Stability problems with inclined temperature gradients.

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STABILITY PROBLEMS WITH

INCLINED TEMPERATURE GRADIENTS

by

Zongchun Qiao

A Dissertation
submitted to the College of Graduate Studies and Research through the Department of Economics, Mathematics and Statistics in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the University of Windsor

Windsor, Ontario, Canada
1999
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0-612-52439-6
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Abstract

In this thesis, we study several stability problems related to mono-diffusive (thermal) convection, or double-diffusive (thermal-solutal) convection, in a porous medium and in a viscous fluid, induced by inclined temperature gradients with mass flow, or through-flow, or with surface tension.

The thesis begins with a brief outline of porous media, the Darcy model, and a brief exposition of linear and nonlinear stability methods. It also briefly discusses the compound matrix method and the Chebyshev tau method for solving eigenvalue problems.

Two chapters deal with the convection problems in a porous medium induced by inclined temperature gradients. Problems of horizontal mass flow or vertical through-flow are discussed, and the effect of including solutal gradient and mass flow is analyzed. In each case, an energy functional with coupling parameter(s) is chosen to establish the differential inequality, which gives the sufficient condition for the nonlinear stability of the basic steady solution. The associated variational problem is formulated from this condition. The compound matrix method with secant method and golden search routine is employed to numerically solve the resulting eigenvalue problem. Numerical results and comparisons are presented to show that a lower-order Galerkin approximation may not be accurate enough to predict the correct results, and that there may be a wide difference between the stability bounds computed by the linear stability theory and the energy stability theory.

The numerical solution of the inclined temperature gradient convection problem in a viscous fluid is also presented. The discussion includes the effect of horizontal mass flow. Both energy and linear stability methods are employed and the
compound matrix method with secant method and golden section search routine is used to solve the eigenvalue problems. This results in an eighth-order differential equation. The effect of mass flow-rate, Prandtl number, and horizontal temperature gradient on critical Rayleigh number are discussed. A comparison between the energy and linear stability results is also reported.

The linear stability problem of convection induced by inclined temperature gradient and surface tension in a viscous flow is also studied. The resulting eigenvalue problem, in this case, turns out to be complex and thus prohibits the use of the compound matrix method. The problem is solved by implementing the Chebyshev tau-QZ method. Numerical results and comparisons are presented to demonstrate that the implementation of the approach and algorithm proposed are correct and powerful. Moreover, this algorithm can also be used to numerically verify whether the principle of exchange of stability is valid or not.
IN MEMORY OF MY PARENTS
Acknowledgments

I would like to acknowledge foremost the constructive guidance of my supervisor, Professor P.N. Kaloni. Without his patience, knowledge and encouragements, this dissertation could not have been completed.

I am grateful to Professor A.S. Wineman, of University of Michigan, for his valuable comments, suggestions and clarification of a few points in this dissertation.

I wish to thank the members of my committee, Professor O.P. Chandna, Professor R. Barron and Professor L. Li, for their helpful comments and encouragement. I am especially grateful to Professor R. Barron for improving the presentation throughout the dissertation. I also wish to thank Professor R. Caron and Professor H. Atkinson, Heads of the Department of Economics, Mathematics and Statistics, for providing graduate assistantship and research facilities.

Postgraduate scholarship from Natural Science and Engineering Research Council (NSERC), and University of Windsor, are gratefully acknowledged.

Finally, I would like to thank Nan Zhang, my wife, and Katherine Qiao, our daughter, whose support and understanding was vital to the completion of my dissertation.
Nomenclature

$a_x$  
period in $x$ direction

$a_y$  
period in $y$ direction

$C$  
concentration in Chapter 4, an arbitrary constant in Chapter 5

$C_0$  
reference concentration in Chapter 4, a constant in Chapter 5

$D$  
differential operator, $\frac{d}{dz}$

$D^j$  
differential operator, $\frac{d^j}{dz^j}, j = 2, 3, 4$

$D$  
the first differentiation matrix defined in Chapter 5

$D^2$  
the second differentiation matrix defined in Chapter 5

$g$  
gravity

$H$  
layer height

$i$  
integer index, or imaginary number $i^2 = -1$

$k$  
unit vector in $z$ direction

$K$  
permeability tensor

$K$  
permeability

$L^2(-1, 1)$  
weighted Hilbert space \{ $f \mid \int_{-1}^{1} (1 - z^2)^{-\frac{1}{2}} f^2 dz < \infty$ \}

$P$  
pressure

$p$  
perturbation of pressure

$p_s$  
basic-state pressure

$q$  
horizontal mass flow rate

$Q$  
dimensionless horizontal mass flow rate

$Q_v$  
Péclet number

$R_E$  
critical vertical Rayleigh number in energy stability analysis
$R_H$  horizontal Rayleigh number

$R_L$  critical vertical Rayleigh number in linear stability analysis

$R_V$  vertical Rayleigh number

$s$  time growth rate

$s_r$  real part of time growth rate $s$

$s_i$  imaginary part time growth rate $s$

$t$  time

$T$  temperature

$T_s$  basic-state temperature

$u$  velocity component in $x$ direction

$u_0$  initial velocity

$u_s$  basic-state velocity vector

$u_z$  basic-state velocity component in $x$ direction

$v$  velocity component in $y$ direction

$v$  dimensionless perturbation of velocity

$v_z$  basic-state velocity component in $y$ direction

$w$  velocity component in $z$ direction

$w_z$  basic-state velocity component in $z$ direction

$\alpha$  overall wave number defined by $\alpha^2 = \alpha_x^2 + \alpha_y^2$

$\alpha_m$  thermal diffusivity

$\alpha_x$  wave number in $x$ direction

$\alpha_y$  wave number in $y$ direction

$\beta_T$  horizontal temperature gradient

$\gamma$  dimensionless perturbation of concentration
\( \gamma_T \) thermal expansion coefficient
\( \eta \) positive coupling parameter
\( \theta \) dimensionless perturbation of temperature
\( \kappa \) thermal diffusivity
\( \mu \) dynamic viscosity
\( \xi \) positive coupling parameter
\( \rho \) density
\( \rho_0 \) reference density
\( \sigma \) time growth rate in Chapter 1, surface tension in Chapter 5
\( \sigma_0 \) reference surface tension
\( \sigma_r \) real part of time growth rate \( \sigma \)
\( \sigma_i \) imaginary part time growth rate \( \sigma \)
\( \phi \) porosity
\( \mathcal{H} \) Hilbert space
\( \mathcal{L} \) linear operator
\( \mathcal{N} \) nonlinear linear operator
Chapter 1

Introduction

There exist a large number of studies dealing with mono-diffusive (thermal) convection, or double-diffusive (thermal-solutal) convection, in a porous medium and in a viscous fluid, bounded by two infinite horizontal layers and induced by a vertical temperature gradient; see, for instance, Chandrasekhar (1961), Drazin and Reid (1981), Straughan (1992), Nield and Bejan (1998), and other references therein. There are, however, very few papers that have dealt with the convection problems induced by a horizontal temperature gradient, or the more general situation of inclined temperature gradients. The latter problems are complicated due to the fact that the basic steady flow is no longer the rest state, and this invariably leads to the associated eigenvalue equations being generally complex. Most previous studies have been based on linear stability method with lower order Galerkin approximation, perturbation method, or shooting method; see, for instance, Weber (1973, 1974, 1978), Smith and Davis (1983), Nield (1990, 1991, 1994a,b, 1998), Parmentier et al. (1993), Nield et al. (1993), and Manole et al. (1994).

This field has attracted considerable research activity in recent years because of its wide range of applications from various thermal engineering designs to geophysical problems. Some specific areas of applications include the insulation of buildings and equipment, energy storage and recovery, geothermal energy extraction and reservoirs, dispersion of pollutants in the environment and in the chemical industry, material and food processing, underground transport of pollutants, circulation in planetary atmospheres, and the growth of metal crystals (Rudraiah et al. 1979, Straughan 1992, Nield and Bejan 1998).

In this thesis, we study several stability problems related to the convection induced by inclined temperature gradients using the nonlinear energy method as well
as the linear stability method. The associated eigenvalue problems are numerically solved by the compound matrix method or Chebyshev tau-QZ algorithm, with the secant method and golden section search routine. Before proceeding further with the details of each chapter, we give below a brief description of various topics relevant to the present work.

A porous medium generally is an extremely complicated network of channels and obstructions. In order to study the flow of fluids through porous media, it is necessary to clarify some properties of porous media and fluids. Below we give only some physical definitions pertinent to our study and then move on to a mathematical description.

In general terms a porous medium is a solid matrix containing voids or pores that are interconnected or unconnected and are dispersed randomly or in an ordered geometry. The solid matrix is either rigid or undergoes very small deformations. Porosity is defined as the ratio of pore volume to total volume of the poruous material. Permeability of porous media is defined as the ability to let fluid flow through its interconnected pore network. It is a measure of fluid conductivity. Historically, the study of the flow of viscous fluids through permeable materials began with Darcy's law (1856), which revealed a proportionality between flow rate and the applied pressure difference. It is the momentum equation in the porous medium analog of the Navier-Stokes equation. In modern notation and in one dimensional form this is expressed by

\[ u = -\frac{K}{\mu} \frac{\partial P}{\partial x} \]  \hspace{1cm} (1.1)

where \( u \) is the average velocity, \( \frac{\partial P}{\partial x} \) is the pressure gradient in the flow direction and \( \mu \) is the dynamic viscosity of the fluid. The coefficient \( K \) is independent of the fluid but dependent on the geometry of the medium, and is called the specific permeability or intrinsic permeability of the medium (abbreviated as permeability).
In three dimensions, the generalization of (1.1) is

$$u = -\mu^{-1} K \cdot \nabla p$$

(1.2)

where now the permeability $K$ is in general a second-order tensor. For the case of an isotropic medium the permeability is a scalar $K$. Also $u = (u, v, w)$ denotes the volume average macroscopic velocity of the fluid over a representative elementary volume. This quantity has been given various names by different authors, such as seepage velocity, filtration velocity, and volumetric flux density.

Darcy's law has been verified by the results of many experiments. Theoretical derivations for it have been obtained by many researchers. Darcy's law has been applied to a vast array of problems involving flow through porous media and has proven to be a reliable model for flow in an infinite porous material or in the interior of finite porous materials. For the details of discussion on Darcy's law, we refer to the book of Nield and Bejan (1998). In this thesis, Darcy's law is employed in the studies of stability problems of mono-diffusive and double-diffusive convection in a porous medium.

In stability theory there are two useful methods, namely, the linear stability method and the energy method. Both methods have been used extensively and, in fact, complement each other. In this thesis, both the energy and linear stability methods are used to deal with several stability problems related to the convection induced by inclined temperature gradients. We now give a brief introduction to these two methods.

In the energy stability method, an energy functional (or a generalized functional) is chosen to establish a differential inequality. Then sufficient conditions are found to ensure the nonlinear stability of the basic solution. This condition, in general, leads to a variational problem and to the determination of an eigenvalue problem. Numerical methods are finally used to solve the associated eigenvalue problem.

Let \( \mathcal{H} \) be a Hilbert space endowed with a scalar product \((\cdot,\cdot)\) and associated norm \(||\cdot|||\). We consider in \( \mathcal{H} \) the following initial value problem

\[
    u_t + \mathcal{L}u + \mathcal{N}(u) = 0, \quad u(0) = u_0
\]  

(1.3)

where \( \mathcal{L} \) represents a linear operator (possibly unbounded), and \( \mathcal{N} \) is a nonlinear operator with \( \mathcal{N}(0) = 0 \) in order that (1.3) admits a null solution. We assume:

(i) \( \mathcal{L} \) is a densely defined closed operator such that \((\mathcal{L} - \lambda \mathcal{I})^{-1}\) is compact for some complex number \( \lambda \) (\( \mathcal{I} \) is the identity operator in \( \mathcal{H} \)), that is, \( \mathcal{L} \) is an operator with compact resolvent.

(ii) The bilinear form associated with \( \mathcal{L} \) is defined (and bounded) in a space \( \mathcal{H}^* \), which is compactly embedded in \( \mathcal{H} \).

(iii) The nonlinear operator \( \mathcal{N} \) satisfies the condition

\[ (\mathcal{N}(u),u) \geq 0, \quad \forall u \in \mathcal{D}(\mathcal{N}), \]

where \( \mathcal{D}(\cdot) \) denotes the domain of the associated operator.

Because of (i), the following result is true (Kato 1976; p.185-187): The spectrum of the operator \( \mathcal{L} \) consists entirely of at most a denumerable number of eigenvalues
of \( \{\sigma_n\}_{n \in \mathbb{N}} \) with finite (both algebraic and geometric) multiplicities and, moreover, such eigenvalues can cluster only at infinity.

Since the operator \( \mathcal{L} \) is in general non-symmetric, the eigenvalues, which satisfy the equation

\[
\mathcal{L}\phi = \lambda\phi
\]

are not necessarily real. They may, however, be ordered in the following manner

\[
\text{Re}(\sigma_1) \leq \text{Re}(\sigma_2) \leq \cdots \leq \text{Re}(\sigma_n) \leq \cdots \tag{1.4}
\]

**Definition 1** The null solution of (1.3) is said to be linearly stable if and only if

\[
\text{Re}(\sigma_1) > 0 \tag{1.5}
\]

**Definition 2** The null solution of (1.3) is said to be nonlinearly stable if and only if for any \( \epsilon > 0 \) there is a \( \delta = \delta(\epsilon) \) such that

\[
\|u_0\| < \delta \Rightarrow \|u(t)\| < \epsilon \tag{1.6}
\]

and there exists a \( \gamma \) with \( 0 < \gamma \leq \infty \), such that

\[
\|u_0\| < \gamma \Rightarrow \lim_{t \to \infty} \|u(t)\| = 0 \tag{1.7}
\]

If \( \gamma = \infty \), we say that the null solution is unconditionally nonlinearly stable (or simply refer to it as being nonlinearly asymptotically stable), otherwise for \( \gamma < \infty \) the solution is conditionally stable. The value of \( \gamma \) is called the size of the attracting radius. When the operator \( \mathcal{L} \) is symmetric, linear stability always implies nonlinear stability, and the converse is also true (Davis 1969a, Galdi and Straughan 1985a).

The linear theory of hydrodynamic stability, however, deals with stability against infinitesimal disturbances (Lin 1966). Generally, the basic flow is assumed to be steady (equilibrium solution). The linear stability method consists of introducing
sinusoidal disturbances on the basic flow, which is the flow whose stability is to be investigated. The analysis can be made in terms of two-dimensional periodic waves of assigned wave numbers. Thus, we ascribe to all quantities describing the disturbance a dependence on \( x, y, \) and \( t \) of the form

\[
\exp(i(\alpha_x x + \alpha_y y) + \sigma t)
\]

where \( \sigma = \sigma_r + i\sigma_i \) is the complex time growth rate, and \( \alpha_x \) and \( \alpha_y \) are the wave numbers in the \( x \) and \( y \) direction, respectively. The method then leads to a linear system of ordinary differential equations with, in general, homogeneous boundary conditions. Existence of nontrivial solutions for such a system leads to an eigenvalue problem, and the behavior of the system to all possible real values of \( \alpha_x \) and \( \alpha_y \) is analyzed. This method, in which the disturbances are resolved into modes, is usually called the method of normal modes. If \( \sigma_r < 0 \) the disturbance to the basic flow is said to be asymptotically stable or stable, if \( \sigma_r = 0 \) the mode is said to be neutrally or marginally stable, and if \( \sigma_r > 0 \) it is said to be unstable. At marginal stability, if \( \sigma_r = 0 \) implies \( \sigma_i = 0 \), then we say that the principle of exchange of stability is valid. If, on the other hand, \( \sigma_i \neq 0 \) in the marginal state then we have an oscillatory mode of stability, sometimes also called overstability.

The approach mentioned above, which is used in this thesis, is actually called temporal stability. Nevertheless, it is worth noting that, within the linear stability theory, there is an alternative approach called spatial stability. In spatial stability calculations, \( \alpha_x \) and \( \alpha_y \) are complex, and \( \sigma \) is pure imaginary. If \( \text{Im}(\alpha_x) < 0 \) or \( \text{Im}(\alpha_y) < 0 \), then the mode grows in space. If such growing modes exist for any pure imaginary \( \sigma \) and for any orientation of disturbances, the flow is spatially unstable (Watson 1962, Nayfeh 1980, Cebeci and Stewartson 1980). The approximate relation between temporal and spatial stability analysis in hydrodynamic stability was discussed by Gaster (1962). An application of the spatial stability analysis with
the use of the Chebyshev method, to Poiseuille flow has been performed by Bramley and Dennis (1982), and Bridges and Morris (1984). It should be noted that, by means of spatial stability analysis, the resulting eigenvalue problem is usually nonlinear. Extra work is thus needed to cope with this difficulty, though this kind of nonlinear eigenvalue equation may be solved with the use of a companion matrix method, or factorization of the lambda matrix, or a transformation method (Moler and Stewart 1973, Bridges and Morris 1984, Pearssten and Goussis 1988).

Since, almost invariably, stability calculations involve determining eigenvalues and eigenfunctions, only a few of the associated eigenvalue problems are solvable analytically. Hence accurate and efficient numerical eigenvalue and eigenfunctions solvers are needed. The numerical calculations performed in this thesis are based on two techniques, namely, the compound matrix method and the Chebyshev tau-QZ method. These two methods are very accurate, and are designed to handle variable coefficients and to avoid round off error and the appearance of spurious eigenvalues. These methods, along with the secant method and golden section search method, are implemented to solve the resulting eigenvalue problems in the present thesis. Below we present a brief description of the two methods with a summary of the advantages and the disadvantages of each method.

Eigenvalue problems for ordinary differential equations are usually treated by first defining a solution matrix which satisfies certain prescribed initial conditions and the required eigenvalues are then obtained as the roots of some minor of the solution matrix. If an attempt is made to evaluate this minor by computing its elements separately, as in a standard shooting method, then there may be a serious loss of numerical accuracy especially when the differential equations are stiff. This difficulty can be avoided, however, by considering the differential equation satisfied by a certain compound matrix whose elements are the minors of the solution ma-
trix, and in this way we can compute the required minor directly. This method is called the compound matrix method designed to avoid round off error in solving stiff differential equations. This method is essentially an adaption of the standard shooting method in the context of linear eigenvalue problems for ordinary differential equations. Its general history may be found in Ng and Reid (1979a), and Drazin and Reid (1981) in which particular attention was paid to the well-known Orr-Sommerfeld equation describing the stability of plane Poiseuille motion. This method has been extended to an inhomogeneous fourth order equation in Ng and Reid (1979b), and to sixth order equations in Ng and Reid (1985). A lucid comprehensive account of this method has been given by Lindsay and Straughan (1992). The compound matrix method has been employed successfully in many eigenvalue problems related to stability problems (Straughan 1992).

One advantage of the compound matrix method is that it is particularly easy to change from problem to problem, usually requiring change in only one subroutine that contains the differential equations for the compound matrix variables. Also, different boundary conditions are easily incorporated with the compound matrix method. Since only one eigenvalue is tracked, the compound matrix method may be quicker if optimization in other parameters is required. However, the disadvantage of this method is that it is not suitable for some cases, particularly if the nature of the spectrum is unknown, or if the principle of exchange of stabilities is inoperative, or when the order of differential equations is too high (Lindsay and Ogden 1992 and Dongarra et al. 1996). The spectral methods play an invaluable role, and offer useful alternatives to the compound matrix method for solving the eigenvalue problems in these cases. The advantages of spectral methods include the production of a global solution, rapid convergence, and avoidance of the Gibbs phenomenon at domain boundaries (Gottlieb and Orszag 1977, Canuto et al. 1988).
A particularly useful spectral method is the Chebyshev tau method. The tau method is credited to Lanczos (1938, 1957), and was developed extensively by Fox (1962) and Fox and Parker (1968). Use of Chebyshev polynomials in hydrodynamic stability problems was advocated by Orszag, and the fundamental paper of Orszag (1971) has been a cornerstone in this field. Orszag (1971) and Lindsay and Ogden (1992) have shown that expansions in Chebyshev polynomials are better suited to the solutions of hydrodynamic stability problems than expansions in other sets of orthogonal functions.

Compared with the compound matrix method, the Chebyshev technique is especially easy to utilize to generate eigenfunctions. In addition, the Chebyshev technique approach is easy to extend to higher order systems. For problems where mode crossing is experienced (Abdullah and Lindsay 1991), i.e., the eigenvalue which is dominant in one area of parameter space is replaced by another in moving to some other domain of parameter space, then a technique like the Chebyshev one, which yields all eigenvalues, is vital. Finally, the Chebyshev tau method is particularly suitable for eigenvalue problems, in which the differential equations contain a singular term (Dongarra et. al. 1996).

Nevertheless, applications of spectral methods have been troubled by the emergence of spurious eigenvalues in situations in which the mathematical solution is known to be stable in the context of hydrodynamic stability (Gottlieb and Orszag 1977). These spurious eigenvalues appear even when a pseudo-Chebyshev-Galerkin approach (in which the basis functions satisfy the boundary conditions and the test functions are Chebyshev polynomials) is used (Zebib 1984). Recently many articles have been concerned with the removal of these spurious eigenvalues; see, for instance, Gardner et al. (1987), Zebib (1987), Pearlstein and Goussis (1988), Goussis and Pearlstein (1989), McFadden et al (1990), Lindsay and Ogden (1992),
Straughan and Walker (1996a), and Dongarra et al. (1996).

The contents of each chapter for this thesis are outlined below.

In Chapter 2, the energy method with a coupling parameter is applied to study the nonlinear stability problem of convection induced by an inclined temperature gradient with horizontal mass flow or vertical through-flow in a shallow horizontal layer of the porous medium. Both linear and nonlinear stability calculations are carried out by the compound matrix method and a comparison is made with the results obtained by previous authors. In addition, a rigorous mathematical proof is presented to show that there is no longitudinal oscillatory mode, in the case of horizontal mass flow.

Chapter 3 discusses the nonlinear stability of convection induced by inclined thermal and solutal gradients with mass flow, in a shallow horizontal layer of a porous medium. A nonlinear stability analysis is performed by the energy method. The resulting eigenvalue problem derived from the associated variational problem is numerically solved by the compound matrix method with golden section search method for the coplanar plane case.

In Chapter 4, we study the linear and nonlinear stability of convection, induced by inclined temperature gradient and horizontal mass flow in a viscous fluid. A nonlinear stability analysis is established, and the associated variational problem is formulated. For the purpose of comparison, computation is carried out for both the nonlinear and the linear stability theory Numerical results shows that there are considerable differences between the two theories.

In Chapter 5, we discuss the linear stability of convection in an infinite horizontal layer of fluid, induced by inclined temperature gradients with surface tension for a general case. We introduce and implement the Chebyshev tau-QZ method to solve the associated complex eigenvalue problem. Several matrices related to the Cheby-
shev polynomials are introduced for the purpose of easy implementation in coding. Numerical results and comparisons are presented to confirm that the approach and algorithm are correct. The approach and algorithm presented in this chapter are expected to be applied to many other stability problems. Moreover, it can be used to numerically verify whether the principle of exchange of stability is valid or not.

Finally, in Chapter 6, some general conclusions and directions for future work are outlined.
Chapter 2
Convection in a Porous Medium
Induced by Inclined Temperature Gradient with
Horizontal Mass Flow or Vertical Through Flow

2.1 Introduction.

Historically speaking, studies of thermal convection in a fluid-saturated porous layer, uniformly heated from below (which is analogous to Rayleigh-Bénard problem in a viscous fluid), was first investigated by Horton and Rogers (1945) and later, independently, by Lapwood (1948). Their linear theory indicates that, for the critical Rayleigh number \( R_a^c > 4\pi^2 \), the layer is convectively unstable to disturbances of small amplitude. The energy method has also been applied to this problem by Westbrook (1969), and Wankat and Schowalter (1970). The energy bound that gives sufficient conditions for stability, in this case, is identical to the linear bound. Thus, for \( R_a^c < 4\pi^2 \), the layer is stable for disturbances of arbitrary amplitude. This is not surprising, but an outcome of a general result since the linearized operator in this case is symmetric (Galdi and Straughan 1985a). The experimental results of Edler (1967) have confirmed these predictions. Thus, a fairly complete theoretical and experimental picture exists for this problem.

One extension to the Horton-Rogers-Lapwood problem is to consider the effects of net horizontal mass flow. In the Darcy model, if the basic flow is changed from rest to a uniform speed in the horizontal direction, the eigenvalue problem of linear stability analysis is then not altered if dispersion is negligible, since all the equations involved are invariant to a change to coordinate axes moving with that uniform speed. The stationary longitudinal modes are, therefore, favoured over other modes (Nield and Bejan 1998).

The effect of net mass flow with uniform speed in the vertical direction, however,
is more significant due to the change of temperature gradient from -1 to a nonlinear function. This kind of extension, similar to the Horton-Rogers-Lapwood problem is called the vertical through-flow. Wooding (1960) calculated linear stability bound for a semi-infinite layer with large through-flow when the upper surface was porous and submerged in a layer of liquid at constant temperature. Sutton (1970) presented results valid only for small through-flow rates. A detailed stability analysis by energy method and linear theory for both strong and weak mass flow, subject to constant velocity and temperature at upper and lower boundaries was discussed by Homsy and Sherwood (1976). Numerical calculations and Galerkin approximations of linear stability bound for the problem that was studied by Homsy and Sherwood, but subject to other different boundary conditions, have been given by Jones and Persichetti (1986), and Nield (1987), respectively.

In a real-world situation, however, strictly uniform heating does not occur. Generally, both horizontal and vertical temperature gradients are present simultaneously. Weber (1974) was the first author to consider linear instability of a convection problem in a porous medium, induced by an inclined temperature gradient. When considering the transition from stable ($\sigma_r < 0$) to unstable ($\sigma_r > 0$) solutions through a neutral state, characterized by $\sigma_r = 0$, where $\sigma_r$ is the real part of the time growth rate, we cannot generally prove that the principle of exchange stabilities is valid in this case, though it is true in the absence of a horizontal temperature gradient. However, by means of a perturbation method, Weber (1974) showed that when $\beta$, the horizontal temperature gradient, is small enough, the preferred mode of disturbance is the longitudinal stationary mode, and the physical explanation for it is that this particular mode minimizes the potential energy. The restriction of small $\beta$ was removed by Nield (1991) with the use of a second-order Galerkin approximation to solve the resulting eigenvalue problem. Nield's numerical results
showed that longitudinal oscillatory modes are never unstable (a rigorous proof given in Section 2.4 confirms this conclusion), and the oscillatory modes are never the favored ones for the onset of instability. Nevertheless, it was realized that the accuracy of the second-order Galerkin approximation was not satisfactory, particularly when $\beta$ increased considerably. Later, Nield (1994a) employed an eighth-order Galerkin approximation and found considerably improved results. In particular, he found that the values of the critical vertical Rayleigh number $R_V$, rather than increasing indefinitely with the increase of the horizontal Rayleigh number $R_H$, now reached a maximum and then decreased to the value zero. Nield applied this new result to predict that Hadley flow in a porous medium, when the circulation is sufficiently intense, becomes unstable even in the absence of an applied vertical temperature gradient. Nield’s result that the onset of instability is with the longitudinal stationary mode was confirmed by Straughan and Walker (1996a).

