University of Windsor

Scholarship at UWindor

Electronic Theses and Dissertations

Theses, Dissertations, and Major Papers

1990

Shape-contour recognition using moment invariants.

Saeid Omar. Belkasim
University of Windsor

Follow this and additional works at: <https://scholar.uwindsor.ca/etd>

Recommended Citation

Belkasim, Saeid Omar., "Shape-contour recognition using moment invariants." (1990). *Electronic Theses and Dissertations*. 3452.

<https://scholar.uwindsor.ca/etd/3452>

This online database contains the full-text of PhD dissertations and Masters' theses of University of Windsor students from 1954 forward. These documents are made available for personal study and research purposes only, in accordance with the Canadian Copyright Act and the Creative Commons license—CC BY-NC-ND (Attribution, Non-Commercial, No Derivative Works). Under this license, works must always be attributed to the copyright holder (original author), cannot be used for any commercial purposes, and may not be altered. Any other use would require the permission of the copyright holder. Students may inquire about withdrawing their dissertation and/or thesis from this database. For additional inquiries, please contact the repository administrator via email (scholarship@uwindsor.ca) or by telephone at 519-253-3000ext. 3208.

National Library
of Canada

Bibliothèque nationale
du Canada

Canadian Theses Service

Service des thèses canadiennes

Ottawa, Canada
K1A 0N4

NOTICE

The quality of this microform is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us an inferior photocopy.

Reproduction in full or in part of this microform is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30, and subsequent amendments.

AVIS

La qualité de cette microforme dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de qualité inférieure.

La reproduction, même partielle, de cette microforme est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30, et ses amendements subséquents.

SHAPE-CONTOUR RECOGNITION USING MOMENT INVARIANTS

by

Saeid Omar Belkasim

A Dissertation

Submitted to the
Faculty of Graduate Studies and Research
through the Department of
Electrical Engineering in Partial Fulfilment
of the requirements for the degree
of Doctor of Philosophy at
the University of Windsor

Windsor, Ontario, Canada
1990

Permission has been granted to the National Library of Canada to microfilm this thesis and to lend or sell copies of the film.

The author (copyright owner) has reserved other publication rights, and neither the thesis nor extensive extracts from it may be printed or otherwise reproduced without his/her written permission.

L'autorisation a été accordée à la Bibliothèque nationale du Canada de microfilmer cette thèse et de prêter ou de vendre des exemplaires du film.

L'auteur (titulaire du droit d'auteur) se réserve les autres droits de publication; ni la thèse ni de longs extraits de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation écrite.

ISBN 0-315-61036-0

11043610

© Saeid Omar Belkasim 1990.

ACKNOWLEDGEMENTS

I would like to express my deep gratitude and appreciation to Dr. M. Shridhar and Dr. M. Ahmadi my supervisors for their guidance, support, and helpful remarks throughout the course of this work.

I also would like to thank Dr. J. J. Soltis, and Dr. D. Watt for serving in my committee.

Finally, I wish to thank all my fellow graduate students and colleagues for their help and support.

ABSTRACT

A comparative evaluation of the effectiveness of moment invariants as shape sensitive features for pattern recognition applications is presented.

Zernike, pseudo Zernike, Hu, Bamieh, and regular moment invariants were compared for their performance in a shape recognition study using two data sets: handwritten numerals and aircraft pictures.

Zernike moment invariants and pseudo Zernike moment invariants have been derived in a new form for the n -th order.

Using a new normalization scheme, it is shown that Zernike moment invariants as derived in this dissertation yielded a better performance.

New algorithms have also been developed for the optimum thresholding and contour extraction of images.

ABBREVIATIONS AND NOTATIONS

BMI : Bamieh moment invariants.

CE : classification efficiency.

DF : discrimination factor.

HMI : Hu moment invariants.

KNNR : K-nearest neighbor rule.

M_{pq} : two dimensional moment of order (p+q).

MI : moment invariants.

NN : nearest neighbor.

NNR : nearest neighbor rule.

NPZM : normalized pseudo Zernike moments.

NPZMI : normalized pseudo Zernike moment invariants.

NZM : normalized Zernike moments.

NZMI : normalized Zernike moment invariants.

PZM : Pseudo Zernike moments.

PZMI : Pseudo Zernike moment invariants.

RM : regular moments.

RMI : regular moment invariants.

RPZM : reduced pseudo Zernike moments.

RPZMI : reduced pseudo Zernike moment invariants.

RZM : reduced Zernike moments.

RZMI : reduced Zernike moment invariants.

TZMI : Teague-Zernike moment invariants.

μ_{pq} : two dimensional scale invariant central moment of order p+q.

ZM : Zernike moments.

ZMI : Zernike moment invariants.

TABLE OF CONTENTS

ABSTRACT	iii
ACKNOWLEDGEMENTS	iv
ABBREVIATIONS AND NOTATIONS	v
LIST OF FIGURES	viii
LIST OF TABLES	xii
Chapter 1: INTRODUCTION	1
1.1 Objectives of the dissertation	7
1.2 Organization of the dissertation	8
Chapter 2: DATA COLLECTION AND PREPROCESSING	9
2.1 Data collection	9
2.2 Data preprocessing	15
2.2.1 Optimum contour detection and automatic thresholding	16
2.2.2 Optimum edge detection and tracking	17
2.2.3 Optimum threshold selection techniques	18
2.2.4 Preprocessing of noisy data	22
2.2.5 Experimental results	23
2.3 Summary	37
Chapter 3: MOMENTS AND MOMENT INVARIANTS	38
3.1 Introduction	38
3.2 Regular moment invariants	40
3.3 Algebraic moment invariants	44
3.3.1 Hu moment invariants	44
3.3.2 Bamieh moment invariants	46
3.4 Zernike and pseudo-Zernike moment invariants	48
3.4.1 Zernike polynomials and Zernike moments	48
3.4.2 Teague-Zernike moment invariants	49
3.4.3 New form of Zernike moment invariants	51
3.4.4 Pseudo-Zernike polynomials and pseudo-Zernike moments	54
3.4.5 Pseudo-Zernike moment invariants	55
3.5 Normalization of Zernike and pseudo-Zernike moment invariants	57
3.5.1 Normalized pseudo-Zernike moment invariants	57
3.5.2 Normalized Zernike moment invariants	59
3.6 Feature vector size reduction	62
3.6.1 Reduced pseudo-Zernike moment invariants	62
3.6.2 Reduced Zernike moment invariants	63
3.7 Moment computation	66
3.7 Summary	67

Chapter 4: CLASSIFICATION AND DISCRIMINATION	69
4.1 Syntactic pattern classification	70
4.2 Statistical pattern classification	70
4.2.1 Parametric classifiers	70
4.2.2 Non-parametric classifiers	71
4.2.2.1 Minimum distance classification	71
4.2.2.2 Distance measure	71
4.2.2.3 Nearest neighbor rule	73
4.2.2.4 K-nearest neighbor rule	73
4.2.2.5 Efficient evaluation of the K-nearest neighbor rule	73
4.3 Weighted-KNNR and multiple membership classification	79
4.4 Discrimination factor	81
4.5 Classification efficiency	82
4.6 Dynamic range reduction of the invariants	83
4.7 Summary	84
Chapter 5: PERFORMANCE EVALUATION OF THE MOMENT INVARIANTS . .	85
5.1 Sample preparation and data processing	85
5.2 Experimental results using handwritten numerals . . .	89
5.3 Experimental results using aircraft pictures . . .	100
5.5 Performance in the presence of noise	110
5.6 Summary	123
Chapter 6: CONCLUSION AND CONTRIBUTIONS.	124
6.1 Suggestions for future work	128
References	129
Appendix A: HIGHER ORDER HU MOMENT INVARIANTS	142
A.1 Hu moment polynomials	143
Appendix B: TEAGUE-ZERNIKE MOMENT INVARIANTS	146
Appendix C: NORMALIZATION OF PSEUDO ZERNIKE MOMENTS	148
Appendix D: NORMALIZED ZERNIKE MOMENTS	149
Appendix E: REDUCED PSEUDO ZERNIKE MOMENT INVARIANTS	150
Appendix F: REDUCED ZERNIKE MOMENT INVARIANTS	153

LIST OF FIGURES

Fig.1.1: The main elements of a pattern recognition system . .	3
Fig.2.1: Samples of the military aircrafts	12
Fig.2.2: Different warloads of F-111, F-4 and A-6	13
Fig.2.3: Samples of the handwritten numerals	14
Fig.2.4: Obtaining t_k using the golden section	19
Fig.2.5: The center pixel (I_o, J_o) and its eight neighbors . .	21
Fig.2.6a: original iamge	26
Fig.2.6b: image histogram	26
Fig.2.6c: thresholded image (method 1)	26
Fig.2.6d: image contour (method 1)	26
Fig.2.6e: thresholded image (method 2)	26
Fig.2.6f: image contour (method 2)	26
Fig.2.6g: thresholded image (histogram)	26
Fig.2.6h: image contour (histogram)	26
Fig.2.7a: original iamge	27
Fig.2.7b: image histogram	27
Fig.2.7c: thresholded image (method 1)	27
Fig.2.7d: image contour (method 1)	27
Fig.2.7e: thresholded image (method 2)	27
Fig.2.7f: image contour (method 2)	27
Fig.2.7g: thresholded image (histogram)	27
Fig.2.7h: image contour (histogram)	27
Fig.2.8a: original iamge	28
Fig.2.8b: image histogram	28
Fig.2.8c: thresholded image (method 1)	28
Fig.2.8d: image contour (method 1)	28
Fig.2.8e: thresholded image (method 2)	28
Fig.2.8f: image contour (method 2)	28
Fig.2.8g: thresholded image (histogram)	28
Fig.2.8h: image contour (histogram)	28
Fig.2.9a: original iamge	29
Fig.2.9b: image histogram	29
Fig.2.9c: thresholded image (method 1)	29
Fig.2.9d: image contour (method 1)	29
Fig.2.9e: thresholded image (method 2)	29

Fig.2.9f: image contour (method 2)	29
Fig.2.9g: thresholded image (histogram)	29
Fig.2.9h: image contour (histogram)	29
Fig.2.10a: original iamge	30
Fig.2.10b: image histogram	30
Fig.2.10c: thresholded image (method 1)	30
Fig.2.10d: image contour (method 1)	30
Fig.2.10e: thresholded image (method 2)	30
Fig.2.10f: image contour (method 2)	30
Fig.2.10g: thresholded image (histogram)	30
Fig.2.10h: image contour (histogram)	30
Fig.2.11a: original iamge	31
Fig.2.11b: image histogram	31
Fig.2.11c: thresholded image (method 1)	31
Fig.2.11d: image contour (method 1)	31
Fig.2.11e: thresholded image (method 2)	31
Fig.2.11f: image contour (method 2)	31
Fig.2.11g: thresholded image (histogram)	31
Fig.2.11h: image contour (histogram)	31
Fig.2.12a: original iamge	32
Fig.2.12b: image histogram	32
Fig.2.12c: thresholded image (method 1)	32
Fig.2.12d: image contour (method 1)	32
Fig.2.12e: thresholded image (method 2)	32
Fig.2.12f: image contour (method 2)	32
Fig.2.12g: thresholded image (histogram)	32
Fig.2.12h: image contour (histogram)	32
Fig.2.13a: original iamge	33
Fig.2.13b: image histogram	33
Fig.2.13c: thresholded image (method 1)	33
Fig.2.13d: image contour (method 1)	33
Fig.2.13e: thresholded image (method 2)	33
Fig.2.13f: image contour (method 2)	33
Fig.2.14a: original iamge	34
Fig.2.14b: image histogram	34
Fig.2.14c: thresholded image (method 1)	34
Fig.2.14d: image contour (method 1)	34

Fig.2.14e: thresholded image (method 2)	34
Fig.2.14f: image contour (method 2)	34
Fig.2.15a: original iamge	35
Fig.2.15b: image histogram	35
Fig.2.15c: thresholded image (method 1)	35
Fig.2.15d: image contour (method 1)	35
Fig.2.15e: thresholded image (method 2)	35
Fig.2.15f: image contour (method 2)	35
Fig.2.16a: original iamge	36
Fig.2.16b: all contours	36
Fig.2.16c: thresholded image	36
Fig.2.16d: contours > 9 pixels	36
Fig.3.1: Classification of moment invariants	39
Fig.3.2: The estimation of the image orientation	43
Fig.3.3: The dynamic range for PZM and NPZM	61
Fig.3.4: The dynamic range for ZM and NZM	61
Fig.4.1: The main approaches for pattern classification	69
Fig.4.2: Efficient evaluation of the KNNR	76
Fig.4.3: Sample elimination	78
Fig.5.1: Numeral discrimination factors for moment invariants	96
Fig.5.2: Numeral error rates for the moment invariants	96
Fig.5.3: Numeral discrimination factors for NPZMI and PZMI	97
Fig.5.4: Numeral error rates for NPZMI and PZMI	97
Fig.5.5: Numeral discrimination factors for NZMI, ZMI and TZMI	98
Fig.5.6: Numeral error rates for NZMI, ZMI and TZMI	98
Fig.5.7: Numeral classification efficiencies	99
Fig.5.8: Aircraft discrimination factors of moment invariants	106
Fig.5.9: Aircraft error rates of moment invariants	106
Fig.5.10: Aircraft discrimination factors of NPZMI and PZMI	107
Fig.5.11: Aircraft error rates of NPZMI and PZMI	107
Fig.5.12: Aircraft discrimination factors of NZMI, ZMI and TZMI	108
Fig.5.13: Aircraft error rates of NZMI, ZMI and TZMI	108
Fig.5.14: Aircraft classification efficiencies	109
Fig.5.15: Numeral discrimination factors of RPZMI and PZMI	114
Fig.5.16: Numeral error rates of RPZMI and PZMI	114
Fig.5.17: Aircraft discrimination factors of RPZMI and PZMI	115
Fig.5.18: Aircraft error rates of RPZMI and PZMI	115

Fig.5.19:	Numeral discrimination factors of RZMI,ZMI and TZMI	116
Fig.5.20:	Numeral error rates of RZMI,ZMI and TZMI	116
Fig.5.21:	Aircraft discrimination factors of RZMI,ZMI and TZMI	117
Fig.5.22:	Aircraft error rates of RZMI,ZMI and TZMI	117
Fig.5.23:	Numeral classification efficiencies	118
Fig.5.24:	Aircraft classification efficiencies	118
Fig.5.25:	Noisy data error rates	121
Fig.5.25:	Noisy data discrimination factors	122

LIST OF TABLES

Table.2.1: Military aircraft types	11
Table.2.2 Results of method 1, method 2 and the histogram analysis	25
Table.3.1: Bamieh moment invariants (BMI).	47
Table.3.2 Feature vector sizes for PZMI and RPZMI of moments up to tenth order	65
Table.3.3 Feature vector sizes for ZMI and RZMI of moments up to tenth order	65
Table.5.1 The effect of reducing the dynamic range on the error percentage	88
Table.5.2 The effect of reducing the dynamic range on the discrimination factor	88
Table.5.3 Comparison between feature vector sizes up to seventh order moments	92
Table.5.4 Arabic-numeral recognition using moment invariants up to second order	92
Table.5.5 Arabic-numeral recognition using moment invariants up to third order	93
Table.5.6 Arabic-numeral recognition using moment invariants up to fourth order	93
Table.5.7 Arabic-numeral recognition using moment invariants up to fifth order	94
Table.5.8 Arabic-numeral recognition using moment invariants up to sixth order	94
Table.5.9 Arabic-numeral recognition using moment invariants up to seventh order	95
Table.5.10 Arabic-numeral classification efficienciess for invariants of moments up to seventh order	95
Table.5.11 Aircraft recognition using moment invariants up to second order	102
Table.5.12 Aircraft recognition using moment invariants up to third order	102
Table.5.13 Aircraft recognition using moment invariants up to fourth order	103

Table.5.14 Aircraft recognition using moment invariants	
up to fifth order	.103
Table.5.15 Aircraft recognition using moment invariants	
up to sixth order	.104
Table.5.16 Aircraft recognition using moment invariants	
up to seventh order	.104
Table.5.17 Aircraft classification efficiencies for	
invariants of moments up to seventh order	.105
Table.5.18 Feature vector sizes for RZMI, RPZMI, ZM and PZMI .	.112
Table.5.19 Numeral-classification efficiencies for RZMI,	
RZMI, ZM and PZMI	.112
Table.5.20 Numeral-classification efficiencies for RZMI,	
RZMI, ZM and PZMI	.113
Table.5.21 Error rate of moment invariants for clean and	
noisy data	.120
Table.5.22 Discrimination factor of moment invariants for clean	
and noisy data	.120

CHAPTER 1

INTRODUCTION

In the last few decades image processing using digital computers has received great attention.

Digital computers offer the potential for replacing some intellectual tasks by machines. These tasks which are sometimes called perception are part of any meaningful human activity.

Although imitating the way humans perceive things, given the present technology, is impossible, machines can still simulate perception using different approaches. The classical example of birds and airplanes is often used to show that both of them use the physical phenomenon of aerodynamic lift but the motion is achieved differently.

One of the major areas which attempts to simulate the perception in human beings using digital computers is contour-shape recognition.

The contour-shape recognition and analysis area includes a wide range of applications such as scene analysis[1,2], contour coding[3,4], biomedical applications[5,6], aircraft identification[7-9], three-dimensional object recognition[9,10], character recognition[12-14]. Each application is governed by a number of important parameters and has different problems.

Creating a practical pattern recognition system is a very difficult task. A system that performs well in any number of samples collected in the research laboratory, might perform poorly

when confronted with the real life samples. Reporting the success of a recognition system has no meaning if it was not based on a carefully prepared standard data set. The other and more quantitative way to convey the degree of difficulty of a recognition problem is to describe the performance of various recognition systems on the same data set. Processing a particular set of data by several recognition systems might make that data set acquire the status of standard data set[15].

The simplest form of a pattern recognition problem is to assign a number of test samples to one of N classes.

A pattern recognition system is shown in Fig.1.1. It consists of transducer, preprocessor, feature extractor and a classifier.

The transducer is usually employed at the front end of the system to transform patterns from their original domain to a representation acceptable by the remainder of the system. The output of the transducer is generally a collection of digital information.

The preprocessor is usually employed next to the transducer as a preparatory stage for the feature extractor. Digital filtering, thresholding, and contour detection are some examples of preprocessing.

The feature extractor extracts from the pattern a feature vector consisting of a number of features. These features are represented as numerical quantities.

The classifier is given the task of assigning the feature vectors of a test sample among the N classes.

Fig.1.1 The main elements of a pattern recognition system

The syntactic, relaxation and statistical approaches are the main approaches used to tackle the problem of contour-shape recognition.

In the syntactic approach, a string of primitives such as line segments and arcs are derived. A set of production rules is used to describe the contour. When a contour of unknown class is given, the string of primitives is analyzed to determine its class label[16-18].

The relaxation approach is based on using a relaxation process to enhance the compatibility of two shapes using pairs of angles[19,20]. Other techniques may involve the use of polygonal approximation to derive the basic contour description[2,21,22], chain codes[23-27], or the structural description of the shape[28-30].

In the statistical approach a number of numerical features are extracted from the boundary or contour of the shape and a statistical classifier is used in the classification process. Several methods are used for the feature extraction process among them; circular autoregressive models, Fourier analysis and moment invariants.

The circular autoregressive technique is based on representing the boundary points by a stochastic autoregressive model. The parameters of the model are invariant to rotation, translation and scaling[31,32].

One of the common techniques based on Fourier analysis is the Fourier descriptors. Granlund[33] in 1971 derived a set of invariants to describe the contours of characters. It is based on

coding the coordinates of the contour elements as the real and imaginary parts of a complex sequence which is then expanded in terms of a Fourier series. The coefficients of the Fourier series expansion are used to derive a set of Fourier shape descriptors. Zahn and Roskies[34] used the arc length and the tangent angle to form another set of Fourier descriptors. Many others, among them Shridhar and Badreldin[35], and Person and Fu[36] used Fourier descriptors to recognize hand written numerals and characters. Lin and Chellappa[37] used Fourier descriptors to classify partial two-dimensional shapes. Fourier coefficients were also used for the reconstruction of complicated contours[4].

The other most commonly used technique is the moment invariants. The invariance properties of moments of 2-D and 3-D shapes have received considerable attention in recent years.

Hu in 1961 was the first to introduce the moment invariants[38]. He derived a set of invariants based on combinations of regular moments using algebraic invariants. These invariants are invariant under change of size, translation, rotation and/or reflection. Hu invariants, which are based on moments up to the third order, were used for the recognition of images of different types of aircraft[7,39]. The same invariants were also used for the recognition of ships[40]. There exist other moment invariants which are also derived from algebraic invariants[10]. They were used in the recognition of 3-D images from their 2-D moments.

The radial and angular moments were used as an alternative approach for the derivation of Hu invariants[41]. Hu invariants

were also modified to include invariance under change in contrast[42].

