Some stability and flow problems of magnetic fluids.

Somasundhar Venkatasubramanian

University of Windsor

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SOME STABILITY AND FLOW
PROBLEMS OF MAGNETIC FLUIDS

by

SOMASUNDHAR VENKATASUBRAMANIAN

A Thesis
submitted to the Faculty of Graduate Studies and Research
through the Department of Mathematics and Statistics
in partial fulfillment of the requirements for the
degree of Master of Science at
the University of Windsor

Windsor, Ontario, Canada
1992
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ABSTRACT

This thesis deals with the study of some stability and flow problems of magnetic fluids. In the first problem, the effect of rotation on the thermo-convective instability of a horizontal layer of a ferromagnetic fluid which is heated from below and influenced by a uniform vertical magnetic field is studied. The problem is analyzed when the fluid is confined between two free boundaries, between two rigid paramagnetic boundaries and between two rigid ferromagnetic boundaries. A qualitative analysis is presented for the instability to occur as oscillatory convection.

An exact solution is obtained for the free boundaries. A Galerkin technique is used to obtain the solutions for the paramagnetic and ferromagnetic boundaries. A comparative study of the convective phenomenon on these boundaries is presented. The effects of buoyancy and magnetic field on the stationary convection is studied for all the three boundaries. The effect of the magnetic susceptibility on the instability is studied for the paramagnetic boundaries.

In the second problem, hodographic study of plane, steady flows of magnetic fluids is presented. Equations obtained, using the low frequency approximation for the magnetization, are transformed and studied in the hodograph plane. The corresponding equations in the polar coordinates are also obtained. The conditions to be satisfied by the magnetic potential and the Legendre transform function, so that a certain flow is physically possible are obtained. Some simple illustrations are provided by assuming different forms for the Legendre transform function.
To my beloved parents for their love and affection
TABLE OF CONTENTS

ABSTRACT iv
DEDICATION v
ACKNOWLEDGEMENTS vi
LIST OF TABLES ix
LIST OF FIGURES x
NOMENCLATURE xii

CHAPTER 1.
INTRODUCTION

1.1 Physical description of magnetic fluids 1
1.2 Continuum description of magnetic fluids 4
1.3 Boundary conditions 8
1.4 Outline of the present work 9

CHAPTER 2.
EFFECT OF ROTATION ON THE THERMO-CONVECTIVE INSTABILITY OF A HORIZONTAL LAYER OF A FERROMAGNETIC FLUID.

2.1 Introduction 12
2.2 Governing equations 13
2.3 Instability analysis 15
2.4 Qualitative analysis on the oscillatory instability 20
2.5 Free boundaries 23

2.5.1 Exact solution for free boundaries 23
2.5.2 Numerical results and discussion

2.6 Two rigid paramagnetic boundaries
   2.6.1 Solution for paramagnetic boundaries
   2.6.2 Numerical results and discussion

2.7 Two rigid ferromagnetic boundaries
   2.7.1 Solution for ferromagnetic boundaries
   2.7.2 Numerical results and discussion

2.8 A comparative study for various boundaries

CHAPTER 3.
HODOGRAPH TRANSFORMATION METHODS
FOR FERROMAGNETIC FLUIDS
   3.1 Introduction
   3.2 Governing equations
   3.3 Equations in hodograph plane
   3.4 Illustrations and discussion

CHAPTER 4 CONCLUSIONS

REFERENCES
APPENDIX
VITA AUCTORIS
LIST OF TABLES

TABLE 2.1:
The critical stability parameters and their wave numbers for
the free boundaries. 61

TABLE 2.2:
The critical stability parameters and their wave numbers for
the paramagnetic boundaries, with $M_3 = 10$. 62

TABLE 2.3:
The critical stability parameters and their wave numbers for
the ferromagnetic boundaries, with $M_3 = 10$. 63
LIST OF FIGURES

FIGURE 2.1:
Effect of rotation on the critical Rayleigh number for free boundaries. \((M_3=1)\) .

FIGURE 2.2:
Effect of rotation on the critical magnetic Rayleigh number for free boundaries. \((M_1\to\infty)\).

FIGURE 2.3:
Effect of rotation on the critical Rayleigh number for rigid paramagnetic boundaries. \((M_3=1, \chi=0)\).

FIGURE 2.4:
Effect of rotation on the critical Rayleigh number for rigid paramagnetic boundaries. \((M_3=1, \chi=9999)\).

FIGURE 2.5:
Effect of rotation on the critical magnetic Rayleigh number for rigid paramagnetic boundaries. \((M_1\to\infty, \chi=0)\).

FIGURE 2.6:
Effect of rotation on the critical magnetic Rayleigh number for rigid paramagnetic boundaries. \((M_1\to\infty, \chi=9999)\).

FIGURE 2.7:
Effect of rotation on the critical magnetic Rayleigh number for rigid paramagnetic boundaries. \((M_1\to\infty, M_3=1)\).
FIGURE 2.8:
Effect of rotation on the critical Rayleigh number
for rigid ferromagnetic boundaries. \((M_3=1)\).

FIGURE 2.9:
Effect of rotation on the critical magnetic Rayleigh number
for rigid ferromagnetic boundaries. \((M_1 \to \infty)\).

FIGURE 2.10:
Effect of rotation on the critical Rayleigh number
for various boundaries. \((M_1=1, M_3=1)\).

FIGURE 2.11:
Effect of rotation on the critical Rayleigh magnetic number
for various boundaries. \((M_1 \to \infty, M_3=1)\).

FIGURE 2.12:
Effect of rotation on the critical Rayleigh number
for various boundaries. \((M_1=1, M_3=10)\).

FIGURE 2.13:
Effect of rotation on the critical magnetic Rayleigh number
for various boundaries. \((M_1 \to \infty, M_3=10)\).
NOMENCLATURE

$\dot{A}$ rate of conversion of external angular momentum to internal angular momentum.

$a$ dimensionless wave number.

$\vec{B}$ $\mu_0(\vec{M} + \vec{H})$, the magnetic flux density.

$b$ subscript for the basic state.

$\vec{C}$ couple stress tensor.

$C$ $C_{V,H} + \frac{\mu_0}{\rho_0} KH_0$.

$C_{V,H}$ heat capacity at constant volume and magnetic field.

$d$ thickness of the fluid layer.

$\frac{D}{Dt}$ material derivative.

$\frac{D^2}{Dt^2}$ $\frac{D^2}{Dt^2} \omega_\star$, the objective stress tensor.

$\vec{F}$ body force vector.

$\vec{G}$ body couple per unit mass.

$\vec{g}$ acceleration due to gravity.
$\vec{H}$
magnetic field.

$H_0$
uniform magnetic field in the vertical direction.

$\vec{I}$
unit dyadic.

$I$
sum of moment of inertia of the particles.

$K$
$-(\frac{\partial^2 \vec{M}}{\partial T^2})T_0$, the pyromagnetic coefficient.

$k_1$
thermal conductivity.

$k_x, k_y$
horizontal wave number in the $x$ and $y$ directions respectively.

$k^2$
$(k_x^2 + k_y^2)$.

$\vec{M}$
magnetization vector.

$M_1$
$\frac{\mu_0 X^2 \beta}{\alpha p_0 (1+\chi)}$, the ratio of magnetic force to buoyancy.

$M_2$
$\frac{\mu_0 T_0 X^2}{(1+\chi) p_0 c}$, a nondimensional parameter.

$M_3$
$\frac{(1+\frac{M_1}{M_0})}{(1+\chi)}$, the measure of nonlinearity of magnetization.

$M_0$
constant mean value of magnetization.

$N$
$RM_1$, the magnetic Rayleigh number.
$p$ pressure.

$P \frac{\nu}{k_1} (\rho_0 C)$, the Prandtl number.

$\bar{q}$ velocity with the components $(u, v, w)$.

$R \frac{\alpha \rho_0 \beta d^4 C}{\nu k_1}$, the Rayleigh number.

$\tilde{S}$ angular momentum density.

$\tilde{T}$ unsymmetric stress tensor.

$T$ temperature.

$T_0$ average or the ambient temperature.

$\Delta T$ temperature difference between the plates.

$t$ time.

$\alpha$ thermal expansion coefficient.

$\beta \frac{\Delta T}{d}$, the temperature gradient.

$\omega \frac{\partial M}{\partial H} H_0$, the magnetic susceptibility.

$\kappa_\perp, \kappa_\parallel$ components of the dynamic magnetic susceptibilities.
\( \lambda \)  
bulk coefficient of viscosity.

\( \lambda' \)  
bulk coefficient of spin viscosity.

\( \mu \)  
shear coefficient of viscosity.

\( \mu' \)  
shear coefficient of spin viscosity.

\( \mu_0 \)  
magnetic permeability of vacuum.

\( \mu_v \)  
vortex viscosity.

\( \nu \)  
\( \frac{\mu}{\rho_0} \), the kinematic viscosity.

\( \Omega \)  
(0, 0, \( \Omega \)), the angular velocity of rotation.

\( \vec{\omega} \)  
(\( \xi, \eta, \zeta \)), the vorticity.

\( \vec{\omega}_p \)  
particle spin rate.

\( \phi \)  
scalar magnetic potential.

\( \Phi_v \)  
viscous dissipation.

\( \rho \)  
density of the magnetic fluid.

\( \rho_0 \)  
density of the fluid at the ambient temperature.

\( \tau \)  
\( \frac{4\Omega d^2}{\nu} \), the Taylor number.

\( \tau_0 \)  
relaxational time constant.

\( \theta \)  
temperature in the normal mode as a function of \( z \) alone.
CHAPTER 1

INTRODUCTION

1.1 Physical description of magnetic fluids.

In the past several decades, the interaction between electromagnetic fields and fluids has attracted increasing attention, due to their diverse applications. This subject can be roughly divided into three major categories (Rosensweig [1985]):

1. Electrohydrodynamics (EHD), the branch of fluid mechanics concerned with electric force effects;

2. Magnetohydrodynamics (MHD), which involves the study of the interaction between magnetic fields and fluid conduction of electricity; and

3. Ferrohydrodynamics (FHD), which deals with the mechanics of fluid motion induced by strong forces of magnetic polarization.

The main differences between the above topics have already been clearly given by Rosensweig [1985]. "In MHD the body force acting on the fluid is the Lorentz force that arises when the electric current flows at an angle to the direction of an impressed magnetic field. However in FHD, there is no electric current flowing in the fluid. The body force in FHD is due to the polarization force, which in turn requires material magnetization in the presence of magnetic field gradients or discontinuities. Likewise the force of interaction arising in EHD is often due to free electric charge acted upon by an electric force field. In comparison, in FHD free electric charge is usually absent and the analog of electric charge, the monopole, has not been found in nature. An analogy between EHD and FHD arises, however, for charge-free electrically polarizable fluids exposed to gradient electric field. A
major difference from FHD is the magnitude of the effect, which is usually much smaller in electrically polarizable media." FHD, basically deals with the study of fluids which have gigantic magnetic response, called magnetic fluids. Several types of magnetic fluids arise with FHD, the principal type being colloidal ferrofluid.

Ferromagnetic fluids do not exist in nature, they must be synthesized. In general, they are suspensions of fine ferromagnetic particles, such as, magnetite in non-conducting liquids (Rosensweig [1985]). These fluids are composed of small magnetite particles (about as many as \(10^{18}\) in \(1\ cm^3\)) coated with a monomolecular layer of a dispersant suspended in a liquid carrier. Thermal agitation keeps the particles suspended because of Brownian motion and the coating prevents the particles from sticking to each other. These fluids are attracted strongly by external magnetic field with forces that are large enough to overcome gravity. The magnetic force originates in the particles and is transmitted to the surrounding carrier fluid by viscous interaction. In the absence of applied field, the particles in a colloidal magnetic fluid are randomly oriented and thus the fluid has no net magnetization. When placed in a homogeneous magnetic field, the latter causes a partial orientation of the magnetic moments. In variable fields, the magnetization is relaxational.

The stabilized colloidal magnetic fluid behaves remarkably like a true liquid having concomitant fluid and magnetic properties. Magnetically, there are two major features that distinguish the ferrofluids from ordinary fluids. These features are, the polarization force and the body couple. The magnetic polarization force attracts a nail to a magnet and vice versa. This force arises from the presence of field gradients and most studies of magnetic fluids concern this interaction. The body couple is a source of angular momentum, that is transmitted from a distance magnetically to influence the flow of a magnetic fluid.
Eventhough magnetic fluids were discovered only in the mid sixties, they have found lots of applications. As magnetic fluids are applied to a broad spectrum of areas, the interest in them is growing rapidly. Some of the popular applications of magnetic fluids are summarized below, without going into the details.

(a) Magnetic fluid seals:
Magnetic fluids can be positioned and shaped by an external magnetic field, moreover they are seen to preserve their viscosity even at very low pressures. This fact has lead to various applications of magnetic fluids in different engineering devices. For example, magnetic fluids are used as zero-leakage rotary shaft seals in computer disc drives.

(b) Magnetic fluid as a lubricant:
The use of lubricants increases the operational reliability and the life expectancy of several mechanisms. It often becomes very difficult to retain the lubricants at the contact of the moving parts due to various reasons such as, the injection due to centrifugal forces, the shape of the geometry etc. As magnetic fluids can be efficiently retained at any required place with the help of an external magnetic field, they have found extensive use as lubricants in various mechanical devices.

(c) Magnetic fluid as a heat carrier:
As thermomagnetic convection may exceed natural convection, the application of magnetic fluids as a heat carrier is efficient in devices which already have strong magnetic fields. One common application of these fluids are as heat removers from loud speaker coils, using enhanced rotational convection, a property observed for magnetic fluids.

(d) Magnetic fluids in medicine:
Magnetic fluids attracted the attention of physicians as a material which can be used as artificial clots that might occlude large blood vessels during operations.
Interest is also shown, in magnetic fluids as carriers of medicinal preparations to the target site. With the aid of a magnetic fluid, a drug can be transported to a required area, such as, a tumour, and can be retained there for as long as necessary.

(c) Magnetic fluids in printing and flow control:

The IBM developments using magnetic fluids as ink have been widely advertised. Magnetic fluids can be very effective in controlling the flow of an ordinary fluid. Since, the viscosity ratio of magnetic to non-magnetic fluids is about 1:100, by coating the streamlined body with a layer of low viscosity magnetic fluids, the drag of a body can be reduced.

1.2 Continuum description of magnetic fluids.

The set of equations describing the flow dynamics of a magnetic fluid may be written as (Rosensweig [1985]):

Conservation of mass:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) = 0
\]

Conservation of linear momentum:

\[
\frac{\partial \rho \vec{q}}{\partial t} + \nabla \cdot (\rho \vec{q} \vec{q}) = \nabla \cdot \vec{T} + \rho \vec{F}
\]

Conservation of internal angular momentum:

\[
\frac{\partial (\rho \vec{S})}{\partial t} + \nabla \cdot (\rho \vec{q} \vec{S}) = \rho \vec{G} + \nabla \cdot \vec{C} + \vec{A}
\]

Here, \((\rho \vec{q})\) is the flux of the linear momentum volumetric density \(\rho \vec{q}\); and \((\rho \vec{q} \vec{S})\) is the flux of the internal angular momentum volumetric density \(\rho \vec{S}\), i.e., \((\rho \vec{q} \vec{S})\) is the
spin flux. Notations are defined in the nomenclature. The stress tensor \( \mathbf{T} \) contains, pressure, viscous and magnetic stresses, where, the viscous stress is not symmetric. Thus,

\[
\mathbf{T} = -p\mathbf{I} + \mathbf{T}_{\text{vs}} + \mathbf{T}_{\text{va}} + \mathbf{T}_{\text{m}}
\]  

(1.4)

where, \( p \) is the usual thermodynamic pressure \( p(\rho, T) \); \( \mathbf{T}_{\text{vs}} \) is the symmetric part of the viscous stress tensor; \( \mathbf{T}_{\text{va}} \) is the antisymmetric part of the viscous stress tensor; \( \mathbf{T}_{\text{m}} \) is the magnetic stress tensor and they are given by,

\[
\mathbf{T}_{\text{vs}} = \lambda(\nabla \cdot \mathbf{q})\mathbf{I} + \mu(\nabla \mathbf{q} + (\nabla \mathbf{q})')
\]

\[
\mathbf{T}_{\text{va}} = \mu_v \epsilon_i (\mathbf{\dot{\omega}} - 2\mathbf{\omega}_p)
\]

\[
\mathbf{T}_{\text{m}} = -\left\{ \int_0^H \mu_0 \left[ \frac{\partial (VM)}{\partial V} \right]_{H,T} dH + \frac{1}{2} \mu_0 H^2 \right\} \mathbf{I} + \mathbf{B} \mathbf{H}
\]

(1.5)

where, \( \epsilon \) is the triadic alternator and \( V \) is the volume.

