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Riabouchinsky flows in magnetohydrodynamics.

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RIABOUCHINSKY FLOWS IN
MAGNETOHYDRODYNAMICS

by

Fotini Labropulu

A Thesis

submitted to the Faculty of Graduate Studies
through the Department of Mathematics and Statistics
in Partial Fulfillment of the Requirements for the
Degree of Master of Science at
the University of Windsor

Windsor, Ontario

1987
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Approved by:

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Dr. P.N. Kaloni

Dr. A. Watson
Life has in store for us many pleasant moments. The completion of this Thesis is one such moment.

In this special moment of joy, two very very special people come to my mind. I would like to express my heartfelt gratitude to my beloved mother Alexandra and my grandfather Nicholas, for their help, support, encouragement and most importantly, their love and belief in me. I respectfully dedicate this thesis to them.

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ABSTRACT

This Thesis is devoted to the mathematical study of steady magnetohydrodynamic flows. Steady plane flows of ordinary viscous as well as second-grade, incompressible and electrically conducting fluids are studied under the assumption that the magnetic field and the velocity field are transverse, aligned or constantly-inclined to each other throughout the flow region.

By transverse flow, we mean that the magnetic field is normal to the velocity field. By aligned flow, we mean that the velocity field and the magnetic field are everywhere parallel to each other in the flow region. In constantly-inclined flow the angle between the velocity vector and the magnetic vector is constant everywhere in the flow.

In this work, the flows mentioned above are investigated under the assumption that the streamfunction, defined by the continuity or conservation of mass equation, is linear with respect to x or y. Flow problems involving such a streamfunction are called Riabouchinsky type problems.
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1.1. Historical Sketch

1.1.1. Introduction

Magneto-hydro-dynamics (MHD) is the study of the macroscopic interactions of electrically-conducting liquids and gases in the presence of a magnetic field. In such a problem, the magnetic field influences the fluid motion and the fluid motion changes in turn the magnetic field. As would be expected, both the equations of fluid mechanics and of electro-magnetism feature in the description of magneto-hydro-dynamic flow: each impressing something of its own distinctive features on the subject.

When a conducting fluid moves through a magnetic field, an electric field and consequently a current may be induced and, in turn, the current interacts with the magnetic field to produce a body force on the fluid.

Such interactions occur both in nature and in new man-made devices. MHD flow occurs in the sun, the earth's interior, the ionosphere, the stars and their atmospheres, to mention a few. In the laboratory many new devices have been made which utilize the MHD interactions directly, such as propulsion
units and power generators. Further, at the engineering level, experiments have been made for electric power generation by passing an ionized gas between the poles of a strong electromagnet so that an electric current would be generated at right angles to the magnetic field and to the direction of flow of the plasma, the current being collected by two spaced electrodes at right angles to the direction of the current flow.

From the historical point of view, the first investigations in this field are due to Faraday in 1836. Faraday attempted to detect the electric field induced by the motion of the conducting water in the Thames River in the presence of the earth's magnetic field. His experiments failed because of the low electrical conductivity of the fluid. A second important stage in the development of this science is represented by the experiments of Hartman and Lazarus in 1937 who pointed out the magnetic field influence on the motion of a fluid. In this work, mercury flowing through a tube in the presence of a strong magnetic field was used as conducting fluid. Although the experimental results confirmed the theoretical studies, no practical reason for continuing these investigations existed at that time.

The first genuine magneto-hydrodynamic problems have been posed by astrophysics. So in 1940, the Swedish scientist H. Alfven proved that new waves, unknown to both fluid mechanics and electromagnetism can be propagated through a conducting fluid in a magnetic field.

But the first important impetus in the development of this
science was the discovery of the plasma and its production on the laboratory scale. A plasma is electrically conducting and has the property of maintaining its overall charge neutrality. Plasma is consequently a compressible, continuous (or discrete) electrically conducting medium. It is obvious that the study of the motion of such a medium represents the object of magneto-hydro-dynamics.

Of more recent origin is the interest in magneto-hydro-dynamics spurred by problems encountered in aeronautics and astronautics. At the flight velocities at which future aircraft and space vehicles are expected to operate, the energy levels involved are such as to cause substantial ionization in the atmospheric environment of the vehicles and/or to require propulsion systems capable of producing particle streams with directed energies much higher than those possible with present-day chemical fuels.
1.1.2. Literature Review.

The mathematical study of magneto-hydro-dynamics is concerned primarily with the partial differential equations which arise from the well known physical conservation laws. Most of the research to date has been restricted to MHD with the additional assumption of infinite electrical conductivity. S. Lunquist(1952) investigated unsteady flows with electrical conductivity. Resler and McCune(1959) worked in linearized MHD and Chandrasekhar(1961) worked in related stability problems.

Because this subject is quite complex and because this is a relatively new branch of dynamics, much of the research has consisted in isolating special flows which are accessible via the existing methods of Fluid Dynamics. H. Grad(1960) established the reducibility for a number of MHD problems to fluid dynamics problems. Aligned flows, Transverse flows, Orthogonal flows, Constantly-Inclined flows and Variably-Inclined flows are examples of such special flows.

In Aligned flows, the magnetic field vector and the velocity field vector are everywhere parallel to one another. Steady aligned plane flows were one of the first flows studied. Chandrasekhar(1959) investigated the stability of an aligned flow problem for the case of inviscid incompressible fluids. Vinokur(1961) obtained a kinematic formulation for three dimensional aligned flows of ideal gases. P. Smith (1963) generalized some of the results of steady rotational flows of
ideal gases to aligned flows. Chandna and Nath (1972) developed a substitution principle, for fluids having an arbitrary equation of state, that corresponds to Prim's (1952) substitution principle, for classical gas flows. Kingston and Power (1968) employed complex variable techniques to study compressible aligned flows. Chandna, Murgai and Srihar (1985) applied hodograph and Charpit's method to investigate the geometries and the solutions when the velocity magnitude is constant on each individual streamline.

Flows are said to be orthogonal if the velocity and the magnetic field vectors are everywhere orthogonal to each other. Many researchers, so far, have studied orthogonal flows of inviscid fluids with infinite electrical conductivity. Ladikov (1962) obtained two Bernoulli equations for such flows. Power and Walker (1965), and Power and Talbot (1969), studied plane compressible orthogonal flows by reducing the problem to that of rotational gas dynamic flows. Kingston and Talbot (1969) completely classified the possible flow configurations for the viscous incompressible fluid. Nath and Chandna (1973) investigated steady plane viscous incompressible MHD flows, using the streamfunction and magnetic flux function as independent variables. Chandna and Nath (1973) obtained a number of geometric results for the case of compressible fluids. Garg and Chandna (1976) applied Martin's method to study Hamel's problem (1916) for orthogonal steady plane viscous incompressible fluids of infinite electrical conductivity. Chandna and Garg (1979) obtained various solutions for the plane viscous incompressible flow of infinite electrical conductivity by
employing the hodograph transformation.

**Constantly-Inclined flows** are defined to be those flows for which the magnetic lines and the streamlines are constantly inclined so that the angle between the velocity vector and the magnetic vector is \( \phi = \) constant and \( \phi = 0 \) everywhere. Until 1973, there appears no mention of such flow in the literature. Waterhouse and Kingston (1973) investigated constantly-inclined flows of incompressible non-viscous fluid with infinite electrical conductivity. Toews and Chandna (1974) considered constantly inclined flows of inviscid compressible fluids. Chandna, Toews and Nath (1975) studied these flows of viscous incompressible fluids.

Plane flows are said to be **transverse** if the magnetic field is normal to the plane of flow. H. Grad (1960) derived two integrals for transverse flows. R.M. Gunderson (1966) studied simple waves for transverse flows. O.P. Chandna (1972) obtained a compatibility equation for such flows. Toews and Chandna (1974) derived a method for solving the transverse flow problem.

A flow is said to be a **variably-inclined flow** if the angle between the magnetic field and the velocity field is varying in the flow region. Chandna, Barron and Chew (1982, 1983) applied hodograph transformation and Martin's method respectively to study steady plane flows of incompressible, viscous electrically conducting fluid having infinite electrical conductivity. Chandna, Soteros and Swaminathan (1984) used magnetograph transformations to obtain certain exact solutions of steady plane flows of incompressible, viscous finitely electrically conducting fluid.
A variety of other problems in magnetohydrodynamics have been studied by K.B. Ranger (1969), J.A. Shercliff (1953) and R.H. Wasserman (1967) among others.
1.2 Outline of Current Work.

In this work, steady plane flows of electrically conducting fluids in the presence of a magnetic field are investigated under the assumption that the streamfunction defined by the continuity equation is linear with respect to $x$ and $y$. Flow problems involving such a streamfunction are called Riabouchinsky type problems. Kaloni and Huschilt (1984) obtained some solutions to these problems for a Non-Newtonian, non-MHD fluid. There are three major parts of this investigation. The first part is devoted to a consideration of transverse viscous incompressible plane flows of finitely and infinitely electrically conducting fluids. The second part is devoted to the study of aligned viscous plane flows of fluids having finite and infinite electrical conductivity. The third part is devoted to the study of constantly inclined viscous incompressible plane flows of infinitely electrically conducting fluids.

We now proceed to give a detailed outline of this thesis:

Chapter 2 consists of some preliminary work. In section 2.1, we give the flow equations of magnetohydrodynamics for both ordinary viscous fluids and second grade fluids. In section 2.2, we consider the solutions of Riabouchinsky type problems for an ordinary, non-MHD viscous fluid. In the last section 2.3, we present the Cardan's method for solving a cubic equation.

In Chapter 3, we consider the plane transverse MHD flows. We study
both finitely and infinitely electrically conducting fluids. In section 3.2, the basic flow equations are transformed into convenient form for this work by employing the definition of transverse flow. Both ordinary viscous fluids and second grade fluids have been studied. In section 3.3, we derive the compatibility or integrability equation for these flows for an ordinary viscous and second grade fluid. In section 3.4, Riabouchinsky type problems of an ordinary viscous fluid are investigated by using the compatibility equation found in the previous section. The last section 3.5 consists of the solution of the Riabouchinsky type problems of a second grade fluid.

In Chapter 4, we consider the plane aligned or parallel flows. We also consider both finitely and infinitely electrically conducting fluids. In section 4.2 of this chapter, the basic equations given in Chapter 2 are transformed into more convenient form for this work by employing the definition of aligned flows. Equations for ordinary and second grade fluids are given. In section 4.3, the derivation of the compatibility equation for this flows is presented. Section 4.4, consists of the solutions of Riabouchinsky type problems of an ordinary viscous fluid by using the compatibility equation. In the last section 4.5, the Riabouchinsky type problems of a second-grade fluid are investigated.

Chapter 5 deals with constantly-inclined viscous incompressible MHD flows with infinite electrical conductivity. In section 5.2, we transform the basic equations of section 2.1 into a more convenient form for this work by using the definition of
constantly-inclined flows. In section 5.3, we derive the compatibility equation for these flows and lastly in section 5.4, we consider the solution of Riabouchinsky type problems for the second-grade fluid.
CHAPTER 2

PRELIMINARIES

2.1. Basic Equations of Magnetohydrodynamics

2.1.1. Ordinary Viscous Fluids

The steady flow of a viscous, incompressible electrically conducting fluid, in the presence of a magnetic field is governed by the following system of equations:

\[
\text{div } \mathbf{V} = 0 \quad (2.1.1)
\]

\[
\rho(\mathbf{V} \cdot \nabla) \mathbf{V} + \nabla p = \mu \mathbf{V}^2 + \mu^* (\nabla \times \mathbf{H}) \times \mathbf{H} \quad (2.1.2)
\]

\[
\nabla \times (\mathbf{V} \times \mathbf{H}) - (1/\mu^* \sigma) \nabla \times (\nabla \times \mathbf{H}) = 0 \quad (2.1.3)
\]

\[
\text{div } \mathbf{H} = 0 \quad (2.1.4)
\]

where \( \mathbf{V} \) denotes the velocity field vector, \( \mathbf{H} \) the magnetic field vector, \( p \) the fluid pressure function, \( \mu \) the coefficient of viscosity, \( \rho \) the constant fluid density, \( \mu^* \) the constant magnetic permeability, \( \sigma \) the electrical conductivity.

Introducing the identity

\[
(\mathbf{V} \cdot \nabla) \mathbf{V} = \nabla [(1/2)|\mathbf{V}|^2] - \mathbf{V} \times (\nabla \times \mathbf{V})
\]

in equation (2.1.2), the system of equations (2.1.1) to (2.1.4) is replaced by the following system:
\[ \text{div } \mathbf{V} = 0 \]  \hspace{1cm} (2.1.5)

\[ \rho \left( \text{grad} \left[ (1/2) |\mathbf{V}|^2 \right] - \mathbf{V} \times (\text{grad} \times \mathbf{V}) \right) = -\text{grad } p + \mu \nabla^2 \mathbf{V} + \\
+ \mu^* (\text{curl } \mathbf{H}) \times \mathbf{H} \hspace{1cm} (2.1.6) \]

\[ \text{curl}(\mathbf{V} \times \mathbf{H}) - (1/\mu^* \sigma) \text{curl } (\text{curl } \mathbf{H}) = 0 \]  \hspace{1cm} (2.1.7)

\[ \text{div } \mathbf{H} = 0 \]  \hspace{1cm} (2.1.8)

This is a system of eight equations in seven unknowns \( \mathbf{V}, \mathbf{H} \) and \( p \). Equation (2.1.8) is an additional condition on \( \mathbf{H} \) expressing the absence of magnetic poles in the flow.

2.1.2. Second Grade Fluids

The basic equations governing the motion of an incompressible second grade and electrically conducting fluid, in the presence of a magnetic field, are given by:

\[ \text{div } \mathbf{V} = 0 \]  \hspace{1cm} (2.1.9)

\[ \text{div } \mathbf{T} + \mu^* (\text{curl } \mathbf{H}) \times \mathbf{H} = \rho \left[ \mathbf{V} \cdot \text{grad } \right] \mathbf{V} \]  \hspace{1cm} (2.1.10)

\[ \text{curl}(\mathbf{V} \times \mathbf{H}) - (1/\mu^* \sigma) \text{curl } (\text{curl } \mathbf{H}) = 0 \]  \hspace{1cm} (2.1.11)

\[ \text{div } \mathbf{H} = 0 \]  \hspace{1cm} (2.1.12)

and the constitutive equation for the Cauchy stress \( \mathbf{T} \) [Coleman and Noll, 1960]

\[ \mathbf{T} = -p \mathbf{I} + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2 \]  \hspace{1cm} (2.1.13)

where \( \mathbf{V} \) denotes the velocity vector, \( \mathbf{H} \) the magnetic field vector, \( p \) the fluid pressure function, \( \rho \) the constant fluid density, \( \mu \) the coefficient of viscosity, \( \mu^* \) the constant magnetic permeability, and \( \sigma \)
the electrical conductivity, \( \mathbf{I} \) is the unit tensor, and \( \alpha_1 \) and \( \alpha_2 \) are the normal-stress moduli.

The Rivlin-Ericksen tensors \( \mathbf{A}_1 \) and \( \mathbf{A}_2 \) are defined as

\[
\mathbf{A}_1 = (\text{grad } \mathbf{V}) + (\text{grad } \mathbf{V})^T
\]

\[
\mathbf{A}_2 = \mathbf{A}_1 + (\text{grad } \mathbf{V})^T \mathbf{A}_1 + \mathbf{A}_1 (\text{grad } \mathbf{V})
\]

where

\[
\mathbf{A}_1 = (\mathbf{A}_1 \cdot \text{grad } \mathbf{V}) \mathbf{V}.
\]

Substituting equation (2.1.13) in (2.1.10) and making use of (2.1.14), we obtain

\[
-\text{grad } p + \mu \text{ div } \mathbf{A}_1 + \alpha_1 \text{ div } ((\text{grad } \mathbf{A}_1) \mathbf{V} + (\mathbf{A}_1 \cdot \text{grad } \mathbf{V}) + \\
(\mathbf{A}_1 \cdot \text{grad } \mathbf{V})^T) + \alpha_1 \text{ div } (\mathbf{A}_1^2) + \mu^*(\text{curl } \mathbf{H}) \times \mathbf{H} = \rho(\mathbf{V} \cdot \text{grad } \mathbf{V}).
\]

(2.1.15)

Making use of the following two identities [Fosdick and Truesdell, 1977]

\[
\text{div } ((\text{grad } \mathbf{V})^T \mathbf{A}_1) = (\text{grad } \mathbf{V})^T \text{div } \mathbf{A}_1 + \mathbf{A}_1 \cdot \text{grad } (\text{grad } \mathbf{V})^T
\]

(2.1.16)

\[
\text{div } ((\text{grad } \mathbf{A}_1) \mathbf{V}) = (\text{grad } \text{div } \mathbf{A}_1) \mathbf{V} + \text{div } (\mathbf{A}_1 \cdot \text{grad } \mathbf{V})^T
\]

(2.1.17)

in equation (2.1.15), we get

\[
-\text{grad } p + \mu \text{ div } \mathbf{A}_1 + \alpha_1 ((\text{grad } \text{div } \mathbf{A}_1) \mathbf{V} + (\text{grad } \mathbf{V})^T \text{div } \mathbf{A}_1 + \\
\mathbf{A}_1 \cdot (\text{grad } (\text{grad } \mathbf{V})^T) + (\alpha_1 + \alpha_2) \text{ div } (\mathbf{A}_1^2) + \\
\mu^*(\text{curl } \mathbf{H}) \times \mathbf{H} = \rho \left(\text{grad } ((1/2)|\mathbf{V}|^2) - \mathbf{V} \times (\text{grad } \mathbf{V})\right)
\]

(2.1.18)
Here we also use the fact that

\[(V \cdot \text{grad}) \, V = \text{grad} \left( \frac{1}{2} |V|^2 \right) - V \times (\text{grad} \times V)\]

Since \( \text{curl}(\text{curl} \, H) = \text{grad} \, (\text{div} \, H) - \nabla^2 H \) and since \( \text{div} \, H = 0 \) by (2.1.12), then equation (2.1.11) reduces to

\[\text{curl} \, (V \times H) + \left( \frac{1}{\mu^2} \right) \nabla^2 H = 0 \quad (2.1.19)\]

Hence, equations of motion (2.1.9) to (2.1.12) reduce to the following system of equations:

\[\text{div} \, V' = 0 \quad (2.1.20)\]

\[-\text{grad} \, p + \mu \text{div} \, A_1 + \alpha_1 (\text{grad} \, \text{div} \, A_1) \, V + (\text{grad} \, V)^T \, \text{div} \, A_1 + A_1 \cdot (\text{grad} \, (\text{grad} \, V)^T) + (\alpha_1 + \alpha_2) \, \text{div} \, [A_1^2] + \mu^2 (\text{curl} \, H) \times H =

= \rho \left[ \text{grad} \left( \frac{1}{2} |V|^2 \right) - V \times (\text{curl} \, V) \right] \quad (2.1.21)\]

\[\text{curl} \, (V \times H) + \left( \frac{1}{\mu^2} \right) \nabla^2 H = 0 \quad (2.1.22)\]

\[\text{div} \, H = 0 \quad (2.1.23)\]

Equations (2.1.20) to (2.1.23) is a system of eight equations in seven unknowns \( V, H, \) and \( p \). Equation (2.1.23) is an additional condition on \( H \) expressing the absence of magnetic poles in the flow.
2.2 Riabouchinsky Type Problem for an ordinary Non-MHD Viscous Fluid (Ratip Berker, 1963)

2.2.1. Generalities

The compatibility equation of an ordinary, Non-MHD viscous fluid is given by

\[ \nu \nabla^4 \psi + \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x, y)} = 0 \quad (2.2.1) \]

where \( \nu = \mu / \rho \), and \( \psi \) is the streamfunction defined by the continuity equation, that is

\[ \frac{\partial \psi}{\partial y} = u \quad \text{and} \quad \frac{\partial \psi}{\partial x} = -v \]

We assume that the streamfunction is linear with respect to \( y \); that is, it has the form

\[ \psi(x, y) = y F(x) + G(x) \quad (2.2.2) \]

where \( F(x) \) and \( G(x) \) are arbitrary functions of \( x \). Substituting (2.2.2) into compatibility equation (2.2.1) we find that the functions \( F(x) \) and \( G(x) \) must satisfy the following differential equations:

\[ \nu F^{(iv)} + F' F'' - F F''' = 0 \quad (2.2.3) \]
\[ \nu G^{(iv)} + G' F'' - F G''' = 0 \quad (2.2.4) \]

Integrating these two equations once, we get

\[ \nu F'''' + F' F'' = A \]
\[ \nu G'''' + F' G' - F G'' = B \]

where \( A \) and \( B \) are arbitrary constants of integration.

On assuming that no external forces are present, we can find, by using the linear momentum equations, that the pressure is
given by
\[ p = \mu F' - (1/2) F^2 - (1/2) \rho A y^2 - B y + p_0 \]  \hspace{1cm} (2.2.5)
where \( p_0 \) is an arbitrary constant.

If the fluid domain is extended to infinity then in order for the pressure to stay bounded there, we must have \( A=B=0 \). Also on interchanging \( x \) and \( y \) in the definition of the streamfunction \( (2.2.1) \) we can handle, with minor changes, the solutions of the form \( \psi = x f(y) + g(y) \).