The efficient computation of moments using optical processors[43-45] or digital filters[46] is considered an attractive feature for real time shape recognition applications. A VLSI chip for the computation of two-dimensional moments of grey level images in real time has also been designed[47].

Teague[48] used Zernike orthogonal polynomials to derive Zernike moment invariants and suggested their use in shape recognition problems.

Based on exhaustive analysis[49,50] it has been shown that most of the information carried by the moments come from the boundary regions in the image. This makes the boundary moments computationally more attractive to use particularly for higher order moment invariants. Additionally, the boundary moments are simpler to compute and do not require contrast normalization. Although the boundary moments have been used in conjunction with the silhouette moments to compute the regular moment invariants and Hu invariants[7,8] they have not been fully analyzed for other types of invariants or for moments of orders larger than four.

A survey of recent literature reveals the need for an objective study on the performance characteristics of the different moment invariants when used in pattern and object recognition applications. There are, however, papers that have discussed the ability of the moments to reconstruct the original image and the amount of new information they carry[48,49,51,52].

It was also recently shown that the regular moments

out-performed the Hu invariants and the Fourier descriptors in an experiment for identifying military airplanes[8,53]. Shape recognition in the presence of noise was also studied[8,53] and it was shown that the regular moments performed better than the Fourier descriptors in that aspect too. It was also found that the use of multiple views of the same sample dramatically reduces the error rate[53]. Another interesting result which was reported in [8] indicated that the accuracy of classification using Hu invariants improved in the presence of noise.

In the author's opinion, these conflicting observations point to a need for an objective study.

1.1 Objectives of the Dissertation

The main objectives of this dissertation are:

- 1- To compare the various moment invariants and evaluate their performance as tools for contour shape recognition.
- 2- Devise some new techniques and ways to improve the recognition and computation efficiency of the moment invariants.
- 3- Due to the importance of the contours of the shape for the computation of the contour moments, a reliable technique is needed to extract the optimum contours.

In this work a new recursive formula is introduced to describe Zernike and pseudo Zernike moments[51,54]. PZMI (pseudo Zernike moment invariants) and ZMI (Zernike moment invariants) are also derived in a new form. A new normalization procedure which reduces the information redundancy in both ZMI and PZMI is also introduced. A new measure of discrimination is proposed for comparing the

performances of the various moment invariants in pattern recognition applications. In this dissertation samples of hand-written numerals and aircraft pictures are used for this study.

1.2 Organization of the Dissertation

This dissertation is laid out as follows:

Chapter 2 describes the data collection, data preprocessing, and new algorithms for the optimum thresholding and contour extraction of the images.

Chapter 3 presents a review of the moments and moment invariants and describes the derivation of Zernike and pseudo-Zernike moment invariants. It also includes details of the new normalization procedure proposed for both Zernike and pseudo-Zernike moment invariants. Chapter 4 discusses a multiple membership classification technique based on the k-nearest neighbor rule, and the derivation of discrimination and classification efficiency measures for comparing the various moment invariants. The results of this study with both numeral and aircraft data are presented in Chapter 5. Relevant conclusions derived from this study and suggestions for future work are presented in Chapter 6.

CHAPTER 2

DATA COLLECTION AND PREPROCESSING

In this chapter the collection and preprocessing of the data base used for testing the various shape recognition techniques are discussed. New algorithms for the optimum thresholding and contour detection are presented with experimental results to test their robustness.

2.1 Data collection

Two data sets were used in this work: photographs of military aircraft and handwritten numerals. The main reason for using the hand written numerals is the wide range of variability they have over other shapes[55].

A video camera was used to capture the images. Each image was digitized and represented by a 128x128 pixel array whose gray levels ranged between 0 and 255 (8 bits/pixel). All the samples were stored in an IBM-AT personal computer.

The number of aircraft types used is 18 and they are listed in Table.2.1. Each aircraft has 4 to 6 representations containing different war loads. The total number of samples for each aircraft type is 16. None of the sixteen samples is identical to the other. They vary in scale, rotation and position. The change in scale has been achieved by changing the distance between the original picture and the video camera. The change in orientation has been accomplished by rotating the original image few degrees in the same plane, and the position has been changed by shifting the original

picture left or right. The total number of the generated samples is 288. A sample of these aircrafts is shown in Fig.2.1. Fig.2.2 shows three types of military aircraft each carrying a different war load.

Two distinctive features make this aircraft data set different from the data sets used by previous researchers[7-9,53]. These features are:

- 1- These military aircrafts are carrying war loads which make them more suitable for resembling real life situations than the ones without the war loads.
- 2- The number of these aircraft classes is eighteen rather than six which has been used in most of the previous studies. The larger number of classes used would allow a more meaningful evaluation of the recognition techniques.

The second data set, the handwritten Arabic numeral set (0 to 9), consists of 32 samples of each numeral. Eight different writers participated in writing the test samples. No restrictions were imposed on the writing. A sample of the numeral set is shown in Fig.2.3.

Table.2.1 Military aircraft types

Aircraft name	Country
1-Boeing B-52	USA
2-Grunman A-6 Intruder	USA
3-Vought A-7 Corsair II	USA
4-General dynamics F-111	USA
5-McDonnell Douglas F-4G Phantom	USA
6-Fairchild Republic A-10A Thunderbolt II	USA
7-North American Rockwell B-1B	USA
8-McDonnell Douglas A-4 Skyhawk	USA
9-Sukhoi Su-7 'Fitter'	USSR
10-Sukhoi Su-24 'Fencer'	USSR
11-Mikoyan Gurevich MIG-27 'Flogger'	USSR
12-Tupolev Tu-16 'Badger'	USSR
13-Tupolev Tu-26 'Backfire'	USSR
14-Sepcat Jaguar	UK
15-BAe Buccaneer	UK
16-BAe Harrier GR.3	UK
17-Dassault-Breguet Mirage IV	France
18-SAAB A.J.37	Sweden

Fig.2.1 Samples of the military aircrafts

Fig.2.2 Different warloads of F-111, F-4 and A-6

Fig 2.3 Samples of the hand written numerals

2.2 Data preprocessing

The preprocessing operation which transforms a pattern from its original form to a new form suitable for further processing, is extremely important in most pattern recognition applications.

One of the essential preprocessing operations is the extraction of features of an object from the picture. The feature extraction process, which is a form of coding and representing the original picture, can be performed through a thresholding and contour extraction operations, provided the image consists of homogeneous objects against a constant background.

The thresholding is normally used to separate the objects from the background while the contour extraction process is used to detect and code the object contours.

In this chapter the images were preprocessed by a new technique of thresholding and contour detection which is discussed in detail in the next section). The technique utilizes an iterative thresholding, to yield a binary image with dark pixels representing the object and white pixels representing the background, in conjunction with a golden-section optimization technique to maximize the ratio of number of contour pixels to number of contours. The technique is used for selecting the optimum threshold as well as for extracting the contours of the binary image.

2.2.1 Optimum contour detection and automatic thresholding

The segmentation of gray level image of a scene continues to be an area of intensive study, owing mainly to the many potential applications such as robot vision, shape recognition, scene interpretation, etc.

In this context, thresholding techniques offer an efficient scheme for image segmentation both in terms of speed of processing as well as the complexity of implementation. A variety of techniques[56-59] has been proposed for obtaining a thresholded image from a gray level image of a scene.

The simplest of these techniques detects the different modes in the gray-level histogram and chooses the gray-level values at the valleys as the thresholds for segmentation[60]. In particular this technique is used for obtaining a binary (two-level) of a scene. Many variations of this technique have been proposed; the most notable among these is the technique proposed by Chow and Kaneko[61]. Another notable technique that has been widely used is the technique proposed by Otsu[62]. Recently, Perez and Gonzalez[63] published an interesting paper on the thresholding of images under non-uniform illumination.

In the subsequent sections a new technique is presented[64]. This technique is based on the premise that the richness of detail in an image is directly related to the edges in the image. Thus, any scheme that maximizes the edges is bound to reveal all the relevant details in an image. This assumption, however, is predicated on the assumption that the image is relatively noise-free.

2.2.2-OPTIMUM EDGE DETECTION AND TRACKING

Detecting edges plays an important role in image processing. The importance of a boundary or an edge comes from the fact that most of the important information about the picture is contained in them. In general edges in a picture are detected by comparing intensity values within a small area of the picture or obtaining the gradient in the direction of the maximum change of intensity. Other methods use higher derivatives or the Laplacian operator for detecting edges. These methods seem unsatisfactory because of their tendency to amplify noise[65].

Edge followers or border followers are used to combine the edge information produced by a local operator into lines that represent the edges. The edge follower is an algorithm that searches and follows the edge. It combines the curvature and direction information to connect adjacent short segments[66-68].

To detect the edges or borders, a global strategy which is based on optimization schemes is needed. One of the optimization techniques is based on dynamic programming[69-72]. The other technique divides the picture into several partitions and uses polynomial functions to approximate these partitions. An optimization technique is then used to find the partition with the minimum approximation error[73]. The drawback of these techniques is that they are computationally involved.

2.2.3. Optimum Threshold Selection Techniques

The optimum threshold is selected using two techniques. The first one utilizes a direct optimization technique such as the golden section method to determine the threshold that maximizes the number of border pixels in the image.

The second technique uses the same optimization to determine the threshold value that maximizes the number of border pixels averaged over the number of contours.

The optimization problem tackled by the first technique can be formulated as:

$$G_1(t) = K(t) + N(t) \quad (2.1)$$

$$N(t) = \sum_{i=1}^{K(t)} n_i(t) \quad (2.2)$$

where $K(t)$ is the number of contours obtained with threshold value t , $n_i(t)$ is the number of pixels in the i -th contour.

The first technique is based on determining the threshold value t that maximizes $G_1(t)$. This technique is suitable for images in a uniform gray level background.

In this optimization technique, the image is thresholded at $t=t_k$, where $t_{\min} \leq t_k \leq t_{\max}$. An edge is detected at the discontinuity of the gray level through image scanning and comparison of gray levels of each two neighboring pixels. The

position of an edge is the same as that of the pixel having the lower gray level. The main scheme combines an edge tracking algorithm[67] with the golden section search technique to obtain the optimum threshold and contours.

The golden-section search technique uses a non uniform spacing of the points at which the objective function is evaluated to arrive at the desired optimum.

The basic golden-section search scheme is as follows:

Given an initial interval $(t_{\min}, t_{\max})$ and objective function $G(t)$, choose a point t_k such that $t_{\min} \leq t_k \leq t_{\max}$ and;

$$\frac{d_{k+1}}{d_k} = \frac{d}{d_{k+1}} \quad (2.3)$$

where $d_{k+1} = t_k - t_{\min}$, $d_k = t_{\max} - t_{\min}$ and $d = t_{\max} - t_k$. A solution to equation (2.3) gives a value, for the ratio d_{k+1}/d_k of 0.618 which can be used to obtain t_k as shown in Fig.2.4.

Fig.(2.4) Obtaining t_k using the golden section.

The basic algorithm can be summarized as:

Given $t_{\max} = 255$, $t_{\min} = 0$, and a tolerance $\epsilon = 1$, initialize a counter k to zero and proceed as follows:

- (1) Increment k by 1 and compute two thresholding points $(t_{a,k}, t_{b,k})$ using:

$$t_{a,k} = t_{\max} - 0.618 d_k, \text{ and } t_{b,k} = t_{\min} + 0.618 d_k$$
 where $d_k = t_{\max} - t_{\min}$.
- (2) Threshold the gray level image at $t_{a,k}$ to obtain a binary image.
- (3) Let (I_0, J_0) be the location of the first edge pixel obtained through scanning the binary image until two neighboring pixels of different gray levels are encountered.
- (4) Let (I_1, J_1) be the location of the non-edge pixel encountered prior to the detection of the first edge.
- (5) Starting at the pixel at (I_1, J_1) the remaining seven neighboring pixels are labeled 2, 3, ..., 8, as in Fig.2.5.
- (6) Select the next edge pixel, the first pixel in a clock-wise traverse around the pixel at (I_0, J_0) , that has the same gray level as that of the pixel at (I_0, J_0) .
- (7) Move to the new pixel selected in step 6 and repeat steps 4, 5 and 6 until the first edge pixel is again encountered.
- (8) Repeat steps 3-7 until the whole image is scanned.
- (9) Use the number of contours and the number of edge pixels to evaluate $G_1(t)$.
- (10) Threshold the gray level image at $t=t_{b,k}$, repeat step 3-9.
- (11) If $G(t_{a,k}) > G(t_{b,k})$, set $t_{\max} = t_{b,k}$, otherwise $t_{\min} = t_{a,k}$.
- (12) Repeat step 1-11 until $d_k < \epsilon$.

A different formulation for the optimization problem is given by:

$$G_2(t) = \frac{N(t)}{K(t)} \quad (2.4)$$

where $K(t)$ and $N(t)$ are the same as defined previously.

The solution for this problem which forms the basis for the second optimization technique, is based on determining a threshold value t which maximizes $G_2(t)$.

It was found out that this formulation is suitable for images in a non-uniform background possibly due to the fact that the objective function is averaging the number of contour pixels over the number of contours. This averaging process tends to be biased to those threshold values which merge smaller non-uniform background contours into larger contours.

The algorithm for the second optimum thresholding technique is similar to the first one with G_1 replaced by G_2 in step 9.

(2)	(3)	(4)
(1) (I_1, J_1)	(I_o, J_o)	(5)
(8)	(7)	(6)

*Fig.2.5 The center pixel (I_o, J_o) and
Its eight neighbors.*

It is noted that the basic idea behind the two techniques is in many ways similar to the optimum edge detection technique described in Ref.[64]. The major difference lies in the fact that contour detection is done on a binary image and hence some of the problems associated with the previous technique, especially the detection of spurious edges, are avoided. In the present techniques the patches which form the spurious edges are usually merged with the background in the thresholding process.

2.2.4 Preprocessing of noisy data

The process of generating the noisy data set is performed as follows[8]:

- 1- A clean image is generated by setting each object point to a gray level value of 96 and each background point to a gray level value of 160.
- 2- A 2x2 local mean filter operation is performed to blur the edges of the image.
- 3- A Gaussian noise with zero mean and standard deviation of 32 is added to the image. The signal to noise ratio is maintained at 2.

Samples of the noisy data are shown in Fig.2.13-Fig.2.15.

Each noisy image is blurred by a 2x2 local mean filter and then preprocessed using the optimum thresholding and edge detection techniques discussed in the previous section.

2.2.5. Experimental results

The two algorithms discussed earlier were applied on different images to find the best threshold value at which the image can be described by two gray levels (0 and 255).

The results of the two algorithms (based on maximizing $G_1(t)$ and $G_2(t)$) are shown in Fig.2.6 to Fig.2.16. These figures show the original, thresholded image, contours and histogram for each image. The optimum threshold values for these images are shown in Table 2.2. This table also contains a comparison between the values of the objective function for the same images, when they were thresholded using the proposed schemes and when the thresholding was based on a histogram analysis. In all the cases thresholding based on the proposed schemes gave the same or higher objective function value than when the thresholding was based on histogram technique.

An interesting observation about the two schemes is that an image can be transformed into a bilevel image even if the histogram is not bimodal. This is very apparent from the histogram shown in Fig.2.9. The first scheme (based on maximizing $G_1(t)$) is geared towards revealing all the details in the image which is a drawback when the background is not constant or contains some noise as shown in Fig.2.10. The second technique (based on maximizing $G_2(t)$) can be used when the background is not constant or contains some shadows or light reflections. A good example which reveals the power of the two techniques is the image shown in Fig.2.12. In Fig.2.12 (c,d) the image was thresholded using the first technique and the shadow of numeral '3' appeared underneath numeral '2',

while the same shadow was merged with the background and disappeared when the second technique was used as shown in Fig.2.12 (e,f). Fig.2.6, Fig.2.10, and Fig.2.11 show the effect of light reflections on the background, where the first technique revealed the contours of these light reflections, while the second technique produced a clean background.

Fig.2.13- Fig.2.16 show the performance of the proposed schemes in the presence of additive Gaussian noise. In Fig.2.13 to Fig.2.16 and Table.2.2 the performance using the histogram analysis has not been provided for the fact that histograms of noisy images do not have distinct modes or valley points which can be used for the thresholding operation. Small noisy contours are removed by discarding all contours having sizes less than 10 pixels. Fig.2.16 shows the effect of removing contours of sizes less than 10 pixel on the output.

Thresholding using the first technique tends to be biased towards that threshold value which reveals the contours of the object as well as the contours produced by the noise in the background as shown in Fig.2.13 and Fig.2.14 (a,b).

The second technique results show that the noise did not affect the background and has the same effect on the object contours as in the first Technique.

Table.2.2 Results of method 1, method 2 and the histogram analysis:

$t = t_1$ corresponds to threshold value obtained using method 1

$t = t_2$ corresponds to threshold value obtained using method 2

$t = t_3$ corresponds to threshold value obtained using the histogram

Fig.	Method 1		Method 2		Histogram		
	t_1	$G_1(t_1)$	t_2	$G_2(t_2)$	t_3	$G_1(t_3)$	$G_2(t_3)$
6	168	1924	161	38	160	1344	38
7	55	927	70	52	56	890	37
8	76	4075	51	48	64	3559	35
9	84	6750	93	88	32,96	405,4188	48, 79
10	33	1991	56	44	56	541	44
11	103	2313	66	92	72	2125	79
12	161	3546	128	97	128	3191	97
13	140	1691	116	24	-	-	-
14	140	1537	112	21	-	-	-
15	106	621	103	14	-	-	-

Fig.2.6 a-original image b-image histogram
 c-thresholded image (method 1) d-image contour (method 1)
 e-thresholded image (method 2) f-image contour (method 2)
 g-thresholded image (histogram) h-image contour (histogram)

Fig.2.7 a-original image b-image histogram
 c-thresholded image (method 1) d-image contour (method 1)
 e-thresholded image (method 2) f-image contour (method 2)
 g-thresholded image (histogram) h-image contour (histogram)

Fig.2.8 a-original image b-image histogram
 c-thresholded image (method 1) d-image contour (method 1)
 e-thresholded image (method 2) f-image contour (method 2)
 g-thresholded image (histogram) h-image contour (histogram)

Fig.2.9 a-original image b-image histogram
 c-thresholded image (method 1) d-image contour (method 1)
 e-thresholded image (method 2) f-image contour (method 2)
 g-thresholded image (histogram) h-image contour (histogram)

Fig.2.10 a-original image b-image histogram
 c-thresholded image (method 1) d-image contour (method 1)
 e-thresholded image (method 2) f-image contour (method 2)
 g-thresholded image (histogram) h-image contour (histogram)

Fig.2.11 a-original image b-image histogram
 c-thresholded image (method 1) d-image contour (method 1)
 e-thresholded image (method 2) f-image contour (method 2)
 g-thresholded image (histogram) h-image contour (histogram)

Fig.2.13 a-original image b-image histogram
 c-thresholded image (method 1) d-image contour (method 1)
 e-thresholded image (method 2) f-image contour (method 2)

Fig.2.14 a-original image b-image histogram
 c-thresholded image (method 1) d-image contour (method 1)
 e-thresholded image (method 2) f-image contour (method 2)

Fig.2.15 a-original image b-image histogram
 c-thresholded image (method 1) d-image contour (method 1)
 e-thresholded image (method 2) f-image contour (method 2)

a	b
c	d

Fig.2.16 a-original image b-all contours
 c-thresholded image (method 1) d-contours > 9-pixels

2.2.5.Summary

Two types of data, handwritten numerals and military aircrafts, were collected and preprocessed in preparation for the feature extraction and classification processes.

The optimum thresholding of images has been shown to be a feasible procedure. The golden section search technique used for maximizing the objective function is shown to be effective in generating a meaningful segmented image. Two new techniques for the automatic thresholding of images have been introduced. The first technique based on maximizing an objective function derives the value of threshold which maximizes the number of detected borders in the whole image as well as the number of pixels in each border. The second technique also based on maximizing an objective function derives the value of threshold which maximizes the ratio of the number of pixels in each border to the number of detected borders in the whole image. These two techniques do not require that the image histogram be bimodal. Results with several images have indicated the feasibility of the techniques for image processing applications.

CHAPTER 3

MOMENTS AND MOMENT INVARIANTS

3.1 Introduction

The idea of using the moments in shape recognition gained prominence in 1961 when Hu[38] derived a set of invariants using the theory of algebraic invariants. The second order moments were also used to parameterize hand written characters and turned out to be very effective in reducing the classification error rate[12].

The term "invariant" denotes an image or a shape feature which remains unchanged if that image or shape undergoes one or combination of the following changes:

- 1-change of size (scale).
- 2-change of position (translation).
- 3-change of orientation (rotation).
- 4-reflection.

The moment invariants can also be divided into skew moment invariants and true moment invariants. The skew moment invariants are invariant under change of size, translation and rotation only while the true moment invariants are invariant under change of size, translation, rotation and reflection.

