THERMAL STRATIFICATION IN LAKES.

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Ottawa, Canada
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Thermal Stratification in Lakes

by

Samir Sami Girgis

A Dissertation
Submitted to the Faculty of Graduate Studies
Through the Department of Mathematics
in Partial Fulfillment of the
Requirements for the Degree
of Doctor of Philosophy at the
University of Windsor

Windsor, Ontario, Canada

1979
ABSTRACT

This dissertation is a study of thermal stratification in lakes and ponds. Exact analytical solutions have been found by a method of variation of parameters, for the vertical temperature distribution in a stagnant lake or tank, subject to surface heating, for a variety of boundary and initial conditions. The propagation of heat has been assumed to be due solely to molecular diffusion, convection and eddy diffusivity being neglected. The results obtained compare favourably with observations made in the laboratory and in nature. The sums of the series were obtained using a TI-59 programmable hand calculator. An analytical solution has been found suitable for deep lakes.

For solar lakes, exact analytical solutions have been found by a method of variation of parameters and construction of orthogonal expansion of functions over two-layered model for the case of perfect thermal contact between the layers. Similarly, a linear density-depth model has been considered and an exact analytical solution have been found by a method of variation of parameters.
Acknowledgements

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NOMENCLATURE

$\phi_0(t)$ = the net solar radiation after the reflection at the water surface (Cal/L$^2$.t).

$k$ = thermal conductivity (Cal/L - t - °C)

$E$ = eddy diffusivity (L$^2$/t)

$\alpha$ = thermal molecular diffusivity (L$^2$/t)

$q$ = velocity of the fluid (L/t)

$H(x,t)$ = rate of heat generated within the region/volume of the region, radiant power absorbed within the reservoir per unit length per unit area (Cal/tL$^3$)

$\beta$ = fraction of effective solar radiation considered absorbed at the water surface, parametric measure of long-wave (infrared) content of solar radiation, $\beta \neq 0.4$ (dimensionless)

$T(x,t)$ = temperature in (°C)

$u(x,t)$ = temperature in (°C)

$\eta$ = average solar radiation absorption coefficient of the fluid medium (L/L)

$\rho$ = density of the fluid medium at constant pressure (m/L$^3$)

$C$ = specific heat of the fluid medium (Cal/gm. °C)

$x$ = distance measured positive downward from water surface (L)

$t$ = time (t)

$J_n(x)$ = Bessel function of the first kind of order n

$\phi_L(t)$ = total rate at which heat is lost to the atmosphere
Chapter I

Introduction

Lakes can be divided into two types, dimictic or pure water and heliothermal or solar. The dimictic or temperate lakes [1], have water surface temperatures above 4°C in summer and below 4°C in winter. Most lakes in Western North America north of about 40° latitude are dimictic. Water temperatures in these lakes are transformed from an isothermal distribution in spring to a distribution containing a well-defined thermocline in the summer and return to an isothermal condition in late fall. Heat transfer through the water surface is essential for the formation of temperature stratification in a dimictic lake.

Heat energy input to the lake is due to:

(1) Net long wave (mostly atmospheric) radiation which is absorbed completely at the water surface;

(2) Net short wave (mostly solar) radiation which is absorbed exponentially with depth with a specified attenuation coefficient;

(3) Losses (or gains) at the water surface by convection, evaporation (condensation) and back radiation.

Radiation can be either calculated or measured. Total incoming radiation is reduced by reflection at the water surface. The remaining radiation penetrating the water is absorbed partially near the water surface and partially within the lake. Figure (1) is an example of the thermal stratification in dimictic lakes.

A recent analytical model was explored by Duke and Harleman [2],
who solved an initial value problem by a superposition method. Conduction
was held responsible for the temperature distribution in the deeper
portions of the lake. The effect of wind was ignored in their analysis
and in this dissertation. However, on the other hand, Henderson and
Sellers [3], suggested that the thermal structure of a fresh water lake
is determined primarily by the wind shear, which induces turbulence in
the lake. They found that in the case of a lake shallower than about
20 m, the stratification is slight or nonexistent.

In Chapter II we will study the thermal stratification in dimictic
lakes. We will use general form of Harleman's model. In Chapter III we
will study deep dimictic lakes:

The second type of lake is the heliothermal or solar lake. Solar
lakes are those with a hot water lens produced by absorption of solar
energy, and occur in all latitudes [4]. They are produced when a low
density layer overlies a higher density, brine layer. The density
difference requirement for these lakes is 0.015 gm/cm$^3$ or greater [4].
The natural brines capable of such high densities contain sulfates or
chlorides brines. The lower density layer acts as a one-way mirror to
solar radiation, and prevents convection of heated lower brine to the
surface, thus limiting evaporative cooling.

A two-layered model of the solar lakes will be considered in
Chapter IV and a linear density-depth model will be considered in
Chapter V.

The problems of energy transfer in stagnant waters have been
comprehensively surveyed by Dake and Harleman [5] and by Ou et alii [6].
The principal natural heat source considered is the sun, whose ultraviolet and infrared radiation are largely absorbed within the first few centimetres of the surface. The visible radiation, on the other hand, penetrates more deeply, carrying significant energy to depths of the order of tens of metres thereby causing vertical variations in density.

Snider and Viskanta [7], have suggested that an analytical solution in closed form is not possible, and used a finite difference method to obtain numerical solutions. Dake and Harleman [5], have remarked that the nature of the heat source term in the governing differential equation appears to make it impossible for a suitable solution to evolve, which also satisfies the surface condition. They considered some special cases of simple time dependent functions for insolation and heat losses on the surface, and used a modified form of Weinberger's approach [8], in order to obtain analytical solutions; namely a superposition of distinct solutions for the temperature distribution due to effective radiation absorbed at the surface, and for the temperature distribution due to virtually absorbed radiation. A complete discussion of these methods, both analytical and experimental, for shallow and deep dimictic lakes, is given in [5].

The analysis presented in this dissertation takes into account general time dependent functions for insolation and surface heat losses, and assumes an exponentially decaying heat source distribution caused by absorbed radiation. Exact analytical solutions for the dimictic and solar lakes will be obtained, using a method of variation of parameters. Numerical calculations for dimictic lakes show that these analytical solutions compare favourably with observations drawn from nature and the laboratory.
Chapter II

Thermal Stratification in Dimictic Stagnant Lakes

Summary

Exact analytical solutions have been found for the vertical temperature distribution in a stagnant lake or tank, subject to surface heating, for a variety of boundary and initial conditions. The propagation of heat has been assumed to be due solely to molecular diffusion; convection and eddy diffusivity being neglected. The results obtained compare favourably with observations made in the laboratory and in nature.

1. Introduction

The problems of energy transfer in stagnant waters have been comprehensively surveyed by Dake and Harleman [5] and by Ou et alii [6]. The principal natural heat source considered is the sun, whose ultraviolet and infrared radiation are largely absorbed within the first few centimetres of the surface. The visible radiation, on the other hand, penetrates more deeply, carrying significant energy to depths of the order of tens of metres thereby causing vertical variations in density.

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The analysis presented in this chapter takes into account general time dependent functions for insolation and surface heat losses, and assumes an exponentially decaying heat source distribution caused by absorbed radiation. Exact analytical solutions of the heat equation will be obtained, using a method of variation of parameters. Numerical calculations show that these analytical solutions compare favourably with observations drawn from nature and the laboratory.

2. Mathematical Formulation

We assume that heat transfer occurs only in the vertical direction and neglect the convective motion of the fluid. The density and specific heat of the water are sensibly constant within the range of temperatures considered. Furthermore, following Dake and Harleman [2], we neglect eddy viscosity and turbulent diffusion. The well known heat equation is then

\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + \frac{H}{\rho c} \]  

(Appendix G) (2.2.1)

The heat source \( H = -\frac{\partial \phi}{\partial x} \) where \( \phi \) is the heat flux per unit horizontal area, and \( x \) is the vertical distance measured downward from the surface of the water. We write [2]
\[ H = -\frac{\partial \phi}{\partial x} = \eta(1-\beta)\phi_0(t)e^{-\eta x} \] (2.2.2)

Substituting (2.2.2) in (2.2.1) we obtain the inhomogeneous parabolic equation

\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + A(t)e^{-\eta x} \] (2.2.3)

where

\[ A(t) = \frac{\eta(1-\beta)\phi_0(t)}{\rho c} \]

To specify a solution of (2.2.3), two boundary conditions (B.C.s) and an initial condition (I.C.) are needed. The first boundary condition, on the surface \( x = 0 \), has the general form

\[ \beta \phi_0(t) = -\rho c a \left[ \frac{\partial T}{\partial x} \right]_{x=0} + \phi_L(t) \]

where \( \phi_L(t) \) is the total rate at which heat is lost to the atmosphere mainly due to evaporation and back radiation. The second boundary condition, at the bottom of the lake \( x = h \), may stipulate the temperature there, for example constant bottom temperature (see section 4) or, possibly \( T_x(h, t) = 0 \), in which case the bottom would act as an insulator, or again \( T_x(h, t) = C(t) \) should the heat flux across the bottom be known.

Dake and Harleman [2] considered a number of special cases of \( \phi_0(t) \) (the net insolation) and \( \phi_L(t) \) (the surface heat losses) applicable to tropical lakes; certain temperate lakes located in regions of generally clear skies; and also to laboratory conditions in which heating lamps were used.
3. Analytical Method

Consider the parabolic equation

\[ u_t - ku_{xx} = q(x,t) \quad (2.3.1) \]

with inhomogeneous boundary conditions (B.C.s)

\[ u_x(0,t) = B(t), \quad u(h,t) = C(t), \quad (2.3.2) \]

where \( B(t) \) and \( C(t) \) are differentiable when \( t > 0 \), and with initial condition (I.C.)

\[ u(x,0) = f(x). \quad (2.3.3) \]

It is well known (see, for example [9] and [10]) that the B.C.s can be made homogeneous by a suitable transformation, and the resultant problem solved in terms of an eigenfunction expansion of a related homogeneous problem.

First let

\[ u(x,t) = V(x,t) + K(x,t) \quad (2.3.4) \]

where

\[ K(x,t) = (x-h)B(t) + C(t). \quad (2.3.5) \]

Equation (2.3.1) then reduces to

\[ V_t - kV_{xx} = r(x,t) \quad (2.3.6) \]

with (B.C.s)

\[ V_x(0,t) = 0, \quad V(h,t) = 0 \quad (2.3.7) \]

and (I.C.)

\[ V(x,0) = g(x) \quad (2.3.8) \]

where

\[ r(x,t) = q(x,t) - (x-h)B'(t) - C'(t) \]

\[ g(x) = f(x) - (x-h)B(0) - C(0). \]

The homogeneous problem related to (2.3.6-8) is
\[ w_t - k w_{xx} = 0 \]

with (B.C.s)
\[ w_x(0,t) = w(h,0) = 0 \]

and (I.C.)
\[ w(x,0) = g(x) \]

with solution
\[ w(x,t) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n kt} \phi_n(x), \]

where
\[ \phi_n(x) = \cos\left(\left(n + \frac{1}{2}\right) \frac{\pi x}{h}\right), \]
\[ \lambda_n = \left(n + \frac{1}{2}\right)^2 \frac{k}{h} \]

and
\[ a_n = \frac{2}{h} \int_0^h g(x) \phi_n(x) dx. \]

The solution of (2.3.6-8) is obtained by a method of variation of parameters (see [9] and [10]) by writing
\[ V(x,t) = \sum_{n=0}^{\infty} T_n(t) \phi_n(x) \]

whence
\[ T_n(t) = \frac{2}{h} \int_0^h V(x,t) \phi_n(x) dx \]

Differentiating (2.3.12) with regard to \( t \), substituting from (2.3.6), integrating by parts and employing (2.3.7) and (2.3.8), it is found that
\[ T_n'(t) + \lambda_n T_n(t) = r_n(t) \]

where
\[ r_n(t) = \frac{2}{h} \int_0^h r(x,t) \phi_n(x) \, dx. \]

Moreover
\[ T_n(0) = \frac{2}{h} \int_0^h g(x) \phi_n(x) \, dx. \]

Then
\[ T_n(t) = T_n(0) e^{-\lambda_n t} + \int_0^t e^{-\lambda_n (t-u)} r_n(u) \, du. \]

Thus
\[ V(x,t) = \sum_{n=0}^{\infty} [T_n(0) e^{-\lambda_n t} + \int_0^t e^{-\lambda_n (t-u)} r_n(u) \, du] \phi_n(x). \]

The complete solution of (2.3.1), subject to the conditions (2.3.2) and (2.3.3), is then
\[ u(x,t) = (x-h)B(t) + C(t) \]
\[ + \sum_{n=0}^{\infty} [T_n(0) e^{-\lambda_n t} + \int_0^t e^{-\lambda_n (t-u)} r_n(u) \, du] \phi_n(x) \]

(2.3.13)

where
\[ \phi_n(x) = \cos\left(\frac{(2n+1)\pi x}{2h}\right) \]

\[ \lambda_n = \left(\frac{(2n+1)\pi}{2h}\right)^2 \]

\[ T_n(0) = \frac{2}{h} \int_0^h g(x)\phi_n(x)\,dx \]

\[ g(x) = f(x) - (x-h)B(0) - C(0) \]

\[ r_n(t) = \frac{2}{h} \int_0^h \hat{r}(x,t)\phi_n(x)\,dx \]

\[ \hat{r}(x,t) = q(x,t) - (x-h)B'(t) - C'(t) \]

4. **Temperature Constant at Bottom of Lake**

The problem to be solved is

\[
\text{(D.E.)} \quad T_t - \alpha T_{xx} = A(t)e^{-\eta x} \\
\text{(B.C.s)} \quad \begin{cases} 
T_x(0,t) = B(t) \\
T(h,t) = C(t) = T_b 
\end{cases} \\
\text{(I.C.)} \quad T(x,0) = f(x) \]

(2.4.1)

where \( T_b \) is a constant.

Applying the method of the previous section, it is found after some calculation that the solution of (2.4.1) is
\[ T(x,t) = (x-h)B(t) + T_b \]
\[ + \sum_{n=0}^{\infty} \left[ R_n e^{-\lambda_n t} + p_n a_n(t) + Q_n b_n(t) \right] \phi_n(x) \]  

where

\[ \lambda_n = \frac{(2n+1)\pi}{2h} \alpha \quad \phi_n(x) = \cos\frac{(2n+1)\pi x}{2h} \]

\[ a_n(t) = \int_0^t \lambda_n(u-t)A(u)du \]

\[ b_n(t) = \int_0^t \lambda_n(u-t)B'(u)du \]

\[ R_n = \frac{2}{h} \int_0^h [f(x) - (x-h)B(0) - T_b] \phi_n(x)dx \]

\[ p_n = \frac{2[\eta h + (-1)^n \frac{\pi(n+\frac{1}{2})e^{-\eta h}}{\eta h^2 + (n+\frac{1}{2})^2\pi^2}]}{\eta h^2 + (n+\frac{1}{2})^2\pi^2} \]

\[ Q_n = \frac{2h}{(n+\frac{1}{2})^2\pi^2} \]

5. Insulated Bottom

The lower boundary condition in this case is

\[ T_x(h,t) = 0, \]

that is to say, the heat flux across the plane \( x = h \) is zero. Thus
we have to find the solution of

\[
\begin{align*}
\text{(D.E.)} & \quad T_t - \alpha T_{xx} = A(t)e^{-px} \\
\text{(B.C.s)} & \quad T_x(0,t) = B(t) \\
\text{(I.C.)} & \quad T(x,0) = f(x) \\
\end{align*}
\]

\[ T_x(h,t) = 0 \]  (2.5.1)

Using methods analogous to those outlined in section 3, we find that the solution of (2.5.1) is

\[
T(x,t) = x[1 - \frac{x}{2h}]B(t) + \sum_{n=0}^{\infty} T_n(t)\phi_n(x) \]  (2.5.2)

where
\[ \phi_n(x) = \cos \left[ \frac{n\pi x}{h} \right] \]

\[ T_n(t) = e^{-\lambda_n t} T_n(0) + K_1 a_n(t) + K_2 b_n(t) \]

\[ T_n(0) = \frac{2}{h} \int_0^h f(x) \phi_n(x) \, dx + \frac{2h}{n^2 \pi^2} B(0) \]

\[ a_n(t) = \int_0^t A(u) e^{\lambda_n (u-t)} \, du \] \hspace{1cm} (2.5.3)

\[ b_n(t) = \int_0^t B'(u) e^{\lambda_n (u-t)} \, du \]

\[ K_1 = \frac{2nh}{\eta h^2 + \pi^2} \left[ 1 + (-1)^{n+1} e^{-\eta h} \right] \]

\[ K_2 = \frac{2h}{n^2 \pi^2} \lambda_n = \frac{n^2 \pi^2}{h^2} \alpha \]

6. **Complete Residual Radiation Absorption at the Bottom**

In this case the condition at \( x = h \) is \( T_x(h,t) = C(t) \). Thus we have to find the solution of

\[ \begin{align*}
(D.E.) & \quad T_t - \alpha T_{xx} = A(t) e^{-\eta x} \\
(B.C.s) & \quad \begin{cases} 
T_x(0,t) = B(t) \\
T_x(h,t) = C(t) 
\end{cases} \\
(I.C.) & \quad T(x,0) = f(x)
\end{align*} \] \hspace{1cm} (2.6.1)
Let
\[ K(x,t) = \frac{x^2}{2h} [C(t) - B(t)] + xB(t) \]
and
\[ T(x,t) = u(x,t) + K(x,t) \] (2.6.2)

