-k\varepsilon_{(0)}^{1/2}$ for the second. We remind the reader that without loss of generality the amplitudes $F_0$ in the Fourier expansion of both Bloch waves can be taken as unity. If we can neglect in the zero-order approximation all the other harmonics, i.e. if the condition $q << 1 \text{ (see (3.39))}$ is valid, we obtain from the infinite exact system (3.36) the approximate system of two equations for each of the Floquet-Bloch waves

$$\left[ k^2\varepsilon_{(0)} - \left( \frac{\xi}{d} \right)^2 \right] F_0 + k^2\varepsilon_{(\pm M)} F_{\mp M} \approx 0, \hspace{1cm} (3.42)$$

$$\left[ k^2\varepsilon_{(0)} - \left( \frac{\xi}{d} \pm \frac{2\pi}{d} M \right)^2 \right] F_{\mp M} + k^2\varepsilon_{(\mp M)} F_0 \approx 0.$$

This is the so-called Bragg two-wave diffraction. The condition that this system has a solution gives the following dispersion equation for the Bloch phase $\xi$

$$\left( k^2\varepsilon_{(0)} - \left( \frac{\xi}{d} \right)^2 \right) \left[ k^2\varepsilon_0 - \left( \frac{\xi - 2\pi M}{d} \right)^2 \right] - k^4\varepsilon_{(\pm M)}\varepsilon_{(-\pm M)} = 0. \hspace{1cm} (3.43)$$

Solving this equation and the system (3.42) itself near a Bragg resonance, when $|\delta| = |k\varepsilon_{(0)}^{1/2} - (\pi/d) M| << (\pi/d) M$, and when the permittivity modulation is small, i.e. $|\varepsilon_{(\pm M)}| << \varepsilon_{(0)}$, we obtain

$$\xi = \pm (\pi M + i\gamma d),$$

$$F_{\mp M} = \frac{iS_{\mp M}}{\gamma - i\delta}, \hspace{1cm} (3.44)$$

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where
\[ \gamma = \left( S^2 - \delta^2 \right)^{1/2}, \quad S^2 = \frac{k^2\varepsilon(M)\varepsilon(-M)}{4\varepsilon(0)}, \quad S_{\pm M} = \frac{k\varepsilon(\pm M)}{2\varepsilon(0)^{1/2}}. \tag{3.45} \]

The properties of the approximate dispersion equation (3.43) for an arbitrary periodic structure are similar to the properties of the exact dispersion equation for a two-layered periodic structure with uniform layers (3.27). The centers of the forbidden bands, where \( \xi \) is complex (\( \gamma \neq 0 \)), are determined by the Bragg condition
\[ k\varepsilon^{1/2}(0)d = M\pi, \tag{3.46} \]

where \( k\varepsilon^{1/2}(0)d = \Omega \) for a two-layered periodic structure. It follows from (3.44) and (3.45) that the width of the forbidden frequency band is equal to twice the constant \( S \), where \( S = |S_{\pm M}| \). Outside the forbidden frequency band, both \( \gamma_0 \), and \( \xi \) in (3.44) are purely real. As the wave number \( k \) departs from the forbidden region, when \( |\delta| \gg S \), the solutions (3.44) for \( \xi \) and \( F_{\pm M} \) take the form of (3.38), but only for the harmonics \( l = \mp M \).

The quantities \( S_{\pm M} \) are usually called the coupling constants. \( S_{-M} \) determines the strength of the diffraction coupling between the leading harmonic \( F_0 \) and the \( (-M) \)-th harmonic of amplitude \( F_{-M} \) in the Fourier expansion of the first Floquet-Bloch wave. \( S_M \) determines the strength of the diffraction coupling between the leading harmonic \( F_0 \) and the \( (M) \)-th harmonic of amplitude \( F_M \) in the Fourier expansion of the second Floquet-Bloch wave. As a result, the total field inside the structure in the two-wave Bragg diffraction approximation is
\[ E^{(0)} = D_1 \exp(-\gamma z) \left[ \exp(i\frac{\pi}{d} M) + \frac{iS_{-M}}{\gamma - i\delta} \exp(-i\frac{\pi}{d} M) \right] + D_2 \exp(\gamma z) \left[ \exp(-i\frac{\pi}{d} M) + \frac{iS_M}{\gamma - i\delta} \exp(i\frac{\pi}{d} M) \right]. \tag{3.47} \]

This result coincides with the result for the total field inside the finite periodic structure which was obtained in [66] in the approximation of the classical coupled-wave theory.
Chapter 4

Green's Function for Layered Structures and Its Application

In this chapter we develop a general method for the construction of the Green's function for finite one-dimensional layered media in the case of normal wave propagation. Using the results of this method the exact analytical Green's functions are found for a step media and two-layered periodic structures. As an example of the application of the obtained Green's functions, the influence of possible defects (perturbations) on the properties of reflection and transmission coefficients of these media are investigated. In particular, using the Green's function obtained for an ideal medium (no defects), we transform the differential wave equation for a real medium (with possible defects) to the Lippmann-Schwinger integral equation. Then, following the standard procedure of solving the Lippmann-Schwinger equation in terms of the Neumann series, we obtain corrections to the reflection coefficient for the ideal structure.

4.1 Construction of Green's Function for Layered Structures

For one-dimensional differential equations we mention two methods for constructing the Green's function. In the first method the Green's function is constructed as a sum involving the eigenfunctions and eigenvalues of the homogeneous differential equation. In the second method the Green's function is constructed from two inde-
dependent solutions of the homogeneous differential equation which satisfy appropriate boundary conditions. For two- and higher-dimensional differential equations, only the first of the above procedures is available. Our approach for the 1-D case is based on the second method and has been described by us in [71]. The following consideration is further development of that approach.

The dielectric permittivity of an unperturbed (no defects) medium is again described by the formula (2.1). Due to the restriction to the case of normal propagation we will use the refractive index instead of dielectric permittivity:

\[
n_h(z) = \begin{cases} 
n_0, & z < 0, \\
n_L(z), & 0 < z < L, \\
n_f, & z > L, 
\end{cases} \tag{4.1}
\]

where the notation \(n_h(z)\) emphasizes that \(n_h(z)\) is the refractive index for the homogeneous equation (unperturbed profile). This homogeneous equation for monochromatic waves, \(E(z, t) = E(z) \exp(-i\omega t)\), is the one-dimensional Helmholtz equation

\[
\frac{d^2 E(z)}{dz^2} + k^2 n_h^2(z) E(z) = 0, \tag{4.2}
\]

which is exactly same as (3.1) except that the function \(n_h(z)\) is an arbitrary rather than periodic. We emphasize that the refractive index can be complex in general, i.e. we include into consideration absorptive media as well.

