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BALBIR SINGH. DHILLON

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THE ANALYSIS OF THE RELIABILITY OF MULTI-STATE DEVICE NETWORKS

by

BALBIR SINGH DHILLON, B. Sc., M. Sc. (Wales)

A Dissertation Submitted to the Faculty of Graduate Studies
Through the Department of Industrial Engineering
in Partial Fulfillment of the Requirements for the Degree of

DOCTOR OF PHILOSOPHY

at the University of Windsor

Windsor, Ontario, Canada

September, 1975
TO MY MOTHER, FATHER AND BROTHER
ABSTRACT

The main objective of this study was to develop some new reliability evaluation and optimization techniques for two- and three-state device networks. Certain two-, three- and four-state devices can be defined in the following manner:

(i) Firstly, a device is said to have only two states if it can only operate successfully or fail;

(ii) Secondly, a device is said to have three states if it operates in its normal mode and fails in either its short or open failure mode (the typical examples of such a device are an electronic diode, a fluid flow valve, an electric switch, etc.). In addition, for example, a rotational mechanical system may jam so that rotation is blocked, or a shaft may shear so that an input rotation causes no output rotation;

(iii) Thirdly, a system or a device is said to have three states if it has successful, partial and catastrophic failure modes;

(iv) Finally, a system or a device is said to have four states if it has successful, partial, complete and catastrophic failure modes.
In this study, eleven new reliability evaluation and optimization techniques were developed to analyse two- and three-state device networks. In addition, reparable three-state device Markov models were developed.

This study includes the mathematical modeling of selected three-state device networks as well as mixed mode networks i.e., networks made up of both two- and three-state devices (normal, and open and short failure modes).

Lastly, it can be concluded that this study has tremendously deepened the candidate's knowledge of multistate device reliability theory. Hopefully, these findings will be beneficial to the technological advancement of mankind.
ACKNOWLEDGMENTS

It is a difficult thing to express gratitude to anyone, because the words cannot possibly do justice to the feelings of admiration and appreciation that the candidate possesses for the encouraging guidance of Dr. C. L. Proctor throughout this study.

The author wishes to express his gratitude to Dr. M. Shridhar and Professor A. A. Danish for their useful comments through the course of this study.

Finally, the author is deeply indebted to Mrs. Judy Assef who so intelligently and neatly edited and typed the manuscript of this dissertation as well as the manuscripts of the numerous other publications which resulted from this study.
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CHAPTER I.

INTRODUCTION

A device is said to have three states if it operates satisfactorily in its normal mode but fails in either of two other modes. Typical examples of such a device are a fluid flow valve and an electric diode. These devices have failure modes which can be described as closed (or shorted) and opened.

A device or a system may also have three states if it functions satisfactorily in its normal mode, operates satisfactorily in its one degraded mode and fails completely in its catastrophic mode. A typical example of such a device or a system is a wrist watch which loses a specified amount of time in its degraded mode.

In the case of a two-state device, it either operates successfully or fails catastrophically. In "reliability Engineering", most of the devices or systems fall into this category.

This chapter presents a rather brief, but selective literature review. A general introduction on the analysis of networks made up of two and three states devices, the problem statement and the organization of the Dissertation are also included.

1.1. Literature Review

Until a few decades ago, the emphasis of system reliability was on increasing component reliability as the means of increasing the system's reliability. Today redundancy plays a dominant role in increasing a system's reliability, especially where the system

*The reliability of a device (or system) may be defined as the probability of a successful operation of the device (or system) in the manner and under the conditions of its intended use.
reliability must be greater than that of the components used.

The literature is reviewed in the following order:

(i) three-state device (i.e., normal, open and short (closed) modes) networks;

(ii) three-state device (i.e., normal, partial and catastrophic modes) networks;

(iii) two-state device networks.

Since most of this study is related to networks made up of the first type of three-state devices, the literature was reviewed accordingly.

In the case of a three-state device having one successful and, two failure modes, the most important literature found began in 1952. In 1952, Gates (1) presented the first serious reasoning on this subject in his treatment of the reliability of redundant systems. Careful consideration of the reliability of three-state devices was presented by Moore and Shannon (2) and Creveling (3) in their 1956 papers on electrical and electronic devices. Creveling in his study developed the reliability and failure equations for a diode quad arrangement, while Moore and Shannon developed formulas for several relay networks.

The year 1957 brought another development when Lipp (4) discussed the topology of switching elements vs. reliability. The following year, Price (5) specifically dealt with the reliability of three-state devices in a parallel configuration and attempted to optimize the number of redundant components. A significant event in reliability optimization took place in 1960, when Barlow and Hunter (6, 17) used the mathematical calculus to optimize the reliability of series, parallel, series-parallel, and parallel-series
networks. They also computed the number of components which maximize the expected system life for these first two types of systems given above assuming component life is exponentially distributed.

In 1962, Sorensen (7) tried to apply the theory established by the previous researchers on three-state device networks to several electronic circuits. His primary approach was very similar to that of Creveling.

In the same year, Cluley (8) published a paper on low level redundancy as a means of improving the reliability of digital computers. Also in 1962, James, Kent and Holloway (16) reviewed the reliability problem and derived some systems reliability equations for redundant three-state device structures. In 1963, Blake (9) extended the work of Moore and Shannon (2) on nets of relay contacts by investigating the open and short circuit failures of hammock networks.

In the same year, Barlow, Hunter and Proschan (18) extended their previous contribution to maximize the expected system life for components having exponential and uniform time to failure distributions. Also, Gorden (19) developed a model similar to their model, but maximizing a somewhat different objective function.

In 1964, Alven (10) edited a reliability engineering book which includes some reliability derivations of the three-state device configurations. This was simply a restatement of Barlow and Hunter's (6, 17) work with the addition of a primitive graphical plot for the optimization of parallel and series configurations.

Furthermore, in 1965 Barlow, Hunter and Proschan (20) published a book entitled "Mathematical Theory of Reliability" which
includes some of their previous contributions on the unconstrained optimization of three-state device networks.

In 1967, Kolesar (21) extended the work of the previous researchers when he optimized a series-parallel three-state device structure under constrained conditions.

Between 1967 and 1970, the subject received very little attention with the exception of the publication of Shooman's (11) text entitled "Probabilistic Reliability" which was published in 1968. Nevertheless, he did little more than mention the topic. In 1970, Misra and Rao (12) developed a signal flow graph approach. Jensen (22) in same year referred to series-parallel-series network using constrained conditions of cost and weight.

During the following two years, only one of the four studies making reference to the subject appears to be important. Evans (13) and Sengupta (23) give a very brief introduction to three-state device reliabilities in their paper and book, respectively. Butler (14) made brief reference to it in his publication. In 1972, Proctor and Smith (15) took a more mathematical approach in their study of reliability optimization of arrays of multi-state devices.

Therefore, it can be said that the development of reliability analysis of three-state device (i.e. a device with one normal mode and two-failure modes) networks has been an evolutionary rather than a revolutionary process.

In the case of a three-state device or a system with successful, partial and catastrophic modes only, two recent publications in 1974 by Tumolillo (24) and Kontoleon et al (25) are considered important. Tumolillo developed methods for calculating
the reliability function for systems subjected to random stresses, and Kontoleon et al analyzed a repairable three-state device system in which they assumed the normal to catastrophic and partial to catastrophic failure rates are same. The system is only repaired from the catastrophic failure states.

Abundant literature is available on two-state device networks. Since the major aim of this research is directed toward the analysis of three-state device (i.e. normal and two failure states) networks, the literature on two-state device systems is omitted. It should be noted, however, that one chapter of this study is essentially on two state device networks.

1.2. General Introduction

The relatively low reliability achieved by the early guide missiles is attributed, in part, to the inherent weakness of the series structure system that was generally used at that time. If a large number of two-state components are included in a series structured system, the reliability of each component must be very close to unity if the system is to have a high reliability. Two approaches to the solution of the system reliability problem have been postulated:

(a) use components of high reliability
(b) use redundant components

However, in the case of a system containing three-state devices (i.e. a device with normal, open and short modes), redundancy may either increase or decrease the system's reliability. This depends upon the components dominant mode of failure, the configuration of the system and the number of redundant components.

An electronic diode and a fluid flow valve are typical examples of three-state devices. Either of these components may fail catastrophically in either the open or closed (shorted) mode. A given three-state device has a probability of failure in the
open-mode and a probability of failure in the closed or shorted mode.

Because a three-state device cannot fail simultaneously in both the opened and closed (shorted) modes, the failures are mutually exclusive events and the failure of any one such device is considered independent of all of the others.

Three-state devices can be arranged in various redundant configurations such as series, parallel, series-parallel, parallel-series and mixed arrangements, etc. As these configurations become more complex, the analysis of networks becomes more cumbersome, and redundancy can result in a decreased overall system reliability. This lower system reliability is created by the redundancy of the dominant adverse mode of failure.

Another type of three-state device equipment or system, is considered to have successful, partial and catastrophic modes. These types of three-state devices are frequently encountered in civil, industrial and military complexes. For example, it could be a wrist watch, a piece of machinery in an industrial complex, or a tank at a military combat unit.

This type of three-state device system can be arranged in several redundant configurations such as series, parallel, series-parallel, parallel-series, etc. As an example, suppose a tank combat unit is defending a vital defence line at several points. Each post requires one tank to defend it, but redundancy is achieved by the use of extra tanks. If anyone of the post falls into the enemy hands, then it is assumed to cause a catastrophic failure of the entire defence line. Therefore, this system can simply be formulated to yield series-parallel network.
1.3. Objective of the Study

The first major objective of this study is to extend the simplified mathematical modeling procedure to include simple and complex networks consisting of three-state devices having two modes of catastrophic failure.

Secondly, it is to further extend this modeling to include systems with three-state devices which have a normal mode of operation, a degraded mode, and a catastrophic failure mode.

Finally, it is to develop some simplified mathematical modeling procedure to analyze complex two-state device networks, including bridges.

Each of the above objectives are further explained as follows:

1.3.1. Two-State Device Networks

To evaluate the reliability of a complex two-state network such as a bridge configuration is a rather difficult task, especially when the designers are from a conventional engineering background. The analogy of a reliability network to that of an equivalent electrical network is considered a valuable tool to the engineering designer even at an approximation level because of its simplicity in reliability evaluation.

1.3.2. Three-State Device (i.e. normal, short and open modes) Networks

From the review of the literature, it was found that for some complex configurations the conventional methods for evaluating and optimizing redundancies to achieve the maximum reliability of three-state device systems are very difficult and cumbersome, if no
nearly impossible to use. The amount of human activity and computing time markedly increases as the configuration of a three-state device network becomes more complex. Furthermore a sound knowledge of three-state reliability theory is required by the engineering designers of such a system.

The optimization of the reliability of systems composed of three-state device sub-systems, each having the same number of components but with each sub-system having a different set of identical devices, was not found in the survey of the literature. Neither was the analysis of mixed two and three-state device networks found in any of the literature reviewed.

Therefore, one of the objectives of this study is to develop mathematical models and simple but new techniques to analyze three-state device networks. Repairs are also included in the models for some three-state device networks in this study.

1.3.3. Three-State Device (i.e. normal, partial and catastrophic modes) Networks

In the literature survey very little work was found on three-state device networks having a normal operating mode and a partial and catastrophic failure mode (24, 25).

1.4. Organization of Study

Chapter 1 includes the literature review of multi-state device networks as well as the objective of the study.

Chapter 2 and 3 outline the development of the non-repairable and repairable, respectively, Markov models for both types of three-state devices which demonstrate some of the complexities in obtaining
system reliability equations.

Chapter 4 through 8 witness the development of some new but simple techniques to analyze three-state device (i.e., normal, short and open modes) networks. Chapter 9 demonstrates the reliability optimization for some newly developed three-state device (i.e., normal, short and open modes) mathematical models. This chapter also contains the system reliability evaluation equations of mixed networks (i.e., the networks made up of both two-and-three-state devices). Chapter 10 presents the developments of two new but uniquely different methods for the reliability evaluation of two-state device networks.

Finally, Chapter 11 summarizes the conclusions and suggested accomplishments of the study and recommendations for further study.
CHAPTER 2
NON-REPARABLE MARKOV MODELS

The first section of this chapter shows the development of system reliability equations for series, parallel, series-parallel, parallel-series and bridge networks made up of the first type of three-state device (i.e. a device with normal, short and open modes). The second part contains the derivations of the series, parallel, series-parallel and parallel-series networks made up of the second type of three-state device (i.e. a device with normal, partial and catastrophic modes). A four-state Markov model is also included. These Markov models show the complexities in the reliability analysis when the time dependence of failures are introduced.

The transitional probabilities are assumed to obey the following:

(i) the probability of transition in finite time interval \( t, t+\Delta t \) from one state to another is generated by the failure rate \( \lambda \) or \( \mu \) times the finite time interval \( \Delta t \);

(ii) the probabilities of occurrence of more than one transition during a finite time interval \( \Delta t \) are very small and as higher order terms can be ignored;

(iii) the failure density probability functions are assumed to be exponential; hence the failure rates \( \lambda \) or \( \mu \) are assumed to be constants.

2.1. Three-State Device Markov-Models* - (i.e. normal, short and open modes)

This section introduces the Markov-modeling concept for

* Derivations are shown in Appendix A

10
three-state devices which are used in the series, parallel, series-parallel, parallel-series and bridge networks.

2.1.1. A Three-State Device Markov Model type II

As mentioned earlier, a three-state device can be an electronic diode or a fluid flow valve, etc. The transition diagram shown in Figure 1 is for an electronic diode or its equivalent. It should be noted that a fluid flow valve has the opposite operational characteristics to that of an electronic diode.

![Transition Diagram](image)

Figure 1. A Three-State Device Transition Diagram

From Figure 1 the normal, open and short mode probability differential equations can be written as follows, respectively:

\[
\frac{dP_0(t)}{dt} + (\lambda + \mu) P_0(t) = 0 \quad (1)
\]

\[
\frac{dP_1(t)}{dt} - \lambda P_0(t) = 0 \quad (2)
\]

\[
\frac{dP_2(t)}{dt} - \mu P_0(t) = 0 \quad (3)
\]

\[P_0(0) = 1, \quad P_1(0) = 0, \quad P_2(0) = 0\]

where \(P\) is the state probability
\(\mu\) is a short mode failure rate
\(\lambda\) is an open mode failure rate
\(t\) is the time
\(\Delta t\) is a finite time interval used in the development.
Solving the above differential equations will yield the following normal, open and short mode probability equations, respectively:

\[ P_0(t) = e^{-(\lambda + \mu)t} \]  \hspace{1cm} (4)

\[ P_1(t) = \frac{\lambda}{\mu + \lambda} (1 - e^{-(\lambda + \mu)t}) \]  \hspace{1cm} (5)

\[ P_2(t) = \frac{\mu}{\mu + \lambda} (1 - e^{-(\lambda + \mu)t}) \]  \hspace{1cm} (6)

The probabilities of equations (4) to (6) are mutually exclusive events; therefore, their addition will yield unity, i.e.,

\[ P_0(t) + P_1(t) + P_2(t) = 1 \]

2.1.2. A Three-State Device Markov Model of a Special Non-reparable Series System

This Markov model demonstrates that if one three-state device (i.e., an electronic diode) in a series system fails in a short mode, it will also have a probability of failure from short to open mode. The transition diagram is shown in Figure 2.

![Transition Diagram](image)

Figure 2. A Markov Model Transition Diagram

The differential equations associated with Figure 2 are:

\[ \frac{dP_0(t)}{dt} + (\mu + \lambda_{so}) P_0(t) = 0 \]  \hspace{1cm} (7)
\[
\frac{dP_2(t)}{dt} + \lambda_{so} P_2(t) = P_0(t) \mu \\
\frac{dP_1(t)}{dt} = P_2(t) \lambda_{so} + P_0(t) \lambda \\
P_0(0) = 1, P_1(0) = 0, P_2(0) = 0
\] (8) (9)

Solving the above family of differential equations with their initial conditions yields the following normal, open and short mode probability equations:

\[
P_0(t) = e^{-(\lambda+\mu)t} \\
P_1(t) = 1 - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} + \frac{\mu \lambda_{so}}{(\lambda+\mu)(\lambda+\mu-\lambda_{so})} e^{-(\lambda+\mu)t} - \frac{\mu e^{-\lambda_{so}t}}{(\lambda+\mu-\lambda_{so})} \\
P_2(t) = \frac{\mu}{\mu+\lambda-\lambda_{so}} (e^{-\lambda_{so}t} - e^{-(\lambda+\mu)t})
\] (10) (11) (12)

As can be observed from the above probability equations, the introduction of the closed to open failure concept, the series system reliability will decrease when compared to the first model introduced.

2.1.3. Series Configuration

In a series system, any one element failing in an open mode will cause a complete system failure; whereas all elements of the system must malfunction in a short mode to induce failure for this system. The transitional diagram for two elements of the first kind connected in series is shown in Figure 3.
Figure 3. A Two-Element Series Network Markov Model

To evaluate the series system reliability, the differential equations associated with Figure 3 are:

\[
\frac{dP_{N_1N_2}(t)}{dt} + (\lambda_1 + \lambda_2 + \mu_1 + \mu_2)P_{N_1N_2}(t) = 0
\]

(13)

\[
\frac{dP_{N_1S_2}(t)}{dt} + P_{N_1S_2}(t) (\lambda_1 + \mu_1) = P_{N_1N_2}(t) \mu_2
\]

(14)

\[
\frac{dP_{S_1N_2}(t)}{dt} + P_{S_1N_2}(t) (\mu_2 + \lambda_2) = P_{N_1N_2}(t) \mu_1
\]

(15)

\(P_{N_1N_2}(0) = 1, \text{ at } t=0 \text{ the other initial condition probabilities are zero.}\)

*All the network configurations of section 2.1 are made up of the three-state device of section 2.1.1.*
where
\[ \lambda_1, \lambda_2 \] are the open mode failure rates of components number one and two
\[ \mu_1, \mu_2 \] are the short mode failure rates of components number one and two
\[ P(\cdot)(\cdot) \] is the state probability
\[ N_i \] is the normal mode state, \( i = 1, 2 \)
\[ S_i \] is the short mode state, \( i = 1, 2 \)
\[ O_i \] is the open mode state, \( i = 1, 2 \)

Solutions of the above differential equations are:

\[ P_{N_1N_2}(t) = e^{-(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)t} \] \hspace{1cm} (16)

\[ P_{N_1S_2}(t) = \frac{\mu_2}{(\lambda_2 + \mu_2)} (e^{-(\lambda_1 + \mu_1)t} - e^{-(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)t}) \] \hspace{1cm} (17)

\[ P_{S_1N_2}(t) = \frac{\mu_1^2}{(\lambda_1 + \mu_1)} (e^{-(\lambda_2 + \mu_2)t} - e^{-(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)t}) \] \hspace{1cm} (18)

The addition of equations (16), (17) and (18) will yield the system reliability for two non-identical elements as follows:

\[ \text{System Reliability} = P_{N_1N_2}(t) + P_{N_1S_2}(t) + P_{S_1N_2}(t) \]

Three elements in series:

The transition diagram of a series system for three elements is shown in Figure 4.
Figure 4. A Three-Element Series Network Markov Model

To draw a three-state device transition diagram to analyse a network, the following associated formulas with Figure 4 are:

Since \( n \) is the number of components,

\[
\text{total number of reliable banks} = i = n + 1
\]  

(19)
The number of incoming transitional rates to each bank mode (for successful transitional modes only) are:

\[ I_j = (i_j - n_j) \]  \hspace{1cm} (20)

where \( n = 1, 2 \ldots i \)
\[ j = i, i-1, i-2, \ldots 1 \]

The number of outgoing transitional rates from each bank node (for successful transitional nodes only) are:

\[ O_k = 2(n_k - i_k) \]  \hspace{1cm} (21)

where \( i = 0, 1, 2, \ldots n \)
\[ k = 1, 2, \ldots i \]

To evaluate series system reliability, the differential equations associated with Figure 4 (i.e., identical components with same open and short mode failure rates) are:

\[ \frac{dP_{N_1N_2S_3}(t)}{dt} + 6\lambda P_{N_1N_2N_3}(t) = 0 \]  \hspace{1cm} (22)

\[ \frac{dP_{N_1N_2S_3}(t)}{dt} + 4\lambda P_{N_1N_2S_2}(t) = P_{N_1N_2N_3}(t)\lambda \]  \hspace{1cm} (23)

\[ \frac{dP_{S_1N_2S_3}(t)}{dt} + 2\lambda P_{S_1N_2S_3}(t) = 2\lambda P_{N_1N_2S_3}(t) \]  \hspace{1cm} (24)

\[ P_{N_1N_2S_3}(t) = P_{S_1N_2N_3}(t) = P_{N_1S_2N_3}(t), P_{S_1N_2S_3}(t) = P_{S_1S_2N_3}(t) = P_{N_1S_2S_3}(t) \]
\[ P_{N_1N_2N_3}(0) = 1, \text{ at } t=0 \text{ other initial condition probabilities are zero.} \]

where

- \( \lambda \) is the constant failure rate
- \( P(\cdot)(\cdot)(\cdot) \) is the state probability
- \( N_i \) is the normal state, \( i = 1, 2, 3 \)
- \( S_i \) is the short mode state, \( i = 1, 2, 3 \)
- \( O_i \) is the open mode state, \( i = 1, 2, 3 \)

The solutions of the differential equations are:

\[ P_{N_1N_2N_3}(t) = e^{-6\lambda t} \tag{25} \]

\[ P_{N_1N_2S_3}(t) = \frac{1}{2} e^{-4\lambda t} - \frac{1}{2} e^{-6\lambda t} \tag{26} \]

\[ P_{S_1N_2S_3}(t) = \frac{1}{2} e^{-6\lambda t} - \frac{1}{2} e^{-4\lambda t} + \frac{1}{2} e^{-2\lambda t} \tag{27} \]

\[ \therefore \text{ System Reliability} = P_{N_1N_2N_3}(t) + 3P_{N_1N_2S_3}(t) + 3P_{S_1N_2S_3}(t) \]

The above equation can be formulated as:

\[
\text{System Reliability} = \left[ 1 - \left( \frac{\lambda}{\lambda+\lambda} \left( 1-e^{-(\lambda+\lambda)t} \right) \right) \right]^3
- \left[ \frac{\lambda}{\lambda+\lambda} \left( 1-e^{-(\lambda+\lambda)t} \right) \right]^3
\tag{28}
\]

The above equation and the system reliability equation extended for \( n \) identical elements for time varying probabilities correspond to the following constant probability equations:

\[ R = (1-q_0)^n - q_n \]
where $q_0$ is the component's open mode unreliability.

$q_S$ is the component's short mode unreliability.

It is easily shown that equation (28) can be rewritten for non-identical elements as follows:

$$R = \prod_{i=1}^{n} \left[ 1 - \frac{\lambda_i}{\mu_i + \lambda_i} (1 - e^{-(\lambda_i + \mu_i) t_i}) \right] + \sum_{i=1}^{n} \left[ \frac{\mu_i}{\mu_i + \lambda_i} (1 - e^{-(\mu_i + \lambda_i) t_i}) \right]$$  \hspace{1cm} (30)

2.1.4. Parallel Network

For a parallel system to fail, all the elements must fail in the open mode or at least one of the elements must fail in a short mode to cause the system to fail completely. The probability transition diagram of a parallel system with two elements is shown in Figure 5.

![Figure 5. A Two-Element Parallel System Transition Diagram](image)
To obtain system reliability, the differential equations pertaining to Figure 5 are:

\[
\frac{dP_{N_1N_2}(t)}{dt} + P_{N_1N_2}(t)(\mu_1 + \mu_2 + \lambda_1 + \lambda_2) = 0 \quad (31)
\]

\[
\frac{dP_{N_1O_2}(t)}{dt} + P_{N_1O_2}(t)(\lambda_1 + \mu_1) = \lambda_2 P_{N_1N_2}(t) \quad (32)
\]

\[
\frac{dP_{O_1N_2}(t)}{dt} + P_{O_1N_2}(t)(\lambda_2 + \mu_2) = \lambda_1 P_{N_1N_2}(t) \quad (33)
\]

\[P_{N_1N_2}(0) = 1, \text{ at } t=0 \text{ other initial condition probabilities are zero.}\]

The following solutions satisfy the above differential equations:

\[P_{N_1N_2}(t) = e^{-(\mu_1 + \mu_2 + \lambda_1 + \lambda_2)t} \quad (34)\]

\[P_{N_1O_2}(t) = \frac{\lambda_2}{\nu_2 + \lambda_2} (e^{-(\mu_1 + \lambda_1)t} - e^{-(\mu_1 + \mu_2 + \lambda_1 + \lambda_2)t}) \quad (35)\]

\[P_{O_1N_2}(t) = \frac{\lambda_1}{\mu_1 + \lambda_1} (e^{-(\nu_2 + \lambda_2)t} - e^{-(\mu_1 + \mu_2 + \lambda_1 + \lambda_2)t}) \quad (36)\]

The addition of equations (34), (35) and (36) yields the two-component parallel system reliability, i.e.,

\[\text{System Reliability} = P_{N_1N_2}(t) + P_{N_1O_2}(t) + P_{O_1N_2}(t) \quad (37)\]

### Three elements in parallel

A parallel redundant system with three elements becomes somewhat more complicated to analyse than the previous two-element
case. The transition diagram for it is shown in Figure 6.

**Figure 6. A Three-Element Parallel System Transition Diagram**

The following equations* and steps were used to construct the transition diagram of Figure 6:

From equation (19) the number of reliability banks = \( n+1 = 3+1 = 4 \).

where \( n \) is the number of elements

*The derived formulas are in section 2.1.3.
Likewise, from equation (21),

the number of outgoing transition rates from the first bank node = 2(3-0) = 6,

the number of outgoing transition rates from the second bank node = 2(3-1) = 4,

the number of outgoing transition rates from the third bank node = 2(3-2) = 2,

the number of outgoing transition rates from the fourth bank node = 2(3-3) = 0.

Similarly, from equation (20), it can be shown that the number of incoming transition rates to the banks will be 0, 1, 2 and 3, respectively.

If \( \lambda_1 = \lambda_2 = \lambda_3 = \mu_1 = \mu_2 = \mu_3 = \lambda \), as would be the case for identical elements, the three-element parallel system reliability associated with Figure 6 would be:

\[
R = \frac{1}{4} e^{-6\lambda t} + \frac{3}{4} e^{-2\lambda t}
\]

(38)

The above equation can be reformulated (if \( \mu = \lambda \)) as:

\[
R = \left[1 - \left(\frac{\mu}{\lambda + \mu}(1-e^{-(\lambda + \mu)t})\right)\right]^3 - \left[\frac{\lambda}{\mu + \lambda}(1-e^{-(\mu + \lambda)t})\right]^3
\]

(39)

Therefore, equations (38) and (39) can be generalised for \( n \) non-identical elements as follows:

\[
R = \prod_{i=1}^{n} \left[1 - \left(\frac{\mu_i}{\lambda_i + \mu_i}(1-e^{-(\lambda_i + \mu_i)t})\right)\right] - \prod_{i=1}^{n} \left[\frac{\lambda_i}{\mu_i + \lambda_i}(1-e^{-(\mu_i + \lambda_i)t})\right]
\]

(40)

*Note \( n=3 \) for equations (38) and (39)*
Similarly, the general equations for the open and short modes of failure can be analytically derived.

2.1.5. Parallel-Series Network

This configuration results from the combination of the previous two configurations. The network configuration is shown in Figure 7.

![Parallel-Series Network Diagram](image)

Figure 7. A Parallel-Series Network

The reliability transitional flow diagram is shown in Figure 8 for this parallel-series four-element case. For this case, the system reliability, system open and short mode failure equations are derived in Appendix B for identical components, each with equal open and short mode failure probabilities.

If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = \lambda$, the system reliability associated with Figure 8 is:

$$R = \frac{3}{2} e^{-2\lambda t} - \frac{1}{2} e^{-6\lambda t}$$

(41)

As in the previous section, equation (41) can be rewritten in the following form (if $\mu = \lambda$):

*See Appendix B*
Figure 8. A Four-Element Parallel-Series Reliability Transition Diagram
\[ R = \left[ 1 - \left( \frac{\mu}{\lambda + \mu} (1 - e^{-(\mu + \lambda)t}) \right)^2 \right] \left[ 1 - \left( \frac{\lambda}{\mu + \lambda} (1 - e^{-(\lambda + \mu)t}) \right)^2 \right] \]  \hspace{1cm} (42)

The above equation corresponds to the following:

\[ R = [1 - q_s^n]^m - [1 - (1 - q_o^n)]^m \]

where \( m = 2 \) is the number of paths and \( n = 2 \) is the number of components per path.

For a fixed number of components in the series path, the general formula obtained for the above equation for non-identical paths can be formulated as follows:

\[
R = \prod_{i=1}^{m} \left[ 1 - \left( \frac{\mu_i}{\lambda_i + \mu_i} (1 - e^{-(\mu_i + \lambda_i)t_i}) \right)^n \right] \left[ 1 - \left( \frac{\lambda_i}{\mu_i + \lambda_i} (1 - e^{-(\lambda_i + \mu_i)t_i}) \right)^n \right] \]  \hspace{1cm} (43)

**Short Mode System Failure**

The system short mode failure equations for a parallel-series system with identical components having equal open and short mode failure rates are derived in Appendix B. The probabilities transitional diagram is shown in Figure 9.

If \( \mu_1 = \mu_2 = \mu_3 = \mu_4 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda \) and \( P_{S_3S_4}(t) = \)

\[ P_s(t) = \text{system short mode unreliaibility}, \]

\[ P_s(t) = \frac{7}{16} - \frac{1}{16} e^{-3\lambda t} + \frac{1}{4} e^{-6\lambda t} + \frac{1}{8} e^{\lambda t} - \frac{3}{4} e^{-2\lambda t} \]  \hspace{1cm} (44)
Figure 9. A Four-Element Parallel-Series Short Mode Failure Transition Diagram
Equation (44) is found to be of the following form (if \( \mu = \lambda \)):

\[
P_S(t) = 1 - \left[ 1 - \frac{\mu}{\lambda + \mu} \left( 1 - e^{- (\lambda + \mu) t} \right) \right] \left( 1 - e^{- (\lambda + \mu) t} \right)^2
\]

(45)

This equation corresponds to the following:

\[
P_s = 1 - (1 - q_s)^m
\]

Similarly, for non-identical elements, the above equation becomes:

\[
P_S(t) = 1 - \prod_{i=1}^{m} \left[ 1 - \frac{\mu_i}{\lambda_i + \mu_i} \left( 1 - e^{- (\lambda_i + \mu_i) t} \right) \right] \left( 1 - e^{- (\lambda_i + \mu_i) t} \right)^n
\]

(46)

**Open mode system failure**

Similar to the short mode failure case \( P_{0102} = P_0(t) \),

\[
P_0(t) = \frac{9}{16} - \frac{3}{4} e^{-2\lambda t} - \frac{1}{8} e^{-4\lambda t} + \frac{1}{4} e^{-6\lambda t} + \frac{1}{16} e^{-8\lambda t}
\]

(47)

Equation (47) can be expressed in the following form (if \( \mu = \lambda \)):

\[
P_0(t) = \left[ 1 - \left( 1 - \frac{\lambda}{\lambda + \mu} \left( 1 - e^{- (\lambda + \mu) t} \right) \right) \right]^2
\]

(48)

This expression corresponds to the following equation*:

\[
P_0 = [1 - (1 - q_0)^m]^n
\]

*Note \( n = m = 2 \) for equation (48)
For non-identical elements, equation (48) yields:

\[
R(t) = \prod_{i=1}^{m} \left[ 1 - \left(1 - \frac{\lambda_i}{\lambda_i + \mu_i} \right) \left(1 - e^{-(\mu_i + \lambda_i)t} \right)^m \right] 
\]

(49)

2.1.6. **Series-Parallel Configuration**

The series-parallel structure is a dual of the parallel-series configuration of the previous section. The structure is shown in Figure 10.