2.2.2. Particular Solution

Equation \((2.2.3)\) admits the following two particular solutions:

\[ F_1 = \frac{-6\nu}{x}, \quad F_2 = \nu a (1 + k e^{ax}) \]  \hspace{1cm} (2.2.6)
where \( a \) and \( k \) are two arbitrary constants.

When \( F = F_1 \), then equation \((2.2.4)\) becomes the Cauchy Euler differential equation.

\[ x^4 G'' + 6 x^3 G'/x - 12 x G' = 0 \]  \hspace{1cm} (2.2.7)
which admits a general solution of the form

\[ G(x) = c_1 + c_2 x^3 + c_3 x^{-1} + c_4 x^{-2} \]  \hspace{1cm} (2.2.8)
where \( c_1, c_2, c_3, c_4 \) are arbitrary constants.

Hence the streamfunction \( \psi(x,y) \) defined by \((2.2.2)\) is given by

\[ \psi(x,y) = \frac{-6\nu}{x} + c_1 + c_2 x^3 + c_3 x^{-1} + c_4 x^{-2} \]  \hspace{1cm} (2.2.9)

When \( F = F_2 \), then equation \((2.2.4)\) becomes
\[ G^{(iv)} - a(1 + k e^x) G''' + a^3 k e^x G' = 0 \] (2.2.10)

Without loss of generality, we can assume \( a = 1 \), so the above equation (2.2.10) takes the form
\[ G^{(iv)} - (1 + k e^x) G''' + k e^x G' = 0 \] (2.2.11)

Employing a series of substitutions of the form
\[ G'(x) = H(x), \quad H(x) = P(x) e^x, \quad P'(x) = R(x) \] (2.2.12)
in (2.2.11), we find that \( R(x) \) satisfies the following equation
\[ R'' e^x + (2 - k e^x) R' e^x + (1 - 2k e^x) R e^x = 0 \] (2.2.13)

Integrating this equation once we get
\[ R' + (1 - k e^x) R = c_1 e^{-x} \] (2.2.14)
where \( c_1 \) is an arbitrary constant.

The general solution of (2.2.14) is given by
\[ R(x) = c_1 \exp(k e^x - x) \int \exp(-k e^x) \, dx + c_2 \exp(k e^x - x) \] (2.2.15)

Using the series of substitutions (2.2.12), we find that
\[ G(x) = c_1 \int e^x \, dx \int \exp(k e^x - x) \, dx \int \exp(-k e^x) \, dx + c_2 \int e^x \, dx \int \exp(k e^x - x) \, dx + c_3 e^x + c_4 \] (2.2.16)
where \( c_3 \) and \( c_4 \) are also arbitrary constants.

Thus, streamfunction \( \psi(x,y) \) is given by
\[ \psi(x,y) = \nu (1 + k e^x) y + c_1 \int e^x \, dx \int \exp(k e^x - x) \, dx \int \exp(-k e^x) \, dx + c_2 \int e^x \, dx \int \exp(k e^x - x) \, dx + c_3 e^x + c_4 \] (2.2.17)
2.2.3. Solution of Riabouchinsky

We assume a special case of solution (2.2.17) where $G(x) = 0$, that is to say $c_1 = c_2 = c_3 = c_4 = 0$, usually attributed to Riabouchinsky.

Employing (2.2.6), we find that the streamfunction is given by

$$\psi(x,y) = \nu (1 + ke^{px}) y$$

(2.2.18)

If we take $k = -1$ and put $\nu a = -U_0$, we obtain the following solution which is due to Riabouchinsky

$$\psi(x,y) = U_0 y [\exp(-U_0^2 x) - 1]$$

(2.2.19)

$$u = U_0 [\exp(-U_0 x/\nu) - 1]$$

(2.2.20)

and

$$v = U_0^2 (y/\nu) \exp(-U_0 x/\nu)$$

When $x$ tends to infinity, $u$ tends to $-U_0$ and $v$ tends to zero; at infinity we then have a uniform stream parallel to $Ox$.

Moreover for $x = 0$ we have $u = 0$ and $v = (U_0^2/\nu) y$.

The vorticity of the flow (2.2.20) is given by the relation

$$\omega = - (U_0^3/\nu^2) y \exp(-U_0 x/\nu) (x \geq 0)$$

For a fixed value of $y$ the vorticity decreases in absolute value when one goes away from the plane $X = 0$. We can say that there is a neighbourhood of this plane in which the vorticity has an appreciable value; outside of this layer the vorticity is essentially nil.

Finally, the pressure is given by the relation

$$p = p_0 - (1/2) \rho U_0^2 \exp(-2(U_0/\nu) x)$$

(2.2.20)

where $p_0$ denotes the pressure at infinity.
2.3. Cardan's Solution of cubic equations (Borofsky, 1950)

General Method

Let the equation be of the form

\[ f(x) = x^3 + a x^2 + b x + c = 0 \quad (2.3.1) \]

where \( a, b, c \) are any complex numbers. For simplicity, we are taking the leading coefficient to be 1.

We first make a transformation to eliminate the second degree term. If the roots of \( f(x) \) are \( x_1, x_2, x_3 \), then we seek an equation without a quadratic term whose roots are

\[ y_1 = x_1 + \alpha, \quad y_2 = x_2 + \alpha, \quad y_3 = x_3 + \alpha \]

where \( \alpha \) is a constant to be determined.

We perform, therefore, the transformation

\[ y = x + (a/3), \quad \text{or} \quad x = y - (a/3) \quad (2.3.2) \]

Hence, the transformed equation is

\[ y^3 + p y + q = 0 \]

where \( p = b - \left( \frac{a^2}{3} \right), \quad q = \left( \frac{2 a^3 - 9 a b + 27 c}{27} \right) \).

The expression \( y^3 + p y + q \) is called the reduced cubic. Its roots are

\[ y = x + (a/3), \quad y = x + \left( \frac{a}{3} \right), \quad y = x + (a/3) \quad (2.3.4) \]

To solve the reduced cubic, we use a transformation

\[ y = z + \left( \frac{\beta}{z} \right) \]

where \( \beta \) is a constant to be determined.

The transformed equation is

\[ z^3 + \left( \beta^3/z^3 \right) + q + (3 \beta + p) z + \beta \left( 3\beta + p \right)/z = 0 \quad (2.3.5) \]
To eliminate as many terms as possible, it is desirable to have $3 \beta + p = 0$, or $\beta = - (p/3)$. With this choice for $\beta$, the transformation is

$$y = z - \left( \frac{p}{3} z \right) \quad (2.3.6)$$

and the transformed equation is

$$z^6 + q z^3 - \left( \frac{p^3}{27} \right) = 0 \quad \text{6} \quad (2.3.7)$$

Equation (2.3.7) is quadratic in $z^3$ with roots given by

$$z^3 = \left( - q \pm \sqrt{q^2 + \frac{4p^3}{27}} \right)/2 \quad (2.3.8)$$

From these expressions for $z^3$ we shall, in general, obtain six values for $z$. For each one of the non-zero values of $z$ we obtain a corresponding value for $y$ by using (2.3.6). For each of the values of $y$ there is a corresponding value for $x$ by using (2.3.2).

**Discriminant of Cubic**

If $f(x)$ is of degree 3 and $x_1, x_2, x_3$ are the roots of

$$f(x) = 0,$$

then

$$D = (x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3) \quad (2.3.9)$$

is called the discriminant of $f(x)$.

**Theorem 1**: If $x^3 + ax^2 + bx + c$ has real coefficients then

a) if $D > 0$, all roots are real and distinct

b) if $D = 0$, all roots are real and there is a multiple root

c) if $D < 0$, there is one real root and two (conjugate) imaginary roots.

**Theorem 2**: If $f(x) = x^3 + ax^2 + bx + c$, and $g(y) = y^3 + py + q$
is the reduced cubic, then \( f(x) \) and \( g(y) \) have the same

discriminants.

**Theorem 3:**

a) The discriminant of \( y^3 + p y + q \) is \(-4p^3 - 27q^2\)

b) The discriminant of \( x^3 + a x^2 + b x + c \) is

\[18abc - 4a^3c + a^2b^2 - 4b^3 - 27c^2\]

---

**Cubics with Three Real Roots**

Whenever the cubic has real coefficients and the
discriminant is positive, the roots are real and distinct, but
using the general Cardan's method mentioned above we find that the
Cardan's formulas necessarily involve imaginary quantities. For,
considering the reduced cubic \( y^3 + p y + q = 0 \), the discriminant
is \( D = -27q^2 - 4p^3 \) and the roots are

\[
\frac{3}{\sqrt[3]{A}} + \frac{3}{\sqrt[3]{B}} , \quad \omega \frac{3}{\sqrt[3]{A}} + \omega^2 \frac{3}{\sqrt[3]{B}} , \quad \omega^2 \frac{3}{\sqrt[3]{A}} + \omega \frac{3}{\sqrt[3]{B}} \quad (2.3.10)
\]

where \( A = \left( -q + i\sqrt{(D/27)} \right)/2 \), \( B = \left( -q - i\sqrt{(D/27)} \right)/2 \)

\[(2.3.11)\]

and \( \omega \) is any of the imaginary cube roots of 1, that is

\[\omega = \left( -1 \pm i\sqrt{3} \right)/2\]

For practical purposes it is sometimes desirable to express
the roots in a form which involves no imaginary quantities. This
can be done with the help of the triple angle formula

\[4\cos^3\theta - 3\cos\theta = \cos3\theta \quad (2.3.12)\]

From this we can see that \( z = \cos\theta \) is one root of
\begin{equation}
\begin{aligned}
z^3 - \left(\frac{3}{4}\right) z - \left(\cos 3\theta / 4\right) = 0
\end{aligned}
\tag{2.3.13}
\end{equation}

The other two roots of (2.3.13) are \(\cos(\theta + 120^\circ)\), and
\(\cos(\theta + 240^\circ)\).

If the reduced cubic has a positive discriminant, by choosing
a suitable \(\lambda\) and letting \(y = \lambda z\) we transform the equation
\(y^3 + p y + q = 0\) into one of the form (2.3.13). The transformed
equation, after dividing by \(\lambda\), is
\begin{equation}
\begin{aligned}
z^3 + \left(\frac{p}{\lambda^2}\right) z + \left(\frac{q}{\lambda^3}\right) = 0
\end{aligned}
\tag{2.3.14}
\end{equation}

Hence, we want to have
\begin{equation}
\begin{aligned}
\left(\frac{p}{\lambda^2}\right) = -\left(\frac{3}{4}\right) \text{ and } \left(\frac{q}{\lambda^3}\right) = -\left(\frac{\cos 3\theta}{4}\right)
\end{aligned}
\tag{2.3.15}
\end{equation}

Since \(D > 0\) and \(p\) and \(q\) are real, then \(p\) is negative. Hence,
a real value for \(\lambda = 2 \frac{\sqrt[3]{-p}}{\sqrt[3]{3}}\) exists. Using this \(\lambda\), we seek
an angle \(\theta\) such that
\begin{equation}
\begin{aligned}
\cos \theta = \left(\frac{3}{2} \frac{q}{p}\right) \frac{\sqrt[3]{3}}{\sqrt[3]{p}}
\end{aligned}
\tag{2.3.16}
\end{equation}

\(D > 0\) implies that \(-4 p^3 > 27 q^2\).

Hence,
\(0 \leq \left(-\frac{27 q^2}{4 p^3}\right) < 1\)
so that an angle \(3\theta\) exists with the desired cosine. Thus, \(\theta\) can be found.

With the \(\lambda\) and \(\theta\) determined, the transformed equation has the
(2.3.13). Since the roots of (2.3.13) are \(\cos \theta\), \(\cos(\theta + 120^\circ)\), and
\(\cos(\theta + 240^\circ)\), then the roots of the reduced cubic are
\begin{equation}
\begin{aligned}
&2 \frac{\sqrt[3]{-p}}{\sqrt[3]{3}} \cos \theta, \quad 2 \frac{\sqrt[3]{-p}}{\sqrt[3]{3}} \cos(\theta + 120^\circ) \\
&\text{and} \quad 2 \frac{\sqrt[3]{-p}}{\sqrt[3]{3}} \cos(\theta + 240^\circ).
\end{aligned}
\end{equation}
Example

Using Cardan's method and the trigonometric method find the roots of

\[ m^3 - 2 m^2 + (1/4) m + (1/2) = 0 \quad (2.3.17) \]

Let \( y = m - (2/3) \) or \( m = y + (2/3) \), then equation (2.3.17) reduces to

\[ y^3 - (13/12) y - (2/9) = 0 \quad (2.3.18) \]

In this case \( p = -(13/12) \) and \( q = -(2/9) \)

then

\[ \lambda = 2 \sqrt{-p/3} = \sqrt{13} / 3 \]

and

\[ \cos 3\theta = \frac{24}{(13 \sqrt{13})} \]

or

\[ \theta = 19.7342 \]

Hence, solutions of equation (2.3.18) are given by

\[ y_1 = \lambda \cos \theta \quad \text{or} \quad y_1 = 1.13126 \]

\[ y_2 = \lambda \cos(\theta + 120^\circ) \quad \text{or} \quad y_2 = -0.917076 \]

\[ y_3 = \lambda \cos(\theta + 240^\circ) \quad \text{or} \quad y_3 = -0.214187 \]

Since, \( m = y + (2/3) \), then the roots of equation (2.3.17) are

\[ m_1 = y_1 + (2/3) \quad \text{or} \quad m_1 = 1.797926 \]

\[ m_2 = y_2 + (2/3) \quad \text{or} \quad m_2 = -0.25041 \]

\[ m_3 = y_3 + (2/3) \quad \text{or} \quad m_3 = 0.452479 \]

NOTE: Section 2.3 of this Chapter is used in Chapter 4 for solving the cubic equation (4.4.46).
CHAPTER 3

PLANE TRANSVERSE FLOWS

3.1. Introduction

Magneto-hydro-dynamic (MHD) plane flows are said to be transverse if the magnetic field vector is normal to the plane of flow, and the flow variables are functions of the rectangular coordinates x and y only. Many authors have studied these flows when electrical conductivity is infinite.

Not much work has been done for viscous finitely conducting transverse flows. However, Chandna and Garg (1976) established some geometrical properties for finitely conducting incompressible transverse flows.

In this chapter, we study the Riabouchinsky type problems for plane transverse flows applying to both ordinary viscous fluids and second grade fluids. Also, both finitely and infinitely conducting cases are considered.

Up to date, there does not exist any work on the Riabouchinsky type problem for magnetohydrodynamics flows.

This chapter is organized as follows. In section 3.2 the basic flow equations for both ordinary viscous and second grade fluids are transformed into more convenient form for this work. Section 3.3 contains the derivation of the compatibility or
integrability equation. Section 3.4 consists of the solution of Riabouchinsky type problems for an ordinary viscous fluid. Finally, section 3.5 contains the solution of Riabouchinsky type problems for a second grade fluid.
3.2. Equations of Motion

The steady plane motion of a viscous, incompressible fluid of finite electrical conductivity, in the presence of a magnetic field, is governed by the following system of equations

\[ \text{div} \mathbf{V} = 0 \]  
\[ \rho (\mathbf{V} \cdot \text{grad}) \mathbf{V} + \text{grad} p = \mu \nabla^2 \mathbf{V} + \mu^* (\text{curl} \mathbf{H}) \times \mathbf{H} \]  
\[ \text{curl} (\mathbf{V} \times \mathbf{H}) + \left( \frac{1}{\mu^* \sigma} \right) \nabla^2 \mathbf{H} = 0 \]

where \( \mathbf{V} \) denotes the velocity vector, \( \mathbf{H} \) the magnetic field vector, \( p \) the fluid pressure function, \( \rho \) the constant fluid density, \( \mu \) the coefficient of viscosity, \( \mu^* \) the constant magnetic permeability, and \( \sigma \) the electrical conductivity.

This is a system of eight equations in seven unknowns. Equation (iv) is an additional condition on \( \mathbf{H} \) expressing the absence of magnetic poles in the flow.

3.2.1. Ordinary Viscous Fluid

Considering the magnetic field to be acting in a constant direction, we take \( \mathbf{V} = (u, v, 0) \), \( \mathbf{H} = (0, 0, H) \) and \( \frac{\partial}{\partial z} = 0 \). Making use of these results, we find that:

\[ \text{curl} \mathbf{H} \times \mathbf{H} = - \text{grad} \left( \frac{(H^2)}{2} \right) \]  
\[ \text{curl} (\mathbf{V} \times \mathbf{H}) = \left[ 0, 0, -(uH)_x - (vH)_y \right] \]  
\[ \nabla^2 \mathbf{H} = \left( 0, 0, H_{xx} + H_{yy} \right) \]
Employing results (3.2.1) to (3.2.3) into the system of equations (i) to (iv), we find that the flow equations governing the motion of a steady plane transverse flow of a viscous, incompressible fluid of finite electrical conductivity, in the presence of a magnetic field, are given by:

\[ u_x + v_y = 0 \] \hspace{1cm} (3.2.4)

\[ \rho \left[ \frac{\partial}{\partial x} \left( \frac{u^2 + v^2}{2} \right) - v \left( v_x - u_y \right) \right] + \frac{\partial}{\partial x} p^* = \mu \nabla^2 u \] \hspace{1cm} (3.2.5)

\[ \rho \left[ \frac{\partial}{\partial y} \left( \frac{u^2 + v^2}{2} \right) + u \left( v_x - u_y \right) \right] + \frac{\partial}{\partial y} p^* = \mu \nabla^2 v \] \hspace{1cm} (3.2.6)

\[ u H_x + v H_y - \frac{1}{\mu^* \sigma} (H_{xx} + H_{yy}) = 0 \] \hspace{1cm} (3.2.7)

\[ p^* = p + (1/2) \mu^* H^2 \] \hspace{1cm} (3.2.8)

Equations (3.2.4) to (3.2.8) is a system of five equations in five unknowns. The unknowns being \( u, v, p, H \) and \( p^* \).

For infinitely conducting fluids, \( \sigma \) tending to infinity implies that \( (1/\mu^* \sigma) \) tends to zero and the diffusion equation (3.2.7) is modified to

\[ u H_x + v H_y = 0 \] \hspace{1cm} (3.2.9)

3.2.2. Second Grade Fluid

Considering the magnetic field to be acting in a constant direction, we take \( V = (u, v, 0) \), \( H = (0, 0, H) \) and
\( \frac{\partial}{\partial z} \) = 0. Employing results (3.2.1) to (3.2.3) into the system of equations (2.1.20) to (2.1.23), we find that the flow equations governing the motion of the steady plane transverse flow of a second grade, incompressible fluid of finite electrical conductivity, in the presence of a magnetic field, are

\[
\begin{align*}
    u_x + v_y &= 0 \quad \text{(3.2.10)} \\
    \rho \left[ \frac{\partial}{\partial x} \left( \frac{u^2 + v^2}{2} \right) - v (v_x - u_y) \right] + \frac{\partial}{\partial x} p^* &= \mu \nabla^2 u + \\
    + a_1 \left[ \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \nabla^2 u + u_x \nabla^2 u + v_x \nabla^2 v \right] + \\
    + (1/4) (3a_1 + 2a_2) \frac{\partial}{\partial x} \left( |A_1|^2 \right) \\
    \rho \left[ \frac{\partial}{\partial y} \left( \frac{u^2 + v^2}{2} \right) + u (v_x - u_y) \right] + \frac{\partial}{\partial y} p^* &= \mu \nabla^2 v + \\
    + a_1 \left[ \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \nabla^2 v + u_y \nabla^2 u + v_y \nabla^2 v \right] + \\
    + (1/4) (3a_1 + 2a_2) \frac{\partial}{\partial y} \left( |A_1|^2 \right) \quad \text{(3.2.11)} \\
    u H_x + v H_y - \frac{1}{\mu^* \sigma} (H_{xx} + H_{yy}) &= 0 \quad \text{(3.2.13)} \\
    p^* &= p + (1/2) \mu^* H \quad \text{(3.2.14)}
\end{align*}
\]

where
\[ | A_1' |^2 = 4 \left( u_x \right)^2 + 4 \left( v_y \right)^2 + 2 \left( v_x + u_y \right)^2. \]

Equations (3.2.10) to (3.2.14) is a system of five equations in five unknowns \( u, v, p, H \) and \( p^* \).

For infinitely conducting fluids, \( \sigma \) tending to infinity implies that \( \left( 1 / \mu^* \sigma \right) \) tends to zero and hence the diffusion equation (3.2.13) reduces to

\[ u H_x + v H_y = 0 \]  \hspace{1cm} (3.2.15)
3.3 Compatibility Equations

In this section, we derive the compatibility or integrability equations for transverse flow of an ordinary viscous fluid as well as for a second grade fluid by employing the equations of motion for these flows in section 3.2. Riabouchinsky type problems are investigated by using the compatibility equations in the next two sections. This fluid dynamic technique of using the compatibility equation is well known and thoroughly documented in Ratip Berker's (1963) work for the investigation of exact solutions in ordinary Non-MHD fluid dynamics.