There are several types of moment invariants; among these are the regular moment invariants, the algebraic moment invariants and the orthogonal moment invariants. The diagram in Fig.3.1 shows the classification of the moment invariants.

Fig.3.1 Classification of moment invariants

3.2 Regular moment invariants

The most commonly used moments are the regular moments, and they are defined as:

$$M_{pq} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x^p y^q f(x,y) dx dy \quad (3.1)$$

M_{pq} is the two dimensional moment of a function $f(x,y)$. The order of the moment is $(p+q)$.

The first order moments are used to locate the centroid of the image, where:

$$\bar{x} = \frac{M_{10}}{M_{00}}, \quad \bar{y} = \frac{M_{01}}{M_{00}} \quad (3.2)$$

The central moments then can be defined as:

$$M_{pq} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - \bar{x})^p (y - \bar{y})^q f(x,y) dx dy \quad (3.3)$$

M_{pq} can be made invariant under change of scale (size) by dividing it by $[M_{20} + M_{02}]^{\gamma}$, where $\gamma = [p+q+2]/4$.

The scale-invariant central moments can then be written as:

$$\mu_{pq} = \frac{1}{[M_{20} + M_{02}]^{\gamma}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - \bar{x})^p (y - \bar{y})^q f(x,y) dx dy \quad (3.4)$$

Although μ_{pq} is invariant under translation and change of size, they are not invariant under rotation or reflection. To complete the invariance properties of the regular moments they also should include the invariance under rotation and reflection. The first step taken to make the regular moments a set of complete invariance is to estimate the angle θ by which image is rotated as shown in Fig.3.2. The angle θ is estimated using the second order moments as[44]:

$$\theta = \frac{1}{2} \tan^{-1} \left[\frac{2M_{11}}{M_{20} - M_{02}} \right] \quad (3.5)$$

It was also shown that[44], if only moments up to the second order were used for reconstruction they would lead to an ellipse fitted around the original image as shown in Fig.3.2.

The regular moment invariants (RMI) can then be written as[44]:

$$(RMI)_{jk} = \sum_{r=0}^j \sum_{s=0}^k (-1)^{k-s} \binom{j}{r} \binom{k}{s} (\cos\theta)^{j-r+s} (\sin\theta)^{k+r-s} \mu_{(j+k-r-s), (r+s)} \quad (3.6)$$

where μ is the scale-invariant central moments. The invariance under reflection can also be achieved using[8]:

$$(RMI)_{pq} = (-1)^q \mu_{pq} \quad (3.7)$$

where the image is assumed to be reflected about the y-axis.

It is noted that the term $(-1)^{k-s}$ is missing in Equation (13)

in Ref.[8] possibly due to typographical errors. The total number of RMIs is $(n^2 + 3n - 6)/2$, where n is the highest order of the moments.

The main difficulty with the RMI lies in estimating the angle θ . The principal axes can not be uniquely defined[38,44,8], particularly when the image is of circular or n -fold rotational symmetry,

Fig 3.2 The estimation of the image orientation.

3.3 Algebraic moment invariants

The algebraic moment invariants are based on the theory of algebraic invariants. Hu moment invariants[38] and Bamieh moment invariants[10] can be classified as algebraic moment invariants.

3.3.1 Hu moment invariants

Using the theory of algebraic invariants Hu[38], derived the following set of moment polynomials:

$$\begin{aligned}
 I_{p-r,r} = & [\mu_{p0} + \binom{r}{1}\mu_{p-2,2} + \binom{r}{2}\mu_{p-4,4} + \dots + \binom{r}{r}\mu_{p-2r,2r}] \\
 & + (-1)^{\binom{p-2}{1}} [\mu_{p-1,1} + \binom{r}{1}\mu_{p-3,3} + \binom{r}{2}\mu_{p-5,5} + \dots + \binom{r}{r}\mu_{p-2r-1,2r+1}] \\
 & + (-1)^2 \binom{p-4}{2} [\mu_{p-2,2} + \binom{r}{1}\mu_{p-4,4} + \binom{r}{2}\mu_{p-6,6} + \dots + \binom{r}{r}\mu_{p-2r-2,2r+2}] \\
 & + \dots \\
 & + (-1)^r \binom{p-2r}{r} [\mu_{2r,p-2r} + \binom{r}{1}\mu_{2r+2,p-2r-2} + \binom{r}{2}\mu_{2r+4,p-2r-4} + \dots + \binom{r}{r}\mu_{0p}]
 \end{aligned}$$

$$\text{where } p-2r > 0. \quad (3.8)$$

and

$$I_{p/2,p/2} = \mu_{p0} + \binom{p/2}{1}\mu_{p-2,2} + \binom{p/2}{2}\mu_{p-4,4} + \dots + \mu_{0p}$$

$$\text{for } p \text{ even.} \quad (3.9)$$

For simplicity we can express $I_{p-r,r}$ as:

$$I_{p-r,r} = |I_{p-r,r}| e^{j\phi_{p-r,r}} \quad (3.10)$$

$$\text{where } \phi_{p-r,r} = \tan^{-1} \left(\frac{\text{real}(I_{p-r,r})}{\text{imaginary}(I_{p-r,r})} \right)$$

The following Hu moment invariants (HMI) can be constructed[38]:

$$(HMI)_0 = |I_{p0}|^2 \quad (3.11)$$

$$(HMI)_r = |I_{p-r,r}|^2 \quad r = 1, 2, 3, \dots \quad p-2r > 0 \quad (3.12)$$

$$(HMI)_{p/2+r} = 2 |I_{p-r,r}| |I_{r-1,p-r+1}| \cos(\phi_{p-r,r} + \phi_{r-1,p-r+1}) \quad (3.13)$$

$$r = 1, 2, 3, \dots \quad p-2r > 0$$

If p is even we also have:

$$(HMI)_{p+1} = 2 |I_{p/2-1,p/2+1}| |I_{20}| \cos(\phi_{p/2-1,p/2+1} + \phi_{20}) \quad (3.14)$$

$$(HMI)_{p/2} = I_{p/2,p/2} \quad (3.15)$$

Or if p is odd we have:

$$(HMI)_{p+1} = 2 |I_{p/2,p/2+1}| |I_{20}| \cos(\phi_{p/2,p/2+1} + \phi_{20}) \quad (3.16)$$

note that when p is odd, the integer part of p/2 is used.

A list of HMI, up to the seventh order, is described in Appendix A.

3.3.2 Bamieh moment invariants

Another set of algebraic moment invariants was derived by Bamieh and De Figueiredo[10]. These moments are invariant under three dimensional rigid body motion which is a combination of translation and rotation about an arbitrary axis.

The main characteristic of these invariants is that their feature vector size is much lower than any other known invariants which makes them computationally very efficient.

Bamieh moment invariants (BMI) were formed using moment tensors[10]. The moment tensor is defined as:

$$M^{ijk\dots} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x^i x^j x^k \dots f(x^1, x^2) dx^1 dx^2 \quad (3.17)$$

where i, j, k take the values of 1 or 2.

The moment tensors can be related to the regular moments by putting $x^1 = x$, and $x^2 = y$.

Combinations of these moment tensors were used by Bamieh and De Figueiredo to derive a set of algebraic moment invariants.

Using the scale-invariant central moments Bamieh moment invariants (BMI) can be listed up to the fourth order moments as in Table.3.1.

Some typographical errors were noted in the invariants shown in Table.1 in Ref.[10]. The following is a correction for these errors:

The second order moment invariant should be written as:

$(M_{02} M_{20} - M_{11}^2)$ and the weight w for the third order moment invariant is -5. The weight w is not shown in Table.3.1 due to the

fact that the scale-invariant central moments are used rather than the regular moments. The weight w is equivalent to γ in equation(3.4).

Table.3.1 *Bamieh moment invariants (BMI)*

<p><i>Second order moment invariant:</i></p> $(BMI)_1 = \mu_{02}\mu_{20} - \mu_{11}^2$
<p><i>Third order moment invariant:</i></p> $(BMI)_2 = (\mu_{03}\mu_{30} - \mu_{21}\mu_{12})^2 - 4(\mu_{03}\mu_{12} - \mu_{21}^2)(\mu_{21}\mu_{30} - \mu_{12}^2)$
<p><i>Fourth order moment invariant:</i></p> $(BMI)_3 = \mu_{40}\mu_{04} - 4\mu_{31}\mu_{13} + 3\mu_{22}^2$ $(BMI)_4 = \mu_{40}\mu_{22}\mu_{04} - 2\mu_{31}\mu_{22}\mu_{13} - \mu_{40}\mu_{13}^2 - \mu_{04}\mu_{31}^2 - \mu_{22}^3$

3.4 Zernike and pseudo Zernike moment invariants

Zernike and pseudo Zernike moment invariants are based on Zernike and pseudo Zernike orthogonal polynomials. In this section, a description of these polynomials is given. Zernike moment invariants as described by Teague[44] are presented. A new form of Zernike and Pseudo Zernike moment invariants are also derived.

3.4.1 Zernike polynomials and Zernike moments

Zernike polynomials are well known and widely used in the analysis of optical systems[76-80]. More recently they were used in shape reconstruction and suggested for recognition of shapes[44,51].

Zernike polynomials are an orthogonal set of polynomials of the following form[54,81]:

$$V_{nL}(x,y) = V_{nL}(r\cos\phi, r\sin\phi) = R_{nL}(r)e^{jL\phi} \quad (3.18)$$

where $V_{nL}(x,y)$ denotes a complete set of complex-valued polynomials, in two real variables x and y , which are orthogonal for the interior of the unit circle $x^2 + y^2 = 1$, n represents the degree of the polynomial, L represents its angular dependence, $R_{nL}(r)$ represents a real-valued set of polynomials orthogonal inside the unit circle, r and ϕ are polar coordinates and $j = \sqrt{-1}$.

The Zernike moments are defined as[44]:

$$A_{nL} = \frac{n+1}{\pi} \int_0^{2\pi} \int_0^\infty f(r \cos \phi, r \sin \phi) R_{nL}(r) e^{-jL\phi} r dr d\phi \quad (3.19)$$

$$n = 0, 1, 2, \dots, \infty \quad n-L \text{ is even, and } n \geq L \geq 0$$

$R_{nL}(r)$ can be expanded in powers of r using:

$$R_{nL}(r) = \sum_{k=L}^n B_{nLk} r^k \quad (3.20)$$

where

$$B_{nLk} = \frac{(-1)^{(n-k)/2} [(n+k)/2]!}{[(n-k)/2]! [(L+k)/2]! [(k-L)/2]!} \quad (3.21)$$

and n denotes the order of the moment.

3.4.2 Teague-Zernike moment invariants

A set of invariants can be derived using the following property[44]:

$$\tilde{A}_{nL} = A_{nL} e^{-jL\theta} \quad (3.22)$$

θ is the angle by which the function $f(x,y)$ is rotated. A_{nL} and $\tilde{A}_{nL}$ are Zernike moments before and after rotation respectively.

Using equation (3.22), Teague-Zernike moment invariants (TZMI), can be written as:

$$(TZMI)_{no} = A_{no} \quad (3.23)$$

$$(TZMI)_{nL} = |A_{nL}|^2 \quad (3.24)$$

$$(TZMI)_{nz} = [A_{nL}^* (A_{mh})^p] \pm [A_{nL}^* (A_{mh})^p]^* \quad (3.25)$$

where $h \leq L$, $p \geq 1$, $p=L/h$, $(L \bmod h)=0$, and $z=pL+h$.

The first two sets of invariants given by equations (3.23) and (3.24) are called primary invariants. The third set of invariants, equation (3.25) is called secondary invariants. These sets of invariants are invariant under rotation and reflection when the positive sign of the secondary invariants is used and invariant under rotation only when the negative sign is used.

There are an infinite number of Zernike moments which can satisfy equation (3.25), but only few of them are functionally independent. The functional independence of the invariants can be defined[44] as follows; a set of invariants is independent if they can be solved for the moments which forms these invariants.

Teague randomly selected moments which satisfy equation (3.25) to derive the invariants. Using these invariants he then solved for the moments to ensure their functional independence. The random selection of the moments to form the invariants does not guarantee their functional independence which makes this method subject to trial and error. The other drawback of this method is that the factor p increases as the order of the moments gets higher, which makes the dynamic range between the primary and the secondary invariants very large. The large dynamic range leads to a feature vector which is dominated by the larger-magnitude invariants. The complexity of this method particularly when the order of moments is large, makes it impractical and difficult to use. A simpler method that can be used to derive Zernike moment invariant up to any order, will be explained in the next section.

TZMIs are listed up to the seventh order in Appendix B.

3.4.3 New form of Zernike moment invariants

Equations (3.23-3.25) can be modified as:

$$(ZMI)_{no} = A_{no} \quad (3.26)$$

$$(ZMI)_{nL} = |A_{nL}| \quad (3.27)$$

$$(ZMI)_{n,n+z} = [A_{mh}^* (A_{nL})^p] \pm [A_{mh}^* (A_{nL})^p]^* \quad (3.28)$$

where $h \leq L$, $m \leq n$, $p = h/L$, $0 \leq p \leq 1$, and $z=L/h$.

The difference between ZMI and TZMI is that equation (3.27) is reduced to the magnitude of the moment only. The major change is in equation (3.28), where more constraints are used to avoid the problems associated with the construction of the secondary invariants of TZMI. The first constraint is used to ensure that only combinations of moments of orders lower than n are used to form the secondary invariants, the second constraint is used to limit the values of the factor p between 0 and 1. This constraint tends to decrease the magnitudes of the secondary invariants as $(n-L)$ increases. This factor is quite different from the one used by Teague[44], where p can take any value ≥ 1 which tends to increase the magnitudes of the secondary invariants. This in turn makes the magnitudes of the secondary invariants more dominant and the feature vector more biased to those features gathered by the secondary invariants.

Equation (3.28) can be written as:

$$[A_{mh}^* (A_{nL})^p] + [A_{mh}^* (A_{nL})^p]^* = 2|A_{mh}| |A_{nL}|^p \cos[p\phi_{nL} - \phi_{mh}] \quad (3.29)$$

$$[A_{mh}^* (A_{nL})^p] - [A_{mh}^* (A_{nL})^p]^* = 2|A_{mh}| |A_{nL}|^p \sin[p\phi_{nL} - \phi_{mh}] \quad (3.30)$$

The number of independent ZM for each order n is $n+1$, hence the number of independent ZMI should also be $n+1$.

For n even: the following ZMI can be constructed;

$$(ZMI)_{n0}, \text{ and } (ZMI)_{nL} \quad L=2, 4, 6, \dots, n \quad (3.31)$$

$$(ZMI)_{n,n+z} = 2|A_{n2}| |A_{nL}|^p \cos[p\phi_{nL} - \phi_{n2}] \quad (3.32)$$

$$L = 4, 6, \dots, n, \quad n = 4, 6, 8, \dots, \quad p=2/L, \text{ and } z=L/2.$$

and

$$(ZMI)_{n,n+1} = 2|A_{n-2,2}| |A_{n2}| \cos[\phi_{n-2,2} - \phi_{n2}] \quad (3.33)$$

$$, n = 4, 6, 8, \dots$$

where ϕ is the phase component of the invariant.

It should be noted that the number of independent ZMI given by $(ZMI)_{n0}$ and $(ZMI)_{nL}$ constitutes $(n/2)+1$ of the invariants. The other $n/2$ invariants are given by: $(ZMI)_{n,n+1}$ and $(ZMI)_{n,n+z}$ where $z=2, 3, 4, \dots, n/2$. It is also noted that the second order moments

have no secondary invariants, due to the fact that the first order moments are either zero or constant.

The previous set of invariants are related through the phase angle ϕ_{nL} . This set of invariants is then independent only if the matrix of the coefficients of the phase angles is non-singular.

In other words the determinant should satisfy:

$$\det \begin{vmatrix} \phi_{n2} & \phi_{n4} & \phi_{n6} & \phi_{nn} \\ -1 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & 0 \\ -1 & 0 & \frac{1}{3} & 0 \\ \cdot & \cdot & \cdot & \cdot \\ -1 & 0 & 0 & \frac{2}{n} \end{vmatrix} \neq 0$$

For n odd: The following independent set of ZMI can be constructed:

$$(ZMI)_{nL} = |A_{nL}| \quad L = 1, 3, 5, \dots, n \quad (3.34)$$

$$(ZMI)_{n,n+L} = 2|A_{n1}| |A_{nL}|^p \cos[p\phi_{nL} - \phi_{n1}] \quad (3.35)$$

$$L = 3, 5, 7, \dots, n \quad \text{and } p = 1/L$$

and

$$(ZMI)_{n,n+1} = 2|A_{n-2,1}| |A_{n1}| \cos[\phi_{n-2,1} - \phi_{n1}] \quad (3.36)$$

It should be noted, here also, that the number of independent ZMI should be complemented by extra $(n+1)/2$ as $(ZMI)_{nL}$ constitute $(n+1)/2$ of the invariants. The other $(n+1)/2$ invariants are given by: $(ZMI)_{n,n+L}$ and $(ZMI)_{n,n+1}$.

It is also noted that the number of ZMI for the third order moments is 3 instead of 4. The reason is that A_{11} , which is used in equation (3.36), is equal to zero. This in turn makes the number of ZMI, for any order higher than 2, always less than TZMI by one invariant.

For independence the following condition must be satisfied:

$$\begin{vmatrix} \phi_{n1} & \phi_{n3} & \phi_{n5} & \phi_{nn} \\ -1 & 0 & 0 & 0 \\ -1 & \frac{1}{3} & 0 & 0 \\ -1 & 0 & \frac{1}{5} & 0 \\ . & . & . & . \\ -1 & 0 & 0 & \frac{1}{n} \end{vmatrix} \neq 0$$

3.4.4 Pseudo Zernike polynomials and pseudo Zernike moments

Pseudo Zernike polynomials are similar to Zernike polynomials and have the same properties. Zernike polynomials are polynomials in x and y only, while pseudo Zernike polynomials are polynomials in x , y and r , where $r = [x^2 + y^2]^{1/2}$. Pseudo Zernike polynomials have the following form[54]:

$$W_{nL}(x, y, r) = W_{nL}(r \cos \phi, r \sin \phi, r) = S_{nL}(r) e^{jL\phi} \quad (3.37)$$

where $W_{nL}(x, y, r)$ denotes a complete set of complex-valued polynomials, in three real variables x , y and r , which are orthogonal in the interior of the unit circle $x^2 + y^2 = 1$, n represents the degree of the polynomial, L represents its angular dependence, $S_{nL}(r)$ represents a real-valued set of polynomials orthogonal inside the unit circle, r and ϕ are polar coordinates and $j = \sqrt{-1}$.

The pseudo Zernike moments (PZM) are defined as:

$$T_{nL} = \frac{n+1}{\pi} \int_0^{2\pi} \int_0^{\infty} f(r \cos \phi, r \sin \phi) S_{nL}(r) e^{-jL\phi} r \, dr \, d\phi \quad (3.38)$$

$$n = 0, 1, 2, \dots, \infty \quad L \leq n$$

It should be noted that in this case $n-L$ need not be even.

PZM can be expressed in powers of r using:

$$S_{nL}(r) = \sum_{k=L}^n C_{nLk} r^k \quad (3.39)$$

where

$$C_{nLk} = \frac{(-1)^{(n-k)} [n+k+1]!}{[n-k]! [L+k+1]! [k-L]!} \quad (3.40)$$

3.4.5 Pseudo Zernike moment invariants

Since pseudo Zernike polynomials have the same rotational invariance as Zernike polynomials, we can construct pseudo Zernike moment invariants (PZMI) as follows:

$$(PZMI)_{n0} = T_{n0} \quad (3.41)$$

$$(PZMI)_{nL} = |T_{nL}| \quad (3.42)$$

$$(PZMI)_{n,n+z} = [T_{mh}^* (T_{nL})^p] \pm [T_{mh}^* (T_{nL})^p]^* \quad (3.43)$$

$$h \leq L, \quad m \leq n, \quad z=L/h \quad \text{and} \quad p=h/L$$

In equation (3.43) when the negative sign is used the invariance property holds under improper transformation only (it is not invariant under reflection). The total number of linearly independent PZM in each order is $2n+1$ and hence the total number of independent PZMI is $2n+1$. With the help of equations (3.41-3.43) we can write PZMI up to an order n as:

$$(PZMI)_{n0}, \text{ and } (PZMI)_{nL} \quad L=1,2,3,4,\dots,n \quad (3.44)$$

Using $(PZMI)_{n0}$ and $(PZMI)_{nL}$ we can get only $n+1$ independent moment invariants. To construct the remaining n -invariants we use:

$$(PZMI)_{n,n+L} = 2|T_{n1}| |T_{nL}|^p \cos[p\phi_{nL} - \phi_{n1}] \quad (3.45)$$

$$L = 2, 3, 4, \dots, n, \text{ and } p=1/L$$

and

$$(PZMI)_{n,n+1} = 2|T_{n-1,1}| |T_{n1}| \cos[\phi_{n-1,1} - \phi_{n1}] \quad (3.46)$$

The phase part of equation (3.45) and equation (3.46) can be expressed as:

$$\phi_{n-1,1} - \phi_{n1} = a_1$$

$$p\phi_{nL} - \phi_{n1} = a_2$$

where a_1 and a_2 are constants. Thus the condition of functional independence of PZMI is given by:

$$\det \begin{vmatrix} \phi_{n1} & \phi_{n2} & \phi_{n3} & \phi_{nn} \\ -1 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & 0 \\ -1 & 0 & \frac{1}{3} & 0 \\ \cdot & \cdot & \cdot & \cdot \\ -1 & 0 & 0 & \frac{1}{n} \end{vmatrix} \neq 0$$

3.5 Normalization of Zernike and pseudo Zernike moment invariants

The main task of an invariant is to represent some features of the shape which remain unchanged under certain transformations. This task is served well when each invariant is conveying different information or features for the same shape. In practice some of the invariants are repeating the same information which makes them redundant and leads to errors in shape recognition.