![Series-Parallel Configuration](image)

Figure 10. A Series-Parallel Configuration

Similar to the parallel-series case, the general equations for the system reliability with short and open modes of failure are as follows:

\[
R(t) = \prod_{i=1}^{n} \left[ 1 - \left(1 - \frac{\lambda_i}{\lambda_i + \mu_i} \right) \left(1 - e^{-(\mu_i + \lambda_i)t} \right)^m \right] 
\]

- \[
Q_s(t) = \prod_{i=1}^{n} \left[ 1 - \left(1 - \frac{\mu_i}{\lambda_i + \mu_i} \right) \left(1 - e^{-(\mu_i + \lambda_i)t} \right)^m \right] 
\]

(50)

and

\[
Q_0(t) = 1 - \prod_{i=1}^{n} \left[ 1 - \left(1 - \frac{\lambda_i}{\lambda_i + \mu_i} \right) \left(1 - e^{-(\mu_i + \lambda_i)t} \right)^m \right] 
\]

(51)
where \( n \) is the number of sub-systems and \( m \) the number of components in a sub-system.

2.1.7. Bridge Network

A bridge network (4) is shown in Figure 11.

![Diagram of a bridge network](image)

Figure 11. A Bridge Network

The element or component numbered '5' shown in Figure 11 can be considered as a critical one for a bridge.

For the bridge case, suppose

- \( f_1(q_s) \) = the probability of short mode failure of the critical element
- \( f_2(q_s) \) = the short mode failure probability of the series-parallel network
- \( f_3(q_s) \) = the short mode failure probability of the parallel-series network
- \( f_3(q_o) \) = the open mode failure probability of the parallel-series network
\( f_2(q_0) \) = the open mode failure probability of the series-parallel network

\( f_1(q_0) \) = the probability of open mode failure of the critical element

If the critical element (i.e., marked '5') of Figure 11 fails in an open mode, the bridge reduces to a parallel-series structure. However, if the critical element fails in a short mode, the bridge network becomes a series-parallel configuration.

As a consequence, the bridge short mode failure probability can be expressed as:

\[
Q_{SB} = f_1(q_s)f_2(q_s) + f_3(q_s)(1-f_1(q_s))
\]

Equations (6), (46) and (51) can be substituted into (53), giving the following results:

\[
f_1(q_s) = \frac{\mu_5}{\mu_5+\lambda_5} (1-e^{-(\mu_5+\lambda_5)t})
\]

\[
f_2(q_s) = \frac{\mu_1\mu_2\mu_3\mu_4(1-e^{-(\lambda_1+\mu_1)t})}{(\mu_1+\lambda_1)(\mu_2+\lambda_2)(\mu_3+\lambda_3)(\mu_4+\lambda_4)} (1-e^{-(\lambda_2+\mu_2)t})
\]

\[
(1-e^{-(\lambda_3+\mu_3)t})(1-e^{-(\lambda_4+\mu_4)t}) + \left( \frac{\mu}{\mu_1+\lambda_1} \right) (1-e^{-(\mu_1+\lambda_1)t})
\]

\[
+ \frac{\mu_2}{\lambda_2+\mu_2} (1-e^{-(\mu_2+\lambda_2)t}) \left( \frac{\mu_3}{\mu_3+\lambda_3} \right) (1-e^{-(\mu_3+\lambda_3)t})
\]

\[
+ \frac{\mu_4}{\mu_4+\lambda_4} (1-e^{-(\mu_4+\lambda_4)t}) - \frac{\mu_3\mu_4}{(\mu_3+\lambda_3)(\mu_4+\lambda_4)} (1-e^{-(\mu_3+\lambda_3)t})
\]
\[ (1-e^{-(\mu_4+\lambda_4)t}) \left( \frac{\lambda_1}{\mu_1+\lambda_1} (1-e^{-(\mu_1+\lambda_1)t}) + \frac{\mu_2}{\mu_2+\lambda_2} (1-e^{-(\mu_2+\lambda_2)t}) \right) \]

\[- \frac{\mu_1\mu_2}{(\mu_1+\lambda_1)(\mu_2+\lambda_2)} (1-e^{-(\mu_1+\lambda_1)t})(1-e^{-(\mu_2+\lambda_2)t}) \left( \frac{\mu_3}{\lambda_3+\mu_3} \right) \]

\[ (1-e^{-(\mu_3+\lambda_3)t}) + \frac{\mu_4}{\lambda_4+\mu_4} (1-e^{-(\mu_4+\lambda_4)t}) \]

and

\[ f_3(q_s) = \frac{\mu_4\mu_2(1-e^{-(\mu_4+\lambda_4)t})}{(\mu_4+\lambda_4)(\mu_2+\lambda_2)} (1-e^{-(\mu_2+\lambda_2)t}) + \frac{\mu_3\mu_1}{(\lambda_3+\mu_3)(\lambda_1+\mu_1)} \]

\[ (1-e^{-(\mu_3+\lambda_3)t})(1-e^{-(\mu_1+\lambda_1)t}) - \frac{\mu_4\mu_2\mu_3\mu_1}{(\mu_1+\lambda_1)(\mu_2+\lambda_2)(\mu_3+\lambda_3)(\mu_4+\lambda_4)} \]

\[ (1-e^{-(\mu_1+\lambda_1)t})(1-e^{-(\mu_2+\lambda_2)t})(1-e^{-(\mu_3+\lambda_3)t})(1-e^{-(\mu_4+\lambda_4)t}) \]

Similarly, the system open mode failure probability for the bridge case is:

\[ Q_{OB} = f_1(q_0)f_2(q_0) + f_3(q_0)(1-f_1(q_0)) \quad (55) \]

\[ \therefore \text{Equations } (5), (49) \text{ and } (52) \text{ can be substituted into } (55) \text{ yielding the following:} \]

\[ f_1(q_0) = \frac{\lambda_5}{\mu_5+\lambda_5} (1-e^{-(\mu_5+\lambda_5)t}) \]

\[ f_2(q_0) = \frac{\lambda_3\lambda_4}{(\lambda_3+\mu_3)(\lambda_4+\mu_4)} (1-e^{-(\lambda_2+\mu_2)t})(1-e^{-(\lambda_4+\mu_4)t}) \]

\[ + \frac{\lambda_1\lambda_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} (1-e^{-(\lambda_2+\mu_2)t})(1-e^{-(\lambda_1+\mu_1)t}) \]

\[- \frac{\lambda_1\lambda_2\lambda_3\lambda_4(1-e^{-(\mu_1+\lambda_1)t})}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)(\lambda_3+\mu_3)(\lambda_4+\mu_4)} (1-e^{-(\mu_2+\lambda_2)t}) \]
\[(1-e^{-(\mu_3+\lambda_3)t})(1-e^{-(\mu_4+\lambda_4)t})\]

\[f_3(q_0) = \left[1-(1-\frac{\lambda_4}{\mu_4+\lambda_4}(1-e^{-(\mu_4+\lambda_4)t}))\left(1-\frac{\lambda_2}{\lambda_2+\mu_2}(1-e^{-(\mu_2+\lambda_2)t})\right)\right]\]

\[\left[1-(1-\frac{\lambda_3}{\mu_3+\lambda_3}(1-e^{-(\mu_3+\lambda_3)t}))\left(1-\frac{\lambda_1}{\mu_1+\lambda_1}(1-e^{-(\mu_1+\lambda_1)t})\right)\right]\]

\[\therefore \text{Bridge Reliability } = 1 - Q_{OB} - Q_{SB}\]

2.2 **Three-State Device Markov Models** (i.e., normal, partial and catastrophic modes) *type II*

This section includes the reliability analysis of three-state device series, parallel, series-parallel and parallel-series networks. Similarly, as in section 2.1., the general reliability equations for this case are derived. General reliability formulas are presented; however, the details of the derivations are not included.

2.2.1. A **Three-State Device Markov Model**

In this case, a three-state device might be a wrist watch, a piece of equipment with several parts, etc. The transition diagram for such a system is shown in Figure 12.

![Figure 12. A Markov Model Transition Diagram](image)
The differential equations associated with Figure 12 are:

\[ \frac{dP_0(t)}{dt} + (\lambda_1 + \lambda_2)P_0(t) = 0 \quad (56) \]

\[ \frac{dP_1(t)}{dt} + \lambda_2 P_1(t) = P_0(t)\lambda_1 \quad (57) \]

\[ \frac{dP_2(t)}{dt} = P_1(t)\lambda_2 + P_0(t)\lambda_3 \quad (58) \]

subject to the initial conditions:

\[ P_0(0) = 1, \quad P_1(0) = 0, \quad P_2(0) = 0 \]

where

- \( P_0(t) \) is the probability of the successful mode
- \( P_1(t) \) is the probability of the partial failure mode
- \( P_2(t) \) is the probability of the catastrophic failure mode
- \( \lambda_1 \) is the partial failure rate
- \( \lambda_2 \) is the catastrophic failure rate from the partial mode
- \( \lambda_3 \) is the catastrophic failure rate

The solutions to the above differential equations are:

\[ P_0(t) = e^{-(\lambda_1 + \lambda_3)t} \quad (59) \]

\[ P_1(t) = 1 + \frac{(\lambda_2 - \lambda_3)}{(\lambda_1 + \lambda_3 - \lambda_2)} e^{-(\lambda_1 + \lambda_3)t} - \frac{\lambda_1}{\lambda_1 + \lambda_3 - \lambda_2} e^{-\lambda_2 t} \quad (60) \]

\[ P_2(t) = \frac{\lambda_1}{(\lambda_1 + \lambda_3 - \lambda_2)} (e^{-\lambda_2 t} - e^{-(\lambda_3 - \lambda_1)t}) \quad (61) \]

As can be seen from Figure 12, the state probabilities \( P_0, P_1, P_2 \) are mutually exclusive; therefore, the addition of equations (59),
(60) and (61) will yield unity.

It is interesting to note that the above equations correspond to those obtained in section 2.1.2. Hence, the system reliability is

\[ P_0(t) + P_1(t) = 1 + \frac{\lambda_1}{(\lambda_1 + \lambda_3 - \lambda_2)}(e^{-(\lambda_1 + \lambda_3)t} - e^{-\lambda_2 t}) \]  

(62)

2.2.2. Series Network

For the series network configuration, the failure of any one device or for the system failure in a catastrophic mode, total system will fail. Therefore

\[ \text{System Reliability} = \prod_{i=1}^{n} \left[ 1 + \frac{\lambda_i}{\lambda_1 + \lambda_3 + \lambda_2} \right] \left( e^{-(\lambda_1 + \lambda_3)t} - e^{-\lambda_2 t} \right) \]  

(63)

where \( n \) is the number of redundant components.

2.2.3. Parallel Network

A three-state device parallel network increases the system reliability. In the case of this particular network, the catastrophic system failure occurs only if all the redundant three-state devices fail catastrophically. The resulting system reliability equation is the following:

\[ R = 1 - \prod_{i=1}^{n} \left[ \frac{\lambda_1^i}{\lambda_1^i + \lambda_3^i - \lambda_2^i} \right] \left( e^{-\lambda_2^i t} - e^{-(\lambda_1^i + \lambda_3^i)t} \right) \]  

(64)
2.2.4. **Series-Parallel Configuration**

If any one of the parallel sub-systems fails in a catastrophic mode, it will cause total system failure. The resulting system reliability equation is the following:

\[
R = \prod_{i=1}^{m} \left[ 1 - \left( \frac{\lambda_1 t}{\lambda_1 + \lambda_3 - \lambda_2} \right) \left( e^{-\lambda_2 t} - e^{-\left(\lambda_1 + \lambda_3\right)t} \right)^n \right]. \tag{65}
\]

2.2.5. **Parallel-Series Configuration**

This is a dual configuration of the series-parallel network. This system will only experience complete failure if all the series paths of the system fail in the catastrophic modes.

The following equation was developed to yield the system reliability:

\[
\text{System Reliability} = 1 - \prod_{i=1}^{m} \left[ 1 - \left( 1 + \frac{\lambda_1}{\lambda_1 + \lambda_3 - \lambda_2} \right) \left( e^{-\left(\lambda_1 + \lambda_3\right)t} - e^{-\lambda_2 t} \right)^n \right]. \tag{66}
\]

2.3. **A Four-State System Markov Model**

The four-state system considered in this section describes a system with normal, partial, complete and catastrophic failure modes.

A mathematical Markov model was developed for the flow graph shown in Figure 12a.
Figure 12a. A Four-State System Markov Model

where
- $P_0$ is the probability of the system in the normal mode
- $P_1$ is the probability of the system in the partial mode
- $P_2$ is the probability of the system in the complete failure mode
- $P_3$ is the probability of the system in the catastrophic failure mode
- $\lambda_i$ is the constant failure rate in question ($i = 1, 2, 3$)
- $\mu_i$ is the constant failure rate in question ($i = 1, 2$)
- $t$ is the time
- $\Delta t$ is the time interval

The state probabilities of an essential four-state system are not just desirable but vital to predict the proper functioning of the equipment used in nuclear power stations and some other crucial industrial complexes. A typical example of a four-state device is an automatic machine used to carry out some operations on an assembly line. The four states of such a system are defined as follows:
(i) Normal state - It is defined as the functioning of the system at its full capacity.

(ii) Partial state - It is defined as the functioning of the system, say tolerable, at 70% of its full capacity, (due to some component failure).

(iii) Complete failure - It is defined as the total failure where a system does not operate at all or operates between 0-70% of its full capacity.

(iv) Catastrophic failure - It is defined as failure where a system or an equipment carries out some unacceptable operations.

The differential equations associated with Figure 12a are:

\[
\frac{dP_0(t)}{dt} = - (\lambda_1 + \lambda_2 + \mu_2) P_0(t) \quad (67)
\]

\[
\frac{dP_1(t)}{dt} = P_0(t) \mu_1 - P_1(t) (\mu_1 + \lambda_3) \quad (68)
\]

\[
\frac{dP_2(t)}{dt} = P_1(t) \mu_1 + P_0(t) \mu_2 \quad (69)
\]

\[
\frac{dP_3(t)}{dt} = P_1(t) \lambda_3 + P_0(t) \lambda_2 \quad (70)
\]

subject to the initial conditions:

\[
P_0(0) = 1, \quad P_1(0) = P_2(0) = P_3(0) = 0
\]

The Laplace transform of the differential equations (67) to (70) can be written as follows:

\[
(s + \lambda_1 + \lambda_2 + \mu_2) P_0(s) + 0 P_1(s) + 0 P_2(s) + 0 P_3(s) = 1
\]
\[-\lambda_1 \ P_0(s) + (s+\mu_1+\lambda_3) \ P_1(s) + 0 \ P_2(s) + 0 \ P_3(s) = 0 \]
\[-\mu_2 \ P_0(s) - \mu_1 \ P_1(s) + s \ P_2(s) + 0 \ P_3(s) = 0 \]
\[-\lambda_2 \ P_0(s) - \lambda_3 \ P_1(s) + 0 \ P_2(s) + s \ P_3(s) = 0 \]

The coefficients of the above equations are as follows:

\[
\begin{array}{cccc|c}
(s+\lambda_1+\lambda_2+\mu_2) & 0 & 0 & 0 & 1 \\
-\lambda_1 & (s+\mu_1+\lambda_3) & 0 & 0 & 0 \\
-\mu_2 & -\mu_1 & s & 0 & 0 \\
-\lambda_2 & -\lambda_3 & 0 & s & 0 \\
\end{array}
\]

The s-domain solution via Cramer's rule yields:

\[P_0(s) = \frac{1}{s+\lambda_1+\lambda_2+\mu_2}\]  \hfill (71)

\[P_1(s) = \frac{\lambda_1}{(s+\lambda_1+\lambda_2+\mu_2)(s+\mu_1+\lambda_3)}\]  \hfill (72)

\[P_2(s) = \frac{s\mu_2+\lambda_1\mu_1+\mu_1\mu_2+\mu_2\mu_3}{s(s+\lambda_1+\lambda_2+\mu_2)(s+\mu_1+\lambda_3)}\]  \hfill (73)

\[P_3(s) = \frac{s\lambda_2+\lambda_1\lambda_3+s\mu_1+\lambda_2\lambda_3}{s(s+\lambda_1+\lambda_2+\mu_2)(s+\mu_1+\lambda_3)}\]  \hfill (74)

The above equations in the time domain become:

\[P_0(t) = e^{-(\lambda_1+\lambda_2+\mu_2)t}\]  \hfill (75)
\[ p_1(t) = \left[ \frac{\lambda_1}{(\mu_1 + \lambda_3 - \lambda_1 - \lambda_2 - \mu_2)} \right] e^{-(\lambda_1 + \lambda_2 + \mu_2)t} \]

\[ - \left[ \frac{\lambda_1}{(\mu_1 + \lambda_3 - \lambda_1 - \lambda_2 - \mu_2)} \right] e^{-(\mu_1 + \lambda_3)t} \]

\[ p_2(t) = \frac{\mu_1 \lambda_1 + \mu_1 \mu_2 + \mu_2 \lambda_3}{(\lambda_2 + \lambda_2 + \mu_2)(\mu_1 + \lambda_3)} + \frac{\mu_1 \lambda_1}{(\mu_1 + \lambda_3)(\lambda_1 + \lambda_3 - \lambda_1 - \lambda_2 - \mu_2)} \]

\[ e^{-(\mu_1 + \lambda_3)t} - \left[ \frac{\mu_1 \lambda_1}{(\mu_1 + \lambda_3)(\lambda_1 + \lambda_3 - \lambda_1 - \lambda_2 - \mu_2)} \right] \]

\[ + \frac{\mu_1 \lambda_1 + \mu_1 \mu_2 + \mu_2 \lambda_3}{(\lambda_1 + \lambda_2 + \mu_2)(\mu_1 + \lambda_3)} \left[ e^{-(\lambda_1 + \lambda_2 + \mu_2)t} \right] \]

\[ p_3(t) = \frac{\lambda_1 \lambda_3 + \lambda_2 \mu_1 + \lambda_2 \lambda_3}{(\lambda_1 + \lambda_2 + \mu_2)(\mu_1 + \lambda_3)} + \frac{\lambda_1 \lambda_3}{(\mu_1 + \lambda_3)(\mu_1 + \lambda_3 - \lambda_1 - \lambda_2 - \mu_2)} e^{-(\mu_1 + \lambda_3)t} \]

\[ - \left[ \frac{\lambda_1 \lambda_3}{(\mu_1 + \lambda_3)(\mu_1 + \lambda_3 - \lambda_1 - \lambda_2 - \mu_2)} + \frac{\lambda_1 \lambda_3 + \lambda_2 \mu_1 + \lambda_2 \lambda_3}{(\lambda_1 + \lambda_2 + \mu_2)(\mu_1 + \lambda_3)} \right] e^{-(\lambda_1 + \lambda_2 + \mu_2)t} \]

2.4 Observations

This chapter illustrates some of the complexities experienced in the development of reliability and unreliability equation derivations as the level of redundancy is increased. The complexities of reliability analysis also increase as the number of system or device states increase.

The major advantage of the Markov model approach is that a system with non-constant failure rates can be modelled.

As this chapter title indicates, only non-reparable Markov models have been considered. The next chapter includes recapability Markov modelling.
CHAPTER 3

REPARABLE MARKOV MODELS

In general, whenever the average cost of minor repairs, including the cost of total downtime of a system, is small compared to its replacement cost, the only logical alternative is to repair the system. This, of course, means that the system, in selecting this alternative, can be corrected to its normal performance with a minimum risk of failure.

To illustrate this concept further, six reparable models are included in this chapter. The first two sections of the chapter include Markov models having three-state devices with normal, short and open modes. The Markov model of sections three and four presents models which contain three-state devices having normal, partial and catastrophic modes (25).

The final two sections of this chapter contain a Markov model made up of both types of three-state devices and one four-state Markov model. The reliability equations are derived for all the mathematical models.

The transition probabilities used to derive the system reliability expressions obey the following assumptions:

(i) the probability of transition in an infinitesimal time increment $\Delta t$ from one state to another is generated by failure rate $\lambda$ (and repair rate $\mu$) time finite time interval $(t, t+\Delta t)$;
(ii) the probabilities of occurrence of more than one transition in a higher order infinitesimal time interval $\Delta t$ are very small and can be ignored;

(iii) the failure and repair densities are exponential;

(iv) $\lambda$ and $\mu$ are constant.

3.1. A Three-State Device Reparable Markov Model*

A typical three-state equipment such as a fluid flow valve would have two different failure and repair rates. The Markov model is formulated as follows to obtain state probabilities and the equipment availability.**

The transition diagram is shown in Figure 13a.

![Transition Diagram](image)

Figure 13a. A Reparable Markov Model

*See-Appendix C

**The availability function $A(t)$ is defined as the probability that the system is operating at time $t$. 
where
P is the probability of the state in question
λ is the constant failure rate in question
µ is the constant repair rate in question
t is the time
Δt is the time interval

The probability derivations of the three-state device concerned are as follows:

From Figure 13a, the resulting differential equations are:

\[ \frac{dP(t)}{dt} = -(λ_1 + λ_2)P_0(t) + 2μ_1P_1(t) + 2μ_2P_2(t) \]  \hspace{1cm} (1)
\[ \frac{dP_1(t)}{dt} = P_0(t)λ_1 - μ_1P_1(t) \]  \hspace{1cm} (2)
\[ \frac{dP_2(t)}{dt} = λ_2P_0(t) - μ_2P_2(t) \]  \hspace{1cm} (3)

P_0(0) = 1, P_1(0) = P_2(0) = 0

The Laplace transform of equations (1) through (3) will yield:

\[ (s + λ_1 + λ_2)P_0(s) - μ_1P_1(s) - μ_2P_2(s) = 1 \]  \hspace{1cm} (4)
\[ -λ_1P_0(s) + (s + μ_1)P_1(s) + 0P_2(s) = 0 \]  \hspace{1cm} (5)
\[ -λ_2P_0(s) + 0P_1(s) + (s + μ_2)P_2(s) = 0 \]  \hspace{1cm} (6)

The coefficient of the above simultaneous equations can be written as follows:
The solution via Cramer's rule yields:

\[
P_0(s) = \frac{(s+\mu_1)(s+\mu_2)}{s[s^2+s(\mu_1+\mu_2+\lambda_1+\lambda_2)+(\mu_1\mu_2+\lambda_1\mu_2+\lambda_2\mu_1)]} \quad (7)
\]

\[
P_1(s) = \frac{\lambda_1(s+\mu_1)}{s[s^2+s(\mu_1+\mu_2+\lambda_1+\lambda_2)+(\mu_1\mu_2+\lambda_1\mu_2+\lambda_2\mu_1)]} \quad (8)
\]

\[
P_2(s) = \frac{\lambda_2(s+\mu_1)}{s[s^2+s(\mu_1+\mu_2+\lambda_1+\lambda_2)+(\mu_1\mu_2+\lambda_1\mu_2+\lambda_2\mu_1)]} \quad (9)
\]

The denominators of equations (7), (8) and (9) become:

\[
k_1, k_2 = \frac{-(\mu_1+\mu_2+\lambda_1+\lambda_2)\pm\sqrt{(\mu_1+\mu_2+\lambda_1+\lambda_2)^2-4(\mu_1\mu_2+\lambda_1\mu_2+\lambda_2\mu_2)}}{2}
\]

Now, equations (7) to (9) can be expended in a partial fraction form:

\[
P_0(s) = \frac{(s+\mu_1)(s+\mu_2)}{s(s-k_1)(s-k_2)}
= \frac{\mu_1\mu_2}{k_1k_2} \frac{1}{s} + \frac{(k_1+\mu_1)(k_1+\mu_2)}{k_1(k_1-k_2)} \frac{1}{s-k_1} - \frac{(k_2+\mu_1)(k_2+\mu_2)}{k_2(k_1-k_2)} \frac{1}{s-k_2} \quad (10)
\]

\[
P_1(s) = \frac{\lambda_1(s+\mu_2)}{s(s-k_1)(s-k_2)}
= \frac{\lambda_1\mu_2}{k_1k_2} + \frac{(\lambda_1k_1+\lambda_1\mu_2)}{k_1(k_1-k_2)} \frac{1}{s-k_1} - \frac{(\mu_2+k_2)\lambda_1}{k_2(k_1-k_2)} \frac{1}{s-k_2} \quad (11)
\]
\[ P_2(s) = \frac{\lambda_2(s+\mu_1)}{s(s-k_1)(s-k_2)} \]

\[ = \frac{\lambda_2\mu_1}{k_1k_2} \left( \frac{k_1k_2}{k_1(k_1-k_2)} \right) \frac{1}{s-k_1} - \frac{\mu_1k_2}{k_2(k_1-k_2)} \frac{1}{s-k_2} \]

(12)

In the time domain, equations (10) to (12) become:

\[ P_0(t) = \frac{\mu_1\mu_2}{k_1k_2} \left( \frac{k_1+k_2}{k_1(k_1-k_2)} \right) e^{k_1t} - \left( \frac{k_2+k_2}{k_2(k_1-k_2)} \right) e^{k_2t} \]

(13)

\[ P_1(t) = \frac{\lambda_1\mu_2}{k_1k_2} \left( \frac{k_1+k_2}{k_1(k_1-k_2)} \right) e^{k_1t} - \frac{\mu_1k_2}{k_2(k_1-k_2)} e^{k_2t} \]

(14)

\[ P_2(t) = \frac{\lambda_2\mu_1}{k_1k_2} \left( \frac{k_2+k_1}{k_1(k_1-k_2)} \right) e^{k_1t} - \frac{\mu_1k_2}{k_2(k_1-k_2)} e^{k_2t} \]

(15)

Since \[ k_1k_2 = \mu_1\mu_2 + \lambda_1\mu_2 + \lambda_2\mu_1 \]
\[ k_1+k_2 = -(\mu_1+\mu_2+\lambda_1+\lambda_2) \]

therefore, the addition of equations (13), (14) and (15) will yield unity, i.e.:

\[ P_0(t) + P_1(t) + P_2(t) = 1 \]

The availability of the equipment from equation (13) is:

\[ \text{Availability} = P_0(t) = \frac{\mu_1\mu_2}{k_1k_2} \left( \frac{k_1+k_2}{k_1(k_1-k_2)} \right) e^{k_1t} - \left( \frac{k_2+k_2}{k_2(k_1-k_2)} \right) e^{k_2t} \]

The availability expression is valid if and only if \( k_1 \) and \( k_2 \) are negative. As \( t \) becomes very large, the following steady-state
availability equation can be expressed as:

\[
\lim_{t \to \infty} P_0(t) = \frac{\mu_1 \mu_2}{k_1 k_2}
\]

3.2. A Reparable Markov Model of Two-Element in Series (open and closed failure modes)

Consider two identical devices with the same open and closed mode failure rates. The repair is performed only when one of the equipments fail in its closed mode, assuming the other one is still operating. Due to the complexity of the analysis of this situation, a preliminary assumption has been made that the repair rate is the same as the closed mode failure rate. The reparable transition diagram is shown in Figure 13b.

\[\text{Figure 13b. A Series Two-Element Reparable Markov Model}\]

The differential equations associated with Figure 13b are:
\[ \frac{dP_{N_1N_2}(t)}{dt} + 4\lambda P_{N_1N_2}(t) = P_{S_1N_2}(t) + \lambda P_{N_1S_2}(t) \lambda \tag{16} \]
\[ \frac{dP_{S_1N_2}(t)}{dt} + 3\lambda P_{N_1N_2}(t) = P_{N_1N_2}(t) \lambda \tag{17} \]
\[ \frac{dP_{N_1S_2}(t)}{dt} + 3\lambda P_{N_1N_2}(t) = P_{N_1N_2}(t) \lambda \tag{18} \]

\[ P_{N_1N_2}(0) = 1, \quad P_{S_1N_2}(0) = 0, \quad P_{N_1S_2}(0) = 0 \]

where

\( P(\cdot,\cdot) \) is the probability of the state in question
\( \lambda \) is the repair rate and also the open and closed mode failure rate
\( N_i \) is the normal state, \( i = 1,2 \)
\( S_i \) is the short mode state, \( i = 1,2 \)

The above differential equations can be written in Laplace domain as follows:

\[ P_{N_1N_2}(s) (s+4\lambda) - \lambda P_{S_1N_2}(s) - \lambda P_{N_1S_2}(s) = 1 \]
\[ -P_{N_1N_2}(s) \lambda + (s+3\lambda) P_{S_1N_2}(s) + 0 P_{N_1S_2}(s) = 0 \]
\[ -P_{N_1N_2}(s) \lambda + 0 P_{S_1N_2}(s) - (s+3\lambda) P_{N_1S_2}(s) = 0 \]

The coefficients of the above simultaneous equations are:
\[
\begin{array}{ccc}
(s+4\lambda) & -\lambda & -\lambda \\
-\lambda & (s+3\lambda) & 0 \\
-\lambda & 0 & (s+3\lambda)
\end{array}
\]

The solutions via Cramer's rule and partial fraction yield:

\[P_{N_1N_2}(s) = \frac{1}{3} \frac{1}{s+2\lambda} + \frac{2}{3} \frac{1}{s+5\lambda}\]

(19)

\[P_{S_1N_2}(s) = \frac{1}{3} \frac{1}{s+2\lambda} - \frac{1}{3} \frac{1}{s+5\lambda}\]

(20)

\[P_{N_1S_2}(s) = \frac{1}{3} \frac{1}{s+2\lambda} - \frac{1}{3} \frac{1}{s+5\lambda}\]

(21)

In time domain, the above equations are:

\[P_{N_1N_2}(t) = \frac{1}{3} e^{-2\lambda t} + \frac{2}{3} e^{-5\lambda t}\]

(22)

\[P_{S_1N_2}(t) = \frac{1}{3} e^{-2\lambda t} - \frac{1}{3} e^{-5\lambda t}\]

(23)

\[P_{N_1S_2}(t) = \frac{1}{3} e^{-2\lambda t} - \frac{1}{3} e^{-5\lambda t}\]

(24)

Since the above state probabilities are mutually exclusive, the system reliability is:

\[R = P_{N_1N_2}(t) + P_{S_1N_2}(t) + P_{N_1S_2}(t) = e^{-2\lambda t}\]

(25)

For an identical parallel network, the above equation will also yield the same system reliability.
3.3. A Three-State Device Markov Model type II

Three-state system or equipment availability knowledge is not just necessary but essential for the equipment or systems to be used in the nuclear power stations or other vital industrial complexes.

For example, suppose a machine has three modes (i.e., normal, partial and catastrophic) where

- the normal mode is defined as the functioning of the machine at its full capacity;
- the partial mode is defined as the functioning of the machine, say tolerable, at .70% of its capacity due to some component failure;
- the catastrophic failure is defined as the complete failure of the machine (i.e., the machine's operational capacity between 0-70%).

In our mathematical model as shown in Figure 1, the repair is carried out on the machine which is still operational at its partial mode to restore the machine to its full capacity. However, while the repair is underway, the machine operating in its partial mode may have a catastrophic failure. At this stage, the repair could be undertaken in two ways simultaneously, provided the replacement for a particular failed component is not available. Both reparable methods are as follows:

- (i) restore the machine to its partial mode capacity with some temporary repairs;
(ii) begin manufacturing or ordering such components
to restore the machine back to its full capacity.

The transition diagram is shown in Figure 14.