The Green’s function \(G(z, z_1)\) for Eq. (4.2) satisfies the equation

\[
\frac{d^2 G(z, z_1)}{dz^2} + k^2 n_h^2(z) G(z, z_1) = \delta(z - z_1) \tag{4.3}
\]

and either one of radiation conditions

\[
G^+(z, z_1) \xrightarrow{z \to -\infty} \exp(-ikn_0 z), \quad G^+(z, z_1) \xrightarrow{z \to +\infty} \exp(ikn_f z), \tag{4.4}
\]

\[
G^-(z, z_1) \xrightarrow{z \to -\infty} \exp(ikn_0 z), \quad G^-(z, z_1) \xrightarrow{z \to +\infty} \exp(-ikn_f z). \tag{4.5}
\]

In these formulae \(G^+(z, z_1)\) and \(G^-(z, z_1)\) are the Green’s functions for the waves which impinge on the structure from the region \(z < 0\) and the region \(z > L\), respectively. Without loss of generality we consider further only the first case of incoming waves and use the notation \(G(z, z_1)\) for \(G^+(z, z_1)\). Equation (4.3) with the radiation
condition (4.4) is the Sturm-Liouville problem. The usual procedure of obtaining the solution of this problem involves the spectral decomposition of \( G(z, z_1) \) in terms of the normalized eigenfunctions of the homogeneous equation (4.2) (see, for example, Ref. [72]), i.e. the first aforementioned method for constructing of the Green's function is used. But the calculations in an exact analytical form appear lengthy and cumbersome even for simple cases where the profile \( n_L(z) \) is either a step [73], quadratic [74], or \( \delta \)-function [75].

The first basic idea of the proposed method for the construction of an analytical Green's function associated with Eq.(4.2) is to use the symmetry property \( G(z, z_1) = G(z_1, z) \) in the whole plane \((z, z_1)\) in an explicit form. Following the well-known theorems for the Green's function in 1D case, we have

\[
G(z, z_1) = \frac{1}{W} \begin{cases}
E_h^t(z)E_h^r(z_1), & z < z_1, \\
E_h^r(z_1)E_h^t(z), & z > z_1,
\end{cases}
\] (4.6)

where \( E_h^t(z) \) and \( E_h^r(z) \) are independent solutions of the homogeneous equation (4.2), and \( W = E_h^t(z)E_h^r(z) - E_h^r(z)E_h^t(z) \) is their Wronskian. The value of the Wronskian \( W \) does not depend on \( z \) since Eq.(4.2) does not contain the first \( z \) derivative.

The second basic idea is to take the scattering solutions of Eq.(4.2) for a plane wave falling on the structure from the region \( z < 0 \) and \( z > L \) respectively as the independent solutions \( E_h^t(z) \) and \( E_h^r(z) \). In accordance with the radiation condition (4.4) these scattering solutions for the profile (4.1) have the form

\[
E_h^t(z) = \begin{cases}
\exp(ikn_0z) + R_h^t \exp(-ikn_0z), & z < 0, \\
A_h^t E_{h_1}^t(z) + B_h^t E_{h_2}^t(z), & 0 < z < L, \\
T_h^t \exp(ikn_f(z - L)), & z > L,
\end{cases}
\] (4.7)

and

\[
E_h^r(z) = \begin{cases}
T_h^r \exp(-ikn_0z), & z < 0, \\
A_h^r E_{h_1}^r(z) + B_h^r E_{h_2}^r(z), & 0 < z < L, \\
\exp(-ikn_f(z - L)) + R_h^r \exp(ikn_f(z - L)), & z > L,
\end{cases}
\] (4.8)

where the constants \( R_h^t, T_h^t, R_h^r \) and \( T_h^r \) are the amplitude reflection and transmission coefficients, see (2.42) - (2.44). The functions \( E_{h_1}^t(z) \) and \( E_{h_2}^t(z) \) are the linearly
independent solutions to the homogeneous equation (4.2) in the region $0 < z < L$. The Wronskian of $E_r(z)$ and $E_l(z)$ can be expressed as

$$W = 2ikn_0 T_h^r = 2ikn_f T_h^l.$$  \hspace{1cm} (4.9)

The third idea of the proposed method is to divide the whole plane $(z, z_1)$ into twelve parts, symmetrical in pairs with respect to the line $z = z_1$, see Figure 4.1.

![Diagram showing the division of the plane into twelve parts.]

**Figure 4.1: Construction of the Green's function**

Now let us introduce the following notation for the Green's function (4.6) in each
of these parts

\[
G(z, z_1) = \begin{cases} 
G_1(z, z_1) & \text{if } z_1 > L, \\
G_2(z, z_1) & \text{if } 0 < z_1 < L, \\
G_3(z, z_1) & \text{if } z < z_1 < L, \\
G_4(z, z_1) & \text{if } z_1 < z, \\
G_5(z, z_1) & \text{if } x_1 > L, \\
G_6(z, z_1) & \text{if } z < z_1 < L, \\
G_7(z, z_1) & \text{if } 0 < z_1 < z, \\
G_8(z, z_1) & \text{if } z_1 < 0, \\
G_9(z, z_1) & \text{if } z_1 > z, \\
G_{10}(z, z_1) & \text{if } z_1 < z, \\
G_{11}(z, z_1) & \text{if } 0 < z_1 < L, \\
G_{12}(z, z_1) & \text{if } z_1 < 0, 
\end{cases} 
\] (4.10)

The expressions for the Green’s function in the upper half-plane follow from (4.7) and (4.8) with the aid of boundary conditions on the internal boundaries of this plane

\[
G_1 = \frac{T_h}{2ikn_0} \exp(iKn_f(z_1 - L)) \exp(-ikn_0z), \\
G_2 = \frac{1}{2ikn_0} \left[ A_hE_{h_1}^L(z_1) + B_hE_{h_2}^L(z_1) \right] \exp(-ikn_0z), \\
G_3 = \frac{1}{2ikn_0} \left[ \exp(ikn_0z_1) + R_h \exp(-ikn_0z_1) \right] \exp(-ikn_0z), \\
G_5 = \frac{1}{2ikn_0} \exp(iKn_f(z_1 - L)) \left[ A_hE_{h_1}^L(z) + B_hE_{h_2}^L(z) \right], \\
G_6 = \frac{1}{2ikn_0T_h} \left[ A_hE_{h_1}^L(z_1) + B_hE_{h_2}^L(z_1) \right] \left[ A_hE_{h_1}^L(z) + B_hE_{h_2}^L(z) \right], \\
G_9 = \frac{1}{2ikn_f} \exp(iKn_f(z_1 - L)) \left[ \exp(-ikn_f(z - L)) + R_h \exp(iKn_f(z - L)) \right]. 
\] (4.11)

This exact analytical result represents a significant development in the Green’s function theory and can be used in a wide range of problems if we can find the fundamental system of solutions \(E_{h_1}^L(z)\) and \(E_{h_2}^L(z)\) of the homogeneous equation (4.2) in the region \(0 < z < L\).

The solution in the lower half-plane follows immediately from symmetry equalities.
\[ G_1(z, z_1) = G_{12}(z_1, z), \quad G_2(z, z_1) = G_{8}(z_1, z), \quad G_3(z, z_1) = G_{4}(z_1, z), \]
\[ G_5(z, z_1) = G_{11}(z_1, z), \quad G_6(z, z_1) = G_{7}(z_1, z), \quad G_9(z, z_1) = G_{10}(z_1, z). \]

Suppose we have found the fundamental solutions \( E_{L_1}^t \) and \( E_{L_2}^t \) by some means, either analytically or numerically. Now let us consider the same dielectric structure but with an embedded defect of refractive index \( n_0(z) \) and with finite width \( \sigma = z'' - z' \) between \( 0 < z < L \). The one-dimensional Helmholtz equation for such a refractive index profile \( n(z) \) can be represented in the form of the inhomogeneous equation

\[
\frac{d^2 E(z)}{dz^2} + k^2 n_0^2(z) E(z) = \begin{cases} 
0, & z < z', z > z'', \\
- k^2 [n_0^2(z) - n_h^2(z)] E(z), & z' < z < z''
\end{cases}
\]

with the corresponding homogeneous equation describing wave propagation through the structure without the defect (4.1).