The couple stress \( \mathbf{C} \), which accounts for the diffusion of the spin angular momentum, is given by,

\[
\mathbf{C} = \lambda'(\nabla \cdot \mathbf{\dot{\omega}}_p)\mathbf{I} + \mu'(\nabla \mathbf{\omega}_p + \nabla \mathbf{\omega}_p')
\]

(1.6)

Moreover the expressions for \( \mathbf{\tilde{A}} \) and \( \mathbf{\tilde{G}} \) are assumed to be

\[
\mathbf{\tilde{A}} = 2\mu_v (\mathbf{\dot{\omega}} - 2\mathbf{\omega}_p)
\]

(1.7)

\[
\rho \mathbf{\tilde{G}} = \mu_0 \mathbf{\tilde{M}} \times \mathbf{H}
\]

(1.8)

In addition to the above the magnetic induction and the magnetic field are described by the magnetostatic limit of the Maxwell's equation in the absence of displacement currents as,

\[
\nabla \times \mathbf{\tilde{H}} = 0, \quad \nabla \cdot \mathbf{\tilde{B}} = 0
\]

(1.9)
Finally, since both $\bar{T}$ and $\bar{G}$ will have dependence on the magnetization vector $\bar{M}$, a relaxation equation for $\bar{M}$ of the form (Shilimos [1974]),

$$\frac{D\bar{M}}{Dt} = \bar{\omega}_p \times \bar{M} - \frac{1}{\tau_0} (\bar{M} - \bar{M}_0)$$  \hspace{1cm} (1.10)

where, $\bar{M}_0 = \frac{M_p}{H} \bar{H}$ is proposed. The equation connecting $\bar{B}$, $\bar{H}$ and $\bar{M}$ is given by,

$$\bar{B} = \mu_0 (\bar{H} + \bar{M})$$  \hspace{1cm} (1.11)

In this thesis only incompressible flow of magnetic fluids are considered. As a result, on substituting equations (1.4) to (1.8) in equations (1.2) and (1.3) and simplifying the resulting expressions, the basic equations take the form:

$$\nabla \cdot \bar{q} = 0$$  \hspace{1cm} (1.12)

$$\rho_0 \frac{D\bar{q}}{Dt} = -\nabla \bar{p} + \bar{\mu} \nabla^2 \bar{q} + \mu_0 (\bar{M} \cdot \nabla) \bar{H} + 2\mu_0 \nabla \times \bar{\omega}_p$$  \hspace{1cm} (1.13)

$$\rho_0 \frac{D\bar{\omega}_p}{Dt} = \mu' \nabla^2 \bar{\omega}_p + 2\mu_0 (\nabla \times \bar{q} - 2\bar{\omega}_p) + \mu_0 \bar{M} \times \bar{H} + \mu' \nabla (\nabla \cdot \bar{\omega}_p)$$  \hspace{1cm} (1.14)

where,

$$\bar{p} = p(\rho, T) + \mu_0 \int_0^H \left( \frac{\partial (VM)}{\partial V} \right)_{H,T} dH, \quad \text{and} \quad \bar{\mu} = \mu + \mu_v$$  \hspace{1cm} (1.15)

Equations (1.12) to (1.14) are seven equations in thirteen unknowns viz., $\bar{p}$, $\bar{q}$, $\bar{M}$, $\bar{H}$ and $\bar{\omega}_p$. These equations are supplemented by three equations (1.9) and three equations (1.10) to yield a determinate system to be used with appropriate initial and boundary conditions.

As the equations are quite complicated, in practice, it is common to simplify the equation (1.10) depending upon the nature of the magnetic field $\bar{H}$ and the corresponding magnetization $\bar{M}$. 

6
One of the simplest and so far the most extensively studied situation is when the magnetization is parallel to the field. Such a situation occurs in flows with slowly shifting orientation of magnetic field relative to the translating and rotating fluid elements. Mathematically this implies that,

$$\vec{\mathcal{M}} \times \vec{H} = 0, \quad \mathcal{M} = \mathcal{M}_0, \quad \vec{A} = 0 \quad \text{and} \quad \vec{C} = 0$$  \hspace{1cm} (1.16)

thus, antisymmetric stress and the couple stress disappear automatically. Furthermore, $2\vec{\omega}_p = \nabla \times \vec{q}$ and thus reducing our system to seven equations, equations (1.12), (1.13) and (1.9), in seven unknowns, $\vec{p}, \vec{q}, \vec{H}$.

The next stage of approximation involves assuming that the characteristic hydrodynamic time as well as the time of the magnetic field variation are in excess of the relaxation time of magnetization. Sometimes, such an approximation is also known as the low frequency approximation. As it turns out, in this case, an equation, linear in dynamic variables can be derived for magnetization. It is recalled that, in general, $\vec{M} = \vec{M}(\rho, T, \vec{H}, \frac{D\vec{H}}{Dt})$. In the state of equilibrium, we have $\frac{D\vec{H}}{Dt} = 0$ and hence $\vec{M} = \vec{M}_0(\rho, T, \vec{H})$. For a state not too far from equilibrium, a linear relation of the form can be assumed,

$$M_i = \chi H_i + \sum_{k=1}^{3} \chi_{ik} \frac{D H_k}{Dt}, \quad i = 1, 2, 3$$  \hspace{1cm} (1.17)

where the objective tensor in the second term of the RHS maintains the invariance under rigid body rotation and $\chi_{ik}$, as given by (Bashtovoy et al., [1988]) is,

$$\chi_{ik} = \kappa_\perp \delta_{ik} - (\kappa_\parallel - \kappa_\perp) \frac{H_i H_k}{H^2}$$  \hspace{1cm} (1.18)

Bashtovoy et al., [1988] has arrived at the same equation by using different arguments. With the use of the Langevin function $L$ and the Langevin argument $\xi$ one
can arrive at the values of $\kappa_\perp$ and $\kappa_\parallel$ in terms of these functions, moreover, in the limit of weak and strong fields it can be verified that,

$$\kappa_\parallel = \kappa_\perp = \kappa_0 = \text{constant} \quad \xi \ll 1$$

$$\kappa_\parallel = 3\kappa_0/\xi^2, \quad \kappa_\perp = 6\kappa_0/\xi \quad \xi \gg 1$$ (1.19)

A detailed derivation of these can however be obtained from Bashtovoy et al., [1988].

1.3 Boundary conditions.

The boundary conditions for the magnetic field and the magnetization are discussed first. The continuity of the vertical component of $\vec{B}$ across the boundaries, implies,

$$\left[ \vec{B} \cdot \hat{n} \right] = 0$$ (1.20)

where $\hat{n}$ is the unit vector normal to the boundary and the brackets indicate the difference across the interface. Thus using equation (1.16), equation (1.20) can be written as,

$$H_n^0 - H_n^i = M_n^i - M_n^0$$ (1.21)

where the subscript $n$ denotes their normal components and the superscripts denote either sides of the boundary.

The continuity of the tangential components of the magnetic field across the boundaries gives,

$$\left[ \vec{H} \cdot \hat{i} \right] = 0, \quad \text{or} \quad \hat{n} \times (\vec{H}^0 - \vec{H}^i) = 0$$ (1.22)

The above boundary conditions are well-defined; while, the boundary conditions for the velocity $\vec{q}$ and the particle spin $\vec{\omega}_p$ are still in an uncertain status. The boundary conditions for the velocity and the particle spin used in all the problems
done so far in the study of magnetic fluids have been of the traditional type. That is, for the fluid velocity, the usual adhesion of a fluid to a solid surface, that is \( \bar{q} = 0 \), is used; for the spin vector \( \bar{w}_p \), the no spin conditions on the solid walls is employed. Rosenwweig [1985] explains the uncertainty of these boundary conditions but he does not provide any alternatives.

Recently, Kaloni [1992] has proposed alternative boundary conditions for both velocity and the spin vectors. In the place of no-slip and no-spin conditions, Kaloni [1992] proposed,

\[
\bar{q} \cdot \hat{n} = \bar{q}_* \cdot \hat{n}
\]

\[
\bar{q} - \bar{q}_* = \frac{1}{\beta_0} \left[ \hat{n} \times \hat{n} \cdot T \times \hat{n} \right]
\]

and

\[
[\bar{w}_p] = \frac{\alpha_0}{2} (\nabla \times \bar{q})_s \quad 0 < \alpha_0 < 1
\]

where, \( \bar{q}_* \) is the velocity of the boundary and \( \beta_0 \) is the coefficient of sliding friction. Using these boundary conditions Kaloni [1992] has explained some experimental observations, which could not be explained with the help of the usual boundary conditions for magnetic fluids. However, as it is not yet possible to make any firm judgements on these new boundary conditions, traditional boundary conditions are retained in this thesis.

1.4 Outline of the present work.

Two different problems in the theory of magnetic fluids are studied in this thesis. The first problem deals with the stability analysis of a convective heat transfer phenomena in magnetic fluids and the second problem deals with obtaining exact solutions for some magnetic fluid flow problems with the help of transformation techniques.
The first problem is discussed in chapter 2. It deals with the stability of a horizontal layer of a ferromagnetic fluid which is heated from below and is acted upon by a vertical uniform magnetic field. This system is set into uniform rotation and the effect of rotation on the instability of this system is analyzed. An exact solution is obtained for the fluid bounded by two stress free boundaries.

For the fluid bounded by two rigid paramagnetic boundaries a Galerkin technique is used and the critical Rayleigh number $R_c$ and the critical Rayleigh magnetic number $N_c$ are obtained. The effects of the non-linearity of the magnetization on the instability is studied for various values of angular velocity. Also, the effects of the magnetic and buoyant forces acting, both, alone as well as in combination are studied. The effect of the magnetic susceptibility on the instability of the system is also studied for various values of the Taylor number. The effect of rotation with reference to all these parameters are studied to provide an overall view of the problem.

For a fluid confined between two rigid ferromagnetic boundaries, a similar analysis is conducted. The effects of the high magnetization of these boundaries in comparison to that of the fluid is studied. The effect of the magnetic and buoyant forces, for this boundary, on the instability is studied. The rotation effects on these boundaries are analyzed. The effect of non-linearity of the magnetization of the fluid, on the convection of this layer under rotation is studied.

The effects of the boundaries on the instability is also studied by comparing the critical stability parameters. For various values of the Taylor number, the problem is analyzed and the effect of rotation on these boundaries are studied.

In chapter 3, transformation techniques are used to study some steady, two dimensional magnetic flow problems. This study is mainly inspired by the work of Chandna, Barron and Smith [1982] who employed hodograph and Legendre trans-
formations to study plane viscous flow problems. The basic flow equations are developed under the low frequency assumption for magnetization. The equations obtained in the physical plane are then transformed into the hodograph plane by swapping the dependent and the independent variables. Legendre transformation is employed to the system of equations in the hodograph plane. This resulting system is also developed in polar coordinates of the hodograph plane. Application of this transformed system of equations, at present is illustrated by solving some simple flow problems.
CHAPTER 2

EFFECT OF ROTATION ON THE

THERMO-CONVECTIVE INSTABILITY OF A

HORIZONTAL LAYER OF A FERROMAGNETIC FLUID

2.1 Introduction.

Heat transfer through ferromagnetic fluids is a technically important subject that has been reviewed in several recent monographs (Rosensweig [1985], Bashtovoy et al., [1988]). The magnetization of magnetic fluids is in general a function of the magnetic field, the temperature and the density of the fluid. Therefore, any variation of these quantities can induce some corresponding spatial distribution of the body force. This force gives rise to convection in magnetic fluids in the presence of a gradient of magnetic field. In particular, the force induced due to a temperature gradient is of importance in phenomena where the buoyancy due to thermal expansion is essential.

The generalization of the classical Rayleigh - Benard problem (Chandrasekhar [1961]) for the magnetic fluids was first studied by Finlayson [1970], using the linear theory. Later, Lalas and Carmi [1971] analyzed the same problem using the energy method. A similar analysis but with the fluid confined between ferromagnetic plates was carried out by Gotoh and Yamada [1982] using the linear analysis.

The study of fluids in rotation is in itself an interesting topic (Greenspan [1968]). Analysis of the Benard problem when the layer is rotating is well known for ordinary viscous fluids (Chandrasekhar [1961]). Ferrofluids are known to exhibit very
peculiar characteristics when set to rotation (Rosensweig [1985]). Thus, studying
the effects of rotation on the convection in ferrofluids is scientifically and techn-
ologically important. The convective instability analysis for a rotating layer of
ferrofluids between two free boundaries was studied by Gupta and Gupta [1979].
These authors mainly confined their analysis to the numerical discussion of the
overstability problem.

In this chapter, the effects of rotation on the Rayleigh - Benard problem for
ferrofluids confined between two free boundaries, two rigid paramagnetic boundaries
and also between two ferromagnetic boundaries is analyzed. Here, the onset of
convection in a horizontal layer of a ferromagnetic fluid rotating about its vertical
axis, heated from below and in the presence of a uniform vertical magnetic field,
is studied using the linear analysis. A qualitative analysis for the instability to
manifest itself as oscillatory convection is conducted. It is also shown analytically
that for the non-rotating case oscillatory instability can never occur. An exact
solution is obtained for the free boundary case and a Galerkin technique is used for
rigid boundaries. Numerical calculations for stationary convection are carried out
to obtain critical Rayleigh numbers and critical wave numbers for various values of
the other parameters. The effect of the rotation on the critical stability parameters
are explained for various values of the other parameters. Our results agree well with
that of Chandrasekhar [1961] and Finlayson [1970] in the limits of zero magnetic
field and zero rotation, respectively.

2.2 Governing equations.

Consider a layer of a ferromagnetic fluid bounded between two horizontal plates,
z = -d/2 and z = d/2, in the presence of a magnetic field applied normal to the
plates. The layer is rotating uniformly about its vertical axis with an angular ve-
velocity $\vec{\Omega}$. A constant temperature gradient is maintained between the plates. The lower and the upper plates are maintained at constant temperatures $T_0 + (\Delta T/2)$ and $T_0 - (\Delta T/2)$ respectively. A cartesian co-ordinate system $(x, y, z)$ is used, with the $z$-axis normal to the plates. Boussinesq approximation on the density is assumed. The governing equations for this system in the rotating frame of reference are (Rosensweig [1985], Gupta and Gupta [1979]) :

Continuity equation,

$$\nabla \cdot \vec{q} = 0$$  \hspace{1cm} (2.1)

Linear momentum equation,

$$\rho_0 \frac{D\vec{q}}{Dt} = -\nabla p + \rho_0 \vec{g} + \mu \nabla^2 \vec{q} + \nabla \cdot (\vec{\Omega} \vec{B}) + 2\rho_0 \vec{q} \times \vec{\Omega} + \frac{\rho_0}{2} \nabla((\vec{\Omega} \times \vec{r}))$$  \hspace{1cm} (2.2)

Temperature equation (Finlayson [1970]),

$$\left[ \rho_0 c_v H - \mu_0 \vec{H} \cdot \left( \frac{\partial \vec{M}}{\partial T} \right)_{v,H} \right] \frac{DT}{Dt} + \mu_0 T \left( \frac{\partial \vec{M}}{\partial T} \right)_{v,H} \cdot \frac{D\vec{H}}{Dt} = k_1 \nabla^2 T + \Phi_v$$  \hspace{1cm} (2.3)

Maxwell's equations for non-conducting fluids with no displacement currents,

$$\nabla \cdot \vec{B} = 0; \quad \nabla \times \vec{H} = 0$$  \hspace{1cm} (2.4a,b)

Further, $\vec{B}$, $\vec{M}$ and $\vec{H}$ are related by

$$\vec{B} = \mu_0 (\vec{M} + \vec{H})$$  \hspace{1cm} (2.5)

The assumption that the magnetization is aligned with the field (Rosensweig [1985]),

$$\vec{M} = \frac{M(H, T)}{H} \vec{H}$$  \hspace{1cm} (2.6)

The fluid equation of state,

$$\rho = \rho_0 [1 - \alpha(T - T_0)]$$  \hspace{1cm} (2.7)
The magnetic equation of state,
\[ M = M_0 + \chi(H - H_0) - K(T - T_0) \]  \hspace{1cm} (2.8)

Equations (2.1) to (2.8) are the governing equations for this problem. The various notations used in these equations are explained in the nomenclature.