![Transition Diagram](image)

**Figure 14. A Reparable Markov Model**

where

- $P_0$ is the probability of the system in the normal mode
- $P_1$ is the probability of the system in the partial mode
- $P_2$ is the probability of the system in the catastrophic mode
- $\lambda$ is the constant failure rate in question
- $\mu$ is the constant repair rate in question
- $t$ is the time

$\Delta t$ is the time interval

Figure 14 yields the following differential equations:

$$\frac{dP_0(t)}{dt} + (\lambda_1+\lambda_2)P_0(t) = \mu_1P_1(t) + \mu_2P_2(t)$$  \hspace{1cm} (26)

$$\frac{dP_1(t)}{dt} + (\mu_1+\lambda_3)P_1(t) = \mu_3P_2(t) + P_0(t)\lambda_1$$  \hspace{1cm} (27)

$$\frac{dP_2(t)}{dt} + (\mu_1+\mu_3)P_2(t) = \lambda_3P_1(t) + \lambda_2P_0(t)$$  \hspace{1cm} (28)
\[ P_0(0) = 1, P_1(0) = 0, P_2(0) = 0 \]

The Laplace transform of equations (26), (27) and (28) can be written as follows:

\[ (s+\lambda_1+\lambda_2) P_0(s) - \mu_1 P_1(s) - \mu_2 P_2(s) = 1 \quad (29) \]
\[ -\lambda_1 P_0(s) + (s+\mu_1+\lambda_3) P_1(s) - \mu_3 P_2(s) = 0 \quad (30) \]
\[ -\lambda_2 P_0(s) - \lambda_3 P_1(s) + (s+\mu_2+\mu_3) P_2(s) = 0 \quad (31) \]

The coefficients of the above equations are as follows:

\[
\begin{array}{cccc}
(s+\lambda_1+\lambda_2) & -\mu_1 & -\mu_2 & 1 \\
-\lambda_1 & (s+\mu_1+\lambda_3) & -\mu_3 & 0 \\
-\lambda_2 & -\lambda_3 & (s+\mu_2+\mu_3) & 0
\end{array}
\]

The solution via Cramer's rule yields:

\[
P_0(s) = \frac{s^2+s(\mu_1+\mu_2+\mu_3+\lambda_3)+\mu_1\mu_2+\lambda_2\mu_2+\mu_1\mu_3}{s^3+sx^2(\mu_1+\mu_2+\mu_3+\lambda_1+\lambda_2+\lambda_3)+sx(\mu_1\mu_2)}
\]

\[
P_1(s) = \frac{s\lambda_2+\lambda_1\mu_2+\lambda_1\mu_3+\mu_3}{s^3+sx^2(\mu_1+\mu_2+\mu_3+\lambda_1+\lambda_2+\lambda_3)+sx(\mu_1\mu_2)}
\]

\[
P_2(s) = \frac{\lambda_2\mu_2+\mu_1\mu_3+\lambda_3+\mu_2\lambda_3+\mu_1\lambda_3+\mu_1\lambda_2+\lambda_2\mu_3+\lambda_2\lambda_3}{s^3+sx^2(\mu_1+\mu_2+\mu_3+\lambda_1+\lambda_2+\lambda_3)+sx(\mu_1\mu_2)}
\]
\[ p_2(s) = \frac{s\lambda_2 + \lambda_1^2 + \mu_1^2 + \lambda_2\lambda_3 + \mu_2^2 + \mu_1\mu_3}{s^2 + s^2(\mu_1 + \mu_2 + \mu_3 + \lambda_1 + \lambda_2 + \lambda_3) + s(\mu_1^2 + \mu_2^2 + \mu_1\mu_3)} \]

The roots of the denominators of equations (32), (33) and (34) become:

\[ c_1, c_2 = \frac{-(\mu_1 + \mu_2 + \mu_3 + \lambda_1 + \lambda_2 + \lambda_3) \pm \sqrt{(\mu_1 + \mu_2 + \mu_3 + \lambda_1 + \lambda_2 + \lambda_3)^2 - 4(\mu_1^2 + \mu_2^2 + \mu_1\mu_3 + \lambda_1 + \lambda_2 + \lambda_3)}}{2} \]

Now, equations (32) through (35) in time domain become:

\[ P_0(t) = \frac{\mu_1^2 + \mu_2^2 + \mu_1\mu_3}{c_1c_2} \]

\[ + \left[ \frac{\mu_1 + \mu_2 + \mu_3 + \lambda_1 + \lambda_2 + \lambda_3 + \mu_1^2 + \mu_2^2 + \mu_1\mu_3}{c_1(c_1 - c_2)} \right] e^{c_1 t} \]

\[ + \left[ 1 - \frac{\mu_1 + \mu_2 + \mu_3 + \lambda_1 + \lambda_2 + \lambda_3}{c_1c_2} \right] e^{c_2 t} \]

\[ P_1(t) = \frac{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_2\lambda_3 + \mu_1\mu_3}{c_1c_2} \]

\[ + \left[ \frac{c_1 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_2\lambda_3}{c_1(c_1 - c_2)} \right] e^{c_1 t} \]

\[ - \frac{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_2\lambda_3}{c_1c_2} + \frac{c_1 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_2\lambda_3}{c_1(c_1 - c_2)} \]
\[ P_2(t) = \left[ \frac{\lambda_1 \lambda_3 + \mu_1 \lambda_2 + \lambda_2 \lambda_3}{c_1 c_2} \right] e^{c_1 t} \left[ \frac{c_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \mu_1 + \lambda_2 \lambda_3}{c_1 (c_1 - c_2)} \right] e^{c_2 t} \]

\[ - \left[ \frac{\lambda_1 \lambda_3 + \mu_1 \lambda_2 + \lambda_2 \lambda_3}{c_1 c_2} + \frac{\lambda_2 c_1 + \lambda_1 \lambda_3 + \mu_1 \lambda_2 + \lambda_2 \lambda_3}{c_1 (c_1 - c_2)} \right] e^{c_2 t} \]

\[ \text{System Availability} = P_0(t) + P_1(t) \]

\[ = \left[ \frac{\mu_1 \nu_2 + \lambda_3 \nu_2 + \mu_1 \nu_3 + \lambda_1 \nu_2 + \lambda_1 \nu_3 + \lambda_2 \nu_3}{c_1 c_2} \right] e^{c_1 t} \]

\[ + \left[ \frac{\mu_1 c_1 + \nu_2 c_1 + \nu_3 c_1 + c_1 \lambda_3 + c_2 + \mu_1 \nu_2 + \lambda_3 \nu_2 + \mu_1 \nu_3 + c_1 \lambda_1 + \lambda_1 \nu_2 + \lambda_1 \nu_3 + \lambda_2 \nu_3}{c_1 (c_1 - c_2)} \right] e^{c_1 t} \]

\[ + \left[ 1 - \left( \frac{\mu_1 \nu_2 + \lambda_3 \nu_2 + \mu_1 \nu_3 + \lambda_1 \nu_2 + \lambda_1 \nu_3 + \lambda_2 \nu_3}{c_1 c_2} \right) \right] e^{c_2 t} \]

\[ - \left[ \frac{\mu_1 c_1 + \nu_2 c_1 + \nu_3 c_1 + c_1 \lambda_3 + c_2 + \mu_1 \nu_2 + \lambda_3 \nu_2 + \mu_1 \nu_3 + c_1 \lambda_1 + \lambda_1 \nu_2 + \lambda_1 \nu_3 + \lambda_2 \nu_3}{c_1 (c_1 - c_2)} \right] e^{c_2 t} \]

The availability expression is valid if and only if \( c_1 \) and \( c_2 \) are negative. As \( t \) becomes very large, the following steady-state availability equation can be expressed as:

\[ \lim_{t \to \infty} (P_0(t) + P_1(t)) = \left[ \frac{\mu_1 \nu_2 + \lambda_3 \nu_2 + \mu_1 \nu_3 + \lambda_1 \nu_2 + \lambda_1 \nu_3 + \lambda_2 \nu_3}{c_1 c_2} \right] \]

3.4. A Reparable Markov Model of Two-Element in Series (partial and catastrophic failure modes)

The devices of this system are identical with the same partial, catastrophic and partial to catastrophic failure rates. The repair rate is also assumed to be the same as for the partial failure rate.
A device is repaired only when it fails in a partial failure mode and the other devices are operating successfully or both devices are functioning in their partial mode. The transition diagram for this particular configuration is shown in Figure 15.

![Diagram](image)

**Figure 15. A Series Two-Element Reparable Markov Model**

where

- $P_1$ is the partial failure state, $i = 1, 2$
- $N_i$ is the normal state, $i = 1, 2$
- $C_i$ is the catastrophic failure state, $i = 1, 2$

The differential equations associated with Figure 15 are:

$$
\frac{dP_{N_1N_2}(t)}{dt} + 4\lambda P_{N_1N_2}(t) = \lambda P_{P_1N_2}(t) + \lambda P_{P_1P_2}(t) + \lambda P_{N_1N_2}(t) \quad (39)
$$

$$
\frac{dP_{P_1N_2}(t)}{dt} + P_{P_1N_2}(t)4\lambda = P_{N_1N_2}(t)\lambda \quad (40)
$$
\[
\frac{dP_{N_1N_2}(t)}{dt} + P_{N_1N_2}(t)4\lambda = P_{N_1N_2}(t)\lambda \\
\frac{dP_{P_1P_2}(t)}{dt} + 3\lambda P_{P_1P_2}(t) = P_{P_1P_2}(t)\lambda + P_{N_1N_2}(t)\lambda
\]

\[P_{N_1N_2}(0) = 1, \; P_{P_1P_2}(0) = 0, \; P_{N_1P_2}(0) = 0, \; P_{P_1P_2}(0) = 0\]

The above differential equations transformed to the s-domain become:

\[
(s+4\lambda) P_{N_1N_2}(s) - \lambda P_{P_1N_2}(s) - \lambda P_{N_1P_2}(s) - \lambda P_{P_1P_2}(s) = 1
\]

\[-\lambda P_{N_1N_2}(s) + (s+4\lambda) P_{P_1N_2}(s) + 0 P_{N_1P_2}(s) + 0 P_{P_1P_2}(s) = 0
\]

\[-\lambda P_{N_1N_2}(s) + 0 P_{P_1N_2}(s) + (s+4\lambda) P_{N_1P_2}(s) + 0 P_{P_1P_2}(s) = 0
\]

\[0 P_{N_1N_2}(s) - \lambda P_{P_1N_2}(s) - \lambda P_{N_1P_2}(s) + (s+3\lambda) P_{P_1P_2}(s) = 0
\]

The coefficients of the above simultaneous equations are:

\[
\begin{array}{cccc|c}
  s+4\lambda & -\lambda & -\lambda & -\lambda & 1 \\
  -\lambda & s+4\lambda & 0 & 0 & 0 \\
  -\lambda & 0 & s+4\lambda & 0 & 0 \\
  0 & -\lambda & -\lambda & s+3\lambda & 0 \\
\end{array}
\]

From the above matrix via Cramer's rule:

\[
P_{N_1N_2}(s) = 2/3 \cdot \frac{1}{(s+5\lambda)} + 1/3 \cdot \frac{1}{(s+2\lambda)}
\]

(43)
\[ p_{1N_2}(s) = \frac{1}{2} \left( \frac{1}{s+2\lambda} + \frac{1}{s+5\lambda} \right) \]

\[ p_{N_1P_2}(s) = \frac{1}{2} \left( \frac{1}{s+2\lambda} + \frac{1}{s+5\lambda} \right) \]

\[ p_{1P_2}(s) = \frac{2}{3} \left( \frac{1}{s+5\lambda} - \frac{1}{s+4\lambda} \right) + \frac{1}{3} \frac{1}{s+2\lambda} \]

In time domain, equations (43) to (46) become:

\[ p_{N_1N_2}(t) = 2/3 e^{-5\lambda t} + 1/3 e^{-2\lambda t} \]

\[ p_{1N_2}(t) = 1/2 e^{-4\lambda t} - 1/2 e^{-5\lambda t} + 1/6 e^{-2\lambda t} \]

\[ p_{N_1P_2}(t) = 1/2 e^{-4\lambda t} - 1/2 e^{-5\lambda t} + 1/6 e^{-2\lambda t} \]

\[ p_{1P_2}(t) = 2/3 e^{-5\lambda t} - e^{-4\lambda t} - 1/3 e^{-2\lambda t} \]

\[ \therefore \text{System reliability} = p_{N_1N_2}(t) + p_{1N_2}(t) + p_{N_1P_2}(t) + p_{1P_2}(t) = 1/3 e^{-5\lambda t} + e^{-2\lambda t} \]

since a partial failure is considered as a reliable state.

3.5. A Mixed Markov Model with Two Three-State Devices (Master-slave relationship)

This particular case suggests that the two elements are modelled mathematically in series. One element has normal, partial and catastrophic states and the other has normal, open and short states. For this situation, repairs are performed only when an equipment fails in its partial mode.
A typical practical example of such a system is a fluid flow valve commanded from an instrumentation control panel where the control panel represents the first type of device and the fluid flow valve represents the second type. Such practical examples are numerous and may often be encountered at a modern electrical power station. The transition diagram for this case is shown in Figure 16.

Figure 16. A Master-Slave Markov Model

The reliable state differential equations which result from Figure 16 are:

\[
\frac{dP_{N_1N_2}(t)}{dt} + (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)P_{N_1N_2}(t) = \mu_1P_{N_1N_2}(t) \tag{52}
\]

\[
\frac{dP_{P_1N_2}(t)}{dt} + (\lambda_3 + \lambda_4 + \lambda_5 + \mu_1)P_{P_1N_2}(t) = \lambda_1P_{N_1N_2}(t) \tag{53}
\]
\[ P_{N_1 N_2}(0) = 1, P_{P_1 N_2}(0) = 0 \]

In Laplace domain, the above differential equations become:

\[ P_{P_1 N_2}(s) (s + \lambda_3 + \lambda_4 + \lambda_5 + \mu_1) - \lambda_1 P_{N_1 N_2}(s) = 0 \]  \hspace{1cm} (54)

\[ -P_{P_1 N_2}(s) \mu_1 + (s + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) P_{N_1 N_2}(s) = 1 \]  \hspace{1cm} (55)

The coefficients of the above simultaneous equations are:

\[
\begin{vmatrix}
-\mu_1 & (s + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) & 1 \\
(s + \lambda_3 + \lambda_4 + \lambda_5 + \mu_1) & -\lambda_1 & 0
\end{vmatrix}
\]

The solution for the roots of the denominator quadratic (i.e., \( \Delta \)) yields:

\[ k_1, k_2 = \frac{-N \pm \sqrt{N^2 - 4AM}}{2A} \]

where

\[ N = \lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + \mu_1 \]

\[ M = \lambda_1 \lambda_3 + \lambda_2 \lambda_3^2 + \lambda_3 \lambda_4 + \lambda_1 \lambda_4 \lambda_2 + \lambda_4 \lambda_2 \lambda_3 + \lambda_4 \lambda_5 + \lambda_2 \lambda_5 + \lambda_3 \lambda_5 + \lambda_4 \lambda_5 + \lambda_2 \mu_1 + \lambda_3 \mu_1 + \lambda_4 \mu_1 \]

\[ A = 1 \]

State probabilities are:

\[ P_{P_1 N_2}(t) = \frac{\lambda_1}{(k_2 - k_1)} e^{k_2 t} - \frac{\lambda_1}{(k_2 - k_1)} e^{k_1 t} \]  \hspace{1cm} (56)
\[ P_{N_1N_2}(t) = \left(1 - \frac{(\lambda_3 + \lambda_4 + \lambda_5 + u_1 + k_2)}{(k_2 - k_1)}\right) e^{k_1t} + \frac{\lambda_3 + \lambda_4 + \lambda_5 + u_1 + k_2}{(k_2 - k_1)} e^{k_2t} \] (57)

System reliability = \[ P_{P_{1N_2}}(t) + P_{N_1N_2}(t) \]

3.6. The Analysis of a Four-State System

The four-state system considered in this section describes a system with normal, partial, complete and catastrophic failure modes. A mathematical Markov model was developed to obtain the partial availability function of such a system.

In nuclear power stations, for instance, the state probabilities and partial availability of certain four-state systems are not only just desirable but vital to predict the functioning of essential equipment used. The same holds for many other crucial systems in various industrial complexes. The four states of such a system may be defined as follows:

(i) Normal state - the successful functioning of the system at its full capacity.

(ii) Partial state - the successful functioning of the system, say tolerable, at 70% of its full capacity (due to some component failure).

(iii) Complete failure - the total failure of the system where a system does not successfully operate at all or operates somewhere between 0-70% of its full capacity.
(iv) Catastrophic failure - the failure of the system where the system or an equipment performs at some unacceptable operating level.

The Markov flow graph model for this case is shown in Figure 16a which simply forms a degraded three-state system. For example, suppose an automatic machine carries out some operations on an item along a conveyor or an assembly line. In its normal state the machine can experience a partial, a complete or catastrophic failure. If it experiences a partial mode failure, it can then be repaired while the machine is operating at some reduced capacity. However, while the repair is underway, the machine operating in its partial mode may have a complete or a catastrophic (i.e., by carrying out unacceptable operations or less than precise operations on the assembly line item) failure.

Figure 16a. A Four-State Reparable Markov Model
where

- $P_0$ is the probability of the system in the normal mode
- $P_1$ is the probability of the system in the partial mode
- $P_2$ is the probability of the system in the complete failure mode
- $P_3$ is the probability of the system in the catastrophic failure mode
- $\lambda$ is the constant failure rate in question
- $\mu$ is the constant repair rate
- $t$ is the time
- $\Delta t$ is the time interval

The differential equations associated with Figure 16a are:

\[
\frac{dP_0(t)}{dt} = -P_0(t)(\lambda_1 + \lambda_2 + \lambda_5) + P_1(t)\mu \tag{58}
\]

\[
\frac{dP_1(t)}{dt} = -P_1(t)(\lambda_3 + \lambda_4 + \mu) + \lambda_1 P_0(t) \tag{59}
\]

\[
\frac{dP_2(t)}{dt} = P_1(t)\lambda_4 + P_0(t)\lambda_5 \tag{60}
\]

\[
\frac{dP_3(t)}{dt} = P_1(t)\lambda_3 + P_0(t)\lambda_2 \tag{61}
\]

$P_0(0) = 1, P_1(0) = P_2(0) = P_3(0) = 0$

The Laplace transform of the differential equations (58), (59), (60) and (61) can be written as follows:
\[ (s+\lambda_1+\lambda_2+\lambda_5)\; P_0(s) - \mu \quad P_1(s) + OP_2(s) + OP_3(s) = 1 \quad (62) \]
\[-\lambda_1 \quad P_0(s) + (s+\lambda_3+\lambda_4+\mu)\; P_1(s) + OP_2(s) + OP_3(s) = 0 \quad (63) \]
\[-\lambda_5 \quad P_0(s) - \lambda_4 \quad P_1(s) + sP_2(s) + OP_3(s) = 0 \quad (64) \]
\[-\lambda_2 \quad P_0(s) - \lambda_3 \quad P_1(s) + OP_2(s) + sP_3(s) = 0 \quad (65) \]

The coefficients of the above equations are as follows:

\[
\begin{array}{cccc}
(s+\lambda_1+\lambda_2+\lambda_5) & -\mu & 0 & 0 \\
-\lambda_1 & (s+\lambda_3+\lambda_4+\mu) & 0 & 0 \\
-\lambda_5 & -\lambda_4 & s & 0 \\
-\lambda_2 & -\lambda_3 & 0 & s \\
\end{array}
\]

The solution via Cramer's rule yields:

\[
P_0(s) = \frac{(s+\lambda_3+\lambda_4+\mu)}{s^2+s(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\mu)+\lambda_3\lambda_1+\lambda_4\lambda_1+2\lambda_2\lambda_3+\lambda_2\lambda_4+2\lambda_2\mu+\lambda_3\lambda_5+\lambda_5\lambda_4+\lambda_5\mu} \quad (66)
\]

\[
P_1(s) = \frac{\lambda_1}{s^2+s(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\mu)+\lambda_3\lambda_1+\lambda_4\lambda_1+2\lambda_2\lambda_3+\lambda_2\lambda_4+2\lambda_2\mu+\lambda_3\lambda_5+\lambda_5\lambda_4+\lambda_5\mu} \quad (67)
\]

\[
P_2(s) = \frac{s\lambda_5+\lambda_1\lambda_4+\lambda_3\lambda_5+\lambda_5\lambda_4+\lambda_5\mu}{s[s^2+s(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\mu)+\lambda_3\lambda_1+\lambda_4\lambda_1+2\lambda_2\lambda_3+\lambda_2\lambda_4+2\lambda_2\mu+\lambda_3\lambda_5+\lambda_5\lambda_4+\lambda_5\mu]} \quad (68)
\]

\[
P_3(s) = \frac{s\lambda_2+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_2\mu+\lambda_1\lambda_3}{s[s^2+s(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\mu)+\lambda_3\lambda_1+\lambda_4\lambda_1+2\lambda_2\lambda_3+\lambda_2\lambda_4+2\lambda_2\mu+\lambda_3\lambda_5+\lambda_5\lambda_4+\lambda_5\mu]} \quad (69)
\]
The roots of the denominator quadratic of equations (66), (67), (68) and (69) are:

\[
k_1, k_2 = \frac{-(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu) \pm \sqrt{(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu)^2 - 4c}}{2}
\]

(70)

where \(c = \lambda_3 \lambda_2 + \lambda_3 \lambda_4 + \lambda_4 \lambda_5 + \lambda_5 \lambda_6 + \lambda_6 \lambda_1 + \lambda_2 \lambda_4 + \lambda_3 \lambda_5 + \lambda_5 \lambda_6 + \lambda_6 \lambda_1\)

Now, equations (66) through (69) in time domain become:

\[
P_0(t) = \left[1 - \frac{(\lambda_3 + \lambda_4 + \mu + k_2)}{(k_2 - k_1)}\right] e^{k_1 t} + \left[\frac{\lambda_3 + \lambda_4 + \mu + k_2}{(k_2 - k_1)}\right] e^{k_2 t}
\]

(71)

\[
P_1(t) = \left[-\frac{\lambda_1}{(k_2 - k_1)}\right] e^{k_1 t} + \left[\frac{\lambda_1}{(k_2 - k_1)}\right] e^{k_2 t}
\]

(72)

\[
P_2(t) = \left[\frac{\lambda_1 \lambda_4 + \lambda_3 \lambda_5 + \lambda_4 \lambda_5 + \lambda_5 \mu}{k_1 k_2}\right] e^{k_1 t} + \left[\frac{\lambda_1 \lambda_4 + \lambda_3 \lambda_5 + \lambda_4 \lambda_5 + \lambda_5 \mu}{k_1 (k_2 - k_1)}\right] e^{k_2 t}
\]

\[
- \frac{(k_1 + k_2)(\lambda_1 \lambda_4 + \lambda_3 \lambda_5 + \lambda_4 \lambda_5 + \lambda_5 \mu)}{k_1 k_2 (k_2 - k_1)} - \frac{\lambda_1 \lambda_4 + \lambda_3 \lambda_5 + \lambda_4 \lambda_5 + \lambda_5 \mu}{k_1 k_2}
\]

\[
+ \left[\frac{\lambda_5}{(k_2 - k_1)} + \frac{k_1 + k_2}{k_1 k_2 (k_2 - k_1)}\right] e^{k_2 t}
\]

(73)

\[
P_3(t) = \left[\frac{\lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \mu \lambda_3 + \lambda_1 \lambda_3}{k_1 k_2}\right] + \left[\frac{\lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \mu \lambda_2 + \lambda_1 \lambda_3}{k_1 (k_2 - k_1)}\right]
\]

\[
(k_1 + k_2)(\lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_1 \lambda_3 + \mu \lambda_2) - \frac{(\lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_1 \lambda_3 + \mu \lambda_2)}{k_1 k_2}
\]

\[
+ \left[\frac{\lambda_2}{(k_2 - k_1)} + \frac{k_1 + k_2}{k_1 k_2 (k_2 - k_1)}\right] e^{k_2 t}
\]

(74)
System partial availability = \( P_0(t) + P_1(t) \)

\[ = \left[ 1 - \left( \frac{\lambda_3 + \lambda_4 + \lambda_1 + \mu + k_2}{k_2 - k_1} \right) \right] e^{k_1 t} + \left[ \frac{\lambda_1 + \lambda_3 + \lambda_4 + \mu + k_2}{k_2 - k_1} \right] e^{k_2 t} \]  

(75)

The full availability function of this system is derived in the next section.

3.7. Availability Function of a Four-State System

The probability transition diagram for the case of a four-state system is shown by Figure 16b.

![Figure 16b. A Four-State System Markov Model](image)

The differential equations resulting from the system described by Figure 16b are:

\[ \frac{dP_0(t)}{dt} = -P_0(t)(\lambda_1 + \lambda_2 + \lambda_5) + \nu_1 P_1(t) + P_2 P_3(t) + \nu_3 P_2(t) \]  

(76)

\[ \frac{dP_1(t)}{dt} = -P_1(t)(\lambda_3 + \lambda_4 + \nu_1) + \lambda_1 P_0(t) \]  

(77)
\[ \frac{dP_2(t)}{dt} = -P_2(t)\mu_3 + P_1(t)\lambda_4 + P_0(t)\lambda_5 \] (78)

\[ \frac{dP_3(t)}{dt} = -P_3(t)\mu_2 + P_1(t)\lambda_3 + P_0(t)\lambda_2 \] (79)

\[ P_0(0) = 1, P_1(0) = P_2(0) = P_3(0) = 0 \]

The Laplace transform of the above equations becomes:

\[(s+\lambda_1+\lambda_2+\lambda_5) \, P_0(s) - \mu_1 \, P_1(s) - \mu_3 \, P_2(s) - \mu_2 \, P_3(s) = 1 \]

\[-\lambda_1 \, P_0(s) + (s+\lambda_3+\lambda_4+\mu_1) \, P_1(s) + 0 \, P_2(s) + 0 \, P_3(s) = 0 \]

\[-\lambda_5 \, P_0(s) - \lambda_4 \, P_1(s) + (s+\mu_3) \, P_2(s) + 0 \, P_3(s) = 0 \]

\[-\lambda_2 \, P_0(s) - \lambda_3 \, P_1(s) + 0 \, P_2(s) + (s+\mu_2) \, P_3(s) = 0 \]

The coefficients of the above equations are:

\[
\begin{pmatrix}
(s+\lambda_1+\lambda_2+\lambda_5) & -\mu_1 & -\mu_3 & -\mu_2 & 1 \\
-\lambda_1 & (s+\lambda_3+\lambda_4+\mu_1) & 0 & 0 & 0 \\
-\lambda_5 & -\lambda_4 & (s+\mu_3) & 0 & 0 \\
-\lambda_2 & -\lambda_3 & 0 & (s+\mu_2) & 0
\end{pmatrix}
\]

The determinant of the above matrix is:
\[ \Delta = s^6 + s^5 (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \mu_2 + \mu_3) + s^2 [(\lambda_1 + \lambda_2 + \lambda_3) (\lambda_3 + \lambda_4 + \mu_1) \\
+ (\mu_3 + \mu_2) (\lambda_3 + \mu_4 + \mu_1) + (\lambda_1 + \lambda_2 + \lambda_3) (\mu_2 + \mu_3)] - \mu_1 \lambda_1 - \mu_3 \lambda_5 \\
- \mu_2 \lambda_2 + s [\mu_3 \lambda_3 + \lambda_1 \mu_4 + \lambda_3 \mu_3 + \mu_3 \lambda_4 \mu_1 + \mu_2 \lambda_4 \mu_1 + \mu_2 \lambda_1 \mu_1 \\
+ \lambda_3 \mu_2 + \lambda_4 \mu_2 + \lambda_5 \mu_2 + \mu_2 \mu_3 + \lambda_3 \mu_3 + \lambda_4 \mu_2 + \lambda_1 \mu_2 + \lambda_1 \mu_3 + \mu_1 \lambda_2 + \mu_3 \lambda_1 \lambda_4] \\
+ 2 \mu_3 \lambda_1 \lambda_4 \mu_2 \]

The following are the factors of the determinant biquadratic equation (27):

\[ i.e., \quad (s-k_1) (s-k_2) (s-k_3) (s-k_4) \]

The solutions obtained by Cramer's rule are the following:

\[ P_0(s) = \frac{(s + \lambda_3 + \lambda_4 + \mu_1)(s + \mu_2)}{(s-k_1)(s-k_2)(s-k_3)(s-k_4)} \quad (80) \]

\[ P_1(s) = \frac{\lambda_1 (s + \mu_3)(s + \mu_2)}{(s-k_1)(s-k_2)(s-k_3)(s-k_4)} \quad (81) \]

\[ P_2(s) = \frac{\lambda_1 \lambda_4 (s + \mu_2) + (s + \lambda_3 + \lambda_4 + \mu_1)(s + \mu_2) \lambda_5}{(s-k_1)(s-k_2)(s-k_3)(s-k_4)} \quad (82) \]

\[ P_3(s) = \frac{\lambda_3 \lambda_3 (s + \mu_3) + \lambda_2 (s + \mu_3)(s + \lambda_3 + \lambda_4 + \mu_1)}{(s-k_1)(s-k_2)(s-k_3)(s-k_4)} \quad (83) \]

By expanding the above equations in partial fractions, and by taking the inverse Laplace transform yields:

\[ P_0(t) = A_1 e^{k_1 t} + B_1 e^{k_2 t} + C_1 e^{k_3 t} + D_1 e^{k_4 t} \quad (84) \]

\[ P_1(t) = A_2 e^{k_1 t} + B_2 e^{k_2 t} + C_2 e^{k_3 t} + D_2 e^{k_4 t} \quad (85) \]

\[ P_2(t) = A_3 e^{k_1 t} + B_3 e^{k_2 t} + C_3 e^{k_3 t} + D_3 e^{k_4 t} \quad (86) \]
\[ P_3(t) = A_4 e^{k_1 t} + B_4 e^{k_2 t} + C_4 e^{k_3 t} + D_4 e^{k_4 t} \]  \hspace{1cm} (87)

where \( A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4, D_1, D_2, D_3 \) and \( D_4 \) are obtained by expanding the above equations in partial fractions.

As a consequence,

\[ \text{System availability} = P_0(t) + P_1(t) \]  \hspace{1cm} (88)

The solution of the system of differential equations (76) to (79) can also be easily obtained by programming them on a digital computer.

3.8. Observations

As can be seen from these six Markov models, the reliability analysis of such models may become a complex process even when the simplest of models are analysed. For a thorough understanding of such modeling processes, it frequently requires considerable knowledge of advanced mathematics and an indepth treatment of reliability theory.

To follow up on this, the major aim of the next chapter is to simplify certain portions of reliability analyses as well as to advance the knowledge of reliability theory.
CHAPTER 4

THE GRAPHICAL RELIABILITY ANALYSIS

The reliability techniques developed in this chapter are somewhat simpler than the techniques developed in the previous chapters of this study. Three graphical techniques have been developed which require a less indepth knowledge of reliability theory and consume far less time in their application than all other methods found in the literature (6).

The first graphical technique presented was developed to evaluate the reliability of three-state device networks having identical components, paths and sub-systems. In a similar way, the second graphical technique is concerned with the reliability evaluation of three-state device networks having either identical or non-identical components, paths and sub-systems. The third technique deals entirely with the reliability optimization of series, parallel, series-parallel, parallel-series, series-parallel-series, and parallel-series-parallel three-state device networks. The three graphical techniques are presented in the next three sections.

4.1 Graphical Evaluation of Three-State Device Networks (Technique I)

To determine the reliability of a redundant configuration having identical components, paths and sub-systems, a method which

*Chapter 1 to Chapter 9 inclusive are concerned only with three-state devices of normal, open and short modes.
makes use of a computer generated graphical technique was developed. This procedure allows the graphical determination of system reliability as a function of the component unreliabilities and the amount of component redundancy. This technique permits the direct determination of a system's unreliability for a given number of three-state redundant components which have specific open and closed mode unreliability values. Figure 17 was developed for a redundant series configuration in which all components are identical. This means that all components have the same open mode unreliability and the same closed mode unreliability. Similarly, Figure 18 was developed for a parallel configuration.

When more complex redundant networks are to be analysed, the system can be broken down into sub-systems which have either series or parallel configurations.

As can be seen from Figures 17 and 18, the failure graphs were plotted for a system containing as many as fifteen components. When more than fifteen elements are incorporated into a system, the reliability of that system is extremely low. The exception would be a system containing only highly reliable components. The plots represented by Figures 17 and 18 verify this.

Development of the Graphical Technique

The unreliability equations (3), (5), (9) and (11) are plotted for series and parallel structures, respectively, up to fifteen redundant elements with the aid of a digital computer. These plots are shown in Figures 17 and 18. A reliability line (the straight line) is superimposed on the plots by joining the grid lines' adjacent corners of the plots to obtain the total system reliability from the
\( n_o = n \) open mode

\( n_s = n \) short mode

Figure 17. A Fixed Redundant Component Series Structure Unreliability Plot
Figure 18. A Fixed Redundant Component Parallel Structure Unreliability Plot
evaluated structure unreliabilities.

Because the graphical technique is based upon the unreliability of the components, the reliability derivations of the structures are obtained from the system's open and short failure equations.

The system reliability, open and short failure probability equations are developed for the following networks.

**Series Structure**

In a series configuration any one component failing in an open mode will cause an overall system failure, whereas all elements of the system must malfunction in a short mode to induce system failure.