3.3.1. Ordinary Viscous Fluid

Introducing the vorticity function

\[ \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \]  \hspace{1cm} (3.3.1)

and the generalized pressure function

\[ h = \frac{1}{2} \rho q^2 + p^* \] \hspace{1cm} (3.3.2)

in equations (3.2.4) to (3.2.8), where \( q^2 = u^2 + v^2 \), we find that these flows are given by

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  \hspace{1cm} (3.3.3)

\[ \mu \frac{\partial \omega}{\partial y} - \rho v \omega = - \frac{\partial h}{\partial x} \]  \hspace{1cm} (3.3.4)

\[ \mu \frac{\partial \omega}{\partial x} - \rho u \omega = \frac{\partial h}{\partial y} \]  \hspace{1cm} (3.3.5)
\[ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad (3.3.6) \]

\[ u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} = \frac{1}{\mu^* \alpha} \left( \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} \right) \quad (3.3.7) \]

\[ p^* = p + \frac{1}{2} \mu^* H^2 \quad (3.3.8) \]

This system has the advantage that the linear momentum equations (3.3.4) and (3.3.5) are of first order. Martin (1971) has, with much success, used a similar reduction in order to study viscous Non-MHD flows.

Let \( \psi (x, y) \) be the streamfunction defined by the continuity equation, that is

\[ \frac{\partial \psi}{\partial y} = u , \quad \frac{\partial \psi}{\partial x} = -v \quad (3.3.9) \]

Using these equations, we can see clearly that the continuity equation (3.3.3) is identically satisfied, since

\[ u_x + v_y = \psi_{xy} - \psi_{yx} = 0 \quad (3.3.10) \]

while, the vorticity function is given by

\[ \omega = -\nabla^2 \psi \quad (3.3.11) \]

Employing (3.3.9) and (3.3.11) into equations (3.3.4), (3.3.5) and (3.3.7), these equations become

\[ \frac{\partial h}{\partial x} - \rho \frac{\partial \psi}{\partial x} \nabla^2 \psi = \mu \frac{\partial}{\partial y} \left( \nabla^2 \psi \right) \quad (3.3.12) \]

\[ \frac{\partial h}{\partial y} - \rho \frac{\partial \psi}{\partial y} \nabla^2 \psi = -\mu \frac{\partial}{\partial x} \left( \nabla^2 \psi \right) \quad (3.3.13) \]

\[ \frac{\partial \psi}{\partial y} \frac{\partial H}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial H}{\partial y} - \frac{1}{\mu^* \alpha} \left( \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} \right) \quad (3.3.14) \]
Differentiating (3.3.12) with respect to 'y' and (3.3.13) with respect to 'x' and employing the integrability condition
\[ \frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x} \],
we find that the compatibility equation is given by
\[ \rho \frac{\partial (\psi, \nabla^2 \psi \rho)}{\partial (x, y)} + \mu \nabla^4 \psi = 0 \quad (3.3.15) \]

Having solved equation (3.3.15), the velocity components are given by equations (3.3.9) and the vorticity function is given by (3.3.11).

Finally, functions \( h, p, p^* \) and \( H \) can be found by solving equations (3.3.2), (3.3.8), (3.3.12), (3.3.13) and (3.3.14).

3.3.2. Second Grade Fluid

Introducing the vorticity function
\[ \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad (3.3.16) \]
we find that
\[ \nabla^2 \psi = \omega_x \quad (3.3.17) \]
and
\[ \nabla^2 u = -\omega_y \quad (3.3.18) \]

Next, we should arrange the \( \alpha_1 \)-terms of equations (3.2.11) and (3.2.12) into more convenient forms. First consider the \( \alpha_1 \)-term of (3.2.11)
\[ a_1 \left[ \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \nabla^2 u + \frac{\partial u}{\partial y} \nabla^2 v + \frac{\partial v}{\partial y} \nabla^2 v \right] \]

\[ = a_1 \left[ \frac{\partial}{\partial x} (u \nabla^2 u) - v \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial v}{\partial x} \nabla^2 v \right] \]

\[ = a_1 \left[ \frac{\partial}{\partial x} (u \nabla^2 u + v \nabla^2 v) - v \nabla^2 \omega \right] \]

Then consider the \( a_1 \)-term of (3.2.12):

\[ a_1 \left[ \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \nabla^2 v + \frac{\partial u}{\partial y} \nabla^2 u + \frac{\partial v}{\partial y} \nabla^2 v \right] \]

\[ = a_1 \left[ \frac{\partial}{\partial y} (v \nabla^2 v) + u \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial u}{\partial y} \nabla^2 u \right] \]

\[ = a_1 \left[ \frac{\partial}{\partial y} (u \nabla^2 u + v \nabla^2 v) + u \nabla^2 \omega \right] \]

Using the above results and introducing the generalized pressure

\[ h = \frac{1}{2} \rho q^2 + p^* - a_1 (u \nabla^2 u + v \nabla^2 v) - \frac{1}{4} \left( 3a_1 + 2a_2 \right) |A_1|^2 \]

(3.3.19)

in equations (3.2.10) to (3.2.14), we find that these flows are given by
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3.3.20)
\]

\[
\frac{\partial h}{\partial x} - \rho v \omega = -\mu \frac{\partial \omega}{\partial y} - \alpha_1 v \nabla^2 \omega \quad (3.3.21)
\]

\[
\frac{\partial h}{\partial y} + \rho u \omega = \mu \frac{\partial \omega}{\partial x} + \alpha_1 u \nabla^2 \omega \quad (3.3.22)
\]

\[
\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega \quad (3.3.23)
\]

\[
u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} - \frac{1}{\mu^* \sigma} \left( \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} \right) = 0 \quad (3.3.24)
\]

\[p^* = p + (1/2) \mu^* H^2 \quad (3.3.25)\]

Let \(\psi(x,y)\) be the streamfunction defined by the continuity equation, that is

\[
\frac{\partial \psi}{\partial y} = u \quad \text{and} \quad \frac{\partial \psi}{\partial x} = -v \quad (3.3.26)
\]

Using this definition, we can easily see that equation (3.3.20) is identically satisfied, while the vorticity function is given by

\[
\omega = -\nabla^2 \psi \quad (3.3.27)
\]

Employing (3.3.26) and (3.3.27) into (3.3.21), (3.3.22) and (3.3.24), we find that these equations reduce to

\[- 34 -\]
\[
\frac{\partial h}{\partial x} - \rho \frac{\partial \psi}{\partial x} \nabla^2 \psi = \mu \frac{\partial}{\partial y} (\nabla^2 \psi) - \alpha_1 \frac{\partial \psi}{\partial x} \nabla^4 \psi
\] (3.3.28)

\[
\frac{\partial h}{\partial y} - \rho \frac{\partial \psi}{\partial y} \nabla^2 \psi = -\mu \frac{\partial}{\partial x} (\nabla^2 \psi) - \alpha_1 \frac{\partial \psi}{\partial y} \nabla^4 \psi
\] (3.3.29)

\[
\frac{\partial \psi}{\partial y} \frac{\partial h}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial h}{\partial y} - \frac{1}{\mu^* \sigma} \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) = 0
\] (3.3.30)

Differentiating (3.3.28) with respect to 'y' and (3.3.29) with respect to 'x' and employing the integrability condition

\[
\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x}
\]

we find that the compatibility equation for the transverse flow of a second grade fluid is given by

\[
\rho \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x, y)} + \mu \nabla^4 \psi - \alpha_1 \frac{\partial (\psi, \nabla^4 \psi)}{\partial (x, y)} = 0
\] (3.3.31)

Having solved equation (3.3.31), the velocity components and the vorticity are given by equations (3.3.26) and (3.3.27) respectively.

Finally, functions h, p, p* and H can be found by using equations (3.3.19), (3.3.28), (3.3.29) and (3.3.30).
3.4. Solution of Riabouchinsky Type Problems.

Ordinary Viscous Fluids

We assume that the streamfunction is linear with respect to $y$, i.e. it has the form

$$\psi (x, y) = y F(x) + G(x)$$  \hspace{1cm} (I)

The velocity field for solution (I) has the following property: the lines of action of the velocity vectors applied at points $M$ on the line $x = x_0$ parallel to the axis $Oy$ intersect at the same point $P$ say. Let $C$ be the curve traced out by point $P$ when $x_0$ varies; we call $C$ the directrix curve for the velocity field considered. It is sufficient to give the curve $C$ in terms of $x_0$ rather than the vector fields. In the case of solution (I) the directrix curve has parametric equations

$$X = x_0 + \frac{F(x_0)}{F'(x_0)} \quad \quad Y = -\frac{G'(x_0)}{F'(x_0)}$$  \hspace{1cm} (Kamke, 1930).

We also point out that by interchanging $x$ and $y$ in (I), we can handle with minor changes, the solutions of the form

$$\psi (x, y) = x f(y) + g(y)$$

where $f(y)$ and $g(y)$ are arbitrary functions of $y$.

3.4.1. Generalities

We assume that the streamfunction has the form

$$\psi (x, y) = y F(x) + G(x)$$  \hspace{1cm} (3.4.1)

- 36 -
where \( F(x) \) and \( G(x) \) are arbitrary functions of \( x \).

Substituting (3.4.1) into compatibility equation (3.3.15), we find that the functions \( F(x) \) and \( G(x) \) must satisfy the following differential equations:

\[
\mu F^{(iv)} + \rho [F'F'' - F F'''] = 0 \quad (3.4.2)
\]

\[
\mu G^{(iv)} + \rho [F'G' - F G'''] = 0 \quad (3.4.3)
\]

These two equations can each be integrated once, and we thus obtain the equations

\[
\mu F'''' + \rho [F'F^2 - F F'''] = A
\]

\[
\mu G'''' + \rho [F'G' - F G'''] = B
\]

where \( A \) and \( B \) are arbitrary constants.

Employing (3.4.1) into linear momentum equations (3.3.12) and (3.3.13), we find that

\[
h = - \frac{1}{2} y^2 [A - \rho F^2] - y [B - \rho F' G'] + \frac{1}{2} \rho G' + \mu F' + p_0 \quad (3.4.4)
\]

where \( p_0 \) is an arbitrary constant.

Substituting (3.4.1) and (3.4.4) into (3.3.2), we obtain

\[
p^* = \mu F' - (1/2) \rho F^2 - (1/2) A y^2 - B y + p_0 \quad (3.4.5)
\]

Employing (3.4.1), diffusion equation (3.3.14) becomes

\[
F H_x - (y F' + G') H_y - (1/\mu^* \sigma) (H_{xx} + H_{yy}) = 0 \quad (3.4.6)
\]

Any solution of this equation is a possible magnetic field. One such solution is given by

\[ - 37 - \]
\[ H(x, y) = L(y) \int \exp(\mu^a F \, dx) \, dx + M(y) \]  \hspace{1cm} (3.4.7)

where \( L(y) \) and \( M(y) \) are such that the following equation is satisfied

\[ \left( y F' + G' \right) H_y + \left( 1/\mu^a \right) H_{yy} = 0 \]

On assuming that magnetic forces are present, we find, by employing (3.4.5) into equation (3.3.8), that the pressure turns out to be

\[ p = \mu F' - \left( 1/2 \right) \rho F^2 - \left( 1/2 \right) A y^2 - B y + p_o - \left( 1/2 \right) \mu^a H^2 \]  \hspace{1cm} (3.4.8)

where \( H \) is given by (3.4.7).

If the fluid domain is considered to extend to infinity, then in order that the pressure remains bounded there, we must take \( A = B = 0 \).

As we mentioned in section 3.2., if the fluid is infinitely conducting, that is, \( \sigma \) tends to infinity then the diffusion equation reduces to

\[ u \, H_x + v \, H_y = 0 \]

Since \( u = \psi_y \) and \( v = -\psi_x \), then the above equation yields

\[ \psi_y H_x - \psi_x H_y = 0 \]

OR

\[ \frac{\partial (H, \psi)}{\partial (x, y)} = 0 \]

Solving this equation, we find that

\[ H(x, y) = \text{constant} \hspace{1cm} \text{OR} \hspace{1cm} H(x, y) = H(\psi) \]
where \( H(\psi) \) is an arbitrary function of \( \psi \).

### 3.4.2. Particular Solution

In this section, we will try to solve equations (3.4.2) and (3.4.3). Since, we could not find the general solution of this system, then we just consider some particular solutions.

Equation (3.4.2) admits the following three particular solutions:

\[
F_1 = \frac{-6 \mu}{\rho x}, \quad F_2 = (\mu/\rho) a(1 + k e^{ax}), \quad F_3 = D x + E \tag{3.4.9}
\]

wherein \( a, k, D, \) and \( E \) are four arbitrary constants.

When \( F = F_1 \), then equation (3.4.3) becomes the Cauchy-Euler differential equation

\[
x^4 G^{(iv)} + 6 x^3 G''' - 12 x G' = 0 \tag{3.4.10}
\]

which admits a general solution of the form

\[
G(x) = c_1 x^3 + c_2 x^{-1} + c_3 x^{-2} + c_4 \tag{3.4.11}
\]

where \( c_1, c_2, c_3 \) and \( c_4 \) are arbitrary constants.

Thus, the streamfunction defined by (3.4.1) is given by

\[
\psi(x,y) = -(6\mu/\rho) (y/x) + c_1 x^3 + c_2 x^{-1} + c_3 x^{-2} + c_4 \tag{3.4.12}
\]

From equation (3.4.8) we can find that the pressure distribution is given by

\[
p = -10 \mu^2 \rho x^{-2} -(1/2) \mu^2 H^2 + p_0 \tag{3.4.13}
\]

where \( H \) is given by equation (3.4.7).

We seek, now, the expression for the streamfunction \( \psi(x,y) \).
which corresponds to the solution \( F_2 \). When \( F = F_2 \), then equation (3.4.3) takes the form

\[
G^{(iv)} - a (1 + k e^{ax}) G'''' + a^3 k e^{ax} G' = 0
\]  
(3.4.14)

We can easily see, on taking \( \zeta = ax \) as the variable, that all solutions of equation (3.4.14) are of the form:

\[
G = \Phi (ax)
\]

and the streamfunction \( \psi (x,y) \) is of the form

\[
\psi = \Psi (ax, ay)
\]

All solutions \( \psi (x,y) \) are therefore deducible from the solution where \( a = 1 \) by similarity.

When, therefore, \( a = 1 \), equation (3.4.14) becomes

\[
G^{(iv)} - (1 + k e^x) G'''' + k e^x G' = 0
\]  
(3.4.15)

Employing a series of substitutions of the type

\[
G'(x) = M(x) \quad M(x) = N(x) e^x \quad N'(x) = R(x)
\]  
(3.4.16)

in equation (3.4.15), we find that the function \( R(x) \) satisfies the following equation:

\[
R'' + (2 - k e^x) R' + (1 - 2k e^x) R = 0
\]  
(3.4.17)

Integrating once equation (3.4.17), we obtain a first order linear differential equation, that is,

\[
R' + (1 - k e^x) R = c_1 e^x
\]  
(3.4.18)

where \( c_1 \) is an arbitrary constant.

The general solution of (3.4.18) is given by

\[
R(x) = c_1 \exp(k e^x - x) \int \exp(-k e^x) \, dx + c_2 \exp(k e^x - x)
\]

where \( c_2 \) is also an arbitrary constant.
Having found $R(x)$, we are going back to the series of substitutions (3.4.16) and we find that

$$
G(x) = c_1 \int e^x \, dx \int \exp(k \, e^x - x) \, dx \int \exp(-k \, e^x) \, dx +
+ c_2 \int e^x \, dx \int \exp(k \, e^x - x) \, dx + c_3 e^x + c_4 
$$

(3.4.19)

wherein $c_3$ and $c_4$ are two arbitrary constants.

Hence the streamfunction $\psi(x, y)$ defined by (3.4.1) is given by

$$
\psi(x, y) = (\mu/\rho) \left(1 + k \, e^x\right) y + c_1 \int e^x \, dx \int \exp(k \, e^x - x) \, dx

\int \exp(-k \, e^x) \, dx + c_2 \int e^x \, dx \int \exp(k \, e^x - x) \, dx +

+ c_3 e^x + c_4
$$

(3.4.20)

Finally, from equation (3.4.8) we find that the pressure distribution turns out to be

$$
p = (1/\rho) \mu^3 k^2 e^{2x} - (1/2)(\mu^2/\rho)(1 + k \, e^x)^2 + p_0 - (1/2) \mu^2 \rho^2
$$

(3.4.21)

where $H$ can be found from equation (3.4.7).

In the special case, where $k = 0$ then

$$
F = (\mu/\rho) a
$$

and the solution of (3.4.18) is given by

$$
R(x) = c_1 x e^{-x} + c_2 e^{-x}
$$

and the streamfunction $\psi(x, y)$ reduces to

$$
\psi(x, y) = (\mu/\rho) a y - (c_1/2) x^2 - (c_1 + c_2) x + c_3 e^x + c_4
$$

(3.4.22)

When $F = F_3$, then equation (3.4.3) takes the form
\[ \mu \, G^{(i, v)} - \rho \left( D_x + E \right) G''' = 0 \]  \hfill (3.4.23)

Letting \( G'''(x) = R(x) \), then above equation reduces to
\[ \mu \, R' - \rho \left( D_x + E \right) = 0 \]  \hfill (3.4.24)

This equation has a general solution given by
\[ R(x) = c \, \exp \left( \frac{\rho}{\mu} \left( \frac{1}{2} \right) D_x^2 + E \right) \]
where \( c \) is an arbitrary constant.

Since \( G'''(x) = R(x) \), then
\[ G(x) = \iiint c \, \exp \left( \frac{\rho}{\mu} \left( \frac{1}{2} \right) D_x^2 + E \right) \, dx \, dx \, dx \]  \hfill (3.4.25)

and the streamfunction \( \psi(x, y) \) is given by
\[ \psi(x, y) = \left( D_x + E \right) y + \iiint c \, \exp \left( \frac{\rho}{\mu} \left( \frac{1}{2} \right) D_x^2 + E \right) \, dx \, dx \, dx \]  \hfill (3.4.26)

### 3.4.3. Solution of Riabouchinsky

We consider the special form of the streamfunction \( \psi(x, y) \) defined in (3.4.1), where \( G = 0 \). Riabouchinsky (1924) was the first one to consider this special form of the streamfunction in order to investigate the solutions of ordinary Non-MHD fluid dynamics.

Physically, such a flow represents a plane stagnation flow in which the flow is separated in the two symmetrical regions by the plane \( x = 0 \).

When \( F = F_1 \), then the streamfunction \( \psi(x, y) \) takes the form
\[ \psi(x, y) = -6 \left( \frac{\rho}{\mu} \right) y \, x^{-1} \]  \hfill (3.4.27)
Using equation (3.4.8) we find that the pressure is given by

\( p = -10 \left( \mu^2 / \rho \right) x^{-2} - \left( 1/2 \right) \mu^* H^2 + p_o \)  \hspace{1cm} (3.4.28)

where \( H \) is given by equation (3.4.7).

When \( F = F_2 \), we find that the streamfunction \( \psi(x,y) \) reduces to

\( \psi(x,y) = (\mu / \rho) \left[ 1 + k e^{ax} \right] y \)  \hspace{1cm} (3.4.29)

On using (3.4.29) and (3.4.4) into (3.3.2), we obtain

\[
p^* = \left[ - (\mu / \nu) + \rho \right] \frac{U^2}{\nu} \exp \left[ - \left( U_0 / \nu \right) x \right] -
\left( 1/2 \right) \rho \frac{U^2}{\nu} \left( \exp \left[ - 2 \left( U_0 / \nu \right) x \right] + 1 \right) + p_o
\]  \hspace{1cm} (3.4.30)

where \( U_0 = - \nu a, \ \nu = \mu / \rho \)

Using (3.4.29) into the diffusion equation (3.4.6), we find that a particular solution of this equation is given by

\( H(x,y) = y \exp \left[ - \left( U_0 / \nu \right) x \right] + \left( 1 / \nu \right) \left[ (1 / \mu^* \sigma) - \nu \right] y \)  \hspace{1cm} (3.4.31)

Employing (3.4.30) and (3.4.31) into (3.3.8), we find that the pressure distribution is given by

\[
p = \left[ - (\mu / \nu) + \rho \right] \frac{U^2}{\nu} \exp \left[ - \left( U_0 / \nu \right) x \right] + p_o -
\left( 1/2 \right) \rho \frac{U^2}{\nu} \left[ \exp \left[ - 2 \left( U_0 / \nu \right) x \right] + 1 \right] -
\left( 1/2 \right) \mu^* \left[ y \exp \left[ - \left( U_0 / \nu \right) x \right] + \left( 1 / \nu \right) \left[ (1 / \mu^* \sigma) - \nu \right] \right]^2
\]  \hspace{1cm} (3.4.32)
When \( F = F_0 \), then equation (3.4.1) with \( G = 0 \) yields

\[
\psi (x, y) = (D x + E)y
\]  \hspace{1cm} (3.4.33)

Using (3.4.33) in equation (3.4.8), we find that the pressure is given by

\[
p = \mu D^2 - (1/2) \rho (D x + E)^2 + p_0 - (1/2) \mu^0 H^2
\]

where \( D \) and \( E \) are arbitrary constants and \( H \) is given by (3.4.7).
3.5. Solution of Riabouchinsky Type Problems.