The initial quiescent state is given by, \( \bar{q} = \bar{0}, \ p = p_b, \ \bar{H}_b = [0, 0, H_b(z)], \ \bar{M} = [0, 0, M_b(z)] \) and \( T = T_b(z) \). Thus \( T_b \) satisfies,
\[ \nabla^2 T_b = 0, \]

Together with the already mentioned conditions on the boundaries. Thus, solving this we get \( T_b = T_0 - \beta z \), where \( \beta = (\Delta T/d) \). Clearly, equation (2.4b) is identically satisfied and equation (2.4a) gives,
\[ H_b(z) + M_b(z) = C_1 \]  \hspace{1cm} (2.9)

\( C_1 \) being a constant. Now, from equations (2.6), (2.8) and (2.9), we get,
\[ H_b(z) = \frac{-K\beta z}{(1 + \chi)} + \frac{C_1 - M_0 + \chi H_0}{(1 + \chi)} \]  \hspace{1cm} (2.10)

Equations (2.9) and (2.10) give,
\[ H_b(z) = H_0 - \frac{K\beta z}{(1 + \chi)} \]  \hspace{1cm} (2.11a)
\[ M_b(z) = M_0 + \frac{K\beta z}{(1 + \chi)} \]  \hspace{1cm} (2.11b)

The solution for pressure is not provided as it is not required for the analysis.

2.3 Instability analysis.

In general, instability can be explained as the inherent inability of the system to sustain itself against small perturbations, to which any physical system is subjected.
to. In considering the stability of a hydrodynamic system, one essentially seeks
to determine the reaction of the system, containing the physical parameters like
velocity, temperature etc., to small disturbances. In short one asks, if the system is
disturbed in any or all of these parameters, will the disturbance gradually die down
(stable) or will the disturbance grow in amplitude so that the system departs from
the initial state and never returns to it (unstable)?

Thus, to analyze the present problem, the initial state is perturbed infinitesimally
and the corresponding equations in the perturbed quantities are obtained
and studied. Let the components of the perturbed magnetization and the magnetic
field be given by \((M_1', M_2', M_3(z) + M_3')\) and \((H_1', H_2', H_3(z) + H_3')\) respectively. The
perturbed temperature \(T\) be \(T_0 + T'\). Use of these in equations (2.6) and (2.8) and
linearization, gives,

\[
H_i' + M_i' = \left(1 + \frac{M_0}{H_0} \right) H_i' \quad i = 1, 2 \tag{2.13}
\]

\[
H_3' + M_3' = (1 + \chi) H_3' - K T'
\]  \tag{2.14}

where, \(K \Delta T \ll (1 + \chi) H_0\) is assumed.

Taking curl of the equation (2.2) and linearizing, the z-component can be written
as,

\[
\rho_0 \frac{\partial \zeta}{\partial t} = \mu \nabla^2 \zeta + 2 \rho_0 \Omega \frac{\partial w}{\partial z}, \tag{2.15}
\]

which is the vorticity transport equation for this problem.

On taking curl curl of equation (2.2), linearizing, using equations (2.13) and
(2.14) and using \(\vec{H}' = \nabla \phi'\), the z - component of the resulting equation can be
written as,

\[
\rho_0 \frac{\partial}{\partial t} (\nabla^2 w) = -2 \rho_0 \Omega \frac{\partial \zeta}{\partial z} - \mu_0 K \beta \frac{\partial}{\partial z} (\nabla^2 \phi') + \rho_0 g \alpha (\nabla^2 T') + \frac{\mu_0 K^2 \beta}{1 + \chi} (\nabla^2 T') + \mu \nabla^4 w \tag{2.16}
\]
Now, the perturbations are analyzed as two dimensional periodic waves characterized by a wave number \( k = \sqrt{k_x^2 + k_y^2} \) and thus assume all the fields vary in the following way (normal mode analysis, Chandrasekhar [1961]):

\[
\begin{align*}
\zeta &= \zeta(z,t)e^{ik_x x + ik_y y} \\
T' &= \theta(z,t)e^{ik_x x + ik_y y} \\
\phi' &= \phi(z,t)e^{ik_x x + ik_y y} \\
w &= w(z,t)e^{ik_x x + ik_y y}
\end{align*}
\]

(2.17)

Using equation (2.17), equations (2.16) and (2.15) become,

\[
\rho_0 \frac{\partial}{\partial t} \left( \frac{\partial^2}{\partial z^2} - k^2 \right) w = -2\rho_0 \Omega \frac{\partial \zeta}{\partial z} - \rho_0 g \kappa k^2 \theta + \frac{\mu_0 K^2 \beta}{(1 + \chi)} \left[ (1 + \chi) \frac{\partial \phi}{\partial z} - K \theta \right] k^2 + \mu \left( \frac{\partial^2}{\partial z^2} - k^2 \right)^2 w
\]

(2.18)

\[
\rho_0 \frac{\partial \zeta}{\partial t} = \mu \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \zeta + 2\rho_0 \Omega \frac{\partial w}{\partial z}
\]

(2.19)

Equation (2.3) is linearized and the resulting equation upon using \( \vec{H}' = \nabla \phi' \) and equation (2.18) gives,

\[
\rho_0 C \frac{\partial \theta}{\partial t} - \mu_0 T_0 K \frac{\partial \phi}{\partial t} \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \theta + \left[ \rho_0 C \beta - \frac{\mu_0 T_0 K^2 \beta}{(1 + \chi)} \right] w
\]

(2.20)

where,

\[
\rho_0 C = \rho_0 C_{V,H} + \mu_0 KH_0 \quad \text{(2.20a)}
\]

Finally equations (2.3) and (2.4), after using equations (2.13), (2.14) and (2.18) become,

\[
(1 + \chi) \frac{\partial^2 \phi}{\partial z^2} - \left[ 1 + \frac{M_0}{H_0} \right] k^2 \phi - K \frac{\partial \theta}{\partial z} = 0
\]

(2.21)

Thus, equations (2.18) to (2.21) are the governing perturbation equations. The form of the above equations are simplified by introducing the following dimensionless quantities,

\[
\begin{align*}
t^* &= \frac{\mu_0 t}{\rho_0 d^2}, & w^* &= \frac{wd}{\nu}, & \zeta^* &= \frac{\zeta d^2}{\nu}, & \phi^* &= \frac{(1 + \chi)k_1 a \sqrt{t^*/\nu}}{K \rho_0 C \beta \nu d^2}
\end{align*}
\]

17
\[ \theta^* = \frac{k_1 a R^{1/2} \theta}{\rho_0 C_\beta \nu d}, \quad a = kd, \quad z^* = \frac{z}{d}, \quad D \equiv \frac{\partial}{\partial z^*} \] (2.22)

Thus equations (2.18) to (2.21) become,

\[ \frac{\partial}{\partial t^*}(D^2 - a^2)w^* = -\tau^{1/2} D \zeta^* + a R^{1/2} [M_1 D \phi^* - (1 + M_1) \theta^*] + (D^2 - a^2)^2 w^* \] (2.23)

\[ \frac{\partial \zeta^*}{\partial t^*} = (D^2 - a^2) \zeta^* + \tau^{1/2} Dw^* \] (2.24)

\[ P \frac{\partial \theta^*}{\partial t^*} - PM_2 \frac{\partial}{\partial t^*}(D \phi^*) = (D^2 - a^2) \theta^* + a R^{1/2}(1 - M_2)w^* \] (2.25)

\[ D^2 \phi^* - a^2 M_3 \phi^* - D \theta^* = 0 \] (2.26)

Here, \( M_1 \) is the ratio of the magnetic to gravitational forces. The parameter \( M_3 \) is a measure of the non-linearity in magnetization and \( M_3 = 1 \) corresponds to linear magnetization. As \( M_2 \) is of very small order (Finlayson [1970]) it is neglected in the subsequent analysis. \( \tau \) is the Taylor number signifying the effect of rotation. \( P \) is the Prandtl number, which is defined a little differently here comparing to the usual viscous fluid case. \( R \) is the Rayleigh number and it is a ratio of buoyant to viscous forces. \( N \) is the magnetic Rayleigh number and it is a ratio of magnetic to viscous forces. All the above parameters affect the stability of the system in one way or the other. The astersisks above the variables will be neglected hereafter, as the subsequent analysis only deals with the dimensionless variables. Now, following Chandrasekhar [1961],

\[ w(z, t) = e^{\sigma t} w(z), \quad \zeta(z, t) = e^{\sigma t} \zeta(z), \]

\[ \phi(z, t) = e^{\sigma t} \phi(z), \quad \theta(z, t) = e^{\sigma t} \theta(z) \] (2.27)

where, \( \sigma \) is a real or a complex constant.
Using equation (2.27), equations (2.23) to (2.26) can be rewritten as,

\[(D^2 - \sigma)(D^2 - a^2)w - \tau^{1/2}D\zeta = -aR^{1/2}\left[M_1 D\phi - (1 + M_1)\theta\right]\]  

(2.28)

\[(D^2 - a^2 - \sigma)\zeta = -\tau^{1/2} Dw\]  

(2.29)

\[(D^2 - a^2 - P\sigma)\theta = -aR^{1/2}w\]  

(2.30)

\[(D^2 - a^2M_3)\phi - D\theta = 0\]  

(2.31)

The above set of equations is a double eigenvalue problem for $R$ and $\sigma$, to be solved with respect to the appropriate boundary conditions. The boundary conditions for all the cases considered for this study are stated here. For a detailed derivation refer appendix.

(a) **Two free, flat boundaries**:

\[w = 0, \quad D^2w = 0, \quad D\zeta = 0, \quad \theta = 0, \quad D\phi = 0 \quad \text{at} \quad z = \pm\frac{1}{2}\]  

(2.32)

Only the case of large susceptibility is considered for this boundary.

(b) **Two rigid paramagnetic boundaries**:

\[w = 0, \quad D^2w = 0, \quad D\zeta = 0, \quad \theta = 0 \quad \text{at} \quad z = \pm\frac{1}{2}\]

\[(1 + \chi)D\phi - a\phi = 0 \quad \text{at} \quad z = \pm\frac{1}{2}\]  

(2.33)

(c) **Two rigid ferromagnetic boundaries**:

\[w = 0, \quad D^2w = 0, \quad D\zeta = 0, \quad \theta = 0, \quad \phi = 0 \quad \text{at} \quad z = \pm\frac{1}{2}\]  

(2.34)
2.4 Qualitative analysis on the oscillatory instability.

If at the onset of instability a stationary pattern of motion prevails, then one says that the instability sets in as stationary, time independent, convection. On the other hand, if at the onset of instability a oscillatory motion prevails, then one says that the convection sets in as oscillatory. In the latter case the infinitesimal perturbation given to the equilibrium provokes the restoring forces so strongly as to overshoot to the corresponding position on the other side of equilibrium. In the classical Benard problem it was shown that oscillatory convection cannot occur. For the rotating case of the corresponding problem, however, oscillatory instability does occur and the corresponding conditions for its occurrence were qualitatively analyzed in Chandrasekhar [1961]. For the case of magnetic fluids such an analysis has not been done even for the non-rotating case. Here, the possibility for the occurrence of instability as oscillatory convection is analyzed and corresponding bounds on the variables are obtained. First, $\phi$ is eliminated from equation (2.28) by operating $(D^2-a^2 M_3)$ on that equation and then using equation (2.31). Thus,

$$(D^2-a^2 M_3)(D^2-a^2-\sigma)(D^2-a^2)w - \tau^{1/2}(D^2-a^2 M_3)D\zeta$$

$$= -a R^{1/2} [M_1 D^2.3 - (1 + M_1)(D^2-a^2 M_3)\theta]$$

Now, multiplying the above equation by $\bar{w}$ (conjugate of $w$) and integrating from $-\frac{1}{2}$ to $\frac{1}{2}$, we obtain,

$$< \bar{w}(D^2-a^2 M_3)(D^2-a^2)(D^2-a^2-\sigma)w > -\tau^{1/2} < \bar{w}(D^2-a^2 M_3)D\zeta >$$

$$= a R^{1/2} < \bar{w} D^2 \theta > -a^3 R^{1/2}(1 + M_1)M_3 < \bar{w} \theta >$$

(2.35)

where, $< f > = \int_{-\frac{1}{2}}^{\frac{1}{2}} f \ dz$ and $\bar{f}$ denotes the conjugate of $f$, for any $f$.  

20
On using the boundary conditions common to all the three cases and repeatedly applying integration by parts and simplifying the first term on the LHS of equation (2.35) is written as,

\[
[D^2 \omega D^3 w - D \omega D^4 w]^{1/2}_{-1} - \langle |D^3 w|^2 + [a^2(2 + M_3) + \sigma] | D^2 w |^2 \\
+ [a^4(2 + M_3) + a^2 \sigma(1 + M_3)] | Dw |^2 + M_3 a^4(a^2 + \sigma) | w |^2 > \tag{2.35a}
\]

where, $|f|^2 = f \bar{f}$. Similarly, the second term on the LHS of equation (2.35) gives upon using equation (2.29),

\[
- \langle |D^2 \zeta|^2 + [a^2(1 + M_3) + \sigma] | D \zeta |^2 + (a^2 + \sigma)a^2 M_3 | \zeta |^2 > \tag{2.35b}
\]

Also, the first term on the RHS of equation (2.35) gives on using equation (2.30),

\[
- \langle |D^2 \theta|^2 + (a^2 + P \sigma) | D \theta |^2 > \tag{2.35c}
\]

Finally, the second term on the RHS of equation (2.35) gives upon using equation (2.30),

\[
-a^2 M_3 (1 + M_3) < |D \theta|^2 + (a^2 + P \sigma) | \theta |^2 > \tag{2.35c}
\]

Use of equations (2.35a-d) in equation (2.35) gives,

\[
[D^2 \omega D^3 w - D \omega D^4 w]^{1/2}_{-1} - \langle |D^3 w|^2 + [a^2(2 + M_3) + \sigma] | D^2 w |^2 \\
+ [a^4(2 + M_3) + a^2 \sigma(1 + M_3)] | Dw |^2 + M_3 a^4(a^2 + \sigma) | w |^2 + | D^2 \zeta |^2 \\
+ [a^2(1 + M_3) + \sigma] | D \zeta |^2 + (a^2 + \sigma)a^2 M_3 | \zeta |^2 >
\]

\[
= - \langle |D^2 \theta|^2 + [P \sigma + a^2(1 + M_3(1 + M_3))] | D \theta |^2 \\
+ a^2 M_3 (1 + M_3)(a^2 + P \sigma) | \theta |^2 >
\]
The process of equating the imaginary parts on both sides of the above equation and simplification gives,

$$-\Im(\sigma) < |D^2 w|^2 + a^2 (1 + M_3) |Dw|^2 + M_3 a^4 |w|^2$$

$$+ P \left[ |D\theta|^2 + a^2 M_3 (1 + M_1) |\theta|^2 \right] >$$

$$-\Im(\sigma) < \left[ |D\zeta|^2 + a^2 M_3 |\zeta|^2 \right] >$$

It is well known that oscillatory instability can occur only if $\Im(\sigma) \neq 0$. Thus oscillatory instability can occur if,

$$|D^2 w|^2 + a^2 (1 + M_3) |Dw|^2 + M_3 a^4 |w|^2 + P \left[ |D\theta|^2 + a^2 M_3 (1 + M_1) |\theta|^2 \right]$$

$$= |D\zeta|^2 + a^2 M_3 |\zeta|^2$$

(2.36)

This result is interesting in two ways.

First, for the non-rotating case, where the perturbation to the $z$-component of vorticity $\zeta$, is taken as zero (Finlayson [1970]), equation (2.36) shows that the instability can never be oscillatory. This observation can be obtained by putting $\zeta = 0$ in equation (2.36). Thus, it is obtained that oscillatory instability for a layer of ferromagnetic fluids heated from below under the influence of a normal magnetic field can occur only if,

$$|D^2 w|^2 + a^2 (1 + M_3) |Dw|^2 + M_3 a^4 |w|^2$$

$$+ P \left[ |D\theta|^2 + a^2 M_3 (1 + M_1) |\theta|^2 \right] = 0$$

This can never happen as the LHS is positive definite. This result will help in avoiding the massive computations of the type done by Gotoh and Yamada [1982], to examine numerically the mode of the onset of instability. In particular, this
observation would have obviated their lengthy process in finding $\sigma$ and hence the critical stability parameter.