Introducing the notation \( \mu(z) = -k^2 [n_0^2(z) - n_h^2(z)] \), let us change the “inhomogeneous” differential equation (4.12) to the equivalent integral Lippmann-Schwinger equation

\[
E(z) = E_h(z) + \int_{z'}^{z''} G(z, z_1) \mu(z_1) E(z_1) dz_1,
\]

where the exact amount of homogeneous solution \( E_h(z) \) is determined by the outgoing scattered wave boundary conditions. One can see that for the wave \( \exp(ik_0 z) \) incoming from the region \( z < 0 \), \( E_h(z) = E_l(z) \) with \( E_l(z) \) from (4.7). For this wave the solution \( E^l(z) \) of the inhomogeneous equation (4.12) itself has also the form of expression (4.7) but with a different fundamental system of solutions \( E_{L_1}^l(z) \), \( E_{L_2}^l(z) \) inside the structure and with different constants, which we denote by \( R^l \), \( T^l \), \( A^l \), \( B^l \). Our final purpose is to find the constants \( R^l \) and \( T^l \), which are the amplitude reflection and transmission coefficients for the wave \( \exp(ik_0 z) \) falling on the structure with an embedded defect from the region \( z < 0 \). If we take into account the fact that \( \mu \neq 0 \) only somewhere within \( 0 < z < L \), we can rewrite (4.13) for the \( E^l \) as

\[
E^l(z) = E_{h}^l(z) + \int_{z'}^{z''} G(z, z_1) \mu(z_1) \left[ A^l E_{L_1}^l(z_1) + B^l E_{L_2}^l(z_1) \right] dz_1.
\]

This integral equation cannot be used directly to find the \( E^l(z) \) because under the integral we have the unknown function \( E_{L_1}^l(z_1) = A^l E_{L_1}^l(z_1) + B^l E_{L_2}^l(z_1) \), which is the function \( E^l(z) \) itself in the interval \( 0 < z < L \). However, the Lippmann-Schwinger
equation (4.14) can be used to develop an approximation scheme known as the Neumann series. The core idea of the scheme for our problem is to use the homogeneous solution \( E_h^t(z_1) = A_h^t E_{h_1}^L(z_1) + B_h^t E_{h_2}^L(z_1) \) as an initial approximation for the inhomogeneous solution \( E^{t_1}(z_1) \) under the integral, and then iterate to find higher order corrections. As a result, we can express the solution \( E^t(z) \) in terms of the Neumann series

\[
E^t(z) = E_h^t(z) + E^{(1)}(z) + E^{(2)}(z) + \ldots + E^{(j)}(z) + \ldots, \tag{4.15}
\]

where the second term

\[
E^{(1)}(z) = \int_{z_1}^{z''} G(z, z_1) \mu(z_1) \left[ A_h^t E_{h_1}^L(z_1) + B_h^t E_{h_2}^L(z_1) \right] dz_1 \tag{4.16}
\]

is the first-order correction to the solution \( E^t \), and an arbitrary \((j + 1)\) term is given by

\[
E^{(j)}(z) = \int_{z_1}^{z''} \ldots \int_{z_1}^{z''} G(z, z_j) \mu(z_j) G(z_j, z_{j-1}) \mu(z_{j-1}) \ldots G(z_2, z_1) \mu(z_1) \left[ A_h^t E_{h_1}^L(z_1) + B_h^t E_{h_2}^L(z_1) \right] dz_1 dz_2 \ldots dz_j. \tag{4.17}
\]

In order to find the constant \( R^t \) we should put \( z = 0 \) in the formulas (4.15) - (4.17) and take into account the equalities, \( E^t(0) = 1 + R^t \) and, \( E_h^t(0) = 1 + R_h^t \). As a result, the corresponding Neumann series for the reflection coefficient takes the form

\[
R^t = R_h^t + E^{(1)}(0) + E^{(2)}(0) + \ldots + E^{(j)}(0) + \ldots, \tag{4.18}
\]

where the incoming field is assumed to be normalized to the unit amplitude. In order to find the constant \( T^t \) we should put \( z = L \) in the same formulas and take into account the equalities, \( E^t(L) = T^t \) and, \( E_h^t(L) = T_h^t \). As a result, the Neumann series for the transmission coefficient takes the form

\[
T^t = T_h^t + E^{(1)}(L) + E^{(2)}(L) + \ldots + E^{(j)}(L) + \ldots. \tag{4.19}
\]

The development of the Neumann series is certainly not the only available procedure for the solution of the integral Lippmann-Schwinger equation. There are also other methods. The most frequently used one is based on the representation of the
inhomogeneous solutions in both sides of the Lippmann-Schwinger equation in terms of reasonable functions with unknown parameters. After the integration we can find these parameters. Such an approach was described in [76] for simple one-dimensional scattering problems. However, the problem of the function fit can be solved only in a few cases. Therefore, the representation of the solution in terms of the Neumann series is a more general approach as it is applicable to any perturbation.

4.2 Explicit Convergence of Neumann Series for Simple Perturbations

Let us consider the problem of the convergence of the Neumann series. According to the general theory of the integral equations, the sufficient convergence condition for the Neumann series (4.15) can be written as

$$C (z'' - z') < 1, \quad C = \max |\mu(z)G(z, z_1)|_{z' < z''}, z'' < z_1 < z''. \quad (4.20)$$

If $\mu(z)$ is constant we can simplify this expression as

$$c\mu (z'' - z') < 1, \quad c = \max |G(z, z_1)|_{z' < z''}, z'' < z_1 < z''. \quad (4.21)$$

These convergence conditions are obviously valid for the already mentioned Born approximation (see the last section of Chapter 3) when either $\mu (z'' - z') << 1$ (weak potentials) or $k (z'' - z') \gg 1$ (fast particles). Moreover, the solution in the Born approximation is simply $E^b(z) = E^l_h + E^{(1)}(z)$.

It is remarkable that we might obtain the solution for the inhomogeneous equation with the aid of the Neumann series even if the convergence condition fails. This is possible if we can find a way to sum all the infinite Neumann series (4.15). The problem is extremely difficult and obviously can not be solved in general. Even the second Born approximation, $E^{ib}(z) = E^l_h + E^{(1)}(z) + E^{(2)}(z)$, is rarely considered in the literature [77, 78]. However, in what follows we found the solution of the inhomogeneous problem by summing the complete Neumann series for some simple cases. Moreover, as we will see the convergence condition is obviously violated in these cases. Therefore, the following examples are not only illustrations of the application
of the Green’s function method for finding solutions in terms of the Neumann series but also have a methodological interest.