A more general extension of the Horton-Rogers-Lapwood problem is the convection induced by both inclined temperature gradients and mass flow. By means of the Galerkin approximation, Nield (1990) performed a linear stability analysis for the convection induced by an inclined temperature gradient with a net horizontal mass flow. As in the case of the absence of mass flow, Nield argued that the favored form of the disturbance is in the form of non-oscillatory longitudinal mode, which was further confirmed by Manole et al. (1994). The convection problem induced by an inclined temperature gradient with a vertical through-flow was investigated by Nield (1998). The nonlinear energy stability analysis for the convection induced by an inclined temperature gradient with horizontal mass flow, or with vertical through flow, have been studied by Kaloni and Qiao (1997a) and Qiao and Kaloni (1997, 1998).

In the present chapter we apply the energy method with a coupling parameter
to study the nonlinear stability problem of convection induced by an inclined temperature gradient with horizontal mass flow or vertical through-flow in a shallow horizontal layer of the porous medium. We carry out both linear and nonlinear stability calculations by the compound matrix method. In addition, we present a rigorous mathematical proof of the result that, in the linear stability analysis in case of horizontal mass flow, there is no longitudinal oscillatory mode. This reaffirms the result predicted by Nield (1990, 1991, 1994a), and confirmed by Straughan and Walker (1996a), based on their numerical calculations.

2.2 Basic Equations.

We assume a porous medium occupies a layer of height $H$. The vertical temperature difference across the boundaries is $\Delta T$. It is assumed that the flow in the porous medium is governed by Darcy's law, which is modified to add the gravitational term to the right-hand side of Darcy equation. The Boussinesq approximation is also assumed to be valid (Nield and Bejan 1998). The Cartesian axes are chosen with the $z^*$-axis vertically upwards and the $x^*$-axis in the direction of applied horizontal temperature gradient $\beta_T$. The superscript asterisks denote dimensional variables. Accordingly, the governing equations are:

1. $\nabla^* \cdot v^* = 0$, 
2. $\frac{\mu}{K} v^* + \nabla^* P^* = \rho_f g k$, 
3. $(\rho c)_m \frac{\partial T^*}{\partial t^*} + (\rho c)_f v^* \cdot \nabla^* T^* = k_m \nabla^* T^*$, 
4. $\rho_f^* = \rho_0 [1 - \gamma_T (T^* - T_0)]$.

Here $v^* = (u^*, v^*, w^*)$, $P^*$ and $T^*$ are the seepage (Darcy) velocity, pressure and temperature, respectively. The subscripts $m$ and $f$ refer to the porous medium and the fluid respectively. Also $\mu$, $\rho$ and $c$ denote viscosity, density and specific heat,
while $K$ is the permeability of the medium, $k_m$ is the thermal conductivity, $\gamma_T$ is the thermal expansion coefficient, $g$ is the gravitational acceleration, and $\mathbf{k}$ is the unit vector in the $z^*$-direction. We discuss two different situations, namely, the flow is subject to either a horizontal mass flow, or a vertical through-flow. The boundary conditions are:

**Case I:** There is a mass flow-rate $q$ along the $x^*$-direction:

$$w^* = 0, \quad T^* = T_0 - (\pm \Delta T/2) - \beta_T x^*, \quad \text{at} \quad z^* = \pm H/2,$$

$$\int_{-H/2}^{H/2} u^* \, dz^* = q. \quad (2.5)$$

Case II: There is a through-flow of velocity $w_v$ in the vertical direction:

$$w^* = w_v, \quad T^* = T_0 - (\pm \Delta T/2) - \beta_T x^* \quad \text{at} \quad z^* = \pm H/2,$$

$$\int_{-H/2}^{H/2} u^* \, dz^* = 0. \quad (2.8)$$

It should be noted that the mass flow condition (2.6), or (2.8) is necessary to obtain a basic steady solution (Nield 1990, 1998). We introduce the following non-dimensional quantities

$$x = x^*/H, \quad t = \alpha_m t^*/(AH^2), \quad v = Hv^*/\alpha_m, \quad Q = qH/\alpha_m, \quad Q_v = w_v H/\alpha_m,$$

$$P = K(\rho_0 g z^*)/(\mu \alpha_m), \quad T = R_V (T^* - T_0) / \Delta T,$$

$$\alpha_m = k_m/(\rho_0 c)_f, \quad A = (\rho c)_m/(\rho_0 c)_f,$$

$$R_V = \frac{\rho_0 g \gamma_T KH \Delta T}{\mu \alpha_m}, \quad R_H = \frac{\rho_0 g \gamma_T KH^2 \beta_T}{\mu \alpha_m}. \quad (2.9)$$

Here $Q$ is the non-dimensional net mass flow rate, $Q_v$ is the Péclet number, and $R_V, R_H$ are referred to the vertical Rayleigh number, and the horizontal Rayleigh number, respectively. Upon substitution of these non-dimensional variables into
Eqs. (2.1)-(2.3), and using (2.4) to eliminate $\rho_f^*$, the non-dimensional governing equations then take the form:

$$\nabla \cdot \mathbf{v} = 0,$$  \hspace{2cm} (2.10)

$$\mathbf{v} + \nabla P = T\mathbf{k},$$  \hspace{2cm} (2.11)

$$\frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla)T = \nabla^2 T.$$  \hspace{2cm} (2.12)

The non-dimensional boundary conditions are:

Case I:

$$w = 0, \quad T = -((\pm R_V/2) - R_H x) \quad \text{at} \quad z = \pm 1/2.$$  \hspace{2cm} (2.13)

Case II:

$$w = Q_v, \quad T = -((\pm R_V/2) - R_H x) \quad \text{at} \quad z = \pm 1/2.$$  \hspace{2cm} (2.14)

For Case I, the basic steady state solution $(u_s, T_s, p_s)$ of Eqs. (2.10)-(2.12) subject to the boundary condition (2.13) is:

$$u_s = R_H z + Q, \quad v_s = 0, \quad w_s = 0,$$  \hspace{2cm} (2.15)

$$T_s = -R_H x - R_V z + R_H^2 f(z) + R_H Q s(z),$$  \hspace{2cm} (2.16)

$$\nabla p_s = T_s k - u_s,$$  \hspace{2cm} (2.17)

where

$$f(z) = \frac{1}{24} (z - 4z^3), \quad s(z) = \frac{1}{8} (1 - 4z^2),$$  \hspace{2cm} (2.18)

and we have imposed the following non-dimensional form of mass flow condition (2.6)

$$\int_{-1/2}^{1/2} u_s dz = Q.$$  \hspace{2cm} (2.19)

For Case II, the basic steady state solution $(u_s, T_s, p_s)$ of Eqs. (2.10)-(2.12) satisfying the boundary condition (2.14) is:

$$u_s = R_H z, \quad v_s = 0, \quad w_s = Q_v.$$  \hspace{2cm} (2.20)
\[ T_s = -R_H x + f(z), \quad (2.21) \]
\[ \nabla p_s = T_s k - u_s, \quad (2.22) \]

where

\[ f(z) = \frac{R_H^2}{8Q_v} (4z^2 - 1) + \frac{R_H^2 z}{Q_v^2} - \frac{(R_H^2 + Q_v^2 R_V)}{2Q_v^2 \sinh(Q_v/2)} [\exp(Q_v z) - \cosh(Q_v/2)], \quad (2.23) \]

and the following non-dimensional form of through-flow condition (2.8) has been used

\[ \int_{-1/2}^{1/2} u_s \, dz = 0. \quad (2.24) \]

2.3 Nonlinear Stability Analysis.

We now perturb the solution as follows:

\[ v = u_s + u, \quad T = T_s + \theta, \quad P = p_s + p. \quad (2.25) \]

The perturbation equations then take the form

\[ \nabla \cdot u = 0, \quad (2.26) \]
\[ u + \nabla p = \theta k, \quad (2.27) \]
\[ \frac{\partial \theta}{\partial t} + (u \cdot \nabla) \theta = \nabla^2 \theta - u_s \cdot \nabla \theta - u \cdot \nabla T_s, \quad (2.28) \]

where \( u_s \) and \( T_s \) are given by Eqs.(2.15)-(2.16) for Case I, and by Eqs.(2.20)-(2.21) for Case II. The corresponding boundary conditions for both cases now become

\[ w = 0, \quad \theta = 0, \quad \text{at} \quad z = \pm 1/2. \quad (2.29) \]

As the linearized system of Eqs.(2.26)-(2.28) is not symmetric, the energy method will give a different result than the linear stability method (Straughan 1992). We define an energy functional as

\[ E(t) = \frac{\xi}{2} \| \theta \|^2, \quad (2.30) \]
where $\xi$ is a positive coupling parameter. We remark that the choice of this parameter is to make $R_E$, the critical vertical Rayleigh number in energy stability analysis, as large as possible. On multiplying Eq. (2.27) by $u$, Eq. (2.28) by $\theta$ and integrating over $V$, and using the boundary conditions and divergence theorem, we find

\[
\frac{1}{2} \frac{d}{dt} \|\theta\|^2 = -\|\nabla \theta\|^2 - < (u \cdot \nabla T_s) \theta >, \tag{2.31}
\]

\[
\|u\|^2 = < \theta w >. \tag{2.32}
\]

Here $V$ denotes a typical periodic cell, i.e., $V = [0, a_x] \times [0, a_y] \times [-\frac{1}{2}, \frac{1}{2}]$, where $a_x$ and $a_y$ are the periods of the periodic functions in $x$ and $y$ directions, respectively, $< \cdot >$ denotes the integration over $V$, and $\| \cdot \|$ denotes the $L^2(V)$ norm. The system of equation (2.31) and (2.32), along with (2.30), can be put in the form

\[
\frac{dE}{dt} = I - D, \tag{2.33}
\]

where

\[
I = -\xi < (u \cdot \nabla T_s) \theta > + < \theta w >, \tag{2.34}
\]

\[
D = \xi \|\nabla \theta\|^2 + \|u\|^2. \tag{2.35}
\]

We now define

\[
m = \max \frac{I}{D}, \tag{2.36}
\]

where $\mathcal{H}$ is the space of admissible solutions. On combining (2.33) with (2.34)-(2.36), and by using the Poincaré inequality (Straughan 1992)

\[
\|u\|^2 \leq \frac{1}{\pi^2} \|\nabla u\|^2
\]

we can infer, for $0 < m < 1$, that

\[
\frac{dE}{dt} \leq -2\pi^2 (1 - m)E. \tag{2.37}
\]
Inequality (2.37) clearly indicates that for \(0 < m < 1\), \(E(t) \to 0\) at least exponentially as \(t \to \infty\). Since \(E(t)\) in (2.30) does not contain the term \(\|u\|^2\), the kinetic energy term for the velocity, it is worthwhile checking as to what happens to \(\|u\|^2\) as \(t \to \infty\). Using the Cauchy-Schwartz inequality on (2.32) implies that

\[
\|u\|^2 \leq \|\theta\|^2. \quad (2.38)
\]

Thus (2.30) and (2.38) clearly indicate that decay of \(E(t)\) ensures the decay of \(\|u\|^2\).

### 2.4 Variational Equations and Eigenvalue Problem.

We now return to (2.36) and consider the maximum problem at the critical argument \(m = 1\). The associated Euler-Lagrange equations become

\[
-\xi \nabla \cdot \mathbf{u} + \omega + 2\xi \nabla^2 \theta = 0, \quad (2.39)
\]

\[
\xi \nabla \cdot \mathbf{u} - \nabla \omega = 0, \quad (2.40)
\]

where \(\omega\) is a Lagrange multiplier introduced since \(u\) is solenoidal. On taking \text{curl} of (2.40) and then taking the third component of resulting equation, we find

\[
2\nabla^2 \omega - \nabla \cdot (h \nabla \theta) + \xi R_H \frac{\partial^2 \theta}{\partial x \partial z} = 0, \quad (2.41)
\]

where \(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\), and

for Case I:

\[
h = 1 + \xi(R_V - R_H^2 f_1 + R_H Q z), \quad f_1 = \frac{1}{24}(1 - 12 x^2). \quad (2.42)
\]

for Case II:

\[
h = (1 - \xi f_1), \quad f_1 = \frac{R_H^2}{Q_V^2}(Q_V z + 1) - \frac{(R_H^2 + Q_V^2 R_V)}{2Q_V \sinh(Q_V/2)} \exp(Q_V z). \quad (2.43)
\]

Also Eqs. (2.39) and the \(x\)-component of (2.40) can be rewritten as

\[
\xi R_H u + h w + 2\xi \nabla^2 \theta = 0, \quad (2.44)
\]

\[
-\xi R_H \theta + 2u = \frac{\partial \omega}{\partial x}. \quad (2.45)
\]
In the case of linear stability analysis, it has been shown numerically by Nield (1990, 1991, 1994a), Manole et al. (1994), and Straughan and Walker (1996a) that a steady longitudinal mode is the most favorable mode of disturbance. Before proceeding further, we now present an analytical proof to show that there is actually no longitudinal oscillatory mode in the case of horizontal mass flow.

With the use of the standard normal mode analysis, we look for the solution of linearized form of Eqs. (2.26)-(2.28) subject to (2.29) in the form

\[ [u, w, \theta, \omega] = [u(z), v(z), w(z), \theta(z), p(z)] \exp [i(\alpha_x x + \alpha_y y) + \sigma t] \]  

(2.46)

The corresponding eigenvalue equations can, after eliminating the variables \( u(z), v(z) \) and \( p(z) \), be written as

\[ (D^2 - \alpha^2)w + \alpha^2 \theta = 0, \]

(2.47)

\[ (D^2 - \alpha^2 - i\alpha_x u_x - \sigma)\theta + i\frac{\alpha_x}{\alpha^2} R_H D w - D T_s w = 0, \]

(2.48)

where \( \alpha^2 = \alpha_x^2 + \alpha_y^2 \), and \( D = \frac{\partial}{\partial z} \). Eqs.(2.47)-(2.48) are solved subject to the boundary conditions as

\[ w = \theta = 0 \quad \text{at} \quad z = \pm \frac{1}{2} \]

(2.49)

From (2.47) and (2.49), we have

\[ D^2 w = 0 \quad \text{at} \quad z = \pm \frac{1}{2} \]

(2.50)

By eliminating \( \theta \), a single equation for \( w \) may then be obtained from Eqs.(2.47)-(2.48), to yield

\[ (D^2 - \alpha^2)^2 w - (i\alpha_x u_x + \sigma)(D^2 - \alpha^2)w - i\alpha_x R_H D w + \alpha^2 D T_s w = 0, \]

(2.51)

subject to the boundary condition on \( w \) of (2.49) and (2.50). Multiplying (2.51) by \( \bar{w} \) (the complex conjugate of \( w \)), and integrating by parts over \([-\frac{1}{2}, \frac{1}{2}]\), and using the boundary conditions, we obtain

\[ \| D^2 w \|^2 + 2\alpha^2 \| Dw \|^2 + \alpha^4 \| w \|^2 + \sigma (\| Dw \|^2 + \alpha^2 \| w \|^2) \]
\[ + i \alpha_x \int_{-\frac{1}{2}}^{\frac{1}{2}} u_s(|Dw|^2 + \alpha^2 |w|^2) dz + \alpha^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} DT_s |w|^2 dz = 0 \]  

(2.52)

where \( |w|^2 = w \bar{w} \), and

\[
\|D^j w\|^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |D^j w|^2 dz \quad j = 0, 1, 2,
\]

and where we have used the result \( Du_s = R_H \) from Eq.(2.15), and

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} u_s D^2 w \bar{w} dz = - \int_{-\frac{1}{2}}^{\frac{1}{2}} Dw(Du_s \bar{w} + u_s D \bar{w}) dz
\]

\[
= - \int_{-\frac{1}{2}}^{\frac{1}{2}} R_H Dw \bar{w} dz - \int_{-\frac{1}{2}}^{\frac{1}{2}} u_s |Dw|^2 dz.
\]

We then take the imaginary part of (2.52) to deduce

\[
\sigma_i(\|Dw\|^2 + \alpha^2 \|w\|^2) = -\alpha_x \int_{-\frac{1}{2}}^{\frac{1}{2}} u_s (|Dw|^2 + \alpha^2 |w|^2) dz,
\]

(2.53)

where \( \sigma = \sigma_r + i \sigma_i \). With the use of mean-value theorem for integrals on (2.53), we have that there exists a \( z_0 \in [-\frac{1}{2}, \frac{1}{2}] \), such that

\[
\sigma_i(\|Dw\|^2 + \alpha^2 \|w\|^2) = -\alpha_x u_s(z_0) \int_{-\frac{1}{2}}^{\frac{1}{2}} (|Dw|^2 + \alpha^2 |w|^2) dz,
\]

(2.54)

or

\[
\sigma_i(\|Dw\|^2 + \alpha^2 \|w\|^2) = -\alpha_x (R_H z_0 + Q)(\|Dw\|^2 + \alpha^2 \|w\|^2)
\]

(2.55)

We thus have

\[
\sigma_i = -\alpha_x (R_H z_0 + Q)
\]

(2.66)

Obviously,

\[
\alpha_x = 0 \quad \implies \quad \sigma_i = 0
\]

(2.67)

Thus, from (2.67), we reach the conclusion that there is no oscillatory longitudinal mode.
Further, we remark that if $R_H = 0$, from (2.66), we have

$$\sigma_i = -\alpha_z Q. \quad (2.68)$$

Due to the symmetric property, we can deduce that (2.68) also hold if $Q$ is replaced by $-Q$, and thus we have

$$\sigma_i = \alpha_z = 0 \quad (2.69)$$

This proves the conclusion that the stationary longitudinal mode is the only possible mode for the convection induced by horizontal mass flow which was drawn by Nield and Bejan (1998).

From now on, we thus restrict our attention to the steady longitudinal mode. By performing the standard normal mode analysis and looking for the solution of (2.41), (2.44) and (2.45) in the form

$$[u, w, \theta, \omega] = [u(z), w(z), \theta(z), \omega(z)] \exp [i\alpha y] \quad (2.70)$$

we derive

$$D^2w(z) = h_1 w(z) + h_2 \theta(z), \quad (2.71)$$

$$D^2\theta(z) = h_3 w(z) + h_4 \theta(z), \quad (2.72)$$

where $D = d/dz$, and the coefficients $h_1, \cdots, h_4$, are given by

$$h_1 = \alpha^2, \quad h_2 = -\frac{\alpha^2}{2} h, \quad h_3 = -\frac{h}{2\xi}, \quad h_4 = \alpha^2 - \frac{\xi}{4} R^2_H, \quad (2.73)$$

and in which $h$ is defined in (2.42) for Case I, or in (2.43) for Case II. The relevant boundary conditions are

$$w = \theta = 0 \quad \text{at} \quad z = \pm 1/2. \quad (2.74)$$
We consider $R_V$ as the eigenvalue with the remaining variables as parameters. The critical vertical Rayleigh number is defined by

$$R_E = \max_{\xi} \min_{\alpha^2} R_V(R_H, \alpha^2, \xi, Q) .$$

(2.75)

For Case II:

$$R_E = \max_{\xi} \min_{\alpha^2} R_V(R_H, \alpha^2, \xi, Q_v) .$$

(2.76)

On letting $x_1 = w$, $x_2 = Dw$, $x_3 = \theta$, $x_4 = D\theta$, the system (2.71)-(2.72) can be written in the matrix form as

$$\dot{X} = AX ,$$

(2.77)

where $X = (x_1, x_2, x_3, x_4)^T$ and $A$ is the coefficient matrix defined by

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
h_1 & 0 & h_2 & 0 \\
0 & 0 & 0 & 1 \\
h_3 & 0 & h_4 & 0
\end{bmatrix}$$

(2.78)

The boundary conditions now take the form

$$x_1 = x_3 = 0, \quad \text{at} \quad z = \pm 1/2.$$

(2.79)

The compound matrix method and the secant method are used to solve for $R_V$, and the maximization and minimization routines are both carried out by the golden section search method. The compound matrix variable entries and the resulting differential equations are provided in the Appendix.

2.5 Numerical Results.

The numerical calculations are carried out for determining the critical values of vertical Rayleigh number $R_V$ for both linear stability and energy stability theories. These are denoted by $R_L$ and $R_E$, respectively. Figure 1 displays our linear and energy stability results via the compound matrix method for the horizontal mass.
flow when $Q=0$. For the purpose of comparison, we also include Nield's linear results via the Galerkin approximation (Nield 1991, 1994a). We note that our linear results for $R_V$ nearly coincide with those obtained by Nield, up to the value of $R_H = 80$ but there is a difference of these values for $R_H$ greater than 90. The critical $R_E$ values obtained via the energy method, overall, are smaller compared to the linear results. This is not surprising since, as was pointed out earlier, the energy method gives the sufficient condition for stability and, as such, the values obtained by this method are generally conservative as compared to the linear stability theory. Figure 2 displays our results for linear theory and gives the plot of $R_L$ against the horizontal Rayleigh number $R_H$ for various values of the mass flow-rate $Q$. We note that as $Q$ increases the instability also increases. For low values of $Q$ we find that instability depends upon the $R_H$ values. Smaller value of $R_H$ has a stabilizing effect whereas higher value has a destabilizing effect. For higher values of $Q$, the effect of $R_H$, however, is immaterial. In fact, for $Q > 15$ we note that the effect is always destabilizing. Figure 3 shows the corresponding energy stability results. Here, in comparison to the linear case, we find that, for comparative values of $R_H$ and $Q$, the stability regions are again much smaller. Figures 4 and 5 show the effect of varying $R_H$ on $R_L$ and $R_E$, respectively. Figure 4 presents the plots of $R_L$ against $Q$, for varying $R_H$, from a linear stability analysis. We have chosen the values of $R_H$ so as to match Nield's calculations. A comparison with Nield's (1990) results clearly indicates the limitation of low-order Galerkin approximations. We note the similarity in the values of our calculations to Nield's calculations for low values of $Q$ but that only our calculations give the complete spectrum of results. We also find that our results are very close to those of Nield at lower values of $R_H$. For higher values of $R_H$ the difference between the two methods of calculation becomes quite apparent. We find that when $R_H$ increases, its stabilizing effect decreases with
an increase in $Q$. As expected the corresponding range of values from the energy method are much smaller even though the overall pattern is similar. This is clearly seen in Figure 5.

Figures 6-8 display our computed results of linear and nonlinear critical vertical Rayleigh numbers for vertical through-flow at different values of the Péclet number $Q_v$. First of all we remark that our linear instability results nearly coincide with those of Nield (1998) for low values of $R_H$. Thus in this range, the effect of increasing $R_H$ and $Q_v$ values is almost additive and has the effect of stabilizing both $R_L$ and $R_E$ values. A further increase in $R_H$ values, however, have the destabilizing effect and we find that destabilization starts earlier for lower values of $Q_v$. We may conclude that higher values of $Q_v$ have a stabilizing effect. Thus, even though the mechanism of through-flow delays the onset of convection it cannot completely control the instability caused by the higher values of horizontal applied temperature gradient. We remark that results presented by Nield (1998), which are reported for the values of $R_H$ up to 40, do not reflect this destabilizing behavior. Indeed the destabilizing in the linear case only starts after $R_H$ has taken values higher than 70. The energy stability results are almost parallel to the linear instability results except that the changes from stabilizing to destabilizing occur at the lower values of $R_H$.

To sum up, we thus conclude that increasing the mass flow-rate $Q$ has a destabilizing effect on $R_L$ and $R_E$ and that, at lower values of $Q$, the effect of $R_H$ on $R_L$ and $R_E$ changes from stabilizing to destabilizing as $R_H$ values are increased. We also find that the stability bound calculated using the energy theory is smaller than the instability bound of linear stability theory and hence there is the possibility of subcritical instability, which needs to be further analyzed. This behavior is observed in all the cases considered. We also observe that, for very high values
of $Q$ and $R_H$, instability is possible even in the absence of a vertical temperature gradient.

![Graph showing stability results for $R_L$ and $R_E$ vs $R_H$ for $Q = 0$.]

Fig. 1 Linear and energy stability results: $R_L$ ($R_E$) vs $R_H$ for $Q = 0$. 

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Fig. 2 Linear stability results: $R_L$ vs $R_H$ for different values of $Q$
Fig. 3 Energy stability results: $R_E$ vs $R_H$ for different values of $Q$
Fig. 4 Linear stability results: $R_L$ vs $Q$ with comparison to Nield's (1990)
Fig. 5  Energy stability results: $R_E$ vs $Q$ corresponding to Fig. 4
Fig. 6 Linear and energy stability results: $R_L(R_E)$ vs $R_H$ for $Q_v = 1$
Fig. 7 Linear and energy stability results: $R_L(R_E)$ vs $R_H$ for $Q_v = 5$
Fig. 8  Linear and energy stability results: $R_L(R_E)$ vs $R_H$ for $Q_v = 8$
Chapter 3
Nonlinear Convection Induced by Inclined
Thermal and Solutal Gradients with Mass Flow

3.1. Introduction.

In Chapter 2, we confined our discussion to the linear and nonlinear thermal (mono-diffusive) convection, induced by inclined temperature gradients with mass or through-flow. In this chapter we turn our attention to the process of combined heat and mass transfer that are driven by buoyancy. The density gradients that provide the driving buoyancy force are induced by the combined effects of temperature and solutal concentration non-uniformities present in the fluid saturated medium. This is called double-diffusive (or thermohaline, if heat and salt are involved) convection in the literature (Nield and Bejan 1998). The effects in double-diffusive convection arise from the fact that heat diffuses more rapidly than a dissolved substance. The double-diffusive generalization of the Horton-Rogers-Lapwood problem was treated by Nield (1968). The results are in agreement with the experiments conducted by Cooper et al. (1997). The effects of horizontal gradients on thermosolutal stability was studied by Parvathy and Patil (1989) and Sarkar and Phillips (1992). The more general case for inclined thermal and solutal gradients was treated by Rudraiah et al. (1987) and Nield et al. (1993). The corresponding nonlinear energy stability analysis was carried out by Guo and Kaloni (1995c). Manole et al. (1994) extended the linear stability analysis of Nield et al. (1993) by including the effect of non-zero net horizontal mass flow, and employed the second-order Galerkin approximation.