Second-Grade Fluids.

3.5.1. Generalities

We assume that the streamfunction is linear with respect to \( y \), that is to say it has the form

\[
\psi(x,y) = y \, F(x) + G(x) \tag{3.5.1}
\]

where \( F(x) \) and \( G(x) \) are two arbitrary functions of \( x \).

Substituting (3.5.1) into compatibility equation (3.3.31), we find that the arbitrary functions \( F(x) \) and \( G(x) \) must satisfy the following two equations

\[
\mu F'''' + \rho \left( F' F'' - F F''' \right) - \alpha_1 \left( F' F^{(v)} - F F^{(v)} \right) = 0 \tag{3.5.2}
\]

\[
\mu G'''' + \rho \left( F'' G' - F G''' \right) - \alpha_1 \left( F'' G^{(v)} - F G^{(v)} \right) = 0 \tag{3.5.3}
\]

Integration of these two equations yields:

\[
\mu F'''' + \rho \left( F' F'' - F F''' \right) - \alpha_1 \left( -F F^{(iv)} + 2 F' F''' - F''^2 \right) = A \n\]

\[
\mu G'''' + \rho \left( F'' G' - F G''' \right) - \alpha_1 \left( -F G^{(iv)} + F' G''' - F'' G'' + F''' G' \right) = B \n\]

where \( A \) and \( B \) are arbitrary constants of integration.

Employing (3.5.1) into linear momentum equations (3.3.28) and (3.3.29), we find that

\[
h = - \frac{1}{2} y^2 \left[ A - \rho F' F'' + 2 \alpha_1 F' F''' - \alpha_1 F''^2 \right] + p_0 - \]

\[
y \left[ B - \rho F' G' + \alpha_1 F' G''' - \alpha_1 F'' G' + \alpha_1 F''' G' \right] + \]

\[ - 45 \]
\[ (1/2) \rho G'\varepsilon + \mu F' - \alpha_1 \frac{G'}{\varepsilon} G' + (1/2) \alpha_1 G'/2 \]  

(3.5.4)

where \( p_0 \) is an arbitrary constant.

Substituting (3.5.1) and (3.5.4) into (3.3.19), we find that

\[ p^* = \mu F' - (1/2) \rho F^2 + \alpha_1 \left( (1/2) y^2 F'/2 + y F'/G' + (1/2) G'/2 \right) \]

\[ + \frac{F F'}{1 + (3 \alpha_1 + 2 \alpha_2)/2} \left[ (1 F^2 + y^2 F'/2 + 2 y F'/G' + G'/2) \right] \]

\[ - \left( (1/2) A y^2 + B y \right) + p_0 \]

(3.5.5)

On assuming the magnetic forces are present, we find, by employing (3.5.5) into (3.3.25), that the pressure distribution is given by

\[ p = \mu F' - (1/2) \rho F^2 + \alpha_1 \left( (1/2) y^2 F'/2 + y F'/G' + (1/2) G'/2 \right) \]

\[ + \frac{F F'}{1 + (3 \alpha_1 + 2 \alpha_2)/2} \left[ (1 F^2 + y^2 F'/2 + 2 y F'/G' + G'/2) \right] \]

\[ - \left( (1/2) A y^2 + B y \right) + p_0 - (1/2) \mu^* H^2 \]

(3.5.6)

where \( H \) can be found by solving the diffusion equation (3.3.30).

If the fluid domain is considered to extend to infinity, then in order that the pressure remains bounded there, we must take \( A = B = 0 \).

If the fluid is infinitely electrically conducting, that is \( \alpha \) tends to infinity then \((1/\mu^* \alpha)\) tends to zero and the diffusion equation reduces to

\[ u H_x + v H_y = 0 \]  

(\( A \))

By the definition of the streamfunction \( \psi (x,y) \), we have

- 46 -
\[ u = \psi_y \quad \text{and} \quad v = -\psi_x \]

Employing these results, equation (A) yields

\[ \frac{\partial (H, \psi)}{\partial (x, y)} = 0 \]

From this equation, either

\[ H = \text{constant} \quad \text{OR} \quad H = H(\psi) \]

where \( H(\psi) \) is an arbitrary function of \( \psi \).

### 3.5.2. Particular Solutions

Equation (3.5.2) admits the following two particular solutions:

\[ F_1 = \left[ \mu (\rho - \alpha_1 a^2) \right] (1 + k e^{ax}), \quad F_2 = D x + E \quad \text{(3.5.7)} \]

where \( a, k, D \) and \( E \) are arbitrary constants.

I) When \( F = F_1 \), then equation (3.5.3) takes the form:

\[ G^{(iv)} - \alpha_1 \left[ \frac{a^5 k e^{ax}}{\rho - \alpha_1 a^2} G' - \frac{a (1 + k e^{ax})}{\rho - \alpha_1 a^2} G^{(v)} \right] + \]

\[ + \rho \left[ \frac{a^3 k e^{ax}}{\rho - \alpha_1 a^2} G' - \frac{a (1 + k e^{ax})}{\rho - \alpha_1 a^2} G^{(v)} \right] = 0 \quad \text{(3.5.8)} \]

Without loss of generality, we can take \( a = 1 \) and equation (3.5.8) becomes

\[ G^{(iv)} - \alpha_1 \left[ \frac{k e^{x}}{\rho - \alpha_1} G' - \frac{1 + k e^{x}}{\rho - \alpha_1} G^{(v)} \right] + \]

\[ + \rho \left[ \frac{k e^{x}}{\rho - \alpha_1} G' - \frac{1 + k e^{x}}{\rho - \alpha_1} G^{(v)} \right] = 0 \quad \text{(3.5.9)} \]
Next, we try to make a series of substitutions of the type
\[ G'(x) = M(x), \quad M(x) = N(x) e^x, \quad N'(x) = R(x) \] (3.5.10)
in order to lower the order of equation (3.5.9) as well as to obtain a solvable form.

Employing substitutions (3.5.10) in (3.5.9), we find that the function \( R(x) \) satisfies the following differential equation
\[
\alpha_1 \left[ (1 + k e^x) R'''' + (3 + 4 k e^x) R'' + (3 + 6 k e^x) R' + (1 + 4 k e^x) R \right] + \rho [R'' + (2 - k e^x) R'] + (1 - 2 k e^x) R = 0
\] (3.5.11)

Multiplying equation (3.5.11) by \( e^x \) and integrating once we can further reduce the order of this equation. The obtained equation is
\[
\alpha_1 (1 + k e^x) R'''' + [2 \alpha_1 + \rho + 2 \alpha_1 k e^x] R' + [\alpha_1 + \rho + (2 \alpha_1 - \rho) k e^x] R = c e^{-x}
\] (3.5.12)

Taking \( c = 0 \), then the solution of (3.5.12), which turns out to be in terms of the hypergeometric function \( F(\alpha, \beta, \gamma, x) \), is given by
\[
R_1(x) = e^{-x} c_1 F(\alpha, \beta, \gamma, me^x)
\] (3.5.13)

\[
R_2(x) = e^{-x} c_2 [m e^x]^{-(\rho/\alpha_1)} F(\alpha + \gamma - 1, \beta + \gamma - 1, 2 - \gamma, me^x)
\]

where \( \alpha = \sqrt{(\rho + \alpha_1)/\alpha_1}, \beta = \sqrt{(\rho - \alpha_1)/\alpha_1}, \gamma = (\rho + \alpha_1)/\alpha_1 \) and \( m = -k \) (3.5.14)

The streamfunction \( \psi(x,y) \) defined by (3.5.1) is, thus, given by

\[ -48 - \]
\[ \psi(x, y) = \phi[\mu/(\rho - \alpha_1)] (1 + k e^x) y + \int e^x \left( \int e^{-x} c_1 F(\alpha, \beta, \gamma, me^x) + e^{-x} c_2 (m e^x)^{-\frac{\rho}{\alpha_1}} F(\alpha+1, \beta+1, 2-\gamma, me^x) \right) dx \right) dx + k_1 e^x + k_2 \]

where \( c_1, c_2, k_1 \) and \( k_2 \) are four arbitrary constants.

In the special case where \( k = 0 \), then \( F = \frac{\mu}{\rho - \alpha_1} \) and equation (3.5.12) is modified to

\[ \alpha_1 R' + (2 \alpha_1 + \rho) R' + (\alpha_1 + \rho) R = c e^{-x} \]  

(3.5.16)

The complementary solution of this equation is

\[ R_c = A_1 e^{-x} + B_1 e^{-(\alpha_1 + \rho)/\alpha_1} x \]  

(3.5.17)

Using the variation of parameters method, we find that the particular solution of equation (3.5.16) is given by

\[ R_p = \frac{c}{\rho} e^{-x} [x - (\alpha_1 / \rho)] \]  

(3.5.18)

wherein \( c \) is a constant.

Hence, the general solution of (3.5.16) is

\[ R(x) = R_c + R_p = A_1 e^{-x} + B_1 e^{-(\alpha_1 + \rho)/\alpha_1} x + \left( \frac{c}{\rho} e^{-x} [x - (\alpha_1 / \rho)] \right) \]  

(3.5.19)

where \( A_1 \) and \( B_1 \) are arbitrary constants.

Recalling substitutions (3.5.10), we find that

\[ G(x) = [- (A_1 + (c/\rho)) x + (c / \alpha_1) x] [x - (c/\rho) \left( x^2 / 2 \right) + \left( B_1 / (\alpha_1 + \rho) \right) (\alpha_1^2 / \rho) e^{-\rho/\alpha_1} x] \]  

(3.5.20)

and, thus, the streamfunction takes the form
\[
\psi(x, y) = [\mu/(\rho - \alpha_1)] y + [- (A_1 + (c/\rho)) x + (c \alpha_1 \rho^2)] x - \\
- (c/\rho) (x^2/2) + [B_1/(\alpha_1 + \rho)] (\alpha_1^2/\rho) e^{-(\rho/\alpha_1)x} \quad (3.5.21)
\]

II) When \( F = F_2 \), then equation (3.5.3) becomes

\[
a_1 (Dx+E) G^{(v)} + \mu G^{(i v)} - \rho (Dx+E) G''' = 0 \quad (3.5.22)
\]

Letting \( G'''(x) = R(x) \), then we find that \( R(x) \) satisfies the following equation

\[
a_1 (Dx+E) R'' + \mu R' - \rho (Dx+E) R = 0 \quad (3.5.23)
\]

The solution of (3.5.23), which turns out to be in terms of the hypergeometric function \( F(\alpha, \beta, x) \), is given by

\[
R_1(x) = \exp[\pm 2 i ((E/\rho D) + x)] c_1 F(\alpha, \beta, \pm 2 i ((E/\rho D) + x)) \quad (3.5.24)
\]

\[
R_2(x) = \exp[\pm 2 i ((E/\rho D) + x)] c_2 F(1-\alpha-\beta, 2-\beta, \pm 2 i ((E/\rho D) + x))
\]

where

\[
\alpha = \mu i/\rho D \quad \beta = - \mu/\rho D \quad (3.5.25)
\]

Since \( G'''(x) = R(x) \), then

\[
G(x) = \iiint [\exp(\pm 2 i ((E/\rho D) + x)) c_1 F(\alpha, \beta, \pm 2 i ((E/\rho D) + x)) + \\
+ c_2 F(1-\alpha-\beta, 2-\beta, \pm 2 i ((E/\rho D) + x))] \, dx \, dx \, dx \quad (3.5.26)
\]

Hence, the streamfunction \( \psi(x, y) \) defined by (3.5.1) has the form

\[
\psi(x, y) = (Dx+E)y + \iiint [\exp(\pm 2 i ((E/\rho D) + x)) c_1 F(\alpha, \beta, \pm 2 i ((E/\rho D) + x)) + \\
+ c_2 F(1-\alpha-\beta, 2-\beta, \pm 2 i ((E/\rho D) + x))] \, dx \, dx \, dx \quad (3.5.27)
\]
3.8.3. Solution of Riabouchinsky

We now consider the particular case of (3.5.1), where \( G = 0 \), usually attributed to Riabouchinsky (1924). Physically such a flow represents a plane stagnation in which the flow is separated in the two symmetrical regions by the plane \( x = 0 \).

When \( F = F_1 \), then streamfunction \( \psi(x,y) \) is given by

\[
\psi(x,y) = \frac{\mu a}{(\rho^2 - \alpha_1^2a^2)} (1 + k e^{ax}) y 
\]  

(3.5.28)

On using (3.5.28) and (3.5.4) into (3.3.19), we find that

\[
p^* = -\left(\frac{\mu}{s\nu} + \frac{(\rho/s^2)}{}\right) u_0^2 \exp(-u_0/\nu)x - (1/2)\rho(u_0^2/s^2)^* 
\]

\[
\times \left[ \exp(-2u_0/\nu)x + 1 \right] + (2\alpha_1 + \alpha_2) y^2(u_0^6/s^2\nu^4) \exp(-2u_0/\nu)x + 
\]

\[
+ (7\alpha_1 + 4\alpha_2) (u_0^4/s^2\nu^2) \exp(-2u_0/\nu)x - \alpha_1 (u_0^4/s^2\nu^2) \exp(-u_0/\nu)x 
\]

\[
+ p_0 
\]  

(3.5.29)

wherein

\[
u = \nu a, \quad \nu = \mu/\rho, \quad s = 1 - (\alpha_1/\rho) a^2 \quad k = -1 
\]  

(3.5.30)

Employing (3.5.28) into the diffusion equation (3.3.30), we find that one particular solution of this equation is given by

\[
H(x,y) = \frac{1}{s} y \exp(-u_0/\nu)x + (1/\nu) \left[ (1/\mu^*a) - (\nu/s) \right] y 
\]  

(3.5.31)

Substituting (3.5.29) and (3.5.31) into (3.3.25), and assuming that the magnetic forces are present, we find that the pressure distribution is given by

- 51 -
\[ p = \left[ \left( \frac{\mu}{s^2} \right) + \left( \frac{\rho}{s^2} \right) \right] \frac{U_0^2}{U_0} \exp\left(-\frac{U_0}{\nu} x \right) - \left( 1/2 \right) \rho \left( \frac{U_0^2}{s^2} \right) \left( \exp\left(-2\left( \frac{U_0}{\nu} \right) x \right) + 1 \right) + \left( 2\alpha_1 + \alpha_2 \right) \left( \frac{U_0^2}{s^2} \nu^4 \right) \exp\left(-2\left( \frac{U_0}{\nu} \right) x \right) + \left( 7\alpha_1 + 4\alpha_2 \right) \left( \frac{U_0^4}{s^2} \nu^2 \right) \exp\left(-2\left( \frac{U_0}{\nu} \right) x \right) - \alpha_4 \left( \frac{U_0^4}{s^2} \nu^2 \right) \exp\left(-\left( \frac{U_0}{\nu} \right) x \right) \]

\[ + p_0 - \left( 1/2 \right) \mu^* \left( \frac{1}{s^2} \right) \frac{y}{\nu} \exp\left(-\left( \frac{U_0}{\nu} \right) x \right) + \left( 1/\nu \right) \left( \frac{1}{\mu^*} \right) - \left( \frac{\nu}{s^2} \right) y \right)^2 \]

\[ (3.5.32) \]

If we set \( k = -1 \) and if we assume that \( U_0 > 0 \), we find that the velocity components, on using (3.5.28) in (3.3.26), can be written as

\[ u = \left( \frac{U_0}{s} \right) \left( \exp\left(-\left( \frac{U_0}{\nu} \right) x \right) - 1 \right), \quad v = \left( \frac{U_0^2}{s\nu} \right) \frac{y}{\nu} \exp\left(-\left( \frac{U_0}{\nu} \right) x \right) \]

\[ (3.5.33) \]

Accordingly, at \( x = 0 \) we have \( u = 0 \) and \( v = \frac{U_0^2}{s\nu} \) where as \( x \) tends to infinity, \( u = -\frac{U_0}{s} \) and \( v \) tends to zero. Thus provided that \( s > 0 \), i.e. \( \rho > \alpha_4 \) we have two streams of fluid coming from infinity in opposite directions towards the plane \( x = 0 \). If the non-Newtonian parameter, however is such that \( \alpha_4 \rho / \alpha^2 \) then the direction of the flow is reversed. Instead of the fluid coming towards the plane \( x = 0 \), the fluid will flow away towards infinity. Thus depending upon the magnitude of \( \alpha_4 \) we have the flow possibilities in both directions which is not the case in the viscous fluids. We remark that the pressure distribution is not affected by the sign of \( s \).

When \( F = F_2 \), the streamfunction \( \psi(x,y) \) takes the form

\[ \psi(x,y) = (Dx + Ey) \]

\[ (3.5.34) \]

On using (3.5.32) in (3.5.5), we find that

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\[ p^* = \frac{1}{2} \rho (Dx + E)^2 + 2(3\alpha_1 + 2\alpha_2)D^2 + p_0, \quad (3.5.35) \]

Employing (3.5.34) into the diffusion equation (3.3.30), we find one particular solution of this equation is given by

\[
H(x,y) = k_1 \int \exp(\alpha^*[(1/2)D x^2 + E x]) \, dx + \\
+ k_2 \int \exp(- \alpha^*(1/2)D y^2) \, dy + k_3 \quad (3.5.36)
\]

wherein \(k_1, k_2\) and \(k_3\) are three arbitrary constants.

Substituting (3.5.35) and (3.5.36) into (3.3.25), we obtain that the pressure distribution turns out to be

\[
p = - \frac{1}{2} \rho (Dx + E)^2 + 2(3\alpha_1 + 2\alpha_2)D^2 + p_0 - \\
- \frac{1}{2} \mu^* \left[ k_1 \int \exp(\alpha^*[(1/2)D x^2 + E x]) \, dx + \\
+ k_2 \int \exp(- \alpha^*(1/2)D y^2) \, dy + k_3 \right]^2 \quad (3.5.37)
\]

Finally, we find that the velocity components, for this choice of \(F\), are

\[
u = -Dy
\]

\[
v = -Dy
\]

Remark: In comparison to the viscous case, that is \(\alpha_1 = \alpha_2 = 0\), done in section 3.4, we note that the pressure distribution is considerably modified in a Non-Newtonian or second grade fluid.
CHAPTER 4

PLANE ALIGNED (PARALLEL) FLOWS

4.1. Introduction

Magneto-hydro-dynamics (MHD) plane flows are said to be aligned or parallel flows if the magnetic field is everywhere parallel to the velocity field and all the flow variables are functions of the rectangular coordinates \( x \) and \( y \) only.

By the definition of aligned flows the magnetic field vector \( \mathbf{H} \) and the velocity vector \( \mathbf{V} \) are related by \( \mathbf{H} = f(x,y) \mathbf{V} \) where \( \mathbf{V} \cdot \mathbf{grad} f = 0 \). We find that, in regions of incompressible flow where \( f^2 - \mu p = 0 \), the two flows must be strictly proportional. Exceptions to this occur when (i) in plane flows streamlines are circles or parallel straight lines, (ii) in axially symmetric flow the flow is purely axial, (iii) in both irrotational spatial and doubly laminar spatial flows the flow is helical.

In this chapter, we study the Riabouchinsky type problems for plane aligned flows applying to both ordinary viscous fluids and second grade fluids. Also, we consider finitely conducting fluids as well as infinitely conducting. This is the first work that has been done on Riabouchinsky type problems for aligned flows.

This chapter contains five sections which are as follows. In section 4.2 the basic equations of motion of an ordinary viscous fluid as well as of a second grade fluid are transformed into
more convenient form for this work. Section 4.3 contains the derivation of the compatibility or integrability equation of an ordinary viscous fluid as well as of a second grade fluid. Section 4.4 consists of the solution of Riabouchinsky type problems, that is, assuming that the streamfunction is linear with respect to x or y, for an ordinary viscous fluid. Lastly, section 4.5 contains the solution of Riabouchinsky type problems for a second-grade fluid.
4.2. Equations of motion

The basic equations governing the steady plane motion of a viscous incompressible fluid of finite electrical conductivity, in the presence of a magnetic field, are

\[ \text{div } \mathbf{V} = 0 \quad (i) \]

\[ \rho \text{(V·grad) V + gradp} = \mu \nabla^2 \mathbf{V} + \mu^* (\text{curl H}) \times \mathbf{H} \quad (ii) \]

\[ \text{curl} (\mathbf{V} \times \mathbf{H}) + (1/\mu^*) \nabla^2 \mathbf{H} = 0 \quad (iii) \]

\[ \text{div } \mathbf{H} = 0 \quad (iv) \]

wherein \( \mathbf{V} \) denotes the velocity field vector, \( \mathbf{H} \) the magnetic field vector, \( p \) the fluid pressure function, \( \rho \) the constant fluid density, \( \sigma \) the electrical conductivity, \( \mu \) the coefficient of viscosity and \( \mu^* \) the constant magnetic permeability.

This a system of eight equations in seven unknowns, being \( \mathbf{V}, \mathbf{H} \) and \( p \). Equation \((iv)\) is an additional equation on \( \mathbf{H} \) expressing the absence of magnetic poles in the flow.