Another factor which contributes to the errors in shape recognition is the large dynamic range of the invariants. The large dynamic range causes the invariants of large magnitude to dominate the invariants of smaller magnitude.

A new normalization procedure aimed at minimizing the errors in shape recognition for both ZMI and PZMI is presented in this section.

3.5.1 Normalized Pseudo Zernike Moment Invariants

In this section a recursive relationship is derived for PZM, that illustrates the dependence of the moment T_{nL} on $T_{n-1,L}$. A new normalization procedure which is aimed at reducing the dynamic range and leading to an improvement in performance for shape recognition applications is introduced.

The pseudo Zernike moments can be expanded in terms of the scale-invariant central moments μ_{pq} as follows:

$$T_{nL} = \frac{n+1}{\pi} \left(\sum_{q=L}^n \alpha_{nq} G_{qL} \right) \quad (3.47)$$

$$\alpha_{nq} = \frac{(-1)^{n-q} (n+q+1)!}{(n-q)! (2q+1)!} \quad (3.48)$$

$$G_{qL} = \beta_{qL} \left\{ \sum_{j=0}^{q-L} \sum_{m=0}^L (-1)^m \binom{q-L}{j} \binom{L}{m} \mu_{(q-j-L+m), (j+L-m)} \right\} \quad (3.49)$$

$$\beta_{qL} = \frac{(2q+1)!}{(q-L)!(q+L+1)!} \quad (3.50)$$

$$T_{nL} = -\left(\frac{n+1}{\pi}\right) \left\{ \sum_{q=L}^{n-1} \alpha_{n-1,q} G_{qL} \left(\frac{n+q+1}{n-q}\right) + G_{nL} \right\} \quad (3.51)$$

$$T_{nL} = -\frac{n+1}{n} T_{n-1,L} - \left(\frac{n+1}{\pi}\right) \left\{ \sum_{q=L}^{n-1} \alpha_{n-1,q} G_{qL} \left(\frac{2q+1}{n-q}\right) + G_{nL} \right\} \quad (3.52)$$

$$L < n, L=0, 1, 2, \dots, n-1$$

$$T_{nn} = \frac{n+1}{\pi} G_{nn} = \frac{n+1}{\pi} \sum_{m=0}^n (-1)^m \binom{n}{m} \mu_{m, n-m} \quad (3.53)$$

From equation (3.52) we can see that every T_{nL} is dependent on $T_{n-1,L}$ and only T_{nn} is an independent set of moments.

We now define a normalization procedure which rationalizes T_{nL} by $T_{n-1,L}$.

The new normalized PZM (NPZM) can then be written as:

$$T'_{nL} = T_{nL} / T_{n-1,L} \quad \text{for } T_{n-1,L} \neq 0 \text{ and } L < n \quad (3.54)$$

and

$$T'_{nL} = T_{nL} \quad \text{for } T_{n-1,L} = 0 \text{ or } L = n \quad (3.55)$$

Using the normalization procedure explained in (3.54) the NPZM can be written up to the seventh order as shown in Appendix C.

The advantage of the previous normalization is very evident in Fig.(3.3), where the magnitudes of the largest ten PZM before normalization are decreasing very sharply, while the magnitude

after normalization remains relatively constant. The magnitude ratio of the largest moment to the tenth moment reduced from 64 to 3 due to this normalization.

Using the normalized pseudo Zernike moments, the normalized pseudo Zernike moment invariants (NPZMI) can be defined using the same set of equations derived in section (3.4.5) by replacing PZM by NPZM.

3.5.2 Normalized Zernike Moment Invariants

The same normalization procedure as in section (3.5.1) can be applied to Zernike moments as follows:

$$A_{nL} = \frac{n+1}{\pi} \left(\sum_{q=L}^n \alpha_{nq} G_{qL} \right) \quad (3.56)$$

$$\alpha_{nq} = \frac{(-1)^{[n-q]/2} ([n+q]/2)!}{([n-q]/2)! q!} \quad (3.57)$$

$$G_{qL} = \beta_{qL} \left(\sum_{j=L}^{[q-L]/2} (-1)^m \binom{[q-L]/2}{j} \binom{L}{m} \mu_{(q-2j-L+m), (2j+L-m)} \right) \quad (3.58)$$

$$\beta_{qL} = \frac{q!}{([q-L]/2)! ([q+L]/2)!} \quad (3.59)$$

$$A_{nL} = -\left(\frac{n+1}{\pi}\right) \left(\sum_{q=L}^{n-2} \alpha_{n-2,q} G_{qL} \left(\frac{n+q}{n-q}\right) + G_{nL} \right) \quad (3.60)$$

$$A_{nL} = -\frac{n+1}{n-1} A_{n-2,L} - \left(\frac{n+1}{\pi}\right) \left(\sum_{q=L}^{n-2} \alpha_{n-2,q} G_{qL} \left(\frac{2q}{n-q}\right) + G_{nL} \right) \quad (3.61)$$

$L < n$, $L = 0, 1, 2, \dots, n-2$ and $n-L$ is even

$$A_{nn} = T_{nn} = \frac{n+1}{\pi} G_{nn} = \frac{n+1}{\pi} \sum_{m=0}^n (-1)^m \binom{n}{m} \mu_{m, n-m} \quad (3.62)$$

$$A'_{nL} = A_{nL} / A_{n-2,L} \quad \text{for } A_{n-2,L} \neq 0 \text{ and } L < n \quad (3.63)$$

and

$$A'_{nL} = A_{nL} \quad \text{for } A_{n-2,L} = 0 \text{ or } L = n \quad (3.64)$$

Using equation (3.63), the NZM (normalized ZM) can be derived for up to the seventh order as shown in Appendix D. The improvement in the dynamic range is once again evident when one examines Fig.(3.4), where the largest ten moments are plotted for both Zernike and normalized Zernike. The normalization also reduced the ratio of the largest moment to the tenth moment from 29 to 5.

The normalized Zernike moment invariants (NZMI) are derived using the same equations which were used to derive ZMI in section (3.4.4) by replacing ZM by NZM.

Figure (3.3) The dynamic range for PZM and NPZM

Figure (3.4) The dynamic range for ZM and NZM

3.6 Feature vector size reduction

A new technique called feature-vector size reduction is described in this section. This technique is based on the removal of the redundant invariants and using only the essential invariants in the recognition process.

The feature vector size refers to the number of invariants used to construct the feature vector used in the recognition process.

Reducing the number of invariants or feature vector size eventually will reduce the number of computations when these invariants are utilized in shape recognition.

3.6.1 Reduced pseudo Zernike moment invariants

A close examination of equation(3.52) leads to the following observations:

- every moment of order n is dependent on a moment of order $n-1$.
- information carried out by moments of order $n-1$ is implicitly included in the moments of order n .
- including the invariants of order n and excluding the invariants of order $n-1$ in the formation of the feature vector will result in only a small loss of information about the shape.

Based on these observations the reduced PZMI (RPZMI) can be formed from the following orders only:

$$n, n-2, n-4, n-6, \dots, 0. \quad (3.65)$$

The invariants which contain the following orders are excluded:

$$n-1, n-3, n-5, \dots, 0 \quad (3.66)$$

Using the invariants listed in Appendix E and equation (3.65),

RPZMI can be constructed up to n-th order. Table (3.2) shows the amount of savings obtained with this reduction. There is no saving for moment orders lower than 3 due to the fact that the moments of orders lower than 2 are either constant or zero.

3.6.2 Reduced Zernike moment invariants

The same reduction procedure can also be implemented on ZMI. Examining equation (3.61) also leads to the following observations about Zernike moments:

- Every moment of order n and n-1 is dependent on a moment of order n-2 and n-3 respectively.
- The information carried out by moments of order n-2 is implicitly included in the moments of order n and the information carried out by moments of order n-3 is implicitly included in the moments of order n-1.
- Including the invariants of order n, n-1 and excluding the invariants of orders n-2 and n-3 in the formation of the feature vector will result in no or a small loss of information about the shape.

Based on the previous observations the reduced ZMI (RZMI) can be formed from the following orders only:

$$n, n-1, n-4, n-5, n-8, n-9, \dots, 0. \quad (3.67)$$

The invariants which contain the following orders are excluded:

$$n-2, n-3, n-6, n-7, n-10, n-11, \dots, 0 \quad (3.68)$$

Using the invariants listed in Appendix F and equation (3.67) RZMI can be constructed up to n -th order.

The Number of invariants for the ZMI and the RZMI is shown in Table (3.3). The reduction in the number of invariants increases as the moment order increases. There is no saving in the number of invariants for moments of order less than four in the case of ZMI. This is due to the fact that the lower orders are either constant or equal to zero.

TABLE.3.2 FEATURE VECTOR SIZES FOR PZMI AND RPZMI OF MOMENTS UP TO TENTH ORDER.

MOMENT ORDER	PZMI	RPZMI
2	4	4
3	11	6
4	20	12
5	31	16
6	44	24
7	59	30
8	76	40
9	95	48
10	116	60

TABLE.3.3 FEATURE VECTOR SIZES FOR ZMI AND RZMI OF MOMENTS UP TO TENTH ORDER.

MOMENT ORDER	ZMI	RZMI
2	2	2
3	5	5
4	10	7
5	16	9
6	23	13
7	31	18
8	40	22
9	50	26
10	61	32

3.7 Moment Computation

For a gray level image of size $N \times M$ a moment of order $p+q$ can be written as:

$$M_{pq} = \sum_{x=0}^N \sum_{y=0}^M x^p y^q f(x,y) \quad (3.69)$$

where $f(x,y)$ represents the gray level of the image.

For a binary image $f(x,y)$ is set to zero for the background and one for the object, which reduces equation(3.69) to:

$$M_{pq} = \sum_{i=1}^L x_i^p y_i^q \quad (3.70)$$

where L represents the number of pixels belonging to the object. The same equation can be used to evaluate the contour moments where L reduces to the number of contour pixels. The scale-invariant central contour moments can also be represented as follows:

$$\mu_{pq} = \frac{1}{[M_{20} + M_{02}]} \sum_{i=1}^L (x_i - \bar{x})^p (y_i - \bar{y})^q \quad (3.71)$$

where x and y represent the position of the contour pixel; $\bar{x}$ and $\bar{y}$ represent the centroid of the image.

3.8 Summary

Review of the two dimensional moments has been given in this chapter. The definitions of the moment invariants have been outlined.

A general form for the regular moment invariant has been described.

Two forms of algebraic moment invariants; Hu moment invariants and Bamieh moment invariants, have been discussed.

An extensive literature survey of the moment invariants revealed that most previous researchers[7,8,38,39,41,42,45,74,75] used the second and third order HMI only. To make this study complete we extended HMI up to the seventh order. These moments invariants are listed in Appendix A.

A review of Zernike and pseudo Zernike polynomial is also presented in this chapter. Teague-Zernike moment invariants (TZMI) have been described.

A new form of Zernike moment invariants (ZMI) has been derived. These new invariants are basically combinations of Zernike moments of order n and $n-2$. This is different from the approach used by Teague, where random combinations of the n -th and lower orders are used. Checking the new form of Zernike moment invariants for functional independence is very simple as compared to Teague-Zernike moment invariants. A general form for TZMI is not possible due to the way these invariants are formulated.

A general form for pseudo Zernike moment invariants (PZMI) has also been derived. Combinations of n and $(n-1)$ of moment orders are used to derive PZMI.

A recursive relationship for describing Zernike moments and pseudo Zernike moments has been presented. It has been shown that some of Zernike moments of n -th order are dependent on Zernike moments of $(n-2)$ -th order. This leads to a normalization procedure which is based on normalizing the n -th order by the $(n-2)$ th order. It has been also shown that pseudo Zernike moments of order n are dependent on moments of order $n-1$, which leads to a normalization of the n -th order by the $(n-1)$ -th order

The normalization procedure has improved the dynamic range as shown in Fig.3.3-Fig.3.4 and shape recognition as will be shown in chapter 5.

Finally in this chapter, the feature-vector size reduction technique which is based on the removal of the redundant invariants from the original PZMI or ZMI has been described and suggested to be used as an alternative for the normalization procedure. The feature size vector reduction technique reduces the number of feature vectors needed for shape recognition particularly for higher order moments.

CHAPTER 4

CLASSIFICATION AND DISCRIMINATION

The classification or discrimination procedure is the most critical step in shape or pattern recognition application. There are two main approaches for the pattern classification, syntactic and statistical. Statistical approach can be subdivided further into parametric and non-parametric[82], as indicated by the diagram in Fig.4.1.

Fig.4.1 The main approaches for pattern classification

4.1 Syntactic pattern classification

The syntactic pattern classification explicitly utilizes the structure of a shape or pattern in the classification process[83]. The syntactic pattern classification is based on obtaining a grammar relating certain strings of patterns to each other the same way the natural grammar relates sentences to each other. The name linguistic pattern classification can also be attributed to the previous fact.

The classification of a given pattern string can be accomplished simply by determining whether the given string can be generated by a certain grammar or not[84-88].

The syntactic pattern classification is usually implemented in structural pattern recognition.

4.2 Statistical pattern classification

The statistical classification is based on a statistical measure of patterns or shapes and they are mainly categorized into two types of classifiers: parametric classifiers and non-parametric classifiers.

4.2.1 Parametric classifiers

A classifier which assumes a functional distribution of given samples is called a parametric classifier. The Bayes classifier is an example of parametric classifiers. The Bayes classifier is based on minimizing the total expected loss incurred by misclassification[83].

4.2.2 Non-parametric classifiers

The non-parametric classifiers do not assume any functional distribution of the given samples. The nearest neighbor rule and the k-nearest neighbor rule are example of these classifiers.

The main focus of this chapter will be on the non-parametric classifiers due to the fact that the distribution of the given samples is not known a priori.

4.2.2.1 Minimum distance classification

Pattern classification using distance measures is one of the earliest concepts in shape and pattern recognition.

A minimum distance classifier computes the distances from a pattern x of unknown classification to several patterns of known classification. The classifier then assigns the unknown pattern the same identity as that of the pattern with the minimum distance.

4.2.2.2 Distance measures

The fundamental purpose of a distance or similarity measure is to induce an order on the set of couples (x_i, x_q) for any i or q . An infinite number of distances or similarities may induce the same order. Some of the distance measures[89,93] are:

$$\begin{aligned} \text{Minkowsky metric:} \quad d(x_i, x_q) &= \left[\sum_{j=1}^n |x_{ij} - x_{qj}|^{1/\lambda} \right]^\lambda \\ \lambda &> 0 \end{aligned} \quad (4.1)$$

$$\text{Camberra metric:} \quad d(x_i, x_q) = \sum_{j=1}^n \frac{|x_{ij} - x_{qj}|}{|x_{ij} + x_{qj}|} \quad (4.2)$$

$$\text{Chebychev metric:} \quad d(x_i, x_q) = \text{MAX}_j |x_{ij} - x_{qj}| \quad (4.3)$$

$$\text{City block metric: } d(x_i, x_q) = \sum_{j=1}^n |x_{ij} - x_{qj}| \quad (4.4)$$

$$\text{Euclidean metric: } d(x_i, x_q) = \left[\sum_{j=1}^n |x_{ij} - x_{qj}|^2 \right]^{1/2} \quad (4.5)$$

$$\text{Mahalanobis metric: } d(x_i, x_q) = (x_i - x_q) C^{-1} (x_i - x_q) \quad (4.6)$$

where C is the pooled within class covariance matrix, x_i and x_q represent the feature vectors for sample i and q respectively. The covariance matrix is of size $n \times n$ where n is the size of the feature vector and the covariance matrix elements are given by:

$$c_{rs} = \frac{1}{L} \sum_{k=1}^L (x_{kr} - \bar{x}_r)(x_{ks} - \bar{x}_s) \quad (4.7)$$

$r, s = 1, 2, \dots, n$

where $\bar{x}_r$ and $\bar{x}_s$ are the mean values of x_r and x_s respectively and L is the number of samples in each class.

The covariance matrix can be approximated by[91]:

$$c_{rr} = \frac{1}{L-1} \sum_{k=1}^L (x_{kr} - \bar{x}_r)^2 \quad (4.8)$$

and the Mahalanobis distance can be written as:

$$d(x_i, x_q) = \sum_{j=1}^n \frac{(x_{ij} - x_{qj})^2}{c_{jj}} \quad (4.9)$$

4.2.2.3 Nearest neighbor rule

The nearest neighbor rule (NNR) is a classification technique that assigns a pattern x of unknown classification to the class of its nearest neighbor[94,95].

The nearest neighbor rule can be formulated as[96]

$$d(s_1, x) = \text{MIN}_i \{d(s_i, x)\} \quad (4.10)$$

where d is any distance measure, the set of samples $\{s_1, s_2, s_3, \dots, s_i, \dots, s_n\}$ is called the learning set, and x is the unknown sample to be classified.

4.2.2.4 K-nearest neighbor rule

The K-nearest neighbor rule (KNNR) is an extension of the nearest neighbor rule, where the unknown sample x is assigned the class of its K-nearest neighbors (KNN). K is an integer greater than one, when $K=1$ the K-nearest neighbor rule reduces to the nearest neighbor rule.

4.2.2.5 Efficient evaluation of the K-nearest neighbor rule

The direct method for evaluating the KNNR requires the search over the learning data set several times, for a learning set of N-samples this amounts to $K[N - (K-1)/2]$ distance evaluations.

Many efficient techniques were proposed to compute the nearest neighbor and the K-nearest neighbor rules[97-104]. Most of these techniques require preprocessing and clustering of the learning sets. The fastest of these techniques uses clustering and prototypes to approximate the features of their class. These

techniques may lead to incorrect classification when the clusters overlap or some of the samples are incorrectly clustered during the clustering process.

Since the main task is to compare the various moment invariants and the clustering is generally biased to one particular set of invariants, an alternative technique is needed.

In many practical situations the samples are already clustered into smaller subsets. To take advantage of this natural clustering the following technique is proposed:

- 1- Given an unknown sample x for which the KNN are to be found, a learning set of N samples which can be divided into m subsets each of size L (e.g. numeral set 0-9 with $m=10$ and $L=32$).
- 2- Find the nearest neighbor and the variance σ between x and each subset, remove those neighbors from their corresponding subsets and reduce L by 1.
- 3- Construct a distance table for the K -nearest neighbors using the subset that has the minimum variance, eliminate this subset from the search and reduce m by 1. The KNN-distance table is arranged in ascending order with its first and last entries corresponding to the nearest neighbor and the K -nearest neighbor respectively.
- 4- Move to the subset with the lowest variance in the $(m-1)$ remaining subsets.
- 5- If the nearest neighbor distance (computed earlier) is greater than the largest distance in the KNN-distance table, eliminate the current subset and reduce m by 1.

Else find the position of the current nearest neighbor in the KNN-distance table, update the table, find a new nearest neighbor from the current subset, eliminate this neighbor, reduce L by 1 and replace the current nearest neighbor with the new nearest neighbor.

- 6- Repeat step-5 until the current subset is eliminated from the search.
- 7- Repeat steps 4-6 until all the subsets are eliminated from the search.

The previous process is illustrated in fig.4.2.

Fig.4.2 Efficient evaluation of the KNNR.

The algorithm achieves its best performance when the clusters do not overlap. Based on this the number of searches is $K[L - (k-1)/2]$ and the amount of saving is $K(N-L)$.

The algorithm can still be made more efficient by eliminating any sample s_1 , which does not satisfy the following rule[97,98]:

$$R_1 > d(s_1, s_p) > R_2$$

where

$$R_1 = d(x, s_p) + d(x, s_n),$$

$$R_2 = d(x, s_p) - d(x, s_n),$$

s_1 is a sample from the learning set, s_p is a previous candidate, s_n is the current nearest neighbor to the unknown sample x , and d is the Euclidean distance between two samples. Fig.4.3 illustrates the previous rule graphically. The nearest neighbor algorithm can be summarized as:

-Preprocessing:

For a sample s_1 from a learning set of L samples, compute and store the distance $d(s_i, s_{i-1})$ for $i=2, 3, 4, \dots, L$.