In this chapter we deal with the convection induced by the inclined thermal and solutal gradients with mass flow in a shallow horizontal layer of a saturated porous medium. We apply the energy method, together with the use of coupling parameters, to study the nonlinear stability problem of Manole et al. (1994). We
employ Darcy's law to represent the porous medium and assume the boundaries to be perfectly conducting. We first establish an unconditional stability result when the steady solution is nonlinearly stable. This condition naturally leads to the associated Euler-Lagrange equations which lead to the determination of an eigenvalue problem. We then use the compound matrix method to solve the associated eigenvalue problem and the golden section search method for determining the maximum and the minimum routines. Because of the large parameter space involved and because of the complexity of determining the eigenvalues of a large system of equations, attention is only given to the situation when imposed thermal and solutal gradients are coplanar as discussed by Nield et al. (1993), Guo and Kaloni (1995c), and Manole et al. (1994). Numerical results show that horizontal concentration gradients have greater destabilizing effect as compared to horizontal temperature gradients. We also observe that the effect of increasing the positive vertical solutal gradient is always destabilizing while increasing the horizontal thermal gradient is first stabilizing and then destabilizing. Increasing mass flow-rate, in all cases, has a destabilizing effect.

In the course of the present investigation we have noted an error in the work of Guo and Kaloni (1995c) involving the mathematical derivation of the eigenvalue problem and associated numerical computations. Thus, apart from providing the nonlinear stability results, complementing the linear instability results of Manole et al. (1994), we also provide the correct nonlinear energy stability results for the convection induced by inclined thermal and solutal gradients in the absence of mass flow as discussed by Guo and Kaloni (1995c).

3.2. Governing Equations.

The Cartesian axes are chosen with the $z^*$-axis vertically upwards and the $x^*$-axis in the direction of the net flow, of magnitude $Q^*$. The porous medium occupies a
horizontal layer of height $H$. The vertical temperature difference across the boundaries is $\Delta T$ and the vertical concentration difference is $\Delta C$. The imposed horizontal thermal and concentration gradient vectors are $(\beta_{T_x}, \beta_{T_y})$ and $(\beta_{C_x}, \beta_{C_y})$, respectively. The Boussinesq approximation is assumed to be valid, and the flow in the porous medium is assumed to be governed by Darcy's law. The basic equations can, thus, be written as

\[
\nabla^* \cdot \mathbf{v}^* = 0 \quad , \tag{3.1}
\]

\[
(\mu/K)\mathbf{v}^* + \nabla^* P^* = \rho_f^* g \mathbf{k} \quad , \tag{3.2}
\]

\[
(\rho \chi)_m \frac{\partial T^*}{\partial t^*} + (\rho \chi)_f (\mathbf{v}^* \cdot \nabla^*) T^* = k_m \nabla^* \nabla^* T^* \quad , \tag{3.3}
\]

\[
\phi \frac{\partial C^*}{\partial t^*} + (\mathbf{v}^* \cdot \nabla^*) C^* = D_m \nabla^* \nabla^* C^* \quad , \tag{3.4}
\]

\[
\rho_f^* = \rho_0 [1 - \gamma_T (T^* - T_0) - \gamma_C (C^* - C_0)] \quad , \tag{3.5}
\]

where $\mathbf{v}^* = (u^*, v^*, w^*)$, $P^*$, $T^*$ and $C^*$ are the seepage velocity vector, pressure, temperature and concentration, and the subscripts $m$ and $f$ refer to the porous medium and the fluid, respectively. The quantities $\mu$ denotes the viscosity, $\rho$ the density, $\chi$ specific heat. $K$ is the permeability, $\phi$ the porosity, $k_m$ the thermal conductivity and $D_m$ the solutal diffusivity of the medium, respectively. $\gamma_T$ is the thermal expansion coefficient, $\gamma_C$ the corresponding solutal quantity, $g$ the gravity acceleration, $\mathbf{k}$ the unit vector in the $z^*$-direction, and $\rho_0$ the density at the reference temperature $T_0$ and the reference concentration $C_0$. The boundary conditions for rigid boundaries take the form

\[
w^* = 0, \quad T^* = T_0 - (\pm \Delta T/2) - \beta_{T_x} x^* - \beta_{T_y} y^* \quad ,
\]

\[
C^* = C_0 - (\pm \Delta C/2) - \beta_{C_x} x^* - \beta_{C_y} y^* \quad , \quad \text{at } z^* = \pm H/2 \quad . \tag{3.6}
\]

The non-dimensional variables are defined as follow:

\[
x = x^*/H, \quad t = \alpha_m t^*/(AH^2), \quad v = H v^*/\alpha_m, \quad Q = Q^*/H/\alpha_m,
\]

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\[ P = K(P^* + \rho_0 g z^*)/(\mu \alpha_m), \quad T = R_z(T^* - T_0)/\Delta T, \quad C = S_z(C^* - C_0)/\Delta C, \]

where

\[ \alpha_m = k_m/(\rho_0 \chi_f), \quad A = (\rho \chi)_m/(\rho_0 \chi)_f, \]

\[ R_z = \rho_0 g \gamma_T KH\Delta T/\mu \alpha_m, \quad S_z = \rho_0 g \gamma_C KH\Delta C/\mu D_m. \tag{3.7} \]

Here \( R_z \) and \( S_z \) are referred to as the vertical thermal Rayleigh number and the vertical solutal Rayleigh number, respectively. Upon substituting these non-dimensional variables into (3.1) – (3.6), the non-dimensional governing equations then take the form.

\[ \nabla \cdot \mathbf{v} = 0, \tag{3.8} \]

\[ \mathbf{v} + \nabla P = (T + L_e^{-1} C)k, \tag{3.9} \]

\[ \frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla) T = \nabla^2 T, \tag{3.10} \]

\[ \alpha_0 \frac{\partial C}{\partial t} + (\mathbf{v} \cdot \nabla) C = L_e^{-1} \nabla^2 C, \tag{3.11} \]

where \( L_e = \alpha_m/D_m \) is the Lewis number, and \( \alpha_0 = \phi/A \). The corresponding non-dimensional form of boundary conditions become

\[ w = 0, \quad T = -(\pm R_z/2) - R_x x - R_y y, \]

\[ C = -(\pm S_z/2) - S_x x - S_y y, \quad \text{at} \quad z = \pm 1/2, \tag{3.12} \]

where \( R_x, R_y, S_x, S_y \) are referred to be the horizontal thermal Rayleigh numbers and the horizontal solutal Rayleigh numbers, and are defined as follow:

\[ R_x = \rho_0 g \gamma_T KH^2 \beta_{T_x}/\mu \alpha_m, \quad R_y = \rho_0 g \gamma_T KH^2 \beta_{T_y}/\mu \alpha_m, \tag{3.13} \]

\[ S_x = \rho_0 g \gamma_C KH^2 \beta_{C_x}/\mu D_m, \quad S_y = \rho_0 g \gamma_C KH^2 \beta_{C_y}/\mu D_m. \tag{3.14} \]
The basic steady state solution \((u_s, T_s, C_s, p_s)\) of Eqs. (3.8)-(3.11) subject to the boundary conditions (3.12) and the following horizontal mass flow conditions

\[
\int_{-1/2}^{1/2} u_s \, dz = Q, \quad \int_{-1/2}^{1/2} v_s \, dz = 0
\]

is:

\[
u_s = \beta_1 z + Q, \quad v_s = \beta_2, \quad w_s = 0 ,
\]

\[
T_s = -R_x x - R_y y - R_z z + \lambda_1 f(z) + \lambda_3 h(z) ,
\]

\[
C_s = -S_x x - S_y y - S_z z + \lambda_2 f(z) + \lambda_4 h(z) ,
\]

\[
\nabla p_s = (T_s + L_e^{-1} C_s) k - u_s ,
\]

where

\[
\beta_1 = R_x + L_e^{-1} S_x , \quad \lambda_1 = R_x^2 + R_y^2 + L_e^{-1} (R_x S_x + R_y S_y) ,
\]

\[
\beta_2 = R_y + L_e^{-1} S_y , \quad \lambda_2 = S_x^2 + S_y^2 + L_e (R_x S_x + R_y S_y) ,
\]

\[
\lambda_3 = Q R_x , \quad \lambda_4 = L_e Q S_x ,
\]

\[
f(z) = \frac{1}{24} (z - 4z^3) , \quad h(z) = \frac{1}{8} (1 - 4z^2)
\]

We now perturb the steady state solution as follow:

\[
v = u_s + u, \quad T = T_s + \theta, \quad C = C_s + c, \quad P = p_s + p
\]

The perturbation equations then take the form

\[
\nabla \cdot u = 0 ,
\]

\[
u + \nabla p = (\theta + L_e^{-1} c) k ,
\]

\[
\frac{\partial \theta}{\partial t} + (u \cdot \nabla) \theta = \nabla^2 \theta - u_s \cdot \nabla \theta - u \cdot \nabla T_s ,
\]

\[
\alpha_0 \frac{\partial c}{\partial t} + (u \cdot \nabla) c = L_e^{-1} \nabla^2 c - u_s \cdot \nabla c - u \cdot \nabla C_s
\]
where $u_s$ is given by (3.16), and

$$
\nabla T_s = (-R_x, -R_y, -R_z + \lambda_1 f'(z) + \lambda_3 h'(z))
$$

$$
\nabla C_s = (-S_x, -S_y, -S_z + \lambda_2 f'(z) + \lambda_4 h'(z))
$$

The associated boundary conditions become

$$
w = \theta = c = 0 \quad \text{at} \quad z = \pm 1/2 \quad .
$$

(3.23)

3.3. Energy Functional and Variational Problem.

To conduct the nonlinear energy stability analysis for the system of equations (3.19)-(3.22) subject to the boundary conditions (3.23), we introduce an energy functional defined by

$$
E(t) = \frac{\xi}{2} ||\theta||^2 + \frac{\eta L_c \alpha_0}{2} ||\gamma||^2 \quad ,
$$

(3.24)

where we have defined $\gamma = L_c^{-1} c$, and $\xi$ and $\eta$ are positive coupling parameters. On multiplying (3.21) by $\theta$, (3.22) by $\gamma$ and (3.20) by $u$, and integrating over $V$, we find (after using the boundary conditions and divergence theorem)

$$
\frac{1}{2} \frac{d}{dt} ||\theta||^2 = -||\nabla \theta||^2 - < (u \cdot \nabla T_s) \theta > \quad ,
$$

(3.25)

$$
\frac{L_c \alpha_0}{2} \frac{d}{dt} ||\gamma||^2 = -||\nabla \gamma||^2 - < (u \cdot \nabla C_s) \gamma > \quad ,
$$

(3.26)

$$
||u||^2 = < \theta w > + < \gamma w > \quad .
$$

(3.27)

Here $V$ denotes a typical periodicity cell, $< \cdot >$ denotes the integration over $V$, and $||\cdot||$ denotes the $L^2(V)$ norm. The system of Eqs. (3.25)-(3.27), along with Eq. (3.24), can be rewritten as

$$
\frac{dE}{dt} = I - D \quad ,
$$

(3.28)

where

$$
I = -\xi < (u \cdot \nabla T_s) \theta > -\eta < (u \cdot \nabla C_s) \gamma > + < \theta w > + < \gamma w > \quad ,
$$

(3.29)

$$
D = \xi ||\nabla \theta||^2 + \eta ||\nabla \gamma||^2 + ||u||^2 \quad .
$$

(3.30)
We next define

\[ m = \max_{\mathcal{H}} \frac{I}{D}, \quad (3.31) \]

where \( \mathcal{H} \) is the space of admissible solutions. On combining (3.31) with (3.24)-(3.30), and using the Poincaré inequality, we can infer, for \( 0 < m < 1 \), that

\[ \frac{dE}{dt} \leq -2\pi^2 (1 - m) \min(1, \frac{1}{Le\alpha_0}) E. \quad (3.32) \]

Inequality (3.32) clearly indicates that for \( 0 < m < 1 \), \( E(t) \to 0 \) at least exponentially as \( t \to \infty \). Using the following inequalities

\[ < \theta w > \leq \|\theta\|\|w\|, \quad < \gamma w > \leq \|\gamma\|\|w\| \]

\[ \|w\| \leq \|u\|, \quad 2\|\theta\|\|\gamma\| \leq \|\theta\|^2 + \|\gamma\|^2 \]

From (3.27), we obtain

\[ \|u\|^2 \leq 2(\|\theta\|^2 + \|\gamma\|^2) \quad (3.33) \]

From Eqs. (3.24) and (3.33), we note that the decay of \( E(t) \) ensures the decay of \( \|u\|^2 \).

We now return to (3.31) and consider the maximum problem at the critical argument \( m = 1 \). The associated Euler-Lagrange equations become

\[ \xi\theta \nabla T_s + \eta \gamma \nabla C_s - (\theta + \gamma) k + 2u = \nabla \omega, \quad (3.34) \]

\[ -\xi u \cdot \nabla T_s + w + 2\xi \nabla^2 \theta = 0, \quad (3.35) \]

\[ -\eta u \cdot \nabla C_s + w + 2\eta \nabla^2 \gamma = 0, \quad (3.36) \]

where \( \omega \) is a Lagrange multiplier introduced since \( u \) is solenoidal.

Before proceeding further, we first explore the variation of \( R_z \) with respect to the coupling parameters \( \xi \) and \( \eta \), which plays an important role in the maximization.
procedure later on. On multiplying (3.35) by \( \theta \), (3.36) by \( \gamma \) and (3.34) by \( u \), and integrating over \( V \), we have

\[
(1 + \xi R_z) < \theta w > = 2\xi \| \nabla \theta \|^2 + \xi < g_1 \theta w > - \xi R_z < \theta u > - \xi R_y < \theta v > , \tag{3.37}
\]

\[
(1 + \eta S_z) < \gamma w > = 2\eta \| \nabla \gamma \|^2 + \eta < g_2 \gamma w > - \eta S_z < \gamma u > - \eta S_y < \gamma v > , \tag{3.38}
\]

\[
(1 + \xi R_z) < \theta w > + (1 + \eta S_z) < \gamma w > = 2\| \nabla u \|^2 + \xi < g_1 \theta w > + \eta < g_2 \gamma w > - \xi R_z < \theta u > - \xi R_y < \theta v > - \eta S_z < \gamma u > - \eta S_y < \gamma v > , \tag{3.39}
\]

where \( g_1 = \lambda_1 f_1 - \lambda_3 z \), \( g_2 = \lambda_2 f_1 - \lambda_4 z \), \( f_1(z) = \frac{1}{24}(1 - 12z^2) \).

From Eqs.(3.37) – (3.39), after performing some mathematical manipulation, we find

\[
\frac{\partial R_z}{\partial \xi} = \frac{(1 - \xi R_z) \| \nabla \theta \|^2 + < g_1 \theta w > - R_z < \theta u > - R_y < \theta v >}{\xi^2 \| \nabla \theta \|^2 + < g_1 \theta w > - R_z < \theta u > - R_y < \theta v >} , \tag{3.40}
\]

\[
\frac{\partial R_z}{\partial \eta} = \frac{2(1 + \xi R_z)(1 - \eta S_z) \| \nabla \gamma \|^2 + < g_2 \gamma w > - S_z < \gamma u > - S_y < \gamma v >}{\xi^2 (1 + \eta S_z) \| \nabla \theta \|^2 + < g_1 \theta w > - R_z < \theta u > - R_y < \theta v >} . \tag{3.41}
\]

Note that, in case \( R_z = R_y = S_z = S_y = 0 \), we have \( g_1 = g_2 = 0 \), and Eqs.(3.40)-(3.41) are thus reduced to

\[
\frac{\partial R_z}{\partial \xi} = \frac{1 - \xi R_z}{2\xi^2} , \quad \frac{\partial R_z}{\partial \eta} = \frac{(1 + \xi R_z)(1 - \eta S_z) \| \nabla \gamma \|^2}{\xi^2 (1 + \eta S_z) \| \nabla \theta \|^2} . \tag{3.42}
\]

When \( R_z > 0 \), we can easily verify that

\[
\frac{\partial R_z}{\partial \xi} = 0 , \quad \frac{\partial^2 R_z}{\partial^2 \xi} < 0 \quad \text{at} \quad \xi = R_z^{-1} , \tag{3.43}
\]

when \( S_z > 0 \),

\[
\frac{\partial R_z}{\partial \eta} = 0 , \quad \frac{\partial^2 R_z}{\partial^2 \eta} < 0 \quad \text{at} \quad \eta = S_z^{-1} , \tag{3.44}
\]

when \( S_z < 0 \),

\[
\frac{\partial R_z}{\partial \eta} > 0 , \quad \text{if} \quad \eta < -S_z^{-1} ; \quad \frac{\partial R_z}{\partial \eta} < 0 \quad \text{if} \quad \eta > -S_z^{-1} . \tag{3.45}
\]
Therefore, in this case, it is clear from Eqs. (3.43) – (3.45) that the optimal choice of \( \xi \) and \( \eta \), in the sense of maximizing \( R_z \), are \( R_z^{-1} \) and \( |S_z^{-1}| \), respectively.

On taking \( \text{curl curl} \) of (3.34) and then taking the third component of the resulting equation, we find

\[
2\nabla^2 w - \nabla_1^2 (G_1 \theta + G_2 \gamma) + \frac{\partial}{\partial z} [\xi(R_x \frac{\partial \theta}{\partial x} + R_y \frac{\partial \theta}{\partial y}) + \eta(S_x \frac{\partial \gamma}{\partial x} + S_y \frac{\partial \gamma}{\partial y})] = 0 \quad , \tag{3.46}
\]

where

\[
\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} , \quad G_1 = 1 + \xi(R_z - g_1) , \quad G_2 = 1 + \eta(S_z - g_2). \tag{3.47}
\]

Also the \( x \) and \( y \) components of (3.34), (3.35) and (3.36) can be written as

\[
-(\xi R_x \theta + \eta S_x \gamma) + 2u = \frac{\partial \omega}{\partial x} \quad , \tag{3.48}
\]

\[
-(\xi R_y \theta + \eta S_y \gamma) + 2v = \frac{\partial \omega}{\partial y} \quad , \tag{3.49}
\]

\[
\xi(R_x u + R_y v) + G_1 w + 2\xi \nabla^2 \theta = 0 \quad , \tag{3.50}
\]

\[
\eta(S_x u + S_y v) + G_2 w + 2\eta \nabla^2 \gamma = 0 \quad . \tag{3.51}
\]

We now perform the standard normal mode analysis and look for the solution of the above system in the form

\[
[u, v, w, \theta, \gamma, \omega] = [u(x), v(x), w(x), \theta(x), \gamma(x), \omega(x)] \exp[i(\alpha_x x + \alpha_y y)] \quad \tag{3.52}
\]

with \((R_x, R_y) \cdot (\alpha_x, \alpha_y) = 0\) and \((S_x, S_y) \cdot (\alpha_x, \alpha_y) = 0\) (Nield 1994, Manole et al. 1994, and Guo and Kaloni 1995c). On employing (3.52) in (3.46) and (3.48)-(3.51) and eliminating the variables \( u, v, \omega \), we derive the corresponding eigenvalue problem which, after some rearrangement of terms, can be written as

\[
D^2 w = h_1 w + h_2 \theta + h_3 \gamma \quad , \tag{3.53}
\]

\[
D^2 \theta = h_4 w + h_5 \theta + h_6 \gamma \quad , \tag{3.54}
\]

\[
D^2 \gamma = h_7 w + h_8 \theta + h_9 \gamma \quad , \tag{3.55}
\]
where the coefficients \( h_1, \cdots, h_9 \) are defined as follow:

\[
\begin{align*}
    h_1 &= \alpha^2, & h_2 &= -\frac{\alpha^2}{2}G_1, & h_3 &= -\frac{\alpha^2}{2}G_2, \\
    h_4 &= -\frac{G_1}{2\xi}, & h_5 &= \alpha^2 - \frac{\xi}{4}(R_x^2 + R_y^2), & h_6 &= -\frac{\eta}{4}(R_xS_x + R_yS_y), \\
    h_7 &= -\frac{G_2}{2\eta}, & h_8 &= -\frac{\xi}{4}(R_xS_x + R_yS_y), & h_9 &= \alpha^2 - \frac{\eta}{4}(S_x^2 + S_y^2)
\end{align*}
\]

and where \( \alpha^2 = \alpha_x^2 + \alpha_y^2 \). The associated boundary conditions are

\[
w = \theta = \gamma = 0 \quad \text{at } z = \pm 1/2.
\]

We point out that when \( Q = 0 \), the present problem is reduced to the one discussed by Guo and Kaloni (1995c). However, there is an error in (48)-(49) of their work, the numerical calculations given below for \( Q = 0 \) correspond to the correct results.

3.4. Numerical Results and Discussion.

We consider \( R_x \) as the eigenvalue with the remaining variables as parameters.

The critical vertical thermal Rayleigh number is defined by

\[
R_E = \max_{\xi} \max_{\eta} \min_{\alpha^2} R_x(\xi, \eta, \alpha^2, R_x, R_y, S_x, S_y, S_z).
\]

On letting \( x_1 = w, x_2 = Dw, x_3 = \theta, x_4 = D\theta, x_5 = \gamma, x_6 = D\gamma; \) the system (3.53)-(3.55) can be written in the matrix form as

\[
\dot{X} = AX,
\]

where \( X = (x_1, x_2, x_3, x_4, x_5, x_6)^T \) and \( A \) is the coefficient matrix defined by

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & h_9 & 0
\end{bmatrix}.
\]
The boundary conditions now take the form

\[ x_1 = x_3 = x_5 = 0, \quad \text{at} \quad z = \pm 1/2 \]  \hspace{1cm} (3.60)

We next employ the compound matrix method and the secant method to solve the system (3.59) subject to (3.60) and carry out the maximization and minimization routines by golden section search for (3.58). The compound matrix variables and associated differential equations and the initial and final condition are presented in the Appendix.

<table>
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<tr>
<th>( \lambda_1 )</th>
<th>( R_E )</th>
<th>( a_E )</th>
<th>( \lambda_1 )</th>
<th>( R_E )</th>
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Tables 1-3 show the effect of the mass flow-rate and the horizontal thermal Rayleigh number on the critical vertical Rayleigh number \( R_E \). The horizontal solutal Rayleigh number, in these tables, is absent. From these tables we see that in the case of \( Q = 0 \), or \( Q \) taking a small value, the vertical critical Rayleigh number \( R_E \) first increases with the increase of \( \lambda_1 \), the resultant horizontal thermal Rayleigh number, and then decreases as \( \lambda_1 \) is increased further. Thus for low values of \( \lambda_1 \),
the effect is stabilizing but it changes to destabilizing as \( \lambda_1 \) increases considerably.

For small values of \( Q \), the variation pattern between \( R_E \) and \( \lambda_1 \) remains similar except now the change over takes place at a lower value of \( \lambda_1 \). For higher value of \( Q \) the effect is always destabilizing, no matter how small the value of \( \lambda_1 \) may be.

