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Robustness of Rotating Stall Control for Axial Flow Compressors

by

Ali Tahmasebi pour

A Thesis

**Submitted to the Faculty of Graduate Studies and Research through the
Department of Electrical and Computer Engineering in partial fulfillment
of the requirements for the Degree of Master of Applied Science at the
University of Windsor.**

**Windsor, Ontario, Canada
2002**

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Abstract

The useful range of operation in axial flow compressors is limited by aerodynamic flow instabilities such as rotating stall. Feedback control has been pursued to address the rotating stall problem in axial flow compressors in order to extend the stable operating range and to improve engine performance. These controllers guarantee the stability of the bifurcated operating solution near the stall point. However, how robust these controllers could be is still not clear. In this thesis, after presenting an introduction to the problem and providing the mathematical background of bifurcated systems and bifurcation stabilization, local robust analysis is applied systematically to evaluate feedback control design for rotating stall. Different combinations of perturbation in the system model have been considered. In particular, the set of admissible uncertainty is characterized analytically in terms of feedback control gain so that it is possible to compare the robustness of controllers with different feedback gains. We are then able to use this knowledge as a selection criteria to choose a more robust controller from a set of existing control laws. The nonlinear feedback controller is also studied for any possible advantage over the linear control laws.

To
My Wife Who Gives Me Reason To Live,
My Mother Who Gave Me Life,
The Memory Of My Father Who Will Always Be My Hero.

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Chapter 1

Introduction

1.1 Historic Review

Down through the years, human needs and desires have required a continued evolution of more and more sophisticated fluid-handling apparatus. In general, fluid-handling involves two problems, fluid transportation and fluid pressurization.

Ancient man was most concerned with liquid transport and storage. Of primary concern was irrigation for agricultural purposes and transport of water to cities.

The Bronze Age, which began at about 3000 B.C., brought with it the requirement of mechanisms for enhancing air supply to hearth furnaces. Air was first introduced in hearths by crude drafts and simple fanning. As time passed, innovation brought improved air supply devices. Hearths were oriented to capture the prevailing winds, and chimneys were added to help draw more air to the furnace.

With the arrival of the Iron Age, which began around 1000 B.C., no longer were single drafting techniques adequate. A much higher hearth temperature required a pressurized air blast. Small foot- and hand-operated bellows were used in the small hearths of the farrier and blacksmith. Five hundred years ago immense bellows were used in Germany to supply the air required for large furnaces. These were ultimately supplemented by piston pumps. Today, rotary compressors are used almost exclusively for this purpose.

The Industrial Revolution and, most recently, the Space Age, have produced an exponential growth in the advancement of turbomachinery, from the simple squirrel cage fan in a car's heater to the liquid fluid pumps used in the space shuttle engines.

Compressors

Compressors are used in a wide verity of applications. These includes turbo-jet engines used in aerospace propulsion, power generation using industrial gas turbines, turbocharging of internal combustion engines, pressurization of gas and fluids in the process industry, transport of fluids in the pipelines and so on.

Definition of Compressor

A compressor is a device that transfers energy to a gaseous fluid for the purpose of raising the pressure of the fluid as in the case where the compressor is the prime mover of the fluid through the process. The purpose may also

include a desired temperature rise to enhance the chemical reaction in the process [22].

Devices that develop less than 5.0 psi, or that affect a 7% density increase from inlet to discharge, are classified as fans or blowers. Above this level, the devices are referred to as compressors. Due to the low density change, fan equations assume constant density, thus simplifying the calculations.

Pumps are very similar to compressors but deal primarily with incompressible hydraulic fluids, whereas compressors generally deal with compressible gaseous fluids.

Types of Compressors

The two basic types of compressors are positive displacement and dynamic.

Positive Displacement Compressor: The positive displacement compressor functions by means of entrapping a volume of gas and reducing that volume, as in the common bicycle pump and the screw compressor. The general characteristics of the positive displacement compressor are constant flow and variable pressure ratio (for a given speed).

Positive displacement compressors include:

- piston compressor
- screw compressor
- vane compressor

- lobe compressor

Dynamic Compressor: The dynamic compressor depends on motion to transfer energy from the compressor rotor to the process gas. The characteristics of the compression vary depending on the type of gas being compressed. The flow is continuous. There are no valves and there is no 'containment' of the gas, as in a positive displacement compressor. Compression depends on the dynamic interaction between the mechanism and the gas.

Dynamic compressors include:

- ejector
- centrifugal compressor
- axial compressor

Ejector An ejector is a very simple device which uses a high-pressure jet stream to compress gas. The momentum of the high-pressure jet stream is transferred to the low-pressure process gas. This type of compressor is commonly used for vacuum applications.

Centrifugal Compressor A centrifugal compressor acts on a rotating impeller. The rotary motion of the gas results in an outward velocity due to centrifugal forces. The tangential component of this outward velocity is then transformed to pressure by means of a diffuser.

Axial Compressor An axial compressor imparts momentum to a gas by means of a cascade of airfoils. The lift and drag coefficients of the airfoil shape

determine the compressor characteristics. A typical axial flow compressor is shown in Figure 1.1.

Figure 1.1: A typical axial flow compressor

Centrifugal and axial compressors, also known as turbocompressors or continuous flow compressors, basically work by the principle of accelerating the fluid to a high velocity and then converting this kinetic energy into potential energy, manifested by an increase in pressure, by decelerating the gas in diverging channels. In axial compressors the deceleration takes place in

the stator blade passages, and in centrifugal compressor it takes place in the diffuser.

One obvious difference between the last two types of compressors is, in axial compressors, the flow leaves the compressor in the axial direction, whereas, in centrifugal compressors, the flow leaves the compressor in a direction perpendicular to the axis of the rotating shaft.

Aerodynamic Components of Axial Compressor

The major components of an axial compressor consist of (See Figure 1.2) (a) inlet nozzle, (b) prewhirl vanes, (c) rotating vanes, (d) stator vanes, (e) dewhirl vanes, and (f) discharge nozzle.

The nozzle guides and accelerates the gas stream into the prewhirl vanes which turn the gas stream to properly align it with the rotating blades. While both the rotating and stationary vanes act as diffusers, the rotating vane's primary function is to add to the total energy of the gas stream. Stator vanes, acting both as diffusers and reversing blades, orient the flow properly for the next row of rotating blades. While the first few rows of stator vanes are generally adjustable to compensate for off-design conditions, most rows of the stator vanes are fixed. The dewhirl vanes remove the swirl from the gas stream before it enters the diffuser section.

The useful range of operation of turbocompressors is limited, by choking at high mass flows when sonic velocity is reached in some component, and at low mass flows by the onset of two instabilities known as surge and rotating

Figure 1.2: Axial compressor

stall. Traditionally, these instabilities have been avoided by using controllers that prevent the operating point of the compression system to enter the unstable regime to the left of the *surge line*, that is the stability boundary. A fundamentally different approach, known as active surge/stall control, is to use feedback to stabilize this unstable regime. This approach will allow operation in the peak efficiency and the pressure rise regions located in the neighborhood of the surge line, as well as an extension of the operating range of the compressor.

1.2 Surge and Rotating Stall in Compressors

Compression systems such as gas turbines can exhibit several types of instabilities: combustion instabilities, aeroelastic instabilities such as flutter and finally aerodynamic flow instabilities, which this study is restricted to.

Two types of aerodynamic flow instabilities can be encountered in compressors. These are known as *surge* and *rotating stall*. The instabilities limit the range in which the compressor can operate. Surge and rotating stall also restrict the performance (pressure rise) and efficiency of the compressor.

Surge

Surge is an axisymmetrical oscillation of the mass flow along the length of the compressor in the axial direction. A simple diagram of surge is shown in Figure 1.3.

Figure 1.3: Axially oscillating flow in surge

Surge is especially harmful to an axial compressor because of the relatively large mass flow rate of gas and relatively thin blades. Besides reverse bending stresses and eventual fatigue, there is a problem of thermal growth. During surge, discharge gas is being forced back through the compressor then recompressed. The compressor is now compressing heated gas and temperatures rise quickly, causing the blades to grow, eventually resulting in a rub.

Surge is characterized by a limit cycle in the compressor characteristics. An example of such characteristics is shown as the S-shaped curve in Figure 1.4. The dotted segment of the curve indicates that this section usually is an approximation of the system, as it is difficult to measure experimentally.

It is common to distinguish between at least two different types of surge: 1) Mild/Classic surge and 2) Deep surge. A combination of surge and rotating stall is known as modified surge.

The first of these types is a phenomenon with oscillations in both pressure and flow in the compressor system, while in the second type, the oscillations in mass flow have such a large amplitude, that flow reversal occurs in the compression system. A drawing of a typical deep surge cycle is shown in Figure 1.4. The cycle starts at (1) where the flow becomes unstable. It then jumps to the reverse flow characteristics (2) and follows this branch of the characteristics until approximately zero flow (3), and then jumps to (4) where it follows the characteristics to (1), and the cycle repeats. Surge can occur in both axial and centrifugal compressors.

Figure 1.4: Compressor characteristics with deep surge cycle

Rotating Stall

Rotating stall can occur in both axial and centrifugal compressors. Although rotating stall is known to happen in centrifugal compressors, there exists little theory on the subject, and according to de Jager [14] its importance is still a matter of debate. In this thesis, only rotating stall in axial compressors will be considered, and when it is referred to rotating stall it is to be understood that an axial compressor is considered.

Rotating stall is an instability where the circumferential flow pattern is disturbed. This is manifested through one or more *stall cells* of reduced flow or stalled, propagate around the compressor annulus at 20-70% of the rotor speed [21], as illustrated in Figure 1.5.

Figure 1.5: Circumferentially nonuniform flow in rotating stall

Rotating stall leads to a reduction of the pressure rise, and in the compressor map this corresponds to the operating on the so called in-stall characteristics, see Figure 1.7. In this state, the compressor operates at a much

lower performance level, which is undesirable.

Figure 1.6: Physical mechanism for inception of rotating stall

The basic explanation of the rotating stall mechanism was given by Emmons *et al.* [15] and can be summarized as follows:

Consider a row of axial compressor blades operating at a high angle of attack, as shown in Figure 1.6. Suppose that there is a non-uniformity in the inlet flow such that a locally higher angle of attack is produced on blade B which is high enough to stall it. The flow now separates from the suction surface of the blade, producing a flow blockage between B and C. This

blockage causes a diversion of the inlet flow away from B towards A and C, resulting in a increased angle of attack on C, causing it to stall. Thus the stall cell propagates along the blade row.

It is common to distinguish between at least two types of rotating stall, full-span and part-span. In full-span stall, the complete height of the annulus is stalled, while in part-span rotating stall a restricted region of the blade passage is stalled. Full-span stall is most likely to happen in high hub/tip ratio axial compressors. In addition we can have various degrees of rotating stall depending on the size of the area of the compressor annulus being blocked.

Because of the extremely low efficiencies associated with rotating stall (below 20%) operation for any substantial period of time in this mode can result in excessive internal temperatures which have an adverse effect on blade life. The situation is further complicated because the blades go in and out of stalled flow, leading to severe vibrations with rapid and catastrophic consequences. Moreover, if a natural frequency of vibration of the blades coincides with the frequency at which the stall cell passes a blade, the result is resonance and possible mechanical failure due to fatigue.

Another consequence of rotating stall is the hysteresis occurring when trying to clear the stall by using the throttle. Once rotating stall is encountered, it may not be possible to return to an unstalled condition only by opening the throttle, because of system hysteresis effects. In this situation, which is called 'stagnation', the only way to come out of stall may be to

decrease the rotational speed considerably or even to shut the engine down and restart it.

Figure 1.7: Schematic drawing of hysteresis caused by rotating stall

The hysteresis phenomena can be explained on the schematic compressor characteristics, Figure 1.7. In this graph, Ψ denotes the pressure rise, Φ is the average flow rate and γ indicates the throttle position. The designed operating point is uniquely determined by the intersection of the throttle line (dashed lines A-B or C-D for two different values of the throttle position) with the compressor performance curve $\psi_c(\cdot)$ which is governed by the equation:

$$\psi_c(\Phi) = c_0 + c_1\Phi + c_3\Phi^3 \quad (1.1)$$

The maximum pressure rise takes place at point A because this is where the derivative of ψ_c equals to zero. Any point on right side of the peak of ψ_c is a stable operating point [35], and the uniform flow is a stable operating point, as indicated by the solid line. However if the throttle position γ decreases, then the flow rate intensity Φ decreases, and the derivative of ψ_c eventually changes its sign into positive. Therefore left side of the peak point A on ψ_c corresponds to unstable operating points, as indicated by dotted line. At the critical value of $\gamma = \gamma_c$, stall cells will be born at point A. Furthermore if the underlying bifurcation associated with rotating stall is subcritical, or unstable, the A-C portion of the stall curve is unstable. The operating point will be changed from A to B quickly which is a stable, although undesirable operating point on the in-stall characteristics Ψ_s , because there is a tremendous drop in both the pressure rise and flow rate. Moreover, increasing the throttle position and flow rate at point B does not increase the pressure rise. Rather the pressure rise becomes even lower before it reaches point C at which, due to again loss of stability, it jumps back to the performance curve ψ_c . The hysteresis loop A-B-C-D is the main cause for the loss of compressor performance, and the potential damage to aeroengines.