-Nearest neighbor:

For an unknown sample x , s_n is the current nearest neighbor and s_p is a previous candidate.

Compute the distance $d(x, s_1)$, set $i=2$, $s_p = s_1$ and $s_n = s_1$.

1- If $d(s_1, s_p) \geq d(x, s_p) + d(x, s_n)$

or $d(s_1, s_p) \leq d(x, s_p) - d(x, s_n)$ then eliminate s_1 , otherwise

compute $d(x, s_1)$ and set $s_n = s_1$ if $d(x, s_1) < d(x, s_n)$, or

$s_p = s_1$ otherwise.

2- Set $i = i+1$ and repeat step-1 until $i = L$.

The main difference between the present algorithm and the ones described in the literature[97,98] is that in the present technique no clustering is used and the search for the nearest neighbor is confined to each class only. The fact that no approximation is used makes this technique very accurate and always converges to the desired neighbor. The other fact is that the K-nearest neighbors are computed independently for each sub class of the m classes which makes the parallel implementation of this algorithm possible.

Fig 4.3 Sample elimination: the samples which do not fall within the dotted area can be eliminated because they are not nearer than the current nearest neighbor s_n .

4.3 Weighted-KNNR and Multiple membership classification

In order to test the proposed invariants for their effectiveness in shape recognition applications, we use the well known cross validation[105,106] testing procedure.

The cross validation method which has a small bias, is widely used for the evaluation of the error rates of the various classifiers.

Let Ω be a set $\{1,2,\dots,n\}$ that contains n number of samples and m is the number of classes which forms the set Ω .

In order to have a testing procedure which is not biased to any particular set of invariants, the testing should be performed in the following manner:

For an element i of the set Ω , find the closest k -neighbors among the rest of the elements belonging to the set Ω by excluding the element i . Assign weights to each neighbor of the k -nearest neighbors as follows:

$$w_{sg} = \left[\frac{c_g}{c_M} \right] e^{-t} \quad (4.11)$$

$$\text{where} \quad t = \left[\frac{d_{s1}^2}{2\sigma^2} \right], \quad \sigma = \frac{1}{k} \left[\sum_{s=1}^k d_{s1}^2 \right]^{1/2}$$

c_g is the population of the nearest neighbors which belongs to a class g , c_M is the population of the largest class among the nearest neighbors, w_{sg} is the weight assigned to the neighbor s , and d_{s1} is the Euclidean distance between the neighbor s and the element i .

Add the weights w_{sg} for each class g of the m classes separately according to:

$$\lambda_g = \frac{1}{k} \left[\sum_{s=1}^k w_{sg} \right]. \quad (4.12)$$

where $g = 1, 2, \dots, m$.

$$0 \leq \lambda_g \leq 1$$

Let λ_1 , λ_2 , and λ_3 be the three largest weights among the m classes, then the unknown sample can be assigned a membership of λ_1 in class 1, λ_2 in class 2 and λ_3 in class 3.

The unknown sample is correctly classified if it belongs to at least one of the three classes.

A sample is rejected if the value of $(\lambda_1 + \lambda_2 + \lambda_3)$ is below a certain tolerance.

4.4 Discrimination factor

The number of misclassifications or the error rate itself is not always an indication of the performance of certain invariants. The need is for a measure which is sensitive to both the error rate as well as the rejection rate. In other words two methods having the same number of errors and different number of rejections should be rated differently.

A factor of discrimination can then be defined as follows:

$$\begin{aligned} DF &= 100 \left(\frac{N_c}{N - N_r} \right) \\ &= 100 \left(1 - \frac{N_r}{N - N_r} \right) \end{aligned} \quad (4.13)$$

where DF is the factor of discrimination, N is the total number of test samples, N_c is the number of correctly classified samples, N_r is the number of incorrectly classified samples, and N_r is the number of rejected samples.

The present form of equation (4.13) fails to discriminate between two invariants for the special case when N_r is zero for both of them. As a remedy for this situation the following adjustments are made:

$$DF = 100 \left(1 - \frac{N_r}{N - N_r + \epsilon} \right) \quad (4.14)$$

where ϵ is a constant and can be taken as $N/1000$ to ensure that it is relatively small compared to N.

4.5 Classification efficiency

The previous measures: error rate, rejection rate and the discrimination factor, are sufficient for comparing the various classification methods only if they all have the same number of invariants. In practice the number of invariants varies from one method to the other as well as from one order to the next.

An efficient classification method is a method which has the lowest number of invariants, lowest error rate and lowest rejection rate. The optimal case will be achieved for a certain method if that method has no error, no rejection and a feature vector size of one. Taking all these factors into account the following formula can be defined:

$$CE = \frac{100}{\log(V)} \left(1 - \frac{N_{\bullet} + N_r}{N} \right) \quad (4.15)$$

$$V = \sqrt{N_I} + 9$$

where CE is the classification efficiency, N_I is the number of invariants or feature vector size, $N_{\bullet}$ is the number of incorrectly classified samples, N_r is the number of rejections and N is the total number of samples.

Using the classification efficiency measure CE we can compare the various methods regardless of their feature vector sizes. The other advantage of this measure is that the optimal order of a certain method can be determined from the classification efficiency measure as will be explained in chapter 5.

4.6 Dynamic range reduction of the invariants

One of the main problems associated with constructing a feature vector for a data set is the large dynamic range of each individual element of the feature vector. Several techniques are used to reduce this dynamic range among them: the logarithm of the invariants (the invariants represent the elements of the feature vector) , the Mahalanobis metric, and the variance balancing technique.

The variance balancing technique[8,83] technique is described as follows:

- 1- Compute a library of feature vectors for all the samples.
- 2- Compute the standard deviation of each of the feature vector elements in the library.
- 3- Divide every feature vector element in the library by its corresponding standard deviation computed in step 2.
- 4- Each element of the feature vector of an unknown object must be scaled using the corresponding standard deviation computed in step 2.

A comparison between these techniques is given in chapter 5.

4.7 Summary

An efficient technique has been described for the evaluation of the K-nearest neighbor rule.

The cross validation testing procedure has been outlined. This testing procedure is based on using the same data set for learning and testing processes without duplicating any of the samples. The main advantages of this testing procedure are:

- Instead of using $N/2$ samples for learning and another $N/2$ samples for testing, $N-1$ samples are used for learning and N samples are used for testing.
- This method is not biased to the order or type of the invariants, due to the fact that all the samples are used for testing as well as for learning. This scheme is quite different from the classical testing technique in which the data is divided into two groups; one for testing and the other for learning. If the learning set classes are well clustered and the overlapping between the classes is minimal for one particular set of invariants, there is no guarantee it will remain the same for the other invariants. For this reason the classical scheme of testing is biased to one particular set of invariants and is not suitable for comparative studies.

The multiple membership classification technique has been described. Each sample is classified based on a weighted KNNR. The membership weight is generally higher for the class of the majority of the K-nearest neighbors.

Two measures have been proposed in this chapter, one for comparing the discrimination power of the various invariants and the other one is for comparing their classification efficiencies.

Chapter 5

PERFORMANCE EVALUATION OF THE MOMENT INVARIANTS

5.1 Sample preparation and data processing

The various types of invariants were tested for their effectiveness in shape recognition using the hand written numeral set (0 to 9), and the military aircraft set. In order for the images to be confined to a unit radius circle, the image was normalized to fit in a 1x1 square. The number of samples used in the numeral set is 320 and in the aircraft set is 288. The number of classes in the numeral set is 10 and for aircraft set the number of classes is 18.

In order to provide consistency, all the samples were used for testing. The moment invariants were computed and stored for all the samples. The Euclidean distance measure was used to compute the minimum distance between each sample and the rest of the set. The normalization against change of size and translation was applied to the regular moments as described in chapter 3. The normalized regular moments were used to compute the various moment invariants.

The testing procedure described in chapter 4 was applied to the following sets of invariants:

- 1- Zernike moment invariants (ZMI)
2. Pseudo Zernike moment invariants (PZMI)
- 3- Normalized Zernike moment invariants (NZMI)
- 4- Normalized pseudo Zernike moment invariants (PZMI)
- 5- Teague-Zernike moment invariants (TZMI)

- 6- Hu moment invariants (HMI)
- 7- Bamieh moment invariants (BMI)
- 8- Regular moment invariants (RMI).

The invariants were evaluated up to the seventh order moments using the formulas given in chapters 3. All the invariants, used in this study, are invariant under change of size, translation, rotation and reflection.

The classification procedure was based on the weighted k-nearest neighbor rule and the multiple membership classification method described in chapter 4. The number of K-nearest neighbors was 9 for the numeral set and 5 for the aircraft set. The tolerance value is chosen such that the rejection is kept minimal. The value 0.35 was found to be suitable for both the numeral and aircraft data sets.

The dynamic range of each invariant was reduced using a variance balancing technique[8,83] described in chapter 4. The variance balancing technique was found to be very effective in reducing the error rates and increasing the discrimination factor for most invariants. The Mahalanobis distance measure also improved the performance of the various invariants but it is computationally burdensome. The logarithm of the invariants improved the dynamic range and reduce the error rates of some invariants. The draw back of the logarithm is that the invariants which have the same magnitude but differ in sign will have the same logarithm, this in turn makes the logarithmic approach unsuitable for the skew invariants where the sign of the invariant plays an important role. A comparison between the various dynamic range reduction techniques

is shown in Tables 5.1 and 5.2.

An interesting observation is noted in both tables concerning NZMI, where the error rate is below 10 and the classification factor is higher than 90 in all cases. This also indicates that NZMI has the top performance among all the invariants.

The variance balancing technique has been used in the rest of the experimental results presented in this chapter.

TABLE.5.1 THE EFFECT OF REDUCING THE DYNAMIC RANGE ON THE ERROR PERCENTAGE.

invariant type	Without dynamic range reduction	With dynamic range reduction		
	Euclidean distance	Mahalanob distance	logarithm	Variance balancing
NZMI	9.38	8.75	8.75	7.81
NPZMI	23.75	9.06	9.69	9.38
RMI	15.94	15.94	15.94	9.69
ZMI	18.13	11.25	11.25	10.73
PZMI	25.13	11.88	12.19	12.19
TZMI	24.69	19.38	12.50	12.81
HMI	13.75	25.31	12.81	13.44
BMI	27.19	24.06	19.06	23.75

TABLE.5.2 THE EFFECT OF REDUCING THE DYNAMIC RANGE ON THE DISCRIMINATION FACTOR.

invariant type	Without dynamic range reduction	With dynamic range reduction		
	Euclidean distance	Mahalanob distance	logarithm	Variance balancing
NZMI	90.60	91.22	91.19	92.19
NPZMI	74.58	90.88	90.16	90.40
RMI	83.54	89.31	83.05	90.10
ZMI	80.86	88.68	88.61	89.21
PZMI	71.91	87.94	87.62	87.10
TZMI	72.95	80.32	87.01	86.80
HMI	85.05	74.53	86.55	86.10
BMI	71.29	75.86	79.53	74.32

5.2 Experimental results using hand written numerals

The performance of the various moment invariants to classify the Arabic numeral data set is presented in this section.

In order to arrive at an objective evaluation of the performance of the various moment invariants, they were compared relative to each other based on the following measures:

1-Discrimination factor

2-Error percentage

3-Rejection percentage

4-Classification efficiency

From the graphs shown in Fig.5.1 and Fig.5.2 it is seen that the discrimination factor and the error rate for NZMI is generally better than for other sets of invariants, particularly for moments of order higher than three.

The feature vector sizes which might be used as an indication for the number of computation needed for the Euclidean distance evaluation, are shown in Table.5.3.

For second order moments, although NPZMI has the highest discrimination factor and the lowest error rate it is not the most efficient in terms of feature vector size. BMI is the most efficient set of moment invariants for that particular order, where its discrimination factor is slightly lower than that of NPZMI, while its feature vector size is much lower than that of NPZMI (1:4 ratio), referring to tables 5.3 and 5.4. The low feature vector size enabled BMI to achieve the highest classification efficiency for the second order moments, as shown in Table.5.10.

For the third order moments also NPZMI is slightly better

than the other sets of invariants, but in terms of efficiency it is not as good as the NZMI which has a much lower feature vector size (5:11 ratio) and approximately the same discrimination factor and error rate, refer to Fig.5.1, Fig.5.2, and tables 5.3-5.5,5.10.

Although the RMI method is a close second in terms of the overall performance, the problems associated with estimating the angle of rotation makes it difficult to use.

The performance of the sets of algebraic moment invariants[38,10]; HMI and BMI, relative to the performance of the other sets of invariants, deteriorates as the moment order increases.

As far as the normalization procedure is considered, it is clear that the normalization procedure proposed by us improves the discrimination factor and reduces the error rate as shown in Fig.5.3-Fig.5.6 and tables 5.5-5.9.

Fig.5.5 and Fig.5.6 present a comparison between Zernike moment invariants as derived by Teague[44] (TZMI) and Zernike moment invariants that we derived in Chapter 4 (ZMI). The discrimination factor for ZMI is higher than that of TZMI in all moment orders. ZMI also has a lower error rate than that of TZMI as shown in Fig.5.6.

The best overall discrimination factor is at the sixth order and was achieved by NZMI, where the error rate is only 5% and the discrimination factor is 94.94, as shown in Table.5.8.

Table.5.10 indicates that NZMI has the best classification efficiency for all moment orders except the second. The best overall classification efficiency is 84.97 and was achieved by NZMI

at the fourth order. The fourth order might be considered as the optimum order for most of the invariant. For NPZMI and BMI the optimum orders are the third and the second respectively. The optimum order for each method can be determined from Figure 7, by finding the moment order at which the classification efficiency is maximum.

TABLE.5.3 COMPARISON BETWEEN FEATURE VECTOR SIZES UP TO SEVENTH ORDER MOMENTS

moment order	feature vector size for					
	BMI	ZMI/ NZMI	TZMI	HMI	RMI	PZMI/ NPZMI
second	1	2	2	2	2	4
third	2	5	6	6	6	11
fourth	4	10	11	11	11	20
fifth	-	16	17	17	17	31
sixth	-	23	24	24	24	44
seventh	-	31	32	32	32	59

TABLE.5.4 ARABIC-NUMERAL RECOGNITION USING MOMENT INVARIANTS UP TO SECOND ORDER.

invariant type	error %	rejection %	df
NPZMI	27.19	6.56	70.90
BMI	28.13	4.06	70.68
RMI	28.44	4.06	70.30
NZMI	27.50	8.44	70.00
ZMI	27.81	9.06	69.40
TZMI	27.81	9.06	69.04
PZMI	30.00	6.56	67.90
HMI	30.63	6.88	67.10

TABLE.5.5 ARABIC-NUMERAL RECOGNITION USING MOMENT INVARIANTS UP TO THIRD ORDER.

invariant type	error %	rejection %	df
NPZMI	10.94	1.25	88.90
NZMI	11.25	2.50	88.46
ZMI	11.25	2.50	88.46
RMI	11.56	1.56	88.20
TZMI	15.31	5.00	83.90
PZMI	16.55	5.31	82.50
RPZMI	16.88	5.63	82.10
HMI	16.88	5.63	82.10
BMI	22.81	4.38	76.14

TABLE.5.6 ARABIC-NUMERAL RECOGNITION USING MOMENT INVARIANTS UP TO FOURTH ORDER.

invariant type	error %	rejection %	df
NZMI	7.81	0.00	92.19
RZMI	8.75	2.43	91.05
NPZMI	9.38	1.88	90.40
RMI	9.69	1.88	90.10
ZMI	10.73	1.56	89.21
PZMI	12.19	5.13	87.10
TZMI	12.81	3.13	86.80
HMI	13.44	3.44	86.10
RPZMI	16.32	4.51	84.70
BMI	23.75	7.50	74.32

TABLE.5.7 ARABIC-NUMERAL RECOGNITION USING MOMENT INVARIANTS UP TO FIFTH ORDER.

invariant type	error %	rejection %	df
NZMI	6.25	1.25	93.67
RMI	7.19	0.94	92.74
RZMI	7.50	0.94	92.42
NPZMI	7.81	3.44	91.90
RPZMI	10.31	3.13	89.35
ZMI	10.94	1.25	88.90
PZMI	10.94	3.44	88.70
HMI	11.56	5.63	87.74
TZMI	12.81	3.13	86.80

TABLE.5.8 ARABIC-NUMERAL RECOGNITION USING MOMENT INVARIANTS UP TO SIXTH ORDER.

invariant type	error %	rejection %	df
NZMI	5.00	1.25	94.94
RMI	5.94	0.94	94.01
RZMI	6.56	0.63	93.39
ZMI	6.88	2.50	92.95
NPZMI	8.44	0.63	91.50
TZMI	8.44	2.50	91.35
RPZMI	12.19	1.88	87.58
PZMI	12.19	1.88	87.58
HMI	11.88	5.94	87.37

TABLE.5.9 ARABIC-NUMERAL RECOGNITION USING MOMENT INVARIANTS UP TO SEVENTH ORDER.

invariant type	error %	rejection %	df
NZMI	6.88	1.56	93.00
RZMI	7.19	0.94	92.74
ZMI	7.81	0.63	92.14
RMI	7.81	1.56	92.06
RPZMI	9.38	3.44	90.29
TZMI	10.31	2.19	89.45
NPZMI	11.25	2.81	88.42
PZMI	12.19	3.13	87.42
HMI	15.31	2.81	84.24

TABLE.5.10 ARABIC-NUMERAL CLASSIFICATION EFFICIENCIES FOR INVARIANTS OF MOMENTS UP TO SEVENTH ORDER

moment order	CE for					
	BMI	NZMI	TZMI	HMI	RMI	NPZMI
second	67.81	62.04	62.04	61.40	66.33	63.62
third	71.55	82.09	75.27	73.19	82.06	80.52
fourth	66.02	84.97	77.08	76.22	81.09	78.57
fifth	-	83.04	75.19	74.07	82.17	76.28
sixth	-	82.26	77.92	71.90	81.47	76.15
seventh	-	78.70	75.04	70.22	77.72	70.31

Fig.5.1 Numeral discrimination factors for the moment invariants

Fig.5.2 Numeral error rates for the moment invariants

Fig.5.3 Numeral discrimination factors for NPZMI and PZMI

Fig.5.4 Numeral error rates for NPZMI and PZMI

Fig.5.5 Numeral discrimination factors for NZMI, ZMI and TZMI.

Fig.5.6 Numeral error rates for NZMI, ZMI and TZMI

Fig.5.7 Numeral classification efficiencies.

5.2 Experimental results using aircraft pictures

The performance of the various moment invariants is evaluated in this section for the classification of the various types of military aircraft using the military-aircraft data set.

Once again it is evident from the graphs shown in Fig.5.8, Fig.5.9, and tables 5.11-5.16 that the discrimination factor and the error rate for NZMI is generally better than for other invariants.

The performance of NPZMI and PZMI relative to the other invariants was not as good as in the case of the Arabic numerals.

The relative performance of the RMI also, was not as good as in the case of the Arabic numerals. The TZMI on the other hand performed better in this case than it did in the case of the Arabic numerals.

ZMI and NZMI generally performed better than the other invariants in the case of the numeral data set as well as the aircraft data set.

The algebraic moment invariants (HMI and BMI), did not perform well as compared to the other invariants.

As in the case of the Arabic numerals, the normalization procedure proposed by us improves the discrimination factor, the classification efficiency and reduces the error rate as shown in Fig.5.10-Fig.5.14 and tables 5.11-5.17.

The third order might be considered as an optimum order for most of the invariants, due to the fact that the feature vector has a relatively moderate size and the discrimination factor is relatively high. This is particularly evident in the case of NZMI and ZMI, where the feature vector size is 5 and the discrimination

factor is at 97.22, as shown in Table.5.13, Table.5.17 and Fig.5.14. The best overall discrimination factor is at the fifth order and is shared by NZMI, ZMI and TZMI, where the error rate is only 0.69% and the discrimination factor is 99.30, as shown in Table.5.14.

NZMI is the most efficient set of invariants for all the moment orders as indicated in Table.5.17 and Fig.5.16.