Table 2. \( S_x = S_y = 0, \ S_z = 10, \ Q = 5, \ R_y = 0, \ \lambda_1 = R^2_z \)

<table>
<thead>
<tr>
<th>( \lambda_1 )</th>
<th>( R_E )</th>
<th>( a_E )</th>
<th>( \lambda_1 )</th>
<th>( R_E )</th>
<th>( a_E )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>4.99599</td>
</tr>
<tr>
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<td>3.12205</td>
<td>70</td>
<td>16.0013</td>
<td>5.49364</td>
</tr>
<tr>
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<td>3.08294</td>
<td>71</td>
<td>14.0303</td>
<td>5.58861</td>
</tr>
<tr>
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<td>32.9829</td>
<td>3.06928</td>
<td>72</td>
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</tr>
<tr>
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<td>73</td>
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</tr>
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<td>74</td>
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<td>5.86472</td>
</tr>
<tr>
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<td>3.52698</td>
<td>75</td>
<td>5.03435</td>
<td>5.95395</td>
</tr>
<tr>
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<td>3.94383</td>
<td>76</td>
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<td>6.04185</td>
</tr>
<tr>
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<td>4.46657</td>
<td>76.95</td>
<td>0.00000</td>
<td>6.12414</td>
</tr>
</tbody>
</table>

Table 3. \( S_x = S_y = 0, \ S_z = 10, \ Q = 10, \ R_y = 0, \ \lambda_1 = R^2_z \)

<table>
<thead>
<tr>
<th>( \lambda_1 )</th>
<th>( R_E )</th>
<th>( a_E )</th>
<th>( \lambda_1 )</th>
<th>( R_E )</th>
<th>( a_E )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>3.50740</td>
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<td>3.57530</td>
</tr>
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</tr>
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<td>3.18259</td>
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<td>3.73398</td>
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<tr>
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<td>3.20513</td>
<td>32</td>
<td>12.6457</td>
<td>3.82540</td>
</tr>
<tr>
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<td>34</td>
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<td>3.92498</td>
</tr>
<tr>
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<td>3.26457</td>
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<td>7.81543</td>
<td>4.03261</td>
</tr>
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<tr>
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<td>3.34418</td>
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<td>4.27036</td>
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<td>22.9041</td>
<td>3.39231</td>
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<td>0.00000</td>
<td>4.35252</td>
</tr>
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</table>
Table 4. $S_x = S_y = 0, S_z = 20, Q = 0, \lambda_1 = R_z^2 + R_y^2$

<table>
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<th>$R_E$</th>
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<th>$\lambda_1$</th>
<th>$R_E$</th>
<th>$a_E$</th>
</tr>
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<td>5.80620</td>
</tr>
<tr>
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<td>2.95749</td>
<td>98</td>
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<td>6.06971</td>
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<tr>
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<td>100</td>
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<td>6.32008</td>
</tr>
<tr>
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<td>2.57860</td>
<td>102</td>
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<td>6.55864</td>
</tr>
<tr>
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<td>2.39848</td>
<td>104</td>
<td>39.1244</td>
<td>6.78604</td>
</tr>
<tr>
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<td>46.6936</td>
<td>2.24751</td>
<td>106</td>
<td>32.7735</td>
<td>7.00274</td>
</tr>
<tr>
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<td>2.22264</td>
<td>108</td>
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<td>7.20923</td>
</tr>
<tr>
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<td>110</td>
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<td>7.40602</td>
</tr>
<tr>
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<td>3.33637</td>
<td>112</td>
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<td>7.59358</td>
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<td>7.80443</td>
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</table>

Table 5. $S_x = S_y = 0, S_z = 20, Q = 5, R_y = 0, \lambda_1 = R_z^2$

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$R_E$</th>
<th>$a_E$</th>
<th>$\lambda_1$</th>
<th>$R_E$</th>
<th>$a_E$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>60</td>
<td>16.8286</td>
<td>4.41246</td>
</tr>
<tr>
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<td>3.12595</td>
<td>65</td>
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<td>4.86139</td>
</tr>
<tr>
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<td>3.10185</td>
<td>66</td>
<td>9.72049</td>
<td>4.95359</td>
</tr>
<tr>
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<td>3.11753</td>
<td>67</td>
<td>8.19428</td>
<td>5.04613</td>
</tr>
<tr>
<td>40</td>
<td>23.0087</td>
<td>3.24900</td>
<td>68</td>
<td>6.56472</td>
<td>5.13881</td>
</tr>
<tr>
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<td>3.40535</td>
<td>69</td>
<td>4.83012</td>
<td>5.23145</td>
</tr>
<tr>
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<td>70</td>
<td>2.98899</td>
<td>5.32387</td>
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<tr>
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<td>4.00163</td>
<td>71.51</td>
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</tbody>
</table>

Tables 4-6 consider the effect of varying the vertical solutal Rayleigh number. We find a similar pattern as in Tables 1-3 except that the change from stabilizing to destabilizing now occurs at slightly lower values of $\lambda_1$. In this case, again, the effect of increasing $Q$ is destabilizing.
Table 6. $S_z = S_y = 0, S_z = 20, Q = 10, R_y = 0, \lambda_1 = R_z^2$

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$R_E$</th>
<th>$a_E$</th>
<th>$\lambda_1$</th>
<th>$R_E$</th>
<th>$a_E$</th>
</tr>
</thead>
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<tr>
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<td>19.3858</td>
<td>3.14473</td>
<td>20</td>
<td>11.4603</td>
<td>3.39965</td>
</tr>
<tr>
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<td>19.1109</td>
<td>3.15397</td>
<td>22</td>
<td>9.91355</td>
<td>3.45014</td>
</tr>
<tr>
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<td>18.6618</td>
<td>3.16885</td>
<td>24</td>
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<td>3.50548</td>
</tr>
<tr>
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<td>3.18880</td>
<td>26</td>
<td>6.41482</td>
<td>3.56589</td>
</tr>
<tr>
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<td>17.2850</td>
<td>3.21336</td>
<td>28</td>
<td>4.44705</td>
<td>3.63151</td>
</tr>
<tr>
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<td>16.3791</td>
<td>3.24221</td>
<td>30</td>
<td>2.32015</td>
<td>3.70248</td>
</tr>
<tr>
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<td>15.3397</td>
<td>3.27521</td>
<td>31</td>
<td>1.19329</td>
<td>3.73999</td>
</tr>
<tr>
<td>16</td>
<td>14.1721</td>
<td>3.31237</td>
<td>32.02</td>
<td>0.00000</td>
<td>3.77959</td>
</tr>
</tbody>
</table>

Table 7. $R_x = R_y = 0, S_z = 10, Q = 0, \lambda_2 = S_z^2 + S_y^2$

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$R_E$</th>
<th>$a_E$</th>
<th>$\lambda_2$</th>
<th>$R_E$</th>
<th>$a_E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>29.4784</td>
<td>3.14159</td>
<td>80</td>
<td>33.2078</td>
<td>3.41219</td>
</tr>
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<td>90</td>
<td>31.2575</td>
<td>3.54912</td>
</tr>
<tr>
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<td>2.97100</td>
<td>100</td>
<td>28.6952</td>
<td>3.80315</td>
</tr>
<tr>
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<td>3.17733</td>
<td>110</td>
<td>24.6721</td>
<td>4.66437</td>
</tr>
<tr>
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<td>7.58429</td>
</tr>
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<td>7.96017</td>
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<tr>
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<td>117.54</td>
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<td>8.34067</td>
</tr>
</tbody>
</table>

Tables 7-9 show the effect of varying $Q$ and the horizontal solutal Rayleigh number, $\lambda_2$, on $R_E$. For $Q = 0$ we note that as $\lambda_2$ increases $R_E$ initially increases, but reaches a maximum and then, as $\lambda_2$ further increases, $R_E$ decreases. A comparison between Tables 1 and 7 shows that the change from stabilizing to destabilizing takes place at much lower values of $\lambda_2$ as compared to the corresponding $\lambda_1$ values. This clearly indicates that the presence of horizontal concentration gradients have
a stronger destabilizing effect. For non-zero values of $Q$, the value of $R_E$ always decreases as $\lambda_2$ increases, which shows that mass flow, again, has a destabilizing effect.

Table 8. $R_x = R_y = 0, S_x = 10, Q = 1, L_x = 10, S_y = 0, \lambda_2 = S_x^2$

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$R_E$</th>
<th>$\alpha_E$</th>
<th>$\lambda_2$</th>
<th>$R_E$</th>
<th>$\alpha_E$</th>
</tr>
</thead>
<tbody>
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<td>19.9303</td>
<td>3.45017</td>
</tr>
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<td>3.14613</td>
<td>24</td>
<td>18.5916</td>
<td>3.50720</td>
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<td>3.57186</td>
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<tr>
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<td>3.17613</td>
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<td>3.64507</td>
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<tr>
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<td>32</td>
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</tr>
<tr>
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<td>3.24997</td>
<td>34</td>
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</tr>
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<td>4.16345</td>
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<tr>
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<td>3.35539</td>
<td>40</td>
<td>3.29527</td>
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<tr>
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</tr>
</tbody>
</table>

Table 9. $R_x = R_y = 0, S_x = 10, Q = 5, L_x = 10, S_y = 0, \lambda_2 = S_x^2$

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$R_E$</th>
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<th>$\lambda_2$</th>
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<th>$\alpha_E$</th>
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<td>3.30017</td>
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<td>3.88409</td>
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<td>0.22843</td>
<td>3.98119</td>
</tr>
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</table>
Tables 10-11 display the effect of simultaneous presence of both horizontal thermal and solutal gradients. In the absence of mass flow these two tables show the variation of $R_E$ with $L_e$, the Lewis number. We find that as $L_e$ increases, $R_E$ first decreases and then increases with increase in $L_e$. However, when $L_e$ is considerably increased, $R_E$ then decreases. As the values of $\lambda_1$ and $\lambda_2$ are increased, the change, of similar nature, appears to begin at lower values of $L_e$.

Table 10. $R_x = R_y = S_x = S_y = 1$, $S_z = 10$

<table>
<thead>
<tr>
<th>$L_e$</th>
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<th>$R_E$</th>
<th>$a_E$</th>
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</tr>
<tr>
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<td>800</td>
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<td>3.33575</td>
</tr>
<tr>
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<td>3.13894</td>
<td>900</td>
<td>14.6327</td>
<td>3.42787</td>
</tr>
<tr>
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<td>33.8215</td>
<td>3.14177</td>
<td>1000</td>
<td>3.69525</td>
<td>3.65335</td>
</tr>
</tbody>
</table>

Table 11. $R_x = R_y = S_x = S_y = 2$, $S_z = 10$

<table>
<thead>
<tr>
<th>$L_e$</th>
<th>$R_E$</th>
<th>$a_E$</th>
<th>$L_e$</th>
<th>$R_E$</th>
<th>$a_E$</th>
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</tr>
<tr>
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<td>100</td>
<td>38.3601</td>
<td>3.17856</td>
</tr>
<tr>
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<td>3.12581</td>
<td>150</td>
<td>34.0520</td>
<td>3.23261</td>
</tr>
<tr>
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<td>200</td>
<td>23.0330</td>
<td>3.33775</td>
</tr>
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<td>16.3821</td>
<td>3.40879</td>
</tr>
<tr>
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<td>3.12639</td>
<td>240</td>
<td>8.19017</td>
<td>3.50577</td>
</tr>
<tr>
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<td>31.4204</td>
<td>3.12765</td>
<td>250</td>
<td>3.43887</td>
<td>3.56597</td>
</tr>
</tbody>
</table>

To sum up, we observe that the effect of horizontal thermal or solutal gradi-
ent is to switch from stabilizing to destabilizing as their magnitude increases, for zero or small values of mass flow rate. For higher values of mass flow-rate, the effect is always destabilizing. The horizontal concentration gradient has a stronger destabilizing effect than the horizontal temperature gradient.
Chapter 4

Convection Induced by Inclined Temperature Gradient and Horizontal Mass Flow in a Viscous Fluid

4.1. Introduction.

In the previous two chapters, we have discussed mono-diffusive (thermal) convection, and double-diffusive (thermal-solutal) convection, induced by inclined temperature gradient (and solutal gradient) with mass flow in a shallow porous medium. In this chapter, we present the analysis of the convection problem induced by inclined temperature gradient with mass flow in a shallow viscous fluid layer. Thus, instead of Darcy's law, the Navier-Stokes equation is now used as the momentum equation, and the order of governing equations is increased accordingly. Moreover, an important new non-dimensional quantity (Prandtl number) appears in our discussion.

It is well known that a horizontal layer of fluid heated uniformly from below exhibits a state of zero motion unless the temperature gradient exceeds a critical value. The presence of a horizontal temperature gradient, however, always leads to motion of the fluid. Many published papers have been reported on convection, in a horizontal fluid layer, induced by imposed temperature gradient that is either vertical or horizontal. The former is the well known Rayleigh-Bénard convection problem, and the related studies are well summarized in the books of Chandrasekhar (1961) and Drazin and Reid (1981). The latter is referred to as Hadley circulation, and has been examined by, for instance, Hart (1972, 1983), Wang and Korpela (1989), Hung and Andereck (1988), and Kuo and Korpela (1988). All these studies, however, are restricted to the case of very small Prandtl number.

Few articles have dealt with the convection problem, when the imposed temperature gradients act simultaneously, both in horizontal as well as in vertical directions. Weber (1973) used a perturbation method to investigate the linear stability of the
convection problem in the presence of both vertical and horizontal temperature gradients. He considered the stress free and perfectly conducting boundaries and assumed the horizontal temperature gradient to be small. His results showed that the critical Rayleigh temperature is always larger than that for the classical Rayleigh-Bénard problem, and the preferred mode of disturbance is longitudinal stationary when the Prandtl number is larger than 5.1. By using a linear mean field approximation (Gill 1974), Sweet et al. (1977) also showed that the oscillatory stability mode is not predicted in higher Prandtl number. Weber (1978) removed the restriction to small horizontal gradients, but his analysis and calculations were restricted to small and moderate values of Prandtl number. Nield (1994b) reformulated the inclined temperature gradient problem so as to consider effect of large Prandtl number and considered only rigid perfectly conducting horizontal boundaries. Both Weber (1978) and Nield (1994b) used low-order Galerkin approximation to solve the resulting eigenvalue problem. Kaloni and Qiao (1996) have briefly discussed the nonlinear stability features of this problem (Nield 1994b) and have pointed out the differences between the nonlinear and linear stability results. Kaloni and Qiao (1996) have also raised doubts on the accuracy of results based upon the low-order Galerkin method. The extension to consider the effect of horizontal mass flow by both the nonlinear energy stability and the linear instability analysis was conducted by Kaloni and Qiao (1997b).

In the present chapter, we study the linear and nonlinear convection induced by inclined temperature gradient and horizontal mass flow in a viscous fluid. We express our results in terms of four parameters, namely, the vertical Rayleigh number $R_V$, the horizontal Rayleigh number $R_H$, the Prandtl number $Pr$ and the non-dimensional net mass flow-rate $Q$. The nonlinear stability analysis for this problem is performed by the energy method together with the use of coupling parameters.
The resulting eigenvalue problem is numerically solved by the compound matrix method with the secant method and the golden section search method. For the purpose of comparisons, the linear stability calculations are also carried out. Numerical results show that there is a wide difference in the stability bound calculated by the nonlinear energy method and linear stability method.

4.2. Basic Equations and Asymptotic Stability.

We assume that the fluid is confined between horizontal planes at a distance \( d \) apart. A Cartesian coordinate system \((\bar{x}, \bar{y}, \bar{z})\) is chosen such that the origin is in the middle of the layer and \( \bar{z} \)-axis is vertically upward. We assume that a constant net horizontal mass flow \( q \) and a constant horizontal temperature gradient \( \beta \) are imposed in the \( \bar{z} \)-direction, and a constant temperature difference \( \Delta T \) is maintained between the two planes. We thus have the governing equations, boundary and horizontal net mass flow conditions as follows:

\[
\hat{\nabla} \cdot \vec{v} = 0 \quad ,
\]

\[
\frac{\partial \vec{v}}{\partial \bar{t}} + (\vec{v} \cdot \hat{\nabla}) \vec{v} = -\hat{\nabla}\bar{P} + \nu \hat{\nabla}^2 \vec{v} + \vec{T} \mathbf{k} \quad ,
\]

\[
\frac{\partial \bar{T}}{\partial \bar{t}} + (\vec{v} \cdot \hat{\nabla}) \bar{T} = \hat{\nabla}^2 \bar{T} \quad ,
\]

\[
\bar{T} = T_0 - \beta \bar{z} \pm \Delta T/2 \quad \text{at} \quad \bar{z} = -\left( \pm \frac{d}{2} \right) \quad ,
\]

\[
\frac{1}{d} \int_{-d/2}^{d/2} \bar{u} \, d\bar{z} = q \quad , \quad \int_{-d/2}^{d/2} \bar{v} \, d\bar{z} = 0 \quad .
\]

It is also assumed that the ratio of the height to the length of the layer is considerably small so that the lateral end effects do not influence the motion in the horizontal central part. For the density variation, the Boussinesq approximation is adopted and we have

\[
\rho = \rho_0 [1 - \alpha_T (\bar{T} - T_0)] \quad .
\]
We non-dimensionalize the quantities as follows:

\[ x = \tilde{x}/d, \quad t = \kappa \tilde{t}/d^2, \quad v = d \tilde{v}/\kappa, \]

\[ T = R_V \tilde{T}/\Delta T, \quad P = d^2 \tilde{p}/\rho_0 \kappa \nu, \quad Q = dq/\kappa, \]

where \( \tilde{x} = (\tilde{x}, \tilde{y}, \tilde{z}) \), \( \tilde{v} = (\tilde{u}, \tilde{v}, \tilde{w}) \) is the velocity vector, \( \tilde{t} \) the time, \( \tilde{p} \) the pressure, and \( \kappa \) and \( \nu \) are the thermal diffusivity, and the kinematic viscosity of the fluid, respectively. Governing non-dimensional equations then take the form

\[ \nabla \cdot v = 0, \quad (4.4) \]

\[ Pr^{-1} \left[ \frac{\partial v}{\partial t} + (v \cdot \nabla)v \right] = -\nabla P + \nabla^2 v + Tk, \quad (4.5) \]

\[ \frac{\partial T}{\partial t} + (v \cdot \nabla)T = \nabla^2 T, \quad (4.6) \]

where \( k \) is the unit vector in the \( z \)-direction, and \( Pr \) is the Prandtl number. The corresponding non-dimensional forms of boundary and mass flow conditions become

\[ v = 0 \; \text{and} \; T = -(\pm R_V/2) - R_H x \; \text{at} \; z = \pm 1/2, \quad (4.7) \]

\[ \int_{-1/2}^{1/2} u \, dz = Q, \quad \int_{-1/2}^{1/2} v \, dz = 0, \quad (4.8) \]

where \( R_V \) and \( R_H \) are vertical and horizontal Rayleigh numbers, respectively, and are defined as

\[ R_V = g\alpha_T d^3 \Delta T/\nu \kappa, \quad R_H = g\alpha_T d^4 \beta/\nu \kappa. \quad (4.9) \]

The basic steady state solution \((u_s, T_s, p_s)\) of equations (4.4)-(4.6) satisfying the boundary and mass flow conditions (4.7)-(4.8) is

\[ u_s = R_H f_1(z) + Qr(z), \quad v_s = 0, \quad w_s = 0 \]

\[ T_s = -R_H x - R_V z + R_H^2 g_1(z) - R_H Qs(z) \quad (4.10) \]

\[ \nabla p_s = \nabla^2 u_s + T_s k, \quad . \]
where

\[ f_1(z) = \frac{1}{24}(z - 4z^3), \quad g_1(z) = \frac{1}{5760}(7z - 40z^3 + 48z^5) \]

\[ r(z) = \frac{3}{2}(1 - 4z^2), \quad s(z) = \frac{1}{4}(3z^2 - 2z^4) \]

(4.11)

We now perturb the steady state solution as follows:

\[ v = u_s + u, \quad T = T_s + \theta, \quad P = p_s + p \]

(4.12)

The perturbation equations then take the form

\[ \nabla \cdot u = 0 \]

(4.13)

\[ Pr^{-1} \left[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right] = -\nabla p + \nabla^2 u + \theta k - Pr^{-1}[(u_s \cdot \nabla)u + (u \cdot \nabla)u_s] \]

(4.14)

\[ \frac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta = \nabla^2 \theta - u_s \cdot \nabla \theta - u \cdot \nabla T_s \]

(4.15)

where \( u_s \) and \( T_s \) are given by (4.10). The corresponding boundary conditions become

\[ u = 0 \quad \text{and} \quad \theta = 0 \quad \text{at} \quad z = \pm 1/2 \]

(4.16)

We define an energy functional as follows

\[ E(t) = \frac{1}{2Pr} \|u\|^2 + \frac{\xi}{2} \|\theta\|^2 \]

(4.17)

where \( \xi \) is a positive coupling parameter. We remark that the energy functional defined here includes the velocity, which is absent in the energy functionals defined in the previous two chapters. On multiplying (4.14) by \( u \), (4.15) by \( \theta \) and integrating over \( V \), we find (after using the boundary conditions and divergence theorem)

\[ \frac{1}{2} \frac{d}{dt} \|\theta\|^2 = -\|\nabla \theta\|^2 - \langle u_s \cdot \nabla T_s \rangle \theta \]

(4.18)

\[ \frac{1}{2Pr} \frac{d}{dt} \|u\|^2 = -\|\nabla u\|^2 + \langle \theta w \rangle - Pr^{-1} \langle u_s \cdot \nabla \rangle u_s \cdot u \]

(4.19)

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Here $V$ denotes a typical periodicity cell, $< \cdot >$ denotes the integration over $V$, and $\| \cdot \|$ denotes the $L^2(V)$ norm. The system of equation (4.18) and (4.19), along with (4.17), can be put in the form

$$\frac{dE}{dt} = I - D$$  \hspace{1cm} (4.20)

where

$$I = -\xi < (u \cdot \nabla T_s) \theta > + < \theta w > - Pr^{-1} < (u \cdot \nabla) u_s \cdot u >$$  \hspace{1cm} (4.21)

$$D = \xi \| \nabla \theta \|^2 + \| \nabla u \|^2$$  \hspace{1cm} (4.22)

We now define

$$m = \max_{\mathcal{M}} \frac{I}{D}$$  \hspace{1cm} (4.23)

where $\mathcal{M}$ is the space of admissible solutions. On combining (4.20) with (4.21)-(4.23) we can infer

$$\frac{dE}{dt} \leq -D(1 - m)$$

which, for $0 < m < 1$, on using the Poincaré inequality, becomes

$$\frac{dE}{dt} \leq -2\pi^2 (1 - m) \min(1, Pr)E$$  \hspace{1cm} (4.24)

Inequality (4.24) clearly indicates that for $0 < m < 1$, $E(t) \to 0$ at least exponentially as $t \to \infty$.

4.3. Eigenvalue Problems of Nonlinear and Linear Stability Analysis.

We now return to (4.23) and use calculus of variation to find the maximum problem at the critical argument $m = 1$. The associated Euler-Lagrange equations become

$$-\xi \nabla T_s \cdot u + w + 2\xi \nabla^2 \theta = 0$$  \hspace{1cm} (4.25)

$$-\xi \nabla T_s \theta + \theta k - Pr^{-1} \frac{\partial u_s}{\partial z}(wi + uk) + 2\nabla^2 u = \nabla \omega$$  \hspace{1cm} (4.26)
where \( \omega \) is a Lagrange multiplier introduced since \( u \) is solenoidal. On taking \( curl \) of (4.26) and then taking the third component of resulting equation, we find

\[
2 \nabla^4 w + \nabla^2 (h \theta - f_2 R_H u) = R_H \frac{\partial^2}{\partial x \partial z} [\xi \theta - f_2 w],
\]

(4.27)

where
\[
\nabla^2_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]

and

\[
f_2 = Pr^{-1} \left[ \frac{1}{24} (1 - 12z^2) - 12 \frac{Q z}{R_H} \right], \quad h = 1 + \xi (R_V - R_H^2 g_2 + R_H Q s_1),
\]

\[
g_2 = \frac{1}{5760} (7 - 120z^2 + 240z^4), \quad s_1 = \frac{1}{2} (3z - 4z^3).
\]

(4.28)

Also (4.25) and the \( x \)-component of (4.26) can be written as

\[
\xi R_H u + hw + 2 \nabla^2 \theta = 0,
\]

(4.29)

\[
\xi R_H \theta - f_2 R_H w + 2 \nabla^2 u = \frac{\partial \omega}{\partial x}.
\]

(4.30)

Following Nield (1994b), for higher value of \( Pr \), we also restrict the analysis to longitudinal disturbances, as it is the most unstable mode at high value of \( Pr \). We now perform the standard normal mode analysis and look for the solution of the above in the form

\[
[u, w, \theta, \omega] = [u(z), w(z), \theta(z), \omega(z)] \exp [i\alpha x].
\]

(4.31)

On substituting (4.31) into (4.27), (4.29)-(4.30), and eliminating the different variables we find that it is possible to reduce the eigenvalue problem to an eighth order system. This system, after considerable rearrangement, can be written as

\[
D^4 w = h_1 w + h_2 D^2 w + h_3 \theta + h_4 D^2 \theta,
\]

(4.32)

\[
D^4 \theta = h_5 w + h_6 D w + h_7 D^2 w + h_8 \theta + h_9 D^2 \theta,
\]

(4.33)

where \( D^i = d^i/dz^i \) \((i = 1, 2, 3, 4)\), and variables \( h_1, \ldots, h_9 \), are given by

\[
h_1 = \frac{\alpha^2}{2\xi} f_2 h - \alpha^4, \quad h_2 = 2\alpha^2, \quad h_3 = \frac{\alpha^2}{2} h - \alpha^4 f_2,
\]

\[
h_4 = \alpha^2 f_2, \quad h_5 = \alpha^2 h - R_H^2 f_2, \quad h_6 = R_H g_2 - R_H Q s_2,
\]

\[
h_7 = -h, \quad h_8 = \frac{\xi R_H^2}{4} - \alpha^4, \quad h_9 = 2\alpha^2,
\]

(4.34)
In the above $f_2$ and $h$ are given by (4.28) and

\[
\begin{align*}
    h_0 &= R_H^2 g_4 - R_H Q s_3, \\
    g_3 &= \frac{1}{576}(-24z + 96z^3), \\
    s_2 &= \frac{1}{2}(3 - 12x^2), \quad (4.35) \quad s_3 = -12z.
\end{align*}
\]

The relevant boundary conditions are

\[
w = Dw = \theta = D^2\theta = 0 \quad .
\]

(4.36)

We consider $R_V$ as the eigenvalue with the remaining variables as parameters. The critical energy vertical Rayleigh number is defined by

\[
R_E = \max_{\alpha^2} \min_{\xi} R_V(R_H, \alpha^2, \xi, Pr, Q) \quad (4.37)
\]

On letting $x_1 = w, x_2 = Dw, x_3 = D^2w, x_4 = D^3w, x_5 = \theta, x_6 = D\theta, x_7 = D^2\theta, x_8 = D^3\theta$, the system (4.32)-(4.33) can be written in the matrix form as

\[
\dot{X} = AX \quad .
\]

(4.38)

where $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)^T$ and $A$ is given by

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
h_5 & h_6 & h_7 & 0 & h_8 & 0 & h_9 & 0
\end{bmatrix}
\]
The boundary conditions now take the form

\[ x_1 = x_2 = x_5 = x_7 = 0, \quad \text{at} \quad z = \pm 1/2 \quad . \quad (4.39) \]

For analyzing the linear instability results, we return to the perturbed equations (4.13)-(4.16), neglecting the nonlinear terms. We perform the standard stationary mode analysis and look for the solution of these equations in the form (4.31). We find that the relevant equations in this case reduce to

\[
D^4 w = -\alpha^4 w + 2\alpha^2 D^2 w + \alpha^2 \theta, \quad (4.40)
\]

\[
D^4 \theta = (D^3 T_s - \alpha^2 D T_s - R_H Pr^{-1} D u_s) w + 2 D^2 T_s D w + DT_s D^2 w - \alpha^4 \theta + 2\alpha^2 D^2 \theta, \quad (4.41)
\]

where \( u_s \) and \( T_s \) are given by (4.10). A quick comparison between equations (4.40)-(4.41) and (4.32)-(4.33) shows that the two set of equations are similar, provided we redefine the variables \( h_1, \cdots, h_9 \) in the present case, as

\[
h_1 = -\alpha^4, \quad h_2 = 2\alpha^2, \quad h_3 = \alpha^2, \quad h_4 = 0, \quad h_5 = D^3 T_s - \alpha^2 D T_s - R_H Pr^{-1} D u_s, \quad h_6 = 2 D^2 T_s, \quad (4.42)
\]

\[
h_7 = DT_s, \quad h_8 = -\alpha^4, \quad h_9 = 2\alpha^2,
\]

The boundary conditions in the present case are the same as (4.36). We again consider \( R_V \) as eigenvalue and \( R_H, Pr \) and \( Q \) as parameters. The critical vertical Rayleigh number, in the linear case, is defined as

\[
R_L = \min_{\alpha^2} R_V(R_H, \alpha^2, Pr, Q) \quad (4.43)
\]

Following the procedure as stated earlier in the energy stability case, we next write the system (4.40)-(4.41) in the matrix form (4.38), with the entries for \( h_1, \cdots, h_9 \) given by (4.42).
4.4 Numerical Results.