Then
\[ \begin{align*}
(B.E.) & \quad u_t - au_{xx} = r(x,t) \\
(B.C.s) & \quad u_x(0,t) = u_x(h,t) = 0 \\
(I.C.) & \quad u(x,0) = g(x)
\end{align*} \] (2.6.3)

where
\[ r(x,t) = \frac{\alpha}{h} [C(t) - B(t)] + A(t)e^{-\eta x} \left( \frac{x^2}{2h} [C'(t) - B'(t)] - xB'(t) \right) \] (2.6.4)

and
\[ g(x) = f(x) - \frac{x^2}{2h} [C(0) - B(0)] - xB(0) \]

The solution of (2.6.3) can be obtained by variation of parameters, from the solution of the associated homogeneous problem (see section 3 and references therein). After some calculation we find that
\[ T(x,t) = \frac{x^2}{2h} [C(t) - B(t)] + xB(t) \]
\[ + \sum_{n=0}^{\infty} \phi_n(x)T_n(t) \] (2.6.5)

where
\[ \phi_n(x) = \cos \left( \frac{n\pi x}{h} \right) \]

\[ \lambda_n = \frac{n^2 \pi^2}{h^2} \alpha \]

\[ T_n(t) = e^{-\lambda_n t} T_n(0) + L_n^{(1)} a_n(t) + L_n^{(2)} b_n(t) + L_n^{(3)} c_n(t) \]

(2.6.6)

where

\[ T_n(0) = \frac{2}{h} \int_0^h f(x) \phi_n(x) dx + L_n^{(2)} B(0) + L_n^{(3)} C(0) \]

and

\[ L_n^{(1)} = \frac{2nh}{n^2 \pi^2 + n^2 \pi^2} \left[ 1 + (-1)^n e^{-\eta h} \right] \]

\[ L_n^{(2)} = \frac{2h}{n^2 \pi^2} \]

\[ L_n^{(3)} = \frac{2h(-1)^{n+1}}{n^2 \pi^2} \]

(2.6.7)

\[ a_n(t) = \int_0^t \lambda_n(u-t) A(u) du \]

\[ b_n(t) = \int_0^t \lambda_n(u-t) B'(u) du \]

\[ c_n(t) = \int_0^t \lambda_n(u-t) C'(u) du \]

(2.6.8)
7. **Complete Reflection of Residual Radiation from the Bottom**

When there is complete reflection of residual radiation from the bottom, which is considered to be an insulator, the heat source becomes

\[ A_1(t)e^{\eta x} + A_2(t)e^{-\eta x} \]

where

\[ A_1(t) = \frac{\eta(1-\beta)\phi_0}{\rho C} e^{-2\eta h} \]

\[ A_2(t) = \frac{\eta(1-\beta)\phi_0}{\rho C} \]  

(2.7.1)

The equation for heat conduction is then

\[ T_c - \alpha T_{\text{xx}} = A_1(t)e^{\eta x} + A_2(t)e^{-\eta x} \]  

(D.E.)

\[ \begin{align*}
T_x(0,t) &= B(t) \\
T_x(h,t) &= 0 
\end{align*} \]  

(B.C.s)

(2.7.2)

(I.C.)

\[ T(x,0) = f(x) \]

(2.7.3)

The method of solution is similar to that employed in section 5. After some calculation we find that the solution is given by:

\[ T(x,t) = x(1-\frac{x}{2h})B(t) + \sum_{n=0}^{\infty} \phi_n(x)T_n(t) , \]

where \[ \phi_n(x) = \cos\left(\frac{\eta nx}{h}\right) \]

(2.7.4)

\[ T_n(t) = e^{-\frac{\eta}{h}t}T_n(0) + p_n(1)a_n^{(1)}(t) + p_n(2)a_n^{(2)}(t) + q_n b_n(t) \]

where
\[ T_n(0) = \frac{2}{h} \int_{0}^{h} f(x) \phi_n(x) \, dx + Q_n B(0) \]

\[ \lambda_n = \frac{n \pi^2 h}{2} \alpha \]

\[ a_n^{(1)}(t) = \int_{0}^{t} A_1(u) e^{\lambda_n(u-t)} \, du \]

\[ a_n^{(2)}(t) = \int_{0}^{t} A_2(u) e^{\lambda_n(u-t)} \, du \]

\[ b_n(t) = \int_{0}^{t} B'(u) e^{\lambda_n(u-t)} \, du \]

\[ p_n^{(1)} = \frac{-2nh}{n^2 \pi^2 + n^2 \pi^2} \left[ 1 + (-1)^{n+1} e^{nh} \right] \]

\[ p_n^{(2)} = \frac{2nh}{n^2 \pi^2 + n^2 \pi^2} \left[ 1 + (-1)^{n+1} e^{-nh} \right] \]

\[ 0_n = \frac{2h}{n^2 \pi^2} \]

8. Comparison with observed values.

Two sets of observations are available in the literature: the first, occurring in nature, consists of temperatures observed in Lake Tahoe, California [11]; and the second, a controlled experiment using a tank in the M.I.T. Hydrodynamics Laboratory [5]. In both cases, the boundary
conditions correspond to those of section 4.

In the case of Lake Tahoe, the values of the parameters assumed were:

\[
\begin{align*}
\phi_0 &= 65 \times 10^5 \text{ Cal/m}^2 \text{ day} \\
\beta &= 0.4 \ , \ \eta = 0.048 \text{ m}^{-1} \\
\alpha &= 0.012416 \text{ m}^2/\text{day} \\
\rho &= 0.998 \times 10^6 \text{ gm/m}^3 \\
C &= 1.0 \text{ Cal/gm}^0 \text{C} \\
\phi_L &= \beta \phi_0 = 26 \times 10^5 \text{ Cal/m}^2 \text{ day}.
\end{align*}
\]

The boundary conditions were:

\[
\begin{align*}
T_x(0,t) &= B(t) = 0 \\
T(400,t) &= C(t) = T_b = 4^0 \text{C}
\end{align*}
\]

and the function \( A(t) \) was assumed to be constant

\[
A(t) = \frac{\eta(1-\beta)\phi_0}{\rho C} = 0.1876.
\]

The initial condition was \( T(x,0) = 4^0 \text{C} \).

Thus, the effective depth of the lake was assumed to be 400 metres, the water initially at \( 4^0 \text{C} \); and the rate of insulation to be a constant equal to the daily radiation averaged over the total duration of 150 days.

Substituting these values of the parameters and functions in (2.4.2), the sum of the series was obtained using a TI-59 programmable hand calculator [12] appendix A. For the convergence of the series see appendix B. The results are plotted in figure 1, and compared with observed
values [[11] p. 1040, [5] p. 114]. Agreement is excellent in the hypolimnion, which indicates that the eddy diffusivity can be neglected in the hypolimnion. In the epilimnion the computed values appear to be too high, possibly due to the neglect of eddy diffusivity, as has been pointed out by Dake and Harleman [2, p.488] and recently by Henderson-Sellers [13].

That the analytical solution (2.4.2) is well posed may be verified numerically by evaluating the solution for small changes in the boundary conditions. For example, let $B(t) = 0.1$ and $-0.1$. The corresponding numerical solutions are plotted in figure 2, from which it is clear that the solution changes by a small amount also.

Two sets of laboratory observations are available [5]. In the first, the insolation was constant, and the loss at the surface assumed proportional to $t^{0.5}$. The numerical values used were:

$$\begin{align*}
\phi_0 &= 0.01 \text{ Cal/cm}^2\text{sec.} \\
\beta &= 0.75, \eta = 0.01 \text{ cm}^{-1} \\
h &= 75 \text{ cm}, \Omega = 0.004 \text{ sec}^{-0.5} \\
T(x,0) = T(75,t) &= 21^\circ C
\end{align*}$$

Thus, the initial temperature was assumed to be uniform and equal to $21^\circ C$, the depth was 75 cm, and the bottom was maintained at $21^\circ C$. Substituting in (2.4.2) as above, the sums of the series were obtained in appendix C and plotted in figure 3, together with observed values taken from [2, p.492].
In the second experiment, the insolation \( \phi_0 \) was a quadratic function of time, and the surface heat loss \( \phi_L \) another such function:

\[
\phi_0 = \phi_{\text{max}} \left[ \frac{2t}{t_0} - \left( \frac{t}{t_0} \right)^2 \right], \quad 0 \leq t \leq 2t_0
\]  \hspace{1cm} (2.8.5)

where \( t_0 = 3 \) hours was the time at which insolation was a maximum and

\[ \phi_{\text{max}} = 0.007 \text{ Cal/cm}^2 \text{ sec.}, \text{ and} \]

\[
\phi_L = \phi_{\text{max}} \left[ \frac{2t}{t_L} - \left( \frac{t}{t_L} \right)^2 \right], \quad 0 \leq t \leq 2t_L
\]

where \( t_L = 5 \) hours.

These values were substituted in (2.4.2), the sums of the series were obtained using the TI-59 programmable hand calculator [12], appendix D. The results are plotted in figure 4, and compared with observed values taken from [2, p.493]. Good agreement was obtained in both cases.

Thus, exact analytical solutions, believed to be new, have been obtained by a method of variation of parameters, for the temperature distribution in a stagnant lake or tank, for a variety of boundary and initial conditions. The numerical results compare favourably with observations obtained both in nature and in the laboratory.
Chapter III

Deep Dimictic Lakes

Summary

For deep lakes the bottom boundary condition appears only as a limit, [see Section 4, Chapter II]. Harleman [5] obtained solutions for constant average insolation and constant average losses due to evaporation and back radiation from the surface. He obtained also solutions for constant average insolation and heat losses from the surface proportional to \( t^{0.5} \). We will consider Laplace Transforms method to obtain an analytical solution for deep lakes.

1. Mathematical Formulation

We assume, as previously, that heat transfer occurs only in the vertical direction and neglect the convective motion of the fluid. The density and specific heat of the water are sensibly constant within the range of temperatures considered. Furthermore, following Dake and Harleman [2] we neglect eddy diffusivity. The well known heat equation for constant cross section area is then

\[
\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2} + \frac{H}{\rho c} . \quad (3.1.1)
\]

The heat source \( H = -\frac{\partial \phi}{\partial x} \), where \( \phi \) is the heat flux per unit horizontal area, and \( x \) is vertical distance measured downward from the surface of the water. We write [2]
\[ H = - \frac{\partial \phi}{\partial x} = \eta [1-\beta] \phi_0(t) \, e^{-\eta x} \]  

(3.1.2)

Substituting (3.1.2) in (3.1.1) we obtain the inhomogeneous parabolic equation

\[ \frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2} + A(t) e^{-\eta x} \]  

(3.1.3)

where

\[ A(t) = \frac{\eta [1-\beta] \phi_0(t)}{\rho C} \]

To specify a solution of (3.1.3), two boundary conditions (B.C.s) and an initial condition (I.C.) are needed. The first boundary condition, on the surface \( x = 0 \), has the general form [2]

\[ \dot{\phi}_0(x,0) = - \rho \alpha \frac{\partial U}{\partial x} \bigg|_{x=0} + \phi_L(t) \]

The second boundary condition is \( U(h,t) = U_b \). The initial condition is \( U(x,0) = F(x) \). The bottom temperature of the lake = \( U_b \).

Dake and Harleman [2] considered a number of special cases of \( \phi_0(t) \) (the net insolation) and \( \phi_L(t) \) (the surface heat losses) applicable to tropical lakes; certain temperate lakes located in regions of generally clear skies.

2. **Laplace Transforms Method**

Consider the parabolic equation

\[ U_L = \alpha U_{xx} + A(t) e^{-\eta x} \]  

(3.2.1)
Together with the boundary and initial conditions

\[ U_x(0,t) = B(t) \]  \hspace{1cm} (3.2.2)

\[ U(h,t) = U_b \]  \hspace{1cm} (3.2.3)

\[ U(x,0) = U_b \]  \hspace{1cm} (3.2.4)

where, \( B(t) \) is a known function of time and \( U_b \) is a known constant.

The boundary conditions can be made homogeneous by a suitable transformation given by

\[ U(x,t) = T(x,t) + K(x,t) \]  \hspace{1cm} (3.2.5)

where

\[ K(x,t) = (x-h)B(t) + U_b \]  \hspace{1cm} (3.2.6)

Equation (3.2.1) then reduces to

\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} - (x-h)B'(t) + A(t)e^{-\eta x} \]  \hspace{1cm} (3.2.7)

with boundary conditions

\[ T_x(0,t) = 0 \]  \hspace{1cm} (3.2.8)

\[ T(h,t) = 0 \]  \hspace{1cm} (3.2.9)

and initial condition

\[ T(x,0) = -(x-h)B(0) \]  \hspace{1cm} (3.2.10)

Equation (3.2.7) can be solved by the Laplace Transform method. The Laplace Transform is defined by

\[ L\{T(x,t)\} = \int_0^\infty e^{-st}T(x,t)dt = \overline{T(x)} \]

After applying the Laplace Transform to (3.2.7), \( x \) will be the only independent variable in the transformed equation. Associated with the Laplace Transform are there formulae [14, 37]
\[ L\{\frac{\partial^2 T}{\partial t^2}\} = \frac{d^2 T}{dx^2} \]

\[ L\{T(x,0)\} = -\frac{(x-h)B(0)}{s} \]

\[ L\{B'(t)\} = \int_0^\infty e^{-st}B'(t)dt \]

\[ = [B(t)e^{-st} + s\int B(t)e^{-st}dt]_0^\infty \]

\[ = -B(0) + s\int_0^\infty e^{-st}B(t)dt \]

By applying the Laplace Transform to (3.2.7) we obtain

\[ ST - T(x,0) + (x-h)[-B(0) + S\int_0^\infty e^{-st}B(t)dt] \]

\[ = \alpha \frac{d^2 T}{dx^2} + \int_0^\infty e^{-st}A(t)e^{-tx}dx \] \hspace{1cm} (3.2.11)

By substituting the value of \( T(x,0) \) from (3.2.10) we obtain

\[ \alpha \frac{d^2 T}{dx^2} = \frac{S}{T} = \frac{S(x-h)}{\alpha} \int_0^\infty e^{-st}B(t)dt - \frac{1}{\alpha} \int \int_0^\infty e^{-st}A(t)e^{-tx}dt \] \hspace{1cm} (3.2.12)
where \( A(t) = \frac{\eta(1-\beta)\phi_0(t)}{\rho C} \). Therefore, equation (3.2.12) becomes

\[
\frac{d^2 \overline{T}}{dx^2} - \frac{s}{\alpha} \overline{T} = \frac{s(x-h)}{\alpha} \int_0^\infty e^{-st}B(t)dt - p
\]

(3.2.13)

where \( p = \frac{\eta(1-\beta)e^{-\eta x}}{\rho C\alpha} \int_0^\infty e^{-st}\phi_0(t)dt \).

Equation (3.2.13) is an ordinary second order differential equation, the solution of which must satisfy the following conditions

\( \overline{T}_x(0) = 0 \) and \( \overline{T}(h) = 0 \)

where \( h \) is the depth of the lake (deep lake). The homogeneous solution of (3.2.13) is

\[
C_1e^{-s\sqrt{\alpha} x}
\]

where \( C_1 \) is a constant. To obtain the particular solution of (3.2.13), we use the operator method. Let \( \frac{d}{dx} \) is equal to the operator \( D \).