Second, this result provides an equality, independent of the boundary conditions, which should be satisfied by the variables so that the oscillatory instability may occur. Equation (2.36) also indicates that the onset of convection as oscillatory instability is very restrictive. Thus for the paramagnetic and ferromagnetic boundaries, the analysis is restricted to stationary convection only. For the case of free boundaries, however, a further inequality for the Taylor number in terms of the Prandtl number is obtained for the instability to occur as oscillatory convection.

2.5 Free boundaries.

In this section, the analysis is conducted for the fluid bounded between two stress free boundaries. Though this case is of less physical interest, mathematically it is important as an exact solution can be obtained, whose properties are vital for analyzing the other problems considered. The problem here is to solve the system of equations (2.28) to (2.31) subject to the boundary conditions, equation (2.32).

2.5.1 Exact solution for free boundaries.

The solution can be separated into even and odd modes and the even modes are expected to give the lowest eigenvalue. Consequently, the solutions in which $w, \theta$ are even and $\phi, \zeta$ are odd are considered. Thus, the exact solutions can be written as,

$$w = A_1 \cos \pi z, \quad \theta = A_2 \cos \pi z,$$

$$\phi = \frac{A_3}{\pi} \sin \pi z, \quad \zeta = \frac{A_4}{\pi} \sin \pi z \quad (2.37)$$

where $A_1, A_2, A_3, A_4$ are constants. Substituting the above equation (2.37) in equations (2.28) to (2.31) and simplifying, a set of four homogeneous equations for the
four constants $A_1, A_2, A_3, A_4$ is obtained,

$$A_1 \left[ (\pi^2 + a^2)(\pi^2 + a^2 + \sigma) \right] + A_2 \left[ -aR^{1/2}(1 + M_1) \right] + A_3 \left[ aR^{1/2}M_1 \right] + A_4 \left[ -\tau^{1/2} \right] = 0 \quad (2.38)$$

$$A_1 \left[ \pi^2\tau^{1/2} \right] + A_4 \left[ \pi^2 + a^2 + \sigma \right] = 0 \quad (2.39)$$

$$A_1 \left[ -aR^{1/2} \right] + A_2 \left[ \pi^2 + a^2 + P\sigma \right] = 0 \quad (2.40)$$

$$A_2 \left[ -\pi^2 \right] + A_3 \left[ \pi^2 + a^2M_3 \right] = 0 \quad (2.41)$$

For the non-trivial eigenvalue to exist, the determinant of the coefficients should vanish and thus this condition gives on simplifying,

$$L_1 \sigma^3 + L_2 \sigma^2 + L_3 \sigma + L_4 = 0 \quad (2.42)$$

where,

$$L_1 = P(\pi^2 + a^2)(\pi^2 + a^2M_3) \quad (2.43)$$

$$L_2 = (\pi^2 + a^2)^2(\pi^2 + a^2M_3)(1 + 2P) \quad (2.44)$$

$$L_3 = 2(\pi^2 + a^2)^3(\pi^2 + a^2M_3) + P \left[ (\pi^2 + a^2)^3 + \pi^2\tau \right] (\pi^2 + a^2M_3) - a^2R \left[ \pi^2 + a^2M_3(1 + M_1) \right] \quad (2.45)$$

$$L_4 = \left[ (\pi^2 + a^2)^3 + \pi^2\tau \right] (\pi^2 + a^2M_3)(\pi^2 + a^2) - a^2R(\pi^2 + a^2) \left[ \pi^2 + a^2M_3(1 + M_1) \right] \quad (2.46)$$

First, the case when the onset of instability is stationary is considered. Putting $\sigma = 0$ in equation (2.42) and simplifying, $R$ can be written as an explicit function of other parameters $(a^2, M_3, M_1, \tau)$,

$$R = \frac{(\pi^2 + a^2M_3) \left[ \pi^2\tau + (\pi^2 + a^2)^3 \right]}{a^2 \left[ \pi^2 + a^2M_3(1 + M_1) \right]} \quad (2.47)$$
or in a simplified notation as,

\[ R_1 = \frac{(1 + zM_3) \left[ \tau_1 + (1 + x)^2 \right]}{x \left[ 1 + zM_3(1 + M_1) \right]} \]  

\[ (2.48) \]

where \( R_1 = \frac{R}{\pi^4} \), \( \tau_1 = \frac{\tau}{\pi^2} \) and \( x = \frac{z^2}{\pi^2} \). Now, the aim here is to find the minimum value of \( R (= R_c \), the critical Rayleigh number) with respect to wave numbers. Thus equation (2.48) is differentiated with respect to \( x \) and equated to zero. A polynomial in \( x \) whose coefficients vary over a field is obtained,

\[ 2M_3(1 + M_1)x^5 + M_3 \left[ 3 + (1 + 3M_3)(1 + M_1) \right] x^4 + 2(1 + 3M_3)x^3 \]

\[ + \left[ 3 - M_3^2(1 + M_1)(1 + \tau_1) - 3M_1M_3 \right] x^2 - 2M_3(1 + M_1)(1 + \tau_1)x \]

\[ -(1 + \tau_1) = 0 \]  

\[ (2.49) \]

The above equation is solved numerically for various values of \( M_1, M_3 \) and \( \tau \) and the minimum value of \( x \) is obtained each time, hence the critical wave number is obtained. Using this in equation (2.47), we obtain the critical Rayleigh number, above which the convection sets in.

For very large \( M_1 \), the results for the magnetic mechanism operating in the absence of buoyancy effects is obtained. Thus, the corresponding magnetic Rayleigh number \( N \) is expressed as follows,

\[ N = RM_1 = \frac{\pi^4(1 + zM_3) \left[ \tau_1 + (1 + x)^2 \right]}{x^2M_3} \]  

\[ (2.50) \]

On proceeding exactly as above, for this case, a fourth degree equation is to be solved for \( x \),

\[ 2M_3x^4 + (3M_3 + 1)x^3 - [3 + M_3(1 + \tau_1)]x - 2(1 + \tau_1) = 0 \]  

\[ (2.51) \]

Thus, in this case an exact solution is obtained for \( x \) in terms of \( M_3 \) and \( \tau \). And for various values of \( M_3 \) and \( \tau \), the critical wave number and hence the critical magnetic Rayleigh number are obtained.
Next, the conditions for the onset of instability as oscillatory convection is analyzed. Substituting $\sigma = \sigma_1$, where $\sigma_1$ is a non-zero real number, in equation (2.42) and equating the real and imaginary parts of the corresponding equation, one obtains,

$$\frac{L_4}{L_2} = \sigma_1^2, \quad \frac{L_3}{L_1} = \sigma_1^2$$  \hspace{1cm} (2.52)

Using equations (2.43) to (2.46) in equation (2.52) and eliminating $\sigma_1$ we obtain $R = R^a$ as an explicit function of the parameters $a^2, \tau, M_1$ and $M_3$,

$$R^a = \frac{2(1 + P)(\pi^2 + a^2M_3)}{a^2 [\pi^2 + a^2M_3(1+M_1)]}\left[\frac{P^2\pi^2 + (\pi^2 + a^2)^2}{(1+P)^2}\right]$$  \hspace{1cm} (2.53)

Now, as $\sigma_1$ is real, oscillatory convection cannot occur if $\sigma_1^2 < 0$. Thus, this condition together with the equations (2.52) and (2.53) provides the fact that oscillatory instability can occur only if

$$0 < P < 1, \quad \tau_1 \geq \frac{(1 + P)}{(1 - P)}(1 + x)^3$$

or if,

$$P = 0, \quad \tau_1 \geq 1 + x - x^2 - x^3$$  \hspace{1cm} (2.54)

These conditions are exactly in the same form as that of the ordinary viscous fluids (Chandrasekhar [1961]), except that $P$ is defined here differently.

2.5.2 Numerical results and discussion.

Though Gupta and Gupta [1979] have studied this case, their analysis does not include any numerical results for stationary convection. For the oscillatory case they have analyzed the effect of $M_3$ alone with only one value of the Taylor number. As the aim of this study is to analyze the effect of rotation on the system, the numerical work had to be conducted for this case also. Thus, solving equation (2.49) for various values of the coefficients, and hence the physical conditions by
using the Maple software, the critical wave numbers are obtained. Substituting these in equation (2.48), the corresponding critical Rayleigh numbers are obtained. The obtained values are tabulated and plotted to show the effect of rotation on the instability. For all the cases, it is seen that as $\tau$ increases the critical stability parameter ($R_c$ or $N_c$) also increases. This shows that rotation has a stabilizing effect on the system. Though this result is analogous to that of the ordinary viscous fluid case, the rate at which the rotation stabilizes the system is of much lesser order in the case of ferromagnetic fluids. This result can be understood from observing the gradient of the curves drawn in figure (2.1) and (2.2). Graphs are plotted for the logarithmic (to the base 10) values of $R_1$ as a function of the logarithmic values of $\tau_1$, varying the other parameters $M_1$ and $M_3$. When the magnetization is linear, $M_3=1$, Figure (2.1) shows the effect of rotation on the instability for various values of $M_1$. $M_1=0$ case corresponds to the case when only the buoyancy forces are in effect. $M_1 \to \infty$ corresponds to the case when the buoyancy forces are negligible and only the magnetic forces contribute to the stability. It is seen that the latter case delays the convection more than the former one. For all other values of $M_1$, that is when both the mechanisms are in effect, it is seen that $R_c$ decreases with the increase in $M_3$. This shows that a coupling takes place between the magnetic and the buoyant mechanisms and moreover they act complimentary to each other. For $\tau = 0$, a relation between the Rayleigh number $R$ and the Rayleigh magnetic number $N$ was obtained by Finlayson [1970], indicating a tight coupling between them. However, such an explicit relation is not possible for all values of the parameters as well as for the present case with rotation.

For the case of negligible buoyancy, Figure (2.2) shows the effect of rotation on the onset of instability for various values of $M_3$. It is observed that as $M_3$ increases, that is as the magnetization becomes nonlinear, $N_c$ decreases, thus implying a
destabilizing effect due to the non-linearity in the magnetization. As $M_3 \to \infty$, the case reduces to that of the classical Rayleigh–Benard problem with rotation.

The values of the critical wave number and the critical stability parameter are tabulated for various values of $M_3=5$, 10 and $M_1=1$, 2, 10 and $\infty$ for a range of values of the Taylor number $\tau$ in Table (2.1). Results are similar to those already explained. It is also noted that as $M_3$ increases, $a_c$ decreases with increase in $\tau$. While, as $M_1$ increases, $a_c$ also increase with increase in $\tau$. This shows that the effect of rotation on the non-linearity of the magnetization is to decrease the wavelength of the disturbance while that on the magnetic forces is to increase the same. Comparing with the ordinary viscous fluid case ($M_1=0$), it is observed that the rate of stabilization due to rotation is less for the case of ferromagnetic fluids. It is also remarked that the results obtained, exactly agrees with that of Chandrasekhar [1961] in the limiting case when $M_1=0$ and with that of Finlayson [1970] in the limit of zero rotation ($\tau=0$).

2.6 Two rigid paramagnetic boundaries.

In this section, the problem is analyzed for the fluid confined between two paramagnetic boundaries. Only the case of stationary convection is considered here. Thus the eigenvalue problem given by equations (2.28) to (2.31) (with $\sigma = 0$), subject to the boundary conditions (2.33) has to be solved. Unlike the free boundary case, an exact solution is not possible. So, one has to adopt an approximate technique to obtain the critical stability parameters and the critical wave numbers. Chandrasekhar [1961] has proposed a variational technique for the ordinary viscous fluids and has obtained good results. Due to the complexity of the problem and the boundary conditions this method becomes very tedious and also computationally less cost efficient. Finlayson [1970], however, has followed a Galerkin technique
and has obtained good results. In this study, a Galerkin technique is used. A convergence is obtained for a third order expansion of the trial functions. Again the solution is found to be in good agreement with the limiting cases.

2.6.1 Solution for paramagnetic boundaries.

The set of equations (2.28) to (2.31), simplified for stationary convection can be conveniently represented in the matrix notation,

\[ L \cdot W = aR^{1/2} M \cdot W \quad (2.55) \]

where,

\[ W = [w, \zeta, \theta, \phi]^T \]

\[ L = \begin{bmatrix}
(D^2 - a^2) & -\tau^{1/2} D & 0 & 0 \\
-\tau^{1/2} D & -(D^2 - a^2) & 0 & 0 \\
0 & 0 & -(D^2 - a^2) & 0 \\
0 & 0 & D & -(D^2 - a^2 M_3) \\
\end{bmatrix} \]

\[ M = \begin{bmatrix}
0 & 0 & (1 + M_1) & -M_1 D \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \]

This is to be solved with respect to the boundary conditions, equation (2.33). The set of equations is invariant under the transformation \( z \to -z, \ w \to w, \ \zeta \to -\zeta, \ \theta \to \theta \) and \( \phi \to -\phi \), so that the power series solution contains only odd powers of \( z \) in the vorticity and magnetic potential and even powers of \( z \) for velocity and temperature. The solution is thus assumed as a power series in the Galerkin functions as,
\[ w = \sum_{i=1}^{n} A_i (z^2 - \frac{1}{4})^{i+1}, \quad \zeta = \sum_{i=1}^{n} B_i \{(z^2 - \frac{1}{4})\}^{2i-1}, \]
\[ \theta = \sum_{i=1}^{n} C_i (z^2 - \frac{1}{4})^i, \quad \phi = \sum_{i=1}^{n} D_i z^{2i-1} \quad (2.56) \]

The velocity, vorticity and temperature trial functions satisfy the boundary conditions (2.33), while the magnetic potential does not satisfy its boundary conditions. So, following Finlayson [1970], the boundary residual technique is used for these functions. The velocity, vorticity and the temperature equations are made orthogonal to each of the corresponding trial functions. For the magnetic potential the boundary residuals are added in the following way. Define the differential equation and boundary residuals as

\[ R_D = (D^2 - a^2 M_3)\phi - D\theta, \]
\[ R_B(\frac{1}{2}) = -\left(D\phi + \frac{a}{1 + \chi}\phi\right)_{z=\frac{1}{2}}, \]
\[ R_B(-\frac{1}{2}) = +\left(D\phi - \frac{a}{1 + \chi}\phi\right)_{z=-\frac{1}{2}} \quad (2.57) \]

The combined inner product is then set to zero,

\[ \langle \phi_j R_D \rangle + \phi_j(\frac{1}{2})R_B(\frac{1}{2}) + \phi_j(-\frac{1}{2})R_B(-\frac{1}{2}) = 0, \]

where, \( \langle f \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} f \, dz \). These manipulations give us a set of homogeneous linear equations in the expansion coefficients of the trial functions. Clearly, this set of equations has a non-trivial solution if and only if the determinant of the coefficients' vanishes. This leads to an equation of the type \( F(R, a^2, M_3, M_1, \tau) = 0 \). This equation is solved for the minimum value of \( R \) with respect to the wave number, for various values of \( M_3, M_1, \tau \). This value is the required critical stability parameter and the corresponding disturbance is the critical wave number.
2.6.2 Numerical results and discussion.

Galerkin method was applied as explained above. The convergence is obtained by using a third order Galerkin expansions of the trial functions. For the buoyancy problem with no magnetic field ($M_1=0$), the successive approximations to the critical Rayleigh number are seen to be 1749.9757, 1708.5498, 1707.7622, compared to the exact solution of 1707.762 (Chandrasekhar [1961]) for the case of no rotation. For the case of $\tau=10$ the approximations 1754.8553, 1712.9440, 1712.5212 compared to the variational solution of 1713.0 obtained by Chandrasekhar [1961] with a second order approximation. For $\tau=0$, $\chi=0$ and $M_1=1$ the critical Rayleigh magnetic number converges to 2588.8036, 2569.9023, 2568.6062 compared to the approximate value 2570 obtained by Finlayson [1970] and recently shown to be 2568.7 by Stiles and Kagan [1990].

Using $n=1$ in equation (2.56) and substituting in equation (2.55) and applying the usual procedure of normalizing a homogenous system of order six is obtained, whose coefficients are functions of $\tau$, $\chi$, $x(=a^2)$, $M_3$, $M_1$ and $R$. Following the same procedure as for the free boundary case the determinant is equated to zero and $R(\chi, M_1, M_3)$ is obtained. Minimizing this with respect to the square of the wave number, critical wave numbers and the corresponding critical stability parameters for various values of $\tau, M_1, M_3$ and $\chi$ are obtained. As the size of the determinant was small this procedure was possible without taking much computing time. Again this was carried out using the Maple software.