We employ the compound matrix method and the secant method, and carry out the maximization and minimization routines by golden section search. We solve the eigenvalue problem (4.38) and (4.39) by fixing values of some parameters. It is noted that the resulting compound matrix variables generate 70 linear first order coupled differential equations, which are provided in the Appendix. Tables 1-9 display the computed energy stability results for the critical values of \( R_V \), for different values of \( Pr, R_H \) and \( Q \). The corresponding results for the linear instability case are documented in Tables 10-18.

<table>
<thead>
<tr>
<th>( R_H )</th>
<th>( R_E )</th>
<th>( a_E^2 )</th>
<th>( R_H )</th>
<th>( R_E )</th>
<th>( a_E^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
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<td>4.6527</td>
</tr>
<tr>
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<td>9.4957</td>
<td>1200</td>
<td>1456.60</td>
<td>4.1491</td>
</tr>
<tr>
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<td>1711.51</td>
<td>9.2418</td>
<td>1300</td>
<td>1330.57</td>
<td>3.7233</td>
</tr>
<tr>
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<td>1400</td>
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<td>3.3636</td>
</tr>
<tr>
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<td>1500</td>
<td>986.972</td>
<td>3.0616</td>
</tr>
<tr>
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<td>1722.16</td>
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<td>1600</td>
<td>767.550</td>
<td>2.8062</td>
</tr>
<tr>
<td>700</td>
<td>1717.44</td>
<td>7.2565</td>
<td>1700</td>
<td>515.606</td>
<td>2.5816</td>
</tr>
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</tr>
<tr>
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<td>1840</td>
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<td>2.3284</td>
</tr>
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<td>0.00000</td>
<td>2.2894</td>
</tr>
</tbody>
</table>

Tables 1-3 show the effect of increasing Prandtl number on the critical value of \( R_V \) (= \( R_E \)) when \( Q \) has a small value. We note that for lower value of \( R_H \), as \( R_H \) increases the value of \( R_E \) also increases but soon it starts decreasing as \( R_H \) values are increased further. This means that for smaller values of \( R_H \), the horizontal temperature gradient has a stabilizing effect on \( R_E \), but as \( R_H \) is increased con-
considerably, it has a destabilizing effect. We also note that for smaller values of \( R_H \), increasing \( Pr \) has a stabilizing effect but for higher value of \( R_H \), the effect of \( Q \) almost becomes negligible and the Prandtl number then has a destabilizing effect.

<table>
<thead>
<tr>
<th>( R_H )</th>
<th>( R_E )</th>
<th>( a^2_E )</th>
<th>( R_H )</th>
<th>( R_E )</th>
<th>( a^2_E )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1100</td>
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</tr>
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<td>1200</td>
<td>1382.92</td>
<td>4.0876</td>
</tr>
<tr>
<td>300</td>
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<td>8.8897</td>
<td>1400</td>
<td>1059.90</td>
<td>3.3130</td>
</tr>
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<tr>
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</tr>
<tr>
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<td>6.5243</td>
<td>1750</td>
<td>169.318</td>
<td>2.4529</td>
</tr>
<tr>
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</tr>
<tr>
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<td>5.1709</td>
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<td></td>
</tr>
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</table>

Table 2. Nonlinear Critical Vertical Rayleigh Number for \( Pr = 50, Q = 1 \)

<table>
<thead>
<tr>
<th>( R_H )</th>
<th>( R_E )</th>
<th>( a^2_E )</th>
<th>( R_H )</th>
<th>( R_E )</th>
<th>( a^2_E )</th>
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<tbody>
<tr>
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<td>1200</td>
<td>1373.57</td>
<td>4.0809</td>
</tr>
<tr>
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<td>9.2509</td>
<td>1300</td>
<td>1225.85</td>
<td>3.6584</td>
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<tr>
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<td>8.8899</td>
<td>1400</td>
<td>1045.36</td>
<td>3.3056</td>
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<tr>
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<td>1500</td>
<td>831.027</td>
<td>3.0086</td>
</tr>
<tr>
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<td>7.8597</td>
<td>1600</td>
<td>582.131</td>
<td>2.7573</td>
</tr>
<tr>
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<td>7.2133</td>
<td>1700</td>
<td>298.165</td>
<td>2.5401</td>
</tr>
<tr>
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<td>6.5177</td>
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<td>142.913</td>
<td>2.4495</td>
</tr>
<tr>
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<td>5.8191</td>
<td>1794</td>
<td>0.00000</td>
<td>2.3666</td>
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<tr>
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Table 4. Nonlinear Critical Vertical Rayleigh Number for $Pr = 10$, $Q = 5$

<table>
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<th>$R_H$</th>
<th>$R_E$</th>
<th>$a_E^2$</th>
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</thead>
<tbody>
<tr>
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<td>1000</td>
<td>1392.45</td>
<td>5.2262</td>
</tr>
<tr>
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<td>1670.69</td>
<td>9.4170</td>
<td>1100</td>
<td>1293.71</td>
<td>4.6515</td>
</tr>
<tr>
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<td>1656.35</td>
<td>9.1384</td>
<td>1200</td>
<td>1168.91</td>
<td>4.1513</td>
</tr>
<tr>
<td>400</td>
<td>1640.55</td>
<td>8.7718</td>
<td>1300</td>
<td>1015.71</td>
<td>3.7266</td>
</tr>
<tr>
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<td>8.3184</td>
<td>1400</td>
<td>832.444</td>
<td>3.3709</td>
</tr>
<tr>
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<td>7.7831</td>
<td>1500</td>
<td>617.996</td>
<td>3.0672</td>
</tr>
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<td>7.1776</td>
<td>1600</td>
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<td>2.8133</td>
</tr>
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<td>6.5249</td>
<td>1700</td>
<td>92.6886</td>
<td>2.5959</td>
</tr>
<tr>
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<td>5.8612</td>
<td>1731</td>
<td>0.00000</td>
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</tr>
</tbody>
</table>

Table 5. Nonlinear Critical Vertical Rayleigh Number for $Pr = 50$, $Q = 5$

<table>
<thead>
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<th>$R_H$</th>
<th>$R_E$</th>
<th>$a_E^2$</th>
<th>$R_H$</th>
<th>$R_E$</th>
<th>$a_E^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1704.14</td>
<td>9.6535</td>
<td>1000</td>
<td>1478.68</td>
<td>5.3533</td>
</tr>
<tr>
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<td>1700.19</td>
<td>9.5032</td>
<td>1100</td>
<td>1383.20</td>
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<tr>
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<td>9.2599</td>
<td>1200</td>
<td>1259.91</td>
<td>4.2239</td>
</tr>
<tr>
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<td>8.9229</td>
<td>1300</td>
<td>1106.22</td>
<td>3.7781</td>
</tr>
<tr>
<td>500</td>
<td>1676.95</td>
<td>8.4921</td>
<td>1400</td>
<td>920.347</td>
<td>3.4040</td>
</tr>
<tr>
<td>600</td>
<td>1660.78</td>
<td>7.9703</td>
<td>1500</td>
<td>701.091</td>
<td>3.0905</td>
</tr>
<tr>
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<td>1636.59</td>
<td>7.3674</td>
<td>1600</td>
<td>447.646</td>
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</tr>
<tr>
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<td>1601.00</td>
<td>6.7043</td>
<td>1700</td>
<td>159.458</td>
<td>2.6089</td>
</tr>
<tr>
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<td>1751</td>
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</tbody>
</table>

Tables 4 – 6 show the effect when $Q = 5$. Here we find increasing $R_H$ always reduces $R_E$. Thus, irrespective of the value of the horizontal temperature gradient, increasing mass flow-rate has a destabilizing effect on $R_E$. This fact is further confirmed by Tables 7-9 where, for $Q = 10$, the $R_E$ values fall much faster as $R_H$ values
increase. We note that for higher values of $Q$, the effect of increasing Prandtl number appears to be stabilizing on $R_E$. This fact is in contrast to no mass flow case, where increasing Prandtl number value leads to a lower critical $R_E$ value. Thus, to summarize, we can conclude that the effect of non-zero $Q$ is destabilizing and that it is tightly coupled with $R_H$ values. In general, as $R_H$ increases, the destabilizing effect also increases. The effect of $Q$ on the critical wave number $a_E$ can now be stated very briefly. We note that as $Q$ increases, for fixed $R_H$ and $Pr$, the value of $a_E$ decreases but for fixed $Q$ and $R_H$, as $Pr$ increases the value of $a_E$ also increases.

Table 6. Nonlinear Critical Vertical Rayleigh Number for $Pr = 100$, $Q = 5$

<table>
<thead>
<tr>
<th>$R_H$</th>
<th>$R_E$</th>
<th>$a_E^2$</th>
<th>$R_H$</th>
<th>$R_E$</th>
<th>$a_E^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
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<td>9.6592</td>
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</tr>
<tr>
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<td>1703.20</td>
<td>9.5140</td>
<td>1100</td>
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</tr>
<tr>
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<td>1699.24</td>
<td>9.2753</td>
<td>1200</td>
<td>1271.08</td>
<td>4.2328</td>
</tr>
<tr>
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<td>8.9420</td>
<td>1300</td>
<td>1117.37</td>
<td>3.7851</td>
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<td>1400</td>
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<td>3.4102</td>
</tr>
<tr>
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<td>7.9942</td>
<td>1500</td>
<td>711.376</td>
<td>3.0968</td>
</tr>
<tr>
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<td>7.3916</td>
<td>1600</td>
<td>457.070</td>
<td>2.8311</td>
</tr>
<tr>
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<td>6.7274</td>
<td>1700</td>
<td>167.740</td>
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</tr>
<tr>
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</table>
### Table 7. Nonlinear Critical Vertical Rayleigh Number for Pr = 10, Q = 10

<table>
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<th>$R_E$</th>
<th>$a_E^2$</th>
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<tbody>
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<tr>
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<td>1100</td>
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<td>5.0307</td>
</tr>
<tr>
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<td>1200</td>
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</tr>
<tr>
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</tr>
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<td>1480</td>
<td>23.8497</td>
<td>3.3297</td>
</tr>
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### Table 8. Nonlinear Critical Vertical Rayleigh Number for Pr = 50, Q = 10

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<th>$R_E$</th>
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<tbody>
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</tr>
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65
Table 9. Nonlinear Critical Vertical Rayleigh Number for \( Pr = 100, \ Q = 10 \)

<table>
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<th>( R_E )</th>
<th>( a_E^2 )</th>
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<tr>
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<td>1300</td>
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<td>4.3129</td>
</tr>
<tr>
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<td>8.8222</td>
<td>1400</td>
<td>612.515</td>
<td>3.8374</td>
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<tr>
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<td>8.4443</td>
<td>1500</td>
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</tr>
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<td>6.8620</td>
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</table>

Table 10. Linear Critical Vertical Rayleigh Number for \( Pr = 10, \ Q = 0 \)

<table>
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<th>( R_H )</th>
<th>( R_L )</th>
<th>( a_L^2 )</th>
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<td>36.301</td>
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<td>9.9290</td>
<td>8000</td>
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<td>39.732</td>
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<td>10.616</td>
<td>8600</td>
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<td>44.869</td>
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<td>11200</td>
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<td>68.645</td>
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</table>

Tables 10-12 show the effect of increasing \( Q \), with constant \( Pr = 10 \), on the critical value of the vertical Rayleigh number \( R_V (= R_L) \). We note that in all three cases as \( R_H \) value increases from zero, the value of \( R_L \) also increases but after a
certain stage an increase in $R_H$ values results in a decrease in $R_L$ values. We find that this decreasing starts earlier as $Q$ values are increased. Thus in the case of $Q = 0$, the decrease starts around a value of $R_H = 8300$, while for $Q = 10$ it occurs around $R_H = 4600$. We also note that as $Q$ increases, the critical vertical Rayleigh number $R_L$ decreases. Thus increasing $R_H$ first has a stabilizing effect, but as the horizontal temperature gradient increases further it has a destabilizing effect. The effect of increasing $Q$, irrespective of $R_H$ values, is always destabilizing.

<table>
<thead>
<tr>
<th>$R_H$</th>
<th>$R_L$</th>
<th>$a_L^2$</th>
<th>$R_H$</th>
<th>$R_L$</th>
<th>$a_L^2$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>6000</td>
<td>15149.6</td>
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<td>6600</td>
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<td>39.986</td>
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<td>7200</td>
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Table 12. Linear Critical Vertical Rayleigh Number for $Pr = 10$, $Q = 10$

<table>
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<td>5600</td>
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</tr>
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</tr>
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<td>43.002</td>
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Table 13. Linear Critical Vertical Rayleigh Number for $Pr = 50$, $Q = 0$

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<th>$a_L^2$</th>
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<td>9.7625</td>
<td>8000</td>
<td>24481.8</td>
<td>38.627</td>
</tr>
<tr>
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<td>9.9337</td>
<td>8600</td>
<td>24422.9</td>
<td>41.625</td>
</tr>
<tr>
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<td>10.284</td>
<td>9200</td>
<td>23963.6</td>
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</tr>
<tr>
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<td>9800</td>
<td>23017.9</td>
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<td>20869.5</td>
<td>53.741</td>
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</tbody>
</table>
Tables 13-15 show the effect when $Pr = 50$ and $Q$ is increased from 0 to 10. In this case, the variation of $R_H$ with $R_L$ is similar to the previous case and the effect of variation of $Q$ is also quite similar. In fact we notice very little overall changes in the values as compared to the $Pr = 10$ case. A somewhat parallel situation is observed by looking at the Tables 16-18. We find that increasing $Pr$ has very little effect on $R_L$ values. In fact we only notice a slight change in the values for $R_H$ when the effect changes from stabilizing to destabilizing.

<table>
<thead>
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<tr>
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<td>7800</td>
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<td>8400</td>
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<tr>
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<td>9200</td>
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<tr>
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<td>9800</td>
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Table 15. Linear Critical Vertical Rayleigh Number for $Pr = 50$, $Q = 10$

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</tr>
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Table 16. Linear Critical Vertical Rayleigh Number for $Pr = 100$, $Q = 0$

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</tr>
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Table 17. Linear Critical Vertical Rayleigh Number for \( Pr = 100, \ Q = 5 \)

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<td>51.264</td>
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</table>

Table 18. Linear Critical Vertical Rayleigh Number for \( Pr = 100, \ Q = 10 \)

<table>
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</table>

We close this section with two important observations. The first of these is about the use of Nield's (1994b) low-order Galerkin method. In both case, \( Pr = 10 \) and
$Pr = 100$, when $Q = 0$, the cases reported by Nield, we find that our results agree with his results but these only represent one end of the spectrum. Nield (1994b) clearly realizes the limitation of his calculation, and suggests exploring the results for higher value of $R_H$. We have, however, not found any other work. Hence any conclusion, solely based upon the low-order Galerkin approximation, must be considered with utmost caution. The second observation concerns the wide difference between the critical values obtained by the two methods. We note that the energy method, though providing the rigorous criteria for stability, is overly conservative. On the other hand, the linear instability criteria may, most likely, be too high. Since there are no experimental results available for comparison at this time, nothing definite can be said about either method.
Chapter 5
Linear Stability Analysis for Convection
Induced by Inclined Temperature Gradient and Surface Tension

5.1. Introduction.

There are two mechanisms responsible for driving the convective instability. The first one is buoyancy driven due to the density variation generated by the thermal expansion of the fluid, and the second cause of instability results from surface tension gradients due to temperature fluctuations at the upper surface of the layer. The instabilities discussed in the previous three chapters are due to buoyancy. In this chapter, we turn our discussion to the convection problem induced by inclined temperature gradients and surface tension by means of the linear stability analysis.

Bénard (1900) was the first to perform systematic observations on a shallow fluid layer heated from below. He observed the occurrence of a stable steady state formed by a pattern of hexagonal cells. Bénard's experiments were interpreted by Rayleigh (1916), who showed that buoyancy was the driving force for cellular convection. Pearson (1958), however, neglected buoyancy and offered a new explanation for the instability. He showed that if the upper surface was free then Bénard type cells could be produced by tractions arising from the variation with temperature of surface tension. He argued that in many of Bénard's experiments the cells observed must have been due to a surface tension effect rather than buoyancy. It is usual in the literature to refer to the instability produced by buoyancy as the Rayleigh-Bénard effect, while the term Marangoni instability is used for surface tension driven convection. When both buoyancy and surface tension are acting simultaneously, it is called Bénard-Marangoni instability.

Nield (1964) was the first author to perform a linear stability analysis for determining the Bénard-Marangoni instability thresholds in the case of quiescent fluid of
horizontal infinite extent subjected to a vertical temperature gradient. Nield's results showed that for steady convection two destabilizing mechanisms reinforce one another. Takashima (1970) showed numerically that overstable convection cannot occur if the free surface is non-deformable. Davis (1969b) analyzed Nield's problem by the energy method, which was further extended by Davis and Homsy (1980) for a deformable free surface. A nonlinear stability analysis for the buoyancy-surface tension effects in a ferromagnetic fluid layer was developed by Qin and Kaloni (1994). The Bénard-Marangoni linear instability analysis in a viscoelastic layer for a Maxwell fluid model was investigated by Dauby et al. (1993), and for the Jeffrey fluid model by Lebon et al. (1994).

The imposed temperature gradient can take two preferred directions: either a pure vertical one or a pure horizontal one. The presence of a horizontal temperature gradient raises several additional difficulties. First, the rest state is not the solution of the balance equations and, therefore, a preliminary study must be done to determine the basic temperature and velocity profiles. The usual linear perturbation techniques can then be used to find the thresholds of the first instability mode. The second difficulty is related to the theorem of exchange of stability, which has not been proven for a horizontal temperature gradient. Experimental and numerical simulations show the presence of oscillatory regimes. In fact, even in the case of absence of horizontal temperature gradient, both Pearson's (1958) and Nield's (1964) linear analysis were based on the assumption that the neutral state is a stationary one, and no analytical proof to validate this assumption has been found to this date. Nevertheless, the exclusion of the possibility of overstability, or the validity of the principle of exchange of stability, was numerically confirmed for Pearson's problem by Vidal and Acrivos (1966), and for Nield's problem by Takashima (1970). Thirdly, the instability thresholds depend strongly on the Prandtl number.
while it is independent of this parameter when a vertical temperature difference alone is acting. In the case of horizontal temperature gradient, a linear analysis of Marangoni instability and Bénard-Marangoni instability for the special cases was performed by Smith and Davis (1983), and Parmentier et al. (1993), respectively. Their numerical results showed the presence of the oscillatory instability at lower Prandtl number, and that the longitudinal mode is the only possible stationary mode for higher Prandtl number.

In this chapter, we consider the linear stability of convection in a horizontal layer of fluid, induced by inclined temperature gradients with surface tension in a general case. The resulting eigenvalue equations are complex. As we have pointed out in Chapter 1, the compound matrix method is not suitable for the cases when the principle of exchange of stability is not valid, or the order of differential equations is too high (Lindsay and Ogden 1992, Dongarra et al. 1996). To cope with this difficulty, we introduce and implement the Chebyshev tau-QZ method to solve the associated complex eigenvalue problem. The algorithm presented for finding the neutral stability curve is not only powerful, but also can be used to numerically verify whether the principle of exchange of stability is valid or not.

5.2. Chebyshev Polynomials.

The Chebyshev polynomials, named after the Russian mathematician Chebyshev, were discovered more than a century ago. Their importance for practical computation, however, was due to Lanczos (1938, 1957). The advent of the digital computer gave further emphasis to their development, and the research literature of numerical mathematics abounds with papers on applications of Chebyshev polynomials and the theory and practice of Chebyshev approximation. Use of Chebyshev polynomials in hydrodynamic stability problems has been discussed by, for instance, Orszag (1971), Lindsay and Ogden (1992), Straughan and Walker (1996a), and Dongarra
et al. (1996). A brief review of this subject has been given in the first chapter of this thesis. In this section, we introduce and define some matrices related to Chebyshev polynomials. These matrices will be used as fundamental block matrices for constructing the generalized eigenvalue equations resulting from the application of the Chebyshev tau method to the differential equations in Section 5.4.

The Chebyshev polynomial of degree $n$, $T_n(z)$, is defined by

$$ T_n(z) = \cos(n \arccos z), \quad -1 \leq z \leq 1, \quad n = 0, 1, \ldots \quad (5.1) $$

With the introduction of $z = \cos \theta$, this yields

$$ T_n(z) = \cos(n \theta), \quad z = \cos \theta, \quad n = 0, 1, \ldots \quad (5.2) $$

Some properties of Chebyshev polynomials (Orszag 1971), which are easily deduced from the definition (5.2), and will be used later, are:

$$ T_n(\pm 1) = (\pm 1)^n, \quad T_n' (\pm 1) = (\pm 1)^{n+1} n^2, \quad n \geq 0 \quad (5.3) $$

$$ 2T_m(z)T_n(z) = T_{m+n}(z) + T_{|m-n|}(z) \quad m \geq 0, \quad n \geq 0 \quad (5.4) $$

$$ \int_{-1}^{1} (1 - z^2)^{-1/2} T_m(z)T_n(z)dz = \frac{\pi c_n}{2} \delta_{mn} \quad m \geq 0, \quad n \geq 0 \quad (5.5) $$

$$ 2xT_n(z) = c_n T_{n+1}(z) + d_{n-1} T_{n-1}(z) \quad n \geq 0 \quad (5.6) $$

$$ 2T_n(z) = \frac{c_n}{n+1} T_{n+1}'(z) - \frac{d_{n-2}}{n-1} T_{n-1}'(z) \quad n \geq 0 \quad (5.7) $$

where $\delta_{mn}$ is the Kronecker delta defined by

$$ \delta_{mn} = \begin{cases} 
1 & \text{if } m = n \\
0 & \text{if } m \neq n 
\end{cases} $$

and the constants $c_n$ and $d_n$ are defined by

$$ \begin{cases} 
c_0 = 2, & d_0 = 1, \\
c_n = 1, & d_n = 1, \quad \text{if } n > 0 \\
c_n = 0, & d_n = 0, \quad \text{if } n < 0 
\end{cases} \quad (5.8) $$
The Chebyshev polynomial $T_n(z)$ represented in terms of polynomials $z^r, r \leq n$, can be generated from definition (5.2) and the recurrence relation (5.6), and the first few successive members are:

\[
\begin{align*}
T_0(z) &= 1 \\
T_1(z) &= z \\
T_2(z) &= 2z^2 - 1 \\
T_3(z) &= 4z^3 - 3z \\
T_4(z) &= 8z^4 - 8z^2 + 1
\end{align*}
\]  \quad (5.9)

The polynomials of even order are obviously even functions of $z$, and the polynomials of odd order are odd functions of $z$. In our applications later, however, it is desirable to express the powers of $z$ in terms of polynomials $T_r(z)$. We can easily find the first few such formulae by reversing (5.9), with the omission of the argument $z$ for simplicity,

\[
\begin{align*}
1 &= T_0 \\
z &= T_1 \\
z^2 &= \frac{1}{2}[T_2 + T_0] \\
z^3 &= \frac{1}{4}[T_3 + 3T_1] \\
z^4 &= \frac{1}{8}[T_4 + 4T_2 + 3T_0]
\end{align*}
\]  \quad (5.10)

and again we note the separation of the even and odd formulae.

From the formulae for derivatives (5.7), by separating even and odd formulae, we have
\[
\begin{align*}
T_1' &= T_0 & T_2' &= 2T_1 \\
\frac{T_3'}{3} - T_1' &= 2T_2 & \frac{T_4'}{4} - T_2' &= 2T_3 \\
\frac{T_5'}{5} - \frac{T_3'}{3} &= 2T_4 & \frac{T_6'}{6} - \frac{T_4'}{4} &= 2T_5 \\
&\vdots & & \vdots & & \vdots \\
\frac{T_n'}{n} - \frac{T_{n-2}'}{n-2} &= 2T_{n-1} & \frac{T_n'}{n} - \frac{T_{n-2}'}{n-2} &= 2T_{n-1}
\end{align*}
\]

(5.11)

where \( n \) is odd in the left column, and even in the right column. By adding each equation in the same column of (5.11), we thus deduce that the derivative of a Chebyshev polynomial can be represented as a linear combination of lower order Chebyshev polynomials, namely,

\[
T_n' = \begin{cases} 
2n(T_1 + T_3 + \cdots + T_{n-1}) = 2n \sum_{k=1}^{\frac{n}{2}} T_{2k-1}, & n \text{ even} \\
2n(T_2 + T_4 + \cdots + T_{n-1}) + nT_0 = 2n \sum_{k=1}^{\frac{n-1}{2}} T_{2k} + nT_0, & n \text{ odd}
\end{cases}
\]

(5.12)

Eq.(5.12) allows us to introduce the first differentiation matrix \( D = [D_{ij}] \), whose non-zero entries are given by

\[
\begin{align*}
D_{0,2j-1} &= 2j - 1, & j &\geq 1 \\
D_{i,i+2j-1} &= 2(i + 2j - 1), & i &\geq 1, j \geq 1
\end{align*}
\]

(5.13)

(5.14)

Note that for any \( M \geq m - 1 \), we have

\[
T_m'(z) = \sum_{k=0}^{M} D_{k,m} T_k(z),
\]

(5.15)

and from (5.15), we further get

\[
T_m'(z) = \sum_{k=0}^{M} D_{k,m} T_k'(z) = \sum_{k=0}^{M} \sum_{j=0}^{M} D_{k,j} D_{j,m} T_k(z) = \sum_{k=0}^{M} D_{k,m}^2 T_k(z)
\]

(5.16)
where

\[ D_{k,m}^2 = \sum_{j=0}^{M} D_{k,j} D_{j,m} \]  \hspace{1cm} (5.17)

Thus, we can introduce the second differentiation matrix \( \mathbf{D}^2 = [D_{ij}^2] \), where \( \mathbf{D}^2 = \mathbf{D} \cdot \mathbf{D} \) in the sense of matrix multiplication due to (5.17). The non-zero entries of the matrix \( \mathbf{D}^2 \) are given by

\[ D_{0,2j}^2 = 4j^3, \hspace{1cm} j \geq 1 \]  \hspace{1cm} (5.18)

\[ D_{i,i+2j}^2 = 4j(i+j)(i+2j), \hspace{1cm} i \geq 1, j \geq 1 \]  \hspace{1cm} (5.19)

The proof of (5.18) and (5.19) is obtained by observing, from (5.13)-(5.14), that,

\[ D_{i,j} = 0, \hspace{1cm} \text{if } i \geq j \text{ or } i + j \text{ is even} \]  \hspace{1cm} (5.20)

\[ D_{1,2j} = D_{3,2j} = \cdots = D_{2j-1,2j} = 4j, \hspace{1cm} j \geq 1 \]  \hspace{1cm} (5.21)

\[ D_{i,i+1} = 2(i+1), \hspace{1cm} i \geq 1 \]  \hspace{1cm} (5.22)

\[ D_{i,i+3} = 2(i+3) = 2(i+1) + 4, \hspace{1cm} i \geq 1 \]  \hspace{1cm} (5.23)

\[ \vdots \]

\[ D_{i,i+2j-1} = 2(i+2j-1) = 2(i+1) + 4(j-1), \hspace{1cm} i \geq 1, j \geq 1 \]  \hspace{1cm} (5.24)

\[ D_{i+1,i+2j} = D_{i+3,i+2j} = \cdots = D_{i+2j-1,i+2j} = 2(i+2j), \hspace{1cm} i \geq 0, j \geq 1 \]  \hspace{1cm} (5.25)

Using (5.20), we can easily deduce that

\[ D_{0,0}^2 = D_{0,2j-1}^2 = 0, \hspace{1cm} j \geq 1 \]  \hspace{1cm} (5.26)

\[ D_{i,j}^2 = 0, \hspace{1cm} j \leq i + 1, \hspace{1cm} i \geq 1 \]  \hspace{1cm} (5.27)

\[ D_{i,i+2j-1}^2 = 0, \hspace{1cm} j \geq 1 \]  \hspace{1cm} (5.28)

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and, along with the use of (5.13)-(5.14) and (5.22)-(5.25), we have,

\[
D_{0,2j}^2 = D_{0,1}D_{1,2j} + D_{0,3}D_{3,2j} + \cdots + D_{0,2j-1}D_{2j-1,2j}
\]
\[
= (1 + 3 + \cdots + 2j - 1) \cdot 4j = 4j^3, \quad j \geq 1
\]  \quad (5.29)

\[
D_{i,i+2j}^2 = D_{i,i+1}D_{i+1,i+2j} + D_{i,i+3}D_{i+3,i+2k} + \cdots + D_{i,i+2j-1}D_{i+2j-1,i+2j}
\]
\[
= 2(i + 2j)\left\{[2(i + 1)] + [2(i + 1) + 4] + \cdots + [2(i + 1) + 4(j - 1)]\right\}
\]
\[
= 4j(i + j)(i + 2j), \quad i \geq 1, \quad j \geq 1
\]  \quad (5.30)

The proof is thus completed due to the results of (5.26)-(5.30).