1.3 Overview on Control of Compressors

Modelling of Compression Systems

An essential step in controller design is to understand the physical phenomena in the system and to develop a mathematical model that describes the most important phenomena. In the literature, various models can be found for rotating stall and surge. Table 1.3 gives an overview of some of these models [46]. The categorization is based on the applied flow description: *one-dimensional models* can only describe axisymmetric flows, whereas *two-dimensional models* can predict flow variations in both axial and circumferential directions [33].

<i>Model</i>	<i>Flow Description</i>	<i>Machine</i>	<i>Instab.</i>
Badmus <i>et al.</i> [6]	Quasi-1-D Compressible	AC	S
Botros [9]	1-D Compressible	AC	S
Greitzer [20]	1-D Incompressible	A	S
Hansen <i>et al.</i> [26]	1-D Incompressible	C	S
Fink <i>et al.</i> [18]	1-D Incompressible	C	S
Moore [36]	2-D Incompressible	A	R
Adomaitis [3]	2-D Incompressible	A	R
Moore-Greitzer [35]	2-D Incompressible	A	SR
Haynes <i>et al.</i> [27]	2-D Incompressible	A	SR
Gravdahl-Egeland [19]	2-D Incompressible	A	SR
<i>Type of machine described by the model:</i> A: axial compressor, C: centrifugal compressor; <i>Instability described by model-</i> S: surge, R: rotating stall.			

Table 1.3- Some of the models for compression system

With the exception of Badmus *et al.* [6], and Botros [9], in all the models mentioned in Table 1.3, the flow in the ducts is assumed to be incompressible.

This assumption only holds for low-speed machines [23]. The hub-to-tip ratio is also supposed to be small, so radial effects can be neglected. For a suitable choice of the time lag associated with unsteady compressor response, the Greitzer model shows reasonable agreement with experiments [20].

For dynamic analysis and controller design in axial compressors, the low-dimensional Greitzer [20], Moore [36], and Moore-Greitzer [35] models can be used. Based on the work of Moore [36], the Moore-Greitzer model was derived in [35]. This three-state model is capable of describing both surge and rotating stall, with the third state being the stall amplitude. The Greitzer and Moore-Greitzer models are capable of predicting transients subsequent to the onset of compressor instability. In [36] and [35], rotating stall is assumed to be the result of growing modal waves. The Moore-Greitzer model is able to predict a mix of rotating stall and surge.

Many authors have extended and modified the Moore-Greitzer model. McCaughan [34] uses an approach from the point of view of bifurcation theory and with a one-mode truncation, reduces the Moore-Greitzer model to a set of three ordinary differential equations. After this modification, despite the severity of the truncation, the agreement with experimental results is excellent. This modified model is going to be used for the research done in this thesis.

Control of Rotating stall in Compressors

Surge/Stall Avoidance:

Until about a decade ago, the state of the art in control of compressor was the method of surge avoidance, which is essentially an open loop strategy. The compressor is prevented from operating in a region near and beyond the surge line. This is achieved by e.g. recirculation of the flow or blowing off flow through bleed valve. As the compressor characteristics and thus the surge line may be poorly known, it will be necessary to have a fairly conservative *surge margin* between the surge line and the *surge avoidance line*. The compressor is then not allowed to operate between these two lines in the map. Possible disturbances can also affect the size of the surge margin.

The drawbacks of the surge avoidance schemes are several: (1) Recycling and bleeding of compressed flow lower the efficiency of the system, (2) maximum efficiency and pressure rise may not be achievable at all, as they usually are achieved for mass flows close to the surge line and (3) the surge margin limits the transient performance of the compressor as acceleration of the machine tends to drive the system towards the surge line.

An alternative to surge avoidance is *surge detection and avoidance*. Using this strategy, the drawbacks of the surge margin can be avoided by the activation of the controller if the onset of instabilities is detected. De Jager [14] concludes that the main disadvantages of this strategy are problems associated with the detection of the instability onset and the necessity of large

control force and fast-acting control systems.

Active Stall Control

The approach of active stall control aims at overcoming the drawbacks of surge avoidance, by stabilizing some part of the unstable area in the compressor map using feedback. This approach was first introduced by Epstein *et al.* [16]. In the last decade, the work on feedback stabilization of compression systems has become extensive. This is partly due to the introduction and the success of the Moore-Greitzer model. The existing results include linearization and complex-valued proportional control (Paduano [38] and Epstein *et al.* [16]), bifurcation stabilization (Liaw and Abed [32], Chen *et al.* [10]), feedback linearization (Badmus *et al.* [5]), Lyapunov methods (Simon and Valavani [39]), backstepping method (Krstic *et al.* [30], Banaszuk *et al.* [8]), and H_∞ (van de Wal *et al.* [44]).

Among several possible actuators for stabilizing compression systems, the throttle position, as studied in Liaw and Abed [32], Wang *et al.* [45], and Chen *et al.* [10] is the most commonly considered. Bleed valves as studied by Eveker and Nett [17] and Murray [37] is another actuator considered in control design. Other possibilities are variable inlet guide vanes as studied in Paduano *et al.* [38], loudspeaker in Ffowcs Williams and Huang [47], tailored structures in Gysling *et al.* [25], recirculation in Balchen and Mummé [7], movable wall as in Epstein *et al.* [16], or air injection as in Day [13] and Behnken and Murray [12].

1.4 Overview of Thesis

The work reported in this thesis is the study of robustness for different rotating stall controllers of axial flow compressors.

A historic review and introduction to compressor instabilities are provided in Chapter 1 along with a review of control strategies available. Chapter 2 presents a section in bifurcation preliminaries followed by stability analysis and feedback control for systems with stationary bifurcations. Chapter 3 explores the Moore-Greitzer model for post-stall behavior of compression systems and the analysis of this model from the bifurcation point of view and then some of the stabilizing controllers proposed in literature are presented. Chapter 4 provides the main results of the research done for this thesis including the robustness of two stabilizing controllers with different combinations of perturbation added to the system model. A nonlinear controller is also examined followed by the simulation results. Chapter 5 presents conclusions and potential future work.

Chapter 2

Bifurcation and Stabilization

McCaughan [34] shows that the two instabilities of surge and rotating stall are associated with bifurcation. This fact and the growing popularity of Moore-Greitzer model, which is viewed as the simplest formulation that captures the physics of stall and surge phenomena, have motivated many studies in surge/stall control using bifurcation stabilization. In this chapter, we introduce some concepts in bifurcation, and discuss the theoretical foundation for the bifurcation stabilization techniques. The projection method is introduced as a tool to analyze local stability of the bifurcated solution and design of stabilizing control for stationary bifurcations.

2.1 Introduction to Bifurcations

Bifurcation in Dynamic Systems

Bifurcations exist in many dynamic systems, for example, a single pendulum, by a linear proportional-derivative (PD) controller, has various bifurcations

[29]. Even a feedback system with a linear plant and a linear controller can produce bifurcations and chaos if a simple nonlinearity (e.g. saturation) exists in the loop [4].

A typical double pendulum can display bifurcation as well as chaotic motions. A hopping robot, even a simple two-degree-of-freedom flexible robot arm, can produce unusual vibrations and undergo period-doubling oscillation which leads to chaos [43]. Dynamics of ships can exhibit bifurcations according to wave frequencies that are close to the natural frequency of the ship, creating oscillations and chaotic motions leading to ship capsizing [31]. Many other systems have bifurcation properties, including cellular neural networks, laser, weather, and biological population dynamics.

Therefore, controlling bifurcation has a tremendous impact on real-world applications and its significance in both analysis and control of dynamic systems is enormous, profound, and far-reaching.

Bifurcation Preliminaries

Before control methods can be discussed, mathematical definitions of various bifurcations are introduced in this section.

For this purpose, it is convenient to consider a parameterized, nonlinear dynamical system,

$$\dot{x} = f(x; p) \tag{2.1}$$

where p is a real variable parameter.

Let $(x^*) = (x^*(p_0))$ be an equilibrium point of the system at $p = p_0$,

satisfying $f(x^*; p) = 0$. If the equilibrium point is stable (resp. unstable) for $p > p_0$ but unstable (resp. stable) for $p < p_0$, then p_0 is a *bifurcation value* of p , and $(x^*; p_0)$ is a *bifurcation point* in the *parameter space* of coordinates (x, p) . A few examples are given below to distinguish several different but typical bifurcations.

2.1.1 Bifurcation of One-Dimensional Maps

The one dimensional system

$$\dot{x} = f(x; p) = px - x^2 \quad (2.2)$$

has two equilibria: $x_1^* = 0$ and $x_2^* = p$. If p is varied, then there are two equilibrium curves (see Fig. 2.1). Since the Jacobian of the system is $J = \partial f / \partial x |_{x=0} = p$, it is clear that for $p < p_0 = 0$, the equilibrium point $x_1^* = 0$ is stable, but for $p > p_0 = 0$ it becomes unstable. Hence, $(x_1^*, p_0) = (0, 0)$ is a bifurcation point. In Figure 2.1 and the following figures, the solid curves indicate stable equilibria and the dashed curves, the unstable ones. Similarly, one can verify that (x_2^*, p_0) is another bifurcation point. This is called a *transcritical bifurcation*.

The one-dimensional system

$$\dot{x} = f(x; p) = px - x^3 \quad (2.3)$$

has two equilibrium branches, $(x_1^*)^2 = \sqrt{p}$ and $(x_2^*)^2 = -\sqrt{p}$, at $p \geq 0$. Its Jacobian is $J = p - 3(x^*)^2$, so $x_1^* = 0$ is unstable for $p > p_0 = 0$ and stable

Figure 2.1: The transcritical bifurcation

for $p < p_0 = 0$. Also, the entire equilibrium curve $(x^*)^2 = p$ is stable for all $p > 0$ (at which the Jacobian is $J = -2p$). This is called a *pitchfork bifurcation*, and is depicted in Fig. 2.2.

Figure 2.2: The pitchfork bifurcation

Note, however, that not all nonlinear dynamical systems have bifurcations. This can be easily verified by similarly analyzing the following example:

$$\dot{x} = f(x; p) = p - x^3.$$

This equation has an entire stable equilibrium curve, $x = p^{1/3}$, and, thus, does not have any bifurcations.

2.1.2 Hopf Bifurcation of Higher-Dimensional Systems

The bifurcation phenomena discussed above for one-dimensional parameterized nonlinear maps are usually referred to as *stationary bifurcations*. In higher-dimensional systems or maps, the situation is more complicated. For instance, there is an additional bifurcation phenomenon for systems of dimension two or higher: the *Hopf bifurcation*, referred to as a *dynamic bifurcation*.

The Hopf bifurcations are classified as *supercritical* (resp. *subcritical*) if the equilibrium is changed from stable to unstable (resp. from unstable to stable). In other words, the periodic solutions have opposite stabilities as the equilibria. Note that the same terminology of supercritical and subcritical bifurcations apply to other non-Hopf types of bifurcations.

2.2 Stability Analysis of Stationary Bifurcation

Local Bifurcation Stability and Projection Method

This section considers the stability issue for bifurcated system using the projection method developed in [28]. The system under consideration is the

following n th order parameterized nonlinear system:

$$\dot{x} = f(\gamma, x), \quad f(\gamma, x_e) = 0 \quad \forall \gamma \in (-\delta, \delta), \quad (2.4)$$

where $x \in \mathbb{R}^n$, γ is a real valued parameter, and $\delta > 0$ is a sufficiently small real number. Without loss of generality, we can assume $x_e = 0$, $f(\gamma, 0) = 0$ in a small neighborhood of $\gamma = 0$, which is called the zero solution. The linearized system at the zero solution is given by

$$\dot{x}_0 = L(\gamma)x_0, \quad L(\gamma) = \left. \frac{df(\gamma, x)}{dx} \right|_{x=x_e=0}. \quad (2.5)$$

If $L(0)$ has one or more eigenvalues on the imaginary axis, then additional nonzero equilibrium solutions or bifurcated solutions will be born at $\gamma = 0$. It is assumed that $f(\cdot, \cdot)$ is sufficiently smooth such that the bifurcated solution $x_e \neq 0$, satisfying $f(\gamma, x_e) = 0$, is a smooth function of γ .