From (3.2.13) we obtain

\[
\frac{1}{D^2} - \frac{s}{\alpha}
\left[ \frac{s(x-h)}{\alpha} \int_0^\infty e^{-st}B(t)dt - \frac{\eta(1-\beta)}{\rho C\alpha} \int_0^\infty e^{-st}\phi_0(t)dt \right]
\]

\[
= -(x-h) \int_0^\infty e^{-st}B(t)dt - \frac{\eta(1-\beta)e^{-\eta x}}{\rho C\alpha(\eta^2 - \frac{s^2}{\alpha})} \int_0^\infty e^{-st}\phi_0(t)dt.
\]

Therefore the solution of (3.2.13) is
\[ \bar{T} = C_1 \frac{\sqrt{\frac{s}{\alpha}}}{\sqrt{\frac{s}{\alpha}}} - (\pi - h) \int_0^\infty e^{-st}B(t)dt - Q \]  

(3.2.14)

where

\[ Q = \frac{\eta (1-\beta) e^{-\eta x}}{\rho C_\alpha (\eta^2 - \frac{s}{\alpha})} \int_0^\infty e^{-st}\phi_0(t)dt. \]

From the condition \( \bar{T}_x(0) = 0 \), we obtain the value of the constant \( C_1 \)

\[ C_1 = -\sqrt{\frac{s}{\alpha}} \int_0^\infty e^{-st}B(t)dt + \frac{\eta^2 (1-\beta) \sqrt{\alpha}}{\rho C_\alpha s (\eta^2 - \frac{s}{\alpha})} \int_0^\infty e^{-st}\phi_0(t)dt. \]  

(3.2.15)

Consider the case of constant insolation \( \phi_0(t) = I \) and \( B(t) = 0 \), which corresponds to the case of equal radiation absorbed at the surface and that lost through the surface. Therefore,

\[ \int_0^\infty e^{-st}\phi_0(t)dt = \int_0^\infty I e^{-st}dt = \frac{I}{s} \]

where \( I \) is a known constant. And \( \int_0^\infty e^{-st}B(t)dt = 0 \). Under these conditions, from (3.2.15) we obtain

\[ C_1 = \frac{\eta^2 (1-\beta) \sqrt{\alpha} I}{\rho C_\alpha (\eta^2 - \frac{s}{\alpha}) s \sqrt{\alpha}}. \]

Equation (3.2.14) becomes

\[ \bar{T} = \frac{\eta^2 (1-\beta) \sqrt{\alpha} I e^{-\sqrt{\frac{s}{\alpha}} x}}{\rho C_\alpha (\eta^2 - \frac{s}{\alpha}) \sqrt{\alpha}} - \frac{\eta (1-\beta) I e^{-\eta x}}{\rho C_\alpha (\eta^2 - \frac{s}{\alpha}) s \sqrt{\alpha}}. \]  

(3.2.16)

To find the inverse Laplace Transform of equation (3.2.16) we use the formulae [37 page 1026]
\[ L^{-1} \left\{ \frac{1}{s(\eta^2 - s^2/\alpha)} \right\} = L^{-1} \left\{ \frac{1}{\eta^2 s} - \frac{1}{\eta^2 s - \alpha n^2} \right\} \]

\[ = \frac{1}{\eta^2} - \frac{1}{\eta^2} e^{\alpha n^2 t} \]

\[ L^{-1} \left\{ \frac{-1 - \eta^2/2}{s} e^{-s \sqrt{s}} \right\} = (4t)^{\frac{n}{2}} \int_0^\infty \text{erfc} \, \frac{a}{2\sqrt{t}} \, du, \quad n = 0, 1, \ldots \]

where

\[ \int_0^\infty \text{erfc} \, x = \int_x^\infty \text{erfc} \, u \, du, \quad n = 1, 2, \ldots \]

\[ \int_0^\infty \text{erfc} \, x = \text{erfc} \, x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} \, du. \]

Therefore,

\[ L^{-1} \left\{ \frac{-x \sqrt{s/\alpha}}{s \sqrt{s}} \right\} = \sqrt{4t} \int_0^\infty \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} \right) \, du \]

\[ L^{-1} \left\{ \frac{1}{\eta^2 - s/\alpha} \right\} = -ae^{\alpha n^2 t} \]

To find the inverse Laplace Transform of \( \frac{e^{-x \sqrt{s/\alpha}}}{s \sqrt{s}} \cdot \frac{1}{\eta^2 - s/\alpha} \), we use the convolution theorem

\[ L^{-1} \{f(s) \cdot g(s)\} = \int_0^t f(u) \cdot g(t-u) \, du \]

where \( f(s) = L[F(t)] \) and \( g(s) = L[G(t)] \). Therefore,
The inverse Laplace transform of (3.2.16) is

\[
T(x, t) = \frac{-n^2(1-\beta)\sqrt{\alpha}}{\rho c} \int_0^t \sqrt{4u} \operatorname{erfc} \left( \frac{x}{2\sqrt{au}} \right) e^{\alpha u^2} du
\]

\[
= \frac{n(1-\beta)e^{-\eta x}}{\rho c \alpha} \left( \frac{1}{\eta^2} - \frac{1}{\eta^2} e^{\alpha n^2 t} \right)
\]

(3.2.17)

erfc \zeta and \operatorname{erfcz} decay very rapidly with its increasing argument \( z \) [37 page 300, 5 page 137]. For large value of \( x \) equation (3.2.17) gives

\[
\frac{I(1-\beta)}{\rho c n} e^{\alpha n^2 t} \left[ e^{-\eta x} - \sqrt{\pi} \operatorname{erfc} \left( \frac{x}{2\sqrt{at}} \right) + 2 \operatorname{erfc} \left( \frac{x}{2\sqrt{at}} \right) \right]
\]

\[
- \frac{(1-\beta)I e^{-\eta x}}{\rho c n} e^{\alpha n^2 t} \frac{2(1-\beta)I \sqrt{T}}{\rho c \sqrt{\alpha}} \operatorname{erfc} \left( \frac{x}{2\sqrt{at}} \right)
\]

Therefore the solution of (3.2.1) is

\[
U(x, t) = U_b + T(x, t)
\]
Chapter IV

Thermal Stratification in Stagnant Saline Lakes (Heloithermal).

Two-Layered Model

Summary

Exact analytical solutions have been found for the vertical temperature distribution in a stagnant saline lake, subject to surface heating. The propagation of heat has been assumed to be due solely to molecular diffusion, convection and eddy diffusivity being neglected. A two-layered mathematical model has been considered in this chapter.

1. Introduction

The problems of energy transfer in stagnant saline lakes have been considered by many researchers. Rychkov [15], was the first to draw attention to peculiar thermal regimes of saline lakes. Kalesinsky [16] observed similar phenomenon in a group of Hungarian lakes. Smith [17], discovered a series of salt pools in which the temperature increased greatly from the surface to the bottom. Anderson [18] discovered the same phenomenon in a shallow saline body of water occupying a former epsom salt excavation in north central Washington. Weinberger [8] studied the physics of the solar pond. Shirtcliffe [19,20] studied the phenomenon in Lake Bonney—Antarctica, which is 31.5 m deep and permanently ice covered. Elata [21] studied the hydraulics of the solar pond. Dzens-Litovskii [22] observed the existence of a two-layered lake Kuchuck in USSR. Hirschmann [23] in a review article reported that the density dependence on depth in some ponds is a linear function. Sonnenfeld [4] reported the existence of these lakes all over the globe. The principal natural heat source considered is the sun, whose ultraviolet and infrared radiation are largely absorbed within the first few centimetres of the surface. The visible radiation,
on the other hand, penetrates more deeply, carrying significant energy to depths of the order of metres.

Dake and Harleman [5] have remarked that the nature of the heat source term in the governing differential equations appears to make it impossible for a suitable solution to evolve, which also satisfies the surface condition. Shirtcliffe [19,20] considered some special case of simple time dependent functions for insolation and a special surface boundary condition which is appropriate to Lake Bonney, Antarctica. His analytical solution was obtained by a superposition of distinct solutions for the temperature distribution due to effective radiation absorption, heat conduction and due to geothermal effect.

In many cases the limnological cycle with periods of inverse temperature profile has not been described in detail. The brine of saline lakes has a greater ability to absorb heat energy than the fresh water. If the surface of the brine of a small saline lake is covered by a thin layer of fresh water, then in the summer months the temperature of the brine is always higher than that of the brackish surface layer. The solar energy produces a greater temperature increase in the brine than in fresh or brackish water. Smith [17], and Dzens-Litovskii [22], observed that the existence of a covering layer of fresh water is essential for this phenomenon to occur. The layer of fresh water protects the brine from evaporation and loss of thermal energy. The brine lower layer does not mix with the upper brackish water and can preserve its higher temperature for a long time. Upon heating a saline solution, the solubility of many salts, for example sodium chloride and magnesium chloride, increases. As a consequence, by maintaining constant a determined degree of saturation, the solution absorbs more salt upon being heated, which prevents the reduction of the density.
A solar lake is an accumulator of solar energy. Because of a natural salt concentration gradient in these lakes, lower regions remain denser even when warmer due to the fact that thermal diffusivity is much greater than salt diffusivity convection is prevented in these lakes. The stable lake water traps the incident solar radiation, heat is lost from the surface mainly due to evaporation and back radiation. On the other hand, heat is lost from the lower regions due to conduction through the poorly conducting water and through the bottom of the lake.

There would appear to be only four possible theories as to the source of the heat. Firstly, it might be said that the bottom water in these lakes arises from a hot saline spring but a very brief review of the existing evidence [17] contradicts this. For instance, the maximum temperature does not always occur at the bottom, as one would expect if the warmer water came from a spring. Again the phenomenon arose in artificial pools where there was no previous indication of either hot or cold springs. Secondly, there is a possibility that the water becomes heated through standing over hot earth, but all the evidence is also against such a theory. Thirdly, there is a possibility that the water becomes heated due to chemical reactions, but the heat corresponding to this is very minute. The only remaining hypothesis is that the water is heated up by direct solar radiation and this appears to agree with the facts very closely.

The optimal depth of a solar pond should be around 1 – 1.2 m [21]. In a deeper pond, too little of the sun's energy would reach the bottom, while in a shallower one too much of the absorbed heat would diffuse upwards.
Dzens-Litovskii [22], observed the existence of a two layered lake. The upper layer of fresh water 15 cm thick over the surface of the light-coloured and dense lower layer of salt water of Lake Kuchuck in USSR.

Shirtcliffe [19] studied Lake Bonney, Antarctica and considered three-layered model, two different water layers and the bottom of the lake.

A solar pond can be used for generation of electric power, desalination, production of high quality salt and health resort.

The analysis presented in this chapter considered a two-layered model and takes into account general time dependent functions for insolation and surface heat losses, and assumes an exponentially decaying heat source distribution caused by absorbed radiation. Exact analytical solutions of the heat equation will be obtained, using a method of variation of parameters.

2. Mathematical Formulation

We assume that heat transfer occurs only in the vertical direction and neglect the convective motion of the fluid. The density and specific heat of each layer are sensibly constant within the range of temperatures considered. Furthermore, following Dake and Harleman [2], we neglect eddy viscosity and turbulent diffusion. The well known heat equations for the two layers are

\[
\frac{\partial T_1}{\partial t} = \alpha_1 \frac{\partial^2 T_1}{\partial x^2} + \frac{H_1}{\rho_1 c_1} , \text{ for } -L \leq x \leq 0 = x_2 , \quad t > 0
\]  \hspace{1cm} (4.2.1)
and
\[
\frac{\partial T_2}{\partial t} = \alpha_2 \frac{\partial^2 T_2}{\partial x^2} + \frac{H_2}{\rho_2 c_2}, \quad x_2 = 0 \leq x \leq h = x_3, \quad t > 0
\]  \hspace{2cm} (4.2.2)

where the subscripts 1 and 2 are for the upper and lower layers respectively. \( x \) is the vertical distance measured positively downward from the interface between the two layers, (figure 5).

\[
H_1 = \eta_1 (1 - \beta) \phi_0(t) e^{-\eta_1 x}, \quad H_2 = \eta_2 (1 - \beta) \phi_0(t) e^{-\eta_1 x} - \eta_2 x
\]

\[
A_1(t) = \frac{\eta_1 (1 - \beta) \phi_0(t)}{\rho_1 c_1}
\]

\[
A_2(t) = \frac{\eta_2 (1 - \beta) \phi_0(t) e^{-\eta_1 x}}{\rho_2 c_2}
\]

\[
\alpha = \frac{k}{\rho c}
\]

Therefore, equations (4.2.1) and (4.2.2) become

\[
\frac{\partial T_1}{\partial t} = \alpha_1 \frac{\partial^2 T_1}{\partial x^2} + A_1(t) e^{-\eta_1 x}
\]  \hspace{2cm} (4.2.3)

and

\[
\frac{\partial T_2}{\partial t} = \alpha_2 \frac{\partial^2 T_2}{\partial x^2} + A_2(t) e^{-\eta_2 x}
\]  \hspace{2cm} (4.2.4)

which can be written in the form

\[
\frac{\partial T_i(x,t)}{\partial t} = \alpha_i \frac{\partial^2 T_i(x,t)}{\partial x^2} + q_i(x,t), \quad i = 1, 2, \quad \text{in} \quad x_1 \leq x \leq x_{i+1}, \quad t > 0
\]  \hspace{2cm} (4.2.5)
where \( q_1(x,t) = A_1(t) e^{-\eta_1 x} \). To specify a solution of (4.2.5), two boundary conditions, an initial condition and a matching condition at the interface \( x_2 \), are needed. For the matching condition we assume perfect thermal contact at the interface \( x_2 \). For perfect thermal contact both the temperature and the heat flux are continuous at \( x_2 \), therefore

\[
T_1(x_2,t) = T_2(x_2,t) \tag{4.2.6}
\]

and

\[
k_1 \frac{\partial T_1(x_2,t)}{\partial x} = k_2 \frac{\partial T_2(x_2,t)}{\partial x} \tag{4.2.7}
\]

The first boundary condition, on the surface \( x = x_1 = -\ell \), has the general form

\[
- \rho_1 c_1 \alpha_1 \left( \frac{\partial T_1}{\partial x} \right)_{x=x_1} = \beta \phi_0(t) - \phi_L(t) \tag{4.2.8}
\]

which takes care of evaporation and back radiation. The second boundary condition, at the bottom of the lake \( x = x_3 = h \), may stipulate the temperature there or, possibly

\[
\frac{\partial T_2}{\partial x}(x_3,t) = 0 \tag{4.2.9}
\]

in which case the bottom of the lake would act as an insulator which is the practical case. The bottom of the lake consists of mud, rocks, or sand which can be considered as an insulator.

The initial condition is
\[ T_i(x, 0) = F_i(x), \quad x_i \leq x \leq x_{i+1}, \quad i = 1, 2 \] when \( t = 0 \). \hspace{1cm} (4.2.10)

At fixed time the lakes are in the isothermal condition and \( F_i(x) \) is a known constant, Cohen [30].

3. Analytical Method

The solution of the heat conduction problem for multilayer symmetrical bodies (i.e., plates, cylinders, spheres) with heat generation within the solid has been solved by Bulavin and Kashcheev [24] by the method of separation of variables and by the construction of orthogonal expansion of functions over multilayered regions, for the case of perfect thermal contact between the layers. The orthogonal-expansion technique and the orthogonality factor used for expansion is similar to the variation of parameters, but the treatment of the eigenvalue problem is somewhat different.

Figure 5 shows the geometry of a one-dimensional composite region consisting of two parallel layers. It is assumed that the thermal properties of each layer are uniform but discontinuous at the interfaces between the layers which are in perfect thermal contact. The initial temperature distribution for each layer is prescribed, i.e., equation (4.2.10). The boundary-value problem of heat conduction is given as (4.2.5) with the surface boundary condition (4.2.8) which can be written in the form

\[
\frac{\partial T_i(x_1, t)}{\partial x} = B(t) \hspace{1cm} (4.3.1)
\]

where \( B(t) \) is a known given function of time. The interface matching
conditions are given by (4.2.6) and (4.2.7), the bottom boundary condition is given by (4.2.9) where, \( T_i(x,t) \) = temperature at the \( i \)th layer in \( x_i \leq x \leq x_{i+1} \), \( c_i \), \( k_i \), \( \alpha_i \), \( \rho_i \) = specific heat, thermal conductivity, thermal diffusivity, density respectively of the \( i \)th layer.

It is well known (see for example [9], and [25]) that the solution of the above problem can be written in the form

\[
T_i(x,t) = \sum_{n=1}^{\infty} X_{in}(x) T_n(t) \quad \text{in} \quad x_i \leq x \leq x_{i+1}, \quad t > 0
\]

(4.3.2)

where \( T_n(t) \) is a function which is to be determined, and \( X_{in}(x) \) are functions satisfy the following eigenvalue problem

\[
\alpha_i \frac{d^2 X_{in}(x)}{dx^2} + \beta_n^2 X_{in}(x) = 0 \quad \text{in} \quad x_i \leq x \leq x_{i+1}, \quad i = 1,2
\]

(4.3.3)

with the boundary conditions

\[
\frac{dX_{in}(x_1)}{dx} = 0
\]

(4.3.4)

\[
X_{in}(x_2) = X_{2n}(x_2)
\]

(4.3.5)

\[
k_1 \frac{dX_{in}(x_2)}{dx} = k_2 \frac{dX_{2n}(x_2)}{dx}
\]

(4.3.6)

\[
\frac{dX_{2n}(x_3)}{dx} = 0
\]

(4.3.7)

where \( \beta_n \) are the eigenvalues.
The solution of equation (4.3.3) is in the form
\[ x_{in}(x) = C_{in} \cos \frac{\beta_n}{\sqrt{\alpha_1}} x + D_{in} \sin \frac{\beta_n}{\sqrt{\alpha_1}} x. \] (4.3.8)

There are two solutions in this form for the two layers, hence we have to determine four arbitrary constants (i.e., \( C_{in} \) and \( D_{in} \) for \( i = 1, 2 \)).

The four equations that are needed to determine the four constants are obtained by substituting the eigenfunctions (4.3.8) into the boundary conditions (4.3.4) - (4.3.7). We obtain four linear homogeneous equations. For nontrivial solution of these homogeneous equations at least one of the coefficients must be non zero, say \( C_{in} \neq 0 \). We find that

\[ C_{2in} = C_{in} \] (4.3.9)
\[ D_{in} = C_{in} \tan \frac{\beta_n x_1}{\sqrt{\alpha_1}} \] (4.3.10)
\[ D_{2in} = C_{in} \tan \frac{\beta_n x_3}{\sqrt{\alpha_2}} \] (4.3.11)
\[ \tan \frac{\beta_n x_1}{\sqrt{\alpha_1}} = \sqrt{\frac{\alpha_1}{\alpha_2}} \frac{k_1}{k_2} \tan \frac{\beta_n x_3}{\sqrt{\alpha_2}}. \] (4.3.12)

From (4.3.12) we can determine \( \beta_n \). The positive roots of (4.3.12) give the eigenvalues

\[ \beta_1 < \beta_2 < \beta_3 < \ldots < \beta_n < \ldots. \]

Hence (4.3.8) takes the form
\[ x_{1n}(x) = C_{1n} \cos \frac{\beta_n x}{\sqrt{\alpha_1}} + C_{1n} \sin \frac{\beta_n x}{\sqrt{\alpha_1}}, \]
\[ x_1 \leq x \leq x_2 \] 

(4.3.13)

and

\[ x_{2n}(x) = C_{1n} \cos \frac{\beta_n x}{\sqrt{\alpha_2}} + C_{1n} \sin \frac{\beta_n x}{\sqrt{\alpha_2}}, \]
\[ x_2 \leq x \leq x_3 \] 

(4.3.14)

To determine the unknown function \( T_n(t) \) in the solution (4.3.2), consider the following orthogonality property of the eigenfunctions \( X_{1n}(x) \) over the entire range of the two layers from \( x_1 \) to \( x_3 \).

\[ \alpha_1 \frac{d^2 X_{1n}(x)}{dx^2} + \beta_n^2 X_{1n}(x) = 0 \] 

(4.3.15)

\[ \alpha_1 \frac{d^2 X_{1n}(x)}{dx^2} + \beta_n^2 X_{1n}(x) = 0. \] 

(4.3.16)

Assume that for \( n \neq n' \), \( \beta_n \neq \beta_{n'} \) (distinct eigenvalues).