For higher order Galerkin approximations (two and three) the size of the determinant increases and so does the complexity of this process. For the two term approximation, a determinant of order ten with elements as functions of the various variables is obtained. $R(x)$, however, was obtained by equating the determinant to zero, keeping all other variables fixed. And this is minimized to obtain the critical
wave and Rayleigh numbers. By a small Maple procedure this process is repeated for other values of $M_1$, $M_3$, $\tau$ and $\chi$. It was observed that the critical wave numbers do not change considerably from the previous approximation, a result which was observed and explained by Finlayson [1970]. As shown earlier the convergence for the critical stability parameter was not satisfactory for this approximation. So, a three term expansion becomes a requirement. The determinant obtained for this approximation is of order fourteen and again the coefficients are functions of the variables. Here, it becomes impossible to solve it using the previous method. Using the information from the corresponding exact solution obtained for the free boundary case, the critical wave number is assumed to be the same and hence the critical stability parameter for various values of $M_1$, $M_3$, $\tau$ and $\chi$ is obtained using a procedure in Maple. It maybe worthy to mention that this method is better than the usual method, as this does not require an approximate range for the wave numbers and a solution procedure to be repeated for each of this wave numbers. Thus, to that extent this method is computationally efficient. The major drawback, however, is that this method cannot be put to use for problems which require a higher order approximation of the trial functions. A judicious choice of the trial functions could still facilitate its use.

Figure (2.3) shows the effect of rotation on the instability for zero susceptibility and for linear magnetization, by plotting $\tau$ (in fact logarithm of $\tau$) against the critical stability parameter, $R_c$ or $N_c$ as the case maybe. The curves are plotted for various values of $M_1$. It can be observed that as the rotation increases, the critical stability parameter also increases, leading to the conclusion that rotation stabilizes the system. The curve $M_1=0$, denotes the case of absence of magnetic forces and the curve $M_1 \to \infty$, depicts the case of absence of buoyant forces. The latter one is seen to delay the convection more than the former. Curves for other values of $M_1$
denote the combined effect of these forces and they help in understanding the tight coupling between the buoyant and the magnetic forces. It is also observed that the rotation affects the system to a marginal extent only after some significant value of the Taylor number \( \Gamma > 10^3 \). This result is similar to that observed for usual viscous fluid case, while the rate at which the system is affected by rotation is much lower for the ferromagnetic case. Figure (2.4) similarly helps in understanding the effects for the case of susceptibility \( \chi = 9999 \). Again, it is observed that the effect of rotation is to stabilize the system and both the magnetic and buoyant forces act complimentary to each other. Also a similar result on the rate of stabilization of rotation is seen.

Figure (2.5) shows the effect of rotation on the instability for zero magnetic susceptibility and for the case when only the magnetic forces are in effect. The curves are plotted for typical values of \( M_3 \). It is seen that increase in \( M_3 \) is destabilizing and again rotation is stabilizing. Similar to the free boundary case, it is expected that as \( M_3 \) tends to infinity the problem returns to that of the classical rotational Benard problem. Figure (2.6) shows the effect of rotation on the system for \( \chi = 9999 \), and for the case of negligible buoyancy. Curves are drawn for typical values of \( M_3 \). It is again seen that rotation is stabilizing and the non-linearity in magnetization is destabilizing.

Figure (2.7) shows the effect of the magnetic susceptibility on the instability of the system and the combined effect due to the rotation and magnetic susceptibility is seen. It is observed that as \( \chi \) increases the magnetic Rayleigh number also increases, thereby reinstating the stabilizing effect of rotation on the system. It is observed that for large values of Taylor number \( \Gamma > 10^5 \), the difference in the susceptibility can be ignored. This can be understood as, for large values of Taylor number, rotation plays a strong role in stabilizing the system so that the effect due
to the susceptibility is subdued.

Critical wave numbers and critical Rayleigh numbers are tabulated for $M_3 = 10$ and for various values of $M_1$, $\tau$ and $\chi$ in Table 2.2. It is observed that as $M_1$ increases $a_c^2$ also increases with increase in $\tau$; as $\chi$ increases $a_c^2$ also increase with increase in $\tau$; and it can be seen that as $M_1$ increases the wave number increases while the critical Rayleigh number decreases, for a fixed $\tau$. This shows that the effect of $M_1$ in stabilizing the system is more than that of rotation. This is true only for smaller values of Taylor number. For large values of Taylor number ($\tau > 10^5$), rotation tends to affect the stability more, subduing the other effects. Though this effect is not explicitly seen from the graphs, it is expected to behave in this fashion from the results for the free boundaries.

The results obtained are in good comparison with that of Chandrasekhar [1961] and Finlayson [1970] in the limiting cases of no magnetic forces and no rotation, respectively.

2.7 Two rigid ferromagnetic boundaries.

In this section, the instability analysis is conducted for the fluid confined between two flat, parallel ferromagnetic plates. The analysis is restricted for stationary convection and thus the eigenvalue problem (with $\sigma = 0$) given by the equations (2.28) to (2.31) subject to the boundary conditions, equation (2.34) has to be solved. Or in the matrix notation, equation (2.55) subject to the boundary conditions, equation (2.34) has to be solved. Gotoh and Yamada [1982] analyzed the similar problem without rotation. They used a Galerkin technique to obtain the critical stability parameters. Their way of non-dimensionalization allowed them to seek the trial functions in the form of Legendre polynomials. Here the trial functions are assumed in the form similar to that of the paramagnetic case discussed in the last
section; though, the trial functions are selected in such a way that they satisfy all the boundary conditions so that they satisfy the boundary conditions for ferromagnetic case. The solution converges for a third order expansion of the trial functions. The solution coincides with Gotoh and Yamada [1982] in the limit of zero rotation.

2.7.1 Solution for ferromagnetic boundaries.

The boundary conditions, allow us to choose the trial functions as follows,

\[ w = \sum_{i=1}^{n} E_i (z^2 - \frac{1}{4})^{i+1}, \quad \zeta = \sum_{i=1}^{n} F_i [z(z^2 - \frac{1}{4})]^{2i-1}, \]

\[ \theta = \sum_{i=1}^{n} G_i (z^2 - \frac{1}{4})^{i}, \quad \phi = \sum_{i=1}^{n} H_i [z(z^2 - \frac{1}{4})]^{2i-1} \quad (2.58) \]

Here, this choice of functions automatically satisfies all the boundary conditions.

Using these expansions in equation (2.55) and making them orthogonal to their corresponding trial functions we obtain a set of homogenous equations in the coefficients \( E_i, F_i, G_i \) and \( H_i \), where \( i \) varies from 1 to the order of approximation, \( n \). This set will have a non-trivial solution iff the determinant of its coefficients vanishes. This provides a equation, of the type, \( g(R, a^2, \tau,M_1,M_3) = 0 \). This is to be minimized for \( R \) with respect to the wave number to obtain the critical Rayleigh number and the corresponding wave number. In the limiting case of \( M_1 \rightarrow \infty \), the critical magnetic Rayleigh number was obtained. A convergence to the solution became apparent for a third order expansion of the trial functions.

2.7.2 Numerical results and discussion.

A Galerkin one term approximation is first carried out. As explained above, a determinant of order four with the elements as functions of the variables \( M_1,M_3,\tau,R \) and \( a = a^2 \) is obtained. This is expanded analytically and an explicit function for \( R \) in terms of the other variables is obtained as,

\[ R = \frac{28(z + 10)(zM_3 + 42) [(x + 42)(x^2 + 24x + 504) + 12\tau]}{9x(x + 42)[98M_1 + 126 + 3zM_3(1 + M_1)]} \quad (2.59) \]
This function is minimized with respect to $x$ and the critical values $a^2_c$ and $R_c$ are obtained for various values of $M_1$, $M_3$ and $\tau$. Such a explicit relation is worthwhile in a way that this suggests that the effect of rotation is to increase $R$, and thus to stabilize the system. This understanding cannot be conclusive as it is got at an intermediate step of an iterative process. Proceeding further, for the next iteration, the second order approximation of the trial functions is taken and the Galerkin type orthogonalization to the set of equations is carried out and a determinant of order eight, whose elements are functions of the various variables is obtained. So, a relation similar to that of the first approximation is difficult and the critical wave number using a Maple procedure for various values of the parameters is obtained. It is observed that the change in wave number is very minimal from that of the previous approximation. This result is consistent with that obtained by Gotoh and Yamada [1982]. Again as the convergence was not satisfactory, a third order expansion is carried out. The equations are made orthogonal to the corresponding trial functions and a homogeneous system of order twelve was obtained. The corresponding determinant, with its elements as functions of the parameters, is evaluated for various values of the parameters with the help of a procedure written with the use of Maple V. The critical wave numbers and the critical stability parameters is obtained for various values of $M_1$, $M_3$ and $\tau$. The effect of each of these parameters on the stability of the system is being analyzed.

Figure (2.8) brings out the effect of rotation on the instability of the system for linear magnetization. Curves are drawn for various values of $M_1$ with $\log_{10} \tau$ against $\log_{10} R_c$. For $M_1$ tending to infinity the values of the critical magnetic Rayleigh number $N_c$ are plotted in the same graph to show the effect of rotation on the stability parameters. $M_1 = 0$ curve is for the case when only the buoyant forces are in effect. Large $M_1$ case is when the magnetic forces subdue the buoyant forces
completely. It is seen that the latter case is more stabilizing than the former. In general it is observed that for all cases as \( \tau \) increases the critical stability parameter increases, suggesting the conclusion that rotation has a stabilizing effect on the system. For the cases when both buoyancy and magnetic forces are in effect the curves with various \( M_1 \) values provide with the observation that these forces are coupled and that they act complementary to each other, a result similar to those observed by both Finlayson [1970] and Gotoh & Yamada [1982].

Figure (2.9) shows the analysis for typical values of \( M_3 \) on the graph of \( \log_{10} \tau \) against \( \log_{10} N_c \). This shows the effect of magnetic forces on the stability. It is observed that as \( M_3 \) increases the critical magnetic Rayleigh number decreases. Thus, increase in the non-linearity of magnetization has a destabilizing effect on the system. Also it is expected that this effect will be pronounced only for low values of the Taylor number. For large values of Taylor number (\( \tau > 10^9 \)), the effect of rotation is expected to subdue these effects, a result though not shown explicitly in the figure, can be understood from the slope of the curves influencing the figure.

Critical wave numbers and critical stability parameters are tabulated for \( M_3 = 10 \), for various values of \( M_1 \) and \( \tau \) in Table (2.3). It is seen that \( a_e^2 \) and \( R_e \) increase with increase in \( \tau \) for a fixed \( M_1 \). Also it is observed that as \( M_1 \) increases \( a_e^2 \) increases, while \( R_e \) decreases for a given \( \tau \). It is again remarked that the results coincide with that of Gotoh and Yamada [1982] in the appropriate limits.

2.8 A comparative study for various boundaries.

In this section, the results obtained earlier are viewed with the perspective of understanding the effect due to the various boundaries on the instability of the system. For the non-rotating case, Gotoh and Yamada has obtained the result that the ferromagnetic boundary has a destabilizing effect in comparison with the
paramagnetic one. So such an analysis is all the more interesting when combined with the rotational effects.

Figure (2.10) shows the comparative study for the case of free boundaries, paramagnetic boundaries with $\chi=0$, paramagnetic boundaries with $\chi=9999$ and ferromagnetic boundaries. The graph is drawn for logarithm of $\tau$ against the logarithm of $R_c$, for $M_1=1$ and linear magnetization. It is seen that the paramagnetic boundary with a large susceptibility ($\chi=9999$) is most stable. The paramagnetic boundary with negligible susceptibility, though is not as stable as one with a large susceptibility, still delays the convection more than that by the ferromagnetic boundaries. Rotation again is seen to have a strong effect on stabilizing all the boundaries, with the rate of stabilization being maximum for the free boundaries, which is incidentally the least stable among those analyzed here.

Figure (2.11) shows the effect of $\tau$ on the critical magnetic Rayleigh number for linear magnetization case. The curve b1 denotes the free boundary, curve b2 denotes the ferromagnetic boundary, the curves b3 and b4 denote the paramagnetic boundaries with $\chi=0$ and $\chi=9999$ respectively. It is seen that for this linear magnetization case, the magnetization of the boundaries has a marginal effect on the instability. Again it is observed that the large susceptibility case is most stabilizing. Moreover, paramagnetic boundaries with zero susceptibility is more stabilizing than the ferromagnetic boundaries. Free boundaries, in comparison are the least stable.

Figure (2.12) depicts the case when the magnetization is non-linear with $M_2=10$, for the various boundaries. The graph is drawn for logarithm of $\tau$ against the logarithm of $R_c$. It is seen that the free boundaries are destabilizing in comparison with other boundaries. The figure shows that there is only little effect due to the magnetization of the boundaries in the stability of the system for this $M_2$. This
result can be explained thus, the effect of non-linearity of the magnetization is so strong that it subdues the effect arising out of the magnetization of the boundaries. From tables (2.2) and (2.3) it can, however be observed that for this case also the paramagnetic boundaries with large susceptibility delays the convection the most, followed by the paramagnetic boundaries with $\chi=0$ and then by ferromagnetic boundaries, the effect being very small. Thus, this explains the reason for the critical stability parameter to be almost the same for the various rigid boundaries considered, with $M_3 = 10$.

From Figure (2.13) it can be observed that the critical Rayleigh magnetic number does not change markedly for the cases of paramagnetic and ferromagnetic boundaries for $M_3 = 10$. The curves b2, b3 and b4 are seen to be very close throughout the interval considered. This is again due to the fact that as the magnetization in the fluid becomes non-linear it affects the stability so strong that the effects arising due to the susceptibility and the magnetization of the boundaries are subdued.
CHAPTER 3

HODOGRAPH TRANSFORMATION

METHODS FOR FERROMAGNETIC FLUIDS

3.1 Introduction.

It has been nearly three decades since the evolution of the physics of magnetic fluids. But still, solutions to even simple flow problems of magnetic fluids are scarce in general. Even among those available solutions, experimental investigations take a lead role. Theoretical investigations available are very selective in their approach and seem to analyze only particular cases under various approximations. This is mainly due to the complexity in the governing equations and boundary conditions of the flow of a magnetic fluid. As a magnetic fluid is structurally a complex medium, it becomes very difficult to obtain a well-defined theoretical model. It is all the more important that, for practical purposes, the model should include the most important features of the fluids while maintaining a reasonable amount of simplicity to be handled theoretically. The most commonly used model so far is the one proposed by Neuringer and Rosensweig [1964], because of its simplicity in structure. Application of this model, however, is limited as it totally neglects the effects of particle rotation. Shiblmos [1974], proposed a model which incorporates the particle rotation of the suspended magnetic particles. Bashtovoy et. al. [1988], explains the low frequency approximation, (see Chapter 1), which provides with a model, though simpler in structure maintains all the major effects of the physics

40
of the flow of these fluids. In this chapter we use this model to explain the flow of magnetic fluids.

Transformation techniques are often employed for solving non-linear partial differential equations. Hodograph transformations, (Ames [1965]), is one of those transformation techniques which has received considerable success in fluid mechanics amongst other areas. Chandna et al., [1982] first used this technique to solve viscous flow problems. On MHD flows there have been tremendous success by applying these transformation techniques. Swaminathan et al., [1983] applied this approach to transverse MHD fluid flows. Nguyen and Chandna [1990] used this approach to obtain non-newtonian MHD aligned flows. For a more detailed reference on these transformation techniques applied to MHD flows, refer to Nguyen [1987]. Second grade flows were solved using these techniques by Siddiqui et al., [1985]. Recently, Moro et al., [1990] has effectively used these transformations to solve plane flow problems of a third grade fluid.

In this chapter, hodograph and Legendre transformations are employed to study the flow problems in magnetic fluids. The flows considered are two dimensional, steady and incompressible with the magnetic field acting in the same plane as that of the flow. The dependent and independent variables are interchanged, a Legendre transform function of the stream function is introduced and all the equations are recast in terms of this transformed function. The conditions to be satisfied by this function are obtained. The equations are also obtained in polar coordinates and the corresponding conditions for the Legendre transform function are determined.

It should be understood that this approach is an inverse method in the sense that a form for the Legendre transform function is selected and then the conditions are used to analyze whether it is possible physically for such a flow to take place for magnetic fluids. Then, the stream function, velocity components and pressure
distribution for different magnetic fields are determined. Some simple applications are shown to highlight the effectiveness of this method. These applications are investigated for equilibrium magnetization case and for uniform magnetic field case.