Definition 2.1. *The nonlinear system in (2.4) is said to have local bifurcation stability if the bifurcated solution is locally asymptotically stable for sufficiently small γ .*

Local Stability for Stationary Bifurcations

For stationary bifurcations, it is assumed that $L(\gamma)$ possesses a simple eigenvalue $\lambda(\gamma)$, depending smoothly on γ , satisfying

$$\lambda(0) = 0, \quad \lambda'(0) = \frac{d\lambda}{d\gamma}(0) \neq 0, \quad (2.6)$$

while all other eigenvalues are stable in a neighborhood of $\gamma = 0$. It implies that the zero solution changes its stability as γ crosses 0. For instance, $\lambda'(0) < 0$ implies that the zero solution is locally stable for $\gamma > 0$ and becomes unstable for $\gamma < 0$. Furthermore additional equilibria $x_e \neq 0$ will be born at $\gamma = 0$ that are smooth functions of γ by the smoothness of $f(\cdot, \cdot)$. Such bifurcated solutions are independent of time t and called stationary bifurcation. Thus $\gamma = 0$ is the critical value of the parameter and $\lambda(\gamma)$ is called the critical eigenvalue. The nonlinear system (2.4) at $\gamma = 0$ is referred to as the critical system. The bifurcated solution of the nonlinear system born at $\gamma = 0$ may or may not be locally stable. For simplicity, only double points [28] will be considered in this thesis. A useful tool to determine local stability of the bifurcated solution and of the critical system is the projection method developed in [28] and advocated in [2, 28].

Let ℓ and r denote the left row and right column eigenvectors of $L(0)$, corresponding to the critical eigenvalue $\lambda(0) = 0$. Then $\ell r = 1$ by suitable normalization. Denote $\varepsilon = \ell x_e$, where $x_e \neq 0$ satisfying $f(\gamma, x_e) = 0$ is also an equilibrium solution of (2.4), or bifurcated solution in a small neighborhood of $\gamma = 0$. Then by [28] there exists a series expansion

$$\begin{bmatrix} x_e(\varepsilon) \\ \gamma(\varepsilon) \end{bmatrix} = \sum_{k=1}^{\infty} \begin{bmatrix} x_{ek} \\ \gamma_k \end{bmatrix} \varepsilon^k. \quad (2.7)$$

Since $f(\gamma, x)$ is sufficiently smooth, there exists a Taylor expansion near the origin of $\mathbb{R}^n$ of the form

$$\dot{x} = f(\gamma, x) = L(\gamma)x + Q(\gamma)[x, x] + C(\gamma)[x, x, x] + \cdots \quad (2.8)$$

where $L(\gamma)x$, $Q(\gamma)[x, x]$, and $C(\gamma)[x, x, x]$ are vector valued linear, quadratic and cubic terms of $f(\gamma, x)$ respectively, and can each be expanded into

$$L(\gamma)x = L_0x + \gamma L_1x + \gamma^2 L_2x + \cdots ,$$

$$Q(\gamma)[x, x] = Q_0[x, x] + \gamma Q_1[x, x] + \cdots ,$$

and

$$C(\gamma)[x, x, x] = C_0[x, x, x] + \gamma C_1[x, x, x] + \cdots ,$$

where L_0 , L_1 , and L_2 are $n \times n$ constant matrices.

Let $\tilde{\lambda}(\gamma)$ be the critical eigenvalue of the linearized system matrix at the new (bifurcated) equilibrium close to the origin. Then $\tilde{\lambda}(0) = \lambda(0) = 0$ at $\gamma = 0$. There exists a series expansion [28]

$$\tilde{\lambda}(\varepsilon) = \sum_{i=1}^{\infty} \tilde{\lambda}_i \varepsilon^i = \tilde{\lambda}_1 \varepsilon + \tilde{\lambda}_2 \varepsilon^2 + \cdots$$

The computation of the first two coefficients of $\tilde{\lambda}$ can proceed as follows [2, 28]:

- Step 1: Calculate $\lambda'(0) = \ell L_1 r$ where λ is a function of γ .
- Step 2: Set $x_{e1} = r$, and calculate $\gamma_1 = -\ell Q_0[r, r]/\lambda'(0)$.
- Step 3: Compute x_{e2} from equations $\ell x_{e2} = 0$ and $L_0 x_{e2} = -Q_0[r, r]$, and γ_2 from

$$\gamma_2 = -\frac{1}{\lambda'(0)} (\gamma_1 \ell L_1 x_{e2} + \gamma_1^2 \ell L_2 r + 2\ell Q_0[r, x_{e2}] + \gamma_1 \ell Q_1[r, r] + \ell C_0[r, r, r]) .$$

- Step 4: Set $\tilde{\lambda}_1 = -\gamma_1 \lambda'(0)$ and $\tilde{\lambda}_2 = -2\gamma_2 \lambda'(0)$.

Local stability of a stationary bifurcation is given by the following lemma [1].

Lemma 2.2. *Suppose that all eigenvalues of L_0 are stable, except one critical eigenvalue. For the case $\gamma_1 \neq 0$, the branch of the bifurcated equilibrium solution is locally stable for γ sufficiently close 0 if $\ell Q_0[r, r]\varepsilon < 0$, and unstable if $\ell Q_0[r, r]\varepsilon > 0$. For the case $\gamma_1 = 0$, the bifurcated solution is locally stable for γ sufficiently close to 0 if $\tilde{\lambda}_2 < 0$, and unstable if $\tilde{\lambda}_2 > 0$, where*

$$\tilde{\lambda}_2 = 2\ell (2Q_0[r, x_{e2}] + C_0[r, r, r]), \quad x_{e2} = -(\ell^T \ell + L_0^T L_0)^{-1} L_0^T Q_0[r, r].$$

2.3 Stabilizing control of Stationary Bifurcations

Let the nonlinear control system be in the form

$$\dot{x} = f(\gamma, x) + g(x, u), \quad y = h(x), \quad x \in \mathbb{R}^n, \quad (2.9)$$

where $f(\gamma, x)$ is the same as in (2.8), and $g(\cdot, \cdot)$ and $h(\cdot)$ are also smooth functions satisfying

$$g(x, 0) = 0 \quad \forall x \in \mathbb{R}^n, \quad h(0) = 0. \quad (2.10)$$

It is assumed that $u \in \mathbb{R}$, and $y \in \mathbb{R}^p$ with $p \geq 1$. Thus the nonlinear system (2.9) has only one control input, but may have more than one output

measurement. Its Taylor series expansion is given by

$$\begin{aligned} \dot{x} = & L_0x + \gamma L_1x + \gamma^2 L_2x + B_1u + u\tilde{L}_1x + Q_0[x, x] + B_2u^2 \\ & + u\tilde{L}_{12}[x, x] + \gamma Q_1[x, x] + u^2\tilde{L}_2x + u^2\tilde{Q}_1[x, x] + C_0[x, x, x] + B_3u^3 + \cdots, \end{aligned} \quad (2.11)$$

where $\tilde{L}_1x$ and $\tilde{Q}_1[x, x]$ are the linear and quadratic terms for the linear control component of $g(x, u)$, $\tilde{L}_2x$ is the linear term for the quadratic control component and $\tilde{L}_{12}[x, x]$ is the quadratic term for the linear control component of $g(x, u)$, and B_1, B_2, B_3 are the coefficient vectors of u, u^2 and u^3 , respectively. It is assumed that L_0 has only one zero eigenvalue with the rest of the eigenvalues stable, and that the bifurcated solution born at $\gamma = 0$ is not locally stable. The assumption on stability of the nonzero eigenvalues of L_0 has no loss of generality. If some of the nonzero eigenvalues of L_0 are unstable, then any linear control method, such as pole placement, can be employed to stabilize those unstable modes corresponding to nonzero eigenvalues. It is the unstable mode corresponding to the critical eigenvalue $\lambda(0) = 0$ that renders linear control methods inadequate because of bifurcations.

We seek a smooth local output feedback control law of the form

$$u = K(y) = K_Ly + K_Q[y, y] + K_C[y, y, y] + \cdots, \quad (2.12)$$

that stabilizes the bifurcated system (i.e., the closed-loop system admits bifurcation stability; See Definition 2.1), where $K_Q[\cdot, \cdot], K_C[\cdot, \cdot, \cdot]$ are the quadratic, and cubic terms of $K(y)$ respectively, satisfying

$$K_Q[0, 0] = K_C[0, 0, 0] = 0.$$

The output has a Taylor series expansion

$$y = h(x) = H_1x + H_2[x, x] + H_3[x, x, x] + \cdots, \quad (2.13)$$

where $H_1x, H_2[x, x], H_3[x, x, x]$ are the linear, quadratic and cubic terms of $h(x)$ respectively, satisfying $H_2[0, 0] = H_3[0, 0, 0] = 0$. Without loss of generality, it is assumed that linear, quadratic and cubic terms of the feedback control law in (2.12) are of the form

$$\begin{cases} K_L y = K_1 h(x) \\ K_Q[y, y] = K_2 \tilde{H}_2[x, x] + K_2 \tilde{H}_3[x, x, x] + \cdots \\ K_C[y, y, y] = K_3 \tilde{\tilde{H}}_3[x, x, x] + \cdots, \end{cases} \quad (2.14)$$

with K_1, K_2 and K_3 some constant matrices, and $\tilde{H}_2[0, 0] = \tilde{H}_3[0, 0, 0] = \tilde{\tilde{H}}_3[0, 0, 0] = 0 \in \mathbb{R}^p$.

Substituting the control law into (2.11), we get the closed-loop system equation in series form:

$$\dot{x} = L_0^*x + \gamma L_1^*x + Q_0^*[x, x] + \gamma^2 L_2^*x + \gamma Q_1^*[x, x] + C_0^*[x, x, x] + \cdots \quad (2.15)$$

where the linear, quadratic and cubic terms are given by

$$\begin{aligned} L_0^* &= L_0 + B_1 K_1 H_1, \quad L_1^* = L_1, \quad L_2^* = L_2, \quad Q_1^*[x, x] = Q_1[x, x], \\ Q_0^*[x, x] &= Q_0[x, x] + B_1 \left(K_1 H_2[x, x] + K_2 \tilde{H}_2[x, x] \right) + K_1 H_1 x \tilde{L}_1 x + B_2 (K_1 H_1 x)^2, \\ C_0^*[x, x, x] &= C_0[x, x, x] + \left(K_1 H_2[x, x] + K_2 \tilde{H}_2[x, x] \right) \tilde{L}_1 x \\ &\quad + 2B_2 K_1 H_1 x \left(K_1 H_2[x, x] + K_2 \tilde{H}_2[x, x] \right) + (K_1 H_1 x)^2 \tilde{L}_2 x \\ &\quad + B_1 \left(K_1 H_3[x, x, x] + K_2 \tilde{\tilde{H}}_3[x, x, x] + K_3 \tilde{\tilde{H}}_3[x, x, x] \right) \\ &\quad + K_1 H_1 x \tilde{L}_{12}[x, x] + K_1 H_1 x \tilde{Q}_1[x, x] + B_3 (K_1 H_1 x)^3. \end{aligned}$$

Abed and Fu studied this problem in [1] for the case of state feedback where the critical mode of the linearized system at $\gamma = 0$ is controllable. In this research, the case is that for the output feedback, the critical mode of L_0 is linearly uncontrollable. It should be clear that in practice, measurement of all state variables is unrealistic. Often only partial, or a nonlinear function of state variables are measurable. Under this circumstance, the critical mode of the linearized system may, or may not be observable based on linearized output measurements.

For rotating stall control, as we will establish later, the nonlinear system has transcritical bifurcation where the critical mode is linearly observable based on output measurement. The results from [24] for stabilization of this system are summarized in the next subsection.