TABLE.5.11 AIRCRAFT RECOGNITION USING MOMENT
INVARIANTS UP TO SECOND ORDER.

invariant type	error %	rejection %	df
NPZMI	14.93	0.35	85.01
BMI	28.82	0.69	70.97
RMI	28.47	0.69	71.32
NZMI	8.33	0.00	91.66
ZMI	8.33	0.00	91.66
TZMI	10.07	0.00	89.93
PZMI	14.93	0.35	85.01
HMI	30.90	2.08	68.44

TABLE.5.12 AIRCRAFT RECOGNITION USING MOMENT
INVARIANTS UP TO THIRD ORDER.

invariant type	error %	rejection %	df
NZMI	2.78	0.00	97.22
ZMI	2.78	0.00	97.22
TZMI	3.13	0.00	96.87
NPZMI	3.82	0.00	96.18
RMI	4.17	0.00	95.83
PZMI	4.51	0.00	95.48
RPZMI	7.29	0.35	92.68
HMI	8.33	0.69	91.60
BMI	19.44	0.00	80.55

TABLE.5.13 AIRCRAFT RECOGNITION USING MOMENT INVARIANTS UP TO FOURTH ORDER.

invariant type	error %	rejection %	df
NZMI	1.04	0.00	98.95
ZMI	1.04	0.00	98.95
TZMI	1.39	0.00	98.61
RZMI	1.74	0.00	98.26
RMI	3.13	0.00	96.87
NPZMI	5.25	0.35	94.77
HMI	5.56	0.00	94.44
PZMI	5.56	0.35	94.42
RPZMI	6.60	0.69	93.35
BMI	9.72	0.35	90.24

TABLE.5.14 AIRCRAFT RECOGNITION USING MOMENT INVARIANTS UP TO FIFTH ORDER.

invariant type	error %	rejection %	df
RZMI	0.35	0.00	99.65
NZMI	0.69	0.00	99.30
ZMI	0.69	0.00	99.30
TZMI	0.69	0.00	99.30
RMI	3.33	0.00	96.87
NPZMI	5.90	0.00	94.09
HMI	6.25	0.00	93.73
PZMI	6.94	0.00	93.05
RPZMI	9.38	0.00	90.62

TABLE.5.15 AIRCRAFT RECOGNITION USING MOMENT INVARIANTS UP TO SIXTH ORDER.

invariant type	error %	rejection %	df
NZMI	0.69	0.00	99.30
RZMI	1.39	0.00	98.61
ZMI	1.39	0.00	98.61
TZMI	1.74	0.00	98.26
RMI	2.43	0.00	97.57
HMI	3.82	0.00	96.18
NPZMI	5.90	0.00	94.09
PZMI	6.94	0.00	93.05
RPZMI	10.42	0.35	89.55

TABLE.5.16 AIRCRAFT RECOGNITION USING MOMENT INVARIANTS UP TO SEVENTH ORDER.

invariant type	error %	rejection %	df
NZMI	1.04	0.00	98.95
ZMI	0.69	0.35	98.95
RZMI	1.39	0.00	98.61
TZMI	1.39	0.00	98.61
RMI	2.78	0.00	97.22
NPZMI	7.64	0.00	92.36
PZMI	8.68	0.00	91.31
RPZMI	10.07	0.00	89.93
HMI	12.15	0.35	87.80

TABLE.5.17 AIRCRAFT CLASSIFICATION EFFICIENCIES FOR
INVARIANTS OF MOMENTS UP TO SEVENTH ORDER

moment order	CE for					
	BMI	NZMI	TZMI	HMI	RMI	NPZMI
second	70.50	90.08	88.37	65.86	69.61	81.35
third	79.16	92.54	91.49	85.93	90.51	88.20
fourth	86.35	91.21	90.43	86.60	88.83	83.58
fifth	-	89.15	88.82	83.85	86.46	80.88
sixth	-	87.13	85.96	84.15	85.36	78.81
seventh	-	85.06	84.57	75.04	83.37	75.57

Fig.5.8 Aircraft discrimination factors for the moment invariants

Fig.5.9 Aircraft error rates for the moment invariants

Fig.5.10 Aircraft discrimination factors for NPZMI and PZMI

Fig.5.11 Aircraft error rates for NPZMI and PZMI

Fig.5.12 Aircraft discrimination factors for NZMI, ZMI and TZMI.

Fig.5.13 Aircraft error rates for NZMI, ZMI and TZMI

Fig.5.14 Aircraft classification efficiencies

5.3 The performance of reduced ZMI and PZMI

The reduced ZMI and PZMI are derived in chapter 3. The main feature of these sets of invariants is that the feature vector size of these new sets of invariants is much less than that of the original ZMI and PZMI, due to the fact that all the redundant moment invariants are excluded.

In this section both the numeral data set and the aircraft data set were used to evaluate the performances of both RZMI and RPZMI. The classification efficiencies of both numeral data set and the aircraft data set are listed in tables 5.19 and 5.20 and also shown in Fig.5.23 and Fig.5.24.

The most surprising result is in the performance of RZMI, where the classification efficiency of RZMI was higher than that of ZMI for all moment orders. Although the difference between RZMI and ZMI is the elimination of some of the invariants, refer to Table.5.18. This elimination process turns out to be the main reason behind the high performances of both the discrimination factor and the classification efficiency of RZMI, which also supports our claim that some of the invariants are redundant and have no or negligible contribution during the classification process. The discrimination factors for RZMI and ZMI are listed in figures 5.17 and 5.22.

Although the classification efficiency of RPZMI was higher than that of PZMI, the discrimination factor and the error rate were not. The main reason for that is the large difference in feature vector sizes for RPZMI and PZMI, refer to tables 5.18-5.20 and figures 5.15-5.18.

The discrimination factors and the error rates for RZMI were always better than that of ZMI and TZMI, despite the fact that the feature vector size of RZMI is lower than both of them, refer to figures 5.19 and 5.20. The discrimination factors and the error rates for the aircraft data do not show a noticeable difference between RZMI, ZMI and TZMI, which can be attributed to the fact that the discrimination factors are relatively high and the error rates are relatively low. Refer to figure 5.21 and figure 5.22.

The classification efficiency measure plays an important role in comparing and evaluating the various methods, particularly in this case where the feature vector sizes of all methods are not the same.

In general the RZMI performed better than the other sets of invariants except NZMI for the fourth order, where NZMI performed better than RZMI. The feature size reduction procedure proved to be very effective in improving the classification efficiency particularly in ZMI.

The main purpose for deriving the RZMI is to increase the efficiency of ZMI and PZMI even if it comes on the expense of slight increase in the error or rejection rates. As it was demonstrated by the previous results, this purpose has been achieved in all cases.

TABLE.5.18 FEATURE VECTOR SIZES FOR RZMI,
RPZMI, ZMI AND PZMI.

moment order	feature vector size for			
	RZMI	ZMI	RPZMI	PZMI
second	-	2	-	4
third	-	5	6	11
fourth	7	10	12	20
fifth	9	16	16	31
sixth	13	23	24	44
seventh	18	31	30	59

TABLE.5.19 NUMERAL-CLASSIFICATION EFFICIENCIES
FOR RZMI, RPZMI, ZMI, AND PZMI.

moment order	Classification efficiency			
	RZMI	ZMI	RPZMI	PZMI
second	-	62.04	-	60.92
third	-	82.09	73.19	71.66
fourth	83.31	80.84	72.26	73.20
fifth	84.84	78.83	77.71	73.60
sixth	84.33	79.51	75.18	71.97
seventh	81.88	78.70	75.11	69.28

TABLE.5.20 AIRCRAFT-CLASSIFICATION EFFICIENCIES
FOR RZMI, RPZMI, ZMI, AND PZMI.

moment order	Classification efficiency			
	RZMI	ZMI	RPZMI	PZMI
second	-	90.08	-	81.35
third	-	92.54	87.23	87.57
fourth	92.16	91.21	84.62	83.31
fifth	92.34	89.15	81.35	80.00
sixth	89.60	86.52	78.37	77.94
seventh	87.90	85.06	77.48	74.72

Fig.5.15 Numeral discrimination factors for RPZMI and PZMI

Fig.5.16 Numeral error rates for RPZMI and PZMI

Fig.5.17 Aircraft discrimination factors for RPZMI and PZMI

Fig.5.18 Aircraft error rates for RPZMI and PZMI

Fig.5.19 Numeral discrimination factors for RZMI, ZMI and TZMI.

Fig.5.20 Numeral error rates for RZMI, ZMI and TZMI

Fig.5.21 Aircraft discrimination factors for RZMI, ZMI and TZMI.

Fig.5.22 Aircraft error rates for RZMI, ZMI and TZMI

Fig.5.23 Numeral classification efficiencies.

Fig.5.24 Aircraft classification efficiencies

5.4 Performance in the presence of noise

The performance of the various moment invariants in the presence of noise is studied using the numeral data samples.

A Gaussian additive noise of zero mean and standard deviation of 32 has been added to the clean image of each numeral as explained in chapter 2. A signal to noise ratio of 2 has been used for all the samples. Samples of the noisy data and their contours are shown in Fig.2.15-Fig.2.17. The noisy samples are processed using the optimum thresholding and contour detection algorithms outlined in chapter 2.

Table.5.21 and Table.5.22 show error rates and discrimination factors for the moment invariants using clean and noisy data sets. The performance is dependent on the moment order as well as the type of the invariant. For the second order the error rates for most of the invariants have deteriorated except for HMI, PZMI and NPZMI which has the best performance at that order. The third order moment invariants did not perform well in the presence of noise except HMI, RPZMI, NPZMI and RMI which has the lowest error rate at that order. RMI has the best performance for the fourth order moment invariants. NZMI has the best performance for the fifth, sixth and seventh order. The lowest overall error rate for noisy data has been achieved by NZMI at the fifth order. The highest discrimination factor has been also achieved by NZMI at the fifth order. Fig.5.25 and Fig.5.26 show the error rates and the discrimination factors of the various invariants for the noisy data.

Table.5.21 Error rate of moment invariants for clean and noisy data (C = clean data, N = noisy data).

I _{INV} - R _{IA} - N _T	MOMENT ORDER											
	2		3		4		5		6		7	
	C	N	C	N	C	N	C	N	C	N	C	N
NZMI	27.5	30.3	11.2	16.5	7.81	11.5	6.25	5.93	5.00	6.25	6.88	5.93
RZMI	-	-	-	-	8.75	13.1	7.50	10.6	6.56	6.87	7.19	8.12
NPZMI	27.2	24.4	10.9	10.3	9.38	10.9	7.81	10.3	8.44	8.12	11.2	11.8
RPZMI	-	-	16.9	15.0	16.3	12.8	10.3	10.3	12.2	12.5	9.38	11.2
ZMI	27.8	30.3	11.2	17.8	10.7	13.7	10.9	10.6	6.88	7.18	7.81	5.93
PZMI	30.0	24.4	16.5	17.1	12.2	15.3	10.9	12.8	12.2	10.9	12.2	10.6
TZMI	27.8	30.3	15.3	18.1	12.8	13.7	12.8	12.8	8.44	10.9	10.3	12.5
RMI	28.4	34.0	11.6	9.68	9.69	7.18	7.19	8.44	5.94	8.12	7.81	9.06
HMI	30.6	29.6	16.9	14.6	13.4	13.4	11.6	13.1	11.9	12.8	15.3	16.5
BMI	28.1	34.0	22.8	24.6	23.7	25.9	-	-	-	-	-	-

Table.5.22 Discrimination factor of moment invariants for clean and noisy data (C = clean data, N = noisy data).

I _{INV} - R _{IA} - N _T	MOMENT ORDER											
	2		3		4		5		6		7	
	C	N	C	N	C	N	C	N	C	N	C	N
NZMI	70.0	67.5	88.4	83.0	92.1	88.1	93.6	94.0	94.9	93.6	93.0	93.9
RZMI	-	-	-	-	91.0	86.7	92.4	89.2	93.3	93.0	92.7	91.8
NPZMI	70.9	73.6	88.9	89.5	90.4	88.5	91.9	89.3	91.5	91.5	88.4	87.6
RPZMI	-	-	82.1	84.5	84.7	86.6	89.3	89.3	87.5	87.2	90.2	88.1
ZMI	69.4	67.5	88.4	81.5	89.2	86.0	88.9	89.2	92.9	92.6	92.1	94.0
PZMI	67.9	73.6	82.5	82.2	87.1	84.4	88.7	86.8	87.5	88.5	87.4	88.9
TZMI	69.0	67.5	83.9	80.9	86.8	85.4	86.8	86.6	91.3	88.8	89.4	87.3
RMI	70.3	64.3	88.2	90.1	90.1	92.6	92.7	91.3	94.0	91.7	92.0	90.8
HMI	67.1	68.6	82.1	84.9	86.1	85.8	87.7	86.4	87.3	86.5	84.2	82.9
BMI	70.6	64.2	76.1	74.4	74.3	72.3	-	-	-	-	-	-

Fig.5.25 Noisy data error rates.

Fig.5.26 Noisy data discrimination factors

5.5 Summary

The evaluation of the various moment invariants attempted in this chapter shows that the normalization and the feature vector size reduction procedures, which have been introduced in chapter 3, are very effective in improving the classification of numeral and aircraft data sets.

The two measures: classification efficiency and discrimination factor are used in the evaluation process. The discrimination factor is important in comparing invariant sets of the same feature vector size while the classification efficiency can be used otherwise. Another advantage of the classification efficiency measure is that it can be used to determine the moment order beyond which the invariants are no more effective in the classification process.

The NZMI proved to be very effective means of shape recognition for moments of order higher than the fourth order for both noisy and clean data. For moment orders lower than the fifth NZMI did not perform as well as RMI or NPZMI for the noisy data.

RZMI is a close second to NZMI at the sixth and seventh order moments, given the fact that they have a much lower feature vector size makes them more efficient.

The normalization and feature reduction techniques have lead to a considerable improvement over the performance of ZMI and PZMI in the recognition of both noisy and clean data.

CHAPTER 6

CONCLUSION AND CONTRIBUTIONS

The primary objective of this dissertation has been the investigation and improvement of the moment invariants as means of shape recognition.

In chapter 2 the optimum thresholding and contour detection of images has been shown to be a feasible procedure. Two new techniques for the automatic thresholding of images have been introduced. The first technique is based on deriving a threshold value that maximizes the number of detected borders in the whole image as well as the number of pixels in each border. The second technique is based on deriving a threshold value that maximizes the ratio of the number of pixels in all the borders to the number of detected borders in the whole image. These two techniques do not require that image histogram be bimodal. They also do not require any interference from the user or a priori information which makes them suitable for automatic thresholding and contour detection applications.

In chapter 3 a new form of Zernike moment invariants is derived (ZMI). These new invariants comprise combinations of Zernike moments of order n and $n-2$. This is different from the approach used by Teague, where random combinations of the n -th and lower order moments are used. Testing the new form of Zernike moment invariants for functional independence is very simple as compared to Teague-Zernike moment invariants. A general form for

TZMI is not possible due to the formulation of these invariants.

A general form for pseudo Zernike moment invariants (PZMI) has also been derived. Combinations of n and $(n-1)$ of moment orders are used to derive PZMI. A recursive relationship for describing Zernike moments and pseudo Zernike moments has been presented. It has been shown that some of Zernike moments of order n are dependent on Zernike moments of order $(n-2)$. This leads to a normalization procedure which is based on normalizing the n -th order by the $(n-2)$ th order. It has been also shown that pseudo Zernike moments of order n are dependent on moments of order $n-1$, which leads to a normalization of the n -th order by the $(n-1)$ -th order.

The feature-vector size reduction technique which is based on the removal of the redundant invariants from the original PZMI or ZMI has been described and suggested for use as an alternative for the normalization procedure. The feature size vector reduction technique reduces the number of feature vectors needed for shape recognition particularly for higher order moments.

In chapter 4, an efficient technique has been described for the evaluation of the K-nearest neighbor rule.

The cross validation testing procedure was outlined. This testing procedure is based on using the same data set for learning and testing processes without duplicating any of the samples. The main advantages of this testing procedure are:

- Instead of using $N/2$ samples for learning and another $N/2$ samples for testing, $N-1$ samples are used for learning and N samples are used for testing.

-This method is not biased to the order or type of the invariants, due to the fact that all the samples are used for testing as well as for learning. This scheme is quite different from the classical testing technique in which the data is divided into two groups; one for testing and the other for learning.

The multiple membership classification technique was described. Each sample is classified based on a weighted KNNR. The membership weight is generally higher for the class of the majority of the K-nearest neighbors.

Two measures have been proposed in the same chapter, one for evaluating the discrimination power of the various invariants and the other one is for comparing their classification efficiencies.

In chapter 5 the evaluation of the various moment invariants shows that the normalization and the feature vector size reduction procedures, are very effective in improving the classification of numeral and aircraft data sets.

The two measures: classification efficiency and discrimination factor have been used in the evaluation process. The discrimination factor is important in comparing invariant sets of the same feature vector size while the classification efficiency can be used otherwise. Another advantage of the classification efficiency measure is that it can be used to determine the optimum order of the moment invariants.

The NZMI proved to be a very effective means of shape recognition for moments of order lower than the fifth order, while the RZMI proved to be more efficient for moments of orders higher than the fourth order.

The NPZMI and RPZMI proved to be inefficient in comparison with the other invariants, particularly for moments of orders higher than the third.

In general the best overall performance is at the sixth order by NZMI and RZMI respectively.

The moment invariants in general did not perform very well in the case of the hand written numeral data set, this is possibly due to the non linear variations, which the moment invariants were not normalized against. The best remedy for that is to combine the moment invariants with some structural recognition techniques such as the one described by Shridhar and Badreldin[35].

The moment invariants performed very well in the case of the military aircraft even though the number of classes (18) is much higher than that used by other researchers[8-9,53,107].

The main contributions of this study lie in the following:

- 1- General expressions have been introduced for deriving ZMI and PZMI up to any order.
- 2- New recursive formulas have been described for expanding Zernike moments and pseudo Zernike moments.
- 3- New normalization procedures aimed at minimizing the effect of information redundancy as well as reducing the dynamic range have been proposed for both ZMI and PZMI.
- 4- Techniques for reducing the feature vector size have been also proposed to improve the efficiency of both ZMI and PZMI.
- 5- Two measures have been introduced for evaluating the performance of the various moment invariants. The first one is used for measuring the discrimination factor and the second one is used

for the classification efficiency measurements.

- 6- A New method for the efficient evaluation of the KNNR has been used in the classification process.
- 7- New algorithms for the optimum thresholding and contour detection have been implemented and used for extracting shape contours.

6.1 Suggestions for future work

Future work in this area might include the following:

The effect of the presence of noise in the data set on the performance of NZMI and ZMI.

A method is needed to make Zernike and pseudo Zernike moment invariants directly invariant under translation and change of size without having to go through the scale-invariant central moments.

Other techniques like structural pattern recognition can be combined with the moments invariants to improve the recognition of hand written numerals or characters.

Airplanes can be mixed with clouds or birds and the discrimination power of the moment invariants can be analyzed on these shapes.

Three dimensional object recognition can be achieved by constructing a library of two dimensional views for the same object at all possible viewing angles.

The hardware implementation of the algorithms presented in this dissertation can be addressed.

REFERENCES

- 1- A. Ambler et al., "A versatile system for computer controlled assembly", Artificial Intell., Vol.6, pp.129-156, 1975.
- 2- B. Bhanu and O. Faugeras, "Shape matching of two-dimensional objects", IEEE Trans. on Pattern Analy. Machine Intell., Vol. PAMI-6, No.2, March 1984., pp.137-155.
- 3- R. Chellapa and R. Bagdazian, "Fourier coding of image boundaries", IEEE Trans. on Pattern Analy. Machine Intell., Vol. PAMI-6, No.1, Jan., 1984., pp.102-105.
- 4- F. Dekking and P. Van Otterloo, "Fourier coding and reconstruction of complicated contours", IEEE Trans on sys. man, sybern., Vol. SMC-16, No.3, May/June 1986, pp.395-404.
- 5- H. Wechsler and J. Sklansky, "Finding the rib cage in chest radiographs", Pattern Recognition, Vol.9, No.1 Jan., 1977.
- 6- J. Quignon and C. Faure, "An hybrid system for the automatic interpretation of electromyogram," Proceedings of the Seventh International Conference on Pattern Recognition, Montreal, Canada, pp. 37-39, July 30- August 2, 1984.
- 7- S. Dudani, K. Breeding and R. McGhee, "Aircraft identification by moment invariants", IEEE Trans. Comput., vol C-26, pp. 39-45, Jan. 1977.
- 8- A. Reeves, R. Prokop, S. Andrews and F. Kuhl, " Three dimensional shape analysis using moments and fourier discriptors", IEEE Trans. Pattern Anal. Machine Intell., vol PAMI-10, no 6, pp.937-943, Nov. 1988.

- 9- L. Gupta and M. Srinath, "Contour classification using invariant moments", Proceedings of The Fifth International Phoenix Conference on Computers and Communications, Scottsdale, Az., USA, pp.482-486, March, 1986.
- 10- B. Bamieh and R. De Figueiredo, " A general moment invariants / attributed-graph method for three dimensional object recognition from a single image", IEEE Journal of robotics and automation, vol RA-2, No 1, pp. 31-41, March 1986.
- 11- D. Cyganski and J. Orr, "Applications of tensor theory to object recognition and orientation determination", IEEE Trans. Pattern Anal. Machine Intell., vol PAMI-7, no 6, pp.662-672, Nov. 1985.
- 12- R. Casey, "Moment normalization of hand printed characters", IBM J. Res. Develp., pp. 548-557, Sept., 1970.
- 13- S. Khan, T. Pavlidis and H. Baird, "On the recognition of printed characters of any font and any size", IEEE Trans. Pattern Anal. Machine Intell., vol. PAMI-9, no 2, pp.274-287, March, 1988.
- 14- K. Ikeda et al., "On line recognition of hand written characters using positional and stroke vector sequences", Pattern Recognition, Vol. 13, no 3, pp.191-206, 1981.
- 15- S. Watanabe (Ed.), "Methodologies of Pattern Recognition", Academic Press, New York, pp.417-436, 1969.
- 16- K. Fu, "Syntactic Pattern Recognition", Englewood Cliffs, NJ, Prentice Hall, 1982.