The resulting reliability and unreliability equations are as follows:

**Reliability**

For identical components, the series system reliability \( R \) can be obtained from equation (1):

\[
R = (1 - q_o)^n - q_s^n
\]  

(1)

where

- \( n \) is the number of three-state components
- \( q_o \) is the probability of open mode failure of components
- \( q_s \) is the probability of short mode failure of components

In general, for the non-identical elements case, the system reliability can be obtained from equation (2):

*See Appendix D*
\[ R = \prod_{i=1}^{n} (1-q_{oi}) - \prod_{i=1}^{n} q_{si} \]  

(2)

**Failure**

For short mode system failure, equation (3) yields:

\[ Q_s = q_s^n \text{ (for identical components only)} \]  

(3)

where

- \( Q_s \) is the probability of short mode system failure.

When elements are non-identical, \( Q_s \) becomes:

\[ Q_s = \prod_{i=1}^{n} q_{si} \]  

(4)

The open mode system failure from equation (5) becomes:

\[ Q_o = 1 - (1-q_o)^n \text{ (for identical elements only)} \]  

(5)

where

- \( Q_o \) is the probability of open mode system failure.

In non-identical element case:

\[ Q_o = 1 - \prod_{i=1}^{n} (1-q_{oi}) \]  

(6)

**Steps Required to Evaluate Reliability of a Series Configuration**

(i) evaluate the number of redundant three-state components having known open and short failure probabilities;
(ii) use the horizontal axis of Figure 17 for the given values of short and open mode component failure. Read off the open and short failure of the system from the vertical scale for the corresponding redundant components.

(iii) add both open and short mode system failure probabilities. Use this total system failure probability on the vertical scale of Figure 17 and read the reliability of system on the horizontal axis where it cuts the straight reliability line.

Example 1: Consider two diodes connected in series with open and short circuit failure probabilities of .2 and .1, respectively. It is required to find the system reliability of the two diodes for this simple arrangement.

In this case, $n = 2$, $q_s = .1$ and $q_o = .2$.

Solution: The system reliability obtained by this graphical technique yields $R = .63$. This system reliability is easily verified by equation (1).

Parallel Structure

For a simple parallel configuration, all the elements must fail in the open mode or any one of the elements must stop functioning in a short mode to cause the system to fail completely. The following reliability, open and short mode failure probability formulas derived are:
Reliability

For the case of identical elements, the network reliability

\[ R = (1-q_S)^m - q_0^m \]  \hspace{1cm} (7)

where

\( m \) is the number of components connected in parallel

In general, for the non-identical element case, equation (7) gives:

\[ R = \prod_{i=1}^{m} (1-q_{S_i}) - \prod_{i=1}^{m} q_{0i} \]  \hspace{1cm} (8)

Failure

The short mode system failure

\[ Q_S = 1 - (1-q_S)^m \]  \hspace{1cm} (identical elements only) \hspace{1cm} (9)

In other non-identical element cases:

\[ Q_S = 1 - \prod_{i=1}^{m} (1-q_{S_i}) \]  \hspace{1cm} (10)

Open mode system failure (identical components only) can be written as:

\[ Q_o = q_0^m \]  \hspace{1cm} (11)

In the case of non-identical components, equation (11) becomes:

\[ Q_o = \prod_{i=1}^{m} q_{0i} \]  \hspace{1cm} (12)

Graphical Technique

For a parallel structure, the procedure described in the section on steps required to evaluate reliability for series network
is also applicable; however, instead of Figure 17, one should use Figure 18. An example for the series case will give the following results for a parallel structure.

**Solution:** Consider a structured reliability value of .65 obtained from its derived reliability formula and compare it with that obtained from the graphical technique. It is easily shown that the same result is obtained.

**Series-Parallel Structure**

This is simply a combination of previous configurations. The reliability, open and short mode failure probability formulas are as follows:

**Reliability**

For identical elements, the system reliability for series-parallel network is formulated as equation (13):

$$R = (1+q_o)^n - (1-(1-q_s)^m)^n$$  \((13)\)

For the case of \(n\) non-identical units, each containing \(m\) elements, equation (13) becomes:

$$R = \left(1 - \prod_{i=1}^{m} q_{oi}\right)^n - \left(1 - \prod_{i=1}^{m} (1-q_{si})\right)^n$$ \((14)\)

**Failure**

The short mode failure (where \(n\) and \(m\) are identical units and elements) formula is given by equation (19a) (if \(k = 1\), where \(k\) is the number of sub-systems):

$$Q_S = (1 - (1-q_s)^m)^n$$ \((19a)\)
Similarly, for open mode failure, equation (18a) is obtained (if \( k = 1 \)):

\[
q_0 = 1 - (1-q_o^m)^n
\]  

(18a)

**Graphical Technique**

To obtain the reliability of this configuration requires the use of both Figures 17 and 18. The following extra steps to the technique must be added:

(i) obtain the open and short probability of failure for parallel units just as for normal three-state device parallel structures; apply rules (i) and (ii) only (use Figure 18);

(ii) re-use the unit open and short probability of failure values obtained from the previous step and repeat the procedure as for series structure, applying steps (i) through (iii) (use Figure 17).

**Example 2:** Consider the evaluation of the reliability of the series-parallel arrays of the identical fluid flow valves with given \( q_o = .2 \), \( q_s = .1 \), \( n = 2 \) and \( m = 4 \).

**Solution:** The system structure is made up of series and parallel networks which can be analyzed according to the steps stated above.

System reliability (graphical technique) = .88

System reliability (normal procedure) = .88
Parallel-Series Structure

The structure is a dual of the series-parallel configuration in the previous section. The dual reliability, open and short mode failure probability equations are as follows:

Reliability

Reliability for identical n elements and m units can be obtained from equation (15) as follows:

\[ R = (1-q_s^n)^m - (1-(1-q_o^n)^m \]  

(15)

If the configuration contains m identical units and n non-identical elements, equation (15) becomes:

\[ R = (1 - \prod_{i=1}^{n} q_{si})^m - (1 - \prod_{i=1}^{n} (1-q_{oi}))^m \]  

(16)

Failure

In the case of n identical elements and m units, the short and open failure mode equations can be obtained from equations (18b) and (19b) (if k = n and n = 1) as follows:

\[ Q_o = (1 - (1-q_o^n)^m \]  

(18b)

\[ Q_s = 1 - (1-q_s^n)^m \]  

(19b)

Graphical Technique

To evaluate the reliability of this network, apply the steps described for the series-parallel structure, but in dual form. The total reliability of the example solved for series-parallel network
is reevaluated for the parallel-series structure as follows:

Solution:

System reliability (graphical technique) = .65
System reliability (normal procedure) = .65

General Three-Dimensional Structure (series-parallel-series structure)

The reliability equation of this structure is derived from the previous equations. The resulting formula can be written for identical elements as follows:

\[ R = (1 - (1 - (1 - q_o)^k)^m)^n - (1 - (1 - q_s)^k)^m)^n \]  \hspace{1cm} (17)

where

- \( k \) is the number of components connected in series

By giving appropriate values for \( k, m \) and \( n \), the reliability formulas of any of the previously discussed structures can be obtained without any difficulty as follows:

If \( k = n, n = m = 1 \), then equation (17) will yield the same results as equation (1).

If \( k = n = 1 \), then equation (17) will yield the same results as equation (7).

If \( k = 1 \), then equation (17) will yield the same results as equation (13).

If \( k = n, n = 1 \), then equation (17) will yield the same results as equation (15).
Failure

Open and short mode failure formulas may be called general formulas because these can be used to obtain the reliability equations for any other series and parallel combinations and mixed structures.

From the previous equations, the following formulas can be written for the three-dimensional, symmetrical and identical element networks for open and short system failure, respectively:

\[ Q_o = 1 - (1 - (1-q_o)k)^n \]  
\[ Q_s = (1 - (1-q_s)^m)^n \]  

For the non-symmetrical and non-identical element network, equations (18) and (19) become:

\[ Q_o = 1 - \prod_{i=1}^{n} (1-C_{mi}) \]

where

\[ A_k = \prod_{i=1}^{k} (1-q_{oi}) \]

\[ C_m = \prod_{i=1}^{m} (1-A_{ki}) \]

and

\[ Q_s = \prod_{i=1}^{n} (1-D_{mi}) \]

where

\[ B_k = \prod_{i=1}^{k} q_{si} \]

\[ D_m = \prod_{i=1}^{m} (1-B_{ki}) \]
Graphical Technique

The graphical technique will solve the network shown in Figure 19 \((q_s = .1, q_o = .2)\) within three steps, as follows:

(i) obtain the open and short mode path probability of failure for the section labelled '1' as shown in Figure 19 (apply the same rules as for series configuration, i.e., steps (i) and (ii) only);

(ii) evaluate the open and short parallel unit failure of section '2' by using the result of the previous step (apply the same rules as for parallel configuration, i.e., steps (i) and (ii) only);

(iii) obtain the reliability of the whole system by making use of the result of step (ii) (apply the same rules as for series configuration).

Figure 19. A Three-Dimensional Structure
Solution:
System reliability (graphical technique) = .76
System reliability (normal procedure) = .76

Mixed Structure
The task to evaluate the mixed structure reliability with the graphical technique becomes a cumbersome process. However, for some mixed structures, this difficulty can be overcome by making use of some simple derived formulas as well as the graphical technique. This theme may be called a Hybrid-Graphical Technique.

Example:
The equation of system reliability for Figure 20 ($q_o = .1$, $q_s = .2$) can be derived by introducing the general formulas of open and short mode failure (see section on General Three-Dimensional Structure Failure).

Figure 20. A Mixed Structure
This reliability equation is as follows:

\[
R = \left[ 1 - (1-(1-q_o)^2)^2 \right] (1-q_o)(1-q_o) - (1-(1-q_s)^2)(1-(1-q_s)^2) q_s
\]  
(20)

**Hybrid Graphical Technique**

As shown in Figure 20, the open and short mode failure of the sections of the network labelled as '1', '2' and '3', are evaluated graphically in steps (as steps described for parallel and series configurations). The evaluation of the overall reliability of the network labelled as '4' is accomplished by using the derived formula for non-identical element series structure (i.e., by substituting the fictitious values of \( Q_o, Q_s \) yielded by the graphical technique).

**Solutions of the Hybrid Graphical Technique**

Consider the system reliability for the Hybrid Graphical Technique with a value of .85 which corresponds to the system reliability with the normal procedure of .85.

4.2. The Graphical Reliability Evaluation of Three-State Device Networks (Technique II)

The approach described in section 4.1 is applicable only to networks having identical components, paths and sub-systems. Therefore, the main objective of this section of this study is to present a means of analysis which will permit the engineering designer to analyse any three-state device network configuration with a minimum amount of knowledge of reliability theory and time required for the analysis.
Although this newly developed graphical technique of this section accomplishes the above requirements, because of the additional complications introduced to handle non-identical components, paths and sub-systems, it is advisable to use the graphical technique presented in section 4.1. for the case when a system contains identical components, paths and sub-systems.

The Mathematical Development of the Graphical Technique

The following system's equations of open and short mode failure for both series and parallel cases are plotted and shown in Figures 21 and 22, respectively.

In the case of the components connected in series, the network open and short unreliabilities become:

\[
Q_{o} = 1 - \prod_{i=1}^{n} (1-q_{oi}) \tag{21}
\]

\[
Q_{s} = \prod_{i=1}^{n} q_{si} \tag{22}
\]

For a parallel network,

\[
Q_{s} = 1 - \prod_{i=1}^{n} (1-q_{si}) \tag{23}
\]

\[
Q_{o} = \prod_{i=1}^{n} q_{oi} \tag{24}
\]

where \( q_{oi} \) and \( q_{si} \) are the component's open and short mode unreliabilities, respectively.
Figure 21. A Series Structure Unreliabilities Plot

$q_{o1}$, $q_{s1}$ (Component unreliabilities)
Figure 22. A Parallel Structure Unreliabilities Plot
Q_o and Q_s are the system open and short mode unreliabilities, respectively.

n is the number of redundant components.

The above equations (21) through (24) are plotted for two non-identical elements only, as shown in Figures 21 and 22, respectively.

In the case of a series system, consider equations (21) and (22) for two non-identical elements, i.e.:

\[ Q_o = 1 - (1-q_{o1})(1-q_{o2}) \]  
(25)

\[ Q_s = q_{s1}q_{s2} \]  
(26)

Both these equations are plotted by holding the q_{o2} and q_{s2} at the interval of .05 and varying q_{o1} and q_{s1} from zero to one, respectively, as shown in Figure 21 with the aid of a digital computer.

A similar procedure is applied for the parallel case but in reverse order as shown in Figure 22. The only difference generated for both series and parallel cases is that their open and short mode unreliability plots are interchanged.

The horizontal scale of both figures is used for the component or system open and short mode failure probabilities (for a number of components greater than two only). The system short and open mode unreliabilities are read on the vertical scale. In the case of a series circuit, the short mode system unreliability is read in normal direction and for open mode failure probability, the reverse scale is used as shown in Figure 21. For the parallel case, the converse of the above sentence is true as shown in Figure 22.
System reliability can be obtained on the vertical scale for both figures which is merely a difference between the system open and short mode probabilities on that axis.

Series Network

For this configuration, any one component failing in an open mode will cause an overall system failure; however, if all components fail in a short mode, it will also cause a complete system failure. System reliability can be calculated from the following equation:

$$ R = \prod_{i=1}^{n} (1-q_{oi}) - \prod_{i=1}^{n} q_{si} \quad (27) $$

where

- $R$ is the system reliability

Graphical Technique

The use of this technique is demonstrated in Figure 21 by solving the following example:

Example 1: Assume a set of three electronic diodes are connected in series with the given data as follows:

- $q_{o1} = .1$
- $q_{o2} = .05$
- $q_{o3} = .2$
- $q_{s1} = .2$
- $q_{s2} = .1$
- $q_{s3} = .3$

Calculate the system reliability.

Steps Required to Evaluate the Series System Reliability

Steps are explained separately for both modes of failure as follows:
System open mode failure probability evaluation rules:

(i) obtain the open mode failure probabilities of the redundant components;

(ii) use the horizontal axis of Figure 21 for the first given value of open mode failure probability;

(iii) read off the system open mode failure probability on the outer reverse direction vertical axis scale for the corresponding second component open mode failure probability;

(iv) if the number of components is greater than two, reuse the system open mode failure value for two components (result of previous step) at the horizontal axis and read off the system open mode failure probability on the outer reverse direction vertical axis for the corresponding third component open mode failure probability;

(v) repeat the cycle if redundant components are four or more as for the previous step.

System short mode failure probability evaluation rules:

(i) obtain the short mode failure probabilities of the redundant components;

(ii) use the horizontal axis of Figure 21 for the first given value of short mode failure probability;
(iii) read off the system short mode failure probability on the inner normal direction vertical axis scale for the corresponding second component short mode failure probability;

(iv) if the number of components is greater than two, reuse the system short mode failure value for two components (result of previous step) at the horizontal axis and read off the system short mode failure probability on the inner normal direction vertical axis scale for the corresponding third component short mode failure probability;

(v) repeat the cycle if redundant components are four or more as for the previous step.

**General Rule to Obtain System Reliability**

To obtain system reliability, find the system open and short mode failure probabilities on the outer and inner vertical axes for the given number of redundant components as described previously and then read off the middle difference between these two failure probabilities on the vertical axis.

**Example Solution:** To solve the above example with the graphical technique, apply the already-established first four rules for both failure modes and the general rule to read off system reliability.

The series system reliability obtained with the graphical technique yields \( R = .68 \) which is verified by equation (27).
Parallel Network

This is a dual configuration of the series case, i.e., if all components fail in open mode or any one component fails in short mode, this will cause the overall system to fail.

The reliability formula for this configuration is as follows:

\[ R = \prod_{i=1}^{n} (1-q_{si}) - \prod_{i=1}^{n} q_{oi} \]  \hspace{1cm} (28)

Graphical Technique

To analyse this type of system, use Figure 22 instead of Figure 21. Apply the same rules as for the series case, but interchange the open and short mode failure probability rules of the series case to the parallel case (i.e., apply open mode failure rules of the series case to the short mode failure of the parallel configuration and short mode failure rules of the series network to the open mode failure of the parallel system).

Example 2: Three relays are connected in parallel with the following open and short mode failure probabilities:

\[ q_{o1} = .2 \quad q_{o2} = .3 \quad q_{o3} = .4 \]
\[ q_{s1} = .1 \quad q_{s2} = .2 \quad q_{s3} = .2 \]

It is required to calculate the system reliability.

Solution: This example is solved by applying the first four rules of the series case for both modes. However, in the case of parallel configuration, the open and short mode failure rules of the series case are applied to the short and open mode failure of this system, respectively.
The end result yielded by the conventional and graphical methods is almost identical ($R = 0.62$).

**Complex Network**

![Diagram of a complex network](image)

Figure 23. A Complex Network

Consider the complex network of Figure 23 made up of series and parallel combinations. The evaluation of the reliability of this type of network poses a more difficult task (i.e., if analytical procedure is used—as can be seen from equation (29)—to derive the reliability equation). But with the graphical technique it is just a matter of reading the graphs a few more times.

$$R = \left[ 1 - \left(1 - (1 - q_{s1}) (1 - q_{s2})^2 \right) \right] \left[ 1 - q_{s3} q_{s4} \right] \left[ 1 - q_{s5} \right]$$

$$- \left[ 1 - \left(1 - q_{s1} q_{s2}\right)^2 \right] \left[ 1 - \left(1 - q_{s3} (1 - q_{s4}) \right) \right] [q_{s5}]$$

(29)

**Example 3:** It is required to evaluate the reliability of the network shown in Figure 23 for the following given short and open mode
failure probabilities of the components:

\[ q_{s1} = 0.1 \quad q_{s2} = 0.3 \quad q_{s3} = 0.3 \quad q_{s4} = 0.5 \quad q_{s5} = 0.3 \]
\[ q_{o1} = 0.2 \quad q_{o2} = 0.1 \quad q_{o3} = 0.4 \quad q_{o4} = 0.2 \quad q_{o5} = 0.2 \]

**Solution:** As it can be seen from Figure 23, this network contains two series and parallel combinations, each marked as '1', '4' and '2', '3', respectively. Therefore, it will be analysed graphically in that order as indicated on the diagram, i.e., series (1), parallel (2), parallel (3) and series (4). This time the graphical technique will require use of both Figures 21 and 22.

**Steps Required for Graphical Analysis**

(i) evaluate the series section (marked as '1' on the diagram) open and short mode failure unreliabilities. Apply rules (i) through (iii) as stated for series configuration for both modes—use Figure 21;

(ii) obtain the parallel section (marked as '2' on the diagram) open and short mode failure unreliabilities by using the results of the previous step. Apply rules (i) through (iii) as stated for parallel network for both modes—use Figure 22;

(iii) evaluate the parallel section (marked as '3' on the diagram) open and short failure unreliabilities. Apply the same rules as for the
parallel case (i.e., rules (i) through (iii)) use Figure 22 again.

(iv) evaluate the final series system reliability (marked as '4' on the diagram) by using the results of the previous steps (i) and (ii) and the unreliability modes' value of the component marked as '5'. Apply the same rules as stated for the series network (i.e., rules (i) through (iv)) and also the final general rule to calculate the system reliability.

The network shown in Figure 23 was analysed according to the above four steps which yielded the system reliability equal to 0.67. This is verified by equation (29).

Complex Configuration with a Bridge

If any configuration contains a bridge, as shown in Figure 24, the reliability evaluation of that network becomes a cumbersome process. However, the graphical technique can be utilized if the bridge network failures are evaluated separately. Then, the graphical technique can be applied as follows:

The element marked '3' of section '1' in Figure 24 is a critical one--if it fails in its open mode, section '1' will be reduced to a parallel-series network as shown in Figure 25. Similarly, if the critical element fails in its short mode, then section '1' will be reduced to a series-parallel system as shown in Figure 26. The open
Figure 24. A Complex Configuration with a Bridge

and short mode unreliabilities of both these networks can be very
easily determined graphically according to the established rules for
series and parallel networks.

Bridge unreliabilities of both modes can be calculated from
the following formulas:

Suppose,

$q_{o3} = \text{open mode failure probability of the critical element}$
$q_{s3} = \text{short mode failure probability of the critical element}$

$Q_{ops} = \text{open mode failure unreliability of parallel-series}$

network
Figure 25. A Parallel-Series Network

Figure 26. A Series-Parallel Network
\[ Q_{sps} = \text{short mode failure unreliability of parallel-series network} \]
\[ Q_{osp} = \text{open mode failure unreliability of series-parallel network} \]
\[ Q_{ssp} = \text{short mode failure unreliability of series-parallel network} \]

where \( Q_{ops}, Q_{sps}, Q_{osp} \) and \( Q_{ssp} \) are obtained graphically for Figures 25 and 26, respectively. Therefore, the bridge open and short mode failure probabilities can be calculated from the following established formulas:

\[ Q_0 = q_{o3}Q_{osp} + (1-q_{o3})Q_{ops} \quad (30) \]
\[ Q_s = q_{s3}Q_{osp} + (1-q_{s3})Q_{sps} \quad (31) \]

With the results of equations (30) and (31), Figure 24 becomes a simple series circuit of two elements which can be analysed by applying the same established rules (i) through (iii) of the series network again, as well as the final general rule to calculate system reliability.

**Example 4:** Suppose that some electronic diodes or relays are connected as shown in Figure 24. If the open and short mode probabilities are given for each component as follows:

\[ q_{o1} = .1, \quad q_{o2} = .1, \quad q_{o3} = .1, \quad q_{o4} = .1, \quad q_{o5} = .1, \quad q_{o6} = .4 \]
\[ q_{s1} = .2, \quad q_{s2} = .2, \quad q_{s3} = .2, \quad q_{s4} = .2, \quad q_{s5} = .2, \quad q_{s6} = .3 \]

calculate the system reliability.
Solution: By applying the previously discussed graphical rules, yield the same results of .56 analytically as well as graphically.

4.3. **Graphical Optimization of Three-State Device Networks**

In the study from which this section resulted, only reliability optimization under unconstrained conditions is of concern. Some authors have already analysed three-state device networks, but their contribution has mainly been to derive the equations for optimizing the number of redundant components for either series or parallel configurations which optimize the network reliability.

This section contains two graphical techniques developed to optimize the number of redundant components in series, parallel, series-parallel, parallel-series configurations and other more complex arrangements. These graphical techniques were developed to provide highly accurate (optimum) results almost at a glance without going into all the details of reliability equation derivations or the use of a digital computer. The details of the graphical techniques are described as follows:

**Graphical Technique I**

This particular technique deals with the optimization of series, parallel-series, series-parallel configurations with specific paths and a fixed number of sub-system components, respectively.

The following equations for optimum reliabilities for series and parallel configurations are plotted with open and short failure probabilities as shown in Figures 27 and 28, respectively.
Figure 27. A Series Network Optimum Plot
Figure 28. A Parallel Network Optimum Plot
For series configuration, the plot results from equation (32):

\[
\log e \left[ \frac{\log e q_s}{\log e (1-q_o)} \right]
\frac{n'}{\log e \left[ \frac{1-q_o}{q_s} \right]}
\]  (32)

and for parallel case, the plot comes from equation (33):

\[
\log e \left[ \frac{\log e q_o}{\log e (1-q_s)} \right]
\frac{m'}{\log e \left[ \frac{1-q_o}{q_s} \right]}
\]  (33)

where
- \( q_o \) is the components' open mode failure probability
- \( q_s \) is the components' short mode failure probability
- \( n' \) is the optimum number of redundant components in series
- \( m' \) is the optimum number of redundant components in parallel

The graph for the series case shown in Figure 27 is plotted by fixing the values for \( q_s \) and varying \( q_o \) and vice versa. In a similar way, the parallel structure graph is obtained where the plots of \( q_o \) and \( q_s \) are simply interchanged. It is of interest to note that from equations (32) and (33) the zero values of \( q_s \) and \( q_o \) must be avoided, otherwise an indeterminate quantity will result. Likewise, the ratios of \((1-q_s)/q_o\) and \((1-q_o)/q_s\) which are equal to unity must be avoided, otherwise an infinite quantity will result. It is for this reason the plots in Figures 27 and 28 vary between the range of .05 and .9.

This technique and the rules established for its use are demonstrated for a series configuration with identical components,
parallel configuration with identical components, a parallel-series configuration and series-parallel configuration.

**Series Network (for identical components only)**

To make the study of redundancy worthwhile, the optimization of a series configuration is not only necessary but essential, otherwise for a specific given component unreliability, an increase in the number of components may decrease the overall system reliability. In some cases, a system with a large number of components may achieve a lower overall reliability than one with a fewer number of components. For the normal series case, system reliability can be evaluated from the following equation:

\[ R = \prod_{i=1}^{n} (1-q_{oi}) - \prod_{i=1}^{n} q_{si} \]  \hspace{1cm} (34)

**Steps Required to Evaluate Graphically the Optimum**

**Reliability of a Series Configuration**

(i) establish the components' known open and short mode unreliabilities;

(ii) use the horizontal axis of Figure 27 for the given values of the components' short or open mode failure probabilities;

(iii) read the optimum number of redundant components from the vertical scale for the corresponding value of the components' open or short mode unreliabilities on the horizontal axis of Figure 27;
(iv) the horizontal axis of Figure 29 is used for the optimum value (from step (iii)) of redundant components. Obtain the open and short failure probabilities of the system from the vertical axis for the corresponding open and short mode component failures (from step (i));

(v) add both the open and short mode system failure probabilities. Use this as the total system failure probability on the vertical scale of Figure 29 and read the reliability of the system on the horizontal axis where it cuts the straight reliability line.

Example: Consider a system with identical three-state components which have open and short mode failure probabilities of .1 and .25, respectively. Suppose the objective is to find the optimum number of redundant components and the maximum reliability when they are to be connected in such a manner as to form a series configuration.

Example Solution: According to the steps established, a series system yields a maximum reliability of .75 with the optimum number of redundant components equal to two. These results are readily verified by equations (34) and (32), respectively.

Parallel Network

Similarly, the optimization of a parallel network with identical components, as for the series configuration, is much simpler if all of the components are identical. Reliability of this parallel
Figure 29. A Fixed Component Series Structure Unreliability Plot

A network system can be calculated from the following equation:

\[ R = \prod_{i=1}^{m} (1-q_{s_i}) - \prod_{i=1}^{m} n_{o_i} \]  

(35)

where

- \( m \) is the number of redundant components
Steps Required to Evaluate the Optimum Reliability of a Parallel System

The same steps as for the series case are applicable; however, now one will use Figures 28 and 30 instead of Figures 27 and 29 for optimizing the number of components and evaluating the maximum system reliability, respectively.

Figure 30. A Fixed Component Parallel Structure Unreliability Plot
Example 2: Consider a number of identical three-state devices which have open and short mode failure probabilities of .05 and .35, respectively. The object is to evaluate the optimum number of redundant components and evaluate the maximum system reliability. The components are to be connected physically to form a parallel configuration.

Example Solution: A maximum system reliability of .73 is found for an optimum number of redundant components equal to three. These results can be readily verified by equations (35) and (33), respectively.

Parallel-Series System (for identical components and paths only)

This configuration is a simple combination of the two previous networks considered. If identical series path components are specified, these redundant identical paths can be optimized graphically for known values of the paths' short and open mode unreliabilities. These latter values are obtained graphically from Figure 29 and then use is made of these series system failure values in Figure 28.

The rules established for the parallel-series configuration are outlined as follows:

(i) establish the fixed number of identical redundant components to be used in a series path and the identical components' open and short mode failure probabilities;

(ii) the horizontal axis of Figure 29 is used for the number of redundant components known from step (i). The open and short failure probabilities
of the series path are obtained from the vertical axis for the corresponding components' open and short mode failures (i.e., known from step (i));

(iii) use the results of step (ii) to evaluate the optimum number of redundant paths for the reduced fictitious parallel network--use Figure 28;

(iv) use the horizontal axis of Figure 30 for the evaluated optimum number of redundant paths from step (iii). The open and short failure probabilities of the system are obtained from the vertical axis for the corresponding series paths open and short mode failure probabilities (i.e., evaluated in step (ii));

(v) both system open and short mode unreliabilities are to be added. Use this total system failure probability as a value on the vertical scale of Figure 30 and read the reliability of the system on the horizontal axis where it intersects the straight reliability line.

Example 3: Consider two identical elements which are connected physically to form a series configuration, with each component having an open and short mode failure probability of .45 and .3, respectively. The object is to evaluate the maximum system reliability and the optimum number of series configuration to be connected in a parallel system.
Example Solution:

The optimum system reliability = 0.46

The optimum number of series configurations = 5

The above results are the same for both analytical and graphical approaches.

**Series-Parallel Network** (for identical components and sub-systems only)

There is a set of identical parallel networks physically connected to form a series configuration. If an identical number of redundant components of the parallel networks are specified and the same, then the number of parallel sub-systems can be optimized graphically for any specified open and short mode failure probability values for the components.

The open and short mode failure probability values for the individual parallel sub-systems can be obtained graphically from Figure 30 and then these values used in Figure 27, to obtain the optimum number of redundant sub-systems.

To obtain optimum system reliability and redundant sub-systems, the following rules are to be observed:

(i) establish the fixed number of redundant components of parallel networks or sub-systems and components' open and short mode failure probabilities;

(ii) use the horizontal axis of Figure 30 for the fixed number of redundant components (known from step (i)) and read off the open and short failure
probabilities of the parallel sub-systems from the vertical scale for the corresponding components' open and short mode unreliabilities (i.e., known from step (i));

(iii) use the results of step (ii) to evaluate the optimum number of redundant sub-systems to be connected in series—use Figure 27;

(iv) use the horizontal axis of Figure 29 for the evaluated optimum number of redundant paths from step (iii). The open and short failure probabilities of the system are obtained from the vertical axis for the corresponding parallel sub-systems' open and short mode failure probabilities (i.e., evaluated in step (ii));

(v) add both system open and short mode failure probabilities. Use this total system failure probability on the vertical scale of Figure 29 and then read the reliability of the system on the horizontal axis where it cuts the straight reliability line.

Example 4: Consider a set of four identical components which are connected in a parallel configuration with known components' short and open mode failure probabilities (i.e., $q_s = .2$, $q_o = .4$). The objective is to find the maximum system reliability and the optimum number of identical parallel sub-systems with identical
components to be connected physically in series to form a network.

Example Solution: According to the rules established, the end results of the above example are as follows:

The optimum number of parallel sub-systems = 2

Maximum system reliability = .58

Both end results are verified analytically by the equations upon which the graphical solutions are based.

Graphical Technique II

The graphical technique developed for both the parallel-series and series-parallel cases are presented separately as follows:

Parallel-Series Case (for identical components and paths only)

![Diagram of Parallel-Series Network]

Figure 31. A Parallel-Series Network

Since this network is made up of series and parallel configurations, it is necessary to minimize the system's probability of short failure for the series paths first followed by the parallel paths' open mode failure as illustrated by segments '1' and '2' of Figure 31. Both the failure mode equations of a series path ('1') with identical elements are as follows:
\[ Q_{OS} = 1 - (1-q_o)^n \] (36)

\[ q_{SS} = q_s^n \] (37)

It can be readily seen from these equations that the series path open mode failure unreliability will increase if the number of redundant components increases and vice versa for the system short mode failure case. Therefore, it is necessary to minimize the series path (marked '1' in Figure 31) short mode probability of equation (37) to maximize the series-parallel system reliability.

i.e., minimize \[ q_s = q_s^n \] (38)

As can be seen from the above equation for \( q_s > 0 \), it is impossible to obtain an absolute minimum point. However, for practical reasons, the minimum point can be approximated for some insignificant value of \( q_s \) (i.e., \( q_s \leq .001 \)). In this way, the system can be minimized in terms of the probability of short failure for the series paths. The parallel network marked '2' in Figure 31 requires minimization of the probability of the following open mode failure equation:

\[ Q_o = Q_{OS}^m \] (39)

Likewise, for open mode failure of the parallel network (marked '2' in Figure 31), provided \( Q_{OS} > 0 \), the minimization of the above equation is not possible. Therefore, it is desirable to approximate the value of \( Q_o \), i.e., \( Q_o \leq .001 \).
The minimization of equations (38) and (39) can be readily approximated by the use of Figures 29 and 30, respectively. The "minimum" value for equations (38) and (39), .001, is used to achieve a good approximation. This approximate method has been shown to generate the system optimum reliability results to within a variance of about ± .01.