4.2.1. Ordinary Viscous Fluids

Assuming that the flows are aligned flows, that is \( \mathbf{H} \) is parallel to \( \mathbf{V} \) everywhere, then

\[ \mathbf{H} = f(x,y) \mathbf{V} \quad (4.2.1) \]

where \( f(x,y) \) is some scalar function.

Making use of \((4.2.1)\), we find that

\[ (\text{curl H}) \times \mathbf{H} = [-f^2v(x,y) - v^2f_x + uvff_y, uf^2v_x - uvff_y, 0] \quad (4.2.2) \]
\[ \text{curl}(\nabla \times \mathbf{H}) = 0 \]

\[ \nabla^2 H = [\nabla^2 (fu), \nabla^2 (fv), 0] \]

Employing (4.2.1) in equation (iv), we obtain

\[ uf_x + vf_y = 0 \]  

Using results (4.2.2) to (4.2.5) in the system of equations (i) to (iv), we find that the flow equations governing the motion of a steady plane aligned flow of a viscous incompressible fluid of finite electrical conductivity, in the presence of a magnetic field, are given by

\[ u_x + v_y = 0 \]

\[ \rho \left[ \frac{\partial}{\partial x} \left( \frac{u^2 + v^2}{2} \right) - v(v_x - u_y) \right] + \frac{\partial}{\partial x} p = \mu \nabla^2 u - \mu^* f (u^2 + v^2) f_x - \mu^* f^2 v (v_x - u_y) \]  

\[ \rho \left[ \frac{\partial}{\partial y} \left( \frac{u^2 + v^2}{2} \right) + u(v_x - u_y) \right] + \frac{\partial}{\partial y} p = \mu \nabla^2 v - \mu^* f (u^2 + v^2) f_y + \mu^* f^2 u (v_x - u_y) \]  

\[ \nabla^2 (fu) = \nabla^2 (fv) = 0, \quad (1/\mu^* \sigma) \neq 0 \]

\[ uf_x + vf_y = 0 \]

\[ \mathbf{H} = f(x,y) \mathbf{V} \]
The diffusion equations (4.2.9) are identically satisfied when the fluid is infinitely electrically conducting so that \((1/\mu^*\sigma)\) tends to zero. However, for finitely conducting flows, the functions \(f, u\) and \(v\) must satisfy these additional equations.

4.2.2. Second-Grade Fluids:

Considering the magnetic field to be acting parallel to the velocity field, then there exists some scalar function \(\Gamma(x,y)\) such that

\[ H = \Gamma(x,y) V \]

Introducing this definition and employing results (4.2.1) to (4.2.5) into the system of equations (2.1.20) to (2.1.23), then the steady plane motion of aligned flows of a second grade fluid of finite electrical conductivity, in the presence of the magnetic field is governed by the system

\[ u_x + v_y = 0 \quad (4.2.12) \]

\[
\rho \left[ \frac{\partial}{\partial x} \left( \frac{u^2 + v^2}{2} \right) - v(v_x - u_y) + \frac{\partial}{\partial x} p \right] = \mu \nu^2 u - \\
- \mu^* f(u^2 + v^2)f_x - \mu^* f^2 v(v_x - u_y) + \alpha_1 \left[ (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}) \nu^2 u + \frac{\partial u}{\partial x} \nu^2 u + \frac{\partial v}{\partial x} \nu^2 v \right] + \frac{1}{4} (3\alpha_1 + 2\alpha_2) \frac{\partial}{\partial x} |A_1|^2 \quad (4.2.13)
\]
\[ \rho \left[ \frac{\partial}{\partial y} \left( \frac{u^2 + v^2}{2} \right) + u(v_y - u_y) \right] + \frac{\partial}{\partial y} p = \mu \nu^2 \nu - \]

\[ - \mu^* \nu(u^2 + v^2) f_y + \mu^* \nu^2 u(v_x - u_y) + \alpha \left[ (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}) \nu^2 \nu + \right. \]

\[ + \frac{\partial u}{\partial y} \nu^2 u + \frac{\partial v}{\partial y} \nu^2 v \right] + \frac{1}{4} (3\alpha_1 + 2\alpha_2) \frac{\partial}{\partial y} |A_1|^2 \quad (4.2.14) \]

\[ \nabla^2 (fu) = \nabla^2 (fv) = 0, \quad (1/\mu^* \nu) \neq 0 \quad (4.2.15) \]

\[ u f_x + v f_y = 0 \quad (4.2.16) \]

\[ H = f(x, y) \nu \quad (4.2.17) \]

where

\[ |A_1|^2 = 4(u_x)^2 + 4(v_x)^2 + (v_x + u_y)^2 \quad (4.2.18) \]

If the fluid is infinitely conducting so that (1/\mu^* \nu) tends to zero, then the diffusion equations (4.2.15) are identically satisfied. However, for finitely electrically conducting flows, the functions f, u and v must satisfy these additional conditions.
4.3. Compatibility Equations

In this section, we derive the compatibility or integrability equation for aligned flows of an ordinary viscous fluid as well as a second grade fluid by employing the flow equations for these flows in section 4.2. Motions where the streamfunction is linear with respect to \( x \) or \( y \) and the solution of Riabouchinsky are investigated by using these compatibility equation in the next two sections.

4.3.1. Ordinary Viscous Fluids.

Introducing the vorticity function

\[
\omega = v_x - u_y
\]

and the generalized pressure function

\[
h = \frac{1}{2} \rho q^2 + p
\]

in equations (4.2.6) to (4.2.11), where \( q^2 = u^2 + v^2 \), then aligned flows are governed by

\[
\begin{align*}
  u_x + v_y &= 0 \quad (4.3.3) \\
  h_x &= -\mu \omega_y + \rho v \omega - \mu^f (u^2 + v^2) f_x - \mu^f v^2 \omega \quad (4.3.4) \\
  h_y &= \mu \omega_x - \rho u \omega - \mu^f (u^2 + v^2) f_y + \mu^f u^2 \omega \quad (4.3.5) \\
  v_x - u_y &= \omega \quad (4.3.6) \\
  u f_x + v f_y &= 0 \quad (4.3.7) \\
  \nabla^2 (fu) = \nabla^2 (fv) &= 0 \quad (4.3.8)
\end{align*}
\]
This system has the advantage that the linear momentum equations (4.3.4) and (4.3.5) are of first order.

Let \( \psi(x,y) \) be the streamfunction defined by the continuity equation, that is

\[
\mathbf{u} = \psi_y, \quad \mathbf{v} = -\psi_x
\]  \hspace{1cm} (4.3.9)

Using this definition of the streamfunction, we can see clearly that equation (4.3.3) vanishes, while the vorticity function is given by

\[
\omega = -\nabla^2 \psi
\]  \hspace{1cm} (4.3.10)

Employing (4.3.9) and (4.3.10) into equations (4.3.4), (4.3.5) and (4.3.7), these equations yields:

\[
\frac{\partial h}{\partial x} - \rho \frac{\partial \psi}{\partial x} \nabla^2 \psi = \mu \frac{\partial}{\partial y} (\nabla^2 \psi) - \mu \mathbf{f} \left[ \left( \frac{\partial \psi}{\partial y} \right)^2 + \left( \frac{\partial \psi}{\partial x} \right)^2 \right] f_x - \mu f^2 \frac{\partial \psi}{\partial x} \nabla^2 \psi
\]  \hspace{1cm} (4.3.11)

\[
\frac{\partial h}{\partial y} - \rho \frac{\partial \psi}{\partial y} \nabla^2 \psi = -\mu \frac{\partial}{\partial x} (\nabla^2 \psi) - \mu \mathbf{f} \left[ \left( \frac{\partial \psi}{\partial y} \right)^2 + \left( \frac{\partial \psi}{\partial x} \right)^2 \right] f_y - \mu f^2 \frac{\partial \psi}{\partial y} \nabla^2 \psi
\]  \hspace{1cm} (4.3.12)

\[
\frac{\partial \psi}{\partial y} \frac{\partial \h}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial h}{\partial y} = 0
\]  \hspace{1cm} (4.3.13)

Differentiating (4.3.11) with respect to 'y' and (4.3.12) with respect to 'x' and employing the integrability condition
\[ \frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x} \] we find that the compatibility equation for the aligned flows of an ordinary viscous fluid is given by

\[ \rho \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x, y)} + \mu \nabla^4 \psi - \mu^f \frac{1}{\partial (x, y)^2} \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x, y)} + \mu^f \frac{\partial (|\text{grad}\psi|^2, f)}{\partial (x, y)} = 0 \]

(4.3.14)

Solving equation (4.3.14), we can find \( \psi(x, y) \). Having found \( \psi(x, y) \), the velocity components and the vorticity can be found by using equations (4.3.9) and (4.3.10) respectively.

Finally, functions \( h, p \) and \( f \) can be found by using equations (4.3.2), (4.3.11), (4.3.12) and (4.3.13).

4.3.2. Second-Grade fluids

Introducing the vorticity function

\[ \omega = \nabla \times \mathbf{v} = \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \]

we find that

\[ \nabla^2 u = -\omega_y, \quad \nabla^2 v = \omega_x \]

(4.3.15)

(4.3.16)

Next, we arrange the \( \alpha_1 \)-terms of equations (4.2.13) and (4.2.14) in order to obtain more convenient forms. First, consider the \( \alpha_1 \)-term of equation (4.2.13)

\[ \alpha_1 [(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}) \nabla^2 u + \frac{\partial u}{\partial x} \nabla^2 u + \frac{\partial v}{\partial x} \nabla^2 v] \]

\[ \begin{aligned} &+ (u \frac{\partial \nabla^2 u}{\partial x} + v \frac{\partial \nabla^2 u}{\partial y}) + \frac{\partial u}{\partial x} \nabla^2 v + \frac{\partial v}{\partial x} \nabla^2 v \end{aligned} \]
\[
\begin{align*}
&= \alpha_1 \left[ \frac{\partial}{\partial x} (u \nabla^2 u) + \frac{\partial v}{\partial x} \nabla^2 v + v \left( -\frac{\partial^2 \omega}{\partial y^2} \right) \right] \\
&= \alpha_1 \left[ \frac{\partial}{\partial x} (u \nabla^2 u + v \nabla^2 v) - v \nabla^2 \omega \right]
\end{align*}
\]

Then consider \( \alpha_1 \)-term of (4.2.14)

\[
\begin{align*}
&= \alpha_1 \left[ (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}) \nabla^2 v + \frac{\partial u}{\partial y} \nabla^2 u + \frac{\partial v}{\partial y} \nabla^2 v \right] \\
&= \alpha_1 \left[ \frac{\partial}{\partial y} (v \nabla^2 v) + u \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial u}{\partial y} \nabla^2 u \right] \\
&= \alpha_1 \left[ \frac{\partial}{\partial y} (u \nabla^2 u + v \nabla^2 v) + u \nabla^2 \omega \right]
\end{align*}
\]

Employing these results and introducing the generalized pressure function

\[
h = \frac{1}{2} \rho q^2 + p - \alpha_1 (u \nabla^2 u + v \nabla^2 v) - \frac{1}{4} (3\alpha_1 + 2\alpha_2) |A_1|^2 \quad (4.3.17)
\]

in equations (4.2.12) to (4.3.18), where \( q^2 = u^2 + v^2 \), then the aligned flows of a second grade fluid are governed by

\[
u_x + v_y = 0 \quad (4.3.18)
\]

\[
\begin{align*}
\nabla^2 u &= \rho v \omega = -\mu \omega_y - \mu^* f(u^2 + v^2) f_x - \mu^* f^2 \omega v - \alpha_1 \nabla^2 \omega \quad (4.3.19) \\
\nabla^2 v &= \rho u \omega = \mu \omega_x - \mu^* f(u^2 + v^2) f_y + \mu^* f^2 u \omega + \alpha_1 u \nabla^2 \omega \quad (4.3.20)
\end{align*}
\]

\[
\nabla^2 (fu) = \nabla^2 (fy) = 0, \quad \frac{1}{\mu^* \sigma} \neq 0 \quad (4.3.21)
\]

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\[ uf_x + vf_y = 0 \quad (4.3.22) \]
\[ v_x - u_y = \omega \quad (4.3.23) \]

Introducing the streamfunction \( \psi(x,y) \) defined by the continuity equation, that is

\[ u = \psi_y, \quad v = -\psi_x \quad (4.3.24) \]

and using the integrability condition \( \frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x} \), the compatibility equation of aligned flows for a second grade fluid is given by

\[
\rho \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x,y)} + \mu \nabla^4 \psi - \mu f^2 \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x,y)} + \mu^* f^2 \frac{\partial |\nabla \psi|^2}{\partial (x,y)} - \alpha_1 \frac{\partial (\psi, \nabla^4 \psi)}{\partial (x,y)} = 0 \quad (4.3.24)
\]

Solving the compatibility equation, we can find the streamfunction \( \psi(x,y) \). Having found \( \psi(x,y) \), the velocity components and the vorticity are given by

\[ u = \psi_y, \quad v = -\psi_x, \quad \omega = -\nabla^2 \psi \quad (4.3.25) \]

Lastly, the functions \( h, p \) and \( f \) can be found by solving equations (4.3.17), (4.3.19), (4.3.20) and (4.3.22).
4.4. Solution of Riabouchinsky Type Problems

Ordinary Viscous Fluids.

Since the compatibility equations are very difficult to solve, different researchers over the past years have assumed different forms of the streamfunction \( \psi(x,y) \) in terms of arbitrary functions and they tried to find these arbitrary functions. In the Riabouchinsky type problems, we assume that the streamfunction is a linear function of \( x \) or \( y \).

For this particular case of aligned flows, recalling equation (4.3.13) we find that

\[
\frac{\partial (f, \psi)}{\partial (x, y)} = 0
\]

From this equation, either

\[ f(x, y) = \text{constant} \quad \text{OR} \quad f = f(\psi) \]

where \( f(\psi) \) is an arbitrary function of \( \psi \).

Hence, we have to consider two different cases.

CASE 1

We take \( f = c \), where \( c \) is an arbitrary constant.

A) Generalities.

Assume that the streamfunction is of the form

\[
\psi(x,y) = yF(x) + G(x)
\]  
(4.4.1)

where \( F(x) \) and \( G(x) \) are arbitrary functions of \( x \).

Substituting (4.4.1) into the compatibility equation (4.3.14) with the assumption that \( f = c \), we find that \( F(x) \) and \( G(x) \) must satisfy the following equations:
\[ \mu F^{(iv)} + (\rho - \mu^* c^2)(F'F'' - FF''') = 0 \]  
(4.4.2) 

\[ \mu G^{(iv)} + (\rho - \mu^* c^2)(F''G' - FG'') = 0 \]  
(4.4.3) 

These two equations can each be integrated once, and we, thus, obtain the following two equations:

\[ \mu F'''' + (\rho - \mu^* c^2)(F''^2 - FF') = A \]  
(4.4.2a) 

\[ \mu G'''' + (\rho - \mu^* c^2)(F''G' - FG'') = B \]  
(4.4.3a) 

where \( A \) and \( B \) are arbitrary constants of integration.

Employing (4.4.1) into (4.3.11) and (4.3.12), we obtain

\[ h_x = \rho(yF' + G')(yF'' + G'') + \mu F'' - c^2 \mu^*(yF' + G')(yF'' + G'') \]  
(4.4.4) 

\[ h_y = \rho F(yF'' + G'') - \mu(yF'''' + G'''') - c^2 \mu^* F(yF'' + G'') \]  
(4.4.5) 

Integrating (4.4.5) with respect to \( y \), after using (4.4.2a) and (4.4.3a), we obtain

\[ h = -\frac{y^2}{2} [A - (\rho - c^2 \mu^*) F''^2] - y[B - (\rho - c^2 \mu^*) F'G'] + g(x) \]  
(4.4.6) 

where \( g(x) \) is an arbitrary function of \( x \).

Differentiating (4.4.6) with respect to \( x \) and using (4.4.4), we get

\[ g(x) = (1/2)(\rho - c^2 \mu^*) G' - F' + p_0 \]  
(4.4.7) 

where \( p_0 \) is an arbitrary constant.

Using (4.4.7) in (4.4.6), we obtain

\[ h = -\frac{1}{2}y^2[A - (\rho - c^2 \mu^*) F''^2] - y[B - (\rho - c^2 \mu^*) F'G'] + 
\] 
\[ + (1/2)(\rho - c^2 \mu^*) G' + F' + p_0 \]  
(4.4.8)
On assuming that the magnetic forces are present, we find, by substituting (4.4.8) into (4.3.2), that the pressure distribution turns out to be

\[ p = \mu F' - (1/2)\rho F^2 - c^2 \mu^2 I(1/2)y^2 F' F + y F' G' + (1/2)G'^2 \] - 
- (1/2)Ay^2 - By + p_0 \tag{4.4.9}

If the fluid domain is considered to extend to infinity, then in order that the pressure remains bounded there, we must take \( A = B = 0 \). We also pointed out that by interchanging \( x \) and \( y \) in (4.4.1), we can, with minor changes, handle the solutions of the form \( \psi(x, y) = xf(y) + g(y) \).

As we mentioned in section 4.2, if the fluid is finitely conducting, then functions \( f, u \) and \( v \) must satisfy the diffusion equations

\[ \nabla^2(fu) = \nabla^2(fv) = 0 \]

Since, in this case \( f = c \), then the above diffusion equations reduce to

\[ \nabla^2 u = \nabla^2 v = 0 \tag{4.4.10} \]

B) Particular Solutions.

Returning to equations (4.4.2) and (4.4.3), we will try to solve them in order to find the arbitrary functions \( F(x) \) and \( G(x) \). Since, it was not possible to find general solutions, then some particular solutions have been considered.

Equation (4.4.2) admits the following three particular
solutions:

\[ F_1 = \frac{-6\mu}{(\rho - \rho^2 \mu^*)}, \quad F_2 = \frac{\mu}{\rho - \rho^2 \mu^*} a(1 + ke^{ax}), \quad F_3 = Dx + E \]

(4.4.11)

where a, k, D and E are arbitrary constants.

I) When \( F = F_1 \), then equation (4.4.3) becomes the Cauchy Euler equation

\[ x^4 G^{(iv)} + 6x^3 G''' - 12xG' = 0 \]

which admits a general solution of the form

\[ G(x) = c_1 x^3 + c_2 x^{-1} + c_3 x^{-2} + c_4 \]

where \( c_1, c_2, c_3 \) and \( c_4 \) are four arbitrary constants.

Hence, the streamfunction \( \psi(x,y) \) defined by (4.4.1) takes the form

\[ \psi(x,y) = \frac{-6\mu}{\rho - \rho^2 \mu^*} \frac{y}{x} + c_1 x^3 + c_2 x^{-1} + c_3 x^{-2} + c_4 \]

(4.4.12)

For the finitely conducting fluids, the diffusion equations (4.4.10) must be satisfied. Putting

\[ \dot{\mu} = \mu_y = -6\mu(x(\rho - \rho^2 \mu^*)x) \]

into \( \nabla^2 u = 0 \), we find that \( \mu = 0 \). But this contradicts the fact that the fluid is viscous, that is \( \mu \neq 0 \). Hence, we conclude that solution (4.4.12) is not a possible solution for finitely conducting fluids while this is a perfectly good solution for the infinitely conducting fluids.

II) Using \( F = F_2 \), equation (4.4.3) yields

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\[ G^{(iv)} - a(1 + ke^{x})G''' + a^3 ke^{x}G' = 0 \quad (4.4.13) \]

Without loss of generality, we can take \( a = 1 \), so equation (4.4.13) reduces to
\[ G^{(iv)} - (1 + ke^{x})G''' + ke^{x}G' = 0 \quad (4.4.14) \]

Employing a series of substitutions of the type
\[ G'(x) = N(x), \quad N'(x) = N(x)e^{x}, \quad N'(x) = R(x) \quad (4.4.15) \]
in the equation (4.4.14), we find that the function \( R(x) \) satisfies the following equation
\[ R'' + (2 - ke^{x})R' + (1 - 2ke^{x})R = 0 \quad (4.4.16) \]

Integrating once equation (4.4.16), we obtain
\[ R' + (1 - ke^{x})R = c_1 e^{-x} \quad (4.4.17) \]

where \( c_1 \) is an arbitrary constant.

The general solution of (4.4.17) is given by
\[ R(x) = c_1 \exp[ke^{x} - x] \int \exp[-ke^{x}] \, dx + c_2 \exp[ke^{x} - x] \]

Having found \( R(x) \), we use it in (4.4.15) and we find that
\[ G(x) = c_1 \int e^{x} \, dx \int \exp[ke^{x} - x] \, dx \int \exp[-ke^{x}] \, dx + c_2 \int e^{x} \, dx \int \exp[ke^{x} - x] \, dx + c_3 e^{x} + c_4 \]

where \( c_2, c_3 \) and \( c_4 \) are arbitrary constants.