Multiplying (4.3.15) by \( X_{1n}(x) \) and (4.3.16) by \( X_{1n}(x) \), then subtracting, we obtain

\[ \alpha_1 \frac{d^2 X_{1n}(x)}{dx^2} - X_{1n}(x) \frac{d^2 X_{1n}(x)}{dx^2} \] 

\[ + X_{1n}(x) X_{1n}(x) [\beta_n^2 - \beta_{n'}^2] = 0 \] 

(4.3.17)

(4.3.17) can be written in the form
\[
\alpha_i \frac{d}{dx} \left[ x_{in}(x) \frac{dx_{in}(x)}{dx} - x_{in}(x) \frac{dx_{in'}(x)}{dx} \right] \\
+ x_{in}(x)x_{in'}(x) \left[ \beta_n^2 - \beta_{n'}^2 \right] = 0 \quad (4.3.18)
\]

Integrating (4.3.18) with respect to \( x \) over the entire layers and using the boundary condition (4.3.4) – (4.3.7) and similar ones with \( n \) replaced by \( n' \), we obtain

\[
\sum_{i=1}^{2} \frac{k_i}{\alpha_i} \int_{x_i}^{x_{i+1}} x_{in}(x)x_{in'}(x)dx = \begin{cases} 
0 & \text{for } n \neq n' \\
\text{const.} = N, n = n' 
\end{cases}
\]

where the norm \( N \) is given by

\[
N = \sum_{i=1}^{n} \frac{k_i}{\alpha_i} \int_{x_i}^{x_{i+1}} x_{in}^2(x)dx
\]

and \( \frac{k_i}{\alpha_i} \) is the discontinuous weight-function for the layer \( i \).

Therefore from (4.3.2) we obtain

\[
T_n(t) = \frac{1}{N} \sum_{i=1}^{n} \frac{k_i}{\alpha_i} \int_{x_i}^{x_{i+1}} T_i(x,t)x_{in}(x)dx. \quad (4.3.19)
\]

We assume that \( \frac{\partial T_i(x,t)}{\partial t} \) is a continuous function in the region \( t > 0 \), \( x_i \leq x \leq x_{i+1} \). Under these circumstances, the integral in (4.3.19) has a derivative with respect to \( t \) which can be calculated by differentiation under the integral sign. We get, referring to (4.2.5)
\[ T_n'(t) = \frac{1}{N} \sum_{i=1}^{N} \frac{k_i}{\alpha_i} \int_{x_i}^{x_{i+1}} \left[ \alpha_i \frac{\partial^2 T_i(x,t)}{\partial x^2} + q_i(x,t) \right] X_{in}(x) dx \]  

(4.3.20)

where prime denotes differentiation with respect to \( t \).

The last term of (4.3.20)

\[ q_n(t) = \frac{1}{N} \sum_{i=1}^{N} \frac{k_i}{\alpha_i} \int_{x_i}^{x_{i+1}} q_i(x,t) X_{in}(x) dx \]

is a known function of \( t \), since \( q_i(x,t) \) is given. The first term of (4.3.20) will be integrated by parts twice and we use the given boundary conditions. To justify this operation, we assume that for every fixed \( t > 0 \), \( T_i(x,t) \), \( \frac{\partial T_i(x,t)}{\partial x} \) and \( \frac{\partial^2 T_i(x,t)}{\partial x^2} \) are continuous on \( x_i \leq x \leq x_{i+1}, \ i = 1, 2 \). Hence

\[ \frac{1}{N} \sum_{i=1}^{N} \frac{k_i}{\alpha_i} \int_{x_i}^{x_{i+1}} \alpha_i \frac{\partial^2 T_i(x,t)}{\partial x^2} X_{in}(x) dx \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \frac{k_i}{\alpha_i} \left[ \frac{\partial T_i(x,t)}{\partial x} X_{in}(x) - T_i(x,t) X'_{in}(x) \right]_{x_i}^{x_{i+1}} \]

\[ + \int T_i(x,t) X''_{in}(x) dx \right]_{x_i}^{x_{i+1}} \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \frac{k_i}{\alpha_i} \left[ \frac{\partial T_i(x_{i+1},t)}{\partial x} X_{in}(x_{i+1}) - \frac{\partial T_i(x_i,t)}{\partial x} X_{in}(x_i) \right] \]

\[ - T_i(x_{i+1},t) X_{in}(x_{i+1}) + T_i(x_i,t) X'_{in}(x_i) \]

\[ + \frac{1}{N} \sum_{i=1}^{N} \frac{k_i}{\alpha_i} \int_{x_i}^{x_{i+1}} T_i(x,t) X''_{in}(x) dx \]  

(4.3.21)
Substituting from (4.3.15), the last term of (4.3.21) becomes

\[- \frac{1}{N} \sum_{i=1}^{N} k_i \int_{x_i}^{x_{i+1}} T_1(x,t) \frac{\beta^2}{\alpha_1} x_{1n}(x) \, dx \]

\[= \frac{-\beta^2}{N} \sum_{i=1}^{N} \frac{k_i}{\alpha_1} \int_{x_i}^{x_{i+1}} T_1(x,t) x_{1n}(x) \, dx . \quad (4.3.22)\]

Substituting (4.3.19) in (4.3.22), we obtain \[-\beta^2 T_n(t) .\]

Hence,

\[- \frac{1}{N} \sum_{i=1}^{N} k_i \int_{x_i}^{x_{i+1}} T_1(x,t) x_{1n}(x) \, dx = -\beta^2 T_n(t) \]

and (4.3.21) becomes

\[\left[ \frac{k_1}{N} \left[ \frac{\partial T_1(x_2,t)}{\partial x} x_{1n}(x_2) - \frac{\partial T_1(x_1,t)}{\partial x} x_{1n}(x_1) \right] \right.\]

\[\left. - T_1(x_2,t) x_{1n}'(x_2) + T_1(x_1,t) x_{1n}'(x_1) \right] \]

\[+ \frac{k_2}{N} \left[ \frac{\partial T_2(x_3,t)}{\partial x} x_{2n}(x_3) - \frac{\partial T_2(x_2,t)}{\partial x} x_{2n}(x_2) \right] \]

\[\left. - T_2(x_3,t) x_{2n}(x_3) + T_2(x_2,t) x_{2n}(x_2) \right] \]

\[- \beta^2 T_n(t) . \quad (4.3.23)\]

Substitute from (4.2.9), (4.3.4) and (4.3.7) in (4.3.23), we obtain
\[
\frac{k_1}{N} \left[ \frac{\partial T_1(x_2, t)}{\partial x} x_{1n}(x_2) - \frac{\partial T_1(x_1, t)}{\partial x} x_{1n}(x_1) \right] \\
- T_1(x_2, t)x_{1n}'(x_2) + \frac{k_2}{N} \left[ - \frac{\partial T_2(x_2, t)}{\partial x} x_{2n}(x_2) \right] \\
+ T_2(x_2, t)x_{2n}'(x_2) - \beta_{nT}^2 T_n(t) .
\]

(4.3.24)

And by using (4.3.13) and (4.3.14), we obtain

\[
x_{1n}(x_2) = C_{1n} \quad \text{and} \quad x_{2n}(x_2) = C_{1n}
\]

where \( x_2 = 0 \), hence from (4.2.7) we obtain

\[
\frac{k_1}{N} \frac{\partial T_1(x_2, t)}{\partial x} x_{1n}(x_2) - \frac{k_2}{N} \frac{\partial T_2(x_2, t)}{\partial x} x_{2n}(x_2) = 0 .
\]

Also (4.3.24) becomes

\[
\frac{k_1}{N} \left[ - \frac{\partial T_1(x_1, t)}{\partial x} x_{1n}(x_1) - T_1(x_2, t)x_{1n}'(x_2) \right] \\
+ \frac{k_2}{N} T_2(x_2, t)x_{2n}'(x_2) - \beta_{nT}^2 T_n(t) .
\]

(4.3.25)

From (4.3.13) and (4.3.12) we have

\[
x_{1n}'(x_2) = C_{1n} \frac{\beta_n}{\sqrt{\alpha_1}} \tan \frac{\beta_n x_1}{\sqrt{\alpha_1}} \\
= C_{1n} \frac{\beta_n}{\sqrt{\alpha_1}} \frac{\alpha_1^{1/2} \alpha_2^{k_1} k_2}{\sqrt{\alpha_1}} \tan \frac{\beta_n x_3}{\sqrt{\alpha_2}} \\
= C_{1n} \frac{\beta_n k_2}{\sqrt{\alpha_2} k_1} \tan \frac{\beta_n x_3}{\sqrt{\alpha_2}} .
\]

From (4.3.14) we have
\[
X'_{2n}(x_2) = c_1 n \frac{\beta_n}{\sqrt{\alpha_2}} \tan \frac{\beta_n x_2}{\sqrt{\alpha_2}}.
\]

Hence from (4.2.6) we obtain
\[
-\frac{k_1}{N} T_1(x_2, t) X'_{1n}(x_2) + \frac{k_2}{N} T_2(x_2, t) X'_{2n}(x_2) = 0
\]
and (4.3.25) becomes
\[
-\frac{k_1}{N} \frac{\partial T_1(x_1, t)}{\partial x} X_{1n}(x_1) = \beta_n^2 T_n(t).
\]

Substituting (4.3.1) in (4.3.26) we obtain
\[
-\frac{k_1}{N} B(t) X_{1n}(x_1) = \beta_n^2 T_n(t).
\]

Therefore the first term of (4.3.20) is given by (4.3.27) and (4.3.20) reduces to
\[
T_n'(t) = -\frac{k_1}{N} B(t) X_{1n}(x_1) - \beta_n^2 T_n(t) + q_n(t)
\]
which can be written in the form
\[
T_n'(t) + \beta_n^2 T_n(t) = r_n(t)
\]

where
\[
r_n(t) = q_n(t) - \frac{k_1}{N} B(t) X_{1n}(x_1).
\]

Equation (4.3.28) is a linear first-order ordinary differential equation for \(T_n(t)\), in which \(r_n(t)\) is a known function given by (4.3.29).

Setting \(t = 0\) in (4.3.19) we get the initial condition
\[ T_n(0) = \frac{1}{N} \sum_{i=1}^{k_1} \frac{\alpha_i}{\alpha_i} \int_{x_i}^{x_{i+1}} F_1(x) X_{in}(x) dx \]

= M (say) \hspace{1cm} (4.3.30)

The solution of (4.3.28) which satisfies (4.3.30) is

\[ T_n(t) = T_n(0) e^{-\beta_n^2 t} + \int_0^t r_n(u) e^{-\beta_n^2 (u-t)} du . \hspace{1cm} (4.3.31) \]

The coefficients \( T_n(t) \) in (4.3.2) are now completely known and the problem has been solved. We have

\[ T_1(x,t) = \sum_{n=1}^{\infty} X_{in}(x) \left[ T_n(0) e^{-\beta_n^2 t} + \int_0^t r_n(u) e^{-\beta_n^2 (u-t)} du \right] \]

where \( x_i \leq x \leq x_{i+1}, i = 1, 2 \) and \( t > 0 \), \( X_{in}(x) \) and \( X_{2n}(x) \) are given by (4.3.13) and (4.3.14) respectively. \( \beta_n \) are the positive roots of the transcendental equation (4.3.12). \( T_n(0) \) is given by (4.3.30), \( r_n(t) \) by (4.3.29), and \( N = \sum_{i=1}^{\frac{k_1}{\alpha_i}} \int_{x_i}^{x_{i+1}} x_i^2 \alpha_i \).

Thus, an exact analytical solution (4.3.32) has been found for the vertical temperature distribution in a stagnant saline lake, subject to surface heating. The propagation of heat has been assumed to be due solely to molecular diffusion as Harleman [2] has suggested, convection and eddy diffusion being neglected. A two-layered mathematical model has been considered. These results can be immediately generalized to apply to any number of interfaces and layers.
4. Experimental Application

The values of the parameters assumed for a controlled experimental tank [5] with some modification appropriate for the two layered model were:

\[
\begin{align*}
\rho_1 &= 1 \text{ gm./cm.}^3 \\
\rho_2 &= 1.2 \text{ gm./cm.}^3 \\
C_1 &= 1 \text{ Cal./gm.}^\circ \text{C} \\
C_2 &= 0.8 \text{ Cal/gm.}^\circ \text{C} \\
\alpha_1 &= 5.184 \text{ cm}^2/\text{hr.} \\
\alpha_2 &= 5.063 \text{ cm}^2/\text{hr.} \\
\eta_1 &= 0.01 \text{ cm}^{-1} \\
\eta_2 &= 0.01 \text{ cm}^{-1} \\
k_1 &= 5.184 \text{ hr}^{-1} \text{ } ^\circ \text{C}^{-1} \\
k_2 &= 4.86 \text{ hr}^{-1} \text{ } ^\circ \text{C}^{-1} \\
x_1 &= -10 \text{ cm.} \\
x_2 &= 0, \ x_3 = 90 \text{ cm.} \\
\beta &= 1 \\
A_1(t) &= A_2(t) = q_1(x,t) = 0.
\end{align*}
\]

The insolation \( \phi_0 \) was a quadratic function of time, and the surface heat loss \( \phi_L \) another such function.

\[
\phi_0 = \phi_{0\max} \left[ \frac{2t}{t_0} - \left( \frac{t}{t_0} \right)^2 \right], \quad 0 \leq t \leq 2t_0 \tag{4.4.1}
\]

\[
\phi_L = \phi_{0\max} \frac{t}{t_{OL}} \left[ \frac{2t}{t_{OL}} - \left( \frac{t}{t_{OL}} \right)^2 \right], \quad 0 \leq t \leq 2t_{OL} \tag{4.4.2}
\]

where

\[ t_0 = 3 \text{ hours was the time at which insolation was a maximum,} \]

\[ t_{OL} = 5 \text{ hours and } \phi_{0\max} = 25.2 \text{ Cal./cm.}^2 \text{ hr.} \] The surface boundary condition (4.3.1) implies that \( B(t) = 0.424t^2 - 2.074t \) and the initial condition (4.2.10) is

\[
F_i(x) = -1^\circ \text{C}, \quad x_i \leq x \leq x_{i+1}, \quad i=1,2.
\]
These values were substituted in (4.3.12) where we obtained the eigenvalues \( \beta_n \) (Appendix E) and from (4.3.32) we obtained the temperature distribution (WATPIV sum program, Appendix E).

\[
\sum_{i=1}^{N} \frac{k_i}{\alpha_i} \int_{x_i}^{x_{i+1}} F_i(x)X_{n}(x)dx = 0 , \text{ hence from (4.3.30) } T_n(0) = 0 . \text{ Now (4.3.32)}
\]

implies that \( T_i(x,t) = \sum_{n=1}^{\infty} X_{n}(x) \int_{0}^{t} r_n(u)e^{-\beta_n^2(u-t)} du . \) From (4.3.29) we obtain \( r_n(t) = -\frac{k_n}{N} \left( \frac{8.194t^2 - 2.074t}{X_{n}(x_1)} \right) . \) Therefore

\[
T_i(x,t) = \sum_{n=1}^{\infty} \frac{X_{n}(x_1)}{N} X_{n}(x)D_n(t)
\]

where

\[
D_n(t) = \frac{-2.198t^2}{\beta_n^2} + \frac{10.677t}{\beta_n^2} + \frac{4.396t}{\beta_n^4} - \left( \frac{10.677}{\beta_n^4} + \frac{4.396}{\beta_n^6} \right) \left( 1 - e^{-\beta_n^2t} \right)
\]

\( X_{n}(x) \) are given by (4.3.13-4.3.14). \( X_{n}(x_1) = \frac{C_{1n}}{\cos(-8.784\beta_n)} \)

\[
N = C_{1n} \left[ 48.2 - \frac{569}{\beta_n} \sin(-8.784\beta_n) + \frac{1.138}{\beta_n} \tan(-8.784\beta_n) \cdot \cos(-8.784\beta_n) + 5 \tan^2(-8.784\beta_n) \cdot \frac{569}{\beta_n} \sin(-8.784\beta_n) + \frac{541}{\beta_n} \sin(79.996\beta_n) \right. \]

\[
\left. \frac{1.081}{\beta_n} \tan(39.998\beta_n) \cdot \cos(79.996\beta_n) + 42.2 \tan^2(39.998\beta_n) - \frac{541}{\beta_n} \tan^2(39.998\beta_n) \cdot \sin(79.996\beta_n) \right]
\]

These results are plotted in figure 6 for \( t = 1 \) and \( t' = 6 \) hours.
Chapter V

Thermal Stratification In Stagnant Saline Lakes.