- 17- C. Bjorklund and T. Pavlidis, "Global shape analysis by syntactic similarity," IEEE Trans. on Pattern Analy. Machine Intell., Vol. PAMI-3, No.2, pp.144-155, Mar. 1981.
- 18- A. Belard and J. Haton, "A syntacatic approach for handwritten mathematical formula recognition," IEEE Trans. on Pattern Analy. Machine Intell., Vol. PAMI-6, No.1, pp.105-111, Jan. 1984.
- 19- R. Hummel, and S. Zucker, "On the foundation of relaxation labeling process," IEEE Trans. on Pattern Analy. Machine Intell., Vol. PAMI-5, No.3, pp.267-286, May 1983.
- 20- L. Davis, "Shape matching using relaxation techniques", IEEE Trans. on Pattern Analy. Machine Intell., Vol. PAMI-1, No.1, pp.60-72, Jan. 1979.
- 21- K. Yamamoto and S. Mori, "Recognition of hand printed characters by outermost point method," Pattern Recognition, Vol 12, pp. 224-236, 1980.
- 22- L. Lam and C. Suen, "Structural classification and relaxation matching of totally unconstrained handwritten zip-code numbers", Pattern Recognition, Vol. 21, no 1, pp.19-31, 1988.
- 23- T. Pavlidis and F. Ali, "Computer recognition of handwritten numerals by polygonal approximations," IEEE Trans. Syst. Man Cybern., vol.SMC-6, no.5, pp.610-614, Nov. 1975.
- 24- H. Freeman and L. Davis, "A corner finding algorithm for chain coded curves," IEEE Trans. Comput. vol.26, pp.297-303, Mar. 1977.

- 25- H. Maurer, G Rozenberg, and E. Welzl, "Using string languages to describe picture languages," Inform. and contr., vol.54, no.3, pp.155-185, 192.
- 26- M. Berthod and J. Maroy, "Learning in syntactic recognition of symbols drawn on a graphic tablet," Computer Graphics and Image Processing, vol.9, pp.166-182, 1979.
- 27- T. Pavlidis, Structural Pattern Recognition, New York, Spring Verlag, 1977.
- 28- M. Shridhar and A. Badreldin, "Recognition of isolated and connected handwritten numerals", Proceedings of the 1984 IEEE International Conference on Systems Man and Cybernetics, pp.142-146, Halifax, Nova Scotia, Oct. 10-12, 1984.
- 29- L. Shapiro and R. Haralick, "Decomposition of two dimensional shapes by graph-theoretic clustering," IEEE Trans. on Pattern Analy. Machine Intell., Vol. PAMI-1, No.1, pp.10-20, Jan. 1979.
- 30- L. Shapiro and R. Haralick, "Structural description and inexact matching," IEEE Trans. on Pattern Analy. Machine Intell., Vol. PAMI-3, No.5, pp.504-519, Sep. 1981.
- 31- R. Kashyap and R. Chellapa, "Stochastic models for closed boundary analysis: Representation and reconstruction", IEEE Trans. on Inform. Theory, Vol. IT-27, No.5, pp. 627-637, 1987.
- 32- S. Dubois and F. Glanz, "An Autoregressive model approach to 2-D shape classification", IEEE Trans. Pattern Analy. Machine Intell., Vol. PAMI-8, No.1, pp.55-56., Jan., 1986.

- 33- G. Granlund, "Fourier preprocessing for handprint character recognition", IEEE Trans. Comput., Vol.C-21, pp.195-201, Feb., 1972.
- 34- C. Zahn and R. Roskies, "Fourier descriptors for plane closed curves recognition", IEEE Trans. Comput., Vol.C-21, pp.269-281, March, 1972.
- 35- M. Shridhar and A. Badreldin, "High accuracy character recognition algorithm using Fourier and topological descriptors", Pattern recognition, Vol.17, No 5, pp.515-524, 1984.
- 36- E. Person and K. Fu, "Shape discrimination using Fourier descriptors", IEEE Trans. Pattern Analy. Machine Intell., Vol.PAMI-8, no 3, pp. ,May, 1986.
- 37- C. Lin and R. Chellapa, "Classification of partial 2-D Shapes using Fourier descriptors", IEEE Trans. Pattern Analy. Machine Intell., Vol.PAMI-9, no 5, pp.686-690, Sept., 1987.
- 38- M. Hu, "Visual pattern recognition by moment invariants", IRE Trans. Inform Theory, vol IT-8, pp. 179-187, Feb. 1962.
- 39- R. Gonzalez and P. Wintz, Digital Image Processing, pp. 354-358, Reading MA: Adison Wesley, 1977.
- 40- D. Casasent and R. Cheatham, "Image segmentation and real image tests for an optical moment-based feature extractor", Optics Communications, vol 51, No 4, pp.227-230, sept.15, 1984.

- 41- S. Reddi, "Radial and angular moment invariants for image identification", IEEE Trans. Pattern Anal. Machine Intell. vol PAMI-3, No 2, pp.240-242, March 1981.
- 42- S. Maitra, "Moment invariants", Proceedings of the IEEE, vol 67, No 4, pp.697-699, April 1979.
- 43- D. Casasent, L. Cheatham and D. Fetterly, "Optical system to compute intensity moments: Design", Applied Optics, vol 21, No 18, pp.3292-3298, Sept. 1982.
- 44- M. Teague, "Calculation of irradiance moments", Applied Optics, vol 19, No 8, pp.1353-1356, April. 1980.
- 45- D. Casasent and D. Psaltis, "Hybrid processor to compute invariant moments for pattern recognition", Optics Letters, vol 5, No 9, pp. 395-397, sept. 1980.
- 46- Z. Budrikis and M. Hatamian, "Moment calculations by digital filters", ATT Bell Labs Tech Jour., vol 63, No 2, pp.217-229, feb. 1984.
- 47- R. Anderson, "Real time grey scale video processing using a moment generating chip", IEEE J. of Robotic & auto., vol RA-1, No.2, pp. 79-85, June, 1985.
- 48- M. Teague, "Image analysis via the general theory of moments", J. Opt. Soc. Amer., vol 70, pp. 920-930, Aug. 1980.
- 49- Y. Abu-Mustafa and D. Psaltis, "Recognitive aspects of the moment invariants", IEEE Trans. Pattern Anal. Machine Intell., vol PAMI-6, pp.698-706, Nov. 1984.

- 50- Y. Abu-Mustafa and D. Psaltis, "Image Normalizations by complex moments", IEEE Trans. Pattern Anal. Machine Intell., vol PAMI-7, No 1, pp.46-55, Jan. 1984.
- 51- C. Teh and R. Chin, "On image analysis by the methods of moments", IEEE Trans. Pattern Anal. Machine Intell. vol 10, No 4, pp.496-512, July 1988.
- 52- C. Lo and H. Don, "3-D moments: Their construction and application to object identification and positioning," IEEE Trans. Pattern Anal. Machine Intell. vol 11, No 10, pp.1053-1064, Oct. 1989.
- 53- A. Reeves and R. Taylor, "Identification of three dimensional objects using range information", IEEE Trans. Pattern Anal. Machine Intell., vol 11, no 4, pp.403-410, April, 1989.
- 54- A. Bhatia and E. Wolf, "On the circle polynomials of Zernike and related orthogonal sets", Proc. Camb. Phil. Soc., vol 50, pp. 40-48, 1954.
- 55- J. Ward and T. Kuklinski, "A model for variability effects in handprinting with implications for the design handwriting character recognition systems," IEEE Trans. Systems, Man and Cybernetics, Vol.18, No.3, pp.438-451, May/June 1988.
- 56- K. Fu and J. Mui "A survey on image segmentation", Pattern Recognition, Vol.13 pp.3-16, 1981.
- 57- R. Haralick "Image segmentation survey" in Fundamental in Computer Vision. O.D. Faugeras. Ed. Cambridge England: Cambridge University Press, pp.209-223, 1983.

- 58- R. kohler " A segmentation system based on thresholding"
Comput. Graphics Image Processing, Vol.15, pp.139-337, 1981.
- 59- J. Wezka "A survey of threshold selection techniques"
Comput. Graphics Image Processing, Vol.7, pp.259-265, 1978.
- 60- R.C. Gonzalez and P.Wintz "Digital Image Processing" 2nd
edition, Reading, M.A.:Adison-Wesley, 1987.
- 61- C. Chow and T. Kaneko "Automatic boundary detection of the
left ventricle from cineangiograms" Computer. Biomed. Res.,
Vol.5, pp.388-410, 1972.
- 62- N. Otsu,"A threshold selection model from gray-level
histogram" IEEE Trans. on Systems, Man, and Cybernetics,
Vol. SMC-9, No.1, pp.62-66, Jan.1979.
- 63- A. Perez and R. Gonzalez "An iterative thresholding
algorithm for image segmentation" IEEE Trans. on Pattern
Analysis and Machine Intelligence, Vol.PAMI-9, No.6,
pp.742-751, Nov.1987.
- 64- S.O. Belkasim, M.Shridhar, and M. Ahmadi "Optimization
techniques for edge and contour detection," Proceedings of
SPIE-The International Society for Optical Engineering,
Vol.956, pp.9-17, Dearborn, Michigan, June 30, 1988.
- 65- M. D. Kelly,""Edge detection in pictures by computer
planning," Machine Intelligence, vol 6, pp.397-409, Edinburgh
Univ. Press. Edinburgh, Scotland 1971.
- 66- A. Rosenfield, "Adjacency in digital pictures",
Information and control, vol 26, pp. 24-33, Sept. 1974.

- 67- A. Chottera, M. Shridhar, "Feature Extraction from Manufactured Parts in the Presence of Spurious Reflections," Canadian Journal of Electrical Engineering, pp.29-23, Oct. 1983.
- 68- D. Cooper and F. Sung "Multiple-window parrallel adaptive boundary finding in computer vision," IEEE Trans. Pattern Analys. Machine Intell., Vol.PAMI-5, No.3, pp. 299-213, May 1983.
- 69- U. Montanari, "On the optimal detection of curves in noisy pictures", Communications of the ACM, Vol.14, No.5, pp.335-345, May 1971.
- 70- D. Ballard and J. Sklansky, "A ladder structured decision tree for recognizing tumors in chest radiographs", IEEE Trans. on Computer, Vol. c-25, No.5, pp.503-512, May 1976.
- 71- D. Ballard and J. Sklansky, "Tumor detection in radio-graphs", Comput. Biomed. Res., Vol.6, pp.299-321, Aug. 1973.
- 72- A. Martelli, "An application of heuristic search methods to edge and contour detection", Ass. Comput. Mach., Vol.19, pp.73-83, Feb. 1976.
- 73- J. beaulieu and M. Goldberg, "Hierarchy in picture segmentation: A step wise optimization approach," IEEE Trans. Pattern Analys. Machine Intell., vol.11, no.2, pp.150-163, Feb. 1989.

- 74- A. Kundu, Y. He and P. Bahl, "Recognition of handwritten word: first and second order hidden Markov model based approach," Pattern Recognition, vol.22, no.3, pp.383-397, 1989.
- 75- T. Hsia, "A note on invariant moments in image processing," IEEE Trans on sys. man, sybern., Vol. SMC-11 No.12, pp.831-834, Dec. 1981.
- 76- A. Prata and W. Rusch, "Algorithm for computation of Zernike polynomials expansion coefficients", Applied Optics, Vol.28, No.4, pp.749-754, February 1989.
- 77- R. Noil, "Zernike polynomials and atmospheric turbulence", J. OPT Soc. Am., Vol.66, No.3, March 1976.
- 78- J. Wang and D. Silva, "Wave-front interpretation with Zernike polynomials", Applied Optics, Vol.19, No.9, May 1980.
- 79- S. Bezdidko, "The use of Zernike polynomials in optics", Sov. J. Opt. Technol., Vol.41, pp.425, 1974.
- 80- S. Bezdidko, "Determination of the Zernike polynomials expansion coefficients of the wave aberration", Sov. J. Opt. Technol., Vol.42, pp.426, 1975.
- 81- M. Born and E. Wolf, Principles of Optics, pp. 767-774, New york: Pergamon, 1975.
82. H. Niemann (Ed.), "Pattern Analysis," Springer-Verlag, pp. 110-119, New York, 1981.
- 83- J. Tou and R. Gonzalez, "Pattern Recognition Principles," Adison-Wesley Pub. Co., Reading, Ma., pp. 250-361, 110-123, 1974.

- 84- K. Fu and B. Bhargava, "Tree systems for syntactic pattern recognition," IEEE Trans. Comput., vol. C-22, pp.1087-1099, 1973.
- 85- M. Thomason and R. Gonzalez, "Error detection and classification in syntactic pattern structures," IEEE Trans. Comput., vol. C-24, pp., 1975.
- 86- Y. Chiang and K. Fu, "Parallel parsing algorithm and VLSI implementations for syntactic pattern recognition," IEEE Trans. Pattern Analys. Machine Intell., vol. PAMI-6, no.3, pp.302-313, May, 1984.
- 87- B. Moayer and K. Fu, "A tree system approach for fingerprint pattern recognition," IEEE Trans. Pattern Analys. Machine Intell., vol. PAMI-8, no.3, pp.376-387, May, 1986.
- 88- K. Fu, "A step towards unification of syntactic and statistical pattern recognition," IEEE Trans. Pattern Analys. Machine Intell., vol. PAMI-8, no.3, pp.398-404, May, 1986.
- 89- K. Fu (Ed.), "Digital Pattern Recognition", Springer-Verlag, Berlin Heidelberg, Germany, pp.52-53, 1980.
- 90- J. Bryant, "A fast classifier for image data," Pattern Recognition, vol.22, no.2, pp.45-48, 1989.
- 91- T. Anderson, Introduction to Multivariate Statistical Analysis, pp.19-27, John Wiley, New York 1958.
- 92- B. Everitt, Cluster Analysis, pp.56-57, Heinemann Educational Books Ltd., London England, 1974.
- 93- D. Morrison, "Measurement problems in cluster analysis", Management Science, vol.13, no.12, pp.B775-B780, August, 1967.

- 94- J. Friedman, F. Baskett and L. shustek, "An algorithm for finding nearest neighbors," IEEE Trans. Comput., vol.C-24, pp.1000-1006, 1975.
- 95- I. Kalanatory and G McDonald, "A data structure and an algorithm for the nearest point problem," IEEE Trans. Software Eng., vol. SE-9, pp.631-634, Sep. 1983.
- 96- T. Young and T. Calvert, "Classification Estimation and Pattern Recognition", American Elsevier Pub. Co., NewYork, N.Y., pp.95-99, 1974.
- 97- E. Vidal et al., "On the use of a metric space search algorithm for fast DTW based recognition of isolated words", IEEE Trans. on Acoustics, speech and signal processing, vol 36, no 5, pp. 651-660, May 1988.
- 98- E. Vidal Ruiz, "An algorithm for finding nearest neighbors in (approximately) constant average time", Pattern Recognition Letters, Vol 4, no 3, pp. 145-147, July 1986.
- 99- B. Komgar-Parsi, and L. Kanal, "An improved branch and bound algorithm for computing K-nearest neighbors", Pattern Recognition Letters, Vol 3, no 1, pp. 7-12, Jan. 1985.
- 100- L. Miclet, and M. Dabouz, "Approximative fast nearest neighbor recognition", Pattern Recognition Letters, Vol 1, no 5-6, pp. 277-284, July 1983.
- 101- C. Feustel and L. Shapiro, "The nearest neighbor problem in an abstract metric space", Pattern Recognition Letters, Vol 1, no 12, pp. 125-128, Dec. 1982.

- 102- I. Sethi, "A fast algorithm for recognizing nearest neighbors," IEEE Trans. on systems man and cybernetics, Vol SMC-11, no 3, pp. 245-246, March 1981.
- 103- T. Yunck, "A technique to identify nearest neighbors", IEEE Trans. on systems man and cybernetics, Vol SMC-6, no 10, pp. 678, Oct. 1976.
- 104- K. Fukunaga and P. Narednra, "A branch and bound algorithm for computing the K-nearest neighbors", IEEE Trans. on Computers, Vol C-24, no 7, pp. 750-753, July 1975.
- 105- P. Lachenbruch and M. Mickey, "Estimation of error rates in discriminant analysis," Technometrics, vol.10, pp.167-178, 1968.
- 106- A. Jain, R. Dubes and C. Chen, "Bootstrap techniques for error estimation," IEEE Trans. Pattern anlays. Machine Intell., vol. PAMI-9, no.5, pp.628-633, Sep. 1987.
- 107- D. Kim, and O. Ersoy, "Image recognition with the discrete rectangular-wave trnasform II," J. Opt. Soc. Amer.A, vol 6, No.6, pp. 835-843, June, 1989.

Appendix A

HIGHER ORDER HU MOMENT INVARIANTS

Hu derived a set of six moment invariants that are invariant under rotation and/or reflection. These invariants are based on moments up to the third order only.

An extension of these invariants to include moments of orders up to the seventh order is presented here. In order to distinguish between the true invariants (invariant under reflection and rotation) and the skew invariants (invariant under rotation only), we can rewrite them as follows:

$$I_{r,p-r} I_{s-q,s}^* + I_{p-r,r} I_{s,s-q}^* \quad (A.1)$$

Since each term of equation (A.1) represents a complex quantity, and

$$I_{r,p-r} = \text{complex conjugate of } (I_{p-r,r})$$

equation (A.1) can be written as:

$$\begin{aligned} & |I_{r,p-r}| |I_{s-q,s}| e^{j(\phi_{r,p-r} + \psi_{s-q,s})} + |I_{p-r,r}| |I_{s,s-q}| e^{-j(\phi_{p-r,r} + \psi_{s,s-q})} \\ &= 2 |I_{p-r,r}| |I_{s,s-q}| \cos(\phi_{p-r,r} - \psi_{s,s-q}) \\ &= 2 |I_{p-r,r}| |I_{s-q,s}| \cos(\phi_{p-r,r} - \psi_{s-q,s}) \quad (3.A.2) \end{aligned}$$

Where $\phi = \tan^{-1}[\text{imaginary}(I)/\text{real}(I)]$.

Equation (A.2) represents a true invariant, to get the skew invariant, replace "cos" with "sin":

$$2 |I_{p-r,r}| |I_{s-q,s}| \sin(\phi_{p-r,r} - \psi_{s-q,s}) \quad (A.3)$$

Equation (A.3) represents a skew invariant.

The following is a list of Hu moment invariants derived from the general form given in chapter 3. The first six invariants are the same as the ones listed in Ref.[38]. This list includes only the true invariants, to get the skew invariants use the same set and change every "sin" to "cos".