Observable Critical Mode

Consider the transcritical bifurcation. Without loss of generality, the branch of $\varepsilon > 0$ is assumed to be unstable for $\gamma > 0$. This is equivalent to $\bar{\lambda}_1 > 0$. In [24], a controller of the form (2.12) is developed, to stabilize the bifurcated solution for $\varepsilon > 0$ without changing the stability property of the zero solution. It is noted that by assumption the eigenvalue $\lambda(0) = 0$ is invariant under feedback control. Thus L_0^* also possesses the critical zero eigenvalue as L_0 does. Denote ℓ^* and r^* the left row and right column eigenvectors for L_0^* corresponding to the critical eigenvalue. Then $\ell^* = \ell$ due to the uncontrollability of the critical eigenvalue. Denote $\bar{\lambda}^*$ as the critical eigenvalue of

L_0^* under feedback. It has the series expansion in the form of

$$\tilde{\lambda}^*(\varepsilon) = \tilde{\lambda}_1^* \varepsilon + \tilde{\lambda}_2^* \varepsilon^2 + \dots \quad (2.16)$$

However, $r^* \neq r$ in general due to $H_1 r \neq 0$ by the observability of the critical mode. The next Theorem concerns the stabilization of the transcritical bifurcation [24].

Theorem 2.3. *Consider the nonlinear control system (2.11) with output feedback control law (2.12). Suppose that the critical mode of the linearized system corresponding to the zero eigenvalue at $\gamma = 0$ is observable. Then for the case $\tilde{\lambda}_1 > 0$, i.e., $\ell Q_0[r, r] > 0$, there exists a nonlinear feedback control law $u = K(y)$ that stabilizes the given branch of the bifurcated solution at $\varepsilon > 0$ if and only if there exists a linear feedback control law that stabilizes the given branch of the bifurcated solution at $\varepsilon > 0$. Moreover there exists a nonsingular matrix $T \in \mathbb{R}^{n \times n}$ such that*

$$T L_0 T^{-1} = \begin{bmatrix} L_{00} & 0 \\ 0 & 0 \end{bmatrix}, \quad T B_1 = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} H_1 \\ \ell \end{bmatrix} T^{-1} = \begin{bmatrix} H_{11} & H_{12} \\ 0 & 1 \end{bmatrix}, \quad (2.17)$$

where $L_{00} \in \mathbb{R}^{(n-1) \times (n-1)}$, $B_{11} \in \mathbb{R}^{(n-1) \times 1}$, and $H_{11} \in \mathbb{R}^{p \times (n-1)}$. Let ℓ_i be the i^{th} element of ℓ , and $r^T Q_{0k} r$ be the k^{th} element of $Q_0[r, r]$ with $Q_{0k} = Q_{0k}^T$ for $k = 1, 2, \dots, n$. Partition

$$(T^{-1})^T \left(\sum_{k=1}^n \ell_k Q_{0k} \right) T^{-1} = \begin{bmatrix} \bar{Q}_{00} & \bar{Q}_{01} \\ \bar{Q}_{10} & \bar{Q}_{11} \end{bmatrix}, \quad \bar{Q}_{10} = \bar{Q}_{01}^T, \quad \bar{Q}_{00} \in \mathbb{R}^{(n-1) \times (n-1)}. \quad (2.18)$$

Then $\bar{Q}_{11} = \bar{\lambda}_1 > 0$. The existence of a stabilizing feedback control law, subject to the same stability property for the zero solution, is equivalent to the existence of $K_1 \neq 0$ such that

- (i) $(1 + K_1 H_{11} L_{00}^{-1} B_{11}) (\bar{\lambda}_1 (1 + K_1 H_{11} L_{00}^{-1} B_{11}) + a K_1 H_{12}) + b (K_1 H_{12})^2 < 0$,
- (ii) $\lambda'(0) (\lambda'(0) + K_1 (H_{11} \lambda'(0) - H_{12} d) L_{00}^{-1} B_{11}) > 0$, and
- (iii) $L_{00}^* = L_{00} + B_{11} K_1 H_{11}$ is stable,

where $a = \bar{d}_0 - 2\bar{Q}_{10} L_{00}^{-1} B_{11}$, $b = \ell B_2 + ((L_{00}^{-1} B_{11})^T \bar{Q}_{00} - \bar{d}) L_{00}^{-1} B_{11}$, $[\bar{d} \ \bar{d}_0] = \ell \bar{L}_1 T^{-1}$ with $\bar{d}_0$ scalar, $d^T = [I_{n-1} \ 0] (\ell L_1 T^{-1})^T$, and $\lambda(\gamma)$ is the critical eigenvalue in (2.6).

For detailed proof, see [24].

Nonsingular matrix T exists, by Kalman decomposition, such that (2.17) holds where the lower triangular Schur form of $T L_0 T^{-1}$ is used. It is shown that

$$r^* = T^{-1} \begin{bmatrix} \eta \\ 1 \end{bmatrix}, \quad \eta = -\frac{L_{00}^{-1} B_{11} K_1 H_{12}}{1 + K_1 H_{11} L_{00}^{-1} B_{11}}. \quad (2.19)$$

Theorem 2.3 indicates that stabilization of a transcritical bifurcation is possible by using just a linear control law. More importantly, the conditions (i) – (iii) also provide explicit formulas for synthesis of a stabilizing linear gain K_1 . Indeed, for the case of $p = 1$, K_1 is scalar. The set of K_1 satisfying each of (i) – (iii) can be easily computed that is either finite intervals, or semi-infinite intervals.

Theorem 2.3 has implications to state feedback control:

$$u = K(x) = K_1 x + K_Q[x, x] + K_C[x, x, x] + \cdots,$$

with K_1 the linear state feedback gain, and $K_Q[\cdot, \cdot]$, $K_C[\cdot, \cdot, \cdot]$ the quadratic and cubic terms respectively. The next result is a direct consequence of Theorem 2.3.

Corollary 2.4. *Suppose that all the hypothesis in Theorem 2.3 hold. Then there exists a nonlinear state feedback control law $u = K(x)$ that stabilizes the given branch of the bifurcated solution if and only if there exists a linear state feedback control law that stabilizes the given branch of the bifurcated solution. Moreover with the same notation as in Theorem 2.3, the existence of stabilizing state feedback control law, subject to the same stability property for the zero solution, is equivalent to the existence of a $K_1 = [K_{11} \ K_{12}] T \neq 0$ such that*

- (i) $(1 + K_{11}L_{00}^{-1}B_{11}) \left(\bar{\lambda}_1(1 + K_{11}L_{00}^{-1}B_{11}) + aK_{12} \right) + bK_{12}^2 < 0,$
- (ii) $\lambda'(0) (\lambda'(0) + (K_{11}\lambda'(0) - K_{12}d) L_{00}^{-1}B_{11}) > 0,$ and
- (iii) $L_{00}^* = L_{00} + B_{11}K_{11}$ is stable.

proof The corollary can be easily proven by setting $K_1 H_{11} \rightarrow K_{11}$, $K_1 H_{12} \rightarrow K_{12}$, and noting that

$$K_1 = [K_{11} \ K_{12}] T.$$

with T the similarity transform, as in Theorem 2.3. ■

Chapter 3

Rotating Stall Control for Axial Compressors

3.1 Moore-Greitzer Model for Post-Stall Dynamics

Several dynamic models for the unstable operation of compression systems have been proposed in the last decade, but the model of Moore and Greitzer [35] stands out in the sense that rotating stall amplitude is included as a state, and not manifested as a pressure drop which is the case in the other models. The low order Moore-Greitzer model captures the post-stall transients of a low speed axial compressor system.

In developing this model, some of the assumptions made are: Large hub-to-tip ratio, irrotational and inviscid flow in the inlet duct, incompressible mass flow, short throttle duct, small pressure rise compared to ambient conditions and constant rotor speed. The three differential equations of the model

arises from a Galerkin approximation of the local momentum balance, the annulus-averaged momentum balance and the mass balance of the plenum.

The moore-Greitzer model modified by McCaughan [34] is in the form of three ordinary differential equations:

$$\dot{\Psi} = \frac{1}{\beta^2} \left(\Phi - (\gamma + u)\sqrt{\Psi} + 1 \right), \quad (3.1)$$

$$\dot{\Phi} = -\Psi + \psi_c(\Phi) + 6c_3\Phi R, \quad (3.2)$$

$$\dot{R} = \sigma R(1 - \Phi^2 - R), \quad (3.3)$$

where $\psi_c(\Phi) = c_0 + c_1\Phi + c_3\Phi^3$, Φ is the average flow rate, Ψ the pressure rise, R the amplitude square of the disturbance flow, γ the throttle position and u the actuating signal implemented with throttle, which are all non-dimensionalized. σ and β are design constants.

3.2 Bifurcation Analysis of Moore-Greitzer Model

It is shown that (see[34]) at the peak of the compressor characteristic curve, $\gamma = \gamma_c$, a transcritical bifurcation occurs which leads to rotating stall, and the eigenvalue corresponding to $\dot{R}$ equation becomes unstable.

An obvious equilibrium (Ψ_e, Φ_e, R_e) for $u = 0$ satisfies

$$R_e = 0, \quad \Psi_e = \psi_c(\Phi_e), \quad \Psi_e = \frac{1}{\gamma}(1 + \Phi_e)^2, \quad \Phi_e = 1. \quad (3.4)$$

It can be shown that there exists a $\gamma_c > 0$ such that the above equilibrium is stable for $\gamma > \gamma_c$, but unstable for $\gamma < \gamma_c$ [34]. Denote

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \Psi - \Psi_e \\ \Phi - \Phi_e \\ R \end{bmatrix},$$

$$g(x) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \frac{\sqrt{\Psi}}{\beta^2} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \frac{\sqrt{x_1 + \Psi_e}}{\beta^2}.$$

Then we have the following equation:

$$\dot{x} = f(\delta\gamma, x) + g(x)u, \quad \delta\gamma = \gamma - \gamma_c, \quad (3.5)$$

where

$$u = K_1 x + K_2[x, x] + K_3[x, x, x] + \dots,$$

$$f(\delta\gamma, x) = \begin{bmatrix} \frac{1}{\beta^2}(x_2 + \Phi_e - (\delta\gamma + \gamma_c)\sqrt{x_1 + \Psi_e} + 1) \\ -x_1 - \Psi_e + \psi_c(x_2 + \Phi_e) + 6c_3(x_2 + \Phi_e)x_3 \\ \sigma x_3(1 - (x_2 + \Phi_e)^2 - x_3) \end{bmatrix}.$$

Thus the equilibrium in (3.4) is the zero solution for $u = 0$, and both Ψ_e and Φ_e are functions of γ . Moreover the linearized system at the origin possesses exactly one zero eigenvalue at $\gamma = \gamma_c$. The equilibrium if $\gamma = \gamma_c$ in (3.4) and γ_c can be found as (see [32, 45]):

$$R_c = 0, \quad \Phi_c = 1, \quad \Psi_c = \Psi_c(\Phi_c) = c_0 + c_1 + c_3 = 11/3,$$

$$\gamma_c = \frac{2}{\sqrt{\Psi_c}} = 2\sqrt{\frac{3}{11}},$$

where $c_0 = 8/3$, $c_1 = 1.5$, and $c_3 = -0.5$.

The transcritical bifurcation associated with rotating stall is subcritical and unstable. It is easy to show that the linearization of the system (3.1), (3.2) and (3.3) around the equilibrium point $(\Psi_e, \Phi_e, 0)$ is uncontrollable, in other words, it is impossible to design a smooth state feedback to stabilize the equilibria beyond the peak of the compressor characteristic curve. Nevertheless, a controller in the form of $u = K.x$ [32] can be designed, such that the bifurcation of the closed loop system is supercritical and stable. By changing the criticality of the bifurcation, the stall inception becomes progressive and more benign and the hysteresis loop will be eliminated.