Thus the streamfunction \( \psi(x,y) \) defined by (4.4.1) takes the form
\[ \psi(x,y) = \mu \left[ \rho - c^2 \mu^2 \right] (1 + ke^{x})y + c_1 \int e^{x} \, dx \int \exp[ke^{x} - x] \, dx \]
\[ + \int \exp[-ke^{x}] \, dx + c_2 \int e^{x} \, dx \int \exp[ke^{x} - x] \, dx + c_3 e^{x} + c_4 \quad (4.4.18) \]
For finitely electrically conducting fluids, diffusion equations (4.4.10) must be satisfied. Substituting \( u = \psi_y \) and \( v = -\psi_x \) into the diffusion equations, we find that

\[
k = 0 \quad \text{and} \quad c_3 = 0
\]

Thus, for finitely conducting flows, the streamfunction (4.4.18) reduces to

\[
\psi(x, y) = \mu\left(\rho - c^2 \mu^*\right)y - (c_1/2)x^2 - [c_1 + c_2]x + c_4 \quad (4.4.19)
\]

Using the values of \( F(x) \) and \( G(x) \), that we found above, in the pressure equation (4.4.9) with \( A = B = 0 \), we find that for \( k = 0 \) the pressure distribution is given by

\[
p = -\left(1/2\right)\rho(\mu^2/(\rho - c^2 \mu^*)^2) - (1/2)c^2 \mu^*\left[c_1 x + (c_1 + c_2)\right]^2 + p_0
\]

III) When \( F = F_3 \), then equation (4.4.3) becomes

\[
\mu G^{(iv)} - (\rho - c^2 \mu^*) (Dx + E)G''' = 0
\]

Letting \( G'''(x) = R(x) \), then we find that \( R(x) \) satisfies the following equation

\[
\mu R' - (\rho - c^2 \mu^*) (Dx + E)R = 0
\]

which admits a general solution of the form

\[
R(x) = k \exp\left((\rho - c^2 \mu^*)/\mu)((D/2)x^2 + Ex)\right) \quad (4.4.20)
\]

Since \( G'''(x) = R(x) \), then

\[
G(x) = k \int \int \int \exp\left((\rho - c^2 \mu^*)/\mu)((D/2)x^2 + Ex)\right) dx \, dx \, dx \quad (4.4.21)
\]

and the streamfunction \( \psi(x, y) \) is given by

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\[ \psi(x, y) = (Dx + E)y + \]
\[ + \int \int \int \exp\left(\frac{\rho}{\mu}(D/2)x^2 + Ex\right) dx \, dx \, dx \]  

(4.4.22)

Diffusion equations (4.4.10) must be identically satisfied, for finitely conducting fluids. Thus, using (4.4.22) in the diffusion equations, we find that \( k = 0 \) and hence the form of the streamfunction (4.4.22) reduces to

\[ \psi(x, y) = (Dx + E)y \]  

(4.4.23)

For this value of the streamfunction, we find that the pressure is given by

\[ p = \mu D - (1/2)\rho[Dx + E]^2 - (1/2)c^2\mu^* y^2 D^2 + p_o \]  

(4.4.24)

C) Solution of Riabouchinsky.

We consider the particular case of (4.4.1) when \( G = 0 \), usually attributed to Riabouchinsky (1924). Physically, such a flow represents a plane stagnation flow in which the flow is separated in the two symmetrical regions by the plane \( x = 0 \).

I) When \( F = F_1 \), then the streamfunction takes the form

\[ \psi(x, y) = -6(\mu/\rho - c^2\mu^*) [y/x] \]  

(4.4.25)

and the pressure \( p \) is given by

\[ p = 6(\mu^2/[(\rho - c^2\mu^*)x^2]) - 18\rho(\mu^2/[(\rho - c^2\mu^*)x^2]) - 
- 18\mu^2c^2(\mu^*/[(\rho - c^2\mu^*)x^4]) + p_o \]  

(4.4.26)

For finitely conducting fluids, in order to satisfy the diffusion equations, we must have \( \mu = 0 \), which contradicts the fact that the fluid is viscous, that is \( \mu \neq 0 \). Thus, we conclude
that (4.4.25) is not a possible solution for the case of finitely conducting fluids.

II) Letting $F = F_2$, then streamfunction is given by

$$\psi(x, y) = \left[ \mu a / (\rho - c^2 \mu^*) \right] (1 + ke^{ax}) y \tag{4.4.27}$$

Substituting (4.4.27) in (4.4.9), we find that the pressure is given by

$$p = - \left( \mu / \rho \right) U_0^2 \exp[-(U_0 / \nu)x] - \left( 1/2 \right) \rho (U_0^2 / \nu^3)(\exp[-(U_0 / \nu)x]) - 1)^2$$

$$- (1/2) c^2 \mu^* y^2 (U_0^4 / s^2 \nu^2) \exp[-2(U_0 / \nu)x] + p_0 \tag{4.4.28}$$

where

$$\nu = \mu / \rho, \quad U_0 = - \nu a, \quad s = 1 - \left[ c^2 \mu^* / \rho \right]$$

For finitely conducting fluids, we find that the streamfunction is given by

$$\psi(x, y) = \left[ \mu a / (\rho - c^2 \mu^*) \right] y \tag{4.4.29}$$

For this value of $\psi$, the diffusion equations (4.4.10) are identically satisfied.

III) When $F = F_3$, equation (4.4.1) with $G = 0$, yields

$$\psi(x, y) = (Dx + E)y \tag{4.4.30}$$

Employing (4.4.30) in equation (4.4.9), we find that

$$p = \mu D^2 - \left( 1/2 \right) \rho (Dx + E)^2 - \left( 1/2 \right) c^2 \mu^* y^2 D^2 + p_0 \tag{4.4.31}$$

Substituting $u = \psi_y$ and $v = - \psi_x$ into the diffusion equations (4.4.10), we see that both of these equations are identically satisfied.
CASE 2

We assume that \( f = f(\psi) \), where \( f(\psi) \) is an arbitrary function of \( \psi \). For simplicity take \( f = \psi \).

A) Generalities.

We take the streamfunction to be linear with respect to \( y \), that is
\[
\psi(x, y) = yF(x) + G(x) \quad (4.4.32)
\]
where \( F(x) \) and \( G(x) \) are arbitrary functions of their argument \( x \).

Employing (4.4.32) into the compatibility equation (4.3.14), with the assumption that \( f = \psi \), we find that \( F(x) \) and \( G(x) \) must satisfy the following equations:

\[
\rho [F' G'' - FG''] + \mu G^{(v)} + \mu \left\{ - F' G^2 G' + FG G''' - 2F' G G'' + 2F^2 G' + 2F G G'' \right\} = 0 \quad (4.4.33)
\]

\[
\rho [F' F'' - FF'''] + \mu F^{(v)} + \mu \left\{ - F' F' G^2 + FF' G'' + 2F^2 G G''' - 4F' G G' - 2F F G + 2F^2 F' + 2F F' G' + 2F^2 G G'' \right\} = 0 \quad (4.4.34)
\]

\[
2F^2 F'' G - F^2 F' G' + F^3 G''' - 2F' G - 4F' G' + 2F^2 F' G' = 0 \quad (4.4.35)
\]

\[
F^2 F' F' + F^3 F''' - 2F F' = 0 \quad (4.4.36)
\]

On assuming that magnetic forces are present, we find, by using (4.4.32) and (4.3.9) in (4.3.2), that the pressure is given by

\[
p = h - \rho \left[ (1/2) F^2 + (1/2) G'^2 + (1/2) F'^2 + y F' G' \right] \quad (4.4.37)
\]

where \( h \) can be found by solving the two linear momentum equations.
(4.3.11) and (4.4.12).

If the fluid is finitely electrically conducting, then functions $f$, $u$ and $v$ must satisfy the diffusion equations (4.3.8), that is

$$
\nabla^2 (fu) = \nabla^2 (fv) = 0
$$

(4.4.38)

We remark that by interchanging $x$ and $y$ in (4.4.32) we can, with minor changes, handle the solutions of the form $\psi = xf(y) + g(y)$.

B) Particular Solutions.

Equation (4.4.36) admits the following two particular solutions

$$
F_1 = ke^{ax}, \quad F_2 = Dy^2
$$

(4.4.39)

where $k$, $a$, $D$ are arbitrary constants and we assume that $a \neq 0$.

1) When $F = F_1$, then equation (4.4.35) yields

$$
G''' + 2ag'' - 5a^2 G' = 0
$$

(4.4.40)

Letting $G'(x) = R(x)$, we find that $R(x)$ satisfies the following equation

$$
R'' + 2aR' - 5a^2 R = 0
$$

which admits a general solution of the form

$$
R(x) = c_1 \exp[a(-1 + \sqrt{5})x] + c_2 \exp[-a(1 + \sqrt{5})x]
$$

(4.4.41)

where $c_1$ and $c_2$ are arbitrary constants.

Hence, the streamfunction $\psi(x, y)$ is given by
\[ \psi(x, y) = yke^{ax} + \frac{c_1}{-a + ay} \exp[ax - 1 + \sqrt{a}] - \frac{c_2}{-a + ay} \exp[-a(1 + \sqrt{a})x] + c_3 \]  

Using \( f = \psi, \ u = \psi_y, \) and \( v = -\psi_x \) into the diffusion equations (4.4.38) we find that, for finitely conducting fluids, the arbitrary constants \( c_1, c_2 \) and \( c_3 \) must satisfy the following equations:

\[ 4a^2yke^{2ax} + \frac{6a^2kc_1}{-a + ay} e^{a\sqrt{a}} x - \frac{6a^2k^2c_2}{a + ay} e^{-a\sqrt{a}} x + c_3ka^2e^{ax} = 0 \]  

\[ 6c_1yka^2e^{a\sqrt{a}} x + 4c_1^2(-a + a\sqrt{a})e^{2(-a + a\sqrt{a})x} + \frac{4a^2c_1c_2}{-a + ay} e^{-2ax} \]  

\[ + c_1c_3(-a + a\sqrt{a})^2e^{-(a + a\sqrt{a})x} + 6c_2^2yke^{-a\sqrt{a}x} + 2ak^2e^{2ax} + \]  

\[ + \frac{4c_1c_2a^2}{-a + ay} e^{-2ax} + 4c_1^2(-a - a\sqrt{a})e^{2(-a - a\sqrt{a})x} + \]  

\[ + c_2c_3(-a - a\sqrt{a})^2 e^{-a(1 + \sqrt{a})x} = 0 \]  

II) Using \( F = F_2 \), we find that equation (4.4.35) becomes the Cauchy-Euler differential equation

\[ x^3G''' + x^2G'' + (3/4)xG' + (1/2)G = 0 \]  

Characteristic equation of (4.4.45) is

\[ m^3 - 2m^2 + (1/4)m + (1/2) = 0 \]  

The roots of equation (4.4.45), using Cardan's method (see section
2.3.4), are
\[ m_1 = 1.797926, \quad m_2 = -0.25041, \quad m_3 = 0.452479 \] (4.4.47)

Hence, solution of (4.4.45) is given by
\[ G(x) = c_1 x^{m_1} + c_2 x^{m_2} + c_3 x^{m_3} \] (4.4.48)

where \( c_1, c_2 \) and \( c_3 \) are arbitrary constants and \( m_1, m_2 \) and \( m_3 \) are given by (4.4.47).

Thus the stream function \( \psi(x,y) \) takes the form
\[ \psi(x,y) = Dvxy + c_1 x^{m_1} + c_2 x^{m_2} + c_3 x^{m_3} \] (4.4.49)

For the finitely conducting fluids, substituting the values of \( f, u \)
and \( v \) into the diffusion equations (4.4.38), we find that the
constants \( c_1, c_2 \) and \( c_3 \) must satisfy the following equations:
\[ c_1 D\frac{m_1}{2}(m_1 - \frac{1}{2})x^{m_1 - (3/2)} + c_2 D\frac{m_2}{2}(m_2 - \frac{1}{2})x^{m_2 - (3/2)} + c_3 D\frac{m_3}{2}(m_3 - \frac{1}{2})x^{m_3 - (3/2)} = 0 \] (4.4.50)

\[ Dc_1 m_1 (m_1 - \frac{1}{2})(m_1 - \frac{3}{2})x^{m_1 - (\frac{5}{2})}y + Dc_2 m_2 (m_2 - \frac{1}{2})(m_3 - \frac{3}{2})x^{m_2 - (\frac{5}{2})} + (1/2)Dc_1 \]

\[ * x^{m_2 - (\frac{5}{2})}y + Dc_3 m_3 (m_3 - \frac{1}{2})(m_3 - \frac{3}{2})x^{m_3 - (\frac{5}{2})} + (1/2)Dc_1 \]

\[ * (m_1 - \frac{1}{2})(m_1 - \frac{3}{2})x^{m_1 - (\frac{5}{2})}y + c_1^2 m_1 (2m_1 - 1)(2m_1 - 2) x^{2m_1 - 3} + c_1 c_2 m_2 (m_1 + m_2 - 2) x^{m_1 + m_2 - 3} + c_1 c_3 m_3 (m_1 + m_3 - 1) \]

\[ * (m_1 + m_2 - 2) x^{m_1 + m_2 - 3} + (1/2)Dc_2 (m_2 - \frac{1}{2})(m_2 - \frac{3}{2})x^{m_2 - (\frac{5}{2})}y \]

\[ + c_1 c_2 m_1 (m_1 + m_2 - 1)(m_1 + m_2 - 2) x^{m_1 + m_2 - 3} + c_2^2 m_2 (2m_2 - 1)(2m_2 - 2) x^{2m_2 - 3} + \]

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\[ + c_2 c_3 m_3 (m_2 + m_3 - 1)(m_2 + m_3 - 2)x^{m_2 + m_3 - 3} + \frac{1}{2}D c_3 (m_3 - \frac{1}{2}) * \\
(\frac{m_3}{2})^{(5/2)}y + c_1 c_3 m_1 (m_1 + m_3 - 1)(m_1 + m_3 - 2)x^{m_1 + m_3 - 3} + \]
\[ + c_2 c_3 m_2 (m_2 + m_3 - 1)(m_2 + m_3 - 2)x^{m_2 + m_3 - 3} + \\
+ c_3^2 m_3 (2m_3 - 1)(2m_3 - 2)x^{2m_3 - 3} + D^2 = 0 \quad (4.451) \]
where \( m_1, m_2 \) and \( m_3 \) are given by (4.4.47).

C) Solution of Riabouchinsky.

We assume that \( G = 0 \), so that the form of the streamfunction \( \psi(x,y) \) given by (4.4.32) reduces to

\[ \psi(x,y) = yF(x) \quad (4.4.52) \]
Using this assumption and letting \( F = F_1 \), then the streamfunction (4.4.52) takes the form

\[ \psi(x,y) = kye^{ax} \quad (4.4.53) \]
Employing \( F = F_1 \) and \( G = 0 \) into (4.4.37), we find that the pressure distribution is

\[ p = \mu ke^{ax} - \frac{1}{2} \rho k^2 e^{2ax} + \mu k \left[ - \frac{15}{4} a^2 y^4 - \frac{7}{16} \right] e^{4ax} \quad (4.4.54) \]
For finitely conducting fluids, the functions \( f = \psi, u = \psi_y \) and \( v = -\psi_x \) must satisfy equations (4.4.38). Employing the values of \( f \) \( u \) and \( v \) in (4.4.38), we find that either

\[ k = 0 \quad \text{OR} \quad y = \pm 1/\left(\sqrt{2} a^2\right) \]
If \( k = 0 \), then streamfunction (4.4.53) reduces to

\[ \psi(x,y) = 0 \]
Since \( \psi = 0 \), then \( u = 0 \) and \( v = 0 \) which means that we have no flow. So we cannot accept this possibility.

If \( y = \pm 1/(\sqrt{\gamma a^2}) \), then the streamfunction (4.4.53) takes the form

\[
\psi(x,y) = \pm \left[ 1/(\sqrt{\gamma a^2}) \right] ke^{ax}
\]  

(4.4.55)

and the velocity components are given by

\[
u = 0 \quad , \quad v = \pm \left[ k/(\sqrt{\gamma a}) \right] e^{ax}
\]  

(4.4.56)

When \( F = F_2 \) and \( G = 0 \), then (4.4.52) yields

\[
\psi(x,y) = D \sqrt{x} y
\]  

(4.4.57)

Substituting the values of \( f \), \( u \) and \( v \) into equations (4.4.38), we find that, for finitely conducting fluids, \( D = 0 \). If \( D = 0 \), then \( \psi = 0 \), which means that we have no flow.
4.5. Solution of Riabouchinsky Type problems.

Second-Grade Fluids.

In order to solve the Riabouchinsky type problems, we assume that the streamfunction is linear with respect to x or y.

Recalling equation (4.3.22) and using the definition of the streamfunction \( \psi(x,y) \), we find that

\[
\frac{\partial(f,\psi)}{\partial(x,y)} = 0
\]

From this equation, either

\[ f = \text{constant} \quad \text{OR} \quad f = f(\psi) \]

where \( f(\psi) \) is an arbitrary function of \( \psi \).

Hence, we have to consider two cases.

CASE 1

We take \( f = c \), where \( c \) is any arbitrary constant.

A) Generalities.

Assume that the streamfunction is of the form

\[
\psi(x,y) = yF(x) + G(x)
\]  \hspace{1cm} (4.5.1)

where \( F(x) \) and \( G(x) \) are arbitrary functions of \( x \).

Employing (4.5.1) into compatibility equation (4.3.24), with the assumption that \( f = c \), we find that \( F(x) \) and \( G(x) \) must satisfy the following equations:

\[
\mu F^{(iv)} + (\rho - c^2 \mu^*) (F'F'' - FF'') - \alpha_1 (F'F^{(iv)} - FF^{(v)}) = 0
\]  \hspace{1cm} (4.5.2)

\[
\mu G^{(iv)} + (\rho - c^2 \mu^*) (F'G' - FG') - \alpha_1 (F^{(iv)}G' - FG^{(v)}) = 0
\]  \hspace{1cm} (4.5.3)

Integration of the above equations yields
\[\mu F'''' + (\rho - c^2 \mu^*) (F''^2 - FF''') - \alpha_1 \left(- FF^{(iv)} + 2F'F'''' - F''^2\right) = A \quad (4.5.4)\]
\[\mu G'''' + (\rho - c^2 \mu^*) (F'G' - FG'') - \alpha_1 \left(- FG^{(iv)} + F'G'''' - F''G'' + F''''G'\right) = B \quad (4.5.5)\]

where A and B are arbitrary constants of integration.

Employing (4.5.1) into (4.3.19) and (4.3.20), we obtain
\[h = y (yF' + G')(yF'' + G'') + \mu F'' - c^2 \mu^* (yF' + G')(yF'' + G'') - \alpha_1 (yF' + G')(yF^{(iv)} + G^{(iv)}) \quad (4.5.6)\]
\[h = y (yF'' + G') - \mu (yF'' + G'') - c^2 \mu^* F(yF'' + G'') - \alpha_1 F(yF^{(iv)} + G^{(iv)}) \quad (4.5.7)\]

Integrating (4.5.7), after using (4.5.4) and (4.5.5), we get
\[h = \left(- \frac{1}{2}y^2 [A - (\rho - c^2 \mu^*) F''^2 + 2\alpha_1 F'F'''' - \alpha_1 F''^2] + g(x) - y[B - (\rho - c^2 \mu^*) F'G' + \alpha_1 F'G'''' - \alpha_1 F''G'' + \alpha_1 F''''G'] \quad (4.5.8)\]

where \(g(x)\) is an arbitrary function of \(x\).

Differentiating (4.5.8) with respect to \(x\) and using (4.5.6), we find that
\[g(x) = \left(\frac{1}{2}\right) (\rho - c^2 \mu^*) G''^2 + \mu F' - \alpha_1 G'G'''' + \left(\frac{1}{2}\right) \alpha_1 G''^2 + p_o \quad (4.5.9)\]

where \(p_o\) is an arbitrary constant.

Using (4.5.9), equation (4.5.8) yields
\[h = -\left(\frac{1}{2}\right) y^2 [A - (\rho - c^2 \mu^*) F''^2 + 2\alpha_1 F'F'''' - \alpha_1 F''^2] - y[B - (\rho - c^2 \mu^*) F'G' + \alpha_1 F'G'''' - \alpha_1 F''G'' + \alpha_1 F''''G'] + \left(\frac{1}{2}\right) (\rho - c^2 \mu^*) G''^2 + \mu F' - \alpha_1 G'G'''' + \left(\frac{1}{2}\right) G''^2 + p_o \quad (4.5.10)\]
On assuming that magnetic forces are present, we find, by using (4.5.10) in (4.3.17), that the pressure is given by

\[
p = \mu F' - (1/2) \rho F^2 + \alpha_1 [(1/2)y^2 F''^2 + yF'G' + FF'' + (1/2)G''] + (1/2)(3\alpha_1 + 2\alpha_2)[4F'^2 + y^2 F''^2 + G'^2 + 2yF'G'] - c^2 \mu^* [(1/2)y^2 F'^2 + yF'G' + (1/2)G'^2] - (1/2)\alpha y^2 - By + p_0
\]

(4.5.11)

If the fluid domain is considered to extend to infinity, then in order that the pressure remains bounded there, we must take \( A = B = 0 \). We also pointed out that by interchanging \( x \) and \( y \) in (4.5.1), we can, with minor changes, handle the solutions of the form \( \psi(x,y) = xf(y) + g(y) \).