### 3.2 Governing equations.

The flow of a ferromagnetic fluid under the low frequency approximation of the magnetization is governed by (Shilimos [1974], Chapter 1),

\[
\nabla \cdot \vec{q} = 0 \tag{3.1}
\]

\[
\rho_0 \frac{D\vec{q}}{Dt} = -\nabla p + \mu \nabla^2 \vec{q} + \mu_0 (\vec{M} \cdot \nabla) \vec{H} + \frac{\mu_0}{2} \nabla \times (\vec{M} \times \vec{H}) \tag{3.2}
\]

\[
\nabla \times \vec{H} = 0, \quad \nabla \cdot \vec{B} = 0 \tag{3.3a,b}
\]

\[
M_i = \chi H_i + \sum_{k=1}^{3} \chi_{ik} \frac{DH_k}{Dt}, \quad i = 1, 2, 3 \tag{3.4}
\]

where,

\[
\vec{B} = \mu_0 (\vec{M} + \vec{H}) \tag{3.5}
\]

\[
\chi_{ik} = \kappa_\perp \delta_{ij} - (\kappa_\parallel - \kappa_\perp) \frac{H_i H_k}{H^2} \tag{3.6}
\]

Considering the flow to be steady and assuming \(\vec{q} = (u, v, 0)\), \(\vec{H} = (H_1, H_2, 0)\) and \(\vec{M} = (M_1, M_2, 0)\), the governing equations can be written in the scalar form as,

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.7}
\]

\[
\rho_0 \left\{ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right\} = -\frac{\partial p}{\partial x} + \mu \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right\} + \mu_0 \left\{ M_1 \frac{\partial H_2}{\partial x} + M_2 \frac{\partial H_2}{\partial y} \right\} + \frac{\mu_0}{2} \left\{ \frac{\partial}{\partial y} (H_2 M_1 - H_1 M_2) \right\} \tag{3.8}
\]

\[
\rho_0 \left\{ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right\} = -\frac{\partial p}{\partial y} + \mu \left\{ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right\} + \mu_0 \left\{ M_1 \frac{\partial H_2}{\partial x} + M_2 \frac{\partial H_2}{\partial y} \right\} + \frac{\mu_0}{2} \left\{ \frac{\partial}{\partial x} (H_2 M_1 - H_1 M_2) \right\} \tag{3.9}
\]
\[ M_1 = \chi H_1 - \kappa_\perp \left( \frac{D H_1}{Dt} + \frac{H_2 \zeta}{2} \right) + (\kappa_{\parallel} - \kappa_\perp) \Phi H_1 \]  
(3.10)

\[ M_2 = \chi H_2 - \kappa_\perp \left( \frac{D H_2}{Dt} - \frac{H_1 \zeta}{2} \right) + (\kappa_{\parallel} - \kappa_\perp) \Phi H_2 \]  
(3.11)

\[ \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} = 0 \]  
(3.12)

\[ \frac{\partial}{\partial x} (H_1 + M_1) + \frac{\partial}{\partial y} (H_2 + M_2) = 0 \]  
(3.13)

where,
\[ \Phi = \frac{1}{2H^2} \frac{D(H^2)}{Dt}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \]  
(3.14)

Thus, a system of seven equations in seven unknowns namely, \( H_1, H_2, M_1, M_2, u, v \) and \( p \), is obtained.

Now, this system is simplified by introducing a modified energy function, \( h \) and the third component of vorticity, \( \zeta \). Thus introducing,

\[ h = \frac{\rho_0}{2} (u^2 + v^2) + p - \frac{\mu_0}{2} \chi H^2 - \frac{\mu_0}{2} (\kappa_{\parallel} - \kappa_\perp) H^2 \Phi \]  
(3.15)

Now, equation (3.12) implies that \( \vec{H} \) can be written as a gradient of a scalar function \( \phi \). Thus,

\[ H_1 = \frac{\partial \phi}{\partial x}, \quad H_2 = \frac{\partial \phi}{\partial y} \]  
(3.16)

Using equations (3.15) and (3.16), the system of equations (3.9) to (3.14), can be written as,

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  
(3.17)

\[ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \zeta \]  
(3.18)
\[ \frac{\partial h}{\partial x} = \rho_0 v \zeta - \mu \frac{\partial \zeta}{\partial y} - \frac{\mu_0}{2} (\kappa_\parallel - \kappa_\perp) \frac{\partial \phi}{\partial x} \left( \frac{\partial \phi}{\partial x} \right)^2 + \frac{\mu_0}{2} \frac{\partial \Psi}{\partial y} \]
\[ -\mu_0 \kappa_\perp \left\{ \nabla \left( \frac{\partial \phi}{\partial x} \right), [\vec q, \nabla](\nabla \phi) \right\} - \frac{\zeta}{2} \frac{\nabla \phi \times \nabla (\frac{\partial \phi}{\partial x})}{.h} \right\} \]
\[ (3.19) \]
\[ \frac{\partial h}{\partial y} = -\rho_0 u \zeta + \mu \frac{\partial \zeta}{\partial x} - \frac{\mu_0}{2} (\kappa_\parallel - \kappa_\perp) \frac{\partial \phi}{\partial y} - \frac{\mu_0}{2} \frac{\partial \Psi}{\partial x} \]
\[ -\mu_0 \kappa_\perp \left\{ \nabla \left( \frac{\partial \phi}{\partial y} \right), [\vec q, \nabla](\nabla \phi) \right\} - \frac{\zeta}{2} \frac{\nabla \phi \times \nabla (\frac{\partial \phi}{\partial y})}{.h} \right\} \]
\[ \left[ (1 + \chi) + (\kappa_\parallel - \kappa_\perp) \Phi \right] \nabla^2 \phi = -\kappa_\perp \left\{ \nabla \cdot [(\vec q, \nabla)(\nabla \phi)] + [\nabla \phi \cdot \nabla^2 \vec q] \right\} \]
\[ + (\kappa_\parallel - \kappa_\perp) \nabla \phi \cdot \nabla \Phi = 0 \]
\[ (3.20) \]

where,
\[ \Psi = -\kappa_\perp \left\{ \frac{\partial \phi}{\partial y} (\vec q, \nabla) \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial x} (\vec q, \nabla) \frac{\partial \phi}{\partial y} + \frac{\zeta}{2} |\nabla \phi|^2 \right\} \]
\[ (3.22) \]

Equations (3.17) to (3.21) are to be solved for \( u(x, y) \), \( v(x, y) \), \( \omega(x, y) \), \( \phi(x, y) \) and \( h(x, y) \). Now the function \( h \) is eliminated from equations (3.19) and (3.20) by using the integrability condition to obtain,
\[ \rho_0 \left[ u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} \right] - \mu \nabla^2 \zeta + \frac{\mu_0}{2} \nabla^2 \Psi - \mu_0 \kappa_\perp \left\{ J_i \left[ \frac{\partial \phi}{\partial x}, \frac{D}{D t} \left( \frac{\partial \phi}{\partial x} \right) \right] \right\} \]
\[ -\mu_0 \kappa_\perp \left\{ J_i \left[ \frac{\partial \phi}{\partial y}, \frac{D}{D t} \left( \frac{\partial \phi}{\partial y} \right) \right] + \frac{\partial \phi}{\partial y} J_i \left[ \frac{\partial \phi}{\partial x}, \frac{\zeta}{2} \right] + \frac{\partial \phi}{\partial y} J_i \left[ \frac{\zeta}{2}, \frac{\partial \phi}{\partial y} \right] \right\} \]
\[ -\mu_0 \kappa_\perp \left\{ \zeta J_i \left[ \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right] \right\} = 0 \]
\[ (3.23) \]

where,
\[ J_i[f, g] = \frac{\partial (f, g)}{\partial (x, y)} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y} \]
\[ (3.24) \]

Now the equations (3.17), (3.18), (3.21) and (3.23) form a set of four equations in four unknowns. Once these four variables \( u, v, \omega \) and \( \phi \) are obtained then from
equation (3.19) and (3.20), the energy function h and hence the pressure p can be obtained.

3.3 Equations in the Hodograph plane.

Now, assuming that the flow variables \( u = u(x, y) \), \( v = v(x, y) \) are such that, in the region of flow under consideration, the Jacobian,

\[
J = \frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \neq 0, \quad |J| < \infty
\]

This condition gives \( x, y \) as functions of \( u \) and \( v \), i.e., \( x = x(u, v) \) and \( y = y(u, v) \).

Thus, the following relations become apparent,

\[
\begin{align*}
\frac{\partial u}{\partial x} &= J \frac{\partial y}{\partial v} \\
\frac{\partial u}{\partial y} &= -J \frac{\partial x}{\partial v} \\
\frac{\partial v}{\partial x} &= -J \frac{\partial y}{\partial u} \\
\frac{\partial v}{\partial y} &= J \frac{\partial x}{\partial u}
\end{align*}
\]

(3.25)

Here, \( J \) is the Jacobian (transformed) as a function of \( u \) and \( v \).

\[
\begin{align*}
\frac{\partial f}{\partial x} : \frac{\partial (f, y)}{\partial (x, y)} &= J \frac{\partial (f, x)}{\partial (u, v)} \\
\frac{\partial f}{\partial y} &= \frac{\partial (x, f)}{\partial (x, y)} = J \frac{\partial (x, f)}{\partial (u, v)}
\end{align*}
\]

(3.26)

Here, it is understood that the function \( f \) is transformed into \( f(u,v) \). Similarly higher order derivatives are obtained. Use of these in the system of equations (3.17), (3.18), (3.21) and (3.23) provides the transformed system of equations in the \( u - v \) plane as,

\[
\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v} = 0
\]

(3.27)
\[
J \left[ \frac{\partial x}{\partial u} - \frac{\partial y}{\partial u} \right] = \zeta
\]
(3.28)

\[
\rho_0 \left\{ \bar{u} \bar{J} [\zeta, y] + \bar{v} \bar{J} [x, \zeta] \right\} - \mu \left\{ \bar{J} [\bar{J} [\zeta, y], y] + \bar{J} [x, \bar{J} [x, \zeta]] \right\}
+ \frac{\mu_0}{2} \left\{ \bar{J} [\bar{J} [\psi, y], y] + \bar{J} [x, \bar{J} [x, \psi]] \right\}
- \mu_0 \kappa_\perp \left\{ \bar{J} [\bar{J} [\phi, y], u \bar{J} [\bar{J} [\phi, y], y] + v \bar{J} [x, \bar{J} [x, \phi]]] \right\}
- \mu_0 \kappa_\perp \left\{ \bar{J} [\bar{J} [\phi, y], x \bar{J} [x, \phi]] + v \bar{J} [x, \bar{J} [x, \phi]] \right\}
- \mu_0 \kappa_\perp \left\{ \bar{J} [\bar{J} [\phi, y], y \bar{J} [\phi, y]] + \bar{J} [\phi, y] \bar{J} \left[ \frac{\zeta}{2}, \bar{J} [x, \phi] \right] \right\}
- \mu_0 \kappa_\perp \left\{ \zeta \bar{J} [\bar{J} [\phi, y], \bar{J} [x, \phi]] \right\}
+ \frac{\mu_0}{2} (\kappa_\parallel - \kappa_\perp) \bar{J} \left[ J^2 [\phi, y] + J^2 [x, \phi] \right], \Phi \right\] = 0
(3.29)

\[
[1 + \chi + (\kappa_\parallel - \kappa_\perp) \Phi] \left\{ \bar{J} [\bar{J} [\phi, y], y] + \bar{J} [x, \bar{J} [x, \phi]] \right\}
- \kappa_\perp \left\{ \bar{J} [u \bar{J} [\bar{J} [\phi, y], y] + v \bar{J} [x, \bar{J} [\phi, y]], y] \right\}
- \kappa_\perp \left\{ \bar{J} [x, u \bar{J} [\bar{J} [x, \phi], y] + v \bar{J} [x, \bar{J} [x, \phi]]] \right\}
- \kappa_\perp \left\{ \bar{J} \left[ \frac{\zeta}{2}, \phi \right], \bar{J} [x, y] \right\}
+ (\kappa_\parallel - \kappa_\perp) \left\{ \bar{J} [\phi, y] \bar{J} [\Phi, y] + \bar{J} [x, \phi] \bar{J} [x, \Phi] \right\} = 0
(3.30)

where,

\[
J = \frac{[\delta(x, y)]^{-1}}{[\delta(u, v)]}
\]

\[
\bar{J} [f, g] = \frac{\partial (f, g)}{\partial (u, v)},
\]
(3.31)

\[
\phi = \phi(x(u, v), y(u, v)) = \phi(u, v)
\]
\[ \Phi = \Phi(u, v) = \frac{J}{2 J^2 \{ J^2 [\phi, y] + J^2 [x, \phi] \}} \times \{ u J [J^2 [\phi, y] + J^2 [x, \phi], y] + v J [x, J^2 [\phi, y] + J^2 [x, \phi]] \} \] (3.32)

\[ \Psi = \Psi(u, v) = -J^2 \kappa \{ J [x, \phi] \{ u J [J J [\phi, y], y] + v J [x, J J [\phi, y]] \} \} \]

\[ + J^2 \kappa \{ J [\phi, y] \{ u J [J J [x, \phi], y] + v J [x, J J [x, \phi]] \} \} \]

\[ - J^2 \kappa \frac{\zeta}{2} \{ J^2 [\phi, y] + J^2 [x, \phi] \} \] (3.33)

Now, equation (3.17) together with equation (3.27) imply the existence of a stream function \( \psi(x, y) \) and the Legendre transform function \( L(u, v) \), such that

\[ d\psi = -v dx + u dy \] (3.34)

\[ dL = -y du + x dv \] (3.35)

which in turn give,

\[ \frac{\partial \psi}{\partial x} = -v, \quad \frac{\partial \psi}{\partial y} = u \] (3.35)

\[ \frac{\partial L}{\partial u} = -y, \quad \frac{\partial L}{\partial v} = x \] (3.36)

and the relation between \( \psi \) and \( L \),

\[ L(u, v) = vx - uy + \psi(x, y) \] (3.37)

Introducing, this Legendre transform function in the set of equations (3.27) to (3.30), it can be observed that equation (3.27) is automatically satisfied and the other three equations become,
\[ J \left[ \frac{\partial^2 L}{\partial u^2} + \frac{\partial^2 L}{\partial v^2} \right] = \zeta \quad (3.38) \]

\[ \rho_0 \left\{ u J \left[ \zeta, - \frac{\partial L}{\partial u} \right] + v J \left[ \frac{\partial L}{\partial v}, \zeta \right] \right\} - \mu \left\{ J \left[ J J \left[ \zeta, \frac{\partial L}{\partial u} \right], \frac{\partial L}{\partial u} \right] + J \left[ \frac{\partial L}{\partial v}, J J \left[ \frac{\partial L}{\partial v}, \zeta \right] \right] \right\} \]

\[ + \frac{\mu_0}{2} \left\{ J \left[ J J \left[ \phi, \frac{\partial L}{\partial u} \right], \frac{\partial L}{\partial u} \right] + J \left[ \frac{\partial L}{\partial v}, J J \left[ \frac{\partial L}{\partial v}, \phi \right] \right] \right\} \]

\[ - \mu_0 \kappa_\perp \left\{ J \left[ J J \left[ \phi, - \frac{\partial L}{\partial u} \right], u J J \left[ J J \left[ \phi, \frac{\partial L}{\partial v} \right], \frac{\partial L}{\partial u} \right] + v J J \left[ \frac{\partial L}{\partial v}, J J \left[ \phi, - \frac{\partial L}{\partial u} \right] \right] \right\} \]

\[ - \mu_0 \kappa_\perp \left\{ J \left[ J J \left[ \frac{\partial L}{\partial v}, \phi \right], u J J \left[ J J \left[ \frac{\partial L}{\partial v}, \phi \right], - \frac{\partial L}{\partial u} \right] + v J J \left[ \frac{\partial L}{\partial v}, J J \left[ \frac{\partial L}{\partial v}, \phi \right] \right] \right\} \]

\[ - \mu_0 \kappa_\perp \left\{ J J \left[ \frac{\partial L}{\partial v}, \phi \right], J \left[ J J \left[ \phi, - \frac{\partial L}{\partial u} \right], \frac{\partial L}{\partial u} \right] + J J \left[ \phi, - \frac{\partial L}{\partial u} \right], J J \left[ \frac{\partial L}{\partial v}, \phi \right] \right\} \]

\[ - \mu_0 \kappa_\perp \left\{ \zeta J \left[ J J \left[ \phi, - \frac{\partial L}{\partial u} \right], J J \left[ \frac{\partial L}{\partial v}, \phi \right] \right] \right\} \]

\[ + \frac{\mu_0}{2} \left( \kappa_\parallel - \kappa_\perp \right) J \left\{ J^2 \left[ \phi, - \frac{\partial L}{\partial u} \right] + J^2 \left[ \frac{\partial L}{\partial v}, J J \left[ \phi, - \frac{\partial L}{\partial u} \right], \frac{\partial L}{\partial u} \right] \right\} = 0 \quad (3.39) \]

\[ [1 + x + (\kappa_\parallel - \kappa_\perp) \Phi] \left\{ J \left[ J J \left[ \phi, - \frac{\partial L}{\partial u} \right], - \frac{\partial L}{\partial u} \right] + J \left[ \frac{\partial L}{\partial v}, J J \left[ \frac{\partial L}{\partial v}, \phi \right] \right] \right\} \]

\[ - \kappa_\perp \left\{ J \left[ u J J \left[ J J \left[ \phi, \frac{\partial L}{\partial u} \right], \frac{\partial L}{\partial u} \right] + v J J \left[ \frac{\partial L}{\partial v}, J J \left[ \phi, - \frac{\partial L}{\partial u} \right] \right], - \frac{\partial L}{\partial u} \right\} \]

\[ - \kappa_\perp \left\{ J \left[ \frac{\partial L}{\partial v}, u J J \left[ J J \left[ \frac{\partial L}{\partial v}, \phi \right], - \frac{\partial L}{\partial u} \right] + v J J \left[ \frac{\partial L}{\partial v}, J J \left[ \frac{\partial L}{\partial v}, \phi \right] \right] \right\} \]

\[ - \kappa_\perp \left\{ J \left[ \frac{\zeta}{2}, \phi \right] \right\} \]

\[ + (\kappa_\parallel - \kappa_\perp) \left\{ J J \left[ \phi, - \frac{\partial L}{\partial u} \right], J J \left[ \phi, - \frac{\partial L}{\partial u} \right], J J \left[ \phi, - \frac{\partial L}{\partial u} \right] \right\} = 0 \quad (3.40) \]