3.3 Feedback Control of Rotating Stall

Clearly the compressor model (3.5) can be expanded into the same form as (2.11) and

$$L_0 = \begin{bmatrix} -\frac{\gamma_c \beta^{-2}}{2\sqrt{\Psi_c}} & \beta^{-2} & 0 \\ -1 & c_1 + 3c_3\Phi_c^2 & 6c_3 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \frac{\sqrt{\Psi_c}}{\beta^2},$$

$$Q_0[x, x] = \begin{bmatrix} \frac{\gamma_c}{8\beta^2\Psi_c^{\frac{3}{2}}}x_1^2 \\ 3c_3\Phi_c x_2^2 + 6c_3x_2x_3 \\ -\sigma x_3^2 - 2\sigma\Phi_c x_2x_3 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} \frac{1}{2\beta^2\sqrt{\Psi_c}}\alpha & 0 & 0 \\ 0 & 6c_3\Phi_c \frac{d\Phi_c}{d\gamma}(\gamma_c) & 0 \\ 0 & 0 & -2\sigma\Phi_c \frac{d\Phi_c}{d\gamma}(\gamma_c) \end{bmatrix},$$

where $\alpha = \frac{\gamma_c \frac{d\Psi_c}{d\gamma}(\gamma_c)}{2\Psi_c} - 1$,

$$\bar{L}_1 = \begin{bmatrix} -\frac{1}{2\beta^2\sqrt{\Psi_c}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C_0[x, x, x] = \begin{bmatrix} -\frac{\gamma_c}{16\beta^2\Psi_c^{\frac{3}{2}}}x_1^3 \\ c_3x_2^3 \\ -\sigma x_2^2x_3 \end{bmatrix}, \quad r = \begin{bmatrix} 6c_3 \\ \frac{3c_3\gamma_c}{\sqrt{\Psi_c}} \\ 1 \end{bmatrix},$$

and $\ell = [0 \ 0 \ 1]$.

By Kalman decomposition:

$$T = \begin{bmatrix} 1 & 0 & -6c_3 \\ 0 & 1 & -\frac{3c_3\gamma_c}{\sqrt{\Psi_c}} \\ 0 & 0 & 1 \end{bmatrix}.$$

By using the relationships $TL_0T^{-1} = \begin{bmatrix} L_{00} & 0 \\ 0 & 0 \end{bmatrix}$ and $TB_1 = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}$:

$$L_{00} = \begin{bmatrix} -\frac{\gamma_c\beta^{-2}}{2\sqrt{\Psi_c}} & \beta^{-2} \\ -1 & c_1 + 3c_3\Phi_c^2 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} -\frac{\sqrt{\Psi_c}}{\beta^2} \\ 0 \end{bmatrix}.$$

From the control law we have:

$$K_1 = [K_\Psi \ K_\Phi \ K_R] = [K_{11} \ K_{12}]T, \quad K_{11} = [k_1 \ k_2], \quad K_{12} = k_3$$

then it can be calculated that

$$k_1 = K_\Psi, \quad k_2 = K_\Phi, \quad k_3 = K_R - 3K_\Psi - \frac{3}{\Psi_c}K_\Phi \quad (3.6)$$

Rotating Stall Controllers

A stabilizing controller $u = Kx = [k_\psi \ k_\phi \ k_R] x$ can be synthesized as (see [24, 45]):

$$k_\psi > -\Psi_c^{-3/2}, \quad k_\phi < \Psi_c^{-1/2}, \quad \frac{6 - \Psi_c}{\Psi_c^{3/2}} < -6k_\psi - k_\phi + 2k_R. \quad (3.7)$$

By taking $k_R = k_\phi = 0$, we obtain $u = k_\psi \delta \Psi$ and

$$-\frac{3}{11} \sqrt{\frac{3}{11}} = -\Psi_c^{-3/2} < k_\psi < \frac{1}{6\sqrt{\Psi_c}} - \Psi_c^{-3/2} = -\frac{7}{66} \sqrt{\frac{3}{11}} \quad (3.8)$$

which is the controller obtained in [10]. Taking $k_\psi = k_\phi = 0$, we obtain

$$u = k_R R, \quad k_R > \frac{6 - \Psi_c}{2\Psi_c^{3/2}} = 0.1175, \quad (3.9)$$

so the controller $u = k_R R = 0.5R$ as developed in [32, 45] is implied in the inequality (3.9).

A nonlinear controller is also obtained in [10]: $u_n = k_n \frac{\Psi - \Psi_c}{\sqrt{\Psi}}$ with

$$-\frac{1}{c_0 - 2c_3} = -\frac{3}{11} < k_n < -\frac{1}{12c_3} - \frac{1}{c_0 - 2c_3} = -\frac{7}{66}. \quad (3.10)$$

We will study the robustness of these controllers in the next chapter.

Chapter 4

Robustness of Rotating Stall Control

4.1 Local Robustness of Bifurcation Stabilization

Consider the nonlinear system F subject to uncertainty Δ as shown in Figure 4.1, where C is the controller. F is governed by the following dynamic equation:

$$\dot{x} = f(\gamma, x) + g(x)u + h(x)w, \quad (4.1)$$

where γ is a scalar parameter, u is a scalar control, w is a scalar disturbance signal and

$$f(\cdot, \cdot) : \mathbb{R}^1 \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad g(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

$$h(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

are all smooth functions with $f(\gamma, 0) = 0$.

Figure 4.1: Nonlinear Control System in Robust Consideration

The system is assumed to satisfy the following assumption (S):

1. $L(\gamma) = \left. \frac{df(\gamma, x)}{dx} \right|_{x=0}$ has a simple eigenvalue (critical mode) $\lambda(\gamma)$ satisfying:

$$\lambda(0) = 0, \quad \lambda'(0) = \left. \frac{d\lambda}{d\gamma} \right|_{\gamma=0} \neq 0,$$

while all other eigenvalues are stable in a neighborhood of $\gamma = 0$.

2. The critical mode of $L(0)$ is not linearly controllable by u and is not linearly affected by w , i.e., for all $\ell, r \neq 0$, $\ell L(0) = 0$, $L(0)r = 0$, we have $\ell g(0) = 0$, $\ell h(0) = 0$, $r^T g(0) = 0$ and $r^T h(0) = 0$.

The assumption (S) is corresponding to the stationary bifurcation. To derive the robustness result, we need the Taylor series expansion of the system (4.1):

$$\begin{aligned} \dot{x} = & L_0 x + \gamma L_1 x + \gamma^2 L_2 x + B_1 u + h_0 w + u \bar{L}_1 x + w h_1 x + Q_0[x, x] \\ & + u \bar{L}_2[x, x] + w h_2[x, x] + \gamma Q_1[x, x] + C_0[x, x, x] + \cdots, \end{aligned} \quad (4.2)$$

where L_0x, L_1x, L_2x and $\tilde{L}_1x$ are vectors of linear forms ; $Q_0[x, x], \tilde{L}_2[x, x]$ and $Q_1[x, x]$ are vectors of quadratic forms; $C_0[x, x, x]$ is a vector of cubic form. The uncertainty w is treated as a smooth memoryless state feedback with the Taylor series expansion:

$$w = P_1x + P_2[x, x] + P_3[x, x, x] + \dots \quad (4.3)$$

Suppose the system (4.1) yields a transcritical bifurcation, i.e., $\tilde{\lambda}_1 = \ell Q_0[r, r] \neq 0$, where ℓ and r are the left row and right column eigenvectors of L_0 corresponding to the critical eigenvalue and are chosen such that $\ell r = 1$. Then a class of linear stabilizing controllers $u = Kx$ can be synthesized[24]. Substituting $u = Kx$ into the system, we get:

$$\dot{x} = f(\gamma, x) + g(x)Kx + h(x)w = f'(\gamma, x) + h(x)w, \quad (4.4)$$

where $f'(\gamma, x) = f(\gamma, x) + g(x)Kx$, $f'(\gamma, 0) = 0$. In this case the Taylor series expansion becomes:

$$\begin{aligned} \dot{x} = & L'_0x + \gamma L'_1x + \gamma^2 L'_2x + h_0w + wh_1x + Q'_0[x, x] \\ & + wh_2[x, x] + \gamma Q'_1[x, x] + C'_0[x, x, x] + \dots, \end{aligned}$$

where

$$L'_0 = L_0 + B_1K, \quad L'_1 = L_1, \quad L'_2 = L_2,$$

$$Q'_0[x, x] = Q_0[x, x] + \tilde{L}_1xKx, \quad Q'_1[x, x] = Q_1[x, x],$$

$$C'_0[x, x, x] = C_0[x, x, x] + \tilde{L}_2[x, x]Kx.$$

We introduce some matrices and constants:

Let T and T' be given as nonsingular matrices such that

$$TL_0T^{-1} = \begin{bmatrix} L_{00} & 0 \\ 0 & 0 \end{bmatrix}, \quad TB_1 = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}, \quad \ell T^{-1} = [0 \quad \dots \quad 0 \quad 1], \quad (4.5)$$

$$T'L'_0T'^{-1} = \begin{bmatrix} L'_{00} & 0 \\ 0 & 0 \end{bmatrix}, \quad T'h_0 = \begin{bmatrix} h_{01} \\ 0 \end{bmatrix}, \quad \ell T'^{-1} = [0 \quad \dots \quad 0 \quad 1]. \quad (4.6)$$

Let ℓ' and r' be the left row and right column eigenvectors of L'_0 corresponding to the critical eigenvalue and $\ell'r' = 1$. Then, $\ell' = \ell$ and r' can be calculated as:

$$r' = T^{-1} \begin{bmatrix} \eta \\ 1 \end{bmatrix}, \quad \eta = -\frac{L_{00}^{-1}B_{11}K_{12}}{1 + K_{11}L_{00}^{-1}B_{11}}.$$

where $u = Kx = [K_{11} \quad K_{12}]Tx$.

Let ℓ_i be the i th element of ℓ , and $r^T Q_{0k} r$ be the k th element of $Q_0[r, r]$ with $Q_{0k} = Q_{0k}^T$ for $k = 1, 2, \dots, n$. Partition

$$(T^{-1})^T \left(\sum_{k=1}^n \ell_k Q_{0k} \right) T^{-1} = \begin{bmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{bmatrix}, \quad q_{10} = q_{01}^T, \quad q_{00} \in \mathbb{R}^{(n-1) \times (n-1)}. \quad (4.7)$$

Let $r'^T Q'_{0k} r'$ be the k th element of $Q'_0[r', r']$ and $r^T Q_{0k} r$ be the k th element of $Q_0[r, r]$. Partition

$$(T'^{-1})^T \left(\sum_{k=1}^n \ell_k Q'_{0k} \right) T'^{-1} = \begin{bmatrix} q'_{00} & q'_{01} \\ q'_{10} & q'_{11} \end{bmatrix}, \quad q'_{10} = q'_{01}^T, \quad q'_{00} \in \mathbb{R}^{(n-1) \times (n-1)}. \quad (4.8)$$

The admissible uncertainty $w = P_1x + P_2[x, x] + P_3[x, x, x] + \dots$ is characterized in the next theorem[11].

Theorem 4.1. *Suppose the system (4.1) satisfies the Assumption (S). Let $u = Kx = [K_{11} \ K_{12}]Tx$ be a stabilizing controller. Then this controller will robustly stabilize the system for all uncertainties $w = P_1x + P_2[x, x] + P_3[x, x, x] + \dots$, $P_1r' \neq 0$, if and only if $P_1 = [P_{11} \ P_{12}]T'$ satisfies:*

- (i) $(1 + P_{11}L_{00}^{\prime -1}h_{01}) \left(\tilde{\lambda}'_1(1 + P_{11}L_{00}^{\prime -1}h_{01}) + a'P_{12} \right) + b'P_{12}^2 < 0$, $P_1r' \neq 0$,
- (ii) $\frac{d\lambda'}{d\gamma}(0) \left(\frac{d\lambda'}{d\gamma}(0) + (P_{11}\frac{d\lambda'}{d\gamma}(0) - P_{12}d')L_{00}^{\prime -1}h_{01} \right) > 0$,
- (iii) $L_{00}^* = L_{00}' + h_{01}P_{11}$ is stable.

where

$$\begin{aligned} \tilde{\lambda}'_1 &= \ell Q_0[r, r] + a(1 + K_{11}L_{00}^{-1}B_{11})^{-1}K_{12} \\ &\quad + b(K_{12})^2(1 + K_{11}L_{00}^{-1}B_{11})^{-2} < 0, \\ a &= \tilde{d}_0 - 2q_{10}L_{00}^{-1}B_{11}, \\ b &= \left((L_{00}^{-1}B_{11})^T q_{00} - \tilde{d} \right) L_{00}^{-1}B_{11}, \quad \begin{bmatrix} \tilde{d} & \tilde{d}_0 \end{bmatrix} = \ell \tilde{L}_1 T^{-1}, \\ a' &= \tilde{d}'_0 - 2q'_{10}L_{00}^{\prime -1}h_{01}, \quad \begin{bmatrix} \tilde{d}' & \tilde{d}'_0 \end{bmatrix} = \ell h_1 T'^{-1}, \\ b' &= \left((L_{00}^{\prime -1}h_{01})^T q'_{00} - \tilde{d}' \right) L_{00}^{\prime -1}h_{01}, \end{aligned}$$

$\tilde{d}'_0$ is a scalar, $d'^T = \begin{bmatrix} I_{n-1} & 0 \end{bmatrix} (\ell L_1 T'^{-1})^T$, $\lambda'(\gamma)$ is the critical eigenvalue of L'_0 and $\frac{d\lambda'}{d\gamma}(0) = \ell L_1 r'$.