As we mentioned in section 4.2, if the fluid is finitely electrically conducting, then the functions \( f, u \) and \( v \) must satisfy the diffusion equations, that is

\[
\nabla^2(fu) = \nabla^2(fv) = 0
\]

(4.5.12)

Keeping in mind that in this case \( f = c \), then equations (4.5.12) reduces to

\[
\nabla^2 u = \nabla^2 v = 0
\]

(4.5.13)

**B) Particular Solutions.**

Equation (4.5.2) admits the following two particular solutions

\[
F_1 = \frac{\mu a}{\rho - c^2 \mu^* - \alpha_1 a^2} (1 + ke^{ax}) \quad F_2 = Dx + E
\]

(4.5.14)

where \( a, k, D \) and \( E \) are arbitrary constants.
1) When \( F = F_1 \), equation (4.5.3) yields

\[
\mu G^{(i \psi)} + (\rho - c^2 \mu^*) \left( \frac{\mu a^3 k e^{a x}}{\rho - c^2 \mu^* - a_1^2} G' - \frac{\mu a (1 + ke^{ax})}{\rho - c^2 \mu^* - a_1^2} G'' \right)
\]

\[
- \alpha_1 \left( \frac{\mu a^3 e^{a x}}{\rho - c^2 \mu^* - a_1^2} G' - \frac{\mu a (1 + ke^{ax})}{\rho - c^2 \mu^* - a_1^2} G'' \right) = 0
\]

(4.5.15)

Without loss of generality, we can assume that \( a = 1 \), so equation (4.5.15) reduces to

\[
\alpha_1 (1 + ke^{x}) G^{(i \psi)} + (\rho - c^2 \mu^* - \alpha_1) G^{(i \psi)} + (\rho - c^2 \mu^* - \alpha_1) ke^{x} G' -
\]

\[
- (\rho - c^2 \mu^*)(1 + ke^{x}) G'' = 0
\]

(4.5.16)

In order to reduce the order of (4.5.16), we make a series of substitutions of the type

\[
G'(x) = H(x) \quad H(x) = M(x)e^{x} \quad M'(x) = R(x)
\]

(4.5.17)

and we find that \( R(x) \) satisfies the following equation

\[
\alpha_1 [(1 + ke^{x}) R'' + (3 + 4ke^{x}) R' + (3 + 6ke^{x}) R + (1 + 4ke^{x}) R] +
\]

\[
(\rho - c^2 \mu^*)[R' + (2 - ke^{x}) R'] + (1 - 2ke^{x}) R] = 0
\]

(4.5.18)

Multiplying equation (4.5.18) by \( e^{x} \) and integrating once, we get

\[
\alpha_1 [(1 + ke^{x}) R' + 2(1 + ke^{x}) R' + (1 + 2ke^{x}) R] +
\]

\[
+ (\rho - c^2 \mu^*)[R' + (1 - ke^{x}) R] = d e^{-x}
\]

(4.5.19)

where \( d \) is an arbitrary constant of integration.

Solving equation (4.5.19), we can find \( R(x) \). Having found \( R(x) \) and using substitutions (4.5.17) we find \( G(x) \). Finally, knowing \( F(x) \) and \( G(x) \), we can find the streamfunction defined by (4.5.1).
Consider now a special case of (4.5.19), where \( k = 0 \). If \( k = 0 \), then equation (4.5.19) reduces to

\[
R'' + (2\alpha_1 + \rho - c^2\mu^*)R' + (\alpha_1 + \rho - c^2\mu^*)R = \rho e^{-x} \tag{4.5.20}
\]

The complementary solution of (4.5.20) is given by

\[
R_c(x) = A_1 e^{-x} + B_1 \exp[(c^2\mu^* - \alpha_1 - \rho)/\alpha_1] x \tag{4.5.21}
\]

where \( A_1 \) and \( B_1 \) are arbitrary constants.

Using the Variation of Parameters method, we find that the particular solution of equation (4.5.20) is

\[
R_p(x) = \frac{d}{\rho - c^2\mu^*} [x - (\alpha_1/(\rho - c^2\mu^*))] e^{-x} \tag{4.5.22}
\]

Hence, the general solution of (4.5.20) is of the form

\[
R(x) = R_c + R_p = A_1 e^{-x} + B_1 \exp[(c^2\mu^* - \rho - \alpha_1)/\alpha_1] x + \frac{d}{\rho - c^2\mu^*} [x - (\alpha_1/(\rho - c^2\mu^*))] e^{-x} \tag{4.5.23}
\]

If \( \alpha = 1 \) and \( k = 0 \), then from (4.5.23) and (4.5.17), we get

\[
G(x) = \left[-A_1 + \frac{d\alpha_1}{(\rho - c^2\mu^*)} - \frac{d}{\rho - c^2\mu^*}\right] x - \frac{d}{2(\rho - c^2\mu^*)} x^2 + \frac{B_1 \alpha^2}{(\alpha_1 + \rho - c^2\mu^*)(\rho - c^2\mu^*)} \exp[-(c^2\mu^*/\alpha_1)] x \tag{4.5.24}
\]

Hence, the streamfunction \( \psi(x,y) \) is given by

\[
\psi(x,y) = \frac{\mu}{\rho - \alpha_1 - c^2\mu^*} y + G(x) \tag{4.5.25}
\]

where \( G(x) \) is given by (4.5.24).

Employing the values of \( F(x) \) and \( G(x) \) in equation (4.5.14), with...
\[ p = - (1/2) \rho \frac{\mu^2}{(\rho - \alpha_1 - c^2 \mu^*)^2} + \left( 2\alpha_1 + \alpha_2 \right) \left[ - \frac{d}{\rho - c^2 \mu^*} x^2 + \right. \\
+ \frac{B_1 \alpha_1}{\alpha_1 + \rho - c^2 \mu^*} \frac{\rho - c^2 \mu^*}{\alpha_1} \exp\left[ \frac{(\rho - \alpha_1 x)}{\alpha_1} \right] \right] - \\
\left. - (1/2) c^2 \mu \left[ \frac{d \alpha_1}{(\rho - c^2 \mu^*)^2} - \frac{d}{\rho - c^2 \mu^*} \right] \frac{d}{\rho - c^2 \mu^*} x - \right] \\
- \frac{\rho - c^2 \mu^*}{\alpha_1 + \rho - c^2 \mu^*} \exp\left[ \frac{(\rho - c^2 \mu^*)/\alpha_1}{\alpha_1} \right] \right] \right] - p_0 \] (4.5.26)

Using \( u = \psi_2 \) and \( v = - \psi_1 \) in (4.5.13), we find that, for finitely conducting fluids, \( B_1 = 0 \) and hence the streamfunction (4.5.25) reduces to

\[ \psi(x,y) = \frac{\mu}{\rho - \alpha_1 - c^2 \mu^*} y + \left[ \frac{d \alpha_1}{(\rho - c^2 \mu^*)^2} - \frac{d}{\rho - c^2 \mu^*} \right] x. \] (4.5.27)

II) Employing \( F = F_2 \) into equation (4.5.3), we obtain

\[ \alpha_1 (Dx + E)G^{(\nu)} + \mu G^{(\nu)} - (\rho - c^2 \mu^*) (Dx + E)G'' = 0 \] (4.5.28)

Letting \( G''(x) = R(x) \), then above equation takes the form

\[ \alpha_1 (Dx + E)R'' + \mu R' - (\rho - c^2 \mu^*) (Dx + E)R = 0 \] (4.5.29)

The solution of equation (4.5.29), which turns out to be in terms of the hypergeometric function \( F(\alpha, \beta, \gamma, z) \), is given by
\[ R_1(x) = \exp \left[ \pm 2i(E/\{(\rho - c^2\mu^*)D\})x \right] C_1 F(\alpha, \beta, \pm 2i(E/\{(\rho - c^2\mu^*)D\}) + x)) \]

\[ R_2(x) = \exp \left[ \pm 2i(E/\{(\rho - c^2\mu^*)D\})x \right] C_2 \]

\[ \star F(1 - \alpha - \beta, 2 - \beta, \pm 2i(E/\{(\rho - c^2\mu^*)D\}) + x)) \]  

where

\[ \alpha = \frac{\mu}{\rho - c^2\mu^*} \quad \beta = \frac{-\mu}{\rho - c^2\mu^*} \]

Since \( G'(x) = R(x) \), then solution of (4.5.28) is

\[ G(x) = \int \int \int \left( \exp \left[ \pm 2i(E/\{(\rho - c^2\mu^*)D\})x \right] \right) \left[ C_1 F(\alpha, \beta, \pm 2i(E/\{(\rho - c^2\mu^*)D\}) + x)) + C_2 F(1 - \alpha - \beta, 2 - \beta, \pm 2i(E/\{(\rho - c^2\mu^*)D\}) + x)) \right] \, dx \, dx \, dx \]  

Hence the streamfunction \( \psi(x,y) \) takes the form

\[ \psi(x,y) = (Dx + E)y + G(x) \]  

where \( G(x) \) is given by equation (4.5.32).

For finitely conducting fluids, using \( u = \psi_y \) and \( v = -\psi_x \) in the diffusion equation (4.5.13), we find that \( C_1 = C_2 = 0 \) and thus the streamfunction (4.5.33) reduces to

\[ \psi(x,y) = (Dx + E)y \]  

(4.5.34)
C) Solution of Riabouchinsky.

We consider the particular case of (4.5.1) where $G = 0$, usually attributed to Riabouchinsky (1924).

When $F = F_1$ and $G = 0$, then the streamfunction takes the form

$$\psi(x, y) = \frac{\mu a}{\rho - \alpha_1 a^2 - c^2 \mu^*} y (1 + ke^{ax}) \quad (4.5.35)$$

Putting $F = F_1$, $G = 0$ and $k = -1$ into the pressure equation (4.5.14), we find that the pressure distribution is given by

$$p = -\left(\frac{\mu}{\nu} U_0^2 \exp\left(-\frac{U_0}{\nu} x\right) - \frac{1}{2} \rho (U_0^2 / s^2) \exp\left(-\frac{U_0}{\nu} x\right) \right)^2$$

$$+ (2\alpha_1 + \alpha_2) y^2 (U_0^4 / s^2 \nu^4) \exp(-2(U_0 / \nu) x) +$$

$$+ \left[7\alpha_1 + 4\alpha_2 - (1/2)c^2 \mu^* y^2 \right] (U_0^4 / s^2 \nu^2) \exp\left(-2(U_0 / \nu) x\right) -$$

$$- \alpha_1 (U_0^4 / s^2 \nu^2) \exp\left(-\frac{U_0}{\nu} x\right) + p_0$$

$$\quad (4.5.36)$$

where

$$\nu = \frac{\mu}{\rho} \quad , \quad U_0 = -\nu a \quad , \quad s = 1 - [(\alpha_1 a^2 + c^2 \mu^*) / \rho]$$

If the fluid is finitely electrically conducting, then the diffusion equations (4.5.13) must be satisfied identically. This happen if and only if $k = 0$ and, hence, the streamfunction (4.5.35) reduces to

$$\psi(x, y) = \frac{\mu a}{\rho - c^2 \mu^* - \alpha_1 a^2} y \quad (4.5.37)$$

Substituting $F = F_2$ and $G = 0$ into (4.5.1), we get

$$\psi(x, y) = (Dx + E)y \quad (4.5.38)$$
and the pressure distribution (4.5.11) takes the form

\[ p = \mu D - \frac{1}{2} \rho (u_x + E)^2 + (6a_1 + 4a_2) D^2 - \frac{1}{2} c^2 \mu^* D^2 y^2 + p_0 \]

(4.5.39)

For finitely conducting fluids, solution (4.5.39) does not change since it satisfies the diffusion equations (4.5.13) identically.
CASE 2

In this case, we take \( f = f(\psi) \) where \( f(\psi) \) is an arbitrary function of \( \psi \). For simplicity, we let \( f = \psi \).

\[ \psi(x,y) = yF(x) + G(x) \quad (4.5.40) \]

where \( F(x) \) and \( G(x) \) are two arbitrary functions of \( x \).

Employing (4.5.40) into the compatibility equation (4.5.24), with the assumption that \( f = \psi \), we find that \( F(x) \) and \( G(x) \) satisfy the following equations

\[ \mu G^{(iv)} + \rho (F''G' - FG''') - \alpha_1 (F^{(iv)}G' - FG^{(iv)}) + \mu^* [- F''G^2G' + F'G^2G'' - 2F'GG'' + 2F^2F'G + 2FGG'G''] = 0 \quad (4.5.41) \]

\[ \mu F^{(iv)} + \rho (F'F''' - FF''') - \alpha_1 (F'F^{(iv)} - (FF^{(iv)}) + \mu^* [- F'F''G^2 + 2F^2G'G''' - 4F^2GG' - 2FF'G'' + 2F^2F' + 2FF'GG' + 2F^2F'G'''] = 0 \quad (4.5.42) \]

\[ 2F^2F'''G - F^2F''G' + F^3G''' - 2F'^3G - 4FF'^2G' + 2F^2F'G''' = 0 \quad (4.5.43) \]

\[ F^2F'F''' + F^3F''' - 2FF'^3 = 0 \quad (4.5.44) \]

Substituting (4.5.40) into the linear momentum equations (4.3.19) and (4.3.20), we obtain
\[ h_x = \rho(yF' + G')(yF' + G') + \mu F' - \mu^*(yF' + G')(yF' + G') \ast \\
* (yF' + G')^2 + (yF' + G')^3(yF + G) + F^2(yF + G)(yF' + G') \sideset{\sum}{2} - \\
\quad - \alpha_1(yF' + G')(yF^{(iv)} + G^{(iv)}) \quad (4.5.45) \]

\[ h_y = \rho F(yF' + G') - \mu (yF' + G') - \mu^*(yF' + G')(yF + G)^2 \\
+ F(yF + G)(yF' + G')^2 - F^3(yF + G) - \alpha_1 F(yF^{(iv)} + G^{(iv)}) \quad (4.5.46) \]

Integrating (4.5.46) with respect to \( y \), we get

\[ h = y[\rho FG' - \mu G' - \mu^*(FGG' + FG^2G') + F^3G_\infty] - \alpha_1 FG^{(iv)} + \\
+ (1/2)y^2[\rho F' - \mu F' - \mu^*(FF'G^2 + 2F^2GG') + 2FF'GG' + F^2G' + F^4] - \alpha_1 FF^{(iv)} - (1/3)y^3\mu^*[F^3G' + 2F^2FG + FF'G + 2F^2GG'] \\
- (1/4)y^4 \mu^*[F^3F' + 2F^2F']^2 + g(x) \quad (4.5.47) \]

where \( g(x) \) is an arbitrary function of \( x \).

Differentiating (4.5.47) with respect to \( x \) and using (4.5.45),
we find that

\[ g(x) = \frac{1}{2} \rho G^{(iv)} + \mu F' - \mu^* \int (G^2G'G' + GG' + F^2GG') \, dx - \\
\quad - \alpha_1 \int G^{(iv)} \, dx \quad (4.5.48) \]

Having found \( g(x) \) and substituting this into (4.5.47), we can find the expression for the generalized pressure \( h \).

Finally, substituting the found value of \( h \) into equation (4.3.17),
we find, on assuming that magnetic forces are present, that the
pressure is given by
\[
p = h - \frac{1}{2} \rho [F^2 + y^2 F' + G''^2 + 2yF'G''] + \alpha_1 [F'F'' + y^2 F'F'''] + \\
y(F'G'' + F'G''' + G'G''') + (1/2)(3\alpha_1 + 2\alpha_2)[4F''^2 + y^2 F''']^2 + \\
+ G''^2 + 2yF'G'']
\] (4.5.49)

For finitely conducting fluids, diffusion equations (4.2.15) must be satisfied identically, that is
\[
\nabla^2 (fu) = \nabla^2 (fv) = 0
\] (4.5.50)

Returning to the system of equations (4.3.41) to (4.5.44), we shall try to find a solution of this system.

Equation (4.5.44) admits the following two particular solutions
\[
F_1 = ke^{\alpha x}, \quad F_2 = Dy^x
\] (4.5.51)
where \(k\), \(\alpha\) and \(D\) are arbitrary constants.

Letting \(F = F_1\) and \(F = F_2\) then equation (4.5.43) yields the same solutions as in case 2 of section 4.4. Hence the streamfunctions are given by equations (4.4.42) and (4.4.49) respectively. The pressure distribution (4.5.49) is considerably modified in this case however, because \(\alpha_1\) and \(\alpha_2\)-terms are present.
CHAPTER 8

PLANE CONSTANTLY-INCLINED FLOWS

6.1. Introduction.

Magnetohydrodynamics (MHD) plane flows are said to be constantly-inclined if the magnetic field vector lies in the plane of flow and the angle between the velocity vector \( \mathbf{V} \) and the magnetic vector \( \mathbf{H} \), at any point, is constant throughout the flow and all the flow variables are functions of the rectangular coordinates \( x \) and \( y \) only.

Waterhouse and Kingston (1973) studied inviscid, constantly inclined plane flows. They obtained Bernoulli-type equations following Ladikov (1962) and classified these flows. Toews and Chandra (1974) carried out the study of constantly inclined compressible flows. Chandra, Toews and Nath (1975) studied viscous incompressible flows to investigate some properties and solutions of these flows.

Up to date there does not exist any work on Riabouchinsky type problems for magnetohydrodynamic flows and especially for constantly inclined flows.

In this chapter, we study the Riabouchinsky type problems for plane constantly inclined flows applying to both ordinary viscous fluids and second grade fluids. Only the infinitely
electrically conducting fluids have been considered.

In section 5.2, the flow equations are transformed into more convenient form for this work. Section 5.3 contains the derivation of the compatibility or integrability equation. Finally, section 5.4 consists of the solution to Riabouchinsky type problems for constantly inclined flows.
5.2 Flow Equations.

The basic equations governing the motion of an incompressible second-grade and electrically conducting fluid, in the presence of the magnetic field, are given by

\[ \text{div } \mathbf{V} = 0 \quad (i) \]
\[ \text{div } \mathbf{T} + \mu^* (\text{curl } \mathbf{H}) \times \mathbf{H} = \rho [\mathbf{V} \cdot \text{grad} ] \mathbf{V} = 0 \quad (ii) \]
\[ \text{curl} (\mathbf{V} \times \mathbf{H}) - (1/\mu^*) \text{curl} (\text{curl } \mathbf{H}) = 0 \quad (iii) \]
\[ \text{div } \mathbf{H} = 0 \quad (iv) \]

and the constitutive equation for the Cauchy stress \( \mathbf{T} \)

\[ \mathbf{T} = -p \mathbf{I} + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2 \quad (v) \]

where \( \mathbf{V} \) denotes the velocity field vector, \( \mathbf{H} \) the magnetic field vector, \( p \) the fluid pressure function, \( \rho \) the constant fluid density, \( \mu^* \) the constant magnetic permeability, \( \mu \) the coefficient of viscosity, \( \sigma \) the electrical conductivity, \( \mathbf{I} \) is the unit tensor, and \( \alpha_1 \) and \( \alpha_2 \) are the normal-stress moduli.

The Rivlin-Ericksen tensors \( \mathbf{A}_1 \) and \( \mathbf{A}_2 \) are defined as

\[ \mathbf{A}_1 = (\text{grad } \mathbf{V}) + (\text{grad } \mathbf{V})^T, \quad \mathbf{A}_2 = \mathbf{A}_1 + (\text{grad } \mathbf{V})^T \mathbf{A}_1 + \mathbf{A}_1 (\text{grad } \mathbf{V}) \quad (vi) \]

where

\[ \hat{\mathbf{A}}_1 = (\mathbf{A}_1 \cdot \text{grad} \mathbf{V} \]

Equations (i) to (iv) is a system of eight equations in seven unknowns, the unknowns being \( \mathbf{V}, \mathbf{H} \) and \( p \). Equation (iv) is an additional condition on \( \mathbf{H} \), expressing the absence of magnetic
poles in the flow.

Since in this chapter, we only study infinitely electrically conducting fluids, that is, $\sigma$ tends to infinity, then equation (iii) above reduces to

$$\text{curl}(V \times H) = 0$$  \hspace{1cm} (5.2.1)

Letting $\phi = 0$ denote the constant angle between the velocity vector $V = (u, v, 0)$ and the magnetic vector $H = (H_1, H_2, 0)$ in the $(x,y)$ plane and employing (5.2.1), we find that

$$uH_2 - vH_1 = qHsin\phi = A$$  \hspace{1cm} (5.2.2)

where $q = u^2 + v^2$ and $H = H_1^2 + H_2^2$ are the magnitudes of velocity and magnetic intensity vectors respectively and $A$ is an arbitrary constant which is non-zero due to the exclusion of aligned flows. Since $\phi$ is constant, then equation (5.2.2) implies the existence of an arbitrary constant $B$ such that

$$uH_1 + vH_2 = qHcos\phi = B$$  \hspace{1cm} (5.2.3)

where $B = Acot\phi$.

The constant $B$ is zero if and only if $\phi = (\pi/2)$ rad.