48
where,

\[
J = \left[ \frac{\partial(x,y)}{\partial(u,v)} \right]^{-1} = \left[ \frac{\partial^2 L}{\partial u^2} \frac{\partial^2 L}{\partial u \partial v} - \left( \frac{\partial^2 L}{\partial u \partial v} \right)^2 \right]^{-1}
\]  

(3.41)

\[
\bar{J}[f,g] = \frac{\partial(f,g)}{\partial(u,v)} = \left[ \frac{\partial f}{\partial u} \frac{\partial g}{\partial v} - \frac{\partial f}{\partial v} \frac{\partial g}{\partial u} \right]
\]

\[
\Phi = \frac{J}{2J^2 \left\{ \bar{J} \left[ \phi, -\frac{\partial L}{\partial u} \right] + \bar{J} \left[ \frac{\partial L}{\partial v}, \phi \right] \right\} \times

\bar{J} \left[ \phi, -\frac{\partial L}{\partial u} \right] + \bar{J} \left[ \frac{\partial L}{\partial v}, \phi \right] \right\}, -\frac{\partial L}{\partial u} \right]}

\bar{J} \left[ \phi, -\frac{\partial L}{\partial u} \right] + \bar{J} \left[ \frac{\partial L}{\partial v}, \phi \right] \right\} \right)
\]

(3.42)

and,

\[
\Psi = \Psi(u,v) = -J^2 \kappa \frac{\zeta}{2} \left\{ \bar{J} \left[ \phi, -\frac{\partial L}{\partial u} \right] + \bar{J} \left[ \frac{\partial L}{\partial v}, \phi \right] \right\}

- J^2 \kappa \left\{ \bar{J} \left[ \frac{\partial L}{\partial v}, \phi \right] \left( \bar{J} \left[ \phi, -\frac{\partial L}{\partial u} \right], -\frac{\partial L}{\partial u} \right) + \bar{J} \left[ \frac{\partial L}{\partial v}, \phi \right] \right\}

+ J^2 \kappa \left\{ \bar{J} \left[ \phi, -\frac{\partial L}{\partial u} \right] \left( \bar{J} \left[ \phi, -\frac{\partial L}{\partial u} \right], -\frac{\partial L}{\partial u} \right) + \bar{J} \left[ \frac{\partial L}{\partial v}, \phi \right] \right\}
\]

(3.43)

Thus, summing up, "if \( \phi(u,v) \) is the transformed magnetic potential and \( L(u,v) \) is the Legendre transform of a stream function in the hodograph plane for a plane, steady incompressible flow of a magnetic fluid, then \( L(u,v) \) and \( \phi(u,v) \) should satisfy the equations (3.39) and (3.40) ".

Now, proceeding further, the equations in the polar coordinates \((q, \Theta)\) in \((u,v)\) plane are obtained as follows. For this transformation define,

\[
u + iv = qe^{i\Theta}
\]

(3.44)
This implies that the following relations are true,

\[
\frac{\partial}{\partial u} = \cos \Theta \frac{\partial}{\partial q} - \frac{\sin \Theta}{q} \frac{\partial}{\partial \Theta},
\]

\[
\frac{\partial}{\partial v} = \sin \Theta \frac{\partial}{\partial q} + \frac{\cos \Theta}{q} \frac{\partial}{\partial \Theta}.
\]

(3.45)

Similar relationships are obtained for higher order derivatives also. Also it is known that,

\[
\tilde{J}[f, g] = \frac{1}{q} \tilde{J}^*[f, g]
\]

(3.46)

where,

\[
\tilde{J}^*[f, g] = \frac{\partial (f, g)}{\partial (q, \Theta)}
\]

(3.47)

Again here it is remarked that the functions \(f, g\) are appropriately transformed into their corresponding functions of \(q\) and \(\Theta\). Thus, the equations (3.39) and (3.40) are obtained in the polar form as,

\[
\rho_0 \left\{ \cos \Theta \tilde{J}^* [\zeta, L_1] + \sin \Theta \tilde{J}^* [L_2, \zeta] \right\}
\]

\[
- \frac{\mu}{q} \left\{ \tilde{J}^* \left[ \frac{J^*}{q} \tilde{J}^* [\zeta, L_1], L_1 \right] + \tilde{J}^* \left[ L_2, \frac{J^*}{q} \tilde{J}^* [L_2, \zeta] \right] \right\}
\]

\[
\frac{\mu_0}{2q} \left\{ \tilde{J}^* \left[ \frac{J^*}{q} \tilde{J}^* [\Psi, L_1], L_1 \right] + \tilde{J}^* \left[ L_2, \frac{J^*}{q} \tilde{J}^* [L_2, \Psi] \right] \right\}
\]

\[
- \frac{1}{q} \mu_0 \kappa_{\perp} \left\{ \tilde{J}^* \left[ \frac{J^*}{q} \tilde{J}^* [\phi, L_1], L_3 \right] \right\}
\]

\[
- \frac{1}{q} \mu_0 \kappa_{\perp} \left\{ \tilde{J}^* \left[ \frac{J^*}{q} \tilde{J}^* [L_2, \phi], L_4 \right] \right\}
\]

\[
- \frac{1}{q} \mu_0 \kappa_{\perp} \left\{ \tilde{J}^* \left[ \frac{J^*}{q} \tilde{J}^* [L_2, \phi], \frac{\zeta}{2} \right] \right\}
\]

\[
- \frac{\mu_0 \kappa_{\perp}}{q} \left\{ \tilde{J}^* \tilde{J}^* [\phi, L_1] \tilde{J}^* \left[ \frac{\zeta}{2}, \frac{J^*}{q} \tilde{J}^* [L_2, \phi] \right] \right\}
\]

50
\[-\frac{1}{q} \mu_0 \kappa_\perp \left\{ \zeta \tilde{J}^* \left[ \frac{J^*}{q} J^* [\phi, L_1], \frac{J^*}{q} J^* [L_2, \phi] \right] \right\} \]
\[+ \frac{\mu_0}{2q} (\kappa_{||} - \kappa_\perp) J^* \left[ \frac{J^*}{q^2} \left\{ J^* [\phi, L_1] + J^* [L_2, \phi] \right\} , \Phi \right] = 0 \quad (3.48)\]

and

\[\left[ 1 + \chi + (\kappa_{||} - \kappa_\perp) \Phi \right] \left\{ \tilde{J}^* \left[ \frac{J^*}{q} J^* [\phi, L_1], L_1 \right] + \tilde{J}^* \left[ L_2, \frac{J^*}{q} J^* [L_2, \phi] \right] \right\} \]
\[-\kappa_\perp \left\{ \tilde{J}^* [L_3, L_1] + J^* [L_2, L_4] + J^* \left[ \frac{\zeta}{2}, \phi \right] \right\} \]
\[+ (\kappa_{||} - \kappa_\perp) \left\{ J^* \tilde{J}^* [\phi, L_1] \tilde{J}^* [\Phi, L_1] + J^* \tilde{J}^* [L_2, \phi] \tilde{J}^* [L_2, \Phi] \right\} = 0 \quad (3.49)\]

where,

\[L_1 = -\cos \Theta \frac{\partial L}{\partial q} + \frac{\sin \Theta}{q} \frac{\partial L}{\partial \Theta} \]
\[L_2 = \sin \Theta \frac{\partial L}{\partial q} + \frac{\cos \Theta}{q} \frac{\partial L}{\partial \Theta} \]
\[L_3 = \cos \Theta J^* \tilde{J}^* \left[ \frac{J^*}{q} J^* [\phi, L_1], L_1 \right] + \sin \Theta J^* \tilde{J}^* \left[ L_2, \frac{J^*}{q} J^* [\phi, L_1] \right], \]
\[L_4 = \cos \Theta J^* \tilde{J}^* \left[ \frac{J^*}{q} J^* [L_2, \phi], L_2 \right] + \sin \Theta J^* \tilde{J}^* \left[ L_2, \frac{J^*}{q} J^* [L_2, \phi] \right] \]

are defined for the convenience in notation.

\[J^* = q^4 \left[ q^2 \frac{\partial^2 L}{\partial q^2} \left( \frac{\partial L}{\partial q} + \frac{\partial^2 L}{\partial \Theta^2} \right) - \left( \frac{\partial L}{\partial \Theta} - q \frac{\partial L}{\partial q} \frac{\partial L}{\partial \Theta} \right)^2 \right]^{-1} \quad (3.50)\]
\[\zeta = \zeta(q, \Theta) = J^* \left[ \frac{\partial^2 L}{\partial q^2} + \frac{1}{q^2} \frac{\partial^2 L}{\partial \Theta^2} + \frac{1}{q} \frac{\partial L}{\partial q} \right] \quad (3.51)\]

Similar to the relations in equations (3.42) and (3.43), the values of \( \Phi \) and \( \Psi \) are obtained as functions of \( (q, \Theta) \).
Thus, recapitulating, "if $L(q, \Theta)$ is the Legendre transform function of a stream function in polar coordinates and $\phi(q, \Theta)$ is the corresponding magnetic potential, for a plane steady incompressible flow of a magnetic fluid, then they have to satisfy, equations (3.48) and (3.49) simultaneously".

Once a value of $L$ is determined, the form of $\phi$ can be solved from the above equations. Moreover, employing the equations (3.44), (3.45) and

$$
y = -\cos \Theta \frac{\partial L}{\partial q} + \sin \Theta \frac{\partial L}{\partial \Theta}, \quad x = \sin \Theta \frac{\partial L}{\partial q} + \frac{\cos \Theta}{q} \frac{\partial L}{\partial \Theta},
$$

the velocity components $u = u(x, y)$ and $v = v(x, y)$ are obtained in the physical plane. Using them in the energy equation the energy function and hence the pressure is obtained.

3.4 Illustrations and discussion.

As some illustrations to the above method of solution, two major kinds of magnetic field are solved. The first one is when the Magnetization is always parallel to the magnetic field. As already mentioned this is the most studied model for magnetic fluids. The second one is when the external magnetic field is uniform, but the magnetization need not be parallel to the field, but is under the low frequency approximation. Both these cases are special cases of the problem studied above, which so far is completely general for any steady, two dimensional flow with the magnetic field acting in the plane of the flow.

Case A : Equilibrium magnetization.

Consider, that $\vec{M} \parallel \vec{H}$ in the flow field, thus from chapter 1 it is known that $\kappa_\parallel = \kappa_\perp = 0$. Using this, the equations (3.38), (3.39) and (3.40) take a simplified form of equations in the transformed hodograph plane. The equations for the Legendre transform function $L(u, v)$ and $\phi(u, v)$ are,
\[ J \left[ \frac{\partial^2 L}{\partial u^2} + \frac{\partial^2 L}{\partial v^2} \right] = \zeta \]  
(3.52)

\[ \rho_0 \left\{ u \tilde{J} \left[ \zeta, -\frac{\partial L}{\partial u} \right] + v \tilde{J} \left[ \frac{\partial L}{\partial v}, \zeta \right] \right\} 
- \mu \left\{ \tilde{J} \left[ \tilde{J} \left[ \zeta, \frac{\partial L}{\partial u} \right], \frac{\partial L}{\partial u} \right] + \tilde{J} \left[ \frac{\partial L}{\partial v}, \tilde{J} \left[ \frac{\partial L}{\partial v}, \zeta \right] \right] \right\} = 0 \]  
(3.53)

\[ \left\{ \tilde{J} \left[ \tilde{J} \left[ \phi, -\frac{\partial L}{\partial u} \right], -\frac{\partial L}{\partial u} \right] + \tilde{J} \left[ \frac{\partial L}{\partial v}, \tilde{J} \left[ \frac{\partial L}{\partial v}, \phi \right] \right] \right\} = 0 \]  
(3.54)

where, the notations are already explained. Also, for this case the energy function \( h(x,y) \) simplifies to (equation (3.15)),

\[ h(x, y) = \frac{\rho_0 u^2}{2} + p - \frac{\mu_0 \chi}{2} \left| \nabla \phi \right|^2 \]  
(3.55)

As an example, the solutions of a flow problem is obtained when the Legendre transform function is of the form,

\[ L(u, v) = f(u) + g(v); \quad f'''(u) \neq 0, \quad g'''(v) \neq 0 \]  
(3.56)

This implies,

\[ J = \frac{1}{f'''(u)g'''(v)}, \quad \zeta = \frac{1}{f''(u)} + \frac{1}{g''(v)} \]  
(3.57)

Using (3.56) and (3.57) in (3.53) and simplifying, it is seen that, the assumed form for \( L \) satisfies (3.53) identically only if (Swaminathan et al. [1983]),

\[ f'''(u) = \frac{1}{k_1(k_1 \nu - u)}, \quad g'''(v) = \frac{1}{k_1(v - k_1 \nu)} \]  
(3.58)

where \( k_1 \neq 0 \) is any real constant.
Employing equations (3.56) and (3.58) in equation (3.36) and solving for the velocity components one gets,

\[ u(x, y) = k_1 \nu - c^{k_1}(v+c_1), \quad v(x, y) = k_1 \nu + c^{k_1}(x-c_2) \]  

(3.59)

where \( c_1 \) and \( c_2 \) are arbitrary integration constants. And the third component of vorticity is obtained from (3.57) as,

\[ \zeta = k_1 \left[ e^{k_1(x-c_2)} + e^{k_1(y+c_1)} \right] \]  

(3.60)

Now using these in the linear momentum equations (3.19) and (3.20) (simplified for this case), it is seen that the energy function is given by,

\[ h(x, y) = \rho_0 k_1 \nu \left[ e^{k_1(x-c_2)} - e^{k_1(y+c_1)} \right] + \frac{\rho_0}{2} \left[ e^{2k_1(x-c_2)} + e^{2k_1(y+c_1)} \right] \]

\[ + \rho_0 e^{k_1(x-c_2+y+c_1)} + c_3 \]  

(3.61)

where \( c_3 \) is an arbitrary constant. Now, the pressure function can be obtained from the energy function \( h(x, y) \) in terms of the magnetic potential \( \phi \). In the physical coordinate system, the equation for \( \phi \) is its Laplacian. The pressure function is given as,

\[ p(x, y) = h(x, y) - \frac{\rho_0}{2} (u^2 + v^2) + \frac{\mu_0 \chi}{2} | \nabla \phi |^2 \]  

(3.62)

where, \( h(x,y) \) is given in equation (3.61), \( u, v \) have already been obtained in equation (3.59). However, the magnetic potential \( \phi \) is an arbitrary function satisfying the Laplace equation in the physical plane. For an example \( \phi = x^2 - y^2 \), is a probable magnetic potential for this flow to happen. Thus, for a magnetic fluid flow with the streamlines given by the family of curves,
\[ \frac{1}{k_1} \left[ e^{k_1(x-c_1)} + c^{k_1(y+c_1)} \right] + k_1 \nu(x - y) = \text{constant}. \]  
(3.63)

one possible solution set \( u(x, y), v(x, y), \phi(x, y) \) and \( p(x, y) \) has been obtained using the Legendre transform method and this set is given by,

\[ \phi = x^2 - y^2, \]
\[ u(x, y) = k_1 \nu - c^{k_1(y+c_1)}, \]
\[ v(x, y) = k_1 \nu + c^{k_1(x-c_2)} \]

and

\[ p(x, y) = h(x, y) - \frac{\rho_0}{2} (u^2 + v^2) + 2\mu_0 \chi(x^2 + y^2). \]

Case B. Uniform Magnetic field:

In this case it is assumed that an uniform magnetic field acts in the plane of the flow. Again this is a particular case for the equations derived in the last section. Though the field is uniform, it is not assumed that the magnetization is parallel to the magnetic field. Now, as the magnetic field is uniform, it is assumed,

\[ \phi = A_1 x + A_2 y \]
(3.64)

where \( A_1 \) and \( A_2 \) are arbitrary constants.