Linear density–depth model.

From the practical point of view, the two-layered model is not an adequate representation. From the experimental observations, the number of the layers changes with time and a series of layers forms in succession, from the bottom up [32, p.262]. On the other hand, linear salinity gradients can be formed easily in the laboratory for experimental purposes [35; 36]. Two tanks of the same shape, containing equal depths of salt solution of different densities and at room temperature are joined by a wide pipe. The liquid in the lighter tank is vigorously stirred while its gradually varying contents are slowly drained into the bottom of the experimental tank. If the rate of flow out of the stirred tank is just twice that into it then the resulting density gradient will be linear with the denser liquid at the bottom and the less dense liquid at the top.

Further attempts to refine the analysis considered the linear density–depth model, the density dependence on depth is a linear function. Although the detailed structure of density distribution in lakes is not clear, Hirschmann [23], Von Kálerczinsky [26], Hoare [27], and Lazier [28] have suggested a linear density–depth distribution. Hoare [27], reported that Lake Vanda is of unusual interest. Its bottom temperature is 25°C in a region the mean annual air temperature is about 20°C. Its strong density stratification caused by a salt concentration which increases with depth from zero at the top up to 10 percent by weight at the bottom.
Consider the density-depth distribution \( \rho = a_0 + b_0x \), as shown in figure 7 where \( a_0 \) and \( b_0 \) are constants. In general \( a_0 \) and \( b_0 \) are functions of time, temperature, pressure, and salt concentration in the lake. For the purpose of analysis, assume that \( a_0 \) and \( b_0 \) are known constants which can be determined from measurements.

**Heat transfer equation with linear density-depth.**

Consider the well known one dimensional heat transfer equation.

\[
\frac{\partial T(x,t)}{\partial t} = \frac{k}{\rho c} \frac{\partial^2 T(x,t)}{\partial x^2} + \frac{H(x,t)}{\rho c}, \quad 0 \leq x \leq h, \quad t > 0
\]  

(5.1)

where

\[
\phi = (1 - \beta)\phi_0 e^{-\eta x}
\]

\[
H = -\frac{\partial \phi}{\partial x} = (1 - \beta)\eta \phi_0 e^{-\eta x}
\]

Therefore, equation (5.1) becomes

\[
\frac{\partial T(x,t)}{\partial t} = \frac{k}{\rho c} \frac{\partial^2 T(x,t)}{\partial x^2} + \frac{(1-\beta)\phi_0 \eta}{\rho c} e^{-\eta x}
\]  

(5.2)

To solve this second order, linear parabolic differential equation, two boundary conditions and one initial condition are required. Consider the surface boundary condition

\[
T(x,0,t) = B(t)
\]  

(5.3)

where \( B(t) \) is a known given function of time, which contains the effect of evaporation and back radiation at the surface. Only a small fraction of the energy absorbed by the water is immediately returned into the atmosphere by longwave radiation from the uppermost few millimeters,
Raschke [29]. The bottom boundary condition is

\[ T_x(h, t) = 0 \] (5.4)

which is corresponding to insulated bed, figure 8. The initial condition is in general

\[ T(x, 0) = f(x) \] (5.5)

where \( f(x) \) is a known function of depth. At fixed time the lakes are in the isothermal condition and \( f(x) \) is a known constant, Cohen [30]. To solve equation (5.2) subjected to the nonhomogeneous conditions (5.3–5), we use the method of parameters [9]. Consider the transformation

\[ T(x, t) = U(x, t) + K(x, t) . \]

Choose \( K(x, t) \) such that the boundary conditions (5.3–4) reduce to homogeneous boundary conditions.

\[ K(x, t) = xB(t)(1 - \frac{x}{2h}) . \]

Therefore

\[ T(x, t) = xB(t)(1 - \frac{x}{2h}) + U(x, t) . \] (5.6)

After using this transformation (5.6) the heat transfer equation (5.2) becomes

\[ \frac{\partial U(x, t)}{\partial t} = \frac{k}{\rho c} \frac{\partial^2 U(x, t)}{\partial x^2} + q(x, t) \] (5.7)

where

\[ q(x, t) = -xB'(t) + \frac{x^2}{2h} B'(t) - \frac{kb(t)}{\rho c h} - \frac{(1 - \beta)\phi_0 n\rho c \tau x}{\rho c h} . \] (5.8)

Primes denote differentiations with regard to \( t \). And the boundary conditions (5.3–4) transform to
\[
\begin{align*}
U_x(0,t) &= 0 \\
U_x(h,t) &= 0
\end{align*}
\] (5.9)
\[U_x(0,t) = 0\] (5.10)

The initial condition becomes
\[U(x,0) = g(x)\] (5.11)

where \(g(x) = f(x) - xB(0) + \frac{x^2}{2h} B(0)\).

The related homogeneous problem is
\[
\begin{cases}
(D.E.) & U_t(x,t) - \frac{U_{xx}}{a + bx} = 0 \\
(B.C.s) & U_x(0,t) = 0 \\
(I.C.) & U(x,0) = g(x)
\end{cases}
\] (5.12)

where \(a = \frac{ca^0}{k}\) and \(b = \frac{cb^0}{k}\).

The solution of the homogeneous system (5.12) by the method of eigenfunction expansions leads to the eigenvalue problem
\[
\begin{align*}
(D.E.) & \quad X'' + \lambda (a + bx) X = 0, \quad 0 \leq x \leq h \\
(B.C.s) & \quad X'(0) = X'(h) = 0
\end{align*}
\] (5.13) (5.14)

Primes denote differentiations with regard to \(x\). \(X\) is the eigenfunction and \(\lambda\) is the corresponding eigenvalue. Change the independent variable from \(x\) to \(V\) by the substitution \(\lambda (a + bx) = V\) (5.15)

Substitute (5.15) in (5.13) we obtain
\[
\frac{d^2 X}{dV^2} + \frac{V}{\lambda^2 b^2} X = 0 \quad \text{where} \quad b \neq 0 \quad \text{and} \quad \lambda \neq 0.
\] (5.16)

Change the dependent variable from \(X\) to \(U\) by the substitution
\[ x = \sqrt{V} U \]  

Substituting (5.17) in (5.16) we obtain

\[ \frac{d^2 U}{dV^2} + \frac{1}{V \frac{dU}{dV}} - \frac{U}{4V^2} + \frac{UV}{\lambda b^2} = 0 . \]  

(5.18)

Let \( W = \frac{2}{\lambda b^3} V^{3/2} \). By the chain rule of differentiation, we obtain

\[ \frac{dU}{dV} = \frac{3}{\lambda^3 b^3} V^{1/2} \frac{dU}{dw} \]

\[ \frac{d^2 U}{dV^2} = \frac{3V^{-1/2}}{2\lambda^3 b^3} \frac{dU}{dw} + \frac{9V}{\lambda^6 b^6} \frac{d^2 U}{dw^2} . \]

Therefore, equation (5.18) becomes

\[ W^2 \frac{d^2 U}{dw^2} + \frac{dU}{dw} + U \left( \frac{\lambda^4 b^4 w^2}{9} - \frac{1}{9} \right) = 0 . \]  

(5.19)

Let \( Z = \frac{\lambda^2 b^2}{3} W \). By the chain rule of differentiation, we obtain

\[ \frac{dU}{dw} = \frac{\lambda^2 b^2}{3} \frac{dU}{dz} \]

\[ \frac{d^2 U}{dw^2} = \frac{\lambda^4 b^4}{9} \frac{d^2 U}{dz^2} . \]

Equation (5.19) becomes

\[ Z^2 \frac{d^2 U}{dz^2} + Z \frac{dU}{dz} + U(z^2 - \frac{1}{9}) = 0 . \]  

(5.20)

The solution of (5.20) in terms of Bessel functions is, (Whittaker [31])

\[ U = AJ_\frac{1}{3} (Z) + BJ_{-\frac{1}{3}} (Z) \]
where $A$ and $B$ are arbitrary constants to be determined.

Therefore the solution of (5.13) is

$$X(x) = Q(x) \sqrt[3]{(a+bx)} .$$  \hspace{1cm} (5.21)

Where $Q(x)$ is

$$AJ \left( \frac{2\sqrt{A}}{3b} (a+bx)^{3/2} \right) + BJ \left( \frac{2\sqrt{A}}{3b} (a+bx)^{3/2} \right).$$

To find $X'(x)$, (where prime denotes differentiation with regard to $x$), we use the following identities

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

and

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x).$$

Let $\frac{2\sqrt{A}}{3b} (a+bx)^{3/2} = y$. Under this substitution equation (5.21) becomes

$$X = \left( \frac{3\lambda b}{2} \right)^{1/3} \left[ Ay \frac{1}{3} J_1(y) + By \frac{1}{3} J_{1}(y) \right].$$  \hspace{1cm} (5.22)

By the chain rule of differentiation

$$X' = \frac{dX}{dy} \cdot \sqrt[3]{(a+bx)} .$$

Therefore

$$X' = \lambda (a+bx) P(x) \hspace{1cm} (5.23)$$

where $P(x)$ is

$$AJ \left( \frac{2\sqrt{A}}{3b} (a+bx)^{3/2} \right) - BJ \left( \frac{2\sqrt{A}}{3b} (a+bx)^{3/2} \right).$$

From the conditions (5.14), we obtain
\[ AJ \left( -\frac{2}{3} \frac{2\sqrt{\lambda}}{3b} a^2 \right) - BJ \left( \frac{2\sqrt{\lambda}}{3} \frac{3}{a^2} \right) = 0 \]  
(5.24)

\[ AJ \left( -\frac{2}{3} \frac{2\sqrt{\lambda}}{3b} \left(a+\frac{bh}{2}\right)^2 \right) - BJ \left( \frac{2\sqrt{\lambda}}{3} \frac{3}{a^2} \left(a+\frac{bh}{2}\right)^2 \right) = 0 \]  
(5.25)

such that \( \lambda \neq 0 \), \( a \neq 0 \), and \( a + bh \neq 0 \). Equations (5.24-5.25) are two homogeneous linear equations in \( A \) and \( B \). For nontrivial solution the following condition must be satisfied:

\[
\begin{vmatrix}
J \left( -\frac{2}{3} \frac{2\sqrt{\lambda}}{3b} a^2 \right) & -J \left( \frac{2\sqrt{\lambda}}{3} \frac{3}{a^2} \right) \\
J \left( -\frac{2}{3} \frac{2\sqrt{\lambda}}{3b} \left(a+\frac{bh}{2}\right)^2 \right) & -J \left( \frac{2\sqrt{\lambda}}{3} \frac{3}{a^2} \left(a+\frac{bh}{2}\right)^2 \right)
\end{vmatrix} = 0
\]

or

\[
\begin{pmatrix}
J \left( -\frac{2}{3} \frac{2\sqrt{\lambda}}{3b} a^2 \right) \\
J \left( -\frac{2}{3} \frac{2\sqrt{\lambda}}{3b} \left(a+\frac{bh}{2}\right)^2 \right)
\end{pmatrix} = \begin{pmatrix}
J \left( \frac{2\sqrt{\lambda}}{3} \frac{3}{a^2} \right) \\
J \left( \frac{2\sqrt{\lambda}}{3} \frac{3}{a^2} \left(a+\frac{bh}{2}\right)^2 \right)
\end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}
\]
(5.26)

The positive roots of this transcendental equation determine the eigenvalues \( \lambda \). There is an infinite number of such roots, yielding an infinite number of eigenvalues. From relation (5.26); \( A \) and \( B \) can be chosen as

\[ A_n = J_2 \left( \frac{2\sqrt{\lambda_n}}{3b} a^2 \right) \quad \text{and} \quad B_n = J_2 \left( -\frac{2\sqrt{\lambda_n}}{3b} a^2 \right) \]  
Therefore, the eigenfunctions

\[ X_n(x) \]  
can be written in the form

\[
-52-
\]
\[ X_n(x) = (Q_{1n}(x) + Q_{2n}(x)) \sqrt{\lambda_n(a+bx)} \]  \hspace{1cm} (5.27)

where

\[ Q_{1n}(x) = J_\frac{2\lambda_n}{3b} \left( \frac{2\lambda_n}{3} \right)^\frac{3}{2} \left( \frac{2\lambda_n}{3b} \right)^\frac{3}{2} \]

and

\[ Q_{2n}(x) = -J \frac{2\lambda_n}{3b} \left( \frac{2\lambda_n}{3} \right)^\frac{3}{2} \left( \frac{2\lambda_n}{3b} \right)^\frac{3}{2} \].

The eigenvalues \( \lambda_n \) are the positive roots of the transcendental equation (5.26). The solution of (5.12) is of the form

\[ u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n^t} X_n(x) \]  \hspace{1cm} (5.28)

where the parameters \( a_n \) are determined by the initial condition. We now use the method of variation of parameters [9], to find a solution of the inhomogeneous problem (5.7-11) in the form of a series like (5.28), but in which the parameters \( a_n \) are replaced by functions of \( t \). The product \( a_n e^{-\lambda_n^t} \) will then be replaced by a function \( T_n(t) \), so that the solution will be a series

\[ u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x) \]  \hspace{1cm} (5.29)

For the eigenfunctions \( X_n(x) \) the normalizing constants can be determined from the orthogonal property.

**Orthogonal property of the eigenfunctions**

Consider the eigenvalue problem (5.13-14)

\[ X_n''(x) + \lambda_n(a+bx)X_n(x) = 0 \hspace{0.5cm} 0 \leq x \leq h \]  \hspace{1cm} (5.30)
\( X_n'(0) = X_n'(h) = 0 \), \( \forall n \) \hspace{1cm} (5.31)

For different \( n \) (say \( n' \)), we have corresponding eigenvalue \( \lambda_{n'} \), 

\( (\lambda_n, \neq \lambda_n) \) and

\( X_n''(x) + \lambda_{n'}(a+bx)X_n'(x) = 0 \), \( 0 < x < h \), \hspace{1cm} (5.32)

\( X_n'(0) = X_n'(h) = 0 \), \( \forall n' \) \hspace{1cm} (5.33)

Multiplying (5.30) by \( X_n(x) \) and (5.32) by \( X_{n'}(x) \) and then subtracting we obtain

\[
\frac{d}{dx} \left( X_n'(x) \frac{dX_n(x)}{dx} - X_{n'}(x) \frac{dX_{n'}(x)}{dx} \right) + (\lambda_n - \lambda_{n'})(a+bx)X_n(x)X_{n'}(x) = 0
\]

Integrating between the limits \( 0, \ h \) and using the boundary conditions (5.31 and 5.33), we obtain

\[
\int_0^h (a+bx)X_n(x)X_{n'}(x)dx = 0 \), \( n \neq n' \).
\]

When \( n = n' \) we obtain

\[
\int_0^h (a+bx)X_n^2(x)dx = \text{Constant (say } C_n)\).
\]

Therefore from (5.29) we obtain

\[
\int_0^h (a+bx)u(x,t)X_n(x)dx = C_nT_n(t)
\]

or
\[ T_n(t) = \frac{1}{c_n} \int_0^h (a+bx)u(x,t)X_n(x)dx. \quad (5.34) \]

We assume that \( u_t(x,t) \) is a continuous function in the region \( t > 0 \), \( 0 \leq x \leq h \). Under these circumstances, the integral in (5.34) has a derivative with regard to \( t \) which can be calculated by differentiation under the integral sign. We get, referring to the D.E. (5.7)

\[ T_n'(t) = \frac{1}{c_n} \int_0^h \left[ q(x,t) + \frac{u_{xx}(x,t)}{a+bx} \right] (a+bx)X_n(x)dx. \]

Prime denotes differentiation with regard to \( t \), \( q(x,t) \) and \( X_n(x) \) are given by (5.8) and (5.27) respectively. That is

\[ T_n'(t) = \frac{1}{c_n} \int_0^h u_{xx}(x,t)X_n(x)dx + \frac{1}{c_n} \int_0^h (a+bx)q(x,t)X_n(x)dx. \quad (5.35) \]

The last term of (5.35)

\[ q_n(t) = \frac{1}{c_n} \int_0^h (a+bx)q(x,t)X_n(x)dx \quad (5.36) \]

is a known function of \( t \), since \( q(x,t) \) is given by (5.8). The first term of (5.35) will be integrated by parts twice and we use the boundary conditions (5.31) and \( u_x(0,t) = u_x(h,t) = 0 \). To justify this operation, we assume that for every fixed \( t > 0 \), \( u(x,t), u_x(x,t) \) and \( u_{xx}(x,t) \) are continuous on the interval \( 0 \leq x \leq h \). We find that

\[ \frac{1}{c_n} \int_0^h u_{xx}(x,t)X_n(x)dx = \frac{1}{c_n} \int_0^h u(x,t)X_n''(x)dx. \quad (5.37) \]
Further, because $X''_n(x) = -\lambda_n(a+bx)X_n(x)$ from (5.30), the integral on the right of (5.37) can be written as
\[
\frac{1}{c_n} \int_0^h u_{xx}(x,t)X_n(x)dx = -\lambda_n \int_0^h u(x,t)(a+bx)X_n(x)dx
\]
\[
= -\lambda_n T_n(t), \text{ from (5.34).}
\]
Therefore, we obtain
\[
T'_n(t) + \lambda_n T_n(t) = q_n(t). \tag{5.38}
\]
Equation (5.38) is a linear first-order ordinary differential equation for $T_n(t)$, in which $q_n(t)$ is a known function given by (5.36). Setting $t = 0$ in (5.34) we get the initial condition
\[
T_n(0) = \frac{1}{c_n} \int_0^h (a+bx)g(x)X_n(x)dx = N \text{ (say)}. \tag{5.39}
\]
The solution of (5.38) which satisfies (5.39) is
\[
T_n(t) = Ne^{-\lambda_n t} + \int_0^t e^{-\lambda_n (t-r)} q_n(r)dr. \tag{5.40}
\]
The coefficients $T_n(t)$ in (5.29) are now completely known and the problem has been solved. We have
\[
u(x,t) = \sum_{n=1}^\infty \left[ Ne^{-\lambda_n t} + \int_0^t e^{-\lambda_n (t-r)} q_n(r)dr \right] X_n(x).
\]
From (5.6) we obtain
\[ T(x,t) = xB(t)(1 - \frac{x}{2h}) + u(x,t) \]  