We now define the matrix

\[
Z^m = [z_{i,j}^{(m)}] \quad m = 1, 2, \cdots
\]  \quad (5.31)

where the entry of the matrix is defined by

\[
z_{i,j}^{(m)} = \frac{2}{\pi c_i} < z^m T_j, T_i >
\]  \quad (5.32)

in which the inner product is defined in the weighted space $L^2(-1, 1)$ by

\[
<f, g> = \int_{-1}^{1} \frac{fg}{\sqrt{(1-z^2)}} dz
\]  \quad (5.33)

The orthogonal property of Chebyshev polynomial (5.5) can be rewritten as

\[
<T_j, T_i> = \frac{\pi c_i}{2} \delta_{ij} \quad i \geq 0, \quad j \geq 0
\]  \quad (5.34)

Since we have shown that the powers of $z$, $z^m$ can be represented as the linear combination of Chebyshev polynomials $T_r(z)$, $r \leq m$, say, let us denote

\[
z^m = \sum_{k=0}^{m} \beta_k T_k, \quad m = 0, 1, \cdots
\]  \quad (5.35)
where $\beta_k$ are constants (see (5.10)). Due to (5.35), (5.4) and (5.34), we have,

$$z_{i,j}^{(m)} = \frac{2}{\pi c_i} < \sum_{k=0}^{m} \beta_k T_k T_j, T_i >$$
$$= \frac{2}{\pi c_i} \sum_{k=0}^{m} \frac{\beta_k}{2} < T_{j+k} + T_{j-k}, T_i >$$
$$= \sum_{k=0}^{m} \frac{\beta_k}{2} (\delta_{i,j+k} + \delta_{i,j-k}) \quad (5.36)$$

These $Z^m$ are small bandwidth matrices with mostly entries zero. In particular, when $m = 0$, $Z^0$ is reduced to the identity matrix $I$. For the purpose of fast implementation, we give the explicit representation for the non-zero entries of the matrix $Z^m$ for $m = 1, 2, 3, \text{ and } 4$, which will be used later on.

$$z_{i,i+1}^{(1)} = \frac{1}{2}, \quad i \geq 0, \quad z_{i+1,i}^{(1)} = \frac{1}{2}, \quad i \geq 1, \quad z_{i,0}^{(1)} = 1$$

$$z_{i,i+2}^{(2)} = \frac{1}{4}, \quad i \geq 0, \quad z_{i+1,i}^{(2)} = \frac{1}{4}, \quad i \geq 1, \quad z_{i,i}^{(2)} = \frac{2}{4} = \frac{1}{2}, \quad i \geq 0, \quad i \neq 1$$
$$z_{1,1}^{(2)} = \frac{3}{4}, \quad z_{2,0}^{(2)} = \frac{1}{4}$$

$$z_{i,i+3}^{(3)} = \frac{1}{8}, \quad i \geq 0, \quad z_{i+3,i}^{(3)} = \frac{1}{8}, \quad i \geq 1, \quad z_{i,i+1}^{(3)} = \frac{3}{8}, \quad i \geq 0, \quad i \neq 1$$
$$z_{i+1,i}^{(3)} = \frac{3}{8}, \quad i \geq 2, \quad z_{3,0}^{(3)} = \frac{2}{8} \quad z_{1,0}^{(3)} = \frac{6}{8}$$
$$z_{2,1}^{(3)} = \frac{4}{8}, \quad z_{1,2}^{(3)} = \frac{4}{8}$$

$$z_{i,i+4}^{(4)} = \frac{1}{16}, \quad i \geq 0, \quad z_{i+4,i}^{(4)} = \frac{1}{16}, \quad i \geq 1, \quad z_{i,i+2}^{(4)} = \frac{4}{16} = \frac{1}{4}, \quad i \geq 0, \quad i \neq 1$$
$$z_{i+2,i}^{(4)} = \frac{4}{16}, \quad i \geq 2, \quad z_{i,i}^{(4)} = \frac{6}{16}, \quad i \geq 0, \quad i \neq 1, 2 \quad z_{4,0}^{(4)} = \frac{2}{16} \quad z_{1,3}^{(4)} = \frac{5}{16}$$
$$z_{2,0}^{(4)} = \frac{8}{16}, \quad z_{3,1}^{(4)} = \frac{5}{16} \quad z_{1,3}^{(4)} = \frac{5}{16}$$
$$z_{1,1}^{(4)} = \frac{10}{16}, \quad z_{2,2}^{(4)} = \frac{7}{16} \quad z_{1,3}^{(4)} = \frac{5}{16}$$
Next, we define the matrix

$$Z^m D = [(z^m D)_{i,j}] = \frac{2}{\pi c_i} [< z^m T_j', T_i >] \quad m \geq 0 \quad (5.37)$$

Due to (5.15) and (5.32), we have

$$\begin{align*}
(z^m D)_{i,j} &= \frac{2}{\pi c_i} < z^m \sum_{k=0}^{M} D_{k,j} T_k, T_i > \\
&= \frac{2}{\pi c_i} \sum_{k=0}^{M} D_{k,j} < z^m T_k, T_i > \\
&= \sum_{k=0}^{M} z_{i,k}^{(m)} D_{k,j}
\end{align*} \quad (5.38)$$

Using (5.31), we obtain

$$Z^m D = Z^m \cdot D \quad m \geq 0 \quad (5.39)$$

in the sense of matrix multiplications. In a similar fashion, we can define and derive that

$$Z^m D^2 = [(z^m D^2)_{i,j}] = \frac{2}{\pi c_i} [< z^m T_j'', T_i >] = Z^m \cdot D^2, \quad m \geq 0 \quad (5.40)$$

In particular, since $Z^m$ is reduced to the identity matrix when $m = 0$, due to (5.37), (5.39) and (5.40), we could thus view that the matrix $D$ and $D^2$ are defined by

$$D = [D_{i,j}] = \frac{2}{\pi c_i} [< T_j', T_i >]$$
$$D^2 = [D^2_{i,j}] = \frac{2}{\pi c_i} [< T_j'', T_i >]$$

Since we have seen that the matrix $Z^m$ is, in fact, a small bandwidth matrix, the explicit representation of matrix of $Z^m D$ ($Z^m D^2$) linked with the matrix $D$ ($D^2$) only may be found for the purpose of fast implementation. The expressions for these matrices, however, are not provided here since they are not relevant to our present studies.
As we will see in the next section, with the introduction of these matrices defined above, we can greatly simplify the representation of the generalized matrix eigenvalue equation associated with the system of differential equations, whose order is not higher than second, with coefficients as the polynomials of independent variables. Consequently, the implementation in coding the program will become more easy and efficient.

5.3. Basic Equations and Steady State Solution.

We consider a fluid layer of infinite horizontal extent confined between a rigid plane \( z = 0 \) and a free surface whose height is located at \( z = d \). Since the deflection of the free surface is generally small (Davis 1969b, and Smith and Davis 1983), we assume that the free surface is flat. The Cartesian axes are chosen with the \( z \)-axis vertically upwards and the \( x \)-axis in the direction of applied horizontal temperature gradient. The fluid is Newtonian and incompressible, and the density variation subject to the Boussinesq approximation is given by

\[
\rho = \rho_0 [1 - \alpha_T (\bar{T} - T_0)].
\]

(5.41)

where \( \rho_0 \) is the density at temperature \( T_0 \) and \( \alpha_T \) the constant coefficient of volumetric expansion. The free upper surface is submitted to a surface tension \( \sigma \), whose equation of state is given by

\[
\sigma = \sigma_0 - \gamma (\bar{T} - T_0).
\]

(5.42)

where \( \sigma_0 \) is the surface tension at temperature \( T_0 \), \( \gamma \) the constant rate of change of surface tension with temperature ( \( \gamma \) is positive for most liquids).

For convenience, the variables are expressed in dimensionless form. Distance is scaled by the thickness of the layer \( d \), the velocity vector \( \mathbf{v} \), time \( t \), pressure \( P \), temperature difference and surface tension \( \sigma \) are scaled by \( \kappa d^{-1}, \kappa^{-1} d^2, \kappa \nu \rho_0 d^{-2} \),
\( \beta_T d \) and \( \sigma_0 \), respectively, where \( \kappa \equiv K C_p^{-1} \rho_0 \) is the diffusivity, \( \nu \equiv \mu \rho_0^{-1} \) is the kinematic viscosity and \( \beta_T \) is a characteristic temperature gradient whose physical meaning will be specified later when the boundary conditions are established. The following dimensionless numbers are also introduced:

\[
Pr \equiv \nu \kappa^{-1}, \quad Ra \equiv g \alpha_T \beta_T d^4 \kappa^{-1} \nu^{-1}
\]

\[
Ma \equiv \gamma \beta_T d^2 \kappa^{-1} \nu^{-1} \rho^{-1}, \quad B \equiv \hbar d K^{-1}
\]

where \( Pr \) is the Prandtl number, \( Ra \) the Rayleigh number, \( Ma \) the Marangoni number and \( B \) the Biot number, with \( h \) the thermal surface conductance. The Rayleigh number is representative of the buoyancy effect while the Marangoni number describes more particularly the thermocapillarity effects. Within Boussinesq's approximation, the governing dimensionless equations are given by the continuity equation, the Navier-Stokes equation, and the energy equation, respectively, as follows (cf. Chapter 4)

\[
\nabla \cdot \mathbf{v} = 0, \quad (5.43)
\]

\[
Pr^{-1} \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla P + \nabla^2 \mathbf{v} + Ra T \mathbf{k}, \quad (5.44)
\]

\[
\frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla) T = \nabla^2 T, \quad (5.45)
\]

where \( \nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \) is the nabla operator, and \( \mathbf{k} \) is the unit vector in the \( z \)-direction.

The fluid layer is submitted to an inclined temperature gradient with an arbitrary orientation with regard to the fluid layer. Let us denote by \( Q_h \beta_T K \) and \( Q_v \beta_T K \) the horizontal \( (x) \) and vertical \( (z) \) components of the imposed heat flux, written in dimensional form, \( Q_h \) and \( Q_v \) are multiplicative dimensionless factors while \( \beta_T \) is the common factor appearing in both components of the imposed heat flux. The corresponding nondimensional form of boundary conditions become:
On the rigid plane \( z = 0 \):

\[
\dot{u} = \dot{v} = \dot{w} = 0
\]  
\[
\frac{\partial T}{\partial z} = B_0(T - T_0^\infty) + Q_v
\]  
\[(5.46)\]  
\[(5.47)\]

Equation (5.46) expresses the no-slip condition while equation (5.47) is the general heat transfer condition. Subscript '0' refers to the lower rigid plane, \( B_0 \) is the Biot heat transfer coefficient, and \( T_0^\infty \) is the temperature of the external surroundings.

On the free top plane \( z = 1 \):

\[
\frac{\partial u}{\partial z} + Ma \frac{\partial T}{\partial x} = \frac{\partial v}{\partial z} + Ma \frac{\partial T}{\partial y} = w = 0
\]  
\[
-\frac{\partial T}{\partial z} = B_1(T - T_1^\infty) + Q_v
\]  
\[(5.48)\]  
\[(5.49)\]

Equations (5.48)-(5.49) are the boundary conditions for a non-deformable flat surface with a temperature dependent surface tension, and the subscript '1' refers to the upper free surface. It is worth noting that in Eqs. (5.47) and (5.49), \( B = \infty \) corresponds to a perfectly heat conduction surface, and \( B = 0 \) to an adiabatic boundary condition.

As the imposed temperature gradient has non-zero horizontal component, i.e., \( Q_h \neq 0 \), a convective motion sets in. Because of the hypothesis of infinite horizontal extent, the only non-zero component of velocity is \( u_s \) which, by continuity, depends only on the \( z \)-coordinate. After eliminating the pressure, the momentum and energy equations in the steady state, respectively, are:

\[
\frac{\partial^3 u_s}{\partial z^3} = Ra \frac{\partial T_s}{\partial x}
\]  
\[
u_s \frac{\partial T_s}{\partial x} = \frac{\partial^2 T_s}{\partial z^2}
\]  
\[(5.50)\]  
\[(5.51)\]

While the pressure \( p_s \) can be determined from

\[
\nabla p_s = \nabla^2 u_s + Ra T_s k
\]  
\[(5.52)\]
though it is not relevant to our present analysis.

The relevant boundary conditions for \( u_s \) and \( T_s \) are:

at \( z = 0 \)

\[
    u_s = 0 \tag{5.53}
\]

\[
    \frac{\partial T_s}{\partial z} = B_0 (T_s - T_0^\infty) + Q_v \tag{5.54}
\]

at \( z = 1 \)

\[
    \frac{\partial u_s}{\partial z} = -Ma \frac{\partial T_s}{\partial x} \tag{5.55}
\]

\[
    -\frac{\partial T_s}{\partial z} = B_1 (T_s - T_1^\infty) + Q_v \tag{5.56}
\]

Finally, the steady state must satisfy the global condition

\[
    \frac{\partial T_s}{\partial x} = -Q_h \tag{5.57}
\]

Moreover, we assume that \( T_0^\infty = -Q_h x \), and \( T_1^\infty = -Q_h x - Q_v \), which are necessary to obtain a steady solution, and accordingly we have

\[
    T_0^\infty - T_1^\infty = Q_v \tag{5.58}
\]

so that \( Q_v \) must be viewed as an external vertical heat flux which is superimposed on the vertical heat flux resulting from the application of the horizontal temperature gradient.

The conservation of zero-mass flux can be expressed by means of the condition

\[
    \int_0^1 u_s dz = 0 \tag{5.59}
\]

The basic steady state solution \((u_s, T_s, p_s)\) of equations (5.50)-(5.52) satisfying the boundary and zero-mass flow conditions (5.53)-(5.59) is

\[
    u_s = Q_h f_1(z), \quad v_s = 0, \quad w_s = 0
\]

\[
    T_s = -Q_h x + C_0 z + Q_h^2 g_1(z) + C \tag{5.60}
\]

\[
    \nabla p_s = \nabla^2 u_s + R_a T_s k
\]
where $C$ is a constant whose value is not relevant because, as will be shown later, $T_s$ will appear only in the form of derivatives, and

$$
C_0 = \begin{cases} 
Q_v, & \text{if } B_0 = 0, b_1 \neq 0 \\
Q_v = 0, & \text{if } B_0 = 0, b_1 = 0 \\
-\frac{b_0 - b_1 + b_0b_1}{b_0 + b_1 - b_0b_1} Q_v - \frac{b_1(3Ra + 20Ma)}{960(b_0 + b_1 - b_0b_1)} Q_h^2, & \text{if } B_0 \neq 0, b_1 \neq B_0 \\
-\frac{b_1}{1920}(3Ra + 20Ma)Q_h^2, & \text{if } B_0 \neq 0, b_1 = B_0
\end{cases}
$$

(5.61)

in which

$$
b_0 = \frac{B_0}{1 + B_0}, \quad b_1 = \frac{B_1}{1 + B_1}.
$$

(5.62)

The functions $f_1(z)$ and $g_1(z)$ are given by

$$
f_1(z) = C_1 z + C_2 z^2 + C_3 z^3
$$

(5.63)

$$
g_1(z) = C_4 z^3 + C_5 z^4 + C_6 z^5
$$

(5.64)

in which

$$
C_1 = -\frac{Ra + 4Ma}{8}, \quad C_2 = \frac{5Ra + 12Ma}{16} \\
C_3 = -\frac{Ra}{6}, \quad C_4 = \frac{Ra + 4Ma}{48} \\
C_5 = -\frac{5Ra + 12Ma}{192}, \quad C_6 = \frac{Ra}{120}.
$$

(5.65)

### 5.4 Linear Perturbation Analysis.

We now perturb the steady state solution as follows:

$$
v = u_s + u, \quad T = T_s + \theta, \quad P = p_s + p.
$$

(5.66)

The linearized perturbation equations then take the form

$$
\nabla \cdot u = 0
$$

(5.67)

$$
Pr^{-1} \frac{\partial u}{\partial t} = -\nabla p + \nabla^2 u + Ra \theta k - Pr^{-1}[(u_s \cdot \nabla)u + (u \cdot \nabla)u_s]
$$

(5.68)

$$
\frac{\partial \theta}{\partial t} = \nabla^2 \theta - u_s \cdot \nabla \theta - u \cdot \nabla T_s
$$

(5.69)
The boundary conditions are
\[ u = v = w = \frac{\partial \theta}{\partial z} - B_0 \theta = 0 \quad \text{at} \quad z = 0 , \] (5.70)
\[ w = \frac{\partial \theta}{\partial z} + B_1 \theta = \frac{\partial u}{\partial z} + Ma \frac{\partial \theta}{\partial x} = \frac{\partial v}{\partial z} + Ma \frac{\partial \theta}{\partial y} = 0 \quad \text{at} \quad z = 1 , \] (5.71)

By using standard norm mode analysis, we seek the solutions of the form
\[ [u,v,w,\theta,p] = [u(z),v(z),w(z),\theta(z),p(z)] \exp[st + i(\alpha_x x + \alpha_y y)] \] (5.72)

where \( s \) is the complex stability parameter
\[ s = s_r + i s_i \] (5.73)

wherein \( s_r \) measures the time growth rate of the disturbance, \( \alpha_x \) and \( \alpha_y \) are disturbance wave numbers in the \( x- \) and \( y- \) axis, respectively. \( \alpha_x = 0 \) corresponds to the longitudinal rolls with axes aligned in the direction the bulk flow, \( \alpha_y = 0 \) to transverse rolls with axes normal to the flow. The direction of propagation of the disturbance with regard to the \( x- \)axis is measured by means of angle \( \Phi \) defined as
\[ \Phi = \tan^{-1} \left( \frac{\alpha_y}{\alpha_x} \right) \] (5.74)

The corresponding phase speed is given by
\[ c = -s_i/\alpha^{-1} \] (5.75)

where \( \alpha \) stands for the overall wave number given by
\[ \alpha^2 = \alpha_x^2 + \alpha_y^2 \] (5.76)

Substitution of (5.72) into the governing equations (5.67)-(5.69), and the application of the boundary conditions, yields the eigenvalue equations
\[ Dw + i\alpha_x u + i\alpha_y v = 0 \] (5.77)
\begin{align}
(D^2 - \alpha^2)\theta - i\alpha_x Q_h f_1(z)\theta + Q_h u - DT_s w &= s\theta \quad (5.78) \\
(D^2 - \alpha^2)u - i\alpha_x p - i\alpha_x Q_h P r^{-1} f_1(z) u - Q_h P r^{-1} D f_1 w &= s P r^{-1} u \quad (5.79) \\
(D^2 - \alpha^2)v - i\alpha_y p - i\alpha_x Q_h P r^{-1} f_1(z) v &= s P r^{-1} v \quad (5.80) \\
(D^2 - \alpha^2)w - D p - i\alpha_x Q_h P r^{-1} f_1(z) w + R_d \theta &= s P r^{-1} w \quad (5.81)
\end{align}

subject to the boundary conditions

\begin{align}
u = v = w &= D\theta - B_0 \theta = 0 \quad \text{at} \quad z = 0, \quad (5.82) \\
w &= D\theta + B_1 \theta = Du + i\alpha_x M a \theta = Dv + i\alpha_y M a \theta = 0 \quad \text{at} \quad z = 1, \quad (5.83)
\end{align}

where \( D = \frac{d}{dz} \).

5.5 Chebyshev Tau-QZ Approximation.

In this section, we present the Chebyshev tau-QZ approximation (Straughan and Walker 1996a, Dongarra et al. 1996) to the associated eigenvalue problem (5.77)-(5.83). To this end, we reset the domain from \([0, 1]\) to \([-1, 1]\) with the coordinate transformation of \( z \) to \( 2z - 1 \), and thus derive

\begin{align}
\mathcal{L}_1(u, v, w, \theta, p) &\equiv (4D^2 - h_1(z))u - i\alpha_x p - h_2(z)w - s P r^{-1} u = 0 \quad (5.84) \\
\mathcal{L}_2(u, v, w, \theta, p) &\equiv (4D^2 - h_1(z))v - i\alpha_y p - s P r^{-1} v = 0 \quad (5.85) \\
\mathcal{L}_3(u, v, w, \theta, p) &\equiv (4D^2 - h_1(z))w - 2D p + R_d \theta - s P r^{-1} w = 0 \quad (5.86) \\
\mathcal{L}_4(u, v, w, \theta, p) &\equiv (4D^2 - h_4(z))\theta + Q_h u - h_3(z)w - s\theta = 0 \quad (5.87) \\
\mathcal{L}_5(u, v, w, \theta, p) &\equiv 2D w + i\alpha_x u + i\alpha_y v = 0 \quad (5.88)
\end{align}

where the functions \( h_j(z), j = 1, 2, 3, 4 \) are given by

\begin{align}
h_1(z) &= \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 \\
h_2(z) &= \alpha_4 + \alpha_5 z + \alpha_6 z^2 \\
h_3(z) &= \alpha_7 + \alpha_8 z + \alpha_9 z^2 + \alpha_{10} z^3 + \alpha_{11} z^4 \\
h_4(z) &= \alpha_{12} + \alpha_{13} z + \alpha_{14} z^2 + \alpha_{15} z^3.
\end{align}
in which the constants $\alpha_j, j = 0, 1, \cdots, 15$ are given by

$$
\begin{align*}
\alpha_0 &= \alpha^2 + i \alpha_x Q_h Pr^{-1} \left( \frac{C_1}{2} + \frac{C_2}{4} + \frac{C_3}{8} \right) \\
\alpha_1 &= i \alpha_x Q_h Pr^{-1} \left( \frac{C_1}{2} + \frac{C_2}{2} + \frac{3C_3}{8} \right) \\
\alpha_2 &= i \alpha_x Q_h Pr^{-1} \left( \frac{C_2}{4} + \frac{3C_3}{8} \right) \\
\alpha_3 &= i \alpha_x Q_h Pr^{-1} \left( \frac{C_3}{8} \right) \\
\alpha_4 &= Q_h Pr^{-1} (C_1 + C_2 + \frac{3C_3}{4}) \\
\alpha_5 &= Q_h Pr^{-1} (C_2 + \frac{3C_3}{2}) \\
\alpha_6 &= Q_h Pr^{-1} \left( \frac{3C_3}{4} \right) \\
\alpha_7 &= C_0 + Q_h^2 \left( \frac{3C_4}{4} + \frac{C_5}{2} + \frac{5C_6}{16} \right) \\
\alpha_8 &= Q_h^2 \left( \frac{3C_4}{2} + \frac{3C_5}{2} + \frac{5C_6}{4} \right) \\
\alpha_9 &= Q_h^2 \left( \frac{3C_4}{4} + \frac{3C_5}{2} + \frac{15C_6}{8} \right) \\
\alpha_{10} &= Q_h^2 \left( \frac{C_5}{2} + \frac{5C_6}{4} \right) \\
\alpha_{11} &= Q_h^2 \left( \frac{5C_6}{16} \right) \\
\alpha_{12} &= \alpha^2 + i \alpha_x Q_h \left( \frac{C_1}{2} + \frac{C_2}{4} + \frac{C_3}{8} \right) \\
\alpha_{13} &= i \alpha_x Q_h \left( \frac{C_1}{2} + \frac{C_2}{2} + \frac{3C_3}{8} \right) \\
\alpha_{14} &= i \alpha_x Q_h \left( \frac{C_2}{4} + \frac{3C_3}{8} \right) \\
\alpha_{15} &= i \alpha_x Q_h \left( \frac{C_3}{8} \right)
\end{align*}
$$

For the purpose of comparison, to be made later on, the system of equations (5.84)-
(5.88) can be written in the matrix form as

\[
\begin{bmatrix}
4D^2 - h_1 & 0 & -h_2 & 0 & -i\alpha_x \\
0 & 4D^2 - h_1 & 0 & 0 & -i\alpha_y \\
0 & 0 & 4D^2 - h_1 & Ra \cdot -2D & \\
Q_h & 0 & -h_3 & 4D^2 - h_4 & 0 \\
i\alpha_x & i\alpha_y & 2D & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
w \\
\theta \\
p
\end{bmatrix}
= s
\begin{bmatrix}
Pr^{-1} & 0 & 0 & 0 & 0 \\
0 & Pr^{-1} & 0 & 0 & 0 \\
0 & 0 & Pr^{-1} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
w \\
\theta \\
p
\end{bmatrix}
\]

(5.91)

The boundary conditions (5.82)-(5.83) become

\[
u = v = w = 2D\theta - B_0\theta = 0 \quad \text{at} \quad z = -1,
\]

(5.92)

\[
w = 2D\theta + B_1\theta = 2Du + i\alpha_x Ma\theta = 2Dv + i\alpha_y Ma\theta = 0 \quad \text{at} \quad z = 1.
\]

(5.93)

Having established the eigenvalue equations (5.91) subject to (5.92)-(5.93), our final step is to solve this eigenvalue problem numerically.