**Rules Established to Determine the Optimum System Reliability**

(i) obtain the components' given open and short mode failure probability;

(ii) the minimum value of the series path (marked '1' on Figure 31) short mode failure probability is assumed to be .001. Therefore, where both the assumed point (i.e., .001) on the vertical axis and the given value of the components' short mode failure unreliability from step (i) intersect, read the corresponding number of optimum redundant components 'n' at the horizontal axis--use Figure 29;

(iii) use the horizontal scale for the optimum number of series path redundant components (i.e., evaluated number of redundant components of step (ii)). Read off the open mode failure of the identical series paths from the vertical scale of Figure 29 for the corresponding series path components' open mode unreliability;
(iv) the minimum value of the parallel system (marked '2' on Figure 31) open mode failure probability is assumed to be .001. Therefore, where both the point on the vertical axis (.001) and the evaluated value of the system open mode failure unreliability (from step (iii)) intersect, read off the corresponding number of optimum redundant paths 'm' on the horizontal scale—use Figure 30.

The fictitious short and open mode failure probabilities are known for the series path (see section '1' of Figure 31) and were obtained by using steps (ii) and (iii), respectively. The optimum number of paths 'm' of section '2' of Figure 31 are known from step (iv).

Therefore, it is advisable to use a parallel network formula of equation (35) to calculate the overall system reliability instead of the graphical method to improve accuracy, because the series paths' fictitious short mode failure probability is very small (i.e., .001).

Example 5: The optimum number of identical components and paths will be used for a parallel-series network which will optimize the overall system's reliability. Obtain the system's reliability for the given identical components' short and open mode unreliabilities of .1 and .2, respectively.

Example Solution: This example is analyzed according to the steps established. According to the graphical technique, the optimum
number of paths and components for the parallel-series system should be 0 and 3, respectively (i.e., n = 3, m = 9), which will yield the overall system reliability of .99. The results were verified analytically by approximating the values of m and n on the digital computer.

Series-Parallel Configuration (for identical components and sub-systems only)

![Diagram of Series-Parallel Network]

Figure 32. A Series-Parallel Network

The section '1' (shown in Figure 32) failure mode equations are as follows:

For short mode failure sub-system probability,

\[ q_S = 1 - (1-q_S)^m \]

and in the case of open mode failure sub-system probability,

\[ q_0 = q_0^m \]  \hspace{1cm} (40)

Therefore, it is necessary to minimize equation (40), i.e.,

minimize \[ Q_o = q_0^m \]

Likewise, for section '2' of Figure 32, both the series system open and short mode equations are as follows, respectively:
\[ Q_{osp} = 1 - (1 - Q_o)^n \]

In the case of short mode failure probability,

\[ Q_{sp} = Q_s^n \]  \hspace{1cm} (41)

Thus, to optimize system reliability, it is necessary to minimize equation (41), i.e.,

\[ \text{minimize } Q_{sp} = Q_s^n \]

Rules Established to Determine the Optimum System Reliability

(i) obtain the components' given open and short mode failure probabilities;

(ii) assume the minimum value of the parallel subsystem (marked '1' on Figure 32) open mode failure probability to be .001. Therefore, where both the assumed minimum point (.001) on the vertical axis and the given value of the components' open mode failure unreliability (from previous step) intersect, read the corresponding number of optimum redundant components 'm' at the horizontal axis—use Figure 30;

(iii) use the horizontal axis scale for the number of optimum redundant components (i.e., evaluated number of optimum redundant components of step (ii)). Read off the short mode failure probability of the identical parallel sub-systems
from the vertical scale of Figure 30 for the corresponding parallel sub-system components' short mode unreliability (i.e., known from step (i));

(iv) the minimum value of the fictitious series system (marked '2' on Figure 32) short mode failure probability is assumed to be approximately .001. Therefore, where both the point on the vertical axis (.001) and the evaluated value of the parallel system short mode unreliability (from step (iii)) intersect, read off the corresponding number of optimum redundant sub-systems 'n' on the horizontal scale—use Figure 29.

Now we know the parameters of the parallel identical sub-systems (i.e., section '1' on Figure 32) such as short and open mode failure probability from steps (iii) and (ii), respectively. Also known is the optimum number of 'n' sub-systems of section '2' of Figure 32.

Therefore, it is advisable to use the series network formula shown as equation (34) to calculate system reliability instead of the graphical technique to improve the accuracy, because the parallel sub-systems' open mode failure probability is very small (i.e., .001).

Example 6: Evaluate the number of optimum identical components and the parallel sub-systems to be used for a series-parallel
configuration which will optimize the overall system's reliability. The components' short and open mode unreliabilities are .1 and .3, respectively. The system's reliability is also to be evaluated.

**Example Solution:** The above example was analyzed according to the steps established which yielded the optimum value of 'm' and 'n' (5 and 7), respectively. The evaluated system reliability from equation (34) is .99. The results were verified analytically by approximating the values of m and n on the digital computer.

**Series-Parallel-Series** (for identical components, paths and sub-systems only)

![Diagram of Series-Parallel-Series Structure](image)

**Figure 33. A Series-Parallel-Series Structure**

This network is shown in Figure 33. Since the failure values of the components' open and short modes are less than .2 (i.e., components' reliability .6 or more), the graphical approach of parallel-series network will facilitate the optimization. However, for each
failure value greater than .2, the replotting of Figures 29 and 30 for the larger number of redundant elements is necessary to optimize parallel-series or series-parallel networks. If one component (i.e., open or short mode) is .5 or more, the parallel-series network can be optimized; but it is more advantageous to optimize the series-parallel-series configuration which will increase the overall system reliability with a smaller number of total redundant components. This was verified by analytically computed results.

The following rules for this configuration are applicable:

(i) apply the same rules as for the parallel-series case (i.e., (i) through (iv));

(ii) since open failure probability is already known (.001) for section '2' in Figure 33 (i.e., first result of step (iv) of the parallel-series case), calculate the short mode unreliability of section '2' by substituting the result of steps (ii) (short failure probability of section '2' in Figure 33) and (iv) (redundant paths 'm') of the parallel-series case in the following equation:

\[ Q_{ps} = 1 - (1-Q_s)^m \]

where \( Q_s \) is the path short mode unreliability of step (ii) (i.e., from the steps established for the parallel-series case) and \( m \) is the number of optimum redundant paths 'm' according to step
(iv) (i.e., from the steps established for the parallel-series case);

(iii) assume the minimum value of the fictitious system (marked '3' in Figure 33) short mode unreliability to be .001. Therefore, where both the assumed minimum point (.001) on the vertical axis and the evaluated value of short mode unreliability (i.e., the final result from the previous step) intersect, read the number of optimum redundant identical sub-systems 'K' at the horizontal scale--use Figure 29;

(iv) use equation (34) to calculate the overall optimum system reliability with known results of steps (ii) (i.e., open and short mode failure unreliabilities of section '2' in Figure 33) and (iii) (i.e., the optimum number of sub-systems 'K' of section '3' in Figure 33).

Example 7: Optimize the values of m, n and K of the structure as shown in Figure 33 with given values of identical components' open and short unreliabilities of .5 and .1, respectively. Calculate the optimum system reliability.

Example Solution: According to the established graphical steps, the values of n, m, K and reliability R are 3, 37, 2 and .9901, respectively.
Parallel-Series-Parallel Network

Likewise, as for the series-parallel-series case, a network such as this can be optimized but in dual steps.

Example 8. Consider an airplane which is to be installed with some autopilots. An autopilot can fail in either of two modes—one by simply having a complete open mode failure and the other by continuously giving a command signal which will lead the aircraft into a destructive maneuver. However, if the autopilots are arranged in a series configuration, the command signal is corrected by a redundant autopilot. Suppose it is desirable to increase the reliability of an autopilot system for this airplane which can be constructed from identical autopilots. If the probability of an autopilot failing in an open mode is 0.45 ($q_o = 0.45$) and the probability of failure by giving a destructive command signal is 0.25 ($q_s = 0.25$), then a graphical determination can be made as to the desirability of redundancy either in series or parallel. Redundancy independently in series and parallel is used to determine graphically the maximum overall system reliability for the autopilots.

Example Solution: The above example is analyzed graphically according to the steps given for the series and parallel configurations. The maximum system reliabilities and the number of autopilots are evaluated.

For the series case

System reliability $= 0.3$

Number of autopilots $= 1$
For parallel configuration

System reliability = .36

Number of autopilots = 2

Therefore, it is preferable to choose the parallel configuration.

4.4. Observations

This chapter outlined three newly developed graphical techniques and demonstrated their validity by solving some numerical examples.

However, in contrast to the graphical techniques, the next chapter includes only a delta-star transformation method to transform complex networks such as bridges. Thereonward, the transformed network can be analysed by some conventional means.
CHAPTER 5

RELIABILITY EVALUATION OF THREE-STATE DEVICE COMPLEX NETWORKS

This chapter describes a technique to evaluate the reliability of three-state device networks as an alternative to other recognized methods. This particular technique makes use of the delta to star transformation which was developed to uncouple the two modes of failure generally encountered in a three-state device network. Some numerical examples are solved and their results compared with the results obtained by conventional means (4).

5.1. Delta to Star Transformation

To evaluate the reliability of a bridge or other complex structures, the existing theories are applicable but were found to be far more cumbersome. For example, a two-state device bridge structure made up of sixteen elements, when using the event space method, involves the calculation of probabilities for 65536 states. Similar arguments can be applied to most other techniques used to evaluate the reliability of complex two- or three-state device configurations or bridge networks. By using a delta to star transformation, the above difficulties are most easily overcome. The network is simply reduced to simple series and parallel combinations, after which a set of formal rules for series and parallel combinations can be applied.

Where the structure such as a bridge configuration contains more than five elements, the application of formal series and parallel techniques may be used to reduce the bridge elements to five; hence, the simple
delta to star transformation can be applied to solve for the reliability of the system.

Development of the Delta to Star Failure Mode Formulas

The delta to star transformation applies to both of the element's open and short failure modes. The relationships of the delta to star transformation for the open and short failure modes follow.

Formulas for Open Mode Failures

The relationship for the transformation of open mode failure is shown in Figure 34.

Figure 34. A Delta to Star Open Failure Mode Equivalent

The equivalent delta to star leg diagrams for Figure 34 are shown in Figure 35.
Figure 35. Delta to Star Equivalent Leg Diagrams

By making use of the following open mode system failure probability equations for the series and parallel structures, the equivalent block diagrams as shown in Figure 35 can be used as an aid in their conversion
into equations (3), (4) and (5).

In the case of a series network the system open mode failure equation is:

\[
Q_o = 1 - \prod_{i=1}^{n} (1-q_{oi})
\]  

(1)

Likewise, for the parallel case:

\[
Q_o = \prod_{i=1}^{n} q_{oi}
\]  

(2)

where

- \( n \) is the number of components
- \( q_o \) is the component's open mode failure probability

\[
[1 - (1-q_oA)(1-q_oC)] = [1 - (1-q_oCB)(1-q_oAB)] q_oAC
\]  

(3)

\[
[1 - (1-q_oA)(1-q_oB)] = [1 - (1-q_oAC)(1-q_oCB)] q_oAB
\]  

(4)

\[
[1 - (1-q_oB)(1-q_oC)] = [1 - (1-q_oAC)(1-q_oAB)] q_oCB
\]  

(5)

From equations (3), (4) and (5) the resulting delta to star transformed equations are as follows:

\[
q_{oA} = 1 - \sqrt{\frac{[1-(1-(1-q_oCB)(1-q_oAB))q_oAC][1-(1-(1-q_oAC)(1-q_oCB))q_oAB]}{[1-(1-(1-q_oCB)(1-q_oAB))q_oAC]}}
\]  

(6)

\[
q_{oB} = 1 - \sqrt{\frac{[1-(1-(1-q_oAC)(1-q_oCB))] [1-(1-(1-q_oAC)(1-q_oAB))q_oCB]}{[1-(1-(1-q_oCB)(1-q_oAB))q_oAC]}}
\]  

(7)
\[ q_{OC} = 1 - \sqrt{\frac{1 - (1 - q_{OCB})(1 - q_{OAB})q_{OAC}}{1 - (1 - q_{OAC})(1 - q_{OCB})q_{OAB}}} \]  

(8)

**Short Mode Failure Formulas**

Similarly, as for the derivation of the open mode failure formulas, the short mode failure equivalent block diagrams are shown in Figures 36 and 37.

![Diagram](image)

**Figure 36. A Short Failure Mode Delta to Star Transformation**

Likewise, the delta to star equivalent leg diagrams for Figure 36 are shown in Figure 37.

By making use of the probability short mode failure equations (9) and (10) of the series and parallel networks, equations (11), (12) and (13) are obtained. The reader will recall that the unrelia-
Figure 37. Delta to Star Equivalent Arm Diagrams

Bilities for the series and parallel system short mode equations are respectively as follows:

\[ Q_s = \prod_{i=1}^{n} q_{si} \] (9)
\[ Q_s = 1 - \prod_{i=1}^{n} (1-q_{s_i}) \]  

(10)

where

\( q_s \) is the components' short mode failure probability

The corresponding delta to star transformed equations for Figure 37 are as follows:

\[ q_{sA} q_{sC} = 1 - (1-q_{sCB} q_{sAB})(1-q_{sAC}) \]  

(11)

\[ q_{sB} q_{sA} = 1 - (1-q_{sAC} q_{sCB})(1-q_{sAB}) \]  

(12)

\[ q_{sB} q_{sC} = 1 - (1-q_{sAC} q_{sAB})(1-q_{sCB}) \]  

(13)

By solving simultaneously the equations:

\[ q_{sA} = \frac{\sqrt{1-(1-q_{sCB} q_{sAB})(1-q_{sAC})}}{1-(1-q_{sAC} q_{sCB})(1-q_{sAB})} \]  

(14)

\[ q_{sB} = \frac{\sqrt{1-(1-q_{sAC} q_{sAB})(1-q_{sCB})}}{1-(1-q_{sCB} q_{sAB})(1-q_{sAC})} \]  

(15)

\[ q_{sC} = \frac{\sqrt{1-(1-q_{sCB} q_{sAB})(1-q_{sAC})}}{1-(1-q_{sAC} q_{sCB})(1-q_{sAB})} \]  

(16)

As can be seen from equations (14), (15) and (16) they are interrelated. Therefore, computing the value of the first equation will also make the computation easier for the remaining, because the former computations can be reused. Similar arguments are applicable to the transformed delta to star open mode failure equations (6), (7) and (8). To
illustrate the use of the delta to star transformation, an example follows.

**Example 1:** The configuration shown in Figure 38 uses the node A, B, C to describe a delta arrangement of five three-state devices. The components' open and short mode failure probabilities are given in Figure 38. This network was transformed to the star equivalent by using equations (6), (7), (8) and (14), (15), (16) for the open and short mode failures, respectively. The transformed structure is given by Figure 39. The end system reliability result is obtained from Figure 39 by applying a formalized method which is as follows:

From Figure 39 the total open and short mode probabilities of failure are:

\[
Q_o = 1 - \left[1 - (1 - q_{o1})(1 - q_{oA})\right]\left[1 - (1 - q_{oB})(1 - q_{o2})\right][1 - q_{oC}] \quad (17)
\]

\[
Q_s = 1 - (1 - q_{s1}q_{sA})(1 - q_{s2}q_{sB})q_{sC} \quad (18)
\]

From equations (17) and (18)

\[
\text{Network Reliability} = 1 - Q_s - Q_o
\]

\[
= 1 - 0.088 - 0.022 = 0.89
\]

In order to compare the results of this approach the solution of over another on this subject was evaluated. The bridge network reliability obtained by a procedure described by Lipp (4) also yielded the value of .89.
Figure 38. A Bridge Network

Figure 39. A Transformed Delta to Star Network
5.2. Observations

The delta to star transformation formulas developed for open and short failure modes yield the same results as those obtained by conventional means. This however, is a very simple and useful method for solving complex networks of three-state devices. To obtain the reliability of a double bridge or any other more complex system containing bridges, one needs only to simply substitute these formulas presented here to obtain the equivalent series and parallel network.
CHAPTER 6

PARAMETRIC RELIABILITY ANALYSIS OF THREE-STATE DEVICE NETWORKS

As the number of three-state devices are increased in a complex network, the more cumbersome and time consuming the reliability evaluation becomes, in particular if handled in a conventional manner (11).

A parametric technique was developed and used to represent probabilities to overcome most of the usual difficulties in network reliability evaluation. This approach represents all of the salient probability values as points in a Cartesian frame of reference. Each of these probability values is then put in parametric form.

This chapter is divided into two parts. The first part is entirely concerned with the reliability evaluation of three-state device networks in which the resulting parametric formulas are derived for series, parallel and series-parallel networks and more complex structures such as a bridge. Several numerical examples dealing with this are solved by both parametric and conventional techniques to show the ease and usefulness of this parametric approach. Close agreement of the results of the two methods were found.

In a similar way, the second part includes the reliability optimization of series and parallel networks. The numerical results obtained are also verified by conventional means.

6.1. Three-Dimensional Parametric Representation—Reliability Evaluation

Suppose that a component exists in either one of three states—a success, open or short failure mode. Hence, according to probability
theory:

\[ P + q_0 + q_s = 1 \]  \hspace{1cm} (1)

or alternatively,

\[ P + q_0 = 1 - q_s = p_s \] \hspace{.5cm} (2)

In order to introduce the parameters \( \theta \) and \( \phi \) for open and short failure states, the relationships are defined as follows:

**Open failure state**

- \( \phi_0 = \tan \theta_0 = \frac{q_0}{p_0} = \frac{1-p_0}{p_0} = \frac{q_0}{1-q_0}, \quad p_o \neq 0 \)

then

- \( p_0 = \frac{1}{1+\phi_0} \) \hspace{1cm} (3)

- \( q_0 = \frac{1}{1+cot \theta_0} \) \hspace{1cm} (4)

**Short failure state**

- \( \phi_s = \tan \theta_s = \frac{q_s}{p_s} = \frac{1-p_s}{p_s} = \frac{q_s}{1-q_s}, \quad p_s \neq 0 \)

then

- \( p_s = \frac{1}{1+\phi_s} \) \hspace{1cm} (5)

- \( q_s = \frac{1}{1+cot \theta_s} \) \hspace{1cm} (6)

where

- \( P \) is the component's probability of success
- \( q_o \) is the component's open mode failure probability
- \( q_s \) is the component's short mode failure probability
- \( P_o \) is the component's probability of success in an open mode
- \( P_s \) is the component's probability of success in a short mode
- \( \phi_o \) is the component's open mode parameter
- \( \phi_s \) is the component's short mode parameter
6.1.1. Series Structure*

The configuration shown in Figure 40 will fail if and only if:

a) all units a, b, ..., n are failed in their short mode

b) either a or b, ..., or n is failed in their open mode

Figure 40. A Series Structure

By inspection, statement (a) corresponds to the connection of a number of two-state device networks in parallel. Likewise, the series structure shown in Figure 40 can be redrawn for the short mode failure case as shown in Figure 41.

Figure 41. A Short Circuit Parallel Configuration

*See Appendix D
By letting the short mode probability of failure for each component be \( q_{s1}, q_{s2}, \ldots, q_{sn} \) and the corresponding parametric values can be expressed as \( \phi_{s1}, \phi_{s2}, \ldots, \phi_{sn} \). From equation (4) (Chapter 4), the system probability of failure in the short mode:

\[
Q_S = q_{s1}q_{s2}\cdots q_{sn}
\]  

(7)

where

- \( n \) is the number of redundant components
- \( Q_S \) is the system short mode failure

In the parametric form of \( \Theta_S \):

\[
(1 + \cot \Theta_S) = \prod_{i=1}^{n} (1 + \cot \Theta_{si})
\]

Since \( \cot \Theta_S = 1/\phi_S \), then

\[
(1 + 1/\phi_S) = \prod_{i=1}^{n} (1 + 1/\phi_{si})
\]

(8)

If the three-state devices have a high \( p_s \) value, equation (8) can be approximated as:

\[
\phi_S = \frac{\phi_{s1}\phi_{s2}\cdots\phi_{sn}}{1+\phi_{s1}+\phi_{s2}+\cdots+\phi_{sn}}
\]

(9)

If further, the sum of the denominator approaches 1, then

\[
\phi_S = \prod_{i=1}^{n} \phi_{si}
\]

(10)

Similarly, the open failure state of these devices correspond to a series configuration consisting of only two-state devices. For this configuration case, the structure shown in Figure 40 may be redrawn
as shown in Figure 42.

![Diagram of an Open Mode Series Network](image)

**Figure 42. An Open Mode Series Network**

Suppose the open mode failure values of the component are $q_{o1}$, $q_{o2}$, ..., $q_{on}$ and their corresponding parametric values are $\phi_{o1}$, $\phi_{o2}$, ..., $\phi_{on}$.

Since from equation (6) (Chapter 4)

$$P_o = P_{o1}P_{o2}...P_{on}$$

and by substituting equation (3)

$$(1+\phi_0) = \prod_{i=1}^{n} (1+\phi_{oi})$$

(11)

If the three-state components have a high $P_o$, then equation (11) can be approximated as:

$$\phi_0 = \phi_{o1} + \phi_{o2} + ... + \phi_{on}$$

(12)

The total failure, $Q_T$, of the corresponding series circuit may be interpreted as follows:

$$Q_T = Q_o + Q_s$$

(13)

From equations (4) and (6) and the corresponding equations (10) and (12):

$$Q_T = \frac{\phi_0}{1+\phi_0} + \frac{\phi_s}{1+\phi_s}$$

(14)
Series system reliability, $R_p$

$$R_p = 1 - \left[ \frac{-\phi_0}{1+\phi_0} + \frac{\phi_s}{1+\phi_s} \right]$$

(15)

A Capsule Form of Parametric Theorem I for Series Structure

Product the short mode failure parametric parameters, i.e., $\phi_s\phi_2\ldots\phi_n$, for the system short failure case and sum the open mode failure parameters for the open failure case (see equations (9) and (12)).

6.1.2. Parallel Structure

The structure shown in Figure 43 will fail if:

a) either a or b ... n are shorted

b) all a, b and ... n are opened

![Diagram of a Parallel Structure](image-url)
In the case of statement (a), the parallel network will behave like a two-state device structure connected in series, as shown in Figure 44.

Figure 44. A Short Failure Series Structure

If the short mode failure values of each component are \( q_{s1} q_{s2} \ldots q_{sn} \), their corresponding parametric values are \( \phi_{s1}, \phi_{s2}, \ldots, \phi_{sn} \).

From equation (10) (Chapter 4):

\[
p_s = p_{s1} p_{s2} \ldots p_{sn}
\]

and by substituting equation (5):

\[
(1+\phi_s) = \prod_{i=1}^{n} (1+\phi_{si}) \tag{16}
\]

If the three-state components have a high \( p_s \), then equation (16) can be approximated as:

\[
\phi_s \approx \phi_{s1} + \phi_{s2} + \ldots + \phi_{sn} \tag{17}
\]

Similarly, in the case of statement (b), the parallel network will behave in the same manner as a two-state device structure connected in parallel as redrawn in Figure 45.
Figure 45. An Open Circuit Parallel Structure

Suppose the open mode failure values of the component are \( q_{01}, q_{02}, \ldots, q_{0n} \) and their corresponding parametric values are \( \phi_{01}, \phi_{02}, \ldots, \phi_{0n} \).

Since from equation (12) (Chapter 4)

\[
Q_{0} = q_{01}q_{02}\ldots q_{0n}
\]

In the form of \( \theta_{0} \):

\[
(1 + \cot\theta_{0}) = \prod_{i=1}^{n} (1 + \cot\theta_{0i}) \tag{18}
\]

since \( \cot\theta_{0} = 1/\phi_{0} \),

\[
(1 + 1/\phi_{0}) = \prod_{i=1}^{n} (1 + 1/\phi_{0i}) \tag{19}
\]

If the three-state components have a high reliability, then equation
(19) can be approximated as:

\[
\phi_0 = \frac{\phi_0\phi_{o2}...\phi_{on}}{1+\phi_0+\phi_{o2}+...+\phi_{on}}
\]  

(20)

If the sum of \(\phi_0 + \phi_{o2} + ... + \phi_{on}\) is very small, then:

\[
\phi_0 = \phi_0\phi_{o2}...\phi_{on}
\]  

(21)

The total failure, \(Q_T\), of the corresponding parallel circuit may be interpreted as follows:

\[
Q_T = Q_0 + Q_s
\]

From equations (6) and (4) and the corresponding equations (10) and (12):

\[
Q_T = \frac{\phi_0}{1+\phi_0} + \frac{\phi_s}{1+\phi_s}
\]  

(22)

\[
R_p = 1 - \left[ \frac{\phi_0}{1+\phi_0} + \frac{\phi_s}{1+\phi_s} \right]
\]  

(23)

A Capsule Form of Parametric Theorem II for Parallel Structure

In the case of a parallel structure, simply product the open failure parametric parameters, i.e., \(\phi_0\phi_{o2}...\phi_{on}\), for the system open mode failure and sum the short mode failure parameters, i.e., \(\phi_{s1}\phi_{s2}...\phi_{sn}\), for the short mode failure state (see equations (21) and (17)).

6.1.3. Complex Structure

To evaluate the reliability of a complex structure such as a bridge, the delta-star transformation is very easily applicable to
transform this configuration into series and parallel combinations. Delta to star equivalent formulas for open and short mode failures are derived in the following two sections.

6.1.3.1. **Short Failure Mode**

![Diagram](image)

*Figure 46. A Delta to Star Transformation I*

The equivalent delta to star transformation diagram is shown in Figure 46, where three components (or sub-systems) with short mode parameters $\phi_{sAC}$, $\phi_{sAB}$ and $\phi_{sCB}$ are connected in a delta configuration, are transformed to their equivalent star configuration with parameters $\phi_{sA}$, $\phi_{sB}$ and $\phi_{sC}$. The steps taken to obtain the equivalent star transformation are illustrated by Figure 47.

By using equations (10) and (17), the following three relations are obtained from Figure 47:
Figure 47. A Delta to Star Transformation II

\[ \phi_{sA}\phi_{sC} = \phi_{sAC} + \phi_{sCB}\phi_{sAB} \]  
\[ \phi_{sA}\phi_{sB} = \phi_{sAB} + \phi_{sAC}\phi_{sCB} \]  
\[ \phi_{sC}\phi_{sB} = \phi_{sCB} + \phi_{sAC}\phi_{sAB} \]
By solving these three simultaneous equations:

\[
\phi_{sA} = \sqrt{\frac{(\phi_{sAC}^+\phi_{sCB}^+\phi_{sAB})^2}{(\phi_{sCB}^+\phi_{sAC}^+\phi_{sAB})}} \tag{27}
\]

\[
\phi_{sB} = \sqrt{\frac{(\phi_{sAB}^+\phi_{sAC}^+\phi_{sCB})^2}{(\phi_{sAC}^+\phi_{sCB}^+\phi_{sAB})}} \tag{28}
\]

\[
\phi_{sC} = \sqrt{\frac{(\phi_{sAC}^+\phi_{sCB}^+\phi_{sAB})^2}{(\phi_{sAB}^+\phi_{sAC}^+\phi_{sCB})}} \tag{29}
\]

6.1.3.2. Open Mode Failure

Figure 48. A Delta to Star Transformation III

The open mode failure case, in a fashion similar to that of the short mode case, the delta-star configuration results in a form shown in Figure 48. The steps taken to obtain such an equivalent
star transformation are indicated in Figure 49.

\[ \text{Star} \]

\[ \text{Between points AB} \]

\[ \text{Between points AC} \]

\[ \text{Between points BC} \]

\[ \text{Delta} \]

\[ \text{Figure 49. A Delta to Star Transformation IV} \]
Using equations (12) and (21), we get the following three relations, from Figure 49:

\[
\phi_{oA} + \phi_{oC} = \frac{(\phi_{oCB} + \phi_{oAB})\phi_{oAC}}{1 + \phi_{oCB} + \phi_{oAB} + \phi_{oAC}} \tag{30}
\]

\[
\phi_{oA} + \phi_{oB} = \frac{(\phi_{oAC} + \phi_{oCB})\phi_{oAB}}{1 + \phi_{oAC} + \phi_{oCB} + \phi_{oAB}} \tag{31}
\]

\[
\phi_{oB} + \phi_{oC} = \frac{(\phi_{oAC} + \phi_{oAB})\phi_{oCB}}{1 + \phi_{oAC} + \phi_{oCB} + \phi_{oAB}} \tag{32}
\]

Solving the above simultaneous equations:

\[
\phi_{oA} = \frac{\phi_{oAC} \phi_{oAB}}{1 + \phi_{oAC} + \phi_{oCB} + \phi_{oAB}} \tag{33}
\]

\[
\phi_{oB} = \frac{\phi_{oCB} \phi_{oAB}}{1 + \phi_{oAC} + \phi_{oCB} + \phi_{oAB}} \tag{34}
\]

\[
\phi_{oC} = \frac{\phi_{oCB} \phi_{oAC}}{1 + \phi_{oAC} + \phi_{oCB} + \phi_{oAB}} \tag{35}
\]

As can be expected by looking at equations (27), (28), (29) and (33), (34), (35), it is difficult to compute the star parameter values.

However, it can also be seen by inspection that the short and open mode failure equations are interrelated. Once the computed values of the first equations of both short and open mode failures are available, the values of other equations can be computed in a very short time without the anticipation of any difficulty.
6.1.4. **Examples**

This section includes several hypothetically solved examples to validate the range of this parametric technique.

6.1.4.1. **Example 1**

Consider the following series structure:

![Series Network Diagram](image)

Figure 50. A Series Network

The reliability of the series structure shown in Figure 50 can be calculated from:

\[
R = 1 - \left[ \frac{\phi_0}{1 + \phi_0} + \frac{\phi_s}{1 + \phi_s} \right]
\]

where

\[
\phi_0 = \phi_{01} + \phi_{02}
\]

\[
\phi_s = \frac{\phi_{s1} + \phi_{s2}}{1 + \phi_{s1} + \phi_{s2}}
\]

The reliability obtained with the parametric technique is .9599 as opposed to .9595 with the classical method.
6.1.4.2. **Example 2**

Consider the following parallel structure:

![Parallel Structure Diagram](image)

**Figure 51. A Parallel Configuration**

The reliability of Figure 51 can be obtained from:

\[
R_p = 1 - \left[ \frac{\phi_0}{1 + \phi_0} + \frac{\phi_s}{1 + \phi_s} \right]
\]

where

\[
\phi_0 = \frac{\phi_01 \phi_02}{1 + \phi_01 + \phi_02}
\]

\[
\phi_s = \phi_{s1} + \phi_{s2}
\]

The reliabilities yielded by parametric and classical techniques are as follows:

\[
R_p = .9413, \quad R_c = .9405
\]

where

\[
R_p^c \quad \text{is the system reliability obtained by parametric means}
\]

\[
R_c \quad \text{is the system reliability obtained by classical means}
\]
6.1.4.3. Example 3

Consider the following parallel-series structure. This is a combination of the previous structures already discussed and evaluated.