(5.41)

where

\[ u(x,t) = \sum_{n=1}^{\infty} \left[ -\lambda_n \frac{t}{n} + \int_{0}^{t} e^{-\lambda_n (t-r)} q_n(r) \, dr \right] \, X_n(x) \]

\[ N = \frac{1}{C_n} \int_{0}^{h} (a+bx) g(x) X_n(x) \, dx \]

\[ g(x) = f(x) - xB(0) + \frac{x^2}{2h} B(0) \]

\[ C_n = \int_{0}^{h} (a+bx) X_n^2(x) \, dx \]

\[ X_n(x) = \left[ Q_{1n}(x) + Q_{2n}(x) \right] \sqrt{\frac{\lambda_n}{a+bx}} \]

\[ Q_{1n}(x) = J_{-\frac{2}{3}} \left( \frac{2\sqrt{\lambda_n}}{3b} \frac{a}{2} \right) J_{\frac{3}{2}} \left( \frac{2\sqrt{\lambda_n}}{3b} (a+bx)^{\frac{3}{2}} \right) \]

\[ Q_{2n}(x) = J_{\frac{2}{3}} \left( \frac{2\sqrt{\lambda_n}}{3b} \frac{a}{2} \right) J_{-\frac{3}{2}} \left( \frac{2\sqrt{\lambda_n}}{3b} (a+bx)^{\frac{3}{2}} \right) \]

\( \lambda_n \) are the positive roots of the transcendental equation.
\[
\begin{align*}
J_2 \left( \frac{2\sqrt{\lambda}}{3b} a^2 \right) & = J_2 \left( \frac{2\sqrt{\lambda}}{3b} (a+b h)^2 \right) \\
J_2 \left( \frac{2\sqrt{\lambda}}{3b} a^2 \right) & = J_2 \left( \frac{2\sqrt{\lambda}}{3b} (a+b h)^2 \right)
\end{align*}
\]

\[
q_n(t) = \frac{1}{c_n} \int_0^h (a+bx)q(x,t)X_n(x)dx
\]

\[
q(x,t) = -xB'(t) + \frac{x^2}{2h} B''(t) - \frac{kB(t)}{\rho Ch} + \frac{(1-B)\phi_0 \eta e^{-\eta x}}{\rho C}
\]

\[
a = \frac{a_0 c}{k}, \quad b = \frac{b_0 c}{k}
\]

\[
\rho = a_0 + b_0 x.
\]

Therefore, exact analytical solutions have been found (5.41), for the vertical temperature distribution in a stagnant saline lake, subject to surface heating. The propagation of heat has been assumed to be due solely to molecular diffusion as Harleman (2) has suggested, convection and eddy diffusion being neglected. Linear density-depth mathematical model has been considered.

**Recommendations for further work**

1. Harleman (5) estimated and measured the value of $\beta$, $\phi_0$, $\eta$, $\alpha$, $\rho$ and $C$ in pure water lakes. These functions and parameters should be determined in the saline lakes.

2. Consideration of the possible density-depth distributions

\[
\rho = A + B(1-e^{-ax}) \quad \text{and} \quad \rho = e^{-\frac{A}{B+x}} \quad \text{where} \quad A, B \text{ and } a \text{ are constants.}
\]
(3) Consideration of the eddy diffusivity in the governing differential equation and boundary condition.

(4) Consideration of heat convection, Turner [32].

(5) Consideration of salt diffusion, Turner [32] and Joseph [33].

**Experimental Application**

As an application of the linear density-depth model we will consider the following case.

\[ a_0 = 1.05 \text{ gm./cm.}^3, \quad b_0 = 0.003 \text{ gm./cm.}^4, \quad h = 100 \text{ cm.}, \quad C = 0.8 \text{ Cal./gm.}^\circ \text{C}, \]

\[ k = 4.86 \text{ hr.}^{\circ} -1, \quad a = \frac{Ca_0}{k} = 0.173, \quad b = \frac{Cb_0}{k} = 0.00049, \quad B(t) = 0, \]

\[ \beta = 1 \quad \text{and} \quad T(x;0) = 1^\circ \text{C}. \]  

Equation (5.41) reduces to

\[ T(x,t) = \sum_{n=1}^{\infty} T_n(0)e^{-\lambda_n t}X_n(x) \]

where \( \lambda_n \) is the solution of (5.26), \( X_n(x) \) is given by (5.27) and \( T_n(0) \) is given by (5.39).

\[ C_n = \int_{0}^{h} (a+bx)X_n^2(x)dx. \]

To evaluate this integral [see 25 P. 494, 37 P. 484] and the result is

\[ C_n = 0.08\lambda_n\left[A_1+A_2+A_3+A_4+A_5\right] - 3.52\lambda_n\left[B_1+B_2+B_3+B_4+B_5\right] \]

where

\[ A_1 = \frac{2\frac{\pi}{3}}{\frac{\pi}{3}}J_1^2(d_2) \]

\[ A_2 = 2J_2^2(d_1)J_2^2(d_2) \]

\[ A_3 = -\frac{2}{3}J_2(d_1)J_1(d_2) \]

\[ A_4 = \frac{2}{3}J_2^2(d_1)J_2(d_2) \]

\[ A_5 = -\frac{1}{3}J_2^2(d_1)J_2(d_2) \]
\[ A_3 = \frac{2}{3d_2^2} \frac{j_2(d_1)}{j_2(d_2)} \frac{j_2(d_1)}{j_2(d_2)} \frac{j_1(d_2)}{j_2(d_2)} \]
\[ A_4 = \frac{j_2^2(d_1)}{\frac{2}{3}} \frac{j_1^2(d_2)}{\frac{1}{3}} \]
\[ A_5 = \frac{2}{3d_2} \frac{j_2^2(d_1)}{\frac{2}{3}} \frac{j_2(d_2)}{\frac{1}{3}} \frac{j_1(d_2)}{\frac{1}{3}} \]
\[ B_1 = \frac{j_2^2(d_1)}{\frac{3}{3}} \frac{j_1^2(d_1)}{\frac{1}{3}} \]
\[ B_2 = \frac{2j_2(d_1)}{\frac{3}{3}} \frac{j_2(d_1)}{\frac{2}{3}} \frac{j_1(d_1)}{\frac{1}{3}} \]
\[ B_3 = \frac{2}{3d_1} \frac{j_2^2(d_1)}{\frac{2}{3}} \frac{j_2(d_1)}{\frac{1}{3}} \]
\[ B_4 = \frac{j_2^2(d_1)}{\frac{3}{3}} \frac{j_1^2(d_1)}{\frac{1}{3}} \]
\[ B_5 = \frac{2}{3d_1} \frac{j_2^2(d_1)}{\frac{2}{3}} \frac{j_2(d_1)}{\frac{2}{3}} \frac{j_1(d_1)}{\frac{1}{3}} \]
\[ d_1 = 97.9\sqrt{\frac{\lambda_n}{\lambda}} \]
\[ d_2 = 14.76\sqrt{\frac{\lambda}{\lambda_n}} \]

To evaluate the values of \( C_n \) we used the table of Bessel Functions of Fractional order [38], and the results are \( C_1 = -0.001, \quad C_2 = -0.00094 \)
and \( C_3 = -0.001 \).
\[ T_n(0) = \frac{1}{c_n} \int_0^h (a+bx)X_n'(x) \, dx \]  
(5.39)

\[ = \frac{1}{c_n} \left( \frac{2}{3} \frac{P_1}{J_2(P_2)} \left[ \frac{2}{3} J_2(P_2) - \frac{2}{3} J_2(P_3) \right] \right) \]

\[ + \frac{1}{c_n} \left( \frac{2}{3} \frac{P_1}{J_2(P_2)} \left[ \frac{2}{3} J_2(P_3) - \frac{2}{3} J_2(P_2) \right] \right) \]

where \( P_1 = \frac{2\sqrt{\lambda_n}}{3b} \), \( P_2 = P_1a^\frac{3}{2} \) and \( P_3 = \frac{2\sqrt{\lambda_n}(a+bb)^2}{3b} \). Equation (5.41)

reduces to

\[ T(x,t) = \sum_{n=1}^{\infty} V_n(x,t) \left[ \frac{J_2(d_1)J_2(d_1)}{3} - \frac{2}{3} \right] \]

\[ J_1(d_3) + J_2(d_1)J_1(d_3) \] \( \cdot Q(n) \)

where

\[ V_n(x,t) = \frac{e^{-\lambda_n t} \sqrt{\lambda_n} (.173+.00049x)}{c_n} \]

\[ Q(n) = J_2(d_1) \cdot \left( \frac{.0004}{2^\frac{3}{2}} \lambda_n^\frac{2}{3} - .173 \right) - J_2(d_4) \cdot \left( \frac{.0003}{2^\frac{3}{2}} \lambda_n^\frac{2}{3} - .222 \right) \]

\[ d_3 = 1360.5\sqrt{\lambda_n} (.173+.00049x)^{\frac{3}{2}} \]

\[ d_4 = 142.3\sqrt{\lambda_n} \]

To find the values of \( \lambda_n \) we solved (5.26).
\[
\frac{\frac{3}{2} J_{\frac{3}{2}}(d_1)}{\frac{3}{2} J_{\frac{3}{2}}(d_2)} = \frac{\frac{3}{2} J_{\frac{3}{2}}(d_1)}{\frac{3}{2} J_{\frac{3}{2}}(d_2)}
\]

We found the points of intersection of \(\frac{\frac{3}{2} J_{\frac{3}{2}}(d_1)}{\frac{3}{2} J_{\frac{3}{2}}(d_1)}\) and \(\frac{\frac{3}{2} J_{\frac{3}{2}}(d_2)}{\frac{3}{2} J_{\frac{3}{2}}(d_2)}\) using the tables of Bessel Functions of Fractional order [38], and the results are

\[
\lambda_0 = 0, \quad \lambda_1 = 0.0027, \quad \lambda_2 = 0.0048, \quad \lambda_3 = 0.0076, \quad \lambda_4 = 0.0097, \\
\lambda_5 = 0.011, \quad \lambda_6 = 0.013 \quad \text{and} \quad \lambda_7 = 0.015. 
\]

A graphical method was used to determine these eigenvalues. To check the accuracy of these values, let \(b\) tend to zero, from (5.13) we obtain

\[
x'' + \lambda a x = 0, \quad 0 \leq x \leq h
\]

(5.42)

which corresponds to the constant density case (Chapter II). From (5.42) and (5.14) the eigenvalues are \(\lambda_n = \left(\frac{\pi n}{h}\right)^2 \frac{1}{a}, \quad n = 1, 2, \ldots\) and

\[
\lambda_{n+1} = (1 + \frac{1}{n})^2 \lambda_n. 
\]

(5.43)

In equation (5.26) let \(\frac{2\sqrt{\lambda}}{3b} \frac{2}{a^2} = x_1\) and \(\frac{2\sqrt{\lambda}}{3b} (a + bh)^2 = x_2\). As \(b\) tends to zero \(x_1\) and \(x_2\) tend to infinity. For large values of \(z\), the asymptotic behavior of the Bessel function \(J_\theta(z)\) is given by

\[
J_\theta(z) \sim \sqrt{\frac{2}{\pi z}} \cos[z - (\theta + \frac{1}{2}) \pi].
\]

Therefore from equation (5.26) we obtain
\[
\cos[x_1 - \left(\frac{2}{3} + \frac{1}{2}\right)\pi] \cos[x_2 - \left(\frac{2}{3} + \frac{1}{2}\right)\pi] - \\
\cos[x_1 - \left(-\frac{2}{3} + \frac{1}{2}\right)\pi] \cos[x_2 - \left(-\frac{2}{3} + \frac{1}{2}\right)\pi] = 0.
\]

(5.44)

After using the relation
\[
\cos a \cos b = \frac{1}{2} [\cos(a+b) + \cos(a-b)]
\]
equation (5.44) becomes
\[
\sin(x_1 - x_2) = 0
\]
which indicates that
\[
\frac{2\sqrt{\lambda_n}}{3b} \frac{3}{a^2} - \frac{2\sqrt{\lambda_n}}{3b} \frac{3}{(a+b)^2} = n\pi.
\]

Therefore
\[
\lambda_n = \left(\frac{3bn\pi}{2}\right)^2 \left[ \frac{3}{2} \frac{1}{a^2} \frac{3}{(a+b)^2} \right]^{\frac{1}{2}}
\]

(5.45)

as \( b \) tends to zero.

(5.45) indicates that \( \lambda_n = \left(\frac{mn}{h}\right)^2 \frac{1}{a} \) which is the same as the constant density case. Equation (5.45) gives \( \lambda_{n+1} = (1 + \frac{1}{n})^2 \lambda_n \) which is similar to (5.43). The method of false position was used (Appendix F) to determine accurate values of the eigenvalues. The results are \( \lambda_1 = .064, \lambda_2 = .233, \lambda_3 = .498, \lambda_4 = .854, \lambda_5 = 1.754 \) and \( \lambda_6 = 2.217 \). The results of the temperature distribution are shown in (Appendix F, SUM).
APPENDIX A

TI-59 programmable hand calculator was used to find the sum of series (2.4.2), for Lake Tahoe. Substituting the values of the functions and parameters: $\phi_0 = 6500000 \text{ Cal/m}^2\text{-day}$, $\eta = .048 \text{ m}^{-1}$, $\beta = .4$, $\alpha = .012416 \text{ m}^2/\text{day}$, $\rho = .998 \times 10^3 \text{ gm/m}^3$, $c = 1.0 \text{ Cal/gm}^\circ\text{C}$, $h = 501 \text{ m}$, $B(t) = 0$, $\phi_L = \beta \phi_0 = 2600000 \text{ Cal/m}^2\text{-day}$, $T(h, t) = T_b = 4^\circ\text{C}$, $\eta(1-\beta)\phi_0 = \frac{\eta(1-\beta)\phi_0}{\rho c} = 1876 ^\circ\text{C/day}$. In (2.4.2), consider $t = 0$ corresponding to March 21 we obtain

$$
T(x, t) = T_b + \sum_{n=0}^{\infty} \phi_n(x) \left[ \int_0^t p_n e^{-\frac{(n+.5)\pi}{h} \alpha(u-t)} du \right] (A.1)
$$

where

$$
\phi_n(x) = \cos(n+.5)\frac{x}{h}, \quad p_n = \frac{2\eta h(1876)}{\eta^2 h^2 + \pi^2 (n+.5)^2}.
$$

After integration and substituting the limits, (A.1) reduces to

$$
T(x, t) = 4 + \sum_{n=0}^{\infty} p_n \phi_n(x) \left[ \frac{h}{(n+.5)\pi} \right]^2 \frac{1}{\alpha} \left. \left\{ 1 - e^{-\frac{(n+.5)\pi}{h} \alpha t} \right\} \right. (A.2)
$$

To find the sum of the series in (A.2) we use the following TI-59 program:

CLR

2nd CP

LRN
To start the execution of the program, substitute the value of \( t \) in
C. Store the value of \( x \) and press A then B. Find the sum of the series corresponding to \( t \) and \( x \) then clear the display. Store another value of \( x \) and press A then B. Find the sum of the series corresponding to the same \( t \) and the new \( x \) then clear the display and etc. Substitute another value of \( t \) in C and find the sum of the series corresponding to different \( x \)'s. The results are shown in Table A-1.
TABLE A-1

Temperature distribution in Lake Tahoe

<table>
<thead>
<tr>
<th>x (depth in m.)</th>
<th>t (time in days)</th>
<th>t = 40</th>
<th>t = 120</th>
<th>t = 150</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>10.764</td>
<td>24.212</td>
<td>29.275</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>8.730</td>
<td>18.205</td>
<td>21.715</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>6.845</td>
<td>12.544</td>
<td>14.686</td>
</tr>
<tr>
<td>30</td>
<td></td>
<td>5.794</td>
<td>9.350</td>
<td>10.739</td>
</tr>
<tr>
<td>40</td>
<td></td>
<td>5.091</td>
<td>7.198</td>
<td>8.099</td>
</tr>
<tr>
<td>50</td>
<td></td>
<td>4.674</td>
<td>6.066</td>
<td>6.583</td>
</tr>
<tr>
<td>60</td>
<td></td>
<td>4.416</td>
<td>5.251</td>
<td>5.564</td>
</tr>
<tr>
<td>70</td>
<td></td>
<td>4.241</td>
<td>4.796</td>
<td>4.995</td>
</tr>
<tr>
<td>80</td>
<td></td>
<td>4.158</td>
<td>4.474</td>
<td>4.593</td>
</tr>
<tr>
<td>90</td>
<td></td>
<td>4.090</td>
<td>4.309</td>
<td>4.386</td>
</tr>
<tr>
<td>100</td>
<td></td>
<td>4.060</td>
<td>4.177</td>
<td>4.222</td>
</tr>
</tbody>
</table>

These values of temperatures are plotted in figure 1. To watch the step by step change in the sum of the series we used the 2nd pause instruction.