4.2.1 Homogeneous Medium with an Infinite Uniform Defect

Let us consider a homogeneous medium with the refractive index \( n_0 \). We suppose that there is an infinitesimal absorption, i.e. \( n_0 = n'_0 + i0 \). This is a standard assumption which allows us to find “physical” solutions for non-absorptive media. The Green’s function for the homogeneous medium is very well-known and can be written in the whole plane \((z, z_1)\) as

\[
G(z, z_1) = \frac{1}{2ikn_0} \exp(ikn_0|z - z_1|).
\]  

(4.22)

The perturbation potential is, as shown in Figure 4.2,

\[
\mu(z) = \begin{cases} 
0, & z < 0, \\
-k^2 \left[ n_f^2 - n_0^2 \right], & z > 0.
\end{cases}
\]  

(4.23)

![Diagram of homogeneous dielectric medium with an infinite uniform defect](image)

Figure 4.2: Homogeneous dielectric medium \( n_0 \) with an infinite uniform defect \( n_f \).

We can see that the convergence condition (4.21) obviously fails as, \( z'' = +\infty \), and, \( z' = 0 \). However, let us formally calculate the Neumann series (4.16). The homogeneous solution \( E^h_k(z) \) in the whole region \(-\infty < z < +\infty\) has the form
\[ E_{j}^{(l)}(z) = \exp(-ikn_{0}z). \] Using the relation
\[
E^{(j+1)}(z) = q \exp(ikn_{0}z) \int_{0}^{z} \exp(-ikn_{0}z_{1})E^{(j)}(z_{1})dz_{1}
+ q \exp(-ikn_{0}z) \int_{z}^{+\infty} \exp(ikn_{0}z_{1})E^{(j)}(z_{1})dz_{1}
\] (4.24)
with parameter
\[
q = \frac{\mu}{2ikn_{0}} = -\frac{k^{2}(n_{f}^{2} - n_{0}^{2})}{2ikn_{0}}
\] (4.25)
we obtain the Neumann series in the form
\[
E^{(0)}(z) = \exp(ikn_{0}z),
\]
\[
E^{(1)}(z) = q \left( z - \frac{1}{2ikn_{0}} \right) \exp(ikn_{0}z),
\]
\[
E^{(2)}(z) = q^{2} \left( \frac{1}{2}z^{2} - \frac{2}{2ikn_{0}}z + \frac{2}{(2ikn_{0})^{2}} \right) \exp(ikn_{0}z),
\]
\[
E^{(3)}(z) = q^{3} \left( \frac{1}{6}z^{3} - \frac{3}{4ikn_{0}}z^{2} + \frac{5}{(2ikn_{0})^{2}}z - \frac{5}{(2ikn_{0})^{3}} \right) \exp(ikn_{0}z),
\] (4.26)
\[
E^{(j)}(z) = q^{j} \left( A_{j}^{(j)}z^{j} + A_{j-1}^{(j)}z^{j-1} + \ldots + A_{0}^{(j)} \right) \exp(ikn_{0}z),
\]
\[
E^{(j+1)}(z) = q^{j+1} \left( A_{j+1}^{(j+1)}z^{j+1} + A_{j}^{(j+1)}z^{j} + \ldots + A_{0}^{(j+1)} \right) \exp(ikn_{0}z),
\]
where the coefficients \( A_{m}^{(j+1)} \) for \( E^{(j+1)}(z) \) can be easily expressed in terms of \( A_{m}^{(j)} \) for \( E^{(j)}(z) \) for an arbitrary \( j \). Substituting in (4.26) \( z = 0 \), we end up with the Neumann series for the transmission coefficient
\[
T^{l} = 1 - q \frac{1}{2ikn_{0}} + q^{2} \frac{2}{(2ikn_{0})^{2}} - q^{3} \frac{5}{(2ikn_{0})^{3}} + q^{4} \frac{14}{(2ikn_{0})^{4}} - \ldots
= \frac{2}{1 + \sqrt{1 + 4 \frac{q}{2ikn_{0}}}} \equiv \frac{2n_{0}}{n_{0} + n_{f}} \equiv t_{0f}.
\] (4.27)
This is the Fresnel transmission coefficient, i.e. we obtain the result that was expected from the beginning.
4.2.2 Step Profile with an Infinite Uniform Defect

The step profile of the refractive index is defined by

\[ n(z) = \begin{cases} 
  n_0, & z < 0, \\
  n_1, & z > 0.
\end{cases} \quad (4.28) \]

In accordance with (4.10) and (4.11) we obtain

\[ G(z, z_1) = \begin{cases} 
  G_1(z, z_1) & \text{if } z_1 > 0, \quad z < 0, \\
  G_3(z, z_1) & \text{if } z < z_1 < 0, \quad z < 0, \\
  G_4(z, z_1) & \text{if } z_1 < z, \quad z < 0, \\
  G_9(z, z_1) & \text{if } z_1 > z, \quad z > 0, \\
  G_{10}(z, z_1) & \text{if } 0 < z_1 < z, \quad z > 0, \\
  G_{12}(z, z_1) & \text{if } z_1 > 0, \quad z < 0.
\end{cases} \quad (4.29) \]

where, taking into account that for the step profile \( T_h^l = t_{01} \exp(ikn_1 L), \ T_h^r = t_{10} \exp(ikn_1 L), \ R_h^l = r_{01}, \) and \( R_h^r = r_{10} \exp(2ikn_1 L), \)

\[ G_1(z, z_1) = \frac{t_{01}}{2ikn_0} \exp(ikn_1 z_1) \exp(-ikn_0 z), \]

\[ G_3(z, z_1) = \frac{1}{2ikn_0} [\exp(ikn_0 z_1) + r_{01} \exp(-ikn_0 z_1)] \exp(-ikn_0 z), \]

\[ G_4(z, z_1) = \frac{1}{2ikn_0} [\exp(ikn_0 z_1) + r_{01} \exp(-ikn_0 z_1)] \exp(-ikn_0 z), \]

\[ G_9(z, z_1) = \frac{1}{2ikn_1} \exp(ikn_1 z_1) [\exp(-ikn_1 z) + r_{10} \exp(ikn_1 z)], \]

\[ G_{10}(z, z_1) = \frac{1}{2ikn_1} [\exp(-ikn_1 z_1) + r_{10} \exp(ikn_1 z_1)] \exp(ikn_1 z), \]

\[ G_{12}(z, z_1) = \frac{t_{10}}{2ikn_1} \exp(-ikn_0 z_1) \exp(ikn_1 z). \]

The exact Green’s function for the step profile can be also obtained with the aid of the spectral decomposition as was done in [73] by Aguiar in the case of quantum potentials. The expressions (4.30) are identical to Aguiar’s results but the steps involved in the calculation are much simpler. It illustrates the advantage of the present approach over the method which is based on the spectral decomposition of the Green’s function.
Let us consider the application of the obtained exact Green's function. Suppose the refractive index has an infinite uniform perturbation, as shown in Figure 4.3.

\[ \mu(z) = \begin{cases} 
0, & z < L, \\
-k^2 \left[ n_f^2 - n_1^2 \right], & z > L > 0.
\end{cases} \quad (4.31) \]

![Diagram showing a step dielectric medium with a defect](image)

Figure 4.3: Step dielectric medium \( n_0 / n_1 \) with an infinite uniform defect \( n_f \).