![Diagram of parallel-series structure]

Figure 52. A Series-Parallel Structure

For Figure 52:

\[ R_p = 1 - \left[ \frac{\phi_0}{1+\phi_0} + \frac{\phi_s}{1+\phi_s} \right] \]

where

\[ \phi_s = \frac{\phi_{s1}\phi_{s2}}{1+\phi_{s1}+\phi_{s2}} + \frac{\phi_{s3}\phi_{s4}}{1+\phi_{s3}+\phi_{s4}} \]

\[ \phi_0 = \frac{(\phi_{01}+\phi_{02})(\phi_{03}+\phi_{04})}{1+(\phi_{01}+\phi_{02})+(\phi_{03}+\phi_{04})} \]

Since \( q_{01} = q_{02} = q_{03} = q_{04} = .02 \) and \( q_{s1} = q_{s2} = q_{s3} = q_{s4} = .03 \),
then \( R_p = .9978 \), \( R_c = .9966 \)

6.1.4.4. Example 4

Consider the following complex series-parallel structure as shown in Figure 53:

![Diagram of series-parallel structure]

(a)

(b)

**Figure 53. A Complex Series-Parallel Structure**

The reliability of this configuration can be obtained as follows:

\[
R_p = 1 - \left[ \frac{\phi_0}{1 + \phi_0} + \frac{\phi_s}{1 + \phi_s} \right]
\]

where

\[
\phi_0 = (\phi_{o3} \phi_{o4} + \phi_{o2}) \phi_{o1} + \phi_{o5}
\]

\[
\phi_s = (\phi_{s3} + \phi_{s4}) \phi_{s2} + \phi_{s1} \phi_{s5}
\]

If \( q_{o1} = .02 \), \( q_{o2} = .04 \), \( q_{o3} = .03 \), \( q_{o4} = .09 \), \( q_{o5} = .04 \) and \( q_{s1} = .03 \), \( q_{s2} = .07 \), \( q_{s3} = .1 \), \( q_{s4} = .06 \), \( q_{s5} = .08 \),
then $R_p = .956$, $R_c = .956$

6.1.4.5. Example 5

Consider the following bridge structure:

![Bridge Structure Diagram]

Figure 54. A Bridge Structure

Using equations (27), (28), (29), (33), (34) and (35), we get:

\[ \phi_{oA} = .009, \quad \phi_{ob} = .003, \quad \phi_{oC} = .001 \]
\[ \phi_{sA} = .183, \quad \phi_{SB} = .306, \quad \phi_{SC} = .251 \]

Now transform the bridge structure as shown in Figure 55. The final equations for the configuration can be written as follows:

Since

\[ R = 1 - \left[ \frac{\phi_o}{1 + \phi_o} + \frac{\phi_s}{1 + \phi_s} \right] \]
Figure 55: A Transformed Structure

where

\[ \phi_0 = (\phi_{0A} + \phi_{01})(\phi_{0B} + \phi_{02}) + \phi_{0C} \]

\[ \phi_s = (\phi_{sA}\phi_{s1} + \phi_{sB}\phi_{s2})\phi_{sC} \]

If \( q_{01} = .02, q_{02} = .08, q_{03} = .05, q_{04} = .02, q_{05} = .06 \) and \( q_{s1} = .03, q_{s2} = .04, q_{s3} = .05, q_{s4} = .04, q_{s5} = .07 \),

then \( R_p = .99, R_e = .99 \)

6.1.4.6. Example 6

This network is analysed by applying all the previous structures' gained knowledge. The structure shown in Figure 56 is reduced to the network shown in Figure 57 by applying the already established parametric theory.
Figure 56. A Complex Network

Since

\[ R_p = 1 - \left[ \frac{\phi_0}{1+\phi_0} + \frac{\phi_s}{1+\phi_s} \right] \]

The parametric and classical reliability values are in very close agreement with each other since \( R_p = .9978 \) and \( R_c = .9977 \).
Figure 57. A Reduced Complex Network

6.1.4.7. Reliability Calculation of the Networks with High Component Failure Values

Example 7

Consider the following series structure:

Figure 58. A Series Network with Low Component Reliability

Since $q_{o1} = .1$, $q_{o2} = .75$ and $q_{s1} = .2$, $q_{s2} = .1$. 
\[ R_p = 0.7578, \quad R_c = 0.763 \]

**Example 8**

Consider the following parallel structure:

![Parallel Structure Diagram](image)

Figure 59. A Parallel Structure with High Failure Mode Values

Since \( q_{o1} = 0.5 \), \( q_{o2} = 0.15 \) and \( q_{s1} = 0.1 \), \( q_{s2} = 0.2 \),

\[ R_p = 0.6604, \quad R_c = 0.645 \]

It may be observed for both series and parallel configurations that as long as the values of the failure modes of the components are widely separated, the parametric approximation becomes quite accurate even for high values of open and short failures. The same arguments are also applicable to the similar complex networks.
6.2. **Parametric Network Optimization**

This part presents parametric optimization of series and parallel networks as follows:

**Series Network**

From equation (15):

\[
R_p = 1 - \frac{\phi_{DT}}{1 + \phi_{DT}} - \frac{\phi_{ST}}{1 + \phi_{ST}}
\]

(36)

where

\[\phi_{DT} = n\phi_o, \quad \phi_{ST} = \phi_s^n\]

Equation (36) can be rewritten as:

\[
1 - \frac{\phi_s^n}{1 + n\phi_o} = \frac{\phi_s^n}{1 + \phi_s^n}
\]

(37)

Differentiating the above equation with respect to 'n' and setting it equal to zero yields:

\[-\phi_o (1 + \phi_s^n)^2 - (1 + n\phi_o)^2 (1 + \phi_s^n) \phi_s^n \log_e \phi_s + (\phi_s^n)^2 (1 + n\phi_o)^2 \log_e \phi_s = 0
\]

(38)

The value of 'n' from the above equation can only be approximated by several numerical iterations. These iterations were computerised. The end result was the same as that obtained from equation (32) of Chapter 4.

6.2.1. **Parallel Network**

From equation (23):

\[
R_p = 1 - \left[ \frac{\phi_{DT}}{1 + \phi_{DT}} + \frac{\phi_{ST}}{1 + \phi_{ST}} \right]
\]

(39)

where

\[\phi_{ST} = n\phi_s, \quad \phi_{DT} = \phi_o^n\]
Equation (39) can be rewritten as:

\[
\frac{1}{1+\phi_0^n} - \frac{n\phi_s}{1+n\phi_s} = 0
\]

(40)

Differentiating the above equation with respect to 'n' and setting it equal to zero yields:

\[-\phi_0^n(1+n\phi_s)^2 \log_e \phi_0 - (1+\phi_0^n)^2 (1+n\phi_s)^2 \phi_s + n\phi_s^2 (1+\phi_0^n)^2 = 0\]

(41)

Similar to the series case, the value of 'n' can be computed only by several numerical iterations. These iterations were programmed on the IBM 360/65. The final results were found to be the same as those obtained from equation (33) of Chapter 4.

Likewise, other networks such as series-parallel and parallel-series, although not presented here, were optimized by this parametric technique.

6.3. Observations

The parametric technique developed in this chapter was found to be much more simple than other conventional techniques. It also yielded the same results for both the reliability evaluation and optimization in the case of a system with fairly reliable (i.e., \(R \geq 0.8\)) three-state components.
CHAPTER 7

RELIABILITY EVALUATION WITH INCOMPLETE BETA DISTRIBUTION FUNCTION
AND GEOMETRIC DISTRIBUTION FUNCTION

This chapter consists of two reliability evaluation techniques developed during this study. The first part of this chapter presents the development of an evaluation technique for the application of the incomplete beta distribution function. A total of twelve three-state device networks are transformed to use the incomplete beta distribution function, after which the configuration reliability can be evaluated by using published incomplete beta distribution tables.

In a similar way, six (6) three-state device structures are transformed to their equivalent geometric distribution function. The system reliability of these networks can then be evaluated by the use of the computer generated geometric distribution plots or by computer generated tables.

7.1. RELIABILITY Evaluations by Means of the Incomplete Beta Distribution* Tables

The three-state device configuration formulas discussed above are expressed in terms of the incomplete beta distribution function for twelve different structures which are composed of various sub-systems and/or components. Each sub-system in itself may include one or more components. This technique provides a simple, useful and

*See Appendix E.1.
alternate way to evaluate the reliability of complex systems which include a large number of components and/or sub-systems where the published tables (26) of the incomplete beta distribution may be utilized.

Three-state device structures are transformed as follows:

7.1.1. **Series-Series Structure (SSS)**

In the series-series structure (SSS) the sub-system components as well as the sub-systems are linked in series as shown in Figure 60.

![Diagram of Series-Series Structure](image)

Figure 60. A Series-Series Structure

In the case of identical sub-systems:

\[
R_{SSS} = (1 - q_0)^{mn} - q_s^{mn}
\]  
\[
(1)
\]

where

- \(m\) is the number of components
- \(n\) is the number of sub-systems
- \(q_0\) is the component's open mode failure probability
- \(q_s\) is the component's short mode failure probability
- \(R_{SSS}\) is the system reliability

By adding and subtracting unity, the above equation can be rewritten in the following form:

\[
R_{SSS} = (1 - q_s^{mn}) - \{1-(1-q_0)^{mn}\}
\]  
\[
(2)
\]
In incomplete beta distribution form:

\[ R_{SSS} = \frac{\int_0^{P_s} t^{a-1}(1-t)^{mn-1} dt}{\int_0^{1} t^{a-1}(1-t)^{mn-1} dt} - \frac{\int_0^{q_s} t^{a-1}(1-t)^{mn-1} dt}{\int_0^{1} t^{a-1}(1-t)^{mn-1} dt} \]  

(3)

where

\[ P_s = 1 - q_s \]

\[ a = 1 \]

The above equation can be rewritten as:

\[ R_{SSS} = I_p(1, mn) - I_q(1, mn) \]  

(4)

where

\[ I_x(\cdot, \cdot) \]

is the incomplete beta distribution

From equation (4), the series system reliability can be readily evaluated by use of the incomplete distribution tables.

7.1.2. Parallel-Parallel System (PPS)

This configuration is composed of parallel sub-systems with their components also linked in parallel, as shown in Figure 61.

![Figure 61. A Parallel-Parallel System](image-url)
As can be readily seen, the

\[ R_{PPS} = (1 - q_s)^{mn} - q_o^{mn} \]  
(5)

By adding and subtracting 1 from equation (5):

\[ R_{PPS} = (1 - q_o^{mn}) - (1 - (1 - q_s)^{mn}) \]  
(6)

By putting equation (6) in incomplete beta distribution form:

\[ R_{PPS} = \frac{\int_0^P t^{a-1}(1-t)^{mn-1} dt}{\int_0^1 t^{a-1}(1-t)^{mn-1} dt} - \frac{\int_0^{q_s} t^{a-1}(1-t)^{mn-1} dt}{\int_0^1 t^{a-1}(1-t)^{mn-1} dt} \]  
(7)

where \( P_0 = 1 - q_o \)

The above equation can be rewritten as follows:

\[ R_{PPS} = I_p(1, mn) - I_q(1, mn) \]  
(8)

In a way similar to the series-series system, the parallel-parallel network reliability can be evaluated by use of the incomplete beta distribution tables.

Furthermore, the system reliability of the more complex networks which follow was obtained in a manner similar to that of the two former structures. The details of these derivations have been omitted.

7.1.3. Parallel-Series System (PSS)

Furthermore, the parallel-series case, the components are linked in series and the resulting sub-systems are connected in parallel as shown in Figure 62.
Figure 62: A Parallel-Series Structure

The system reliability which results for this case is:

$$R_{PSS} = [I_{P_s}(1,m)]^n - [I_{Q_o}(1,m)]^n \quad (9)$$

For the case of non-identical sub-systems:

$$R_{PSS} = \prod_{i=1}^{n} I_{P_{Si}}(1,m) - \prod_{i=1}^{n} I_{Q_{Oi}}(1,m) \quad (10)$$

7.1.4. Series-Parallel System (SPS)

For the series-parallel case the sub-systems consisting of parallel components are connected to form a series configuration as shown in Figure 63.

Figure 63: A Series-Parallel Structure
The system reliability which results for this case is:

\[ R_{SPS} = [I_p(1,m)]^n - [I_{qs}(1,m)]^n \]  \hspace{1cm} (11)

For the case of non-identical sub-systems:

\[ R_{SPS} = \prod_{i=1}^{n} I_{p_i}(1,m) - \prod_{i=1}^{n} I_{q_{si}}(1,m) \]  \hspace{1cm} (12)

7.1.5. Three-Dimensional System (TDS)

Here sub-systems with components linked in series are connected repeatedly in series as shown in Figure 64.

![Diagram of a three-dimensional system](image)

Figure 64. A Three-Dimensional Configuration
The system reliability which results for this case is:

\[ R_{TDS} = \left[ 1 - (1-m)^n \right]^k - \left[ 1 - (1-p_s)^n \right]^k \]  

(13)

At \( k = 1 \):

\[ R_{TDS} = [I_p(1,m)]^n - [I_q(1,m)]^n \]  

(14)

This is the same as the parallel-series system (PSS) reliability.

At \( k = 1, n = 1 \):

\[ R_{TDS} = I_p(1,m) - I_q(1,m) \]  

(15)

This is the same as the series system reliability.

At \( k = n, n = m, m = 1 \):

\[ R_{TDS} = (I_p(1,m))^n - [I_q(1,m)]^n \]  

(16)

Since:

\[ I_q(1,1) = q_o = 1 - p_o \]

\[ I_p(1,1) = p_s = 1 - q_s \]

\[ 1 - (1 - q_s)^m = I_{q_s}(1,m) \]

\[ 1 - (1 - p_o)^m = I_{p_o}(1,m) \]

The resulting system reliability is the same as that for the series-parallel structure (SPS).

For equation (16) with \( n = 1 \):

\[ R_{TDS} = I_p(1,m) - I_q(1,m) \]  

(17)

The resulting system reliability is the same as that for the parallel-parallel case.
7.1.6. **Bridge Structure (BS)**

For this case the system components are connected to form a bridge configuration as shown in Figure 65.

![Diagram of a bridge system](image)

**Figure 65. A Bridge System**

For the case of identical components, the system unreliabilities are:

\[
Q_s = 2I_{q_s}(5,1) - 5I_{q_s}(4,1) + 2I_{q_s}(3,1) + 2I_{q_s}(2,1) \tag{18a}
\]

\[
Q_o = 2I_{q_o}(5,1) - 5I_{q_o}(4,1) + 2I_{q_o}(3,1) + 2I_{q_o}(2,1) \tag{18b}
\]

\[
R_{BS} = 1 - [Q_s + Q_o] \tag{18}
\]

where

- \(Q_s\) is the system short mode unreliability.
- \(Q_o\) is the system open mode unreliability.
7.1.7. A Series Bridge Structure (SBS)

For a series bridge structure the sub-system components are connected to form a bridge configuration. The complete system is formed by connecting the sub-systems in series. The system reliability which results for this case is:

\[ R_{\text{SBS}} = \left[ 1 - (2I_{\text{o}} (5,1) - 5I_{\text{o}} (4,1) + 2I_{\text{o}} (3,1) + 2I_{\text{o}} (2,1)) \right]^n \]

\[ - (2I_{\text{s}} (5,1) - 5I_{\text{s}} (4,1) + 2I_{\text{s}} (3,1) + 2I_{\text{s}} (2,1))^n \] (19)

For the case of non-identical sub-systems:

\[ R_{\text{SBS}} = \prod_{i=1}^{n} \left[ 1 - (2I_{\text{o}} (5,1) - 5I_{\text{o}} (4,1) + 2I_{\text{o}} (3,1) + 2I_{\text{o}} (2,1)) \right] \]

\[ \prod_{i=1}^{n} (2I_{\text{s}} (5,1) - 5I_{\text{s}} (4,1) + 2I_{\text{s}} (3,1) + 2I_{\text{s}} (2,1)) \] (20)

7.1.8. Parallel-Bridge System (PBS)

For this case, the bridge sub-systems are joined to form a parallel configuration. The system reliability which results for this case from equations (18a) and (18b) is:

\[ R_{\text{PBS}} = (1 - q_s)^n - q_o^n \]

\[ R_{\text{PBS}} = \left[ 1 - (2I_{q_s} (5,1) - 5I_{q_s} (4,1) + 2I_{q_s} (3,1) + 2I_{q_s} (2,1)) \right]^n \]

\[ - (2I_{q_o} (5,1) - 5I_{q_o} (4,1) + 2I_{q_o} (3,1) + 2I_{q_o} (2,1))^n \] (21)

For the case of non-identical sub-systems:
\[ R_{PBS} = \prod_{i=1}^{n} (1-Q_{si}) - \prod_{i=1}^{n} Q_{oi} \]

7.1.9. Series Components k-out-of-n Sub-system (SKN)

The components of the sub-systems are classed in series with a k-out-of-n configuration as shown in Figure 66.

![Diagram of a series components k-out-of-n system]

Figure 66. A Series Components k-out-of-n System

The system reliability which results for this case is:

\[ R_{SKN} = I_{R_{SSS}}^{(k,n-k+1)} \]

(22)

where

\[ R_{SSS} = 1 - [I_{q_o}(1,m)+I_{q_s}(m,1)] \]

(23)

where

- \( k \) is the number of sub-systems.

7.1.10. k-out-of-n Sub-systems Components in Parallel (PKN).

Here the sub-systems with components in parallel are joined together to form a k-out-of-n network, as illustrated by Figure 67.
Figure 67. A Parallel Component k-out-of-n System

The system reliability which results for this case is:

\[ R_{PKN} = I_{R_{PPS}} (k,n-k+1) \]  \hspace{1cm} (24)

where

\[ R_{PPS} = I_{p_0} (1,m) - I_{q_s} (1,m) \]  \hspace{1cm} (25)

7.1.11: k-out-of-n Sub-systems with Parallel-Series Grouped Components (PSKN)

In a similar fashion, the sub-system components are linked in parallel-series together to form a k-out-of-n configuration.

The system reliability which results for this case is:

\[ R_{PSKN} = I_{R_{TDS}} (k,n-k+1) \]  \hspace{1cm} (26)
where at \( k = 1, n = \lambda \):

\[
R_{TDS} = \left[1 - q_s^m \right]^\lambda - \left[1 - (1 - q_o)^m \right]^\lambda
\]  

(27)

where

\( \lambda \) is the number of sub-systems

7.1.12. k-out-of-n Sub-systems for Series-Parallel Linked Components (SPKN)

For this case, the k-out-of-n networks are linked to form n series-parallel sub-systems.

The system reliability which results for this case is:

\[
R_{SPKN} = R_{SPS}^{(k,n-k+1)}
\]  

(28)

where

\[
R_{SPS} = \left[1 - \left\{I_{q_o}(m,1)\right\}^\lambda + \left\{I_{q_s}(1,m)\right\}^\lambda \right]
\]

Table I gives values of system reliability for several component failure probabilities in systems 7.1.1 to 7.1.12.

7.2 Reliability Evaluation Using the Geometric Distribution*

The main drawback of most other conventional techniques is that as the number of elements increases, the evaluation and computation of the system reliability becomes a very cumbersome process, even in the case of identical elements.

By use of the geometric distribution function table, as given by Table II and the plot in Figure 68, the reliability for transformed

*See Appendix E.2.
At given \( n = 4, \; m = 2, \; k = 2, \; \alpha = 4 \)

<table>
<thead>
<tr>
<th>( K = 2 )</th>
<th>( \alpha = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 4 )</td>
<td>( m = 2 )</td>
</tr>
</tbody>
</table>

**TABLE 1.** The Computed System Reliability for Two-Stage Structures
Figure 68: A Geometric Distribution Plot
network equations was found to be very easy to evaluate. The transformation of the series, parallel, series-parallel, parallel-series and the three-dimensional network was acquired as follows (one numerical example is included).

7.2.1. Series Structure

This structure is shown in Figure 69.

![Figure 69. A Series Structure](image)

System reliability \( R_S = (1-q_0)^n - q_S^n \) \hspace{1cm} (29)

Add and subtract 1 to equation (29):

\[ R_S = (1-q_S^n) - (1-(1-q_0)^n) \] \hspace{1cm} (30)

Since \( F_q(n) = \sum_{i=1}^{n} f(t) = 1-q^n \) where \( f(t) \) is the density function,

\[ R_S = \sum_{i=1}^{n} f_{q_S}(t) - \sum_{i=1}^{n} f_{p_0}(t) \] \hspace{1cm} (31)

\[ = F_{q_S}(n) - F_{p_0}(n) \] \hspace{1cm} (32)

where \( p_0 = 1-q_0 \). \( F_q(n) \) is the geometric distribution function.

The value of the above equation can be evaluated from Table II or Figure 68.
7.2.2. Parallel Structure

This network is shown in Figure 70.

\[ R_p = (1-q_s)^n - q_0^n \]  \hspace{1cm} (33)

Add and subtract 1 to equation (33):

\[ \therefore R_p = (1-q_0^n) - (1-(1-q_s)^n) \]  \hspace{1cm} (34)

As for the series case, the above equation can be rewritten as:

\[ R_p = \sum_{i=1}^{n} f_q(t) - \sum_{i=1}^{n} f_p(t) \]  \hspace{1cm} (35)

where

\[ P_S = 1-q_s \]

\[ = F_q(n) - F_p(n) \]  \hspace{1cm} (36)
The system reliability of equation (36) can be evaluated from Table I or Figure 68. Likewise, the three-state device networks which follow are transformed to the equivalent geometric distribution function as for the series and parallel cases.

7.2.3. **Series-Parallel Structure** (Figure 63)

For identical elements in parallel, the series-parallel system reliability can be expressed as:

\[ R_{SP} = (1-q_o^m)^n - (1-(1-q_s)^m)^n \]  \hspace{1cm} (37)
\[ = (F_{q_o}(m))^n - (F_{p_s}(m))^n \]  \hspace{1cm} (38)

Now, let \( s = F_{q_o}(m), \ D = F_{p_s}(m) \)

\[ R_{SP} = F_D(n) - F_S(n) \]  \hspace{1cm} (39)

In the case of 'n' non-identical units:

\[ R_{PS} = \prod_{i=1}^{n} F_{q_{o_i}}(m) - \prod_{i=1}^{n} F_{p_{s_i}}(m) \]  \hspace{1cm} (40)

7.2.4. **Parallel-Series Structure** (Figure 62)

With an identical series element, the parallel-series structure reliability, \( R_{PS} \):

\[ R_{PS} = (1-q_s^n)^m - (1-(1-q_o)^n)^m \]
\[ = (F_{q_s}(n))^m - (F_{p_o}(n))^m \]  \hspace{1cm} (41)

Now, let \( K = F_{q_s}(n), \ C = F_{p_o}(n) \)
\[ R_{PS} = F_C(m) - F_K(m) \]  \hspace{1cm} (42)

For non-identical series units 'm':

\[ R_{PS} = \prod_{i=1}^{m} F_{q_{si}}(n) - \prod_{i=1}^{m} F_{p_{oi}}(n) \]  \hspace{1cm} (43)

7.2.5. Three-Dimensional Structure

The structure is shown in Figure 64. The reliability of this three-dimensional structure for identical elements can be obtained from the following equation:

\[ R_{TD} = [1-(F_{p_{o}}(k))^m]^n - [1-(F_{p_{s}}(k))^m]^n \]  \hspace{1cm} (44)

This equation can be rewritten in the geometric distribution form as follows:

\[ R_{TD} = [1-(F_{p_{o}}(k))^n]^m - [1-(F_{p_{s}}(k))^n]^m \]  \hspace{1cm} (45)

By giving certain conditional values for \( k, m \) and \( n \) elements, the series, parallel, series-parallel and parallel-series structure reliability can be obtained as follows:

At \( k=n, n=1 \)

\[ R_{TD} = R_{PS} = (F_{q_{si}}(n))^m - (F_{p_{oi}}(n))^m \]

This is the same as the parallel-series system reliability.

At \( k=n, n=1, m=1 \)

\[ R_{TD} = R_{S} = F_{q_{si}}(n) - F_{p_{oi}}(n) \]
This is the same as the series structure reliability.

At \( k = 1 \)

\[
R_{TD} = R_{SP} = (F_{q_0}(m))^n - (F_{p_s}(m))^n
\]

This is the same as the series-parallel system reliability.

At \( k = 1, n = 1, m = n \)

\[
R_{TD} = R_F = F_{q_0}(n) - F_{p_s}(n)
\]

This is the same as the parallel structure reliability.

7.2.6. Example 1

Consider a parallel-series structure with the following data:

\( n = 2, m = 4, q_0 = 0.3, q_s = 0.2 \)

Evaluate the system reliability by the geometric distribution function means.

From equation (41) the system reliability which results for this case is:

\[
R_{PS} = F_{q_0}(m) - F_K(m)
\]

where

\[
K = F_{q_s}(n) = F_{2}(2) = 0.96 \text{ (from Figure 68)}
\]

\[
C = F_{p_0}(n) = F_{7}(2) = 0.51 \text{ (Ditto.)}
\]

for equation (42):

Parallel-series system reliability

\[
R_{PS} = F_{51}(4) - F_{96}(4)
\]

\[
= 0.93 - 0.15 = 0.78 \text{ (from Figure 68)}
\]
7.3. Observations

This author contends that the two techniques presented in this chapter are simpler and less time consuming while yielding very accurate end results than any method found in the literature.

Other complex networks, such as those discussed in section 7.2, can also be transformed to their equivalent geometric distribution function, and in a similar way to the Incomplete Beta Distribution Function.
CHAPTER 8

THE GENERALIZED FORTRAN IV PROGRAMS TO EVALUATE THE RELIABILITY
OF THREE-STATE DEVICE NETWORKS

Nowadays at major industrial or other establishments, the access to a digital computer is usually within the reach of every engineer. In view of this, this chapter is completely devoted to the development of generalized computer programs to evaluate the reliability of systems. This chapter is divided into two parts. In the first part, a generalized computer program is developed to evaluate reliability of any type of three-state device system network composed of identical components, paths and sub-systems. The second part deals with a more generalized computer program to evaluate the reliability of non-identical three-state device configurations.

8.1. A Generalized Computer Program to Evaluate Reliability of Networks Composed of Identical Three-State Components, Paths and Sub-Systems

A computer program for this case is shown in Figure 71. To assist in the evaluation of the reliability of three-state device configurations, the networks may be broken down into sub-systems as shown in Figure 72.

The data cards can be fed into the computer with the following information:

(i) the total number of sub-systems (i.e., N)
(ii) the number of components in each sub-system (i.e., A(I))
(iii) the open and short failure probabilities of the component (i.e., 00, QS)
Figure 71. A Generalized Computer Program for Identical Elements and Paths
Figure 72. A Three-State Device Complex Network

(iv) the configuration of the first sub-system which is
designated by 'S' (e.g., series configuration is
identified with 'S' equal to '1' and parallel
configuration with 'S' equal to '0')

In addition, the order of short and open mode probability
computations of the sub-systems is indicated as shown in Figure 72
(i.e., segments '1' to '5'). In a similar way, the number of com-
ponents of each sub-system can be listed in an array A(I).

Two pertinent points to be considered are:
(i) punch 'O' in the first and last locations of the array \(A(I)\) and add two to \(N\) if, and only if, the configuration of the first sub-system results in a parallel-arrangement.

(ii) punch 'O' in the last location of the array \(A(I)\) and add one to \(N\) if, and only if, the configuration of the first sub-system results in a series arrangement. As an example in Figure 72, the values of \(A(I)\) are listed in this order (i.e., 3, 3, 2, 2, 2, 0).

The overall reliability of the network shown in Figure 72 is evaluated for the given values of open and short mode unreliabilities of .2 and .3, respectively. The computer printout for this is shown in Figure 73. The resulting system reliability is enclosed with a square.

This author contends that this generalized computer program approach is more simple, straightforward, more efficient, less time consuming and minimizes the required amount of knowledge of reliability theory to evaluate the reliability of three-state device networks than other existing methods.

8.2. A Generalized Computer Program to Evaluate Reliability of Networks Composed of Non-Identical Three-State Devices

A generalized computer program for this more general case is shown in Figure 74. To assist in the evaluation of reliability of more generalized three-state device networks, the configurations are broken down into sub-systems as shown in Figure 75.
THE TOTAL NUMBER OF PARALLEL SERIES COMBINATIONS ARE 6
THE FIRST COMBINATION IS 1

THE NUMBER OF COMPONENTS FOR ASCENDING ORDER COMBINATION ARE 3

THE NUMBER OF COMPONENTS FOR ASCENDING ORDER COMBINATION ARE 3

THE NUMBER OF COMPONENTS FOR ASCENDING ORDER COMBINATION ARE 2

THE NUMBER OF COMPONENTS FOR ASCENDING ORDER COMBINATION ARE 2

THE NUMBER OF COMPONENTS FOR ASCENDING ORDER COMBINATION ARE 0

THE SHORT CIRCUIT UNRELIABILITY IS 0.30000
THE OPEN CIRCUIT UNRELIABILITY IS 0.20000

THE FICTICIOUS SHORT CIRCUIT UNRELIABILITY IS 0.00015

THE FICTICIOUS OPEN CIRCUIT UNRELIABILITY IS 0.09356

THE SYSTEM RELIABILITY IS 0.90624

Figure 73. The Computer Printout
Figure 74. A Generalized Computer Program for Non-identical Elements and Paths
Figure 74 (Cont.)

Figure 75. A Complex Three-State Device Network
The computer input data cards need to be fed into the computer with the following information:

(i) the total number of network components (i.e., NA)
(ii) the total number of network sub-systems (i.e., NJ)
(iii) the open and short failure probabilities of the components (i.e., ON(I), OS(I))
(iv) the type of network which is identified by 'IS'. In the case of series and parallel networks, the assigned value of 'IS' is '1'; otherwise, it is equal to '0'
(v) the 'IPS' also indicates the type of network configuration. In the case of a series-parallel network, the labelled value of 'IPS' is '2'; otherwise, it is equal to '0'
(vi) the number of components of each sub-system is stored in an array called F(I) in a cumulative order (i.e., 3, 5, 6 for Figure 75)

Furthermore, the order of short and open mode unreliability computations of the sub-systems is indicated as shown in Figure 75 (i.e., segments '1' to '3'). In this order, the values of steps (iii) and (vi) are punched on the input data cards. The system reliability of Figure 75 was computed for the following open and short failure probability values:

\[ q_{o_1} = .1, \ q_{o_2} = .2, \ q_{o_3} = .2, \ q_{o_4} = .2, \ q_{o_5} = .4, \ q_{o_6} = .1 \]
\[ q_{s_1} = .2, \ q_{s_2} = .3, \ q_{s_3} = .1, \ q_{s_4} = .2, \ q_{s_5} = .3, \ q_{s_6} = .2 \]
The computed system reliability result of .78104 was also verified by an accepted conventional approach.

This author contends that this generalised computer program is also simple, efficient and less time consuming for complex networks (within its limitations) than any other known method.

8.3. Observations

The only major difference between the generalised computer programs of sections 8.1 and 8.2 is that the former can handle only symmetrical networks, whereas the latter is an extension to non-symmetrical networks.
CHAPTER 9

RELIABILITY MODELING

In contrast to the preceding chapters, this chapter is devoted entirely to mathematical modeling rather than to reliability evaluation and optimization techniques. Again, this chapter is also divided into two parts. The first part is concerned only with the development of optimization equations for six complex structures of three-state devices. The main objective of part two is to develop reliability equations for mixed structures composed of both two- and three-state devices.

9.1. Reliability Optimization of Coherent System Models of Three-State Devices

The optimization of unconstrained models was studied with their resulting sub-systems connected in complex combinations of series and parallel configurations. But when the three-state sub-systems are not identical, the optimization of the system generally poses a very difficult problem. This particular problem is somewhat simplified in this study. Here, the optimal number of components in non-identical sub-systems are evaluated for a known number of sub-systems. Each sub-system is non-identical but has the same number of redundant components and/or paths. The specified sub-systems are connected in six different configurations. Their optimum path and component equations are established (22) as follows:
9.1.1. **Series-Series System**

Non-identical sub-systems with the same number of identical components

![Diagram of Series-Series Network](image)

Figure 76. A Series-Series Network

This system is composed of non-identical sub-systems connected in series as shown in Figure 76. Each sub-system considered has an identical number of components connected in a series configuration. The number of components connected in series is the same for each sub-system. If a particular sub-system fails in its open mode, then the whole system will fail. In contrast, if all sub-systems fail in a short mode, then the overall series-series system will fail catastrophically. The reliability equation of the series-series system can be written as follows for the non-identical sub-systems:

\[
R = \prod_{i=1}^{k} (1-q_{oi})^n - \prod_{i=1}^{k} q_{si}^n
\]

(1)

where

- \( k \) is the number of sub-systems
- \( n \) is the number of identical components in a sub-system
- \( q_{oi} \) is the open mode unreliability of a component
- \( q_{si} \) is the short mode unreliability of a component
For identical sub-systems:

\[ R = (1-q_o)^{kn} - q_s^{kn} \]  \hspace{1cm} (2)

To obtain the optimum number of redundant components \( n \) for a given number of non-identical sub-systems, differentiate equation (1) with respect to \( n \) and set it equal to zero. Therefore, the optimum value of \( n \) can be obtained from the following equation:

\[ n = \frac{\log_e \left[ \sum_{i=1}^{k} \log_e q_{si} \right] / \sum_{i=1}^{k} \log_e (1-q_{oi})}{\sum_{i=1}^{k} \log_e \left( \frac{1-q_{oi}}{q_{si}} \right)} \]  \hspace{1cm} (3)

For identical sub-systems, the above equation reduces to the following:

\[ n = \log_e \left[ \frac{\log_e q_s}{\log_e (1-q_o)} \right] / k \log_e \left( \frac{1-q_o}{q_s} \right) \]  \hspace{1cm} (4)

To obtain the optimum reliability, substitute the results of equations (3) and (4) in (1) and (2), respectively, for a given number of sub-systems.