Due to the difficulty of the application of the compound matrix method, we resort to the Chebyshev tau-QZ method for solving the eigenvalue problem (5.91-5.93). We expand the solution of (5.91-5.93) in a finite series of Chebyshev polynomials as

\[
(u, v, w, \theta, p) = \sum_{k=0}^{N+2} (u_k, u_k, w_k, \theta_k, p_k)T_k(z)
\]

(5.94)

Though we have five equations, the system of equations (5.91) is actually an eighth order equation subject to eight boundary conditions. Since equation (5.88) does not contain the eigenvalue variable s, we can treat equation (5.88) as a constraint
condition on the solution \((u, v, w, \theta, p)\). By means of the Chebyshev tau method, we turn to solve the following equations
\[
\mathcal{L}_j(u, v, w, \theta, p) = \tau_{2j-1}T_{N+1}(z) + \tau_{2j}T_{N+2}(z), \quad j = 1, 2, 3, 4
\]
\[
\mathcal{L}_5(u, v, w, \theta, p) = 0
\] (5.95)
subject to the boundary conditions (5.92)-(5.93). Due to (5.34), i.e., the Chebyshev polynomials are orthogonal in the space \(L^2(-1, 1)\), and from (5.95), we obtain \(5N+7\) equations
\[
< \mathcal{L}_j(u, v, w, \theta, p), T_i > = 0, \quad i = 0, 1, \ldots, N \quad j = 1, 2, 3, 4
\]
\[
< \mathcal{L}_5(u, v, w, \theta, p), T_i > = 0, \quad i = 0, 1, \ldots, N + 2
\] (5.96)
There are eight further conditions which arise from (5.95),
\[
< \mathcal{L}_j(u, v, w, \theta, p), T_{N+k} >= \tau_{2(j-1)+k}||T_{N+k}||^2, \quad k = 1, 2, j = 1, 2, 3, 4
\] (5.97)
and these may effectively be used to calculate the \(\tau\)'s, which may be used to measure the error associated with the truncation of an infinite series. Here \(|| \cdot ||\) is the associated norm in the weighted \(L^2(-1, 1)\) space defined by (5.33). The remaining eight equations found from the boundary conditions (5.92)-(5.93) with the use of (5.3), are
\[
\sum_{j=0}^{N+2} (-1)^j u_j = 0, \quad \sum_{j=0}^{N+2} \left[ j^2 u_j + \frac{i\alpha_x Ma}{2} \theta_j \right] = 0, \quad \sum_{j=0}^{N+2} \left[ j^2 v_j + \frac{i\alpha_y Ma}{2} \theta_j \right] = 0,
\] (5.98)
\[
\sum_{j=0}^{N+2} (-1)^j v_j = 0, \quad \sum_{j=0}^{N+2} \left[ j^2 v_j + \frac{i\alpha_y Ma}{2} \theta_j \right] = 0, \quad \sum_{j=0}^{N+2} \left[ j^2 + \frac{B_0}{2} \right] \theta_j = 0,
\] (5.99)
\[
\sum_{j=0}^{N+2} (-1)^j w_j = 0, \quad \sum_{j=0}^{N+2} w_j = 0,
\] (5.100)
\[
\sum_{j=0}^{N+2} (-1)^j \left[ j^2 + \frac{B_0}{2} \right] \theta_j = 0, \quad \sum_{j=0}^{N+2} \left[ j^2 + \frac{B_1}{2} \right] \theta_j = 0,
\] (5.101)
The above procedure yields \(5(N+3)\) equations for the coefficients \(u_i, v_i, w_i, \theta_i\, and \, p_i\). The difficulty associated with this approach is that the boundary conditions are
all on \( u_i, v_i, w_i, \) and \( \theta_i, \) and none on \( p_i. \) Thus, we can not remove the boundary condition rows by removing \( u_{N+1}, u_{N+2}, v_{N+1}, v_{N+2}, w_{N+1}, w_{N+2}, \theta_{N+1}, \theta_{N+2}, p_{N+1}, \) and \( p_{N+2}. \) Instead, we write the boundary conditions as the rows of the matrix. This technique has been used by McFadden (1990), Lindsay and Odgen (1992), and Dongarra et al. (1996).

With the use of the matrices defined in Section 5.2, the system of equations (5.96) and (5.98)-(5.101) can be rewritten in the matrix form

\[
Ax = sBx
\]

where \( x = (u_0, \ldots, u_{N+2}, v_0, \ldots, v_{N+2}, w_0, \ldots, w_{N+2}, \theta_0, \ldots, \theta_{N+2}, p_0, \ldots, p_{N+2}), \)

and

\[
A = \begin{bmatrix}
4D^2 - H_1 & 0 & -H_2 & 0 & -i\alpha_x I \\
0 & 4D^2 - H_1 & 0 & 0 & -i\alpha_y I \\
0 & 0 & 4D^2 - H_1 & Ra I & -2D \\
Q_h I & 0 & -H_3 & 4D^2 - H_4 & 0 \\
i\alpha_x I & i\alpha_y I & 2D & 0 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
Pr^{-1}I & 0 & 0 & 0 & 0 \\
0 & Pr^{-1}I & 0 & 0 & 0 \\
0 & 0 & Pr^{-1}I & 0 & 0 \\
0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

and in which

\[
H_1 = \alpha_0 I + \alpha_1 Z + \alpha_2 Z^2 + \alpha_3 Z^3
\]

\[
H_2 = \alpha_4 I + \alpha_5 Z + \alpha_6 Z^2
\]

\[
H_3 = \alpha_7 I + \alpha_8 Z + \alpha_9 Z^2 + \alpha_{10} Z^3 + \alpha_{11} Z^4
\]

\[
H_4 = \alpha_{12} I + \alpha_{13} Z + \alpha_{14} Z^2 + \alpha_{15} Z^3
\]
Here each fundamental block matrix $D, D^2, Z, Z^2, Z^3$ and $Z^4$ defined in Section 5.2 is $N+3$ by $N+3$, and $I, O$ stand for the identity and zero matrices of $N+3$ by $N+3$, respectively. and the $(N+2)$'s, $(N+3)$'s, $(2N+5)$'s, $(2N+6)$'s, $(3N+8)$'s, $(3N+9)$'s, $(4N+11)$'s, $(4N+12)$'s equations of (5.102) are overwritten by the boundary conditions (5.98)-(5.101), respectively.

We remark that if we ignored the processing of boundary conditions in (5.102), we could see the greater similarity between the eigenvalue equation of the differential form (5.91) with the corresponding discrete form (5.102). In fact, with the use of the Chebyshev tau approximation and the matrices defined in Section 5.2, we could write down immediately the discrete algebraic eigenvalue equations from its corresponding differential equations whose order is not higher than second with coefficients as the polynomials of independent variables. This is the reason and the advantage for introducing those matrices in Section 5.2.

We now need to solve a generalized eigenvalue equation like (5.102), which can be handled most efficiently by using the QZ algorithm of Moler and Stewart (1973), and this has been implemented in several standard libraries. In our implementation, the ZGEGV routine in LAPACK library (Anderson et al. 1995) is employed for solving the complex generalized eigenvalue equation (5.102). However, care must be taken with the implementation since $B$ is singular due to the nature of the differential equation (5.88) and boundary conditions (5.92)-(5.93), which do not involve the eigenvalue variable $s$. It has been noted that the occurrence of spurious eigenvalues is due to rows of zero's in $B$, see, for instance, McFadden et al.(1990), and Dongarra (1996). By this we mean a number which is seen in the eigenvalue list but is not a solution to the differential equation. To elucidate on this, the QZ algorithm of Moler and Stewart (1973) does not produce the eigenvalues $s_j$, but reduces $A$ and $B$ to upper triangular form with diagonal elements $\lambda_j$ and $\omega_j$. The eigenvalue are
\[ s_j = \frac{\lambda_j}{\omega_j} \] when division make sense. We thus need to filter out those \( \omega_j = 0 \) yielded by the QZ algorithm in our implementation.

We now describe our algorithm for calculating the neutral stability curve. We consider \( s \) as the eigenvalue, and \( Ra \) (or \( Ma \)) as the interested parameter to be explored with respect to all the remaining parameters \( Ma \) (or \( Ra \)), \( B_0, B_1, Pr \) being given. Since the Chebyshev tau-QZ algorithm gives as many eigenvalues as possible, our numerical algorithm for finding the neutral stability curve may thus be summarized as follows:

Step 1. For fixed values of \( \alpha_x \) and \( \alpha_y \) (or \( \alpha \) and \( \Phi \), since \( \alpha_x = \alpha \cos(\Phi) \) and \( \alpha_y = \alpha \sin(\Phi) \) due to Eqs. (5.74) and (5.76)), we provide two estimated trial values \( Ra_1 \) and \( Ra_2 \), (or \( Ma_1 \) and \( Ma_2 \)) and use the QZ algorithm to find the corresponding leading eigenvalue \( s \), which is defined as the eigenvalue with the largest real part, among the whole set of eigenvalues of the generalized eigenvalue equation (5.102) (after filtering out the spurious eigenvalues which occurs because the matrix is \( \mathbf{B} \) singular).

Step 2. Use the secant method to search for the \( Ra \) (or \( Ma \)) such that the real part of the leading eigenvalue \( s \) approaches zero corresponding to the same fixed \( \alpha_x \) and \( \alpha_y \) in Step 1, until the solution satisfies with the predefined accuracy.

The result of Steps 1 and 2 is to produce one point on the neutral stability curve. Accordingly, the critical Rayleigh number \( R_c \) (or \( Ma \)) with the corresponding critical wave number \( \alpha_x \) and \( \alpha_y \), can be defined as

\[
R_c = \min_{\alpha_x} \min_{\alpha_y} R(s_c; Ma, Pr, B_0, B_1) \tag{5.103}
\]

or

\[
Ma_c = \min_{\alpha_x} \min_{\alpha_y} R(s_c; Ra, Pr, B_0, B_1) \tag{5.104}
\]

where \( s_c \) is the imaginary part of the leading eigenvalue of Step 2 since the corresponding real part equals zero now. The minimization of (5.103) or (5.104) may
then be resolved by means of the golden section search routine.

It is worth noting that if the imaginary part of the leading eigenvalue, \( s_c \), happens to be (or approach) zero as well when its real part approaches zero by our algorithm, then the stability is stationary. Otherwise, the stability is oscillatory. Thus, one additional advantage of the approach presented here is that it can also be used to numerically verify whether the principle of exchange of stability is valid or not.

For the purpose of verifying our implementation of the approach and the algorithm described above, we will first consider the problem treated by Nield (1968) in the next section, since it is a special case of the problem discussed in Section 5.3. For this reason, we briefly discuss here some relevant equations related to Nield's problem, which can be obtained by setting the following three parameters as

\[ Q_h = 0, \quad Q_v = 1, \quad B_0 = \infty \quad (5.105) \]

In this case, by eliminating the variables \( u, v, \) and \( p \), the eigenvalue equations (5.77)-(5.81) and the boundary conditions (5.82)-(5.83) can be reduced to a sixth order differential equation

\begin{align*}
(D^2 - \alpha^2)^2 w - \alpha^2 Ra \theta &= s Pr^{-1} (D^2 - \alpha^2) w \quad (5.106) \\
(D^2 - \alpha^2) \theta + w &= s \theta \quad (5.107)
\end{align*}

with boundary conditions

\[ w = Dw = \theta = 0 \quad \text{at} \quad z = 0 \quad (5.108) \]
\[ w = D\theta + B_1 \theta = D^2 w - \alpha^2 Ma \theta = 0 \quad \text{at} \quad z = 1 \quad (5.109) \]

We note that, under the assumption of \( s = 0 \), the eigenvalue problem (5.106)-(5.109) was solved by Nield (1968), and by Pearson (1958) (with one more assumption; \( Ra = 0 \) in (5.106)), by means of Fourier series method, respectively.
With the introduction of \( \phi = (D^2 - \alpha^2)w \), followed by the application of the same procedures employed above, we can write down the discrete eigenvalue problem of (5.106)-(5.109) in the generalized eigenvalue equation of the form (5.102) as

\[
A_1 x_1 = s B_1 x_1 \quad (5.110)
\]

where 
\[ x_1 = (w_0, \ldots, w_{N+2}, \theta_0, \ldots, \theta_{N+2}, \phi_0, \ldots, \phi_{N+2}) \]

and

\[
A_1 = \begin{bmatrix}
4D^2 - \alpha^2 I & 0 & -I \\
I & 4D^2 - \alpha^2 I & 0 \\
0 & -\alpha^2 Ra I & 4D^2 - \alpha^2 I \\
0 & 0 & 0 \\
0 & I & 0 \\
0 & 0 & Pr^{-1} I
\end{bmatrix}
\]

\[
B_1 = \begin{bmatrix}
0 & 0 & 0 \\
0 & I & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

with the \((N+2)\)'s, \((N+3)\)'s, \((2N+5)\)'s, \((2N+6)\)'s, \((3N+8)\)'s, \((3N+9)\)'s rows of the equation overwritten by the following six boundary conditions

\[
\begin{align*}
\sum_{j=0}^{N+2} (-1)^j w_j &= 0 \\
\sum_{j=0}^{N+2} (-1)^{j+1} j^2 w_j &= 0, \\
\sum_{j=0}^{N+2} (-1)^j \theta_j &= 0 \\
\sum_{j=0}^{N+2} w_j &= 0 \\
\sum_{j=0}^{N+2} [j^2 + \frac{B_1}{2}] \theta_j &= 0 \\
\sum_{j=0}^{N+2} [\phi_j - \alpha^2 Ma \theta_j] &= 0
\end{align*} \quad (5.111)
\]

If we follow Niels's assumption that the principle of exchange of stability is valid, i.e., \( s = 0 \), we can further simplify the calculations by considering \( Ra \) as the eigenvalue, and the generalized eigenvalue equation in this case is

\[
A_2 x_1 = Ra B_2 x_1 \quad (5.112)
\]

where

\[
A_2 = \begin{bmatrix}
4D^2 - \alpha^2 I & 0 & -I \\
I & 4D^2 - \alpha^2 I & 0 \\
0 & 0 & 4D^2 - \alpha^2 I
\end{bmatrix}
\]
\[ B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha^2I & 0 \end{bmatrix} \]

with the \((N+2)\)'s, \((N+3)\)'s, \((2N+5)\)'s, \((2N+6)\)'s, \((3N+8)\)'s, \((3N+9)\)'s rows of equation overwritten by the six boundary conditions (5.111). Accordingly, our algorithm needs to be modified by searching for the leading eigenvalue \(Ra\), which is now redefined as the one with the smallest real part, such that its imaginary part approaches zero.

5.6. Numerical Results and Discussion.

The numerical calculations presented in this section are performed with 30 Chebyshev polynomials, the predefined accuracy of the search iteration is set to \(10^{-6}\), and a value \(10^{20}\) is treated as the infinity limit. These settings are sufficient to show convergence of the eigenvalues based on our computation.

To demonstrate the correctness of the approach and the algorithm described in the previous section, we first make comparison with the results of Nield (1968). As we have seen, Nield's problem, in our approach, can be described by the generalized eigenvalue equation (5.102), or (5.110), or (5.112). Note that Eqs. (5.110) and (5.112) are independent of \(\Phi\). Moreover, Eq. (5.112) is also independent of the Prandtl number \(Pr\). For the purpose of comparison, we have implemented the codes based on these three generalized eigenvalue equations.

For Nield's problem \((Q_h = 0, Q_v = 1, B_0 = \infty)\) when \(B_1 = 100, Ma = 0,\) and \(\alpha = 2.672\), Tables 1-3 list the first three leading eigenvalues of (5.112), of (5.110) when \(Pr = 1, 10,\) and of (5.102) when \(Pr = 1, 10\) and \(\Phi = 0, \frac{\pi}{4}, \frac{\pi}{2}\), respectively. As expected, each calculation gives the result \(Ra = 1085.90\), which was also obtained by Nield (1968). The results in Tables 2 and 3 computed from (5.110) and (5.102) are also in agreement with our previous analysis that they are independent of \(Pr\).
and $\Phi$. Moreover, they also numerically confirm the assumption that the stability is stationary since $s_i = 0$. In fact, our calculations for other parameters also show these properties. These results thus confirm that our implementation is correct.

**Table 1. The first 3 eigenvalues of (5.112) for $B_1 = 100$, $Ma = 0$, $\alpha = 2.672$**

<table>
<thead>
<tr>
<th>$Ra_r$</th>
<th>$Ra_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1085897880E+04</td>
<td>0.0000000000E+00</td>
</tr>
<tr>
<td>0.2055654256E+05</td>
<td>0.0000000000E+00</td>
</tr>
<tr>
<td>0.1630177910E+06</td>
<td>0.0000000000E+00</td>
</tr>
</tbody>
</table>

**Table 2. The first 3 eigenvalues of (5.110) for $B_1 = 100$, $Ma = 0$, $\alpha = 2.672$**

<table>
<thead>
<tr>
<th>$Pr$</th>
<th>$Ra$</th>
<th>$s_r$</th>
<th>$s_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1085897917E+04</td>
<td>-0.3592998756E+02</td>
<td>0.0000000000E+00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.4496501308E+02</td>
<td>0.0000000000E+00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.3583065721E-06</td>
<td>0.0000000000E+00</td>
</tr>
<tr>
<td>10</td>
<td>0.1085897897E+04</td>
<td>-0.4297164879E+02</td>
<td>0.0000000000E+00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.9337737613E+02</td>
<td>0.0000000000E+00</td>
</tr>
</tbody>
</table>
Table 3. The first 3 eigenvalues of (5.102) for $B_1 = 100$, $Ma = 0$, $\alpha = 2.672$

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$Pr$</th>
<th>$Ra$</th>
<th>$s_r$</th>
<th>$s_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>1</td>
<td>0.1085897962E+04</td>
<td>0.2085772200E-06</td>
<td>-0.3760997245E-10</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.9606985155E+01</td>
<td>-0.9684941585E-09</td>
<td>-0.0135216568E-07</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.2934619397E+02</td>
<td>-0.1035216568E-07</td>
<td></td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>10</td>
<td>0.1085897908E+04</td>
<td>0.3235118315E-06</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.4297164882E+02</td>
<td>0.0000000000E+00</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.9337737615E+02</td>
<td>0.0000000000E+00</td>
<td></td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>1</td>
<td>0.1085897932E+04</td>
<td>0.2525790795E-06</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.9606985164E+01</td>
<td>0.0000000000E+00</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.2934619397E+02</td>
<td>0.0000000000E+00</td>
<td></td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>10</td>
<td>0.1085897889E+04</td>
<td>0.3971446686E-06</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.4297164881E+02</td>
<td>0.0000000000E+00</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.9337737614E+02</td>
<td>0.0000000000E+00</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0.1085897960E+04</td>
<td>0.2130803656E-06</td>
<td>-0.176052445E-09</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.9606985176E+01</td>
<td>-0.1465810084E-07</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.2934619396E+02</td>
<td>0.3797119671E-08</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>10</td>
<td>0.1085897908E+04</td>
<td>0.3308648620E-06</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.4297164882E+02</td>
<td>0.0000000000E+00</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.9337737615E+02</td>
<td>0.0000000000E+00</td>
<td></td>
</tr>
</tbody>
</table>

In contrast with the imposed temperature gradient having a vertical component ($Q_h = 0$ and $Q_v = 1$), Smith and Davis (1983) discussed the instability when the imposed temperature gradient has a horizontal component ($Q_h = 1$ and $Q_v = 0$) without buoyancy effect ($Ra = 0$). The problem studied by Smith and Davis (1983) corresponds to the case when parameters in our discussion are set as

$$Q_h = 1, \quad Q_v = 0, \quad B_0 = 0, \quad Ra = 0.$$  \hspace{1cm} (5.113)

Table 4 lists the computed results for this case when $Pr = 1$, $\alpha_x = 0$ for different
values of $B_1$ and $\alpha_y$. These results are again in good agreement with Smith and Davis (1983), by comparing with Figure 10 of their paper. Note that in Table 4, $s_i \neq 0$, which means that the stability is oscillatory in the form of longitudinal traveling hydrothermal waves.

<table>
<thead>
<tr>
<th>$B_1$</th>
<th>$\alpha_y$</th>
<th>$Ra$</th>
<th>$s_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2422759035E+03</td>
<td>± 0.3263762812E+02</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.1962649111E+03</td>
<td>± 0.4647202513E+02</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.2570204678E+03</td>
<td>± 0.7798574068E+02</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.3665558793E+03</td>
<td>± 0.1300835380E+03</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.5000215726E+03</td>
<td>± 0.1997406596E+03</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.3160006535E+03</td>
<td>± 0.3966463257E+02</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.2511275282E+03</td>
<td>± 0.5568260314E+02</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.3169753248E+03</td>
<td>± 0.9138161221E+02</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.4327489673E+03</td>
<td>± 0.1476333187E+03</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.5723392011E+03</td>
<td>± 0.2214188482E+03</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.3923256860E+03</td>
<td>± 0.4623740061E+02</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.3091306513E+03</td>
<td>± 0.6476475019E+02</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.3789015239E+03</td>
<td>± 0.1044715736E+03</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.5004198119E+03</td>
<td>± 0.1647743351E+03</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.6459406527E+03</td>
<td>± 0.2426602718E+03</td>
<td></td>
</tr>
</tbody>
</table>

We now present some computed results subject to the inclined temperature gradients for the parameters values

$$Q_v = Q_h = 1, \quad B_0 = B_1 = 10^{20},$$

(5.114)

In this case, the imposed temperature gradients have the same components in both horizontal and vertical directions, and the infinite horizontal fluid layers are bounded
by perfectly heat conduction boundaries.

<table>
<thead>
<tr>
<th>$Pr$</th>
<th>$\alpha$</th>
<th>$Ra$</th>
<th>$s_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5027355943E+03</td>
<td>$\pm$ 0.2579144030E+01</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.3369318660E+03</td>
<td>$\pm$ 0.2744399672E+01</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>3</td>
<td>0.3463960967E+03</td>
<td>$\pm$ 0.3078379318E+01</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0.4151466527E+03</td>
<td>$\pm$ 0.3418594976E+01</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>0.5188145301E+03</td>
<td>$\pm$ 0.3544249271E+01</td>
</tr>
<tr>
<td>1</td>
<td>0.5799461019E+03</td>
<td>$\pm$ 0.4823489688E+01</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.4046561880E+03</td>
<td>$\pm$ 0.2703167923E+01</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0.3382979738E+03</td>
<td>0.0000000000E+00</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0.3523226067E+03</td>
<td>0.0000000000E+00</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>0.4067986067E+03</td>
<td>0.0000000000E+00</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.5111792297E+03</td>
<td>0.0000000000E+00</td>
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<td>2</td>
<td>2</td>
<td>0.3064565576E+03</td>
<td>0.0000000000E+00</td>
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<tr>
<td>10</td>
<td>3</td>
<td>0.2859660029E+03</td>
<td>0.0000000000E+00</td>
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<tr>
<td>4</td>
<td>4</td>
<td>0.3201541539E+03</td>
<td>0.0000000000E+00</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>0.3830027745E+03</td>
<td>0.0000000000E+00</td>
</tr>
</tbody>
</table>

Table 5 and 6 display $Ra$ against $\alpha$ for $Pr = 0.1, 1, 10$ and $\Phi = 0.5\pi, 0.45\pi$ respectively, without the surface tension effect ($Ma = 0$). In this case, the behavior of the stability is largely dependent on the values of the Prandtl number $Pr$ and the orientation of the disturbance. Table 5 shows that, when $\Phi = 0.5\pi$ ($\alpha_x = 0$) the instability is stationary when $Pr = 10$. However, it is oscillatory when $Pr = 0.1$, and either stationary or oscillatory when $Pr = 1$ depending on the value of $\alpha$. Table 6 shows that, when $\Phi = 0.45\pi$ ($\alpha_x \neq 0$), the instability is always oscillatory no matter what the values of $Pr$ may be. This fact is further confirmed by Tables 7
and 8 for $Ma = 10$. Other calculations also show this similar behavior. Thus we may conclude that the longitudinal mode ($\alpha_x = 0$) is the only possible stationary mode for higher Prandtl number $Pr$, and that no stationary stabilities have been displayed at lower Prandtl number $Pr$. This also confirms the conclusions drawn by Davis and Smith (1983) and Parmentier et al. (1993) in the presence of a horizontal temperature gradient.