To study the change in the solution (2.4.2) due to small changes in the boundary conditions we considered the case B(t) = .1 and B(t) = -.1 for t = 150 days.
For $B(t) = 0.1$,

(2.4.2) reduces to

$$T(x, t) = 4 + 0.1(x-501) + \sum_{n=0}^{\infty} \phi_n(x)[a_n e^{-b_n} + c_n (1-e^{-n})]$$

(A.3)

where

$$\phi_n(x) = \cos \left( \frac{(n+.5)\pi x}{501} \right), \quad a_n = \frac{10.1523826}{(n+.5)^2}$$

$$b_n = 0.000432314(n+.5)^2$$

$$c_n = \frac{18481439.52}{578.306304(n+.5)^2 + \pi^2(n+.5)^4}$$

$$d_n = 0.000732314(n+.5)^2$$

To find the sum of the series in (A.3) we use the following TI-59 program

CLR

2nd CP

LRN

[A] 2nd Lbl A

STO 05 \( (x) \)

R/S

[B] 2nd Lbl B

0 STO 01 \( (n) \)

CLR STO 02

[C] 2nd Lbl C

$$(((RCL01+.5) \times 180 \times RCL05 \div 501) \times 2nd \cos x(10.1523826 \times (0.000432314 \times (RCL01+.5)x^2))$$

$$\div (RCL01+.5)x^2) + (((RCL01+.5) \times 180 \times RCL05 \div 501) \times 2nd \cos x((18481439.32$$

$$\times (1-(0.000732314 \times (RCL01+.5)x^2) \times \text{INV } 1nx)) \div (578.306304 \times (RCL01+.5)x^2)$$

$$+ 9.869604401 \times (RCL01+.5)y^{x/4})) = \text{STO } 04$$
To start the execution of the program store the value of $x$ and press A then B. Find the sum of the series corresponding to $t = 150$ days and $x$ then clear the display. Store another value of $x$ and press A then B. Find the sum of the series corresponding to the new $x$ then clear the display and so on. The results are shown in table A-2.

For $B(t) = -1$

(2.4.2) reduces to

$$T(x, t) = 4 - 1(x-501) + \sum_{n=0}^{\infty} \phi_n(x) [a_n e^{-b_n} + c_n (1-e^{-d_n})]$$

(A.4)

where

$$\phi_n(x) = \cos \left( \frac{(n+.5)\pi x}{501} \right)$$

$$a_n = -\frac{10.1523826}{(n+.5)^2}$$

$$b_n = .0000432314(n+.5)^2$$

$$c_n = \frac{18481439.52}{578.306304(n+.5)^2 + \pi^2(n+.5)^4}$$

$$d_n = .0000732314(n+.5)^2$$
To find the sum of the series in (A.4) we use the following TI-59 program.

```
CLR
2nd CP
LRN

[A] 2nd Lbl A
STO 05
R/S

[B] 2nd Lbl B
0 STO 01
CLR STO 02

[C] 2nd Lbl C

(((RCL01+.5)×180×RCL05÷501)2nd cos × (((+10.1523826×(+.0000432314 × (RCL01+.5)x^2)INV ln x)) ÷ (RCL01+.5)x^2)) + (((RCL01+.5) × 180×RCL05 ÷ 501) 2nd cos x ((18481439.32×(1+(+.0000732314×(RCL01+.5)x^2))INV ln x)) ÷ (578.306304×(RCL01+.5)x^2 + 9.8699604401 × (RCL01+.5)y^4))) = STO 04
RCL 04 SUM 02
1 SUM 01
RCL 02
2nd Pause
Go To C
LRN.
```

The execution of the program as in previous case. The results are shown in Table A-2.
TABLE A-2

Sensitivity of computed values of temperatures to small changes in the surface boundary condition for Lake Tahoe, t = 150 days

<table>
<thead>
<tr>
<th>x (depth in m.)</th>
<th>T(x,t) (temperature in °C)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B(t)</td>
</tr>
<tr>
<td></td>
<td>-0.1</td>
</tr>
<tr>
<td>0</td>
<td>29.656</td>
</tr>
<tr>
<td>10</td>
<td>21.690</td>
</tr>
<tr>
<td>30</td>
<td>10.610</td>
</tr>
<tr>
<td>40</td>
<td>8.100</td>
</tr>
<tr>
<td>50</td>
<td>6.537</td>
</tr>
<tr>
<td>60</td>
<td>5.610</td>
</tr>
<tr>
<td>70</td>
<td>4.950</td>
</tr>
<tr>
<td>80</td>
<td>4.640</td>
</tr>
<tr>
<td>90</td>
<td>4.303</td>
</tr>
<tr>
<td>100</td>
<td>4.205</td>
</tr>
</tbody>
</table>

These values of temperatures are plotted in figure 2.


APPENDIX B

For the convergence of the series (2.4.2), substitute the values of the functions and parameters

\[ \phi_n(x) = \cos \frac{(2n+1)\pi x}{2h}, \quad T_b = f(x) = 4, \]

\[ 0 \leq x \leq h, \quad t > 0 \quad \text{and} \quad B(t) = 0 \] we obtain

\[ T(x,t) = 4 + \sum_{n=0}^{\infty} C_n [1 - e^{-k_1(n+\frac{1}{2})^2 t}] \phi_n(x) \]  

(B.1)

where

\[ C_n = \frac{k_2}{n^2 h^2 (n+\frac{1}{2})^4 + \pi^2 (n+\frac{1}{2})^4} \]

\[ k_1 = \frac{\alpha h^2}{2} \quad \text{and} \quad k_2 = \frac{2\pi h^3}{\alpha n^2}. \]

In the region \( 0 \leq x \leq h, \ t \geq t_0, \ t_0 > 0 \)

\[ |C_n [1 - e^{-k_1(n+\frac{1}{2})^2 t_0}] \phi_n(x)| \leq C_n [1 - e^{-k_1(n+\frac{1}{2})^2 t_0}] \leq C_n \]

for each \( n \) and \( 0 \leq x \leq h, \ t \geq t_0, \ t_0 > 0 \). By the comparison test \( \sum_{n=0}^{\infty} C_n \) is a convergent series of positive constants. It follows by the Weierstrass M-test [34] that the series (B.1) converges uniformly in the given region.

In like manner we see that

\[ \frac{\partial T(x,t)}{\partial t} = \sum_{n=0}^{\infty} k_1(n+\frac{1}{2})^2 C_n e^{-k_1(n+\frac{1}{2})^2 t} \phi_n(x) \]  

(B.2)

converges uniformly in the region \( 0 \leq x \leq h, \ t \geq t_0, \ t_0 > 0 \), because
\[ |k_1 (n+\frac{1}{2})^2 c_n e^{-k_1 (n+\frac{1}{2})^2 t} \phi_n(x)| \leq \sqrt{k_1 (n+\frac{1}{2})^2 c_n e^{-k_1 (n+\frac{1}{2})^2 t_0}} = P_n \]

for each \( n \) and \( 0 \leq x \leq h \), \( t \geq t_0 \), \( t_0 > 0 \). By the ratio test

\[ \sum_{n=0}^{\infty} P_n \]

is a convergent series of positive constants. It follows by the Weierstrass M-test that (B.2) converges uniformly in the given region.

Similarly

\[ \frac{\partial^2 T(x,t)}{\partial x^2} = \sum_{n=0}^{\infty} \frac{-(n+\frac{1}{2})^2 \pi^2}{h^2} c_n [1-e^{-(\frac{1}{2})^2 \frac{n+\frac{1}{2})^2 t}{h^2}} \cdot \phi_n(x) \]

converges uniformly in the given region.
APPENDIX C

Calculation of the sum of the series (2.4.2) by the TI-59 programmable hand calculator for the M.I.T. Hydrodynamics laboratory tank [see Section 8 Chapter II] for the case of constant insolation and heat loss at the surface proportional to $t^{0.5}$.

The numerical values used were

$$\phi_0 = 0.01 \text{ Cal/cm}^2 \text{ sec}, \quad \phi_L = \beta \phi_0 t^{0.5}, \quad \beta = 0.75, \quad \eta = 0.01 \text{ cm}^{-1},$$

$$h = 75 \text{ cm}, \quad \Omega = 0.004 \text{ sec}^{-1.5}, \quad T(x,0) = f(x) = 21^\circ \text{C}, \quad T(75,t) = T_b = 21^\circ \text{C},$$

$$\rho = 1 \text{ gm/cm}^3, \quad C = 1 \text{ Cal/gm} \cdot \text{C}, \quad \alpha = 0.00144 \text{ cm}^2/\text{sec},$$

$$B(t) = -\frac{1}{\rho \alpha} (\beta \phi_0 - \beta \phi_0 t^{0.5}) = 5.20833(0.004t^{1.5} - 1)$$

$$B(0) = -5.20833, \quad A(t) = \frac{\eta(1-\beta)\phi_0}{\rho C} = 0.000025.$$

From (2.4.2) we obtain

$$T(x,t) = T_b + (x-h)B(t) + \sum_{n=0}^{\infty} \phi_n(x) \cdot \left[ \frac{-\lambda_n t}{(n+.5)^2 \pi^2} \right] + \int_0^{t} \left[ \frac{2h B'(u)}{(n+.5)^2 \pi^2} + p_n A(u) \right] e^{\frac{\lambda_n (u-t)}{\pi^2}} du \right]$$

(C.1)

where

$$\phi_n(x) = \frac{\cos \left( \frac{(n+.5)\pi x}{h} \right)}{h}$$

$$\lambda_n = \frac{(n+.5)^2 \pi^2 \alpha}{h^2} = 0.0000025266(n+.5)^2$$

$$p_n = \frac{2\eta h + 2\pi(n+.5) e^{-\eta h} \sin(n+.5)\pi}{\eta^2 h^2 + (n+.5)^2 \pi^2}$$
After integration and substituting the limits in (C.1) we obtain

\[ T(x, t) = 21 + 5.20833(0.004 t^{-5} - 1)(x-75) \]

\[ + \sum_{n=0}^{\infty} \phi_n(x)[a_n e^{-\lambda_n t} + b_n t e^{-\lambda_n t} + c_n t e^{-\lambda_n t} + d e^{-\lambda_n t} + e] \]

(C.2)

where

\[ a_n = \frac{-79.15712406}{(n+.5)^2} \]
\[ b_n = \frac{-3166284962}{(n+.5)^2} \]

\[ d = 0.0000002667 \]

\[ Q_n = \frac{14.84208027 + 29.36719884(n+.5)\sin(n+.5)\pi}{0.5625(n+.5)^2 + (n+.5)^4 \pi^2} \]

For \( t = \frac{1}{2} \) hr. = 1800 sec., the series in (C.2) reduces to

\[ \sum_{n=0}^{\infty} \phi_n(x)[g_n e^{-r_n} + 0.0203672209 e^{-r_n} + Q_n(1-e^{-r_n})] \]

(C.3)

where

\[ g_n = \frac{-65.72371465}{(n+.5)^2} \]
\[ r_n = 0.00454788(n+.5)^2 \]

To find the sum of (C.3) we use the following TI-59 program

CLR
2nd CP
LRN
[A] 2nd Lbi A
STO 05 (store x)
R/S
[B] 2nd Lbl B
  0 STO 01. (store n)
  CLR STO 02

[C] 2nd Lbl C.

\[
\left(\left((RCL01+.5)\times180\times RCL05 \div 75\right)^{2}\right)\times(1+0.00454788\times(RCL01+.5)^{2}) \text{INV} \ln x + ((RCL01+.5)\times180\times RCL05 \div 75)^{2}\times\cos x (0.0203672209-(65.72371465 \div (RCL01+.5)^{2})) = \\
\left((1-(1+0.00454788\times(RCL01+.5)^{2})\text{INV} \ln x)^{2} + 9.869604401 \times (RCL01+.5)^{y^{4}}\right)
\]

STO 04

RCL04 SUM02

1 SUM 01

RCL 02

2nd Pause

Go To C

LRN

To start the execution of the program, store \( x \), and press A then B.

Find the sum of the series corresponding to \( x \) then clear the display.

Store another \( x \) and press A then B. Find the sum of the series

corresponding to the new \( x \) then clear the display and etc. Substitute

another value of \( t \) in (C.2) and find the corresponding sum of the series

as previous. The results are shown in table C-1.
TABLE C-1

Computed values of temperature in M.I.T. Hydrodynamics laboratory tank: constant insolation and heat loss $a t^{0.5}$

<table>
<thead>
<tr>
<th>$T(x,t)$ (temperature in °C)</th>
<th>1800</th>
<th>7200</th>
<th>14400</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$ (depth in cm.)</td>
<td>1800</td>
<td>7200</td>
<td>14400</td>
</tr>
<tr>
<td>0</td>
<td>29.169</td>
<td>34.748</td>
<td>37.405</td>
</tr>
<tr>
<td>1</td>
<td>26.423</td>
<td>31.582</td>
<td>34.838</td>
</tr>
<tr>
<td>2</td>
<td>23.648</td>
<td>28.953</td>
<td>32.552</td>
</tr>
<tr>
<td>3</td>
<td>21.802</td>
<td>26.833</td>
<td>30.540</td>
</tr>
<tr>
<td>4</td>
<td>21.415</td>
<td>25.173</td>
<td>28.792</td>
</tr>
<tr>
<td>6</td>
<td>21.077</td>
<td>22.992</td>
<td>26.029</td>
</tr>
<tr>
<td>8</td>
<td>21.043</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>21.041</td>
<td>21.415</td>
<td>22.873</td>
</tr>
<tr>
<td>20</td>
<td>21.037</td>
<td>21.149</td>
<td>21.322</td>
</tr>
<tr>
<td>75</td>
<td>21</td>
<td>21</td>
<td>21</td>
</tr>
</tbody>
</table>

These values of temperatures are plotted in figure 3.
APPENDIX D

Calculation of the sum of the series (2.4.2) by the TI-59 programmable hand calculator for the M.I.T. Hydrodynamics laboratory tank [see Section 8 Chapter II] for the case of insolation \( \phi_0 \) and loss \( \phi_L \) is a quadratic function of time given by

\[
\phi_0 = \phi_{\text{max}} \left[ \frac{2t}{t_0} - \left( \frac{t}{t_0} \right)^2 \right], \quad 0 \leq t \leq 2t_0
\]

\[
\phi_L = \frac{t_0}{t_L} \phi_{\text{max}} \left[ \frac{2t}{t_L} - \left( \frac{t}{t_L} \right)^2 \right], \quad 0 \leq t \leq 2t_L
\]

where \( t_0 = 3 \text{ hr.}, \ t_L = 5 \text{ hr.} \)

\[
\phi_{\text{max}} = 25.2 \text{ Cal/cm}^2 \cdot \text{hr}
\]

\[
\phi_0 = 16.8t - 2.8t^2
\]

\[
\phi_L = 6.048t - 0.6048t^2
\]

\( \rho = 1 \text{ gm./cm}^3, \ C = 1 \text{ cal./gm.}^\circ \text{C}, \ \alpha = 5.184 \text{ cm}^2/\text{hr.} \)

\( \beta = 1, \ h = 75 \text{ cm.}, \ \eta = 0.01 \text{ cm}^{-1} \)

\( T(75,t) = T_b = 24.5^\circ \text{C}, \ T(x,t) = f(x) = 24.5^\circ \text{C} \)

\( B(t) = -\frac{1}{\rho \alpha} [\beta \phi_0 - \phi_L] \)

\[
= 0.4234567901t^2 - 2.074074074t
\]

\( A(t) = \frac{\eta(1-\beta)\phi_0}{\rho C} = 0 \)

From (2.4.2) we obtain
\[
T(x,t) = 24.5 + (x-75)B(t) + \sum_{n=0}^{\infty} \phi_n(x)[a_n(t) - b_n(1 - e^{-\lambda_n t})]
\]  

(D.1)

where

\[
\phi_n(x) = \cos \frac{(n+5)\pi x}{75}
\]

\[
a_n(t) = \frac{1415.104134t}{(n+.5)^4}
\]

\[
b_n = \frac{3465.561145}{(n+.5)^4} + \frac{155577.2853}{(n+.5)^6}
\]

\[
\lambda_n = \frac{\alpha n^2 (n+.5)^2}{h^2} = 0.090958274(n+.5)^2
\]

The rise in temperature is equal to \( T_1(x,t) = T(x,t) - 24.5 \).