9.1.2. Parallel-Parallel System

The system shown in Figure 77 is similar in characteristics to the system shown in Figure 76; however, the only major difference is that its components and sub-systems are connected in the parallel-parallel arrangement instead of the series-series configuration. The reliability equation of the parallel-parallel configuration can be
written as follows for non-identical sub-systems:

\[ R = \prod_{i=1}^{k} (1-q_{si})^n - \prod_{i=1}^{k} q_{0i}^n \]  

(5)

where

\[ n \]  

is the number of identical components

Thus, for identical sub-systems:

\[ R = (1-q_s)^{kn} - q_0^{kn} \]  

(6)

The optimum number of redundant components \( n \) can be obtained for the non-identical sub-systems by differentiating equation (5) with respect to \( n \) and then equating it to zero as follows:

\[ n = \frac{\log_e \left[ \frac{\sum_{i=1}^{k} \log_e q_{0i}}{\sum_{i=1}^{k} \log_e (1-q_{si})} \right]}{\sum_{i=1}^{k} \log_e \left( \frac{1-q_{si}}{q_{0i}} \right)} \]  

(7)
Hence, the above equation for identical sub-systems reduces to the following form:

\[
    n = \frac{\log_e \frac{q_0}{\log_e (1-q_s)}}{k \log_e \frac{1-q_s}{q_0}}
\]

(8)

The optimum parallel-parallel system reliability can be obtained by substituting the final results of equations (7) and (8) in equations (5) and (6), respectively, for the known number of sub-systems.

9.1.3. Series-Series-Parallel System

Non-identical sub-systems with the same number of identical components

Figure 78. A Series-Series-Parallel Network

This configuration consists of the non-identical series-parallel sub-systems connected in a series arrangement as shown in Figure 78. The number of components and paths are connected in a series-parallel configuration for each sub-system. For each sub-system, components and paths are identical. The sub-systems consist of the same number of components and paths.
In the case of the structure shown in Figure 78, all of the sub-systems must fail in the short mode or any one of the specified sub-systems must stop functioning in an open mode to cause the system to fail completely. The system reliability formula can be written as follows:

\[ R = \prod_{i=1}^{k} (1-q_{oi}^m)^n - \prod_{i=1}^{k} (1-(1-q_{si})^m)^n \]  

(9)

where

\[ m \] and \[ n \] are identical components and paths, respectively.

In the case of identical sub-systems:

\[ R = (1-q_o^m)^kn - (1-(1-q_s)^m)^kn \]  

(10)

The reliability expression of equation (9) can be optimized over \[ m \] and \[ n \] by setting the partial derivatives, with respect to \[ m \] and \[ n \], equal to zero and then solving for \[ m \] and \[ n \]. The resulting equations for \[ n \] and \[ m \] are:

\[ n = \frac{\sum_{i=1}^{k} \log_e (1-(1-q_{si})^m)}{\sum_{i=1}^{k} \log_e 1-(1-q_{si})^m} \]  

(11)

and for \[ m \]:

\[ \log_e \left[ \frac{\sum_{i=1}^{k} \log_e (1-(1-q_{si})^m)}{\sum_{i=1}^{k} \log_e (1-q_{oi}^m)} \right] - \log_e \left[ \frac{\sum_{i=1}^{k} \log_e (1-q_{si}^m) \log_e 1-(1-q_{si})^m}{\sum_{i=1}^{k} q_{oi}^m \log_e q_{oi} (1-q_{oi}^m)} \right] = 0 \]  

(12)
As can be seen from the above equation, the value of \( m \) can only be approximated for a given value of \( k \). To obtain optimum reliability, the end results of equations (11) and (12) are to be substituted in expression (9).

9.1.4. Parallel-Parallel-Series Structure

![Diagram of Parallel-Parallel-Series Structure]

Figure 79. A Parallel-Parallel-Series Structure

The configuration shown in Figure 79 consists of non-identical parallel-series sub-systems arranged in a parallel form. In the case of this arrangement, if any one sub-system fails in a short mode, it will cause an overall system failure, whereas all sub-systems of the system must malfunction in an open mode to induce total system failure.

The system reliability of this type of configuration can be obtained from the following formula:
\[ R = \prod_{i=1}^{k} (1-q_{si}^n)^m - \prod_{i=1}^{k} (1-(1-q_{oi})^n)^m \quad (13) \]

For identical sub-systems, the above equation is reduced to:

\[ R = (1-q_s^n)^{km} - ((1-q_o)^n)^{km} \quad (14) \]

The optimum equations for \( m \) and \( n \) can be written from equation (13), respectively:

\[ m = \frac{\sum_{i=1}^{k} \log_e(1-(1-q_{oi})^n)}{\sum_{i=1}^{k} \log_e(1-q_{si}^n)} \quad (15) \]

and for \( n \):

\[ \log_e \left[ \frac{\sum_{i=1}^{k} \frac{(1-q_{oi})^n \log_e (1-q_{oi})}{1-(1-q_{oi})^n}}{\sum_{i=1}^{k} \frac{q_{si}^n \log_e q_{si}}{1-q_{si}^n}} \right] - \log_e \left[ \frac{\sum_{i=1}^{k} \log_e(1-(1-q_{oi})^n)}{\sum_{i=1}^{k} \log_e(1-q_{si}^n)} \right] = 0 \quad (16) \]

According to the above equations, for a given value of \( k \), the values of \( m \) and \( n \) can only be approximated. To obtain optimum system reliability, substitute the optimum values for \( m \) and \( n \) in equation (13).

9.1.5. **Series-Parallel-Parallel Structure**

The configuration shown in Figure 80 has \( m \) identical components connected in parallel which constitute a sub-system. Each non-identical
Figure 80. A Series-Parallel-Parallel Structure

sub-system is connected in a parallel arrangement to form a path, whereas all the identical paths are connected in a series configuration.

The system will experience an overall system failure if any one of the paths fails in an open mode or all the identical paths fail in a short mode. The resulting reliability equation of the system is as follows:

\[
R = \left(1 - \prod_{i=1}^{k} q_{oi}^m\right)^n - \left(1 - \prod_{i=1}^{k} (1-q_{si})^m\right)^n
\]

(17)

where

m is the number of identical components

n is the number of identical paths

k is the number of non-identical sub-systems

In the case of identical sub-systems:
\[ R = (1-q_o)^n - (1-(1-q_s)^m)^n \]  
(18)

The optimum equations for \( n \) and \( m \) from equation (17) can be written, respectively:

\[
n = \frac{\log_e \left[ \prod_{i=1}^{k} (1-q_{s_i})^m \right]}{\log_e \left( \prod_{i=1}^{k} q_{oi}^m \right)}
\]

(19)

and for \( m \):

\[
m = \frac{\log_e \left[ \prod_{i=1}^{k} (1-q_{s_i})^m \right]}{\log_e \left( \prod_{i=1}^{k} q_{oi}^m \right)}
\]

(20)

According to the above equations, the values of \( n \) and \( m \) can be approximated provided the number of sub-systems is known. To obtain the optimum system reliability, substitute the optimum values of \( m \) and \( n \) in equation (17).
9.1.6. Parallel-Series-Series Structure

Non-identical sub-systems with the same number of identical components

Figure 81. A Parallel-Series-Series Structure

The structure shown in Figure 81 has \( n \) identical components arranged in series to form a sub-system. The non-identical sub-systems with the same number of components are connected in a series form. The paths constituted by the sub-systems arranged in a series configuration are connected in parallel to form a complete system. The system will experience a total failure only if any path composed of sub-systems connected in series fails in a short mode or if all paths experience an open mode failure. The overall system reliability equation can be obtained from the following equation for non-identical sub-systems:

\[
R = (1 - \prod_{i=1}^{k} q_{si}^n)^m - (1 - \prod_{i=1}^{k} (1-q_{oi})^n)^m \quad (21)
\]

For identical sub-systems:

\[
R = (1-q_s^{kn})^m - (1-(1-q_o)^{kn})^m \quad (22)
\]
Equation (21) yields the following optimum equations for \( m \) and \( n \), respectively:

\[
m = \frac{\log_e \left( \prod_{i=1}^{k} (1-q_{oi})^n \right)}{\log_e \left( \prod_{i=1}^{k} q_{si}^n \right)}
\]

\[
= \frac{1 - \prod_{i=1}^{k} q_{si}^n}{1 - \prod_{i=1}^{k} (1-q_{oi})^n}
\]

Equations (23) and (24) can only be approximated for a known number of identical or non-identical sub-systems. To obtain optimum system reliability, substitute the end results of equations (23) and (24) in (21).

The existence of maximum system reliability for all six reliability models was tested by programming the digital 360/65 computer.
9.2. Reliability Modeling of Mixed Two- and Three-State Device Networks

The system reliability evaluation of general networks consisting of two-state devices has been described by several authors. Similarly, as in the case of three-state device networks, their system reliability formulas are rather well established. However, none of these authors have yet established system reliability formulas of the general configurations made up of both two- and three-state devices. These networks exist quite commonly in practice. To help fill this void and need, formulas for series, parallel, series-parallel and parallel-series configurations have been developed and are presented in this section.

9.2.1. Series Network

Figure 82. A Mixed Two- and Three-State Device Series Network

A series network configuration is shown in Figure 82; but in this case some of the elements in the network can only fail in one state while the remaining ones can fail in either of two states. The resulting reliability equation for a network consisting of non-identical two- and three-state elements is as follows:

\[
R = \prod_{i=1}^{m} (1 - q_{o i})
\]  
(25)
where

- \( m \) is the total number of elements
- \( k \) is the number of two-state devices
- \( q_0 \) is the open mode failure probability of an element (the failure probability of a two-state device is also considered as an open mode failure probability)

As can be seen from equation (25), this relationship is the same as for a series network for two-state devices. It should be noted that in comparison to the series network of three-state devices, the overall reliability of this series system increases as two-state devices are added.

9.2.2. Parallel Network

![Diagram of a mixed two- and three-state device parallel network]

Figure 83. A Mixed Two- and Three-State Device Parallel Network

The system reliability of a parallel network as shown in Figure 83 for non-identical two- and three-state devices results in
equation (26):
\[ R = \prod_{i=1}^{m-k} (1-q_{Si}) - \prod_{i=1}^{m} q_{0i} \]  
(26)

where:
- \( m \) is the total number of components
- \( k \) is the number of two-state devices
- \( m-k \) is the number of three-state devices
- \( q_s \) is the short mode failure probability of an element
- \( m > k \) and \( m, k > 1 \)

It is obvious that for a three-state device network (i.e., \( k=0 \)), equation (26) will yield the following reliability equation:
\[ R = \prod_{i=1}^{m} (1-q_{Si}) - \prod_{i=1}^{m} q_{0i} \]  
(27)

In the case of only two-state devices, equation (26) (i.e., if \( k=m \)) becomes:
\[ R = 1 - \prod_{i=1}^{m} q_{Si} \]  
(28)

9.2.3. **Parallel-Series Network**

The system reliability equation of the parallel-series network (as shown in Figure 84) for \( n \) identical elements and \( m \) non-identical paths can be written as follows:
\[ R = \prod_{i=1}^{m-k} (1-q_{Si}^n) - \prod_{i=1}^{m} (1-(1-q_{0i}^n)) \]  
(29)
Figure 84. A Mixed Two- and Three-State Device Parallel-Series Network

where

- $m$ is the total number of series paths
- $k$ is the number of two-state device paths
- $m-k$ is the number of three-state device paths
- $n$ is the number of elements of each path

If $k=0$, the above equation will be reduced to an equation of a three-state device parallel-series network. Likewise, for $k=m$, equation (5) will yield the system reliability of a two-state device parallel-series configuration.

9.2.4. Series-Parallel Network

Figure 85. A Mixed Two- and Three-State Device Series-Parallel Network
where
\[ m \text{ is the total number of components} \]
\[ k \text{ is the number of two-state devices} \]
\[ m-k \text{ is the number of three-state devices} \]
\[ n \text{ is the number of identical mixed two- and three-state device sub-systems} \]

The series-parallel configuration is shown in Figure 85. In the case of \( n, m \) identical and non-identical, respectively, the system reliability becomes as follows:

\[
R = \left[ m-k \prod_{i=1}^{m} (1-q_{si}) \prod_{i=1}^{m} q_{oi} \right]^{n} \tag{30}
\]

9.2.5 Bridge Network

To evaluate the reliability of a mixed two- and three-state device network, the delta to star transformation may be used to transform the bridge network into a simple parallel and series relationship. After this is done, the next thing to do is to apply the simple set of conventional rules to evaluate the reliability of the transformed system.

9.3 Observations

It is felt by the author that the major contribution of the first part of this section is due to the development of the reliability optimization of systems with known non-identical sub-systems.

It is interesting to note that when a number of two- and three-state devices are connected to form a series configuration, the resulting reliability equation for the configuration remains the same as it
would for a two-state device series network, provided the two-state components have only open mode failures.
CHAPTER 10

RELIABILITY EVALUATION TECHNIQUES FOR TWO-STATE DEVICE NETWORKS

This chapter considers only reliability evaluation techniques for two-state device networks. The main objective of the first reliability evaluation technique presented was to develop analogous electrical networks to evaluate the system reliability (10).

The second and third techniques are parallel to the three-state device techniques developed in Chapters 5 and 7 (section 7.3), respectively, the delta to star transformation and geometric distribution technique.

10.1. Reliability Evaluations Using a Model Analogous to an Electrical Network

Presently there exist several methods to evaluate the reliability of redundant networks or systems. In this study, to make the two-state device reliability evaluation approach even more meaningful and useful, an analogous electrical model theory was developed and applied.

A two-state device is defined as one that either operates successfully or fails. The usefulness of the analogous network model stems from the fact that a system's unreliability can be measured by calculating an equivalent resistance or simply connecting an ohmmeter across an actual equivalent resistive circuit. In a similar way, the system reliability can be evaluated in terms of capacitance for the case of a capacitive configuration. Note, the unreliability can also
be obtained simply by applying the standard resistive and capacitive rules for electrical networks.

The electrical analogy (resistive and capacitive) for series, parallel, bridge and some other complex networks is developed in the section which follows.

10.1.1. Resistive Analogy

This analogy was developed for both parallel and series network models.

10.1.1.1. Series Network Model*

If any one component of the system fails, the network will experience an overall failure.

\[ \text{Series system unreliability} = 1 - \prod_{i=1}^{n} (1-q_{ci}) \]  \hspace{1cm} (1)

where
\[ q_c \] is the component's unreliability
\[ n \] is the number of components

Let the equivalent parametric component resistance

\[ R = \frac{q_c}{R_c} = \frac{q_c}{1-q_c} \]  \hspace{1cm} (2)

provided \( R_c \neq 0 \)

since \( R_c + q_c = 1 \),

where
\[ R_c \] is the component reliability,

*See Appendix F
\[ q_c = \frac{R}{1+R} \quad (3) \]

and

\[ R_c = \frac{1}{1+R} \quad (3a) \]

Therefore, if the unreliability of each series component is replaced with the equivalent parametric resistance, \( R \), for each component from equation (2), the series unreliability network can be transformed to an analogous series parametric resistive circuit, provided one ohm resistor, \( R_u \), is connected in parallel with this resistive circuit. The series network model will become an analogous resistive circuit as shown in Figure 86 (where component unreliability corresponds to component parametric resistance).

![Diagram of '1' (analogous parametric circuit)](image)

(a) (b)

**Figure 86. A Series Network Resistive Analogy**

By applying the standard resistive rules to segment '1' of Figure 86(b):

\[ \text{Total parametric resistance} = \sum_{i=1}^{n} R_i \quad (4) \]

Similarly, the total resistance of Figure 86(b) is:

\[ \frac{1}{Q_s} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{R_i} + \frac{1}{R_u} \]
the total circuit resistance, \( Q_c \), is:

\[
Q_c = \frac{\left( \sum_{i=1}^{n} R_i \right) R_u}{R_u + \sum_{i=1}^{n} R_i}
\]  

(5)

where

\[ R_u = 1 \]

In the case of very small values of \( R \), the above equation will yield the same results as equation (1), whereas equation (5) represents the total resistance of Figure 86(b). If the resistive units are ignored, it will also yield series network unreliability.

\[
\text{System reliability} = 1 - \frac{\left( \sum_{i=1}^{n} R_i \right) R_u}{R_u + \sum_{i=1}^{n} R_i}
\]

(6)

10.1.1.2. **Parallel Network Model**

This is a dual of the series case. The system will only fail if all the components malfunction.

\[
\text{System unreliability} = \prod_{i=1}^{n} Q_{ci}
\]

(7)

This parallel model can be transformed to the analogous resistive circuits as shown in Figures 87 and 88 (where component unreliabilities correspond to the analogous parametric circuit of segment '1').

Consider segment '1' of Figure 87(b):
Total parametric resistance, $R_T = \frac{R_1 R_2}{1 + R_1 + R_2}$  \hspace{1cm} (8)

If $R_1$ and $R_2$ are very small,

$R_T = R_1 R_2$ \hspace{1cm} (9)

Thus, the unreliability of Figure 87(b):

\[
\frac{1}{Q_s} = \frac{1}{R_T} + \frac{1}{R_u}, \quad Q_s = \frac{R_T R_u}{R_u + R_T} \hspace{1cm} (10)
\]

where

$R_u = 1\Omega$

Similarly, consider segment '1' of Figure 88(b):

\[
R_T = \frac{R_1 R_2 R_3}{1 + R_1 + R_2 + R_3} \hspace{1cm} (11)
\]
For small values of $R_1$, $R_2$ and $R_3$,

$$R_T = R_1R_2R_3$$

(12)

Unreliability $Q_s = \frac{R_T R_u}{R_u + R_T}$

(13)

From equations (8), (11) and (9), (12), the general formulas become:

$$R_T = \prod_{i=1}^{n} R_i$$

$$1 + \prod_{i=1}^{n} R_i$$

(14)

$$R_T = \prod_{i=1}^{n} R_i$$

(15)

Reliability $= 1 - \frac{R_T R_u}{R_u + R_T}$

(16)

10.1.2. Capacitive Analogy

This is developed for the series and parallel network models as follows.

10.1.2.1. Series Network Model

Let the equivalent parametric component capacitance

$$C = \frac{q_c}{R_c} = \frac{q_c}{1-q_c}$$

(17)

$$\therefore q_c = \frac{C}{1+C}$$

(18)
Similarly, as for resistive analogy, the unreliability of each series component is to be replaced with the equivalent parametric capacitance, 'c', from equation (17). Therefore, the series unreliability network is transformed to an analogous parallel parametric capacitive circuit as shown in segment '1' of Figure 89(b). However, if one microfarad capacitor is connected in series with the parametric parallel capacitive circuit, this circuit becomes an analogous capacitive one as shown in Figure 89 (where analogous component unreliability corresponds to component parametric capacitance).

![Diagram of a series network capacitive analogy](image)

Figure 89. A Series Network Capacitive Analogy

From segment '1' of Figure 89(b), the parametric capacitance,

\[ c_p = \sum_{i=1}^{n} c_i \]  

(19)

The total capacitance of Figure 89(b) from equation (20) is:

\[ \frac{1}{Q_s} = \frac{1}{\sum_{i=1}^{n} c_i} + \frac{1}{c_u} \]

\[ \therefore Q_s = \frac{\left( \sum_{i=1}^{n} c_i \right) c_u}{\sum_{i=1}^{n} c_i + c_u} \]  

(20)
where

\[ c_u = 1 \]

If the capacitive units are ignored, equation (20) also represents series network unreliability.

\[ \therefore \text{Network reliability} = 1 - Q_s \] (21)

10.1.2.2. Parallel Network Model

This network can be transformed to an analogous series capacitive circuit as shown in Figures 90 and 91 (where components' unreliabilities correspond to the analogous parametric circuits of segment '1').

![Figure 90. A Two-Element Parallel Network Analogy](image)

![Figure 91. A Three-Element Parallel Network Analogy](image)

The parametric capacitance associated with segment '1' of Figure 90(b) is (i.e., by applying the standard capacitive rules):
\[
\frac{1}{c_p} = \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_1 c_2} \\
\therefore c_p = \frac{c_1 c_2}{1 + c_1 + c_2} \quad (22)
\]

If \( c_1 \) and \( c_2 \) are very small:

\[
c_p = c_1 c_2 \quad (23)
\]

\[
\therefore \text{Total capacitance of Figure 90(b) is:}
\]

\[
Q_p = \frac{c_p c_u}{c_u + c_p} \quad (24)
\]

where

\[
c_u = 1
\]

Equation (24) will also yield unreliability of the network if the capacitive units are ignored.

\[
\therefore \text{Network unreliability} = 1 - Q_p \quad (25)
\]

Similarly, for segment '1' of Figure 91(b):

\[
\text{Parametric capacitance } c_p = \frac{c_1 c_2 c_3}{1 + c_1 + c_2 + c_3} \quad (26)
\]

For very small \( c_1, c_2, c_3 \):

\[
\therefore c_p \approx c_1 c_2 c_3 \quad (27)
\]

\[
\therefore \text{Total capacitance of Figure 91(b) is:}
\]

\[
Q_p = \frac{c_p c_u}{c_p + c_u} \quad (28)
\]
This also is the unreliability of the parallel network.

\[
\text{Reliability} = 1 - Q_p
\]  

Therefore, equations (22), (26) and (23), (27) can be written in the following general form:

\[
c_p = \frac{\sum_{i=1}^{n} c_i}{n + \sum_{i=1}^{n} c_i}
\]  

and

\[
c_p \approx \prod_{i=1}^{n} c_i
\]

10.1.2.3. Bridge Network Model

This network is shown in Figure 92.

![Bridge Network Diagram]

Figure 92. A Bridge Network

In this network transform the components' unreliability of the bridge to the parametric resistance \( R \) or the parametric capacittance \( C \). Then transform the parametric capacitive or resistive bridge to a series-parallel-series one by using the following derived parametric delta to star formulas.
The following formulas for a parametric capacitive bridge are:

\[ C_A = \frac{C_{AC}C_{AB}}{1 + C_{AC} + C_{AB} + C_{BC}} \]  \hspace{1cm} (32)

\[ C_B = \frac{C_{BC}C_{AB}}{1 + C_{BC} + C_{AB} + C_{AC}} \]  \hspace{1cm} (33)

\[ C_C = \frac{C_{BC}C_{AC}}{1 + C_{AC} + C_{BC} + C_{AB}} \]  \hspace{1cm} (34)

In the case of a parametric resistive bridge, the delta to star equations will be the same as for the parametric capacitive network. However, C should be replaced with R.

Example 1: Assume that the bridge network shown in Figure 92 has the following component unreliabilities:

\[ q_{C_1} = .14, \quad q_{C_3} = .1, \quad q_{C_5} = .12, \quad q_{C_2} = .09, \quad q_{C_4} = .06 \]

Calculate the bridge reliability through resistive analogy.

Bridge Network Rules:

(i) transform the reliability of each element of the bridge to the analogous parametric resistance by using equation (2);

(ii) transform the parametric resistive bridge to the series-parallel-series resistive parametric network as shown in Figure 93 by using equations (32) through (34);

(iii) transform series segments '1' and '2' of Figure 93 from segment '1' of Figure 86;
(iv) transform parallel segment '3' of Figure 93 to the corresponding resistive circuit of segment '1' of Figure 87(b);

(v) as for step (iii), replace $R_c$ with a resistor of the same value in segment '4' of Figure 93;

(vi) connect a resistor $R_u$ of value one ohm across segment '4' of Figure 93;

(vii) now apply standard rules to calculate the total resistance of the resistive circuit shown in Figure 94;

(viii) subtract the final result of step (vii) from unity and ignore its units.

\[
\frac{R}{(R_A+R_2)(R_B+R_5)}
\]

\[R_u = 1\Omega\]

Figure 94. An Analogous Resistive Circuit

**Example Solution:** According to the above steps, the bridge network model reliability is .97895 which is verified by
the classical method (.9790).

Example 2 (A Complex Network): This network is shown in Figure 95.

![Figure 95. A Complex Network](image)

Steps to Follow

(i) transform the unreliability of each element of the network shown in Figure 95 to the analogous parametric capacitance by using equation (17);

(ii) transform parallel segment '1' of Figure 95 from equation (20) or from segment '1' of Figure 90(b);

(iii) transform series segment '2' of Figure 95 from equation (19) or from segment '1' of Figure 89(b);

(iv) transform parallel segment '3' of Figure 95 from equation (22) or from segment '1' of Figure 90(b);

(v) transform series segment '4' of Figure 95 from equation (19) or from segment '1' of Figure 89(b);
(vi) connect one microfarad capacitor, $c_u$, in series with the transformed capacitive circuit (i.e., the steps performed from (i) to (v));

(vii) calculate the total capacitance of the transformed network by applying the standard capacitance evaluation rules. Subtract the total capacitance from unity to obtain network reliability. Ignore the capacitive units.

If the elements of Figure 95 have the following unreliabilities, then transform this network to an analogous capacitive circuit and calculate the system reliability.

$$ q_{c_1} = 0.14, \quad q_{c_2} = 0.07, \quad q_{c_3} = 0.08, \quad q_{c_4} = 0.05, \quad q_{c_5} = 0.02 $$

Example Solution: The analogous capacitive circuit as shown in Figure 96 is transformed according to the stated steps. System reliability yielded by the capacitive circuit of $0.9699$ is verified by the conventional method.

![Figure 96. A Capacitive Circuit](image-url)
10.2. Delta to Star Transformation

To evaluate the reliability of a bridge or other such complex structures, the existing theories are applicable but usually are far more cumbersome to use. For example, in the system reliability evaluation of a two-state device bridge structure made up of sixteen elements, using the event space method involves calculating probabilities for their corresponding 65536 states.

To analyse a complex structure such as a bridge, the delta to star transformation is very easily applicable to transform the configuration into corresponding series and parallel combinations. The delta to star equivalent formulas were derived in the following way.

A typical delta to star reliability transformation is shown in Figure 97.

![Figure 97. A Delta to Star Reliability Equivalent](image-url)
From Figure 97 each delta to star equivalent reliability leg diagram is shown for its corresponding configuration in Figure 98.

![Diagram](image)

Figure 98. A Delta to Star Equivalent Arm Diagram

By applying the probability rule, assuming independent for components connected in parallel and series networks, the equivalent block diagrams for Figure 98 yields equations (37), (38) and (39).
10.2.1. **Series Network**

For a simple two-state device series configuration,

\[
\text{Series network reliability} = \prod_{i=1}^{n} R_i^n
\]

(35)

where

- \( n \) is the number of components
- \( R \) is the components' reliability

10.2.2. **Parallel Network**

For a simple two-state device parallel configuration,

\[
\text{Parallel network reliability} = 1 - \prod_{i=1}^{n} (1-R_i^n)
\]

(36)

\[
R_A R_B = 1 - (1-R_{AB})(1-R_{AC}R_{CB})
\]

(37)

\[
R_B R_C = 1 - (1-R_{CB})(1-R_{AC}R_{AB})
\]

(38)

\[
R_A R_C = 1 - (1-R_{AC})(1-R_{CB}R_{AB})
\]

(39)

From the simultaneous equations (37), (38) and (39), the system reliability resulting from the delta to star transformed equations are:

\[
R_A = \sqrt{\frac{[1-(1-R_{AC})(1-R_{CB}R_{AB})] [1-(1-R_{CB})(1-R_{AC}R_{AB})]}{[1-(1-R_{AB})(1-R_{AC}R_{CB})]}}
\]

(40)

\[
R_B = \sqrt{\frac{[1-(1-R_{AB})(1-R_{AC}R_{CB})] [1-(1-R_{CB})(1-R_{AC}R_{AB})]}{[1-(1-R_{AC})(1-R_{CB}R_{AB})]}}
\]

(41)

\[
R_C = \sqrt{\frac{[1-(1-R_{AC})(1-R_{CB}R_{AB})] [1-(1-R_{AB})(1-R_{AC}R_{CB})]}{[1-(1-R_{CB})(1-R_{AC}R_{AB})]}}
\]

(42)
Example 1: Consider the bridge structure shown in Figure 99.

![Bridge Structure Diagram]

Figure 99. A Two-State Device Bridge Structure

Note that the salient nodes of the delta configuration are marked A, B and C. From the equations (40), (41) and (42) the equivalent star values of $R_A$, $R_B$ and $R_C$ are acquired:

$$R_A = .9948$$
$$R_B = .9930$$
$$R_C = .9954$$

Now the network shown in Figure 99 is redrawn to its equivalent shown in Figure 100:
As a result, the reliability equation of the structure shown in Figure 100 is:

$$R_T = \left[1-(1-R_1 R_A)(1-R_2 R_B)\right] R_C$$

(43)

Hence, for the given component reliabilities, the total bridge reliability:

$$R_T = .987$$

By analysing the same structure with the event space method, the result also yields $R_T = .987$.

In conclusion, it is seen that equations (40), (41) and (42) are interrelated. Therefore, by computing the value of the first equation, the computing time required for the other two equations is minimized.
10.3. **Reliability Evaluation with Geometric Distribution Function**

There were found to be several other techniques available to evaluate the reliability for two-state device networks. The major drawback found for all these techniques is that as the number of elements tend to increase, the evaluation of the system reliability becomes a cumbersome process and the computation time rapidly increases, even in the case of identical elements.

By making use of the geometric distribution table as shown in Table II and the corresponding plot in Figure 68, the reliability of the transformed network equations is very easily evaluated. The transformation of a series, parallel, series-parallel, parallel-series and the three-dimensional network is presented in the following section.

10.3.1. **Series Structure**

The network for a series structure is shown in Figure 101.

![Figure 101. A Series Network](image)

The resulting reliability is given by equation (44):

\[
\text{Series system reliability} = R_s = \prod_{i=1}^{n} R_i
\]  

(44)

Equation (44) put in geometric distribution form becomes (i.e., for identical elements):
\[
R_s = 1 - \sum_{i=1}^{n} f_R(t) = 1 - F_R(n)
\] (45) (46)

The reliability value for equation (46) can be evaluated from the unreliability values obtained from Table II or Figure 68.

10.3.2. Parallel Structure

The block diagram shown in Figure 102 is the model for a parallel structure.

![Parallel Network Diagram]

**Figure 102. A Parallel Network**

The reliability resulting for this parallel structure (i.e., for identical units) is given by equation (47):

Parallel system reliability \( R_p = 1 - q^n \) (47)
Equation (47) transformed to geometric distribution form becomes equation (48):

$$R_p = \sum_{i=1}^{n} f_q(t)$$  

(48)

where

$q$ is the components' unreliability

Similarly, the reliability value obtained from equation (48) and the network reliability equations in the sections which follow can be evaluated by using Table II or Figure 68.

10.3.3. *Series-Parallel Structure*

The series-parallel structure is formed by placing the parallel sub-system structures in series as shown in Figure 103. The total system reliability equation was derived, resulting in equation (49) (for identical components).

![Diagram of Series-Parallel Structure]

*Figure 103. A Series-Parallel Structure*

The series-parallel structure reliability

$$R_{sp} = (1-q^n)^m$$  

(49)
Equation (49) can be rewritten as:

$$R_{SP} = [F_q(n)]^m$$

$$= 1 - F_{R_p}(m)$$

where

$q$ is the components' unreliability

$R_p$ is the parallel system reliability

In the case of non-identical units:

$$R_{SP} = \prod_{i=1}^{m} F_{q_i}(n)$$

10.3.4. Parallel-Series Configuration

The parallel-series configuration is made up of series substructures connected in parallel as shown in Figure 104.

![Diagram of a Parallel-Series Structure](image)

Figure 104. A Parallel-Series Structure

In the case of identical elements for the configuration shown by Figure 104:
\[ R_{PS} = 1 - (1 - R^n)^m \]  \hspace{1cm} (53)

If the system reliability of this arrangement is transformed by using the geometric distribution function it becomes:

\[ R_{PS} = F_{1 - R_S}(m) \]  \hspace{1cm} (54)

And for non-identical sub-structures:

\[ R_{PS} = \prod_{i=1}^{m} (1 - F_{R_i}(n)) \]  \hspace{1cm} (55)

10.3.5. Three-Dimensional Structure

Structures such as the series-parallel-series configuration shown in Figure 105 can be considered to spread over three dimensions. The main advantage of this concept is that it can be used to arrive at the reliability of the last four configurations presented. Therefore, this approach may be considered a generalised procedure for solving series and parallel structures.