The convergence condition (4.21) fails again as \( z'' = +\infty \) and \( z' = L \). However, starting with the homogeneous solution \( \phi_h(z) = t_{01} \exp(ikn_1 z) \) \( z > L \) and repeating calculations in the same manner as for the previous example, we obtain the Neumann series for the reflection coefficient in the form

\[ T^l = t_{01} \exp(ikn_1 L) \left( 1 - q \frac{a}{2ikn_1} + q^2 \frac{1}{(2ikn_1)^2} (a^2 + a) - q^3 \frac{1}{(2ikn_1)^3} (a^3 + 2a^2 - a) + \ldots \right), \]

where

\[ q = \frac{\mu}{2ikn_1} \quad a = 1 + r_{10} \exp(2ikn_1 L). \quad (4.33) \]

The series (4.32) is the Taylor expansion respect to \( q \) of the function

\[ T^l = t_{01} \exp(ikn_1 L) \frac{1}{1 - 1/2q(1 + \sqrt{1 + 4q})a} = \frac{t_{01}t_{1f} \exp(ikn_1 L)}{1 + r_{01}r_{1f} \exp(2ikn_1 L)}, \quad (4.34) \]

which is also expected result, see (2.54).
4.3 Application to Two-Layered Periodic Structure with Fluctuations in Layer Thicknesses

In this section we apply the general theory developed in the two previous sections to a finite two-layered periodic dielectric structure with random fluctuations in the width of the layers due to, for example, inhomogeneous growing conditions. The ideal structure is shown in Figure 3.3. In this section we again suppose that \( n_0 = n_f \), and there is no absorption at all, i.e. the refractive index is real in all regions \(-\infty < z < +\infty\). Then

\[
n_L(z) = \begin{cases} 
n_1, & (m-1)d < z < md_1, \\
n_2, & md_1 < z < md,
\end{cases}
\]  

(4.35)

where \( m = 1, 2 \ldots N \) is the number of the current period.

In order to describe random fluctuations, we will use a normal distribution model with mean 0 and standard deviation \( \delta r_1 \) for the fluctuations in the width of the layers with \( n(z) = n_1 \), and standard deviation \( \delta r_2 \) for the fluctuations in the width of the layers with \( n(z) = n_2 \). As a result, the actual widths of the layers with the refractive indexes \( n_1 \) and \( n_2 \) of an arbitrary period \( m \) are \( d_1 + \delta d_{1m} \) and \( d_2 + \delta d_{2m} \) where \( \delta d_{1m} \) and \( \delta d_{2m} \) are either positive or negative random numbers from the above normal distributions, as shown in Figure 4.4.

![Diagram of two-layered periodic dielectric structure with fluctuations in layer thicknesses](image)

Figure 4.4: Two-layered periodic dielectric structure with fluctuations in layer thicknesses: dashed line, ideal structure (no fluctuations); solid line, real structure (fluctuations in layer thicknesses with Gaussian distribution).

It is convenient to introduce two groups of embedded defects inside the struc-
ture. The first group consists of $N$ defects with thickness, boundary points, and the refractive index of the defect in the $m^{\text{th}}$ period determined by the formulas

$$
\sigma_m^{(1)} = \sum_{j=1}^{m} \delta d_{1j} + \sum_{j=1}^{m-1} \delta d_{2j},
$$

$$
z' = z_m^{(1)} \equiv (m-1)d + d_1,
$$

$$
z'' = z_m^{(1)} + \sigma_m^{(1)},
$$

$$
n_\sigma(z) = \begin{cases} n_1, & \text{if } \sigma_m^{(1)} > 0, \\ n_2, & \text{if } \sigma_m^{(1)} < 0. \end{cases}
$$

(4.36)

As a result, the perturbation potential $\mu(z)$ of the defect of this group in the $m^{\text{th}}$ period takes the form

$$
\mu_m^{(1)} = \begin{cases} -k^2 (n_2^2 - n_1^2), & \text{if } \sigma_m^{(1)} > 0, \\ -k^2 (n_1^2 - n_2^2), & \text{if } \sigma_m^{(1)} < 0. \end{cases}
$$

(4.37)

The second group consists of $N-1$ defects with thickness, boundary points, and the refractive index of the defect on the boundary between the $m^{\text{th}}$ and $m+1^{\text{st}}$ periods determined by the formulas

$$
\sigma_m^{(2)} = \sum_{j=1}^{m} \delta d_{1j} + \sum_{j=1}^{m} \delta d_{2j},
$$

$$
z' = z_m^{(2)} \equiv md,
$$

$$
z'' = z_m^{(2)} + \sigma_m^{(2)},
$$

$$
n_\sigma(z) = \begin{cases} n_2, & \text{if } \sigma_m^{(2)} > 0, \\ n_1, & \text{if } \sigma_m^{(2)} < 0. \end{cases}
$$

(4.38)

As a result, the perturbation potential $\mu(z)$ for each defect of this group takes the form

$$
\mu_m^{(2)} = \begin{cases} -k^2 (n_2^2 - n_1^2), & \text{if } \sigma_m^{(2)} > 0, \\ -k^2 (n_1^2 - n_2^2), & \text{if } \sigma_m^{(2)} < 0. \end{cases}
$$

(4.39)

Now let us find the amplitude reflection coefficient $R_l$ of this structure for the case of a plane wave $\exp(ikn_0z)$ incoming from the region $z < 0$. We restrict ourselves to the lowest order Born approximation, i.e. we suppose that

$$
R_l \approx R_{h} + E^{(1)}(0),
$$

(4.40)
where, as it was already mentioned, the incoming field is assumed to be normalized to the unit amplitude. The amplitude reflection coefficient \( R_k \) for the ideal two-layered periodic structure is determined by the formula (3.30). In accordance with (4.16) in order to find \( R^{(1)} \) we should know the scattering solutions (4.7) and (4.8) for the ideal structure in the explicit form. There are two approaches to find these solutions.

First, we can use the transfer matrix method, in particular the Abeles modification, taking into account the property (3.14) for the unimodular matrix \([I_2 I_1]\) with the interface matrices \([I_2] \) and \([I_1]\) from (2.52).

Second, according to Chapter 3, the field inside the ideal structure is a superposition of two Floquet-Bloch waves (3.6). Therefore, we can consider \( E_{h_1}^L(z) \) as the first Bloch wave and \( E_{h_2}^L(z) \) as the second one. It was already mentioned in Chapter 3 that in [62, 63] we found the exact analytical expressions for the Bloch waves in the case of the two-layered periodicity. They can be written in the layers with \( n(z) = n_1 \) as

\[
E_{h_1}^{n_1}(z) = \sin \left( k n_1 (z - (m - 1) d) - \frac{1}{2} k n_1 d_1 + \varphi \right) e^{ik(z-m-1)},
\]

\[
E_{h_2}^{n_1}(z) = \sin \left( k n_1 (z - (m - 1) d) - \frac{1}{2} k n_1 d_1 - \varphi \right) e^{-ik(z-m-1)},
\]

and in the layers with \( n(z) = n_2 \) as

\[
E_{h_1}^{n_2}(z) = \frac{1}{2} \left( 1 + \frac{n_1}{n_2} \right) \sin \left( k n_2 (z - m d) - \frac{1}{2} k n_1 d_1 + \varphi \right) e^{ikz}
- \frac{1}{2} \left( 1 - \frac{n_1}{n_2} \right) \sin \left( k n_2 (z - m d) + \frac{1}{2} k n_1 d_1 - \varphi \right) e^{ikz},
\]

\[
E_{h_2}^{n_2}(z) = \frac{1}{2} \left( 1 + \frac{n_1}{n_2} \right) \sin \left( k n_2 (z - m d) - \frac{1}{2} k n_1 d_1 - \varphi \right) e^{-ikz}
- \frac{1}{2} \left( 1 - \frac{n_1}{n_2} \right) \sin \left( k n_2 (z - m d) + \frac{1}{2} k n_1 d_1 + \varphi \right) e^{-ikz}.
\]