For \( t = 1 \text{ hr.} \), we obtain the rise in temperature from (D.1)

\[
T_1(x,1) = -1.650617284(x-75) + \sum_{n=0}^{\infty} \phi_n(x)[a_n(1) - b_n(1 - e^{-\lambda_n})]
\]  

(D.2)

To find the sum of the series in (D.2) we use the following TI-59 program

CLR
2nd CP
LRN

[A] 2nd Lbl A
·STO 05  (store x)
R/S

[B] 2nd Lbl B
0 STO 01
CLR STO 02
[C] 2nd Lbl C

\(((\text{RCL01}+.5)\times 180 \times \text{RCL05} \div 75)\times \cos(x) (1415.104134 \div ((\text{RCL01}+.5)y^4)) -
((\text{RCL01}+.5)\times 180 \times \text{RCL05} \div 75)\times \cos(x) (3465.561145 \div ((\text{RCL01}+.5)y^4)) +
(155577.2853 \div ((\text{RCL01}+.5)y^6))) \times (1 - (+0.0090958274)(\text{RCL01}+.5)x^2) \text{INV} \ln x)\)

= STO 04

RCL 04 SUM 02

RCL 02

1 SUM 01

2nd Pause

Go To C

LRN

The execution of the program as in the previous appendix and the results are shown in Table D-1.
TABLE D-1

Computed values of temperature in M.I.T. Hydrodynamics laboratory tank: Parabolic insolation and heat loss.

\( T_1(x, t) \)

(rise in temperature in °C)

<table>
<thead>
<tr>
<th>x</th>
<th>( T_1(t) ) (time in hr.)</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.196, 6.867, 6.966, 6.966, .590</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>.5</td>
<td>2.432, 5.710, 7.0202, 1.861</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.789, 4.692, 6.968, 2.877</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>.845, 3.055, 6.569, 4.191</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>.296, 1.897, 5.854, 4.674</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>.087, 1.135, 4.941, 4.535</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>.406, 3.031, 3.311</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>.0572, .881, 1.275</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>.016, .168, .324</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>.009, .014</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>75</td>
<td>0, 0, 0, 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

These are plotted in figure 4.
Calculation of the eigenvalues $\beta_n$ (4.3.12) and the sum of the
series (4.3.32) for the two layered model.
Very poor print
<table>
<thead>
<tr>
<th>T</th>
<th>1.00</th>
<th>1.00</th>
<th>1.00</th>
<th>2.00</th>
<th>2.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>-10.0</td>
<td>-3.0</td>
<td>-6.0</td>
<td>-10.0</td>
<td>-3.0</td>
</tr>
<tr>
<td>SUM</td>
<td>2.39</td>
<td>1.65</td>
<td>0.50</td>
<td>-0.06</td>
<td>0.96</td>
</tr>
<tr>
<td>X</td>
<td>-10.0</td>
<td>-4.0</td>
<td>-3.0</td>
<td>-10.0</td>
<td>-3.0</td>
</tr>
<tr>
<td>SUM</td>
<td>0.80</td>
<td>0.16</td>
<td>0.96</td>
<td>0.13</td>
<td>0.96</td>
</tr>
</tbody>
</table>
\[
\begin{array}{ll}
T = 2.00 & x = -3.0 \\
\text{SUM} = & 2.37332500 \\
T = 2.00 & x = 0.0 \\
\text{SUM} = & 1.00477700 \\
T = 3.00 & x = -10.0 \\
\text{SUM} = & -0.10243800 \\
T = 3.00 & x = -10.0 \\
\text{SUM} = & 10.43803000 \\
T = 3.00 & x = 0.0 \\
\text{SUM} = & 9.44721100 \\
T = 3.00 & x = 0.0 \\
\text{SUM} = & 4.72520900 \\
T = 3.00 & x = 0.0 \\
\text{SUM} = & 2.15136400 \\
T = 3.00 & x = 0.0 \\
\text{SUM} = & -0.07492000 \\
T = 4.00 & x = -10.0 \\
\text{SUM} = & 12.12822000 \\
T = 4.00 & x = -3.0 \\
\text{SUM} = & 10.31111000 \\
T = 4.00 & x = -3.0 \\
\text{SUM} = & 5.71856800 \\
T = 4.00 & x = -3.0 \\
\text{SUM} = & 3.95092700 \\
T = 4.00 & x = 0.0 \\
\text{SUM} = & 0.19273430 \\
T = 5.00 & x = -10.0 \\
\text{SUM} = & 10.43823000 \\
T = 5.00 & x = -3.0 \\
\text{SUM} = & 9.51711000 \\
T = 5.00 & x = 0.0 \\
\text{SUM} = & 9.72913500 \\
T = 5.00 & x = -4.0 \\
\text{SUM} = & 4.76413000 \\
T = 5.00 & x = 0.0 \\
\text{SUM} = & 2.52753400 \\
T = 6.00 & x = -10.0 \\
\text{SUM} = & 4.09912000 \\
T = 6.00 & x = -3.0 \\
\text{SUM} = & 5.19294000 \\
T = 6.00 & x = -3.0 \\
\end{array}
\]
| SUM  | 5.77943400 |
| SUM  | 4.4743100 |
| SUM  | 0.98941030 |
| SUM  | -0.00002750 |
| SUM  | -0.16032630 |
| SUM  | 0.00029350 |
| SUM  | -0.16329250 |
| SUM  | 0.0784329 |
| SUM  | -0.13283600 |
| SUM  | -0.37417450 |
| SUM  | -0.01347530 |
| SUM  | -0.36757630 |
| SUM  | 0.11493130 |
| SUM  | -0.09490000 |
| SUM  | -0.59381880 |
| SUM  | -0.0292075 |
| SUM  | -0.67213330 |
| SUM  | 0.14710680 |
| SUM  | 0.14229430 |
| SUM  | -0.74230990 |
| SUM  | 0.00000000 |

**VERY POOR PRINT**
\[ \text{SUM} = -0.00000224 \]
\[ T = 4.00 \quad x = 0.0 \]
\[ \text{SUM} = -0.65594457 \]
\[ T = 4.00 \quad x = 0.0 \]
\[ \text{SUM} = -0.12014159 \]
\[ T = 5.00 \quad x = 0.0 \]
\[ \text{SUM} = 0.079835 \]
\[ T = 5.00 \quad x = 0.0 \]
\[ \text{SUM} = -0.70352093 \]
\[ T = 5.00 \quad x = 0.0 \]
\[ \text{SUM} = -0.0001407 \quad \text{STD} \]
\[ T = 6.00 \quad x = 0.0 \]
\[ \text{SUM} = 0.01014305 \]
\[ T = 6.00 \quad x = 0.0 \]
\[ \text{SUM} = 0.19029152 \]
\[ T = 7.00 \quad x = 0.0 \]
\[ \text{SUM} = -0.2041238 \]
\[ T = 7.00 \quad x = 0.0 \]
\[ \text{SUM} = -0.30002153 \]
\[ T = 8.00 \quad x = 0.0 \]
\[ \text{SUM} = -0.00000125 \]
\[ T = 9.00 \quad x = 0.0 \]
\[ \text{SUM} = -0.0076388 \]
\[ T = 10.0 \quad x = 0.0 \]
\[ \text{SUM} = 0.07432342 \]

**Statements Executed:** 5247
APPENDIX F

Calculation of the eigenvalues $\lambda_n$ (5.26) and the sum of the series (5.41)
for the linear model.
$T = 0.0000 \quad x = 0.0000$

$NJAUF\ UFRHMS= 5 \quad TEMP(x,t) = -0.06044750000$

$NJAUF\ UFRHMS= 5 \quad TLWP(x,t) = -0.06044750000$

$T = 0.0000 \quad x = 10.0000$

$NJAUF\ UFRHMS= 5 \quad TEMP(x,t) = -0.0103535150000$

$NJAUF\ UFRHMS= 5 \quad TLWP(x,t) = -0.0103535150000$

$T = 0.0000 \quad x = 20.0000$

$NJAUF\ UFRHMS= 5 \quad TEMP(x,t) = -0.009130520000$

$NJAUF\ UFRHMS= 5 \quad TLWP(x,t) = -0.009130520000$

$T = 0.0000 \quad x = 30.0000$
NUM OF TERMS = 5
NUM OF TERMS = 6
T = 2.0000
X = 40.0000
NUM OF TERMS = 5
NUM OF TERMS = 6
T = 2.0000
X = 50.0000
NUM OF TERMS = 5
NUM OF TERMS = 6
T = 2.0000
X = 60.0000
NUM OF TERMS = 5
NUM OF TERMS = 6
T = 2.0000
X = 70.0000
NUM OF TERMS = 5
NUM OF TERMS = 6
T = 2.0000
X = 80.0000
NUM OF TERMS = 5
NUM OF TERMS = 6
T = 2.0000
X = 90.0000
NUM OF TERMS = 5
NUM OF TERMS = 6
T = 2.0000
X = 100.0000
NUM OF TERMS = 5
NUM OF TERMS = 6
T = 2.0000
X = 100.0000
TIME = 2.0000

TEMP(X,T) = -0.107304860000
TEMP(X,T) = -0.107536100000
TEMP(X,T) = -0.128569320000
TEMP(X,T) = -0.128603200000
TEMP(X,T) = -0.153329000000
TEMP(X,T) = -0.153376600000
TEMP(X,T) = -0.182072700000
TEMP(X,T) = -0.182129300000
TEMP(X,T) = -0.215310400000
TEMP(X,T) = -0.215377400000
TEMP(X,T) = -0.253613400000
TEMP(X,T) = -0.253692300000
TEMP(X,T) = -0.297613800000
TEMP(X,T) = -0.297704000000
TEMP(X,T) = -0.347987400000
TEMP(X,T) = -0.348095700000

VERY POOR PRINT
NUM OF TERMS = 5

\[ T_{\text{mp}}(x, t) = -0.056795150000 \]

NUM OF TERMS = 6

\[ T_{\text{mp}}(x, t) = -0.036795500000 \]

\[ T = 4.0000 \quad x = 40.0000 \]

NUM OF TERMS = 5

\[ T_{\text{mp}}(x, t) = -0.103992900000 \]

NUM OF TERMS = 6

\[ T_{\text{mp}}(x, t) = -0.103993300000 \]

\[ T = 4.0000 \quad x = 50.0000 \]

NUM OF TERMS = 5

\[ T_{\text{mp}}(x, t) = -0.124028000000 \]

NUM OF TERMS = 6

\[ T_{\text{mp}}(x, t) = -0.124028300000 \]

\[ T = 5.0000 \quad x = 60.0000 \]

NUM OF TERMS = 5

\[ T_{\text{mp}}(x, t) = -0.147281400000 \]

NUM OF TERMS = 6

\[ T_{\text{mp}}(x, t) = -0.147282100000 \]

\[ T = 4.0000 \quad x = 70.0000 \]

NUM OF TERMS = 5

\[ T_{\text{mp}}(x, t) = -0.174170700000 \]

NUM OF TERMS = 6

\[ T_{\text{mp}}(x, t) = -0.174171500000 \]

\[ T = 4.0000 \quad x = 80.0000 \]

NUM OF TERMS = 5

\[ T_{\text{mp}}(x, t) = -0.205157700000 \]

NUM OF TERMS = 6

\[ T_{\text{mp}}(x, t) = -0.205158600000 \]

\[ T = 4.0000 \quad x = 90.0000 \]

NUM OF TERMS = 5

\[ T_{\text{mp}}(x, t) = -0.240751600000 \]

NUM OF TERMS = 6

\[ T_{\text{mp}}(x, t) = -0.240752600000 \]

\[ T = 4.0000 \quad x = 100.0000 \]

NUM OF TERMS = 5

\[ T_{\text{mp}}(x, t) = -0.281506400000 \]

NUM OF TERMS = 6

\[ T_{\text{mp}}(x, t) = -0.281507700000 \]

\[ \text{TIme} = 4.0000 \]
NUM OF TERMS = 5  
NUM OF TERMS = 6  
T = 6.0000  
X = 40.0000  
TEMP(X,T) = -0.072563840000  
TEMP(X,T) = -0.072563840000  
NUM OF TERMS = 5  
NUM OF TERMS = 6  
T = 6.0000  
X = 50.0000  
TEMP(X,T) = -0.087268230000  
TEMP(X,T) = -0.087268230000  
NUM OF TERMS = 5  
NUM OF TERMS = 6  
T = 6.0000  
X = 60.0000  
TEMP(X,T) = -0.103337400000  
TEMP(X,T) = -0.103337400000  
NUM OF TERMS = 5  
NUM OF TERMS = 6  
T = 6.0000  
X = 70.0000  
TEMP(X,T) = -0.123506700000  
TEMP(X,T) = -0.123506700000  
NUM OF TERMS = 5  
NUM OF TERMS = 6  
T = 6.0000  
X = 80.0000  
TEMP(X,T) = -0.145820200000  
TEMP(X,T) = -0.145820200000  
NUM OF TERMS = 5  
NUM OF TERMS = 6  
T = 6.0000  
X = 90.0000  
TEMP(X,T) = -0.171764600000  
TEMP(X,T) = -0.171764600000  
NUM OF TERMS = 5  
NUM OF TERMS = 6  
T = 6.0000  
X = 100.0000  
TEMP(X,T) = -0.201566400000  
TEMP(X,T) = -0.201566400000  
NUM OF TERMS = 5  
NUM OF TERMS = 6  
TIME = 6.0000

VERY POOR PRINT
\textbf{NUM OF TERMS = 5}
\hspace{1cm} \textbf{NUM OF TERMS = 6}
\hspace{1cm} \textbf{NUM OF TERMS = 6}
\hspace{1cm} \textbf{NUM OF TERMS = 6}
\hspace{1cm} \textbf{NUM OF TERMS = 6}

\begin{align*}
T &= 5.000 \quad \text{TEMP}(x, T) = -0.0519104000000 \\
T &= 5.000 \quad \text{TEMP}(x, T) = -0.0519104000000 \\
& \quad \quad x = 40.0000 \\
T &= 5.000 \quad \text{TLMP}(x, T) = -0.0741790500000 \\
T &= 5.000 \quad \text{TLMP}(x, T) = -0.0741790500000 \\
& \quad \quad x = 50.0000 \\
T &= 5.000 \quad \text{TEMP}(x, T) = -0.0884721200000 \\
T &= 5.000 \quad \text{TEMP}(x, T) = -0.0884721200000 \\
& \quad \quad x = 60.0000 \\
T &= 5.000 \quad \text{TEMP}(x, T) = -0.0950610000000 \\
T &= 5.000 \quad \text{TEMP}(x, T) = -0.0950610000000 \\
& \quad \quad x = 70.0000 \\
T &= 5.000 \quad \text{TEMP}(x, T) = -0.1242440000000 \\
T &= 5.000 \quad \text{TEMP}(x, T) = -0.1242440000000 \\
& \quad \quad x = 80.0000 \\
T &= 5.000 \quad \text{TEMP}(x, T) = -0.1463503000000 \\
T &= 5.000 \quad \text{TEMP}(x, T) = -0.1463503000000 \\
& \quad \quad x = 90.0000 \\
T &= 5.000 \quad \text{TEMP}(x, T) = -0.1717436000000 \\
T &= 5.000 \quad \text{TEMP}(x, T) = -0.1717436000000 \\
& \quad \quad x = 100.0000 \\
T &= 5.000 \quad \text{TEMP}(x, T) = -0.2003185000000 \\
T &= 5.000 \quad \text{TEMP}(x, T) = -0.2003185000000 \\
& \quad \quad x = \text{TIME} = 8.0000
\end{align*}
APPENDIX G

To obtain a non-dimensional form of the heat equation (2.2.1)

\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + \frac{H(x,t)}{\rho C} \]  \hspace{1cm} (G.1)

Consider the non-dimensional quantities \( x' \) and \( t' \) given by

\[ x' = \frac{x}{h}, \quad t' = \frac{t}{t_1} \]

where \( h \) and \( t_1 \) are characteristics depth and time. (G.1) becomes

\[ \frac{\partial T}{\partial t'} = \alpha \frac{\partial^2 T}{\partial x'^2} + G(x',t') \]  \hspace{1cm} (G.2)

where \( \frac{t_1}{h^2} \alpha = 1 \)

\( h \) is the depth of the lake

\[ t_1 = \frac{h^2}{\alpha} \]

\[ G(x',t') = \frac{h^{2*}}{\rho C} \frac{H(hx', \frac{h^2 t'}{\alpha})}{} \]

(G.2) is the non-dimensional form of the heat equation.

From (2.2.2),

\[ H(x,t) = \eta(1-\beta)\phi_0(t)e^{-\eta x}. \]

Then

\[ G(x',t') = \frac{\eta t_1 (1-\beta)\phi_0(t't_1)}{\rho C} e^{-\eta x'h}. \]
Figure 1
-  ○  t = 40 days
-  ■  120 days
-  ×  150 days
-  ○  ■  computed values
-  ○  ×  observed values
Figure 2
- B(t) = 0.0
- 0.1
- 0.1

$t = 150 \text{ days}$

---

**computed values**
Figure 3

- $t = 1/2$ hr.
- 2 hrs.
- 4 hrs.

- Computed values
- Observed values
Figure 4

- \( t = 1 \text{ hr.} \)
- \( 2'' \)
- \( 5'' \)
- \( 6'' \)
- \( \text{computed values} \)
- \( \text{observed values} \)
Figure 5
FIGURE 5

t=1 hr

t=6 hrs

0 1 2 3 4 5 6 °C

-10 0 10 20 30 40 50 cm
\[ \rho = a_o + b_o x \]

Figure 7
$T(x, t) = B(t)$

Figure 8
References


   Bacterial distribution and production", Limnology and Oceanography,
   Vol. 22, No. 4, July 1977, pp. 621-634.
   Press.
34. Churchill, R. V., "Fourier Series and Boundary Value Problems",
36. Turner, J. S., "The behaviour of a stable salinity gradient heated
   1965, Dover.
38. Tables of Bessel Functions of Fractional Order, Vol. I, N.Y., Columbia
   University Press, 1948.
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