The reliability equation for this structure with identical elements can be written in the form of equation (56):

\[ R_{TD} = (1 - (F_{R(n)})^m)^k \]  \hspace{1cm} (56)

Equation (56) can be designated as a general equation for this case. Therefore, the reliability of the series, parallel, series-parallel and parallel-series structures is obtained by knowing the values for \( m \), \( n \) and \( k \). This is discussed in some detail in the following sections.
10.3.5.1. **Parallel-Series Structure**

The parallel-series structure reliability is obtained by letting $k=1$ in equation (56):

$$R_{PS} = 1 - (F_R(n))^m = F_{(1-R_S)}(m)$$  \hspace{1cm} (57)

10.3.5.2. **Series Structure**

The series structure reliability is obtained by letting $k=1$, $m=1$ in equation (56):

$$R_S = 1 - F_R(n)$$  \hspace{1cm} (58)
10.3.5.3. **Series-Parallel Structure**

The series-parallel structure reliability is obtained by letting \( n=1, k=m, m=n, R=q \) in equation (56):

\[
R_{SP} = (1-q^n)^m = (F_q(n))^m = 1-F_p(m)
\]  

(59)

10.3.5.4. **Parallel Structure**

The parallel structure reliability is obtained by letting \( m=1 \) in equation (59):

\[
R_p = F_q(n)
\]  

(60)

10.3.6. **Bridge Structure**

In the case of a bridge structure, the network can be transformed into a series and parallel combination by using the delta to star transformation (see section 10.2). After this, it can be rewritten in the geometric distribution form.

10.4. **Observations**

This chapter presented three new developments on networks composed of two-state devices rather than of three-state devices as presented earlier. This author contends that his contribution in this chapter was in the development of techniques to solve complex two-state device networks which are simpler and more straightforward than those that exist in the literature.
CHAPTER 11

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER STUDY

11.1. Conclusions

This dissertation presented and explored the application of some new solution methods for solving the system reliability problem of complex configurations of multistate devices. In this study a total of eleven different methods were developed to analyze the reliabilities of multistate systems. In this study a set of unique mathematical formulas was developed to analyze mixed mode networks consisting of both two- and three-state devices.

Initially in this study the networks were analyzed as being Markov processes (28, 29, 35) but the complexity of this method of analysis suggested some other approach to certain types of three-state device networks. It was found that there were trade-offs, depending upon the system's complexity, in using a Markov modelling approach. Because of the difficulties encountered with Markov modelling, other mathematical modelling methods were used.

Three graphical modelling techniques (30, 31) resulted from this mathematical modelling of the various configurations of three-state devices. It was found that some of these graphical techniques were also applicable to configurations of two-state devices as well. In presenting these three graphical techniques, several examples were introduced to demonstrate their applicability to rather complex problems and their ease of solution.
A delta to star transformation method (32) for the analysis of bridge networks consisting of two- and/or three-state devices was developed during this study. This approach was found to yield an immediate solution to problems which tended to be too difficult to solve by any other means.

In this study a parametric technique was also developed to analyze three-state device networks. Although this is an approximation method, it offers a very rapid and accurate solution to networks having highly reliable non-identical three-state components. A new approach was taken in applying both the incomplete beta and geometric distribution functions to the transformation of three-state device networks. Such a transformation is extremely useful, since ready use can be made of their tabulated values.

Because of the accessibility of the computer to most practicing engineers, a computer program was developed for the reliability evaluation of complex networks of three-state devices. This computer program (36) is expected to be extremely useful to the practicing engineer who is analyzing certain complex systems where highly accurate reliabilities are required.

An optimization technique (33) for the evaluation of the system reliability of several complex networks consisting of non-identical three-state devices was developed. This approach also led to the development of some generalized formulas to evaluate the reliabilities of systems containing both two- and three-state devices. An electrical analog method (34) was also developed as a very simple approach to
evaluate a complex system's reliability. A system's reliability can either be evaluated analytically by means of this analog technique, or it can be measured by placing an instrument across an electrical network in which its electrical circuit is an analog model of the reliability problem.

Although any one method developed in this study may not be a panacea in itself collectively, the author contends that they make a substantial contribution to knowledge in the field of reliability engineering.

11.2. Suggestions for Further Work

During the course of this study, several areas were found that merited further work, but this was not feasible due to time limitations. The following is a list of recommendations for further work which can modify or expand the results of this research:

a) It would be highly desirable to develop a technique for the optimization of system reliabilities of complex networks composed of identical or non-identical three-state devices (which have a normal and an open and short failure modes) subject to constraints such as cost, weight and/or volume. Dynamic programming and integer programming approaches are suggested as means of attack for this class of problems.
b) Furthermore, it would be useful if a technique were developed for the optimization of system reliabilities of networks of identical and/or non-identical three-state devices (but with successful, and partial and catastrophic failure modes) subject to the same set of constraints given in (a) above—cost, weight and/or volume.

c) The structure of optimization of system reliabilities for complex networks (where analog and parametric representation is used) under constrained conditions of cost, weight and/or volume needs to be examined.
12. References


13. Appendix

A. Appendix A

A.1. A Three-State Device Markov Model

From the transitional diagram of Figure 1, the derivation of the normal, open and short mode probability equations are as follows.

A.1.1. Normal Mode Probability Equation

\[ P_0(t + \Delta t) = P_0(t)(1 - \lambda \Delta t)(1 - \mu \Delta t) \]

\[ \Rightarrow \lim_{\Delta t \to 0} \frac{P_0(t + \Delta t)}{\Delta t} = -P_0(t)\lambda - P_0(t)\mu \]

\[ \frac{dP_0(t)}{dt} = - (\lambda + \mu) P_0(t) \]

By taking the Laplace transform of this equation:

\[ sP_0(s) - P_0(0) = - (\lambda + \mu) P_0(s) \]

Since the initial conditions are

\[ P_0(0) = 1, \ P_1(0) = 0, \ P_2(0) = 0 \]

\[ P_0(s) = \frac{1}{s + (\lambda + \mu)} \]  \hspace{1cm} (1)

The inverse Laplace transform of equation (1) is simply:

\[ P_0(t) = e^{-(\lambda + \mu)t} \]  \hspace{1cm} (2)

A.1.2. Open Mode Failure Probability Equation

\[ P_1(t + \Delta t) = P_0(t)\lambda \Delta t + P_1(t)\Delta t \]
\[
\lim_{{\Delta t \to 0}} \frac{P_1(t+\Delta t)-P_1(t)}{\Delta t} = P_0(t)\lambda
\]

\[
\Rightarrow \quad \frac{dP_1(t)}{dt} = P_0(t)\lambda
\]  

(3)

By taking the Laplace transform of equation (3):

\[
sP_1(s) - P_1(0) = P_0(s)\lambda
\]

Since \( P_1(0) = 0 \),

\[
P_1(s) = \frac{P_0(s)\lambda}{s}
\]  

(4)

Substitute equation (1) in (4):

\[
\Rightarrow \quad P_1(s) = \frac{\lambda}{(s+\mu+\lambda)} \cdot \frac{1}{s}
\]

The inverse Laplace transform is:

\[
P_1(t) = \frac{\lambda}{\mu+\lambda} (1-e^{-(\lambda+\mu)t})
\]  

(5)

A.1.3. **Short Mode Failure Probability Equation**

\[
P_2(t+\Delta t) = P_2(t) \cdot 1. + P_0(t)\mu \Delta t
\]

\[
\lim_{{\Delta t \to 0}} \frac{P_2(t+\Delta t)-P_2(t)}{\Delta t} = P_0(t)\mu
\]

\[
\Rightarrow \quad \frac{dP_2(t)}{dt} = P_0(t)\mu
\]  

(6)

By taking the Laplace transform, equation (6) becomes:

\[
sP_2(s) - P_2(0) = P_0(s)\mu
\]
\[ P_2(s) = \frac{P_0(s)}{s} \quad (7) \]

By substituting equation (1) into (7) yields:

\[ P_2(s) = \frac{\mu}{(s+\mu+\lambda)} \cdot \frac{1}{s} \]

The inverse Laplace transform is:

\[ P_2(t) = \frac{\mu}{\mu+\lambda} \left( 1 - e^{-(\mu+\lambda)t} \right) \quad (8) \]

Similarly, the other derivations of Chapter 2 were developed.
B. Appendix B

Parallel-Series Network Markov Model

The parallel-series network equations were derived for the system reliability and system short mode of failure as presented below.

B.1. System Reliability

The differential equations associated with Figure 8 are:

\[ \frac{dP_{N_1 N_2}}{dt} = -8\lambda P_{N_1 N_2} \]  \hspace{1cm} (1)

\[ \frac{dP_{N_2}}{dt} = \lambda P_{N_1} - 6\lambda P_{N_2} \]  \hspace{1cm} (2)

\[ \frac{dP_{N_3}}{dt} = 2\lambda P_{N_2} - 4\lambda P_{N_3} \]  \hspace{1cm} (3)

\[ \frac{dP_{N_4}}{dt} = 3\lambda P_{N_3} - 2\lambda P_{N_4} \]  \hspace{1cm} (4)

subject to

\[ P_{N_1}(0) = 1, P_{N_2}(0) = P_{N_3}(0) = P_{N_4}(0) = 0 \]
By solving this system of differential equations:

\[ P_{N_1}(t) = e^{-8\lambda t} \]  
\[ P_{N_2}(t) = 4e^{-6\lambda t} - 4e^{-8\lambda t} \]  
\[ P_{N_3}(t) = \frac{18}{4} e^{-8\lambda t} - 9e^{-6\lambda t} - \frac{18}{4} e^{-4\lambda t} \]  
\[ P_{N_4}(t) = \frac{3}{2} e^{-2\lambda t} - \frac{3}{2} e^{-8\lambda t} + \frac{9}{2} e^{-6\lambda t} - \frac{9}{2} e^{-4\lambda t} \]

The addition of the above equations yields system reliability:

\[ R = \frac{3}{2} e^{-2\lambda t} - \frac{1}{2} e^{-6\lambda t} \]

B.2. System Short Mode Failure

The differential equation associated with Figure 9 is:

\[
\frac{dP_S(t)}{dt} = 4\lambda P_{N_2}(t) + 16\lambda P_{N_3}(t) + 12\lambda P_{N_4}(t)
\]

In the s-domain, equation (10) becomes:

\[ P_S(s) = \left[ \frac{4\lambda^2}{(s+8\lambda)(s+6\lambda)s} + \frac{32\lambda^3}{(s+8\lambda)(s+6\lambda)(s+4\lambda)s} \right. 
\]
\[ + \left. \frac{74\lambda^4}{(s+8\lambda)(s+6\lambda)(s+4\lambda)(s+2\lambda)s} \right] 
\]

The inverse transform of the above equation yields:

\[ P_S(t) = \frac{7}{16} e^{-8\lambda t} + \frac{1}{16} e^{-6\lambda t} + \frac{1}{8} e^{-4\lambda t} - \frac{3}{4} e^{-2\lambda t} \]
C. Appendix C

C.1. A Reparable Three-State Device Markov Model

From Figure 13, the derivation of normal, open and short mode probability equations follows and their developments are given below.

C.1.1. Normal Mode Probability Equation

\[
P_0(t+\Delta t) = (1-\lambda_1 \Delta t)(1-\lambda_2 \Delta t)P_0(t)+\mu_1P_1(t)\Delta t+\mu_2P_2(t)\Delta t
\]

\[
\lim_{\Delta t \to 0} \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -(\lambda_1+\lambda_2)P_0(t)+\mu_1P_1(t)+\mu_2P_2(t)
\]

\[
\frac{dP_0(t)}{dt} = -(\lambda_1+\lambda_2)P_0(t)+\mu_1P_1(t)+\mu_2P_2(t)
\]  \( \text{(1)} \)

From initial conditions:

\[
P_0(0) = 1, P_1(0) = 0, P_2(0) = 0
\]

The Laplace transform of equation \( \text{(1)} \) yields:

\[
sP_0(s)+\lambda_1\lambda_2P_0(s)-\mu_1P_1(s)-\mu_2P_2(s) = 1
\]  \( \text{(2)} \)

C.1.2. Open Mode Probability Equation

\[
P_1(t+\Delta t) = P_0(t)\Delta t\lambda_1 + (1-\mu_1 \Delta t)P_1(t)
\]

\[
\lim_{\Delta t \to 0} \frac{P_1(t+\Delta t) - P_1(t)}{\Delta t} = P_0(t)\lambda_1 - \mu_1P_1(t)
\]

\[
\frac{dP_1(t)}{dt} = P_0(t)\lambda_1 - \mu_1P_1(t)
\]  \( \text{(3)} \)

From known initial conditions:

\[
P_1(0) = 0
\]
Taking the Laplace transform of equation (3) yields:

\[ sP_1(s) + \mu_1 P_1(s) - P_0(s)\lambda_1 = 0 \]  

(4)

C.1.3. Short Mode Probability Equation

\[ P_2(t+\Delta t) = \lambda_2 \Delta t P_0(t) + (1-\mu_2 \Delta t) P_2(t) \]

\[ \lim_{\Delta t \to 0} \frac{P_2(t+\Delta t)-P_2(t)}{\Delta t} = \lambda_2 P_0(t) - \mu_2 P_2(t) \]

(5)

From the initial conditions:

\[ P_2(0) = 0 \]

The Laplace transform of equation (5) yields:

\[ sP_2(s) + \mu_2 P_2(s) - \lambda_2 P_0(s) = 0 \]

(6)

Equations (1), (4) and (6) can be rewritten as:

\[
\begin{align*}
(s+\lambda_1+\lambda_2) P_0(s) - \mu_1 P_1(s) - \mu_2 P_2(s) &= 1 \\
-\lambda_1 P_0(s) + (s+\mu_1) P_1(s) + 0 P_2(s) &= 0 \\
-\lambda_2 P_0(s) + 0 P_1(s) + (s+\mu_2) P_2(s) &= 0
\end{align*}
\]

The coefficients of the above simultaneous equation are:

\[
\begin{array}{ccc|c}
(s+\lambda_1+\lambda_2) & -\mu_1 & -\mu_2 & 1 \\
-\lambda_1 & (s+\mu_1) & 0 & 0 \\
-\lambda_2 & 0 & (s+\mu_2) & 0
\end{array}
\]
By Cramer's rule:

\[
\Delta = \begin{vmatrix}
    s + \lambda_1 + \lambda_2 & -\mu_1 & -\mu_2 \\
    -\lambda_1 & s + \mu_1 & 0 \\
    -\lambda_2 & 0 & s + \mu_2
\end{vmatrix}
\]

\[
\Delta = (s + \lambda_1 + \lambda_2)(s + \mu_1)(s + \mu_2) - \mu_1 \lambda_1 (s + \mu_2) - \mu_2 \lambda_2 (s + \mu_1)
\]

\[
\Delta_p_0(s) = \begin{vmatrix}
    1 & -\mu_1 & -\mu_2 \\
    0 & s + \mu_1 & 0 \\
    0 & 0 & s + \mu_2
\end{vmatrix} = (s + \mu_1)(s + \mu_2)
\]

\[
P_0(s) = \frac{\Delta_p_0(s)}{\Delta} = \frac{(s + \mu_1)(s + \mu_2)}{(s + \lambda_1 + \lambda_2)(s + \mu_1)(s + \mu_2) - \mu_1 \lambda_1 (s + \mu_2) - \mu_2 \lambda_2 (s + \mu_1)}
\] (7)

Similarly:

\[
P_1(s) = \frac{\lambda_1(s + \mu_2)}{\Delta}
\] (8)

\[
P_2(s) = \frac{\lambda_2(s + \mu_1)}{\Delta}
\] (9)

The solution for the roots of the denominator quadratic equation is as follows:

\[
s^3 + s^2 (\mu_1 + \mu_2 + \lambda_1 + \lambda_2) + s (\mu_1 \mu_2 + \lambda_1 \mu_2 + \lambda_2 \mu_1) = 0
\]
\[ k_1, k_2 = \frac{-\mu_1 \mu_2 \lambda_1 \lambda_2 \pm \sqrt{(\mu_1 + \mu_2 + \lambda_1 + \lambda_2)^2 - 4(\mu_1 \mu_2 + \lambda_1 \lambda_2)}}{2} \]

Equations (7), (8) and (9) can be rewritten as:

\[ p_0(s) = \frac{(s+\mu_1)(s+\mu_2)}{s(s-k_1)(s-k_2)} \quad (10) \]

\[ p_1(s) = \frac{-\lambda_1 (s+\mu_2)}{s(s-k_1)(s-k_2)} \quad (11) \]

\[ p_2(s) = \frac{\lambda_2 (s+\mu_1)}{s(s-k_1)(s-k_2)} \quad (12) \]

Equation (10) in partial fraction form becomes:

\[ \frac{(s+\mu_1)(s+\mu_2)}{s(s-k_1)(s-k_2)} = \frac{A_1}{s} + \frac{B_1}{s-k_1} + \frac{C_1}{s-k_2} \quad (13) \]

Similarly, equations (11) and (12) were written in partial fraction form:

\[ \frac{\lambda_1 (s+\mu_2)}{s(s-k_1)(s-k_2)} = \frac{A_2}{s} + \frac{B_2}{s-k_1} + \frac{C_2}{s-k_2} \quad (14) \]

\[ \frac{\lambda_2 (s+\mu_1)}{s(s-k_1)(s-k_2)} = \frac{A_3}{s} + \frac{B_3}{s-k_1} + \frac{C_3}{s-k_2} \quad (15) \]

Equations (13), (14) and (15) are rewritten in time domain as follows:

\[ p_1(t) = \frac{\lambda_1 \mu_2}{k_1 k_2} + \frac{\lambda_1 k_1 + \lambda_2}{k_1 (k_1 - k_2)} e^{k_1 t} - \frac{(\mu_2 + k_2) \lambda_1}{k_2 (k_1 - k_2)} e^{k_2 t} \quad (16) \]

\[ p_0(t) = \frac{\mu_1 \mu_2}{k_1 k_2} + \frac{(k_1 + \mu_1)(k_1 + \mu_2)}{k_1 (k_1 - k_2)} e^{k_1 t} - \frac{(k_2 + \mu_1)(k_2 + \mu_2)}{k_2 (k_1 - k_2)} e^{k_2 t} \quad (17) \]
\[ p_2(t) = \frac{\lambda_2 \mu_1}{k_1 k_2} + \frac{\lambda_1 k_1 + \lambda_2 \mu_1}{k_1(k_1-k_2)} e^{k_1 t} - \frac{(\mu_1 + k_2) k_2}{k_2(k_1-k_2)} e^{k_2 t} \] (18)

In a similar way, the other derivations of Chapter 3 were developed.
D. Appendix D

All appendix derivations are based upon the binomial expansion \((P+q)^n\). In the case of three-state device structures, the expansion is modified to \((P+q_0+q_s)^n\), where \(P\) is the component probability of success.

D.1. Series Structure

D.1.1. Reliability Derivation

Let \(n = 2\):

\[ (P+q_0+q_s)^2 = 1 \]

Thus:

\[ P^2 + 2Pq_0 + 2Pq_s + q_0^2 + 2q_0q_s + q_s^2 = 1 \]

Number of state combination for \((n=2) = 3^2 = 9\)

The state combination truthtable may be manipulated as follows:

<table>
<thead>
<tr>
<th></th>
<th>NN</th>
<th>NO</th>
<th>NS</th>
</tr>
</thead>
<tbody>
<tr>
<td>ON</td>
<td>O0</td>
<td>OS</td>
<td></td>
</tr>
<tr>
<td>SN</td>
<td>S0</td>
<td>SS</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE III**

where

- \(N\) is the normal mode of the device (success)
- \(O\) is the open mode failure state
- \(S\) is the short mode failure state

The reliability terms by inspection from Truthtable III =

\[ PP + Pq_s + Pq_s = P^2 + 2Pq_s \]

(by using known probability theorems)*

*These probability theorems are also applied to the system reliability and failure equation derivations to follow.
Since \( P = 1 - q_s - q_o \),
\[
R = (1 - q_o^2) - q_s^2 \quad (2)
\]

Now let \( n = 3 \):
\[
(P + q_o + q_s)^3 = 1
\]
Thus:
\[
p^3 + 3p^2q_o + 3p^2q_s + 3pq_oq_s + 3pq_s^2 + 6pq_oq_s + 3q_o^3 + 3q_o^2q_s + 3q_s^2q_o + q_s^3 = 1
\]

Number of state combinations for \((n=3) = 3^3 = 27\)
The new state combinations truth table is as follows:

<table>
<thead>
<tr>
<th>NNN</th>
<th>SSS</th>
<th>OOO</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOO</td>
<td>S00</td>
<td>OSS</td>
</tr>
<tr>
<td>OON</td>
<td>00S</td>
<td>SSO</td>
</tr>
<tr>
<td>NON</td>
<td>OS0</td>
<td>O0S</td>
</tr>
<tr>
<td>ONO</td>
<td>SOS</td>
<td>NNO</td>
</tr>
<tr>
<td>NSS</td>
<td>SNN</td>
<td>NSO</td>
</tr>
<tr>
<td>SSN</td>
<td>NNS</td>
<td>NOS</td>
</tr>
<tr>
<td>SNS</td>
<td>NSN</td>
<td>SNO</td>
</tr>
<tr>
<td>SON</td>
<td>OSN</td>
<td>ONS</td>
</tr>
</tbody>
</table>

**TABLE IV**

The reliability terms by inspection from Table IV =

\[
Pp^3 + p^2q_s q_o + p^2 q_s q_o + p^2 q_s + p^2 q_s + p^2 q_s + p^2 q_s
\]

\[
R = p^3 + 3p^2q_s^2 + 3p^2q_s^2
\]

\[
(3)
\]
By using the relationship of equation (1) in equation (3):

$$R = (1-q_o)^3 - q_s^3$$  \hspace{1cm} (4)

Obviously, from equations (2) and (4), the general form for the reliability of the series structure can be written as follows. For identical components:

$$R = (1-q_o)^n - q_s^n$$  \hspace{1cm} (5)

and in the case of non-identical components:

$$R = \prod_{i=1}^{n} (1-q_{o_i}) - \prod_{i=1}^{n} q_{s_i}$$  \hspace{1cm} (6)

D.1.2. Failure Derivations

At $n = 2$

From Table III open mode failure terms $= q_o P + P q_o + q_o q_s + q_s q_o + q_o q_s$

Since $P = 1 - q_o - q_s$,

$$Q_o = 1 - (1-q_o)^2$$  \hspace{1cm} (7)

and similarly for short mode failure:

$$Q_s = q_s^2$$  \hspace{1cm} (8)

At $n = 3$

According to Table IV, open mode failure terms

$$= 3Pq_o^2 + 3P^2q_o + 6Pq_oq_s + 3q_o^2q_s + 3q_s^2q_o + q_o^3$$  \hspace{1cm} (9)

By substituting for $P$ in equation (10), open mode failure
and in the same way for short mode failure:

\[ Q_s = q_s^3 \]  

(12)

In the case of identical components, according to equations (8) and (11), the general form of open mode system failure, \( Q_o \)

\[ = 1 - (1 - q_o)^n \]  

(13)

Similarly, for non-identical components:

\[ Q_o = 1 - \prod_{i=1}^{n} (1 - q_{o_i}) \]  

(14)

In the case of short mode system failure for identical components:

\[ Q_s = q_s^n \]  

(15)

and for the non-identical elements case:

\[ Q_s = \prod_{i=1}^{n} q_{s_i} \]  

(16)

D.2. Parallel Structure

D.2.1. Reliability Derivation

Let \( m = 2 \)

\[ (p + q_o + q_s)^2 = 1 \]

Thus:

\[ p^2 + 2pq_o + 2pq_s + q_o^2 + 2q_oq_s + q_s^2 = 1 \]
The reliability terms with inspection from Table III

\[ PP + q_0 P + Pq_0 = p^2 + 2Pq_0 \]  \hspace{1cm} (17)

Since \( P = 1 - q_s - q_o \),

\[ R = (1-q_o)^2 - q_s^2 \]  \hspace{1cm} (18)

At \( m = 3 \)

\[ (P + q_o + q_s)^3 = 1 \]

Thus:

\[ p^3 + 3p^2q_o + 3p^2q_s + 3pq_o^2 + 3pq_s^2 + 6pq_o q_s + q_o^3 + 3q_o^2q_s + 3q_o q_s^2 + q_s^3 = 1 \]

By inspecting Table IV, system reliability =

\[ p^3 + 3q_o^2 p + 3p^2q_o \]  \hspace{1cm} (19)

Replace \( P \) with equation (7); therefore:

\[ R = (1-q_s)^3 - q_o^3 \]  \hspace{1cm} (20)

With the aid of equations (18) and (20), the general system reliability formula for identical elements connected in the parallel configuration becomes as follows:

\[ R = (1-q_s)^m - q_o^m \]  \hspace{1cm} (21)

The above equations, for non-identical elements may be rewritten as:

\[ R = \prod_{i=1}^{m} (1-q_{s_i}) - \prod_{i=1}^{m} q_{o_i} \]  \hspace{1cm} (22)
D.2.2. Failure Derivations

For $m = 2$

Collected short mode failure terms from Table III

$$q_s^2 + q_s q_0 + q_s^2 q_s + q_o q_s + Pq_s = q_s^2 + 2Pq_s + 2q_o q_s$$

Thus:

$$q_s = 1 - (1 - q_s)^2$$  \hspace{1cm} (23)

Similarly, for open mode failure:

$$Q_o = q_o^2$$  \hspace{1cm} (24)

At $m = 3$

Short mode system failure terms from Table IV

$$q_s^3 + 3Pq_s^2 + 6Pq_o q_s + 3q_s q_o^2 + 2q_o q_o^2 + 3q_s q_o^2 + 3q_s^2 q_o$$

Thus:

$$Q_s = 1 - (1 - q_s)^3$$  \hspace{1cm} (25)

Similarly, in the case of open mode failure:

$$Q_o = q_o^3$$  \hspace{1cm} (26)

As seen from equations (23) and (25), the short mode failure equation for identical elements can be generalized as follows:

$$Q_s = 1 - (1 - q_s)^m$$  \hspace{1cm} (27)

Similarly, for the non-identical elements case:
\[ Q_s = 1 - \prod_{i=1}^{m} (1-q_{s_i}) \] (28)

The corresponding generalized open mode system failure:

(identical components) \[ Q_o = q_0^m \] (29)

and

(non-identical components) \[ Q_o = \prod_{i=1}^{m} q_{o_i} \] (30)

D.3. Series-Parallel Structure

D.3.1. Reliability

In the case of identical elements, by substituting equations (27) and (29) for open and short mode failure in equation (5), the total reliability, \( R \):

\[ R = (1-q_0^m)^n - (1-(1-q_s^m)^n) \] (31)

For \( n \) identical units, each containing \( m \) elements:

\[ R = \left( \prod_{i=1}^{m} q_{o_i} \right)^n - \left( \prod_{i=1}^{m} (1-q_{s_i}) \right)^n \] (32)

D.4. Parallel-Series Structure

The reliability equation for identical components and units, by substituting equations (13) and (14) in equation (21), is as follows:

\[ R = (1-q_s^n)^m - (1-(1-q_{o}^n)^m) \] (33)

for \( m \) identical units:

\[ R = \left( \prod_{i=1}^{n} q_{s_i} \right)^m - \left( \prod_{i=1}^{n} (1-q_{o_i}) \right)^m \] (34)
D.5. **General Three-Dimensional Structure**

D.5.1. **Failure Derivations** (for symmetrical units and elements)

In the case of an open mode system failure, substitute equation (13) in equation (29):

\[
Q_o = (1 - (1 - q_o)^n)^m
\]  

(35)

By substituting the above equation in (13):

\[
-Q_o = 1 - (1 - (1 - q_o)^k)^m
\]

(36)

where

\[k = n\]

Similarly, for the short mode failure, substituting equation (15) in equation (27) yields:

\[
Q_s = 1 - (1 - q_s^n)^m
\]  

(37)

By replacing \(q_s\) of equation (15) with equation (37):

\[
Q_s = (1 - (1 - q_s^k)^m)^n
\]

(38)

where

\[k = n\]

D.5.2. **Reliability Derivation** (identical elements and units only)

Adding equations (36) and (38) and subtracting from '1' yields reliability:

\[
R = \left[1 - (1 - q_o^k)^m\right]^n + \left[1 - (1 - q_s^k)^m\right]^n
\]
E. Appendix E

E.1. Incomplete Beta Function

\[ B_x(a,b) = \int_0^x t^{a-1} (1-t)^{b-1} dt \]

where

\( a \) and \( b \) are the integers
\( x \) is the probability of success

Incomplete Beta Function

The incomplete beta function is the ratio of an arbitrary beta function to the beta function evaluated as being between an interval of zero and unity.

\[ \frac{B_x(a,b)}{B(a,b)} = I_x(a,b) \]

E.2. Derivation of the Geometric Distribution Function

In the binomial distribution, the number of trials are specified in advance, and the random variable observed is the number of successes (or failures). In the case of negative binomial distribution, the opposite is done, e.g., the number of successes \( c \) is fixed; and the number of trials \( n \) are made until \( c \) successes occur.

As a result, the number of trials \( n \), until \( c \) successes, becomes the random variable. The probability distribution of the negative binomial can be written as:

\[ P(k) = \binom{n}{k} p^c (1-p)^{k} \]

where

\( k = n-c \) and \( k = 0,1,2,\ldots \)
When $c = 1$ yields the geometric distribution, which makes it a special case of the negative binomial distribution:

$$
\therefore \ P(n) = pq^{n-1} = q^{n-1}
$$

where

$k = n-1$ when $n = 1, 2, 3...$

The above distribution is characterized by the parameter $P$ since $q = 1 - P$.

The sum of an infinite series is as:

$$
\sum_{n=1}^{\infty} P(n) = \frac{p}{1-q} = \frac{P}{P} = 1
$$

Therefore, by the same token, the geometric distribution function can be written as:

$$
F(n) = \sum_{t=1}^{n} P(t) = 1 - q^n \text{ or } 1 - p^n
$$
Appendix F

F.1. Series Network Model

From equation (1) of Chapter 10:

\[
\text{System reliability} = \prod_{i=1}^{n} R_{ci} \tag{1}
\]

By substituting equation (3a) of Chapter 10 into equation (1):

\[
\frac{1}{1+R_s} = \frac{1}{\prod_{i=1}^{n} (1+R_i)}
\]

\[
1+R_s = \prod_{i=1}^{n} (1+R_i) \tag{2}
\]

where

\( R_s \) is the series system parametric resistance

From the above equation for small values of \( R_i \):

\[
R_s = \sum_{i=1}^{n} R_i \tag{3}
\]

F.2. Parallel Network Model

From equation (7) of Chapter 10:

\[
\text{System reliability} = \prod_{i=1}^{n} q_i \tag{4}
\]

By substituting equation (3) of Chapter 10 in equation (4):

\[
\frac{R_p}{1+R_p} = \prod_{i=1}^{n} \left( \frac{R_i}{1+R_i} \right)
\]
The above equation can be rewritten as:

\[ \frac{1}{1 + \frac{1}{R_p}} = \frac{1}{\prod_{i=1}^{n} \left(1 + \frac{1}{R_i}\right)} \]

\[ 1 + \frac{1}{R_p} = \prod_{i=1}^{n} \left(1 + \frac{1}{R_i}\right) \]

where \( R_p \) is the parallel network parametric resistance.

From equation (5):

\[ R_p = \frac{R_1 R_2 R_3 \ldots R_n}{1 + R_1 + R_2 + R_3 + \ldots + R_n} \]
1947 Born in Jhaj, Hoshiarpur, Punjab, India on August 5.


1968 Completed five year technical training and Polytechnic education to become an instrumentation electronic engineer in England.

1969 The candidate fulfilled the requirements of I.E.E. and I.E.R.E. (British) to become a chartered engineer (C.Eng.) and worked as an electrical engineer.

1972 Earned a B.Sc. in electrical and electronic engineering from the University of Wales, Swansea, U.K.

1973 From October 1972 to October 1973 the candidate was awarded the M.Sc. in Industrial and System Engineering (thesis degree) from the University of Wales, Swansea, U.K.

1975 Currently, a candidate for the degree of Doctor of Philosophy in Industrial Engineering at the University of Windsor.