The complex phase \( \varphi \) can be expressed as

\[
\varphi = \frac{1}{2} \arccos \left( \frac{r_{21}^{-1} \sin \Omega + r_{21} \sin \Delta}{2 \sin(kn_2d_2)} \right),
\]

where according to (3.4)

\[
r_{21} = \frac{n_2 - n_1}{n_2 + n_1}.
\]
Using the boundary conditions at the points \( z = 0 \) and \( z = Nd \), we can define analytically all constants \( A^l_h, B^l_h, A^r_h, \) and \( B^r_h \) of the scattering solutions (4.7) and (4.8) as the constants \( R^l_h, R^r_h, T^l_h, \) and \( T^r_h \) are already known, see (3.30) and (3.31).

A knowledge of the scattering solutions (4.7) and (4.8) allows us to find the Green’s function \( G(z, z_1) \) for the finite ideal two-layered periodic structure in all twelve parts (4.10) of the plane \( (z, z_1) \).

Finally, the net contribution of the fluctuations to the first-order correction term (4.37) for the reflection coefficient can be written as

\[
E^{(1)}(0) = \sum_{m=1}^{N} \frac{\mu_m^{(1)}}{2i\kappa n_0} \int_{z_m^{(1)}}^{z_m^{(1)}+\sigma_m^{(1)}} dz_1 \left\{ \begin{array}{ll}
C_6^{n_2^2}(0, z_1) E_h^{n_2^2}(z_1), & \text{if } \sigma_m^{(1)} > 0 \\
C_6^{n_1^2}(0, z_1) E_h^{n_1^2}(z_1), & \text{if } \sigma_m^{(1)} < 0
\end{array} \right.
\]

\[
+ \sum_{m=1}^{N-1} \frac{\mu_m^{(2)}}{2i\kappa n_0} \int_{z_m^{(2)}}^{z_m^{(2)}+\sigma_m^{(2)}} dz_1 \left\{ \begin{array}{ll}
C_6^{n_1^1}(0, z_1) E_h^{n_1^1}(z_1), & \text{if } \sigma_m^{(2)} > 0 \\
C_6^{n_2^2}(0, z_1) E_h^{n_2^2}(z_1), & \text{if } \sigma_m^{(2)} < 0
\end{array} \right.
\]

where

\[
C_6^{n_1^1,2}(0, z_1) = \frac{1}{2i\kappa n_0} \left[ A^l_h E_{h_1}^{n_1^1,2}(z_1) + B^l_h E_{h_2}^{n_1^1,2}(z_1) \right],
\]

and

\[
E_h^{n_1^1,2}(z_1) = A^l_h E_{h_1}^{n_1^1,2}(z_1) + B^l_h E_{h_2}^{n_1^1,2}(z_1).
\]

Using Eqs. (4.46) and (4.47), we can express \( E^{(1)}(0) \) in the more explicit form

\[
E^{(1)}(0) = \sum_{m=1}^{N} \frac{\mu_m^{(1)}}{2i\kappa n_0} \int_{z_m^{(1)}}^{z_m^{(1)}+\sigma_m^{(1)}} dz_1 \left\{ \begin{array}{ll}
\left[ A^l_h E_{h_1}^{n_1^2}(z_1) + B^l_h E_{h_2}^{n_2^2}(z_1) \right]^2, & \text{if } \sigma_m^{(1)} > 0 \\
\left[ A^l_h E_{h_1}^{n_1^1}(z_1) + B^l_h E_{h_2}^{n_1^2}(z_1) \right]^2, & \text{if } \sigma_m^{(1)} < 0
\end{array} \right.
\]

\[
+ \sum_{m=1}^{N-1} \frac{\mu_m^{(2)}}{2i\kappa n_0} \int_{z_m^{(2)}}^{z_m^{(2)}+\sigma_m^{(2)}} dz_1 \left\{ \begin{array}{ll}
\left[ A^l_h E_{h_1}^{n_1^1}(z_1) + B^l_h E_{h_2}^{n_1^1}(z_1) \right]^2, & \text{if } \sigma_m^{(2)} > 0 \\
\left[ A^l_h E_{h_1}^{n_2^1}(z_1) + B^l_h E_{h_2}^{n_2^1}(z_1) \right]^2, & \text{if } \sigma_m^{(2)} < 0
\end{array} \right.
\]

All integrals in the above expression can be evaluated analytically without any difficulties, since the functions under the integral are simply superpositions of the sinusoidal functions (4.41) and (4.42). However, the final answer is very cumbersome and we do not present it here in detail. Instead, in the next section, we consider the numerical application of (4.48) to two-layered periodic dielectric structures with some specific parameters suitable for the construction of optical switching systems. Note that we can obtain any other correction term \( E^{(j)}(0) \) by applying formula (4.17).
4.4 Optical Switching Systems Based on Two-Layered Periodic Structure

As mentioned in the Introduction, layered periodic dielectric structures have a wide variety of applications in optics and acousto-optics where they have been extensively used for more than 30 years. However, some new possibilities for their practical use are still available. In recent papers [22, 79] it was suggested that two-layered periodic dielectric structures could be used as a dielectric omnidirectional reflector. Such an application does indeed follow directly from a thorough analysis of forbidden and allowed bands of optical frequencies in the case of oblique incidence. Note that in Chapter 3 we obtained exact analytical expressions for such bands in the case of normal incidence, see formulas (3.27) and (3.28).

In this section we demonstrate another new possible application of two-layered periodic dielectric structures. In particular, our suggestion is to use such structures for the creation of a new optical switching system. Different types of switching systems are widely used in various optical devices to control and govern laser radiation, as recently reviewed in Refs. [80, 81].

The main idea for a proposed optical switch is to vary the material parameters of the two-layered periodic dielectric medium so as to change significantly (up to 80%) the reflection coefficient for incident electromagnetic waves with a specific wavelength \( \lambda \) by the application of an elastic stress of reasonable size. The proposal differs from existing acousto-optic filters and switches in that it is the thicknesses of the basic layers \( d_1 \) and \( d_2 \) that are changed, rather than their indices of refraction \( n_1 \) and \( n_2 \).

Several requirements must be met in the choice of materials for the practical realization of such a switch. First, the reflection coefficient (as a function of wavelength) must have a well defined structure of forbidden regions, where the reflection coefficient reaches almost unity, and allowed regions, where the reflection coefficient drops to nearly zero. Second, the medium should be constructed from alternating layers of materials with low Young's modulus in one layer and high compressive yield strength (low limit of plastic deformations) in both in order to produce a perceptible
change in the thicknesses $d_1$ and $d_2$ of the basic layers, while avoiding the appearance of irreversible deformation. Third, the parameters should be chosen to minimize the influence of unavoidable random fluctuations $\delta d_{1m}$ and $\delta d_{2m}$ on the performance characteristics of the switch due to imperfect growing conditions. The Green’s function obtained in the previous section is particularly useful in studying the influence of random fluctuations in order to satisfy this last requirement.