A.1 HU moment polynomials

Using equation (3.8) the following set of complex polynomials can be derived:

$$\begin{aligned}
I_{2,0} &= [\mu_{20} - \mu_{02}] - j[2\mu_{11}] \\
I_{1,1} &= [\mu_{20} + \mu_{02}] \\
I_{3,0} &= [\mu_{30} - 3\mu_{12}] - j[3\mu_{21} - \mu_{03}] \\
I_{2,1} &= [\mu_{30} + \mu_{12}] - 2j[\mu_{21} + \mu_{03}] \\
I_{4,0} &= [\mu_{40} - 6\mu_{22} + \mu_{04}] + 4j[\mu_{13} - \mu_{31}] \\
I_{3,1} &= [\mu_{40} - \mu_{04}] - 2j[\mu_{31} + \mu_{13}] \\
I_{2,2} &= \mu_{40} - 2\mu_{22} + \mu_{04} \\
I_{5,0} &= [\mu_{50} - 10\mu_{32} + 5\mu_{14}] - j[5\mu_{41} - 10\mu_{23} + \mu_{05}] \\
I_{4,1} &= [\mu_{50} + \mu_{32} - 2(\mu_{32} + \mu_{14})] + j[-3(\mu_{41} + \mu_{23}) + \mu_{23} + \mu_{05}] \\
I_{3,2} &= [\mu_{50} + 2\mu_{32} + \mu_{14}] - j[(\mu_{41} + 2\mu_{23} + \mu_{05})] \\
I_{6,0} &= [\mu_{60} - 15\mu_{42} + 15\mu_{24} - \mu_{06}] - j[6\mu_{51} - 20\mu_{33} + 6\mu_{15}] \\
I_{5,1} &= [\mu_{60} - 2(\mu_{42} + \mu_{24}) + \mu_{06}] - j[4\mu_{51} + 2\mu_{33} - 2\mu_{15}] \\
I_{4,2} &= [\mu_{60} + \mu_{42} - \mu_{24} - \mu_{06}] - 2j[\mu_{51} + 2\mu_{33} + 2\mu_{15}] \\
I_{3,3} &= \mu_{60} + 3\mu_{42} + 3\mu_{24} + \mu_{06} \\
I_{7,0} &= [\mu_{70} - 21\mu_{52} + 35\mu_{34} - 7\mu_{16}] - j[7\mu_{61} - 35\mu_{43} + 21\mu_{25} - \mu_{07}] \\
I_{6,1} &= [\mu_{70} + \mu_{52} + 35\mu_{34} - 7\mu_{16}] - j[5\mu_{61} + 2\mu_{43} + 2\mu_{25} - \mu_{07}] \\
I_{5,2} &= [\mu_{70} + 3\mu_{34} + \mu_{16}] - j[3\mu_{61} + 5\mu_{43} + \mu_{25} - \mu_{07}] \\
I_{4,3} &= [\mu_{70} + 3\mu_{52} + 3\mu_{34} + \mu_{16}] - j[\mu_{61} - 3\mu_{43} - 3\mu_{25} - \mu_{07}]
\end{aligned}$$

A.2 Hu moment invariants

Second order

$$\begin{aligned}(\text{HMI})_1 &= I_{1,1} \\ (\text{HMI})_2 &= |I_{2,0}|^2\end{aligned}$$

Third order

$$\begin{aligned}(\text{HMI})_3 &= |I_{3,0}|^2 \\ (\text{HMI})_4 &= |I_{2,1}|^2 \\ (\text{HMI})_5 &= 2|I_{3,0}||I_{2,1}|^3 \cos(\phi_{3,0} - 3\phi_{2,1}) \\ (\text{HMI})_6 &= 2|I_{2,0}||I_{2,1}|^2 \cos(\phi_{2,0} - 2\phi_{2,1})\end{aligned}$$

Fourth order

$$\begin{aligned}(\text{HMI})_7 &= |I_{4,0}|^2 \\ (\text{HMI})_8 &= |I_{3,1}|^2 \\ (\text{HMI})_9 &= I_{2,2} \\ (\text{HMI})_{10} &= 2|I_{3,1}||I_{2,0}| \cos(\phi_{3,1} - \phi_{2,0}) \\ (\text{HMI})_{11} &= 2|I_{4,0}||I_{2,0}|^2 \cos(\phi_{4,0} - 2\phi_{2,0})\end{aligned}$$

Fifth order

$$\begin{aligned}(\text{HMI})_{12} &= |I_{3,0}|^2 \\ (\text{HMI})_{13} &= |I_{4,1}|^2 \\ (\text{HMI})_{14} &= |I_{3,2}|^2 \\ (\text{HMI})_{15} &= 2|I_{4,1}||I_{3,0}| \cos(\phi_{4,1} - \phi_{3,0}) \\ (\text{HMI})_{16} &= 2|I_{3,2}||I_{2,1}| \cos(\phi_{3,2} - \phi_{2,1}) \\ (\text{HMI})_{17} &= 2|I_{2,0}||I_{3,2}|^2 \cos(\phi_{2,0} - 2\phi_{3,2})\end{aligned}$$

Sixth order

$$\begin{aligned}
 (\text{HMI})_{18} &= |I_{6,0}|^2 \\
 (\text{HMI})_{19} &= |I_{5,1}|^2 \\
 (\text{HMI})_{20} &= |I_{4,2}|^2 \\
 (\text{HMI})_{21} &= I_{3,3} \\
 (\text{HMI})_{22} &= 2|I_{5,1}||I_{4,0}| \cos(\phi_{5,1} + \phi_{4,0}) \\
 (\text{HMI})_{23} &= 2|I_{4,2}||I_{3,1}| \cos(\phi_{4,2} + \phi_{3,1}) \\
 (\text{HMI})_{24} &= 2|I_{2,0}||I_{4,2}|^2 \cos(\phi_{2,0} - 2\phi_{4,2})
 \end{aligned}$$

Seventh order:

$$\begin{aligned}
 (\text{HMI})_{25} &= |I_{7,0}|^2 \\
 (\text{HMI})_{26} &= |I_{6,1}|^2 \\
 (\text{HMI})_{27} &= |I_{5,2}|^2 \\
 (\text{HMI})_{28} &= |I_{4,3}|^2 \\
 (\text{HMI})_{29} &= 2|I_{6,1}||I_{5,0}| \cos(\phi_{6,1} - \phi_{5,0}) \\
 (\text{HMI})_{30} &= 2|I_{5,2}||I_{4,1}| \cos(\phi_{5,2} - \phi_{4,1}) \\
 (\text{HMI})_{31} &= 2|I_{4,3}||I_{3,2}| \cos(\phi_{4,3} - \phi_{3,2}) \\
 (\text{HMI})_{32} &= 2|I_{2,0}||I_{4,3}|^2 \cos(\phi_{2,0} - 2\phi_{4,3})
 \end{aligned}$$

APPENDIX B

Teague-Zernike Moment Invariants

The following is a list of Teague-Zernike moment invariants (TZMI) derived from equations (3.23-3.25) in chapter 3. These invariants are the same as the ones listed in Ref.[44]. This list includes only the true invariants, to get the skew invariants use the same set and change every sin to cos.

Second order

$$(TZMI)_{20} = A_{20}$$

$$(TZMI)_{22} = |A_{22}|^2$$

Third order

$$(TZMI)_{31} = |A_{31}|^2$$

$$(TZMI)_{33} = |A_{33}|^2$$

$$(TZMI)_{39} = |A_{33}| |A_{31}|^3 \cos(3\phi_{31} - \phi_{33})$$

$$(TZMI)_{26} = |A_{22}| |A_{31}|^2 \cos(2\phi_{31} - \phi_{22})$$

Fourth order

$$(TZMI)_{40} = A_{40}$$

$$(TZMI)_{42} = |A_{42}|^2$$

$$(TZMI)_{44} = |A_{44}|^2$$

$$(TZMI)_{4,10} = |A_{44}| |A_{42}|^2 \cos(2\phi_{42} - \phi_{44})$$

$$(TZMI)_{47} = |A_{42}| |A_{22}| \cos(\phi_{22} - \phi_{42})$$

Fifth order

$$\begin{aligned}
 (TZMI)_{51} &= |A_{51}|^2 \\
 (TZMI)_{53} &= |A_{53}|^2 \\
 (TZMI)_{55} &= |A_{55}|^2 \\
 (TZMI)_{57} &= |A_{51}| |A_{31}| \cos(\phi_{31} - \phi_{51}) \\
 (TZMI)_{59} &= |A_{53}| |A_{33}| \cos(\phi_{33} - \phi_{53}) \\
 (TZMI)_{5,15} &= |A_{55}| |A_{31}|^5 \cos(5\phi_{31} - \phi_{55})
 \end{aligned}$$

Sixth order

$$\begin{aligned}
 (TZMI)_{60} &= A_{60} \\
 (TZMI)_{62} &= |A_{62}|^2 \\
 (TZMI)_{64} &= |A_{64}|^2 \\
 (TZMI)_{66} &= |A_{66}|^2 \\
 (TZMI)_{6,14} &= |A_{66}| |A_{33}|^2 \cos(2\phi_{33} - \phi_{66}) \\
 (TZMI)_{6,11} &= |A_{64}| |A_{44}| \cos(\phi_{44} - \phi_{64}) \\
 (TZMI)_{69} &= |A_{62}| |A_{22}| \cos(\phi_{22} - \phi_{62})
 \end{aligned}$$

Seventh order

$$\begin{aligned}
 (TZMI)_{71} &= |A_{71}|^2 \\
 (TZMI)_{73} &= |A_{73}|^2 \\
 (TZMI)_{75} &= |A_{75}|^2 \\
 (TZMI)_{77} &= |A_{77}|^2 \\
 (TZMI)_{7,21} &= |A_{77}| |A_{31}|^7 \cos(7\phi_{31} - \phi_{77}) \\
 (TZMI)_{7,13} &= |A_{75}| |A_{55}| \cos(\phi_{55} - \phi_{75}) \\
 (TZMI)_{7,11} &= |A_{73}| |A_{33}| \cos(\phi_{33} - \phi_{73}) \\
 (TZMI)_{79} &= |A_{71}| |A_{31}| \cos(\phi_{31} - \phi_{71})
 \end{aligned}$$

Appendix C

Normalization of pseudo Zernike moments

Using the normalization procedure explained in chapter 3 the NPZM can be written up to the seventh order as follows:

$$T'_{76} = T_{76} / T_{66}$$

$$T'_{75} = T_{75} / T_{65}$$

$$T'_{74} = T_{74} / T_{64}$$

$$T'_{73} = T_{73} / T_{63}$$

$$T'_{72} = T_{72} / T_{62}$$

$$T'_{71} = T_{71} / T_{61}$$

$$T'_{70} = T_{70} / T_{60}$$

$$T'_{77} = T_{77}$$

$$T'_{65} = T_{65} / T_{55}$$

$$T'_{64} = T_{64} / T_{54}$$

$$T'_{63} = T_{63} / T_{53}$$

$$T'_{62} = T_{62} / T_{52}$$

$$T'_{61} = T_{61} / T_{51}$$

$$T'_{60} = T_{60} / T_{50}$$

$$T'_{66} = T_{66}$$

$$T'_{54} = T_{54} / T_{44}$$

$$T'_{53} = T_{53} / T_{43}$$

$$T'_{52} = T_{52} / T_{42}$$

$$T'_{51} = T_{51} / T_{41}$$

$$T'_{50} = T_{50} / T_{40}$$

$$T'_{55} = T_{55}$$

$$T'_{43} = T_{43} / T_{33}$$

$$T'_{42} = T_{42} / T_{32}$$

$$T'_{41} = T_{41} / T_{31}$$

$$T'_{40} = T_{40} / T_{30}$$

$$T'_{32} = T_{32} / T_{22}$$

$$T'_{31} = T_{31} / T_{21}$$

$$T'_{30} = T_{30} / T_{20}$$

$$T'_{33} = T_{33}$$

$$T'_{21} = T_{21}$$

$$T'_{20} = T_{20} / T_{10}$$

$$T'_{22} = T_{22}$$

$$T'_{10} = T_{10} / T_{00}$$

$$T'_{11} = T_{11} = 0$$

$$T'_{00} = T_{00}$$

Appendix D

Normalized Zernike moments

Using equation (3.63) the NZM (normalized ZM) can be derived for up to the seventh order as follows:

$$A'_{75} = A_{75} / A_{55}$$

$$A'_{73} = A_{73} / A_{53}$$

$$A'_{71} = A_{71} / A_{51}$$

$$A'_{77} = A_{77}$$

$$A'_{64} = A_{64} / A_{44}$$

$$A'_{62} = A_{62} / A_{42}$$

$$A'_{60} = A_{60} / A_{40}$$

$$A'_{66} = A_{66}$$

$$A'_{53} = A_{53} / A_{33}$$

$$A'_{51} = A_{51} / A_{31}$$

$$A'_{55} = A_{55}$$

$$A'_{42} = A_{42} / A_{22}$$

$$A'_{40} = A_{40} / A_{20}$$

$$A'_{44} = A_{44}$$

$$A'_{31} = A_{31}$$

$$A'_{33} = A_{33}$$

$$A'_{20} = A_{20} / A_{00}$$

$$A'_{22} = A_{22}$$

$$A'_{11} = A_{11} = 0$$

$$A'_{00} = A_{00}$$

APPENDIX E

Reduced-pseudo-Zernike moment invariants

The following is a list of reduced pseudo-Zernike moment invariants (RPZMI) derived from equations (3.41, 3.42, 3.45, and (3.46) in chapter 3. These invariants can be combined to form a feature vector according to equation (3.67). This list includes only the true invariants, to get the skew invariants use the same set and change every sin to cos.

Second order

$$\begin{aligned} (RPZMI)_{20} &= T_{20} \\ (RPZMI)_{21} &= |T_{21}| \\ (RPZMI)_{22} &= |T_{22}| \\ (RPZMI)_{26} &= |T_{21}| |T_{22}|^{\frac{1}{2}} \cos(\frac{1}{2}\phi_{22} - \phi_{21}) \end{aligned}$$

Third order

$$\begin{aligned} (RPZMI)_{30} &= T_{30} \\ (RPZMI)_{31} &= |T_{31}| \\ (RPZMI)_{32} &= |T_{32}| \\ (RPZMI)_{33} &= |T_{33}| \\ (RPZMI)_{35} &= |T_{31}| |T_{32}|^{\frac{1}{2}} \cos(\frac{1}{2}\phi_{32} - \phi_{31}) \\ (RPZMI)_{36} &= |T_{31}| |T_{33}|^{\frac{1}{3}} \cos(\frac{1}{3}\phi_{33} - \phi_{31}) \end{aligned}$$

Fourth order

$$(RPZMI)_{40} = T_{40}$$

$$(RPZMI)_{41} = |T_{41}|$$

$$(RPZMI)_{42} = |T_{42}|$$

$$(RPZMI)_{43} = |T_{43}|$$

$$(RPZMI)_{44} = |T_{44}|$$

$$(RPZMI)_{46} = |T_{41}| |T_{42}|^{\frac{1}{2}} \cos(\frac{1}{2}\phi_{42} - \phi_{41})$$

$$(RPZMI)_{47} = |T_{41}| |T_{43}|^{\frac{1}{3}} \cos(\frac{1}{3}\phi_{43} - \phi_{41})$$

$$(RPZMI)_{48} = |T_{41}| |T_{44}|^{\frac{1}{4}} \cos(\frac{1}{4}\phi_{44} - \phi_{41})$$

Fifth order

$$(RPZMI)_{50} = T_{50}$$

$$(RPZMI)_{51} = |T_{51}|$$

$$(RPZMI)_{52} = |T_{52}|$$

$$(RPZMI)_{53} = |T_{53}|$$

$$(RPZMI)_{54} = |T_{54}|$$

$$(RPZMI)_{55} = |T_{55}|$$

$$(RPZMI)_{57} = |T_{51}| |T_{52}|^{\frac{1}{2}} \cos(\frac{1}{2}\phi_{52} - \phi_{51})$$

$$(RPZMI)_{58} = |T_{51}| |T_{53}|^{\frac{1}{3}} \cos(\frac{1}{3}\phi_{53} - \phi_{51})$$

$$(RPZMI)_{59} = |T_{51}| |T_{54}|^{\frac{1}{4}} \cos(\frac{1}{4}\phi_{54} - \phi_{51})$$

$$(RPZMI)_{5,10} = |T_{51}| |T_{55}|^{\frac{1}{5}} \cos(\frac{1}{5}\phi_{55} - \phi_{51})$$

Sixth order

$$\begin{aligned}
 (RPZMI)_{60} &= T_{60} \\
 (RPZMI)_{61} &= |T_{61}| \\
 (RPZMI)_{62} &= |T_{62}| \\
 (RPZMI)_{63} &= |T_{63}| \\
 (RPZMI)_{64} &= |T_{64}| \\
 (RPZMI)_{65} &= |T_{65}| \\
 (RPZMI)_{66} &= |T_{66}| \\
 (RPZMI)_{69} &= |T_{61}| |T_{62}|^{\frac{1}{2}} \cos(\frac{1}{2}\phi_{62} - \phi_{61}) \\
 (RPZMI)_{8,10} &= |T_{61}| |T_{63}|^{\frac{1}{3}} \cos(\frac{1}{3}\phi_{63} - \phi_{61}) \\
 (RPZMI)_{8,11} &= |T_{61}| |T_{64}|^{\frac{1}{4}} \cos(\frac{1}{4}\phi_{64} - \phi_{61}) \\
 (RPZMI)_{8,12} &= |T_{61}| |T_{65}|^{\frac{1}{5}} \cos(\frac{1}{5}\phi_{65} - \phi_{61}) \\
 (RPZMI)_{8,13} &= |T_{61}| |T_{66}|^{\frac{1}{6}} \cos(\frac{1}{6}\phi_{66} - \phi_{61})
 \end{aligned}$$

Seventh order

$$\begin{aligned}
 (RPZMI)_{70} &= T_{70} \\
 (RPZMI)_{71} &= |T_{71}| \\
 (RPZMI)_{72} &= |T_{72}| \\
 (RPZMI)_{73} &= |T_{73}| \\
 (RPZMI)_{74} &= |T_{74}| \\
 (RPZMI)_{75} &= |T_{75}| \\
 (RPZMI)_{76} &= |T_{76}| \\
 (RPZMI)_{77} &= |T_{77}| \\
 (RPZMI)_{79} &= |T_{71}| |T_{72}|^{\frac{1}{2}} \cos(\frac{1}{2}\phi_{72} - \phi_{71}) \\
 (RPZMI)_{7,10} &= |T_{71}| |T_{73}|^{\frac{1}{3}} \cos(\frac{1}{3}\phi_{73} - \phi_{71}) \\
 (RPZMI)_{7,11} &= |T_{71}| |T_{74}|^{\frac{1}{4}} \cos(\frac{1}{4}\phi_{74} - \phi_{71}) \\
 (RPZMI)_{7,12} &= |T_{71}| |T_{75}|^{\frac{1}{5}} \cos(\frac{1}{5}\phi_{75} - \phi_{71}) \\
 (RPZMI)_{7,13} &= |T_{71}| |T_{76}|^{\frac{1}{6}} \cos(\frac{1}{6}\phi_{76} - \phi_{71}) \\
 (RPZMI)_{7,14} &= |T_{71}| |T_{77}|^{\frac{1}{7}} \cos(\frac{1}{7}\phi_{77} - \phi_{71})
 \end{aligned}$$

Appendix F

Reduced-Zernike moment invariants

The following is a list of reduced Zernike moment invariants (RZMI) derived from equations (3.31- 3.33, 3.34-3.36) in chapter 3. These invariants can be combined to form a feature vector according to equation (3.67). This list includes only the true invariants, to get the skew invariants use the same set and change every sin to cos.

Second order

$$(RZMI)_{20} = A_{20}$$

$$(RZMI)_{22} = |A_{22}|$$

Third order

$$(RZMI)_{31} = |A_{31}|$$

$$(RZMI)_{33} = |A_{33}|$$

$$(RZMI)_{39} = |A_{33}|^{\frac{1}{3}} |A_{31}| \cos(\phi_{31} - \frac{1}{3}\phi_{33})$$

Fourth order

$$(RZMI)_{40} = A_{40}$$

$$(RZMI)_{42} = |A_{42}|$$

$$(RZMI)_{44} = |A_{44}|$$

$$(RZMI)_{4,10} = |A_{44}|^{\frac{1}{2}} |A_{42}| \cos(\phi_{42} - \frac{1}{2}\phi_{44})$$

Fifth order

$$(RZMI)_{51} = |A_{51}|$$

$$(RZMI)_{53} = |A_{53}|$$

$$(RZMI)_{55} = |A_{55}|$$

$$(RZMI)_{57} = |A_{51}| |A_{53}|^{\frac{1}{3}} \cos(\phi_{51} - \frac{1}{3}\phi_{53})$$

$$(RZMI)_{59} = |A_{51}| |A_{55}|^{\frac{1}{5}} \cos(\phi_{51} - \frac{1}{5}\phi_{55})$$

Sixth order

$$(RZMI)_{60} = A_{60}$$

$$(RZMI)_{62} = |A_{62}|$$

$$(RZMI)_{64} = |A_{64}|$$

$$(RZMI)_{66} = |A_{66}|$$

$$(RZMI)_{6,14} = |A_{62}| |A_{64}|^{\frac{1}{2}} \cos(\phi_{62} - \frac{1}{2}\phi_{64})$$

$$(RZMI)_{6,11} = |A_{62}| |A_{66}|^{\frac{1}{3}} \cos(\phi_{62} - \frac{1}{3}\phi_{66})$$

Seventh order

$$(RZMI)_{71} = |A_{71}|$$

$$(RZMI)_{73} = |A_{73}|$$

$$(RZMI)_{75} = |A_{75}|$$

$$(RZMI)_{77} = |A_{77}|$$

$$(RZMI)_{7,21} = |A_{71}| |A_{73}|^{\frac{1}{3}} \cos(\phi_{71} - \frac{1}{3}\phi_{73})$$

$$(RZMI)_{7,13} = |A_{71}| |A_{75}|^{\frac{1}{5}} \cos(\phi_{71} - \frac{1}{5}\phi_{75})$$

$$(RZMI)_{7,11} = |A_{71}| |A_{77}|^{\frac{1}{7}} \cos(\phi_{71} - \frac{1}{7}\phi_{77})$$

VITA AUCTORIS

- 1953 Born on Sept. 19-th.
- 1971 Completed high school at Benghazi High School, Benghazi
Libya.
- 1976 Graduated from Alfateh University with the degree of
B.Sc. in Electrical Engineering, Tripoli, Libya.
- 1985 Graduated from McGill University with the degree of
M. Eng. in Electrical Engineering, Montreal, Quebec.
- 1986 Candidate for the degree of Doctor of Philosophy in
Electrical Engineering at the University of Windsor,
Windsor, Ontario, Canada.