Substituting this in equations (3.42) and (3.43), it is obtained that,

\[ \Phi = 0, \quad \Psi = -\kappa \zeta (A_1^2 + A_2^2) \]
(3.65)

Using these, the equations in the Legendre transform function become,

\[ J \left[ \frac{\partial^2 L}{\partial u^2} + \frac{\partial^2 L}{\partial v^2} \right] = \zeta \]
(3.66)
\[
\rho_0 \left\{ u \bar{J} \left[ \zeta, -\frac{\partial L}{\partial u} \right] + v \bar{J} \left[ \frac{\partial L}{\partial v}, \zeta \right] \right\} \\
- \bar{\mu} \left\{ \bar{J} \left[ \bar{J} \left[ \zeta, \frac{\partial L}{\partial u} \right], \frac{\partial L}{\partial u} \right] + \bar{J} \left[ \frac{\partial L}{\partial v}, \bar{J} \left[ \frac{\partial L}{\partial v}, \zeta \right] \right] \right\} = 0 \quad (3.67)
\]

\[
A_2 \bar{J} \left[ \zeta, -\frac{\partial L}{\partial u} \right] - A_1 \bar{J} \left[ \frac{\partial L}{\partial v}, \zeta \right] = 0 \quad (3.68)
\]

where, \( \bar{\mu} = [\mu + \frac{\mu_n}{4}(A_1^2 + A_2^2)] \).

From this form it is observed that the governing equations are in the same form as that of the usual viscous fluid case, however, the coefficient of viscosity has an additional term due to the magnetization of the fluid. The additional term in viscosity, thus obtained, is similar to those obtained by Bashtovoy [1988] using a different approach. This fact of an apparent viscosity affecting the flow of a magnetic fluid has been studied earlier with the help of different theories and experiments (Bashtovoy [1988]).

The equation (3.67) is the same as that of the viscous fluid case, except that the viscosity here has some effects due to the magnetic susceptibility. Though it may seem that all the flows for the viscous fluids should hold true here also, such a conclusion is not valid because of the additional condition given by the equation (3.68). On the other hand, for those flows for which viscosity plays no role in the solution and equation (3.68) is satisfied, like circular flows, elliptic flows etc., it can be stated that the magnetic fluid flows with an uniform magnetic field are similar to the viscous flows. It is important to add the remark that, the pressure will have an extra term due to the uniform magnetic field as the energy function still possesses some extra terms.

Eventhough only very simple illustrations have been carried out here, it has provided a good understanding of the flow characteristics of magnetic fluid flows.
CHAPTER 4

CONCLUSIONS

In this thesis some stability and flow problems of magnetic fluids were analyzed.

Stability problems.

The effects of rotation on the thermo-convective instability of a horizontal layer of a magnetic fluid and heated from below in the presence of a uniform vertical magnetic field is studied for the fluid bounded by two stress free boundaries, two rigid paramagnetic boundaries and two rigid ferromagnetic boundaries. The results can be summarized as follows:

In general, the effect of rotation is to stabilize the system, that is, rotation helps to delay the onset of convection in the system.

The rigid paramagnetic boundaries with a large magnetic susceptibility is seen to make the system most stable in comparison to the other rigid boundaries and free boundaries considered here.

In general, rigid paramagnetic boundaries should be preferred to the ferromagnetic boundaries whenever a mechanism in uniform rotation requires the convection to be controlled.
The non-linearity of magnetization of the fluid reduces the influence of the magnetization of the rigid boundaries. Thus for any particular angular velocity of rotation and for a non-linear magnetization, the different boundaries considered here had only a negligible difference among them.

It was seen that when the system is under the influence of the magnetic forces alone, in the absence of buoyant forces, the system is most stable. That is, the convection process can be delayed the maximum when the rotating ferromagnetic fluid is very thin.

When the magnetic forces are negligible, the fluid acts as an ordinary viscous fluid and thus the effect is independent of the magnetization of the boundaries.

When both the magnetic and buoyant mechanisms affect the system, a tight coupling between them makes them to act complementarily to each other. Thus, an increase in magnetic force reduces the critical Rayleigh number but increases the critical magnetic Rayleigh number.

In general, the non-linearity in the magnetization of the fluid has a destabilizing effect on the system, the rate of which depends on the type of the bounding surfaces.

For the fluid bounded by rigid paramagnetic boundaries, the increase in the magnetic susceptibility delays the convection, in general.

In general, though rotation is known to have a stabilizing effect even for an ordinary viscous fluids, the rate at which it stabilizes is far less in the case of ferromag-
netic fluids. Thus to obtain a same amount of delay in a convection process, one has
to rotate a layer of ferromagnetic fluid faster than that of an ordinary viscous fluid.
Also, the rate at which the rotation stabilizes the system is strongly dependent on
type of the boundary, amongst others.

In general, for large values of the angular velocity, the effects due to all other
parameters become negligible in comparison with that of the strong stabilization
effect due to the rotation of the system.

It is qualitatively shown that oscillatory instability can never occur, for the non-
rotating case. Also, for the case when the layer is rotating, bounds have been
obtained for the instability to occur as oscillatory convection.

The numerical results obtained for the critical stability parameters and the crit-
ical wave numbers are compared with the available literature. For the limiting case
when \( M_1 = 0 \), these results, coincide with the results obtained by Chandrasekhar
[1961] for the ordinary viscous fluids. For the limiting case when \( \tau = 0 \), the nu-
umerical results obtained compare well with Finlayson [1970] for the paramagnetic
boundaries and with Gotoh and Yamada [1982] for the case of ferromagnetic bound-
daries.

**Flow problems using transformation techniques.**

Application of the hodograph transformation to obtain exact solutions for a few
steady, plane flows of magnetic fluids is considered. A low frequency approximation for the magnetization is used to model the magnetic fluid flow. The conditions to be satisfied by the magnetic potential and the Legendre transform function are obtained for both the rectangular coordinate system and the polar coordinate system of the hodograph plane.

Some simple examples are worked out to illustrate the developed theory. The effect of an apparent viscosity, observed earlier by others for the case of uniform magnetic field, is also explained with the help of this technique.
TABLE 2.1:
The critical stability parameters and their wave numbers for free boundaries.

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<th>$M_1 = 2$</th>
<th>$M_1 = 10$</th>
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<td>$R_c$</td>
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TABLE 2.2:
The critical stability parameters and their wave numbers for paramagnetic boundaries, with $M_3 = 10$.

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<th>$M_1 = 2$</th>
<th>$M_1 = 10$</th>
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</tr>
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TABLE 2.3 :
The critical stability parameters and their wave numbers for ferromagnetic boundaries, with $M_2 = 10$.

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<th></th>
<th>$M_1 = 10$</th>
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<th>$M_1 \to \infty$</th>
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<td>24.4261</td>
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FIGURE 2.1: EFFECT OF ROTATION ON THE CRITICAL RAYLEIGH NUMBER FOR FREE BOUNDARIES. (M₃ = 1)
FIGURE 2.2: EFFECT OF ROTATION ON THE CRITICAL MAGNETIC RAYLEIGH NUMBER FOR FREE BOUNDARIES (M_i→∞)
FIGURE 2.3: EFFECT OF ROTATION ON THE CRITICAL RAYLEIGH NUMBER FOR RIGID PARAMAGNETIC BOUNDARIES ($M_3 = 1, \gamma = 0$)
FIGURE 2.4: EFFECT OF ROTATION ON THE CRITICAL RAYLEIGH NUMBER FOR RIGID PARAMAGNETIC BOUNDARIES (M₂ = 1, x = 9999)
Figure 2.5: Effect of rotation on the critical magnetic Rayleigh number for rigid paramagnetic boundaries. ($M_i \rightarrow \infty$, $\gamma = 0$)
FIGURE 2.6: EFFECT OF ROTATION ON THE CRITICAL MAGNETIC RAYLEIGH NUMBER FOR RIGID PARAMAGNETIC BOUNDARIES ($M_r = \infty, \chi = 9999$)
FIGURE 2.7: EFFECT OF ROTATION ON THE CRITICAL MAGNETIC RAYLEIGH NUMBER FOR RIGID PARAMAGNETIC BOUNDARIES ($M_1=\infty, M_3=1$)
FIGURE 2.8: EFFECT OF ROTATION ON THE CRITICAL RAYLEIGH NUMBER FOR RIGID FERROMAGNETIC BOUNDARIES. ($M_3 = 1$)
FIGURE 2.9: EFFECT OF ROTATION ON THE CRITICAL MAGNETIC RAYLEIGH NUMBER FOR RIGID FERROMAGNETIC BOUNDARIES ($M_1 = \infty$)
INDEX FOR THE CURVES:

b1 : FREE BOUNDARIES
b2 : FERROMAGNETIC BOUNDARIES
b3 : PARAMAGNETIC BOUNDARIES, x = 0
b4 : PARAMAGNETIC BOUNDARIES, x = 9999

FIGURE 2.10: EFFECT OF ROTATION ON THE CRITICAL RAYLEIGH NUMBER FOR VARIOUS BOUNDARIES. (M1 = 1, M3 = 1)
INDEX FOR THE CURVES:

b1 : FREE BOUNDARIES
b2 : FERROMAGNETIC BOUNDARIES
b3 : PARAMAGNETIC BOUNDARIES, \( \alpha = 0 \)
b4 : PARAMAGNETIC BOUNDARIES, \( \alpha = 9999 \)

FIGURE 2.11: EFFECT OF ROTATION ON THE CRITICAL MAGNETIC RAYLEIGH NUMBER FOR VARIOUS BOUNDARIES. (\( M_1 \to \infty \), \( M_3 = 1 \))
INDEX FOR THE CURVES:

- b1: FREE BOUNDARIES
- b2: FERROMAGNETIC BOUNDARIES
- b3: PARAMAGNETIC BOUNDARIES, x = 0
- b4: PARAMAGNETIC BOUNDARIES, x = 9999

FIGURE 2.12: EFFECT OF ROTATION ON THE CRITICAL RAYLEIGH NUMBER FOR VARIOUS BOUNDARIES. (M1 = 1, M3 = 10)
INDEX FOR THE CURVES:

- b1: FREE BOUNDARIES
- b2: FERROMAGNETIC BOUNDARIES
- b3: PARAMAGNETIC BOUNDARIES, x = 0
- b4: PARAMAGNETIC BOUNDARIES, x = 9999

**Figure 2.13:** Effect of Rotation on the Critical Magnetic Rayleigh Number for Various Boundaries. ($M_1 \to \infty, M_3 = 10$)
REFERENCES


APPENDIX

BOUNDARY CONDITIONS FOR

THE MAGNETIC POTENTIAL

The boundary conditions for the perturbations in the velocity, the vorticity and
the temperature for the present problem are the same as that for the classical
Rayleigh - Benard problem with rotation (Chandrasekhar [1961]). However, for the
magnetic potential \( \phi \), a derivation of the boundary conditions is essential. Though,
Finlayson [1970], Stiles and Kagan [1990], have already obtained these conditions
for the sake of clarity it is again derived here. Physical considerations imply that the
tangential component of the magnetic field \( \vec{H} \) is continuous across the boundaries
\( z = \pm \frac{d}{2} \) and the normal component of the magnetic induction \( \vec{B} \) is continuous across
the boundaries \( z = \pm \frac{d}{2} \).

Inside the ferrofluid layer the convective perturbation to the magnetic potential
is assumed to be

\[ \phi'(z, t) = e^{\sigma t} \phi(z) \quad (A1) \]

Outside the fluid layer we take the perturbation to be given by

\[ \phi'(z, t) = \bar{\phi}(z)e^{\sigma t} \quad (A2) \]

Thus the continuity of the magnetic field at the boundaries gives

\[ \phi(\pm \frac{d}{2}) = \bar{\phi}(\pm \frac{d}{2}) \quad (A3) \]

The continuity of the vertical component of \( \vec{B} \) across the boundaries gives,

\[ D\bar{\phi}(\pm \frac{d}{2}) = (1 + \chi)D\phi(\pm \frac{d}{2}) - K\theta(\pm \frac{d}{2}) \quad (A4) \]
As the temperature perturbation vanishes on both the boundaries, equation (A4) simplifies to

$$D\bar{\phi}(\pm \frac{d}{2}) = (1 + \chi)D\phi(\pm \frac{d}{2})$$  \hspace{1cm} (A5)

Outside the fluid layer the magnetization is absent and hence,

$$\nabla^2 \phi' = 0$$  \hspace{1cm} (A6)

which on using equation (2.17) reduces to

$$(D^2 - k^2)\bar{\phi} = 0$$  \hspace{1cm} (A7)

Solving equation (A7) we get,

$$\bar{\phi}(z) = Ce^{kz} + De^{-kz}$$  \hspace{1cm} (A8)

where $C$ and $D$ are constants. Now using the fact that for $z > 0$, the magnetic potential $\bar{\phi}$ remains finite as $z \to \infty$, in equation (A8) we get, $C = 0$. Now using equations (A3) and (A5) at $z = \frac{d}{2}$, we get

$$(1 + \chi)D\phi(\frac{d}{2}) + k\phi(\frac{d}{2}) = 0$$  \hspace{1cm} (A9)

Similarly for $z < 0$, the magnetic potential $\bar{\phi}$ is finite as $z \to \infty$, we get $D = 0$.

Then using the equations (A3) and (A5) at $z = -\frac{d}{2}$, we get,

$$(1 + \chi)D\phi(-\frac{d}{2}) - k\phi(-\frac{d}{2}) = 0$$  \hspace{1cm} (A10)

The equations (A9) and (A10) provide us with the required boundary conditions for the magnetic potential $\phi$.

Now, these in the dimensionless variables take the form,

$$(1 + \chi)D\phi - a\phi = 0 \quad \text{at} \quad z = -\frac{1}{2}$$  \hspace{1cm} (A11)
\[(1 + \chi)D\phi + a\phi = 0 \quad \text{at} \quad z = +\frac{1}{2} \quad \text{(A12)}\]

Now, when \(\chi \rightarrow \infty\), the above two reduces to,

\[D\phi(\pm \frac{1}{2}) = 0 \quad \text{(A13)}\]

This is precisely the boundary conditions used by Finlayson [1970] for free boundaries and by Gupta and Gupta [1979]. Using similar arguments, Gotoh and Yamada [1982] has derived the boundary conditions for the case when the boundaries are electrically conducting. As the procedure is exactly in the same lines as above the derivation is not repeated here. However, in the limiting case of the boundaries being ferromagnetic, the conditions for the magnetic potential take the form,

\[\phi = 0 \quad \text{at} \quad z = \pm \frac{d}{2} \quad \text{(A14)}\]

This is the boundary condition used in the analysis for the ferromagnetic boundaries.
VITA AUCTORIS

NAME: Somasundhar Venkatasubramanian

PLACE OF BIRTH: Madurai, Tamilnadu, India.

YEAR OF BIRTH: 1969

EDUCATION:
University of Windsor, Windsor, Canada
Master of science in Mathematics, 1991-

Anna University, Madras, India

Madras University, Madras, India
Bachelor of Science in Mathematics, 1986-89.