There are two ways to satisfy the first requirement. A well defined structure of allowed and forbidden regions can be obtained either by using a rather large number $N$ of layers consisting of alternating materials with close indices of refraction [62, 63], or by using a small number of layers, but with materials having large differences in refractive index. The second option is preferable because it simultaneously helps to satisfy the third requirement—the smaller the number of layers, the smaller the number of terms in the sum (4.48) over random fluctuations in the calculation of corrections to the reflection coefficient. As for the second requirement, the most suitable material is a range of polymers with high optical transparency. Note that almost all optically transparent polymers have refractive indices in the range 1.33 to 1.50. Consequently, we cannot construct the operating medium from alternating layers of two polymers if we want to keep the number of layers small and still have a well defined structure of allowed and forbidden regions for the reflection coefficient. We are therefore forced to choose the second material to be a glass with high refractive index $n > 2$. Such two-layered periodic structures consisting of alternating polymer/glass layers with a large difference in refractive indices have been also studied in the already mentioned Refs. [22, 79].

Let us consider a concrete example of such a structure. Consider a two-layered periodic structure consisting of $N = 5$ periods of fluorinated ethylene propylene polymer (FEP) with $n_1 = 1.344$, $d_1 = 3.88 \mu m$, Young’s modulus $E_1 = 380$ N/mm$^2$, Poisson’s ratio $\sigma_1 = 0.48$ alternating with chalcogenide glass based on GaS$_3$–La$_2$S$_3$ with $n_2 = 2.4$, $d_2 = 2.17 \mu m$, and $E_2 = 78.4 \times 10^3$ N/mm$^2$ [82]. The structure is assumed to be surrounded by the same chalcogenide glass.
The two curves in Figure 4.5 represent the dependence of the reflection coefficient on the wavelength \( \lambda \) of the incident radiation. The dashed curve characterizes reflection from an ideal structure corresponding to the formula \( R^i = R^i_h \) with \( R^i \) from (3.29). The solid curve characterizes the reflection from a realistic structure with random fluctuations in layer thickness, calculated at the Born approximation, using the formula (4.40) with \( E^{(1)}(0) \) obtained from a numerical evaluation of formula (4.48). For the standard deviations \( \delta r_{1,2} \) of the normal distribution we use 0.25% of the corresponding width of the ideal layers. This figure is in accordance with the actual precision of the procedure used for the preparation of thin films of polymers [84]. We can see that for \( \lambda = 0.633 \, \mu m \) (He-Ne laser) the reflection coefficient approximately equals 100% (no transmission) for both ideal and real structures. Note that, according to the perturbation theory, we can use formula (4.40) only if the numerical value of \( E^{(1)}(0) \) obtained from (4.48) is less than the numerical value of \( R^i_h \) obtained from (3.29), which is true for the two-layered periodic structure above.

Figure 4.5: Dependence of \( R \) on \( \lambda \) for \( N = 5, n_0 = 2.4, n_1 = 1.344, n_2 = 2.4, d_1 = 3.88 \, \mu m, d_2 = 2.17 \, \mu m \); dashed line - ideal structure (no fluctuations in layer thicknesses), solid line - realistic structure (dispersion of fluctuations in layer thicknesses is 0.25%).
If we apply an external compressive stress \( p \) to the boundaries of the structure only in the direction of the periodicity \( z \) (such a deformation is called a simple compression), the homogeneous decrease in the thickness of each basic layer is determined by [84]

\[
\delta d_{1,2} = \frac{p}{E_{1,2}} d_{1,2},
\]

(4.49)

where \( p \) is the applied stress in the \( z \)-direction and \( E_{1,2} \) are Young’s module of the layers. In order to evaluate the variations in the refractive indices \( \delta n_{1,2} \) under the applied stress, we use as a starting point the Lorentz-Lorenz equation, which relates the index of refraction \( n \) to the molecular polarizability \( \alpha \) for isotropic and cubic materials:

\[
\frac{n^2 - 1}{n^2 + 2} = \frac{4\pi}{3} \frac{N_A \rho}{M} \alpha,
\]

(4.50)

where \( N_A \) is Avogadro’s number, \( M \) is the molecular weight, and \( N_A \rho / M \) is the number of molecules per unit volume. If we assume that the polarizability \( \alpha \) changes with the changes in the density of material \( \rho \), as \( \delta \alpha / \alpha = \Lambda_0 \delta \rho / \rho \) [83], by straightforward differentiation of the expression (4.47) we obtain the relation

\[
\frac{\delta n}{\delta \rho} = \frac{(n^2 - 1)(n^2 + 2)}{6n\rho} (1 - \Lambda_0),
\]

(4.51)

where \( \Lambda_0 \) is the phenomenological strain polarizability constant. If, for example, \( \Lambda_0 \) is equal to zero, the changes in the refraction index \( n \) are produced only by changes in the density of material \( \rho \). In most cases, however, \( \Lambda_0 \) is not zero. For the typical polymer, such as polystyrene \( \Lambda_0 = 0.4 \pm 0.1 \) [83].

For our case of a simple compression, we can express the relative changes in densities of our materials, as \( \delta \rho_{1,2} / \rho_{1,2} = p (1 - 2\sigma_{1,2}) / E_{1,2} \). As a result, the variations in the values of the refractive indexes can be written in the form of

\[
\delta n_{1,2} = \frac{(n_{1,2}^2 - 1)(n_{1,2}^2 + 2)}{6n_{1,2}} \frac{p (1 - 2\sigma_{1,2})}{E_{1,2}} (1 - \Lambda_0).
\]

(4.52)

For the relatively small stress \( p = 9 \) N/mm\(^2\), which is far enough from the compressive yield point (lower limit of plastic deformation) of FEP (15 N/mm\(^2\)), we can obviously neglect the variations in parameters of the glass \( n_2 \) and \( d_2 \) because of its
high value of Young modulus $E_2$ in comparison with Young modulus $E_1$ for the FEP. As for variations in the parameters of the FEP layers, using (4.52) and (4.49), we obtain $\delta n_1 = 0.0002$, $\delta d_1 = -0.09\mu m$. The results are summarized in Figure 4.6.

![Figure 4.6: Dependence of $R$ on $\lambda$ for $N = 5$, $n_0 = 2.4$, $n_1 = 1.3442$, $n_2 = 2.4$, $d_1 = 3.79\mu m$, $d_2 = 2.17\mu m$ (the previous structure under applied stress $p = 9$ N/mm$^2$); dashed line - ideal structure (no fluctuations in layer thicknesses), solid line - realistic structure (dispersion of fluctuations in layer thicknesses is 0.25%).](image)

The dashed line again represents the reflection coefficient dependence for an ideal compressed structure, i.e. for the structure with refractive indexes of the basic layers $n_1 + \delta n_1$, $n_2$ and the widths of the basic layers $d_1 + \delta d_1$, $d_2$. The solid line represents the first-order approximation to the reflection coefficient dependence for the realistic compressed structure, i.e. for the structure with the width of the basic layers $d_1 + \delta d_1 + \delta d_{1m}$, $d_2 + \delta d_{2m}$. We can see now that for $\lambda = 0.633\mu m$ $R$ for both structures is less than 10%, i.e. we have almost full transmission.