We will use Theorem 4.1 to calculate the robustness of different stabilizing controllers for rotating stall.

4.2 Robustness of Rotating Stall Control with Perturbation of Pressure Rise

If the uncertainty is only added into the pressure rise equation in the compressor model:

$$\begin{aligned}\dot{\Psi} &= \frac{1}{\beta^2} \left(\Phi - (\gamma + u)\sqrt{\Psi} + 1 \right) + w(x), \\ \dot{\Phi} &= -\Psi + \psi_c(\Phi) + 6c_3\Phi R, \\ \dot{R} &= \sigma R(1 - \Phi^2 - R).\end{aligned}$$

This yields the following robust control problem:

$$\dot{x} = f(\delta\gamma, x) + g(x)u + h(x)w(x), \quad \delta\gamma = \gamma - \gamma_c, \quad (4.9)$$

where $h(x) = [1 \ 0 \ 0]^T$ and

$$u = K_1x + K_2[x, x] + K_3[x, x, x] + \dots$$

It is clear that:

$$\begin{aligned}h_0 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\ h_1 &= h_2 = \dots = 0.\end{aligned}$$

As mentioned before:

$$r' = T^{-1} \begin{bmatrix} \eta \\ 1 \end{bmatrix}, \quad \eta = -\frac{L_{00}^{-1}B_{11}K_{12}}{1 + K_{11}L_{00}^{-1}B_{11}},$$

then we can derive:

$$\eta = \begin{bmatrix} 0 \\ \frac{k_3\sqrt{\Psi_c}}{1-k_2\sqrt{\Psi_c}} \end{bmatrix},$$

$$r' = \begin{bmatrix} \frac{6c_3}{1 - k_2\sqrt{\Psi_c}} + \frac{3c_3\gamma_c}{\sqrt{\Psi_c}} \\ 1 \end{bmatrix}. \quad (4.10)$$

Therefore:

$$T' = \begin{bmatrix} 1 & 0 & -6c_3 \\ 0 & 1 & -\left(\frac{k_3\sqrt{\Psi_c}}{1 - k_2\sqrt{\Psi_c}} + \frac{3c_3\gamma_c}{\sqrt{\Psi_c}}\right) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -\frac{k_3\sqrt{\Psi_c}}{1 - k_2\sqrt{\Psi_c}} + \frac{3}{\Psi_c} \\ 0 & 0 & 1 \end{bmatrix} \quad (4.11)$$

Other vectors can then be calculated as follows:

$$L'_0 = L_0 + B_1 K_1 = \begin{bmatrix} -\frac{\gamma_c + 2K_\Psi \Psi_c}{2\beta^2 \sqrt{\Psi_c}} & \frac{1 - K_\Phi \sqrt{\Psi_c}}{\beta^2} & -\frac{K_R \sqrt{\Psi_c}}{\beta^2} \\ -1 & 0 & 6c_3 \\ 0 & 0 & 0 \end{bmatrix},$$

$$T' L'_0 T'^{-1} = \begin{bmatrix} L'_{00} & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow L'_{00} = \begin{bmatrix} -\frac{\gamma_c + 2K_\Psi \Psi_c}{2\beta^2 \sqrt{\Psi_c}} & \frac{1 - K_\Phi \sqrt{\Psi_c}}{\beta^2} \\ -1 & 0 \end{bmatrix},$$

$$h_{01} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\ell \tilde{L}_1 T^{-1} = [0 \ 0 \ 0] \Rightarrow \tilde{d} = [0 \ 0], \quad \tilde{d}_0 = 0.$$

If $w = P_1 x + P_2 [x, x] + P_3 [x, x, x] + \dots$: $P_1 = [P_{11} \ P_{12}] T' = [P_\Psi \ P_\Phi \ P_R]$

then: $P_{11} = [p_1 \ p_2]$, $P_{12} = p_3$,

$$p_1 = P_\Psi, \quad p_2 = P_\Phi, \quad p_3 = P_R - 3P_\Psi + P_\Phi \left(\frac{k_3\sqrt{\Psi_c}}{1 - k_2\sqrt{\Psi_c}} - \frac{3}{\Psi_c} \right) \quad (4.12)$$

Using partition 4.7 we have:

$$Q_0[r, r] = \begin{bmatrix} \frac{\gamma_c}{8\beta^2 \Psi_c^{1.5}} r_1^2 \\ 3c_3 \Phi_c r_2^2 + 6c_3 r_2 r_3 \\ -\sigma r_3^2 - 2\sigma \Phi_c r_2 r_3 \end{bmatrix}$$

therefore:

$$Q_{01} = \begin{bmatrix} \frac{\gamma_c}{8\beta^2 \Psi_c^{1.5}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_{02} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3c_3 \Phi_c & 3c_3 \\ 0 & 3c_3 & 0 \end{bmatrix},$$

$$Q_{03} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sigma\Phi_c \\ 0 & -\sigma\Phi_c & -\sigma \end{bmatrix},$$

and

$$q_{00} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad q_{10} = q_{01}^T = [0 \quad -\sigma\Phi_c], \quad q_{11} = \frac{-6c_3\gamma_c\sigma\Phi_c}{\sqrt{\Psi_c}} - \sigma. \quad (4.13)$$

By doing more calculations we can derive:

$$a = \bar{d}_0 - 2q_{10}L_{00}^{-1}B_{11} = -2\sigma\Phi_c\sqrt{\Psi_c},$$

$$b = ((L_{00}^{-1}B_{11})^T q_{00} - \bar{d}) L_{00}^{-1}B_{11} = 0,$$

$$\ell h_1 T'^{-1} = [0 \quad 0 \quad 0] \Rightarrow \bar{d}' = [0 \quad 0], \quad \bar{d}'_0 = 0,$$

and

$$\begin{aligned} \bar{\lambda}'_1 &= \ell Q_0[r, r] + a(1 + K_{11}L_{00}^{-1}B_{11})^{-1}K_{12} \\ &\quad + b(K_{12})^2(1 + K_{11}L_{00}^{-1}B_{11})^{-2} \\ &= -\sigma - \frac{6c_3\gamma_c\sigma\Phi_c}{\sqrt{\Psi_c}} - \frac{2\sigma\Phi_c k_3\sqrt{\Psi_c}}{1 - k_2\sqrt{\Psi_c}}, \end{aligned} \quad (4.14)$$

we can simplify 4.14 by substituting the values: $\Psi_c = \frac{11}{3}$, $c_3 = -0.5$, $\Phi_c = 1$ and $\frac{\gamma_c}{\sqrt{\Psi_c}} = \frac{2}{\Psi_c}$ and come up with:

$$\bar{\lambda}'_1 = \sigma \left(\frac{7}{11} - \frac{2k_3\sqrt{\Psi_c}}{1 - k_2\sqrt{\Psi_c}} \right). \quad (4.15)$$

By using partition 4.8:

$$Q'_0[x, x] = Q_0[x, x] + \bar{L}_1 x K_1 x = \begin{bmatrix} \left(\frac{\gamma_c}{8\beta^2\Psi_c^{1.5}} - \frac{K_\Psi}{2\beta^2\sqrt{\Psi_c}} \right) x_1^2 - \frac{K_\Phi}{2\beta^2\sqrt{\Psi_c}} x_1 x_2 - \frac{K_R}{2\beta^2\sqrt{\Psi_c}} x_1 x_3 \\ 3c_3\Phi_c x_2^2 + 6c_3 x_2 x_3 \\ -\sigma x_3^2 - 2\sigma\Phi_c x_2 x_3 \end{bmatrix}$$

therefore

$$q'_{00} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad q'_{10} = q'^T_{01} = \begin{bmatrix} 0 & -\sigma \end{bmatrix}, \quad q'_{11} = -2\sigma \left(\frac{k_3\sqrt{\Psi_c}}{1 - k_2\sqrt{\Psi_c}} - \frac{3}{\Psi_c} \right) - \sigma. \quad (4.16)$$

Now we are able to derive:

$$a' = \bar{d}'_0 - 2q'_{10}L'^{-1}_{00}h_{01} = -\frac{2\sigma\beta^2}{K_\Phi\sqrt{\Psi_c} - 1},$$

$$b' = \left((L'^{-1}_{00}h_{01})^T q'_{00} - \bar{d}' \right) L'^{-1}_{00}h_{01} = 0.$$

At this point we can apply Theorem 4.1 to calculate the admissible uncertainty set. Condition (iii) of Theorem 4.1 implies that if the controller $u = Kx = [K_{11} \ K_{12}]Tx$ is to reach stabilization for the system with uncertainty $w = P_1x + P_2[x, x] + P_3[x, x, x] + \dots$, $L'^*_0 = L'_0 + h_{01}P_{11}$ needs to be stable. This leads to the following results:

$$L'^*_0 = \begin{bmatrix} -\frac{\gamma_c + 2K_\Psi\Psi_c}{2\beta^2\sqrt{\Psi_c}} + p_1 & \frac{1 - K_\Phi\sqrt{\Psi_c}}{\beta^2} + p_2 \\ -1 & 0 \end{bmatrix}, \quad (4.17)$$

$$\det[L'^*_0 - \lambda I] = 0 \Rightarrow \lambda = \text{Re}[\lambda] \pm \sqrt{\Delta}$$

$$\text{Re}[\lambda] < 0 \Rightarrow p_1 - \frac{\gamma_c + 2K_\Psi\Psi_c}{2\beta^2\sqrt{\Psi_c}} < 0$$

$$\Rightarrow p_1 < \frac{3}{11\beta^2} + \frac{K_\Psi}{\beta^2} \sqrt{\frac{11}{3}} \quad (4.18)$$

and

$$\Delta > 0 \Rightarrow \frac{1 - K_\Phi\sqrt{\frac{11}{3}}}{\beta^2} + p_2 > 0$$

$$\Rightarrow p_2 \frac{\beta^2}{K_\Phi\sqrt{\frac{11}{3}} - 1} < 1. \quad (4.19)$$

By applying these results, the following Theorems can be stated [40, 41]:

Theorem 4.2. *For the robust rotating stall control problem stated in (4.9), let the stabilizing control be $u = k_\Psi(\Psi - \Psi_e)$, then u will also achieve rotating stall control for any uncertainty $w = P_1x + P_2[x, x] + P_3[x, x, x] + \dots$ satisfying*

$$p_\Psi < \frac{1}{\beta^2} \left(\frac{3}{11} + \sqrt{\frac{11}{3}} k_\Psi \right), \quad p_\Phi > -\frac{1}{\beta^2},$$

$$2p_R - 6p_\Psi - p_\Phi < -\frac{1}{\beta^2} \left(\frac{7}{11} + 6\sqrt{\frac{11}{3}} k_\Psi \right),$$

where $P_1 = \begin{bmatrix} p_\Psi & p_\Phi & p_R \end{bmatrix}$.

Proof If the controller is $u = k_\Psi \delta \Psi$, then:

$$K_\Phi = K_R = 0, \quad k_3 = -3K_\Psi, \quad k_2 = 0, \quad k_1 = K_\Psi. \quad (4.20)$$

By substituting 4.12 and 4.20 into 4.18 we have:

$$p_\Psi < \frac{3}{11\beta^2} + \frac{K_\Psi}{\beta^2} \sqrt{\frac{11}{3}} = \frac{1}{\beta^2} \left(\frac{3}{11} + k_\Psi \sqrt{\frac{11}{3}} \right).$$

Applying 4.12 and 4.20 into 4.19 will result:

$$-p_2\beta^2 < 1 \Rightarrow -p_\Phi\beta^2 < 1 \Rightarrow p_\Phi > -\frac{1}{\beta^2}$$

Condition (i) of Theorem 4.1 applies that:

$$\left(\frac{7}{11} + 6K_\Psi \sqrt{\frac{11}{3}} \right) (1 + p_2\beta^2) + 2p_3\beta^2 < 0.$$

After simplifying and regrouping this inequality we have:

$$2p_R - 6p_\Psi - p_\Phi < -\frac{1}{\beta^2} \left(\frac{7}{11} + 6\sqrt{\frac{11}{3}} k_\Psi \right).$$

■

Corollary 4.3. *For the robust rotating stall control problem stated in (4.9), the feedback stabilizing controller $u = k_\Psi(\Psi - \Psi_e) = -\frac{12.5}{66}\sqrt{\frac{3}{11}}(\Psi - \Psi_e)$ will achieve stabilizing rotating stall control for any uncertainty $w = P_1x + P_2[x, x] + P_3[x, x, x] + \dots$ if:*

$$p_\Psi < \frac{1}{12\beta^2}, \quad p_\Phi > -\frac{1}{\beta^2}, \quad 2p_R - 6p_\Psi - p_\Phi < \frac{1}{2\beta^2},$$

$$\text{where } P_1 = \begin{bmatrix} p_\Psi & p_\Phi & p_R \end{bmatrix}.$$

Proof Note that $k_\Psi = -\frac{12.5}{66}\sqrt{\frac{3}{11}}$ is the medium value in

$$-\frac{3}{11}\sqrt{\frac{3}{11}} = -\Psi_c^{-3/2} < k_\Psi < \frac{1}{6\sqrt{\Psi_c}} - \Psi_c^{-3/2} = -\frac{7}{66}\sqrt{\frac{3}{11}}.$$

and can be applied directly to Theorem 4.2. ■

The result of Corollary 4.3 is illustrated in Figure 4.2. The gray area in this normalized graph shows the acceptable range of uncertainty tolerated by this controller.