<table>
<thead>
<tr>
<th>$Pr$</th>
<th>$\alpha$</th>
<th>$Ra$</th>
<th>$s_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4768584034E+03</td>
<td>-0.2745984640E+01</td>
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<tr>
<td>2</td>
<td>0.3266386930E+03</td>
<td>-0.3059048784E+01</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>3</td>
<td>0.3610143844E+03</td>
<td>-0.3785363205E+01</td>
</tr>
<tr>
<td>4</td>
<td>0.5458333590E+03</td>
<td>-0.4567613090E+01</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.6212160560E+03</td>
<td>-0.3379979426E+01</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.4247469920E+03</td>
<td>0.2831455839E+00</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.5832036034E+03</td>
<td>0.4608334189E+01</td>
<td></td>
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<tr>
<td>1</td>
<td>3</td>
<td>0.3931628535E+03</td>
<td>0.3004894272E+01</td>
</tr>
<tr>
<td>4</td>
<td>0.3463767926E+03</td>
<td>0.6723501533E+00</td>
<td></td>
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<tr>
<td>5</td>
<td>0.3644705551E+03</td>
<td>0.3879012366E+00</td>
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<tr>
<td>1</td>
<td>0.5216671332E+03</td>
<td>0.5373126498E+00</td>
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<td>2</td>
<td>0.3128458842E+03</td>
<td>0.5584177120E+00</td>
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<tr>
<td>10</td>
<td>3</td>
<td>0.2930547746E+03</td>
<td>0.7045146450E+00</td>
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<tr>
<td>4</td>
<td>0.3301554126E+03</td>
<td>0.9840930169E+00</td>
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</tr>
<tr>
<td>5</td>
<td>0.3982713675E+03</td>
<td>0.1411699364E+01</td>
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</tbody>
</table>

Table 6. $Ra$ against $Pr$ and $\alpha$ for $Ma = 0$, $\Phi = 0.45\pi$
Table 7. $Ra$ against $Pr$ and $\alpha$ for $Ma = 10$, $\Phi = 0.5\pi$

<table>
<thead>
<tr>
<th>Pr</th>
<th>$\alpha$</th>
<th>$Ra$</th>
<th>$s_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4946445393E+03</td>
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<tr>
<td>2</td>
<td>0.3310603968E+03</td>
<td>$\pm$ 0.2982522978E+01</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>3</td>
<td>0.3425723038E+03</td>
<td>$\pm$ 0.3386894083E+01</td>
</tr>
<tr>
<td>4</td>
<td>0.4128201317E+03</td>
<td>$\pm$ 0.3779338646E+01</td>
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</tr>
<tr>
<td>5</td>
<td>0.517822632E+03</td>
<td>$\pm$ 0.3966671384E+01</td>
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</tr>
<tr>
<td>1</td>
<td>0.5788466453E+03</td>
<td>$\pm$ 0.5277423021E+01</td>
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</tr>
<tr>
<td>2</td>
<td>0.4041570302E+03</td>
<td>$\pm$ 0.4084774292E+01</td>
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<td>3</td>
<td>0.3602423230E+03</td>
<td>$\pm$ 0.0000000000E+00</td>
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<td>0.3652287414E+03</td>
<td>$\pm$ 0.0000000000E+00</td>
<td></td>
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<tr>
<td>5</td>
<td>0.4159190264E+03</td>
<td>$\pm$ 0.0000000000E+00</td>
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</tr>
<tr>
<td>10</td>
<td>3</td>
<td>0.5204442971E+03</td>
<td>$\pm$ 0.0000000000E+00</td>
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<td>0.3123541300E+03</td>
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<td>$\pm$ 0.0000000000E+00</td>
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<td>4</td>
<td>0.3243792106E+03</td>
<td>$\pm$ 0.0000000000E+00</td>
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<tr>
<td>5</td>
<td>0.3869183135E+03</td>
<td>$\pm$ 0.0000000000E+00</td>
<td></td>
</tr>
</tbody>
</table>

Table 9 displays $Ra$ against $\alpha$ for $Q_h = 0.0, 0.01, 0.1, 0.5$ and $\Phi = 0.5\pi, 0.45\pi$ respectively, when $Q_v = 1$, $B_0 = B_1 = \infty$, $Ma = 100$ and $Pr = 10$. In this case, the value of $Ra$ is decreased with increase of the values of $Q_h$, the longitudinal stability mode is always stationary, and the non-longitudinal stability mode is always oscillatory except for the case of $Q_h = 0$.

We note that, when $\alpha_x = 0$ ($\Phi = 0.5\pi$), the imaginary part of the eigenvalues appear either as zero or in pairs (positive and negative) as shown in Tables 4, 5, 7 and 9. This behavior is due to the fact that if $s$ is an eigenvalue corresponding to the eigenfunctions $(u, v, w, p, \theta)$, then $\bar{s}$ (the complex conjugate) is also an eigenvalue corresponding to the eigenfunctions $(\bar{u}, -\bar{v}, \bar{w}, \bar{p}, \bar{\theta})$. This could be readily shown
from (5.77)-(5.88) under the assumption of $\alpha_z = 0$. Thus, the complex eigenvalues appear in conjugate pairs.

<table>
<thead>
<tr>
<th>$Pr$</th>
<th>$\alpha$</th>
<th>Ra</th>
<th>$s_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>0.4670745753E+03</td>
<td>-0.2957033274E+01</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0.3221949005E+03</td>
<td>-0.3460532943E+01</td>
</tr>
<tr>
<td>0.1</td>
<td>3</td>
<td>0.3710086614E+03</td>
<td>-0.4397229526E+01</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.5893666078E+03</td>
<td>-0.472233607E+01</td>
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<td>5</td>
<td>0.6438226820E+03</td>
<td>-0.4015288079E+01</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>0.5851304126E+03</td>
<td>0.3802588127E+03</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0.3993603147E+03</td>
<td>0.4015942096E+01</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.3682143031E+03</td>
<td>0.1181446174E+01</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.3802588127E+03</td>
<td>0.5785114679E+00</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.4375996344E+03</td>
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We close this section with some remarks. One of the main objectives of this chapter is to present a numerical algorithm that can be used to deal with the complicated complex eigenvalue problems resulting from the linear stability analysis. Numerical results with comparison to previous work show that our implementation is correct, which thus can be applied with confidence to many other stability problems. Based on the observations on the numerical calculations performed, though no optimization has been carried out in this section to find the critical point of the neutral stability curve, we can conclude that the behavior of the stability may be in
the form of stationary, or oscillatory depending upon the Prandtl number $Pr$ and the disturbance orientation $\Phi$.

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Chapter 6

Conclusions

In this thesis, we have studied five stability problems of mono-diffusive (thermal) convection, or double-diffusive (thermal-solutal) convection, in a porous medium and in a viscous fluid, induced by inclined temperature gradients with mass flow, or through-flow, or surface tension. Some general conclusions and directions for future work can now be drawn.

The stability analysis of convection problems induced by inclined temperature (and solutal) gradients involves exploration of a large parameter space. These may include the nondimensional parameters, such as Biot number, Rayleigh numbers, Marangoni number, Prandtl number, Péclet number, Lewis number and mass flow-rate. The compound matrix method and Chebyshev tau-QZ method are both very useful in their studies of stability problems and give quite accurate results in the respective domain of applicability. The lower-order Galerkin approximation does not give accurate results. Moreover, the results obtained by this method often do not cover the entire spectrum of values. When the linearized operator is not symmetric, the predictions of the two theories (linear and energy) will be different, indicating the possibility of subcritical instability. The results obtained by the energy method are quite conservative as compared to the linear stability method.

More specifically the following conclusions can be made.

In the porous medium problems:
1. Initially as the value of $R_H$ is increased the value of $R_V$ also increases but a further increase in $R_H$ value decreases the $R_V$ value. The favoured mode is stationary longitudinal type.
2. Inclusion of horizontal mass flow has a destabilizing effect on $R_L$ and $R_E$.
3. A vertical through-flow has a stabilizing effect. Vertical through-flow delays the
onset of convection.

4. As Lewis number $L_e$ increases, $R_E$ first decreases and then increases. For much higher values of $L_e$, $R_E$ always decreases.

In viscous fluid problems:

5. The longitudinal mode is the preferred mode only for high values of Prandtl number $Pr$. For low values of $Pr$, the preferred mode can be oscillatory.

6. For high values of $R_H$ and $Q$, convection can be driven even without any vertical temperature gradient.

7. The Chebyshev tau-$QZ$ method can be used to numerically verify whether the principle of exchange of stability is valid or not.

With regard to the scope of further research we point out the following:

1. All of the above problems can be extended to Brinkman or Brinkman-Forchheimer model. The equation, of course, then become of higher order by two and could create difficulty in solving the corresponding eigenvalue problems.

2. Some of the above problems can be studied for parallel shear flows or for other complicated geometries. The nature of the equations again would become complicated and would require better techniques to handle them.

3. The role of anisotropy of porous media could be incorporated but this would increase the number of variables considerably and hence might become very difficult to analyze.

4. With considerable care, some of these problems can be studied in simple viscoelastic fluid model.
Appendix


In this appendix, we list the compound matrix variables along with the associated differential equations, and the initial and final conditions they need to satisfy, for the system of Eq.(2.77), Eq.(3.59), and Eq.(4.38), respectively. We remark that the compound matrix is an adaptation of the shooting method in the context of linear eigenvalue problems for ordinary differential equations. It aims to overcome the instability which occurs with the standard shooting method when target conditions are computed from determinants of the solutions of the differential system. This technique circumvents these problems by directly computing the target conditions in an additive way, but at the expense of solving a potentially much larger system of new differential equations which are generated from the original system. The compound matrix method is thus designed to avoid round off error and works well if the system of differential equations is stiff. Its general history may be found in Drazin and Reid (1981) and Ng and Reid (1979a,1979b, 1985). A lucid comprehensive account of this method (and the notation used below) has been given by Lindsay and Straughan (1992).

The order of differential equations (2.77), (3.59), and (4.38) is even and each is subject to the evenly distributed boundary conditions. According to the method of compound matrix, we need to generate \( (\frac{3}{2}) = 6 \), \( (\frac{5}{3}) = 20 \), and \( (\frac{7}{5}) = 70 \) compound matrix variables, respectively. The detail results for each case are listed as follows:

For the system of equations (2.77), the six compound matrix variables are:

\[
y_1 = (1,2) \quad y_2 = (1,3) \quad y_3 = (1,4) \\
y_4 = (2,3) \quad y_5 = (2,4) \quad y_5 = (3,4)
\]

These six compound matrix variables satisfy the differential equations as follow:
\[
\begin{align*}
\dot{y}_1 &= h_2 y_2 \\
\dot{y}_2 &= y_4 + y_3 \\
\dot{y}_3 &= y_5 + h_4 y_2 \\
\dot{y}_4 &= h_1 y_2 + y_5 \\
\dot{y}_5 &= h_1 y_3 + h_2 y_6 - h_3 y_1 + h_4 y_4 \\
\dot{y}_6 &= -h_3 y_2
\end{align*}
\]

For the boundary conditions (2.79), all the \( y \)'s are initially zero except \( y_5(0) = 1 \), and the appropriate final condition to be satisfied is \( y_2(1) = 0 \).

For the system of equations (3.59), the twenty compound matrix variables are:

\[
\begin{align*}
 y_1 &= (1, 2, 3) & y_2 &= (1, 2, 4) & y_3 &= (1, 2, 5), & y_4 &= (1, 2, 6) \\
y_5 &= (1, 3, 4) & y_6 &= (1, 3, 5) & y_7 &= (1, 3, 6), & y_8 &= (1, 4, 5) \\
y_9 &= (1, 4, 6) & y_{10} &= (1, 5, 6) & y_{11} &= (2, 3, 4), & y_{12} &= (2, 3, 5) \\
y_{13} &= (2, 3, 6) & y_{14} &= (2, 4, 5) & y_{15} &= (2, 4, 6), & y_{16} &= (2, 5, 6) \\
y_{17} &= (3, 4, 5) & y_{18} &= (3, 4, 6) & y_{19} &= (3, 5, 6), & y_{20} &= (4, 5, 6)
\end{align*}
\]

These twenty compound matrix variables satisfy the differential equations as follows:

\[
\begin{align*}
\dot{y}_1 &= -h_3 y_6 + y_2 \\
\dot{y}_2 &= h_2 y_5 - h_3 y_8 + h_5 y_1 + h_6 y_3 \\
\dot{y}_3 &= h_2 y_6 + y_4 \\
\dot{y}_4 &= h_2 y_7 + h_3 y_{10} + h_8 y_1 + h_9 y_3 \\
\dot{y}_5 &= y_{11} + h_6 y_6 \\
\dot{y}_6 &= y_{12} + y_8 + y_7 \\
\dot{y}_7 &= y_{13} + y_9 + h_6 y_6
\end{align*}
\]
\[ \dot{y}_8 = y_{14} + h_5 y_6 + y_9 \]
\[ \dot{y}_9 = y_{15} + h_5 y_7 + h_6 y_{10} - h_8 y_5 + h_9 y_8 \]
\[ \dot{y}_{10} = y_{16} - h_8 y_6 \]
\[ \dot{y}_{11} = h_{11} y_5 + h_{32} y_{17} + h_4 y_1 + h_6 y_{12} \]
\[ \dot{y}_{12} = h_{12} y_6 + y_{14} + y_{13} \]
\[ \dot{y}_{13} = h_{13} y_7 - h_{33} y_{19} + y_{15} + h_7 y_1 + h_9 y_{12} \]
\[ \dot{y}_{14} = y_{17} - h_{43} y_3 + h_5 y_{12} + y_{15} \]
\[ \dot{y}_{15} = h_{15} y_9 + h_2 y_{18} - h_3 y_{20} - h_4 y_4 + h_5 y_{13} + h_6 y_{16} + h_7 y_2 - h_8 y_{11} + h_9 y_{14} \]
\[ \dot{y}_{16} = h_{16} y_{10} + h_2 y_{19} + h_7 y_3 - h_8 y_{12} \]
\[ \dot{y}_{17} = -h_4 y_6 + y_{18} \]
\[ \dot{y}_{18} = -h_4 y_7 + h_6 y_{19} + h_7 y_5 + h_9 y_{17} \]
\[ \dot{y}_{19} = y_{20} + h_7 y_{6} \]
\[ \dot{y}_{20} = h_4 y_{10} + h_5 y_{19} + h_7 y_8 + h_8 y_{17} \]

For the boundary conditions (3.60), all the \( y \)'s are initially zero except \( y_{15}(0) = 1 \), and the appropriate final condition to be satisfied is \( y_6(1) = 0 \).

For the system of equations (4.38), the seventy compound matrix variables are:

\[ y_1 = (1, 2, 3, 4) \quad y_2 = (1, 2, 3, 5) \quad y_3 = (1, 2, 3, 6) \quad y_4 = (1, 2, 3, 7) \]
\[ y_5 = (1, 2, 3, 8) \quad y_6 = (1, 2, 4, 5) \quad y_7 = (1, 2, 4, 6) \quad y_8 = (1, 2, 4, 7) \]
\[ y_9 = (1, 2, 4, 8) \quad y_{10} = (1, 2, 5, 6) \quad y_{11} = (1, 2, 5, 7) \quad y_{12} = (1, 2, 5, 8) \]
\[ y_{13} = (1, 2, 6, 7) \quad y_{14} = (1, 2, 6, 8) \quad y_{15} = (1, 2, 7, 8) \quad y_{16} = (1, 3, 4, 5) \]
\[ y_{17} = (1, 3, 4, 6) \quad y_{18} = (1, 3, 4, 7) \quad y_{19} = (1, 3, 4, 8) \quad y_{20} = (1, 3, 5, 6) \]
\[ y_{21} = (1, 3, 5, 7) \quad y_{22} = (1, 3, 5, 8) \quad y_{23} = (1, 2, 6, 7) \quad y_{24} = (1, 3, 6, 8) \]
\[ y_{25} = (1, 3, 7, 8) \quad y_{26} = (1, 4, 5, 6) \quad y_{27} = (1, 4, 5, 7) \quad y_{28} = (1, 4, 5, 8) \]
\[ y_{29} = (1, 4, 6, 7) \quad y_{30} = (1, 4, 6, 8) \quad y_{31} = (1, 4, 7, 8) \quad y_{32} = (1, 5, 6, 7) \]
\[ y_{33} = (1, 5, 6, 8) \quad y_{34} = (1, 5, 7, 8) \quad y_{35} = (1, 6, 7, 8) \quad y_{36} = (2, 3, 4, 5) \]
\[ y_{37} = (2, 3, 4, 6) \quad y_{38} = (2, 3, 4, 7) \quad y_{39} = (2, 3, 4, 8) \quad y_{40} = (2, 3, 5, 6) \]
\[ y_{41} = (2, 3, 5, 7) \quad y_{42} = (2, 3, 5, 8) \quad y_{43} = (2, 3, 6, 7) \quad y_{44} = (2, 3, 6, 8) \]
\[ y_{45} = (2, 3, 7, 8) \quad y_{46} = (2, 4, 5, 6) \quad y_{47} = (2, 4, 5, 7) \quad y_{48} = (2, 4, 5, 8) \]
\[ y_{49} = (2, 4, 6, 7) \quad y_{50} = (2, 4, 6, 8) \quad y_{51} = (2, 4, 7, 8) \quad y_{52} = (2, 5, 6, 7) \]
\[ y_{53} = (2, 5, 6, 8) \quad y_{54} = (2, 5, 7, 8) \quad y_{55} = (2, 6, 7, 8) \quad y_{56} = (3, 4, 5, 6) \]
\[ y_{57} = (3, 4, 5, 7) \quad y_{58} = (3, 4, 5, 8) \quad y_{59} = (3, 4, 6, 7) \quad y_{60} = (3, 4, 6, 8) \]
\[ y_{61} = (3, 4, 7, 8) \quad y_{62} = (3, 5, 6, 7) \quad y_{63} = (3, 5, 6, 8) \quad y_{64} = (3, 5, 7, 8) \]
\[ y_{65} = (3, 6, 7, 8) \quad y_{66} = (4, 5, 6, 7) \quad y_{67} = (4, 5, 6, 8) \quad y_{68} = (4, 5, 7, 8) \]
\[ y_{69} = (4, 6, 7, 8) \quad y_{70} = (5, 6, 7, 8) \]

These seventy compound matrix variables satisfy the differential equations as follow:

\[ \dot{y}_1 = h_{3y_2} + h_{4y_4} \]
\[ \dot{y}_2 = y_6 + y_4 \]
\[ \dot{y}_3 = y_7 + y_4 \]
\[ \dot{y}_4 = y_8 + y_5 \]
\[ \dot{y}_5 = y_9 + h_{8y_2} + h_{9y_4} \]
\[ \dot{y}_6 = y_7 + y_{16} + h_{2y_2} - h_{4y_{11}} \]
\[ \dot{y}_7 = y_8 + y_{17} + h_{2y_3} + h_{3y_{10}} - h_{4y_{13}} \]
\[ \dot{y}_8 = y_9 + y_{18} + h_{2y_4} + h_{3y_{11}} \]
\[ \dot{y}_9 = y_{19} + h_2y_5 + h_3y_{12} + h_4y_{15} - h_7y_1 + h_6y_6 + h_9y_8 \]
\[ \dot{y}_{10} = y_{20} + y_{11} \]
\[ \dot{y}_{11} = y_{21} + y_{13} + y_{12} \]
\[ \dot{y}_{12} = y_{22} + y_{14} - h_7y_2 + h_9y_{11} \]
\[ \dot{y}_{13} = y_{23} + y_{14} \]
\[ \dot{y}_{14} = y_{24} + y_{15} - h_7y_3 - h_8y_{10} + h_9y_{13} \]
\[ \dot{y}_{15} = y_{25} - h_7y_4 - h_8y_{11} \]
\[ \dot{y}_{16} = y_{36} + y_{17} - h_4y_{21} \]
\[ \dot{y}_{17} = y_{37} + y_{18} + h_3y_{20} - h_4y_{23} \]
\[ \dot{y}_{18} = y_{38} + y_{19} + h_3y_{21} \]
\[ \dot{y}_{19} = y_{39} + h_3y_{22} + h_4y_{25} + h_6y_1 + h_8y_{16} + h_9y_{18} \]
\[ \dot{y}_{20} = y_{40} + y_{26} + y_{21} \]
\[ \dot{y}_{21} = y_{41} + y_{27} + y_{23} + y_{22} \]
\[ \dot{y}_{22} = y_{42} + y_{28} + y_{24} + h_6y_2 + h_9y_{21} \]
\[ \dot{y}_{23} = y_{43} + y_{29} + y_{24} \]
\[ \dot{y}_{24} = y_{44} + y_{30} + y_{25} + h_6y_3 - h_8y_{20} + h_9y_{23} \]
\[ \dot{y}_{25} = y_{45} + y_{31} + h_6y_4 - h_8y_{21} \]
\[ \dot{y}_{26} = y_{46} + y_{27} + h_2y_{20} + h_4y_{32} \]
\[ \dot{y}_{27} = y_{47} + y_{28} + y_{29} + h_2y_{21} \]
\[ \dot{y}_{28} = y_{48} + y_{30} + h_2y_{22} - h_4y_{34} + h_6y_6 + h_7y_{16} + h_9y_{27} \]
\[ \dot{y}_{29} = y_{49} + y_{30} + h_2y_{23} + h_3y_{32} \]
\[ \dot{y}_{30} = y_{50} + y_{31} + h_2y_{24} + h_3y_{33} - h_4y_{35} + h_6y_7 + h_7y_{17} - h_8y_{26} + h_9y_{29} \]
\[ \dot{y}_{31} = y_{51} + h_{2}y_{25} + h_{3}y_{34} + h_{6}y_{8} + h_{7}y_{18} - h_{8}y_{27} \]
\[ \dot{y}_{32} = y_{52} + y_{33} \]
\[ \dot{y}_{33} = y_{53} + y_{34} + h_{6}y_{10} + h_{7}y_{20} + h_{9}y_{32} \]
\[ \dot{y}_{34} = y_{54} + y_{35} + h_{6}y_{11} + h_{7}y_{21} \]
\[ \dot{y}_{35} = y_{55} + h_{6}y_{13} + h_{7}y_{23} + h_{8}y_{32} \]
\[ \dot{y}_{36} = y_{37} + h_{1}y_{2} - h_{4}y_{41} \]
\[ \dot{y}_{37} = y_{38} + h_{1}y_{3} + h_{3}y_{40} - h_{4}y_{43} \]
\[ \dot{y}_{38} = y_{39} + h_{1}y_{4} + h_{3}y_{41} \]
\[ \dot{y}_{39} = h_{1}y_{5} + h_{3}y_{42} + h_{4}y_{45} - h_{5}y_{1} + h_{8}y_{36} + h_{9}y_{38} \]
\[ \dot{y}_{40} = y_{46} + y_{41} \]
\[ \dot{y}_{41} = y_{47} + y_{43} + y_{42} \]
\[ \dot{y}_{42} = y_{48} + y_{44} - h_{5}y_{2} + h_{9}y_{41} \]
\[ \dot{y}_{43} = y_{49} + y_{44} \]
\[ \dot{y}_{44} = y_{50} + y_{45} - h_{5}y_{3} - h_{8}y_{40} + h_{9}y_{43} \]
\[ \dot{y}_{45} = y_{51} - h_{5}y_{4} - h_{8}y_{41} \]
\[ \dot{y}_{46} = y_{56} + y_{47} - h_{1}y_{10} + h_{2}y_{40} + h_{4}y_{52} \]
\[ \dot{y}_{47} = y_{57} + y_{49} + y_{48} - h_{1}y_{11} + h_{2}y_{41} \]
\[ \dot{y}_{48} = y_{58} + y_{50} - h_{1}y_{12} + h_{2}y_{42} - h_{4}y_{54} - h_{5}y_{6} + h_{7}y_{36} + h_{9}y_{47} \]
\[ \dot{y}_{49} = y_{59} + y_{50} - h_{1}y_{13} + h_{2}y_{43} + h_{3}y_{52} \]
\[ \dot{y}_{50} = y_{60} + y_{51} - h_{1}y_{14} + h_{2}y_{44} + h_{3}y_{53} - h_{4}y_{55} - h_{5}y_{7} + h_{7}y_{37} - h_{8}y_{46} + h_{9}y_{49} \]
\[ \dot{y}_{51} = y_{61} - h_{1}y_{15} + h_{2}y_{45} + h_{3}y_{54} - h_{5}y_{8} + h_{7}y_{38} - h_{8}y_{47} \]
\[ \dot{y}_{52} = y_{62} + y_{53} \]
\[
\begin{align*}
\dot{y}_{53} &= y_{63} + y_{54} - h_5 y_{10} + h_7 y_{40} + h_9 y_{52} \\
\dot{y}_{54} &= y_{64} + y_{55} - h_5 y_{11} + h_7 y_{41} \\
\dot{y}_{55} &= y_{65} - h_5 y_{13} + h_7 y_{43} + h_8 y_{52} \\
\dot{y}_{56} &= y_{57} - h_1 y_{20} + h_4 y_{62} \\
\dot{y}_{57} &= y_{58} + y_{59} - h_1 y_{21} \\
\dot{y}_{58} &= y_{60} - h_1 y_{22} - h_4 y_{64} - h_5 y_{16} - h_8 y_{36} + h_9 y_{57} \\
\dot{y}_{59} &= y_{60} - h_1 y_{23} + h_3 y_{62} \\
\dot{y}_{60} &= y_{61} - h_1 y_{24} + h_3 y_{63} - h_4 y_{65} - h_5 y_{17} - h_8 y_{36} + h_9 y_{59} \\
\dot{y}_{61} &= -h_1 y_{25} + h_3 y_{64} - h_5 y_{18} - h_6 y_{38} - h_8 y_{57} \\
\dot{y}_{62} &= y_{66} + y_{63} \\
\dot{y}_{63} &= y_{67} + y_{64} - h_5 y_{20} - h_6 y_{40} + h_9 y_{62} \\
\dot{y}_{64} &= y_{68} + y_{65} - h_5 y_{21} - h_6 y_{41} \\
\dot{y}_{65} &= y_{69} - h_5 y_{23} - h_6 y_{43} + h_8 y_{62} \\
\dot{y}_{66} &= y_{67} + h_1 y_{32} + h_2 y_{62} \\
\dot{y}_{67} &= y_{68} + h_1 y_{33} + h_2 y_{63} + h_4 y_{70} - h_5 y_{26} - h_6 y_{46} - h_7 y_{56} + h_9 y_{66} \\
\dot{y}_{68} &= y_{69} + h_1 y_{34} + h_2 y_{64} - h_5 y_{27} - h_6 y_{47} - h_7 y_{57} \\
\dot{y}_{69} &= h_1 y_{35} + h_2 y_{65} + h_3 y_{70} - h_5 y_{29} - h_6 y_{49} - h_7 y_{59} + h_8 y_{66} \\
\dot{y}_{70} &= -h_5 y_{32} - h_6 y_{52} - h_7 y_{62}
\end{align*}
\]

For the boundary conditions (4.39), all the \(y\)'s are initially zero except \(y_{60}(0) = 1\), and the appropriate final condition to be satisfied is \(y_{11}(1) = 0\).
A2. The Poincaré and Cauchy-Schwarz inequalities.

The Poincaré inequality  (Straughan 1992).

Let $V$ be a cell in three dimensions. Suppose for simplicity $V$ is the cell $0 \leq x \leq 2a_1$, $0 \leq y \leq 2a_2$, $0 < z < 1$, and suppose $U$ is a function periodic in $x, y$ of period $2a_1, 2a_2$, respectively, and $u = 0$ on $z = 0, 1$. Then Poincaré inequality may be written as

$$\|u\|^2 \leq \frac{1}{\pi^2} \|\nabla u\|^2$$

The Cauchy-Schwarz inequality.

In the Hilbert space $L^2(a, b)$, the inner product is defined as

$$<f, g> = \int_a^b f \, g \, dx$$

with $\| \cdot \|$ as its associated norm. The Cauchy-Schwarz inequality may be written as

$$<f, g> \leq \|f\| \|g\|$$

which can be found in any standard textbook of functional analysis.
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