Considering (5.2.2) and (5.2.3) as two linear algebraic equations in the unknowns $H_1$ and $H_2$, we solve these in terms of $u, v$ and $\phi$ to have

$$H_1 = \frac{A[u\cot\phi - v]}{u^2 + v^2}, \quad H_2 = \frac{A[v\cot\phi + u]}{u^2 + v^2}$$  \hspace{1cm} (5.2.4)

Substituting equation (v) and making use of (vi), (5.2.1), (5.2.2), (5.2.3) and (5.2.4) into the system of equations (i) to
(iv), we find that the plane constantly inclined infinitely conducting flows are governed by

\[ u_x + v_y = 0 \quad (5.2.5) \]

\[ \rho \left[ \frac{\partial}{\partial x} \left( \frac{u^2 + v^2}{2} \right) - v(v_x - u_y) \right] + \frac{\partial}{\partial x} p = \mu \nu^2 u - 
\]

\[ - \mu A_j \frac{u + v \cot \phi}{u^2 + v^2} + \alpha_1 \left[ (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}) \nu^2 u + \frac{\partial u}{\partial x} \nu^2 u + 
\]

\[ + \frac{\partial v}{\partial x} \nu^2 v \right] + \frac{1}{4} \left( 3\alpha_1 + 2\alpha_2 \right) \frac{\partial}{\partial x} |A_1|^2 \quad (5.2.6) \]

\[ \rho \left[ \frac{\partial}{\partial y} \left( \frac{u^2 + v^2}{2} \right) + u(v_x - u_y) \right] + \frac{\partial}{\partial y} p = \mu \nu^2 v + 
\]

\[ + \mu A_j \frac{u \cot \phi - v}{u^2 + v^2} + \alpha_1 \left[ (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}) \nu^2 v + \frac{\partial u}{\partial y} \nu^2 u + 
\]

\[ + \frac{\partial v}{\partial y} \nu^2 v \right] + \frac{1}{4} \left( 3\alpha_1 + 2\alpha_2 \right) \frac{\partial}{\partial y} |A_1|^2 \quad (5.2.7) \]

\[ (2uv - 2u^2 \cot \phi) u_x + (v^2 - u^2 - 2uv \cot \phi) u_y + 
\]

\[ (v^2 - u^2 - 2uv \cot \phi) v_x - (2uv + 2v^2 \cot \phi) v_y = 0 \quad (5.2.8) \]

\[ \frac{\partial}{\partial x} \left[ \frac{u + v \cot \phi}{u^2 + v^2} \right] - \frac{\partial}{\partial y} \left[ \frac{u \cot \phi - v}{u^2 + v^2} \right] = \frac{j}{A} \quad (5.2.9) \]

where \[ |A_1|^2 = 4(u_x)^2 + 4(v_y)^2 + 2(v_x + u_y)^2 \quad (5.2.10) \]

Equations (5.2.5) to (5.2.9) is a system of five equations in four unknowns \( u, v, p \) and current density \( j = \text{curl} \mathbf{H} \).
5.3. Compatibility Equation.

In this section, we derive the compatibility equation for constantly-inclined flows by employing the above equations of motion for these flows. Riabouchinsky type problems are investigated by using the compatibility equation in the next section.

Introducing the vorticity function

\[ \omega = v_x - u_y \]  

(5.3.1)

we find that

\[ \nabla^2 v = \omega_x, \quad \nabla^2 u = -\omega_y \]  

(5.3.2)

Arranging the \( \alpha_1 \)-terms of equations (5.2.6) and (5.2.7) into more convenient form and introducing the generalized pressure function

\[ h = \frac{1}{2} \rho q^2 + p - \alpha_1 [u v^2 u + v v^2 v] - \frac{1}{4} (3\alpha_1 + 2\alpha_2) |A_1|^2 \]  

(5.3.3)

in equations (5.2.5) to (5.2.9), we find that the constantly inclined flows are given by

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  

(5.3.4)

\[ \frac{\partial h}{\partial x} + \rho v \omega = -\mu \frac{\partial \omega}{\partial y} - \mu^* A_j \left[ \frac{u + v \cot \phi}{u^2 + v^2} \right] - \alpha_1 v \nabla^2 \omega \]  

(5.3.5)

\[ \frac{\partial h}{\partial y} + \rho v \omega = \mu \frac{\partial \omega}{\partial x} + \mu^* A_j \left[ \frac{u \cot \phi - v}{u^2 + v^2} \right] + \alpha_1 u \nabla^2 \omega \]  

(5.3.6)
\[(2uv - 2u^2\cot \phi) u_x + (v^2 - u^2 - 2uv\cot \phi) u_y \]
\[+ (v^2 - u^2 - 2uv\cot \phi) v_x - (2uv + 2v^2\cot \phi) v_y = 0 \quad (5.3.7)\]

\[\frac{\partial}{\partial x} \left[ \frac{u + v\cot \phi}{u^2 + v^2} \right] - \frac{\partial}{\partial y} \left[ \frac{u\cot \phi - v}{u^2 + v^2} \right] = \frac{J}{A} \quad (5.3.8)\]

\[\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad (5.3.9)\]

Equations (5.3.3) to (5.3.9) is a system of seven equations in six unknowns \(u, v, p, \omega, h\) and \(j\).

We next, introduce the streamfunction \(\psi(x,y)\) defined by the continuity equation (5.3.4), that is

\[\psi_y = u \quad , \quad \psi_x = -v \quad (5.3.10)\]

We can easily see that equation (5.3.4) is identically satisfied, while the vorticity is given by

\[\omega = -\nabla^2 \psi \quad (5.3.11)\]

Employing (5.3.10) and (5.3.11) into equations (5.3.5) and (5.3.6), we find that these equations take the form

\[\frac{\partial h}{\partial x} - \rho \frac{\partial \psi}{\partial x} \nabla^2 \psi = \mu \frac{\partial}{\partial y} (\nabla^2 \psi) - \mu^* A j \left[ \frac{u + v\cot \phi}{u^2 + v^2} \right] - \]

\[- \alpha_1 \frac{\partial \psi}{\partial x} \nabla^4 \psi \quad (5.3.12)\]

\[\frac{\partial h}{\partial y} - \rho \frac{\partial \psi}{\partial y} \nabla^2 \psi = -\mu \frac{\partial}{\partial x} (\nabla^2 \psi) - \mu^* A j \left[ \frac{u\cot \phi - v}{u^2 + v^2} \right] - \]

\[- \alpha_1 \frac{\partial \psi}{\partial y} \nabla^4 \psi \quad (5.3.13)\]

\[\nabla^4 \psi \]

\[\nabla^2 \psi \]
Differentiating equation (5.3.12) with respect to 'y' and (5.3.13) with respect to 'x' and using the integrability condition

\[
\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x},
\]

we find that the compatibility equation for constantly inclined flows of a second grade fluid is given by

\[
\rho \frac{\partial}{\partial x} \left[ \frac{\partial \psi}{\partial x} \right] \frac{\partial^2 \psi}{\partial x \partial y} - \mu \frac{\partial}{\partial y} \left[ \frac{\partial \psi}{\partial y} \right] \frac{\partial^2 \psi}{\partial y \partial x} - \mu A \left[ \left[ \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right]^2 \psi \right]
\]

\[
= \frac{\partial}{\partial x} \left[ \frac{\partial \psi}{\partial x} \right]^2 \cot \phi - \frac{\partial}{\partial y} \left[ \frac{\partial \psi}{\partial y} \right]^2 \mu A \left[ \left[ \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right]^2 \psi \right] - \alpha \frac{\partial}{\partial x} \left[ \frac{\partial \psi}{\partial x} \right]^2 \frac{\partial^2 \psi}{\partial x \partial y} = 0
\]  

(5.3.14)

Having solved equation (5.3.14), the velocity components and the vorticity are given by equations (5.3.10) and (5.3.11) respectively. Finally, functions h, p, H_1 and H_2 can be found by solving equations (5.3.3), (5.3.12), (5.3.13) and (5.2.4).
6.4. Solution of Riabouchinsky type problems.

In the Riabouchinsky type problems, we assume that the streamfunction is linear with respect to $x$ or $y$.

**A) Generalities.**

Assume that the streamfunction is of the form

$$\psi(x,y) = yF(x) + G(x)$$  \hspace{1cm} (5.4.1)

where $F(x)$ and $G(x)$ are arbitrary functions of $x$.

Substituting (5.4.1) into equation (5.3.7), we find that functions $F(x)$ and $G(x)$ must satisfy the following differential equations:

\[ [-4F'G'' - G'G'' + F^2G'] + \cot\phi [-2F^2F' - 2FG'G'' + 2F'G'^2] = 0 \hspace{1cm} (5.4.2) \]

\[ [-4F'^2 - 2FG'G' - F'G'^2 + F^2F''] + \cot\phi [-2F'G'' - 2FF'G' \\
+ 4F'^2G'] = 0 \hspace{1cm} (5.4.3) \]

\[ [-F'^2G'' - 2F'F''G'] + \cot\phi [-2FF'F'' + 2F'^3] = 0 \hspace{1cm} (5.4.4) \]

\[ F'^2F'' = 0 \hspace{1cm} (5.4.5) \]

From equation (5.4.5), either $F' = 0$ or $F'' = 0$.

Assuming that $F'' = 0$ and $F' \neq 0$ and using this assumption into the above equations and into compatibility equation (5.3.14), we find that $F(x)$ and $G(x)$ must satisfy the following equations:

\[ -G'' + 2\cot\phi F' = 0 \hspace{1cm} (5.4.6) \]

\[ 2F' + \cot\phi G'' = 0 \hspace{1cm} (5.4.7) \]

\[ \rho G^{(i)} - \rho G'd' + \alpha F^{(i)} = 0 \hspace{1cm} (5.4.8) \]

\[ -99 - \]
\[ \cot \phi \left( -10F'G'' - F^2G''' - 16F'G'' + 8F'G'' - 4F'F''^2 - 2FG''G'' + 6G''^2 - 24F'G'G'' = 0 \right) \] 

\[ (5.4.10) \]

\[ -12F'F''G'' - 6FG'G''^2 + 6F^2F'G'' - 18G'G''^2 - 3G'F'G'' = 0 \] 

\[ (5.4.11) \]

\[ \cot \phi \left( 12F^3F'G'' - F^4G'' - 30F'G''^2G'' - 8F'G''^3 - 2F^2G''^2G'' - 6F^2G''^2G'' + 6F^2F'G' + 12FG'G''^2 \right) - 4F^3F'G'' \]

\[ -2F^3G'' - 12F'F''G''^2 - 6FG'G''^2 - 2FG'G''^2G'' + 6G''^2G''^2 - 24F'G'G''^2 + 12F^2F'G'G'' - 24F'G'G''^3 = 0 \] 

\[ (5.4.12) \]

\[ \cot \phi \left( 2F^2F'G'G'' + 2F^4F''^2 - F^4G'' - F^4G'G'' - 10F'G''^3G'' - 3F'G''^4 - F^2G''^3G'' + 6F^2F'G'G'' + 6FG'G''^3 \right) - 4F^3G''^2G'' \]

\[ -2F^3G'G''^2 + F^3G''^2G'' - 4F'G''^2G''^2 - 2FG''^3G''^2 + F''G''^2G'' + 6F^2F'G'G''^2 + 6G''^2G''^2 - 6F'G'G''^4 = 0 \] 

\[ (5.4.13) \]

Solving equations (5.4.6), (5.4.7) and (5.4.9), we find that 
\[
F' = 0, \]

which contradicts our assumption that \( F' \neq 0 \).

Hence, we must take \( F' = 0 \).

Assuming \( F' = 0 \), then \( F(x) = D \) where \( D \) is any arbitrary constant.

Using this result in equations (5.4.2) to (5.4.5) and into the compatibility equation (5.3.14), we find that \( G(x) \) must satisfy the following equations:
\[
\mu \left[ D^8G^{(iv)} + 4D^6G'G^{(iv)} + 6D^4G'4G^{(iv)} + 4D^2G'G^{(iv)} + D^0G^{(iv)} \right]
- \alpha \left[ - D^2G^{(v)} - 4D^7G'/2G^{(v)} - 6D^4G'G^{(v)} - 4D^3G'G^{(v)} - 2D^0G^{(v)} \right] + \\
+ 2\mu A^2(1 + \cot^2\phi) \cot\phi \left[ - D^4G'/2 - D^4G'/2G'/2 - D^2G'/2G'/2 - D^0G'/2G'/2 + 6D^2G'/2G'/2 \right] + \\
2A^2(1 + \cot^2\phi) \left[ - 2D^3G'/2G'/2 - D^3G'/2G'/2 - 2D^3G'/2G'/2 - 2D^3G'/2G'/2 + 6G'/2G'/2 \right] + \\
- 2D^2G'/2G'/2 + 2G'/2G'/2G'/2 + 6G'/2G'/2 \right] + \rho \left[ - D^9G'/2 - 4D^7G'/2G'/2 - G'/2G'/2 - 2G'/2G'/2 - 2\cot\phi \right] = 0 \tag{5.4.14}
\]

\[
- G'/2G'/2 + D^2G'/2 - 2\cot\phi \frac{DG'/2}{G'/2} = 0 \tag{5.4.15}
\]

Integrating equation (5.4.15) once, we get

\[
\frac{1}{1/3}G'/3 + D^2G'/2 - \cot\phi \frac{DG'/2}{G'/2} = C_1
\]

OR

\[
G'\left[ - G'/2 + 3D^2 - \cot\phi \frac{DG'/2}{G'/2} \right] = 3C_1 \tag{5.4.16}
\]

If we take \( C_1 = 0 \), then equation (5.4.16) reduces to

\[
G'\left[ - G'/2 + 3D^2 - 3\cot\phi \frac{DG'/2}{G'/2} \right] = 0
\]

If \( \dot{G'} = 0 \), then

\[
- G'/2 + 3D^2 - 3\cot\phi \frac{DG'/2}{G'/2} = 0 \tag{5.4.17}
\]

Solving (5.4.17), we find that

\[
G'(x) = - 3\cot\phi D + \sqrt{9\cot^2\phi + 12}
\]

OR

\[
G(x) = (- 3\cot\phi D + \sqrt{9\cot^2\phi + 12})x + M \tag{5.4.18}
\]

where \( M \) is an arbitrary constant.

Hence, we find that the streamfunction \( \psi(x, y) \) is given by
\[ \psi(x, y) = Dy + G(x) \]  
(5.4.19)

where \( G(x) \) is given by (5.4.18).

Employing (5.4.19) into the linear momentum equations (5.3.12) and (5.4.13), we find that

\[ \frac{\partial h}{\partial x} = 0 \]  
(5.4.20)

and

\[ \frac{\partial h}{\partial y} = 0 \]  
(5.4.21)

Solving equations (5.4.20) and (5.4.21), we find that

\[ h = p_0 \]  
(5.4.22)

where \( p_0 \) is an arbitrary constant.

On assuming that magnetic forces are present, we find, by using (5.4.22) into (5.3.3), that the pressure distribution is given by

\[ p = p_0 - \rho \left[ (1/2)D^2 - (1/2A)(3\cot \phi \pm (9\cot^2 \phi + 12)^{1/2}) \right] \]  
(5.4.23)

Since \( u = \psi_y \) and \( v = -\psi_x \), we find that

\[ u = D \]  
(5.4.24)

and

\[ v = - ( - 3\cot \phi \pm (9\cot^2 \phi + 12)^{1/2}) \]  
(5.4.25)

Finally, for these values of \( u \) and \( v \), equation (5.3.7) is identically satisfied.

Next, if we consider an ordinary viscous fluid, that is if \( \alpha_1 = \alpha_2 = 0 \), then the form of the streamfunction (5.4.19) remains the same. Also the pressure distribution remains unchanged.
B) Solution of Riabouchinsky.

We consider the special case of (5.4.1) where $G = 0$, usually attributed to Riabouchinsky (1924). Physically such a flow represents a plane stagnation flow in which the flow is separated in the two symmetrical regions by the plane $X = 0$.

Under this assumption, the streamfunction $\psi(x,y)$ takes the form

$$\psi(x,y) = Dy \quad (5.4.26)$$

On assuming that magnetic forces are present, we find that the pressure distribution is given by

$$p = p_0 - (1/2)\rho D^2 \quad (5.4.27)$$

where $p_0$ is an arbitrary constant.

Using (5.4.26), we find that the velocity components are

$$u = D \quad (5.4.28)$$
$$v = 0$$

For these values of $u$ and $v$, equation (5.3.7) is satisfied identically.
CONCLUSION

In the present thesis, Riabouchinsky type problems of plane steady-state MHD flows of a viscous, incompressible fluid have been studied when the flow is assumed to be transverse, aligned and constantly-inclined. Both ordinary viscous fluids and second-grade fluids have been considered.

Solutions of the compatibility equations were obtained and from these solutions the velocity components and the pressure distribution were found. For the transverse and aligned flows the pressure distribution is considerably modified in a Non-Newtonian or second-grade fluid in comparison to the ordinary viscous fluid. However in the constantly-inclined flows the pressure distribution remains the same for both cases.

Beside known existing solutions to the compatibility equation, a number of new solutions were obtained for the different types of flows considered. Some of the new solutions are given by equations (3.4.26), (3.4.33), (3.5.27), (4.4.19), (4.4.22), (4.4.42), (4.4.49), (4.5.35), (5.4.19).

Furthermore, a number of particular solutions to the diffusion equation were found such as equations (3.4.7), (3.4.31), (3.5.31) to mention a few.

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# REFERENCES

<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Title</th>
<th>Publisher/Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alféñ, H. (1950)</td>
<td>&quot;Cosmical electrodynamics&quot;</td>
<td>Oxford University Press</td>
</tr>
</tbody>
</table>
Chandna, O.P. and Swaminathan (1984)


Chandrasekhar, S. (1956)


"An Approximation theorem for functionals with applications in Continuum Mechanics". Arch. Rat. M.6

Fosdick, R.L. and Truesdell, C. (1977)

"Universal flows in the simplest theories of fluids". Annali Scholer Nor. Superiore, Sr. 4.323

Grad, H. (1960)

"Reducible problems in magnetofluid dynamic steady flows". Rev. Mod. Phys.; 32, pp. 830-847.

Grad, H. and Chandna, O.P. (1976)


Gunderson, R.M.  
(1966)  "Steady two-dimensional magnetohydrodynamic flow".  

Hamel, G.  
(1916)  "Spiralförmige bewegungen zaher flüssigkeiten".  
Jber. Deutch. Math.-Verein, 25, p. 34-60  
(English translation: "Spiral Motions of Viscous Fluids". NACA Tech. Men 1342)

Kaloni, P.N.  
and Huschilt, K.  
Int. J. Non-Linear. Mech., Vol. 19,  
No. 4, pp. 373-381

Kamke, E.  
(1930)  "Differentialgleichungen reeller Funktionen".  
pp. 29-34. Leipzig.

(1951)  Differentialgleichungen. Lösungsmethoden und Lösungen".  

Kingston, J.G.  
and Power, G.  
(1968)  "An analysis of two-dimensional aligned field Magnetogasdynamics flows"  

Kingston, J.G.  
and Talbot, R.  
(1969)  "The solutions to a class of Magnetohydrodynamics flows with orthogonal magnetic and velocity field distributions".  

Ladicov, I.P.  
(1962)  "Properties of plane and axisymmetrical stationary flows in magnetohydrodynamics".  

Lundquist, S.  
(1952)  "Studies in Magnetohydrodynamics".  
Arkiv F. Fysik, 5, pp. 297-347
Martin, M.H.  
(1971)  
"The flow of a viscous fluid".  
Arch. Rat. Mech. Anal., 41, pp. 266-286  

Power, G.  
and Talbot, R.  
(1969)  
"Magnetogasdynamics flows in two dimensions with orthogonal magnetic and velocity field distributions".  

Power, G.  
and Walker, D.  
(1965)  
"Plane gasdynamic flows with orthogonal magnetic and velocity field distribution".  

Prim, R.C.  
(1952)  
"Steady rotational flow of ideal gases".  

Ranger, K.B.  
(1969)  
"Exact solutions for rotating bodies in a viscous conducting fluid".  
Phys. Fluids, 12, pp. 776-777  

Resler, E.L.  
and McCune J.E.  
(1959)  
"Some exact solutions in linearized magnetoaerodynamics for arbitrary magnetic Reynold numbers".  
Rev. Mod. Phys., 32, pp. 843-854  

Riabouchinsky, D.  
(1924)  
"Some considerations regarding plane irrotational motion of a liquid".  

Shercliff, J. A.  
(1953)  
"Steady motion of conducting fluids in pipes under transverse magnetic fields".  
Pro. Comb. Phil. Soc., 55, pp. 141-143  

Smith, P.  
(1963)  
"Substitution principle for MHD flows"  
J. Math. Mech., 12, pp. 505-520  

Toews, H.  
and Chandna, O.P.  
(1974)  
"Steady transverse plane Magnetogasdynamic flows".  
Tensor, 28, pp. 184-188
(1974) Vinokur, M.  
"Plane magnetofluidodynamic flows with constantly inclined magnetic and velocity fields".  
Can. J. Phys., 52, pp. 753-758  
(1961)  
Waterhouse, J.S. and Kingston, J.G.  
"Kinematic formulation of rotational flow in magnetogasdynamics".  
Lochheed Aircraft Corp., Tech. Report 6-90-61-10  
(1973)  
"Plane magnetohydrodynamics flows with constantly inclined magnetic and velocity fields".  
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