Theorem 4.4. *For the robust rotating stall control problem stated in (4.9), let the stabilizing control be $u = k_R R$, then u will also achieve rotating stall control for any uncertainty $w = P_1x + P_2[x, x] + P_3[x, x, x] + \dots$ with:*

$$p_\Psi < \frac{3}{11\beta^2}, \quad p_\Phi > -\frac{1}{\beta^2},$$

$$2p_R - 6p_\Psi - p_\Phi < -\frac{1}{\beta^2}\left(\frac{7}{11} - 2\sqrt{\frac{11}{3}}k_R\right),$$

Figure 4.2: Acceptable uncertainty region for controller $u = k_\psi \delta\psi = -0.0989\delta\psi$

where $P_1 = \begin{bmatrix} p_\Psi & p_\Phi & p_R \end{bmatrix}$.

Proof If the controller is $u = k_R R$, then:

$$K_\Psi = K_\Phi = 0, \quad k_3 = K_R, \quad k_1 = k_2 = 0. \quad (4.21)$$

By Substituting 4.12 and 4.21 into 4.18 we have:

$$p_\Psi < \frac{3}{11\beta^2}.$$

And by combining 4.19, 4.12 and 4.21:

$$-p_2\beta^2 < 1 \Rightarrow p_\Phi > -\frac{1}{\beta^2}.$$

Now we apply Condition (i) of Theorem 4.1 and derive:

$$\left(\frac{7}{11} - 2K_R\sqrt{\Psi_c}\right)(1 + p_2\beta^2) + 2p_3\beta^2 < 0.$$

After simplifying:

$$2p_R - 6p_\Psi - p_\Phi < -\frac{1}{\beta^2}\left(\frac{7}{11} - 2\sqrt{\frac{11}{3}}k_R\right).$$

■

Corollary 4.5. *For the robust rotating stall control problem stated in (4.9), the feedback stabilizing controller $u = k_R R = 0.5R$ [45] will achieve rotating stall control for any uncertainty $w = P_1x + P_2[x, x] + P_3[x, x, x] + \dots$ with:*

$$p_\Psi < \frac{3}{11\beta^2}, \quad p_\Phi > -\frac{1}{\beta^2},$$

$$2p_R - 6p_\Psi - p_\Phi < \frac{1}{\beta^2}\left(\sqrt{\frac{11}{3}} - \frac{7}{11}\right),$$

where $P_1 = \begin{bmatrix} p_\Psi & p_\Phi & p_R \end{bmatrix}$.

Proof: This is the special case of Theorem 4.4. Simply by substituting the value of $K_R = 0.5$ in the inequalities, we can derive the results for this corollary.

■

The result of the Corollary 4.5 is illustrated in Figure 4.3.

Figure 4.3: Acceptable uncertainty region for controller $u = k_R R = 0.5R$

By comparing the results in Corollary 4.3 and 4.5 we come to the conclusion that the controller $u = K_R R = 0.5R$ can tolerate a larger set of uncertainties than $u = k_\Psi \delta\Psi = -0.0989\delta\psi$ and hence is more robust. The same conclusion can be reached by comparing Figure 4.2 with Figure 4.3 which has a larger grey area, i.e., acceptable uncertainty [40].

4.3 Robustness of Rotating Stall Control with Perturbation of Both Pressure Rise and Flow Rate

In this section we consider the case where the uncertainty perturbations are added into both pressure rise and flow rate equations with arbitrary coefficients n_1 and n_2 :

$$\begin{aligned}\dot{\Psi} &= \frac{1}{\beta^2} \left(\Phi - (\gamma + u)\sqrt{\Psi} + 1 \right) + n_1 w(x), \\ \dot{\Phi} &= -\Psi + \psi_c(\Phi) + 6c_3\Phi R + n_2 w(x), \\ \dot{R} &= \sigma R(1 - \Phi^2 - R).\end{aligned}$$

This actually yields the following robust control problem:

$$\dot{x} = f(\delta\gamma, x) + g(x)u + h(x)w(x), \quad \delta\gamma = \gamma - \gamma_c, \quad (4.22)$$

where $h(x) = [n_1 \ n_2 \ 0]^T$ and

$$u = K_1 x + K_2[x, x] + K_3[x, x, x] + \dots$$

It is clear that in this case we have:

$$\begin{aligned}h_0 &= \begin{bmatrix} n_1 \\ n_2 \\ 0 \end{bmatrix}, \quad h_1 = h_2 = \dots = 0, \\ h_{01} &= \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}.\end{aligned}$$

Also for the system model 4.22 it can be calculated that:

$$a' = \bar{d}'_0 - 2q'_{10}L'^{-1}_{00}h_{01} = \frac{\sigma n_2(\gamma_c + 2K_\Psi \Psi_c) - 2\sigma n_1\beta^2\sqrt{\Psi_c}}{\sqrt{\Psi_c}(K_\Phi\sqrt{\Psi_c} - 1)},$$

$$b' = \left((L'^{-1}_{00}h_{01})^T q'_{00} - \bar{d}' \right) L'^{-1}_{00}h_{01} = 0.$$

Condition (iii) of Theorem 4.1 implies that for the vector:

$$L'^*_{00} = \begin{bmatrix} -\frac{\gamma_c + 2K_\Psi \Psi_c}{2\beta^2\sqrt{\Psi_c}} + n_1p_1 & \frac{1 - K_\Phi\sqrt{\Psi_c}}{\beta^2} + n_1p_2 \\ n_2p_1 - 1 & n_2p_2 \end{bmatrix}, \quad (4.23)$$

the condition of stability should be true such that if:

$$\det[L'^*_{00} - \lambda I] = 0 \Rightarrow \lambda = \text{Re}[\lambda] \pm \sqrt{\Delta}$$

then

$$\begin{aligned} \text{Re}[\lambda] < 0 &\Rightarrow n_1p_1 + n_2p_2 - \frac{\gamma_c + 2K_\Psi \Psi_c}{2\beta^2\sqrt{\Psi_c}} < 0 \\ &\Rightarrow n_1p_1 + n_2p_2 < \frac{3}{11\beta^2} + \frac{K_\Psi}{\beta^2} \sqrt{\frac{11}{3}} \end{aligned} \quad (4.24)$$

and

$$\begin{aligned} \Delta > 0 &\Rightarrow -n_2p_2 \left(\frac{3}{11\beta^2} + \frac{K_\Psi}{\beta^2} \sqrt{\frac{11}{3}} \right) - n_2p_1 \left(\frac{1}{\beta^2} - \frac{K_\Phi}{\beta^2} \sqrt{\frac{11}{3}} \right) + \\ &\quad + \frac{1 - K_\Phi\sqrt{\frac{11}{3}}}{\beta^2} + n_1p_2 > 0 \\ &\Rightarrow n_2p_1 + p_2 \frac{11n_1\beta^2 - 3n_2 - 11n_2K_\Psi\sqrt{\frac{11}{3}}}{11(K_\Phi\sqrt{\frac{11}{3}} - 1)} < 1. \end{aligned} \quad (4.25)$$

By applying these results, the following Theorems can be stated [41, 42]:

Theorem 4.6. *For the robust rotating stall control problem stated in (4.22), let the stabilizing control be $u = k_\Psi(\Psi - \Psi_e)$, then u will also achieve rotating stall control for any uncertainty $w = P_1x + P_2[x, x] + P_3[x, x, x] + \dots$ satisfying*

$$\begin{aligned} n_1 p_\Psi + n_2 p_\Phi &< \frac{1}{\beta^2} \left(\frac{3}{11} + k_\Psi \sqrt{\frac{11}{3}} \right), \\ n_2 p_\Psi + p_\Phi \left(\frac{3}{11} n_2 + n_2 k_\Psi \sqrt{\frac{11}{3}} - n_1 \beta^2 \right) &< 1, \\ 2p_R(n_1 \beta^2 - n_2 k_\Psi \sqrt{\frac{11}{3}} - \frac{3}{11} n_2) - p_\Psi(6n_1 \beta^2 - n_2) - p_\Phi(n_1 \beta^2 - \frac{3}{11} n_2 - n_2 k_\Psi \sqrt{\frac{11}{3}}) \\ &< -\left(\frac{7}{11} + 6k_\Psi \sqrt{\frac{11}{3}} \right). \end{aligned}$$

where $P_1 = \begin{bmatrix} p_\Psi & p_\Phi & p_R \end{bmatrix}$.

Proof If the controller is $u = k_\Psi \delta \Psi$, then:

$$K_\Phi = K_R = 0, \quad k_3 = -3K_\Psi, \quad k_2 = 0, \quad k_1 = K_\Psi. \quad (4.26)$$

Substituting 4.12 and 4.26 into 4.24 will result to:

$$n_1 p_\Psi + n_2 p_\Phi < \frac{3}{11\beta^2} + \frac{K_\Psi}{\beta^2} \sqrt{\frac{11}{3}} = \frac{1}{\beta^2} \left(\frac{3}{11} + k_\Psi \sqrt{\frac{11}{3}} \right).$$

By applying 4.12 and 4.26 into 4.25:

$$\begin{aligned} n_2 p_1 - p_2 \frac{11n_1 \beta^2 - 3n_2 - 11n_2 K_\Psi \sqrt{\frac{11}{3}}}{11} &< 1 \Rightarrow \\ n_2 p_\Psi + p_\Phi \left(\frac{3}{11} n_2 + n_2 k_\Psi \sqrt{\frac{11}{3}} - n_1 \beta^2 \right) &< 1. \end{aligned}$$

Condition (i) of Theorem 4.1 applies that:

$$\left(\frac{7}{11} + 6K_\Psi \sqrt{\frac{11}{3}} \right) \left(1 - n_2 p_1 + p_2 \frac{11n_1 \beta^2 - 3n_2 - 11n_2 K_\Psi \sqrt{\frac{11}{3}}}{11} \right) -$$

$$-p_3 \frac{n_2(\gamma_c + 2K_\Psi \Psi_c) - 2n_1\beta^2\sqrt{\Psi_c}}{\sqrt{\Psi_c}} < 0.$$

After simplifying and regrouping this inequality we have:

$$\begin{aligned} 2p_R(n_1\beta^2 - n_2k_\Psi\sqrt{\frac{11}{3}} - \frac{3}{11}n_2) - p_\Psi(6n_1\beta^2 - n_2) - p_\Phi(n_1\beta^2 - \frac{3}{11}n_2 - n_2k_\Psi\sqrt{\frac{11}{3}}) \\ < -(\frac{7}{11} + 6k_\Psi\sqrt{\frac{11}{3}}). \end{aligned}$$