Figure 4.7 illustrates the general dependence of the reflection coefficient for $\lambda = 0.633\mu m$ on the applied stress for the ideal structure (no fluctuations). The curve to the right side from a stress of 15 N/mm$^2$ onwards has only a representative meaning
as irreversible plastic deformations occur in this region.

![Graph showing the dependence of reflection on wavelength for different materials.]

Figure 4.7: Dependence of $R$ on $p$ for $\lambda = 0.633$ $N = 5$, $n_0 = 2.4$, $n_1 = 1.344$, $n_2 = 2.4$, $d_1 = 3.88 \mu m$, $d_2 = 2.17 \mu m$.

As a second example, we consider a two-layered periodic dielectric structure consisting of $N = 15$ periods of polystyrene ($n_1 = 1.58$, $E_1 = 2900 \text{ N/mm}^2$, $\sigma_1 = 0.33$) with $d_1 = 7.55 \mu m$ and chlorotellurite glass ($n_2 = 2.00$, $E_2 = 42500 \text{ N/mm}^2$ [85]) with $d_2 = 5.9 \mu m$. Outside the structure there is also polystyrene. In Figure 4.8, as in the previous example, the dashed curve characterizes reflection from an ideal structure and the solid curve characterizes the reflection from a realistic structure with random fluctuations in layer thickness having the standard deviations $\delta r_{1,2} = 0.1\%$ of the corresponding width of the ideal layers. The increase in the number of layers forced us to toughen the requirements on the size of unavoidable fluctuations from 0.25% to 0.1%.
Figure 4.8: Dependence of $R$ on $\lambda$ for $N = 15$, $n_0 = 1.58$, $n_1 = 1.58$, $n_2 = 2.00$, $d_1 = 7.55 \mu m$, $d_2 = 5.9 \mu m$; dashed line - ideal structure (no fluctuations in layer thicknesses), solid line - realistic structure (dispersion of fluctuations in layer thicknesses is 0.1%).

For the stress $p = 15$ N/mm$^2$, which is very far from the compressive limit of the plastic deformation of polystyrene (90 N/mm$^2$), we neglect the variations in parameters of the glass $n_2$ and $d_2$ because of its high value of Young modulus $E_2$ in comparison with Young modulus $E_1$ for the polystyrene. As for variations in the parameters of the polystyrene layers we obtain $\Delta n_1 = 0.001$, $\Delta d_1 = -0.04 \mu m$. The results are summarized in Figure 4.9.
Figure 4.9: Dependence of $R$ on $\lambda$ for $N = 15$, $n_0 = 1.58$, $n_1 = 1.581$, $n_2 = 2.00$, $d_1 = 7.51 \mu m$, $d_2 = 5.9 \mu m$ (the previous structure under applied stress $p = 15$ N/mm$^2$); dashed line - ideal structure (no fluctuations in layer thicknesses), solid line - realistic structure (dispersion of fluctuations in layer thicknesses is 0.1%).

We can see again that for $\lambda = 0.633 \mu m$ the reflection coefficient almost equals 100% (no transmission) for both ideal and realistic structures before stress applied and is decreased almost by a factor of 10 under stress $p = 15$ N/mm$^2$. Figure 4.10 illustrates the general dependence of the reflection coefficient for $\lambda = 0.633 \mu m$ on the applied stress for an ideal structure (no fluctuations). We can see that using relatively small stresses, $0 < p < 30$ N/mm$^2$ we can change the reflection and transmission from 0 to almost 100%.
Figure 4.10: Dependence of $R$ on $p$ for $\lambda = 0.633$, $N = 15$, $n_0 = 1.58$, $n_1 = 1.58$, $n_2 = 2.00$, $d_1 = 7.55 \mu m$, $d_2 = 5.9 \mu m$.

These two examples demonstrate the possibility of using a realistic (with fluctuations in layer thicknesses) two-layered periodic dielectric structure as the operating medium for optical switching devices based on shifting an electromagnetic wave from a forbidden to an allowed band with the aid of compressive elastic stress.
Chapter 5

Conclusions and Future Work

In the preceding chapters we have considered in detail the theory of electromagnetic wave propagation through layered dielectric materials. The major part of the second chapter constitutes a review of the many mathematical methods currently in use for description and prediction of wave propagation in layered materials. The multiple reflection approach is then applied to develop a new approximate analytic method for the characterization of the reflection and transmission properties of layered structures. What remains to be done here is to find analytic criteria under which it is reasonable to use only two first approximations of the method.

In the third chapter we considered a special case of layered structures - layered periodic structures. The relation between the classical Floquet theory of wave propagation in periodic structures and the transfer matrix method for layered media was shown. Then, the main features of electromagnetic fields in layered periodic dielectrics were investigated using a two-layered periodic structure as a basic model. In particular, for this model we have found the width of forbidden and allowed bands in an exact analytical form and investigated in detail the behavior of the reflection coefficient in these bands. What remains to be done here is to find the width of forbidden and allowed bands in an exact analytical form for the case of multi-layered periodic structures (when each period contains several rather than two sub-layers). In order to do this we should first obtain the dispersion equation for the Bloch phase (see expression (3.27), as an example of this equation for the case of two sub-layers). This was very recently done in [86] for the case of normal propagation of acoustic waves.
In the fourth chapter we developed a new general method for the construction of the Green's function for 1-D problems. Then we have shown how to apply this function for the calculation of reflection properties in terms of the Neumann series. Finally, the developed theory was applied to the problem of the propagation of electromagnetic waves through a two-layered periodic structure with random Gaussian fluctuations in the thicknesses of the basic layers. In particular, we found the first-order correction to the reflection coefficient of an ideal structure due to fluctuations in layer thicknesses. As a result, we predicted a new possible application of two-layered periodic structures. In particular, an optical switching technique based on the effect of the compression of a two-layered periodic structure with a relatively small number of periods but with high optical modulation, i.e., consisting of materials having a large difference in refractive indexes, is feasible despite the influence of unavoidable fluctuations in layer thicknesses.

There is a wide range of problems that can be solved once the Green's function for the basic structure is known in analytical form. As it was mentioned, they include acousto-optical coupling, nonlinear phenomena etc. I plan to continue the work, currently under development, on the method of constructing of the Green's function. In particular, it is reasonable to further separate the region $0 < z < L$, $0 < z_1 < L$ of the plane $(z, z_1)$ into quadratic parts, provided we know fundamental system of solutions in these parts. Then, we can use the power of the transfer matrix method in this region to write the detailed representation of the Green's function inside. A somewhat similar approach for the construction of the exact form of the Green's function in the case of the quantum 1-D problems was recently published in [87]. The other possible direction of further research is the calculation of the total Born (Neumann) series in more complicated cases than were presented in this work. In order to do this I plan to find the correspondence between the Neumann and the multiple reflection series. It would then be possible to join the two techniques for the calculation of the reflection and transmission properties of waves of different nature, including the acoustic and electron waves, in 1-D structures.
Bibliography


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