■

Corollary 4.7. *For the robust rotating stall control problem stated in (4.22), the feedback stabilizing controller $u = k_\Psi(\Psi - \Psi_e) = -\frac{12.5}{66}\sqrt{\frac{3}{11}}(\Psi - \Psi_e)$ will achieve stabilizing rotating stall control for any uncertainty $w = P_1x + P_2[x, x] + P_3[x, x, x] + \dots$ if:*

$$n_1p_\Psi + n_2p_\Phi < \frac{1}{12\beta^2}, \quad n_2p_\Psi + p_\Phi(\frac{1}{12}n_2 - n_1\beta^2) < 1,$$

$$2p_R(n_1\beta^2 - \frac{1}{12}n_2) - p_\Psi(6n_1\beta^2 - n_2) - p_\Phi(n_1\beta^2 - \frac{1}{12}n_2) < \frac{1}{2}.$$

$$\text{where } P_1 = \begin{bmatrix} p_\Psi & p_\Phi & p_R \end{bmatrix}.$$

Proof Note that $k_\Psi = -\frac{12.5}{66}\sqrt{\frac{3}{11}}$ is the medium value in

$$-\frac{3}{11}\sqrt{\frac{3}{11}} = -\Psi_c^{-3/2} < k_\Psi < \frac{1}{6\sqrt{\Psi_c}} - \Psi_c^{-3/2} = -\frac{7}{66}\sqrt{\frac{3}{11}}.$$

and can be applied directly into Theorem 4.6

■

Theorem 4.8. *For the robust rotating stall control problem stated in (4.22), let the stabilizing control be $u = k_R R$, then u will also achieve rotating stall control for any uncertainty $w = P_1 x + P_2[x, x] + P_3[x, x, x] + \dots$ with*

$$n_1 p_\Psi + n_2 p_\Phi < \frac{3}{11\beta^2}, \quad n_2 p_\Psi + p_\Phi \left(\frac{3n_2}{11} - n_1 \beta^2 \right) < 1,$$

$$2p_R \left(n_1 \beta^2 - \frac{3n_2}{11} \right) - p_\Psi (6n_1 \beta^2 - 2k_R n_2 \sqrt{\frac{11}{3}} - n_2) - p_\Phi \left(n_1 \beta^2 - \frac{3}{11} n_2 \right) < 2k_R \sqrt{\frac{11}{3}} - \frac{7}{11}.$$

$$\text{where } P_1 = \begin{bmatrix} p_\Psi & p_\Phi & p_R \end{bmatrix}.$$

Proof If the controller is $u = k_R R$, then:

$$K_\Psi = K_\Phi = 0, \quad k_3 = K_R, \quad k_1 = k_2 = 0. \quad (4.27)$$

By Substituting 4.12 and 4.27 into 4.24 we have:

$$n_1 p_\Psi + n_2 p_\Phi < \frac{3}{11\beta^2}.$$

And by combining 4.25, 4.12 and 4.27:

$$n_2 p_1 - p_2 \frac{11n_1 \beta^2 - 3n_2}{11} < 1 \Rightarrow n_2 p_\Psi + p_\Phi \left(\frac{3n_2}{11} - n_1 \beta^2 \right) < 1.$$

Now we apply Condition (i) of Theorem 4.1 and derive:

$$\left(\frac{7}{11} - 2K_R \sqrt{\Psi_c} \right) \left(1 - n_2 p_1 + p_2 \frac{11n_1 \beta^2 - 3n_2}{11} \right) - p_3 \frac{n_2 \gamma_c - 2n_1 \beta^2 \sqrt{\Psi_c}}{\sqrt{\Psi_c}} < 0.$$

After simplifying:

$$2p_R \left(n_1 \beta^2 - \frac{3n_2}{11} \right) - p_\Psi (6n_1 \beta^2 - 2k_R n_2 \sqrt{\frac{11}{3}} - n_2) - p_\Phi \left(n_1 \beta^2 - \frac{3}{11} n_2 \right) < 2k_R \sqrt{\frac{11}{3}} - \frac{7}{11}.$$

■

Corollary 4.9. *For the robust rotating stall control problem stated in (4.22), the feedback stabilizing controller $u = k_R R = 0.5R$ [45] will achieve rotating stall control for any uncertainty $w = P_1 x + P_2[x, x] + P_3[x, x, x] + \dots$ with*

$$n_1 p_\Psi + n_2 p_\Phi < \frac{3}{11\beta^2}, \quad n_2 p_\Psi + p_\Phi \left(\frac{3n_2}{11} - n_1 \beta^2 \right) < 1,$$

$$2p_R \left(n_1 \beta^2 - \frac{3n_2}{11} \right) - p_\Psi (6n_1 \beta^2 - n_2 \sqrt{\frac{11}{3}} - n_2) - p_\Phi \left(n_1 \beta^2 - \frac{3}{11} n_2 \right) < \sqrt{\frac{11}{3}} - \frac{7}{11}.$$

$$\text{where } P_1 = \begin{bmatrix} p_\Psi & p_\Phi & p_R \end{bmatrix}.$$

Proof: This is the special case of Theorem 4.8. Simply by substituting the value of $K_R = 0.5$ in the inequalities, we can derive the results for this corollary. ■

4.4 Nonlinear Controller

In [10] a nonlinear controller u_n is proposed as:

$$u_n = k_n \frac{\Psi - \Psi_e}{\sqrt{\Psi}},$$

with the condition for k_n provided in 3.10.

At this point, we prove that this nonlinear controller is equivalent to some linear control law.

Theorem 4.10. *Let $u_n = k_n \frac{\Psi - \Psi_e}{\sqrt{\Psi}}$ be the stabilizing controller. Then it is equivalent to the linear controller $u = k_\Psi (\Psi - \Psi_e)$ with $k_\Psi = \frac{k_n}{\sqrt{\Psi_e}}$.*

Proof: First, if u_n is a stabilizing controller, then we have [10]:

$$-\frac{1}{c_0 - 2c_3} = -\frac{3}{11} < k_n < -\frac{1}{12c_3} - \frac{1}{c_0 - 2c_3} = -\frac{7}{66}.$$

Substituting $k_n = k_\psi \sqrt{\Psi_c} = \sqrt{\frac{11}{3}} k_\psi$, we obtain

$$-\frac{3}{11} \sqrt{\frac{3}{11}} = -\Psi_c^{-3/2} < k_\psi < \frac{1}{6\sqrt{\Psi_c}} - \Psi_c^{-3/2} = -\frac{7}{66} \sqrt{\frac{3}{11}}.$$

Hence, $u = k_\psi(\Psi - \Psi_e)$ is a stabilizing controller.

On the other hand, if $u = k_\psi(\Psi - \Psi_e)$ is a stabilizing controller, then

$$-\frac{3}{11} \sqrt{\frac{3}{11}} = -\Psi_c^{-3/2} < k_\psi < \frac{1}{6\sqrt{\Psi_c}} - \Psi_c^{-3/2} = -\frac{7}{66} \sqrt{\frac{3}{11}}.$$

Note that the linearized part of $u_n = k_n \frac{\Psi - \Psi_e}{\sqrt{\Psi}}$ is $u_{nl} = \frac{k_n}{\sqrt{\Psi_c}}(\Psi - \Psi_e) = k_\psi(\Psi - \Psi_e)$.

Therefore

$$-\frac{3}{11} \sqrt{\frac{3}{11}} = -\Psi_c^{-3/2} < \frac{k_n}{\sqrt{\Psi_c}} < \frac{1}{6\sqrt{\Psi_c}} - \Psi_c^{-3/2} = -\frac{7}{66} \sqrt{\frac{3}{11}},$$

or

$$-\frac{3}{11} < k_n < -\frac{7}{66},$$

which means that u_n is also a stabilizing controller. ■

Theorem 4.10 suggests that the nonlinear stabilizing controller is not superior/inferior to the linear one and hence should yield same robustness as that of the linear stabilizing controller.

4.5 Simulation Results

The Moore-Greitzer compressor model with following parameters ([10, 32, 35, 45]):

$$\lambda = 1.75, \quad H = 0.18, \quad W = 0.25, \quad B = 2, \quad a = 1/3.5,$$

$$c_0 = 8/3, \quad c_1 = 1.5, \quad c_3 = -0.5, \quad l_c = 8, \quad l_F = \infty,$$

is used to illustrate the application of our results on robustness analysis. Figure 4.4 shows four simulation plots for the uncontrolled compression system where (a) is the bifurcation diagram of R vs. γ with solid line for stable and dotted line for unstable regions. It shows a subcritical pitchfork bifurcation associated with the hysteresis loop in rotating stall. The bifurcation diagrams in Figure 4.4(b)-4.4(d) are obtained from Figure 4.4(a) using the relations satisfied for steady equilibrium solutions.

We now apply control law $u = k_\Psi \delta \Psi = -\frac{12.5}{66} \sqrt{\frac{3}{11}} \delta \Psi = -0.0989 \delta \psi$ with no uncertainty assumed. The bifurcation plots are shown in Figure 4.5. This control law changes the pitchfork bifurcation in Figure 4.4(a) from subcritical into supercritical as shown in Figure 4.5(a). The bifurcation diagrams in Figure 4.5(b)-4.5(d) show that the adverse effects resulting from the hysteresis loop are eliminated, and (Ψ_e, Φ_e, R_e) are all single-valued functions of γ .

Figure 4.6 shows the bifurcation diagrams after applying the control law $u = k_R R = 0.5R$. It can be seen from these diagrams that this controller stabilizes the critical equilibria of the compression system.

Figure 4.4: Bifurcation diagrams without feedback control

The bifurcation diagrams for the system with nonlinear controller $u_n = k_n \frac{\Psi - \Psi_e}{\sqrt{\Psi}}$, $k_n = -0.1894$ is shown in Figure 4.7.

Now we consider uncertainties added to the system model. Figure 4.8 shows the bifurcation diagrams for the system with controller $u = k_\psi \delta \Psi = -0.0989 \delta \psi$ with an uncertainty within the acceptable range mentioned in Corollary 4.3. These diagrams show that the critical equilibrium point is still stable. Figure 4.9 illustrates the bifurcation diagrams for the system

Figure 4.5: Bifurcation diagrams with feedback controller $u = k_\psi \delta\psi$

Figure 4.6: Bifurcation diagrams with feedback controller $u = k_R R$

with controller $u = k_R R = 0.5R$ and an uncertainty considered within the acceptable range given in Corollary 4.5. The stability of the system can be seen in these diagrams.

Figure 4.10 and Figure 4.11 are examples of the system with the disturbance w considered to be outside the acceptable range. We can see that in these cases the bifurcation is changed back to unstable subcritical and the applied control law is unable to stabilize the system.

Figure 4.7: Bifurcation diagrams with nonlinear feedback controller

Figure 4.8: Bifurcation diagrams with feedback controller $u = k_\psi \delta\psi$ and acceptable uncertainty

Figure 4.9: Bifurcation diagrams with feedback controller $u = k_R R$ and acceptable uncertainty

Figure 4.10: Bifurcation diagrams with feedback controller $u = k_\psi \delta\psi$ and unacceptable uncertainty

Figure 4.11: Bifurcation diagrams with feedback controller $u = k_R \cdot R$ and unacceptable uncertainty

Chapter 5

Conclusions and Future Work

5.1 Conclusions and Remarks

By observing the results provided in previous chapter, we can make the following remarks, summarizing the most important conclusions of this research:

1. Only linear terms in the Taylor series expansion of w are shown in the characterization of w so higher order terms will not change the control result.
2. The significance of the approach described in this thesis is that it can be applied to selecting a robust rotating stall control from the existing stabilizing controllers. For example, in Corollary 4.3, we obtain the robustness of the central controller $u = -\frac{12.5}{66}\sqrt{\frac{3}{11}}\delta\Psi$ and in Corollary 4.5 we obtain the robustness of the controller $u = 0.5R$. Since the second inequality is exactly same and for the first and third inequalities

we have

$$\frac{1}{12\beta^2} < \frac{3}{11\beta^2}, \quad \frac{1}{2\beta^2} < \frac{1}{\beta^2}(\sqrt{\frac{11}{3}} - \frac{7}{11}),$$

clearly, the controller in Corollary 4.5 is more robust than the one in Corollary 4.3 for the case $n_1 = 1$, $n_2 = 0$ therefore $u = 0.5R$ is a better choice for the controller than $u = k_\psi \delta \Psi = -0.0989\delta\psi$ in terms of robustness. The same conclusion can be made by comparing Figure 4.2 and Figure 4.3.

3. Nonlinear controllers are equivalent to linear controllers, therefore using more complicated nonlinear control laws has no advantage/disadvantage over the linear ones.
4. The robustness result presented in this thesis can be applied to computing a ‘robust rotating stall control’ from the stabilizing controllers. This is extremely important for practical compressor control design since it is very difficult to conduct a computable direct robust rotating stall control design.

5.2 Suggestions for the Future Work

A distributed model can be examined to study the robustness of stabilizing controllers and possible advantage or disadvantage over the use of simplified McCaughan model.

Also other non-bifurcation control strategies can be studied for robustness and a method for selecting a more robust controller from the existing control laws can be developed.

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